User's guide for package ALAMODE (ver. 0.9.0)

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1 Introduction

Program package ALAMODE is designed for estimating anharmonic force constants of solids based on the supercell approach and subsequent calculations of anharmonic phonon properties, such as Grüneisen parameter, phonon self-energy and lattice thermal conductivity.

1.1 What ALAMODE can do

The package includes two main programs, **alm** and **anphon**, and subsidiary small programs and scripts for analyzing results.

- alm extracts harmonic and anharmonic interatomic force constants (IFCs) from the given displacement-force data set by the supercell approach.
- anphon can compute the following quantities using the IFCs extracted by alm:
 - Phonon dispersion and phonon DOS
 - Vibrational free energy, entropy, and internal energy
 - Group velocity of phonon
 - Grüneisen parameter
 - Anharmonic phonon lifetime and mean-free-path of phonons
 - Lattice thermal conductivity

1.2 Before installation

This package does not calculate atomic forces that are necessary to estimate force constants. Users have to get and install a first-principles package (such as VASP, Wien2k, Quantum-ESPRESSO, and xTAPP) or other force field packages (such as LAMMPS) by themselves.

1.3 Installation

1.3.1 Prerequisite

- C++ compiler (Intel compiler is highly recommended.)
- LAPACK library
- MPI library (Either OpenMPI, MPICH2, and IntelMPI)
- Boost C++ library
- Eigen3 package (Optional)
- Python, Numpy, and Matplotlib (Optional)

1.3.2 How to install

- 1. Download the package from this link.
- 2. Change directory to the location of the downloaded file and untar the file as follows:

```
$ tar -xvzf alamode-x.y.z.tar.gz
```

This will create a new directory alamode-x.y.z/ which contains the following subdirectories:

- alm/: Source files for alm
- anphon/: Source files for anphon
- external/: Third-party include files
- include/: Commonly used include files
- tools/: Small programs and scripts

3. Edit the Makefiles

The directories alm/, anphon/, and tools/ contain seperate Makefiles. Please modify the Makefiles appropriately. To enable OpenMP parallelization, please add -openmp or -fopenmp (depends on the compiler) to the CXXFLAGS.

If one wishes to use the Eigen3 package for alm, please add -D_USE_EIGEN to the CXXFLAGS in alm/Makefile. Eigen3 package is not necessary, but it may speed up the calculation especially when the number of force constants are fairly large.

4. Generate executables by make command.

1.4 How to use

Step 1. Prepare an input file for 'alm' Program alm estimates harmonic and anharmonic IFCs based on the *supercell approach*. Each user needs to determine the size of supercell and prepare the input file, say alm.in, for the structure. For details of input variables for alm, please refer to the list of input variables for alm below. Once the input files is properly prepared with MODE = suggest, necessarily displacement patterns may be generated by executing alm as follows:

\$ alm alm.in > alm.log

This produces the following files containing atomic displacements.

- PREFIX.HARMONIC_pattern
- PREFIX.ANHARM?_pattern (If NORDER ≥ 2)

Step 2. Perform first-principles calculations to generate displacement-force data set The next step is to calculate atomic forces for the displaced configurations using a DFT package. Once the atomic forces are calculated, please collect the atomic displacements and atomic forces to separate files, say $disp_all.dat$ and $force_all.dat$. Atomic displacements and forces should be stored in units of Bohr and Ryd/Bohr, respectively. The details of file format can be found here.

Step 3. Fitting In order to perform fitting, please change the variable MODE of the input file alm.in to MODE = fitting. In addition please add the &fitting entry with appropriate NDATA, DFILE, and FFILE. Then, IFCs can be estimated by executing

\$ alm alm.in > alm.log2

which makes the following two files in the working directory.

- PREFIX.fcs : The list of force constants
- PREFIX.xml : XML file containing necessarily information for subsequent phonon calculations

Step 4. Prepare an input file for 'anphon' To perform phonon calculations and thermal conductivity calculations, one needs to prepare another input file, say anphon.in, for the program anphon. If one wishes to perform (harmonic) phonon calculations, one should write MODE = phonons in the &general entry of anphon.in. Please make sure that FCSXML variable being set to the XML file generated by alm. If one wishes to perform thermal conductivity calculations instead of usual phonon calculations, please switch to MODE = RTA with appropriate FCSXML containing cubic IFCs. For details of input variables of anphon, please refer to the list of input variables for anphon.

Step 5. Execute 'anphon' Phonon calculations can be executed by

\$ anphon anphon.in > anphon.log

or

\$ mpirun -np NP anphon anphon.in > anphon.log

This command generates various files in the working directory depending on the given input variables.

