Sensor Fusion with Kalman Filter

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April 14, 2017

1 Basic Kalman Filter

The Kalman filter estimates a process by using a feedback control. The filter predicts the system state at some time and then obtains feedback based on measurement. The time update (prediction) function of the basic Kalman filter is governed by the linear stochastic equation

$$\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k-1} + \mathbf{w}, \tag{1}$$

where $\mathbf{x}_k \in \Re^n$ is the state vector and $\mathbf{w} \sim N(0, \mathbf{Q})$ is the process noise with $\mathbf{Q}$ the covariance matrix. The measurement function is written as:

$$\mathbf{z_k} = \mathbf{H}\mathbf{x}_k + \nu, \tag{2}$$

where $\mathbf{z}_k \in \Re^m$ is the measurement vector and $\nu \sim N(0, \mathbf{R})$ is the process noise with $\mathbf{R}$ the covariance matrix.

The equations used in the Kalman filter are listed in Table 1 [1]. Here $\mathbf{P}_k^-$ and $\mathbf{P}_k$ are the priori and posteriori error covariance matrix, respectively, $\mathbf{z}_k - \mathbf{H}\mathbf{x}_k^-$ is called the measurement innovation, and $\mathbf{K}_k$ is chosen to minimize the posteriori error covariance.

Table 1: Discrete Kalman filter time and measurement update equations.

Time update (predict)	Measurement update (correct)
$\mathbf{x}_k^- = \mathbf{F}\mathbf{x}_{k-1}$ $\mathbf{P}_k^- = \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^\mathbf{T} + \mathbf{Q}$	$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{H}^\mathbf{T}(\mathbf{H}\mathbf{P}_k^-\mathbf{H}^\mathbf{T}+\mathbf{R})^{-1}$ $\mathbf{x}_k = \mathbf{x}_k^- + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\mathbf{x}_k^-)$ $\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_k^-$

In the LIDAR measurement, the state vector $\mathbf{x}_k$ is given by

$$\mathbf{x}_k = \begin{bmatrix} p_x & p_y & v_x & v_y \end{bmatrix}^T. \tag{3}$$

Assuming a linear motion, we have

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

where Δt is the time step. The measurement vector is $\mathbf{z}_k = \begin{bmatrix} p_x & p_y \end{bmatrix}^T$, and the measurement matrix is simply

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{5}$$

2 Extended Kalman filter

The extended Kalman filter is used to solve the problem in which the state function

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1},\ \mathbf{w}_{k-1}) \tag{6}$$

and/or the measurement function

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k, \boldsymbol{\nu}_k). \tag{7}$$

are nonlinear.

The equations used in the extended Kalman filter are summarized in Table 2 [1]. Here, $\mathbf{F}$ is the Jacobian matrix of partial derivatives of $\mathbf{f}$ with respect to $\mathbf{x}$, which gives

$$F_{i,j} = \frac{\partial f_i}{\partial x_j}. \tag{8}$$

$\mathbf{W}$ is the Jacobian matrix of partial derivatives of $\mathbf{f}$ with respect to $\mathbf{w}$, which gives

$$W_{i,j} = \frac{\partial f_i}{\partial w_j}. \tag{9}$$

$\mathbf{H}$ is the Jacobian matrix of partial derivatives of $\mathbf{h}$ with respect to $\mathbf{x}$, that is

$$H_{i,j} = \frac{\partial h_i}{\partial x_j}. \tag{10}$$

$\mathbf{V}$ is the Jacobian matrix of partial derivatives of $\mathbf{h}$ with respect to $\boldsymbol{\nu}$, that is

$$V_{i,j} = \frac{\partial h_i}{\partial \nu_j}. \tag{11}$$

It must be noted that in the extended Kalman filter the noise is no longer normal after the nonlinear transformation.

Table 2: Extended Kalman filter time and measurement update equations.

Time update (predict)	Measurement update (correct)
$\mathbf{x}_k^- = \mathbf{f}(\mathbf{x}_{k-1}, 0)$	$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{H}^\mathbf{T}(\mathbf{H}\mathbf{P}_k^-\mathbf{H}^\mathbf{T}+\mathbf{V}_k\mathbf{R}_k\mathbf{V}_k^T)^{-1}$
$\mathbf{P}_k^- = \mathbf{F}\mathbf{P}_{k-1}\mathbf{F}^\mathbf{T}+\mathbf{W_k}\mathbf{Q}_{k-1}\mathbf{W}_k^T$	$\mathbf{x}_k = \mathbf{x}_k^- + \mathbf{K}_k(\mathbf{z}_k - \mathbf{h}(\mathbf{x}_k^-, 0))$
	$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}_k^-$

The RADAR measurement returns

$$\mathbf{z_k} = \begin{bmatrix}\rho_k & \varphi_k & \dot{\rho}_k\end{bmatrix}^T \tag{12}$$

in the polar coordination system, which means the mapping from the state vector $\mathbf{x_k}$ to the measurement vector $\mathbf{z_k}$ is no longer linear. The transform equations from the Cartesian coordinate system to the polar coordinate system ($\mathbf{h}$) are given by

$$\rho = \sqrt{p_x^2 + p_y^2},\ \varphi = \arctan(\frac{p_y}{p_x}),\ \dot{\rho} = \frac{p_x v_x + p_y v_y}{\sqrt{p_x^2 + p_y^2}}. \tag{13}$$

