Sensor Fusion with Kalman Filter

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1. Basic Kalman Filter

The Kalman filter estimates a process by using a feedback control. The filter predicts the system state at some time and then obtains feedback based on measurement. The time update (prediction) function of the basic Kalman filter is governed by the linear stochastic equation

$$\boldsymbol{x}_k = \boldsymbol{F}\boldsymbol{x}_{k-1} + \boldsymbol{w}, \tag{1}$$

where $\boldsymbol{x}_k \in \Re^n$ is the state vector and $\boldsymbol{w} \sim N(0, \boldsymbol{Q})$ is the process noise with $\boldsymbol{Q}$ the covariance matrix. The measurement function is written as:

$$\boldsymbol{z_k} = \boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{\nu}, \tag{2}$$

where $\boldsymbol{z}_k \in \Re^m$ is the measurement vector and $\boldsymbol{\nu} \sim N(0, \boldsymbol{R})$ is the process noise with $\boldsymbol{R}$ the covariance matrix.

The equations used in the Kalman filter are listed in Table 1. Here $\boldsymbol{P}_k^-$ and $\boldsymbol{P}_k$ are the priori and posteriori error covariance matrix, respectively, $\boldsymbol{z}_k - \boldsymbol{H}\boldsymbol{x}_k^-$ is called the measurement innovation, and $\boldsymbol{K}_k$ is chosen to minimize the posteriori error covariance.

Table 1 Discrete Kalman filter time and measurement update equations.

Time update (predict)	Measurement update (correct)
$\boldsymbol{x}_k^- = \boldsymbol{F}\boldsymbol{x}_{k-1}$ $\boldsymbol{P}_k^- = \boldsymbol{F}\boldsymbol{P}_{k-1}\boldsymbol{F}^{\boldsymbol{T}} + \boldsymbol{Q}$	$\boldsymbol{K}_k = \boldsymbol{P}_k^-\boldsymbol{H}^{\boldsymbol{T}}\left(\boldsymbol{H}\boldsymbol{P}_k^-\boldsymbol{H}^{\boldsymbol{T}} + \boldsymbol{R}\right)^{-1}$ $\boldsymbol{x}_k = \boldsymbol{x}_k^- + \boldsymbol{K}_k(\boldsymbol{z}_k - \boldsymbol{H}\boldsymbol{x}_k^-)$ $\boldsymbol{P}_k = (\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{H})\boldsymbol{P}_k^-$

In 2D space, the state vector $\boldsymbol{x}_k$ is given by

$$\boldsymbol{x}_k = [x \quad y \quad \dot{x} \quad \dot{y}]^T. \tag{3}$$

Assuming a linear motion in the LIDAR measurement, $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

where Δt is the time step. The measurement vector is $\boldsymbol{z}_k = [x \quad y]^T$, and the measurement matrix is simply

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{5}$$

2. Extended Kalman filter

The RADAR measurement returns

$$\boldsymbol{z_k} = [\rho_k \quad \varphi_k \quad \dot{\rho}_k]^T \tag{6}$$

in the polar coordination system, which means the mapping from the state vector $\boldsymbol{x_k}$ to the measurement vector $\boldsymbol{z_k}$ is no longer linear. A Kalman filter that linearizes about the current mean and covariance is referred to as an extended Kalman filter (EKF).

Generally, assuming the process is governed by the non-linear stochastic equation

$$\boldsymbol{x}_k = \boldsymbol{f}(\boldsymbol{x}_{k-1}, \boldsymbol{w}_{k-1}) \tag{7}$$

with a measurement being

$$\boldsymbol{z}_k = \boldsymbol{h}(\boldsymbol{x}_k, \boldsymbol{v}_k). \tag{8}$$

The equations used in the extended Kalman filter are given in Table 2. $\boldsymbol{F}$ is the Jacobian matrix of partial derivatives of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$, which gives

$$F_{i,j} = \frac{\partial f_i}{\partial x_j}. \tag{9}$$

$\boldsymbol{W}$ is the Jacobian matrix of partial derivatives of $\boldsymbol{f}$ with respect to $\boldsymbol{w}$, which gives

$$W_{i,j} = \frac{\partial f_i}{\partial w_j}. \tag{10}$$

$\boldsymbol{H}$ is the Jacobian matrix of partial derivatives of $\boldsymbol{h}$ with respect to $\boldsymbol{x}$, that is

$$H_{i,j} = \frac{\partial h_i}{\partial x_j}. \tag{11}$$

$\boldsymbol{V}$ is the Jacobian matrix of partial derivatives of $\boldsymbol{h}$ with respect to $\boldsymbol{v}$, that is

$$V_{i,j} = \frac{\partial h_i}{\partial v_j}. \tag{12}$$

It must be noted that in the extended Kalman filter the noise is no longer normal after the nonlinear transformation.

Table 2 Extended Kalman filter time and measurement update equations.

Time update (predict)	Measurement update (correct)
$\boldsymbol{x}_k^- = \boldsymbol{f}(\boldsymbol{x}_{k-1}, 0)$	$\boldsymbol{K}_k = \boldsymbol{P}_k^- \boldsymbol{H}^T \left(\boldsymbol{H} \boldsymbol{P}_k^- \boldsymbol{H}^T + \boldsymbol{V}_k \boldsymbol{R}_k \boldsymbol{V}_k^T\right)^{-1}$

$\boldsymbol{P}_k^- = \boldsymbol{F}\boldsymbol{P}_{k-1}\boldsymbol{F}^{\boldsymbol{T}} + \boldsymbol{W}_{\boldsymbol{k}}\boldsymbol{Q}_{k-1}\boldsymbol{W}_k^T$	$\boldsymbol{x}_k = \boldsymbol{x}_k^- + \boldsymbol{K}_k\big(\boldsymbol{z}_k - \boldsymbol{h}(\boldsymbol{x}_k^-, 0)\big)$ $\boldsymbol{P}_k = (\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{H})\boldsymbol{P}_k^-$

The transform equations from the Cartesian coordinate system to the polar coordinate system ($\boldsymbol{h}$) are given by

$$\rho = \sqrt{x^2 + y^2}, \varphi = \arctan\left(\frac{y}{x}\right), \dot{\rho} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}}. \tag{13}$$

In the RADAR measurement, the mapping from the state vector to the measurement vector is governed by

$$\boldsymbol{H} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 & 0 \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 & 0 \\ \frac{y(y\dot{x} - x\dot{y})}{(x^2+y^2)^{\frac{3}{2}}} & \frac{x(x\dot{y} - y\dot{x})}{(x^2+y^2)^{\frac{3}{2}}} & \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}. \tag{14}$$

Since the linear motion assumption still holds, $\boldsymbol{F}$ is still given by equation (4). Furthermore, we assume that the process and measurement noises are both static. Therefore, $\boldsymbol{W}$ and $\boldsymbol{V}$ are both identity matrices.