The Jacobian matrix of $\mathbf{h}$ is

$$\mathbf{H} = \begin{bmatrix} \frac{p_x}{\sqrt{p_x^2+p_y^2}} & \frac{p_y}{\sqrt{p_x^2+p_y^2}} & 0 & 0 \\ -\frac{p_y}{p_x^2+p_y^2} & \frac{p_x}{p_x^2+p_y^2} & 0 & 0 \\ \frac{p_y(p_y v_x - p_x v_y)}{(p_x^2+p_y^2)^{\frac{3}{2}}} & \frac{p_x(p_x v_y - p_y v_x)}{(p_x^2+p_y^2)^{\frac{3}{2}}} & \frac{p_x}{\sqrt{p_x^2+p_y^2}} & \frac{p_y}{\sqrt{p_x^2+p_y^2}} \end{bmatrix}. \tag{14}$$

Since the linear motion assumption still holds, $\mathbf{F}$ is given by equation (4). Furthermore, we assume that the process and measurement noises are both static. Therefore, $\mathbf{W}$ and $\mathbf{V}$ are both identity matrices.

3 Uncented Kalman filter

The uncented Kalman filter (UKF) is a derivative-free alternative to the extended Kalman filter (EKF). The uncented transform is a method for calculating the statistics of a random variable which undergoes a nonlinear transformation. Considering propagating a 1D state vector $\mathbf{x}$ through a nonlinear function $\mathbf{y} = \mathbf{f}(\mathbf{x})$. To calculate the statistics of $\mathbf{f}$, we form a matrix $\mathbf{X}$ consisting of $2L$+1 sigma vectors [2]

$$\mathbf{X_0} = \bar{\mathbf{x}} \tag{15}$$

$$\mathbf{X_i} = \bar{\mathbf{x}} + \sqrt{(\lambda + L)\mathbf{P_x}}, i = 1, ..., L \tag{16}$$

$$\mathbf{X_i} = \bar{\mathbf{x}} - \sqrt{(\lambda + L)\mathbf{P_x}}, i = L + 1, ..., 2L \tag{17}$$

where $\lambda = \alpha^2(L + \kappa) - L$ is a scaling parameter. The constant $\alpha \in (0, 1]$ determines the spread of the sigma points around $\bar{\mathbf{x}}$. Another constant κ is usually set to either 0 or $3 - L$.

These sigma vectors are propagated through the same nonlinear function

$$\mathbf{\Gamma_i} = \mathbf{f}(\mathbf{X_i}), i = 0, ..., 2L \tag{18}$$

The mean and covariance of $\mathbf{\Gamma}$ are given by

$$\bar{\mathbf{\Gamma}} = \sum_{i=0}^{2L} W_i^{(m)} \mathbf{\Gamma_i}, \tag{19}$$

where the weights W_i is given by

$$\begin{cases} W_i^{(m)} & = \lambda/(L + \lambda) \\ W_i^{(c)} & = \lambda/(L + \lambda) + 1 - \alpha^2 + \beta \\ W_i^{(m)} & = W_i^{(c)} = 0.5/(L + \lambda), i = 1, ..., 2L \end{cases}, \tag{20}$$

where β is related to the distribution of $\mathbf{x}$ and $\beta = 2$ for Gaussian distribution.

3.1 UKF in CTRV model

The CTRV (Constant Turn Rate and Velocity) model

The state vector is given by

$$\mathbf{x} = \begin{bmatrix} p_x \\ p_y \\ v \\ \psi \\ \dot{\psi} \end{bmatrix}. \tag{21}$$

The time update function can be written as

$$\mathbf{x}_k = \mathbf{x}_{k-1} + \int_{t_{k-1}}^{t_k} \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{v} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} dt \approx \mathbf{x}_{k-1} + \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ 0 \\ \dot{\psi}\Delta t \\ 0 \end{bmatrix} \tag{22}$$

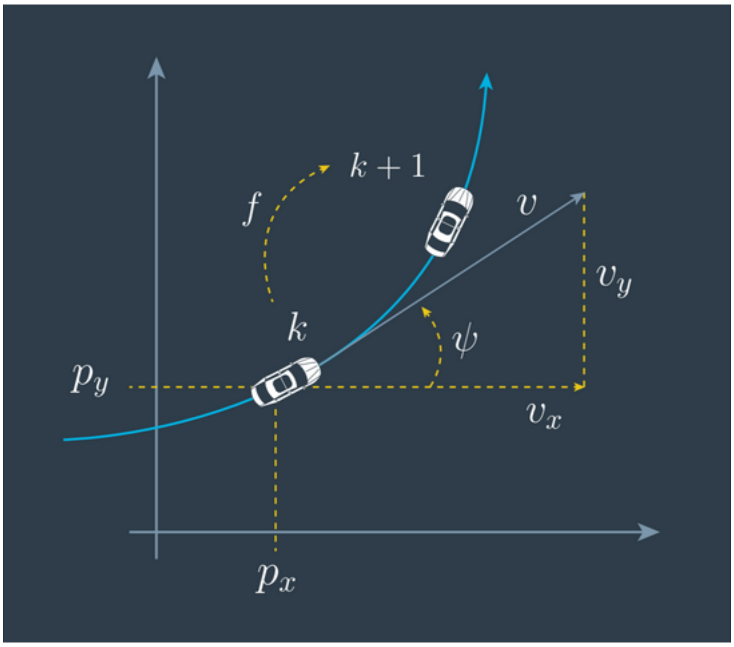

Figure 1: Illustration of the CTRV model

References

[1] Greg Welch and Gary Bishop. *An introduction to the Kalman filter.* ACM, Inc. 2001.

[2] Eric A. Wan and Rudolph van der Merwe, *The uncented Kalman filter.* Kalman filtering and neural networks, John Wiley & Sons, Inc., 2002.