- PREFIX.bands: Phonon dispersion along the designated path
- PREFIX.dos: (Atom projected) phonon DOS
- PREFIX.thermo: Thermodynamical quantities
- PREFIX.rmsd: Atomic displacement parameters
- PREFIX.phvel: Phonon group velocity along the designated path
- PREFIX.phvel_all: Phonon group velocity on a uniform k grid.
- PREFIX.result: Phonon lifetimes and group velocities at irreducible k points.
- PREFIX.gruneisen: Grüneisen constants along the designated path
- \bullet PREFIX.gru_all: Grüneisen constants on a uniform k grid
- PREFIX.kl: Lattice thermal conductivity tensor
- PREFIX.axsf: AXSF format file for visualizing atomic motions (Can be visualized by Xcrys-Den)
- PREFIX.evec: Eigenvector (polarization vector) of phonons

Step 6. Analyze the result (optional) We provide some small programs and scripts in the tools/ directory. For example, plotband.py may be useful for visualizing phonon dispersion relations, which can be executed by

\$./plotband.py target.bands

Command line options can be displayed by ./plotband.py -h. Another script analyze_phonons.py may be useful for plotting phonon lifetimes at a given temperature. For example, phonon lifetimes and mean-free-path at 300 K can be extracted by

\$./analyze_phonons.py --calc tau --temp 300 target.result

Another available options can be seen by ./analyze_phonons.py -h.

1.5 Input variables

- List of input variables for alm.
- List of input variables for anphon.

1.6 License

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1.7 Author

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2 Program alm

Program **alm** (stands for anharmonic lattice model) extracts harmonic and anharmonic force constants based on the supercell approach. The code fully employs the crystal symmetry to reduce the number of IFCs.

2.1 Brief introduction to force constants

The starting point of the computational methodology is to approximate the potential energy of interacting atoms by a Taylor expansion with respect to atomic displacements by

$$U - U_0 = \sum_{n=1}^{N} U_n = U_1 + U_2 + U_3 + \cdots,$$
(1)

$$U_n = \frac{1}{n!} \sum_{\substack{\ell_1 \kappa_1, \dots, \ell_n \kappa_n \\ \mu_1, \dots, \mu_n}} \Phi_{\mu_1 \dots \mu_n}(\ell_1 \kappa_1; \dots; \ell_n \kappa_n) \ u_{\mu_1}(\ell_1 \kappa_1) \dots u_{\mu_n}(\ell_n \kappa_n). \tag{2}$$

Here, $u_{\mu}(\ell\kappa)$ is the atomic displacement of κ th atom in the ℓ th unit cell along μ th direction, and $\Phi_{\mu_1...\mu_n}(\ell_1\kappa_1;...;\ell_n\kappa_n)$ is the *n*th-order interatomic force constant (IFC).

2.2 Structure of the input file

The input file consists of 6 different entry fields: &general, &interaction, &cell, &cutoff, &fitting, and &position. Each entry field start from the key label &field and ends at the terminate character "/". For example, general field should be given as the following:

```
&general
  # Comment line
  PREFIX = prefix
  MODE = fitting
/
```

Multiple entries can be put in a single line. Also, any characters put on the right of sharp ("#") character will be neglected. Therefore, the above example is equivalent to

```
&general
  PREFIX = prefix; MODE = fitting # Comment line
/
```

Each entry should be written inside the appropriate entry field.

2.3 List of input variables for alm

The list of input variables acceptable is explaind below. The variables of **bold face** are always necessarily, whereas other variables have default values.

2.3.1 "&general"-field

PREFIX-tag: Job prefix to be used for names of output files.

Default: None
Type: String

MODE-tag = fitting | suggest

Default: None

Type: String

fitting $\;\;$ Perform fittings to estimate harmonic and anharmonic IFCs. This mode

requires appropriate DFILE and FFILE.

suggest This mode suggests the displacement patterns necessary to estimate

harmonic and anharmonic IFCS.

NAT-tag: Number of atoms in the supercell

Default: None
Type: Integer

NKD-tag: Number of atomic species

Default: None
Type: Integer

KD-tag = Name[1],...,Name[NKD]

Default: None

Type: Array of strings

Example: In the case of GaAs with NKD = 2, KD should be KD = Ga As.

 $NSYM-tag = 0 \mid 1 \mid nsym$

Default: 0

Type: Integer

The program automatically generates the crystal symmetry operations (rotational and translational parts). When PRINTSYM = 1, symmetry operations will be saved in the file "SYMM_INFO".

1 Only the identity operation will be considered.

nsym "nsym" symmetry operations will be read from "SYMM_INFO" file.

TOLERANCE-tag: Tolerance to estimate symmetry operations

Default: 1.0e-8
Type: Double

 $\mathtt{PRINTSYM-tag} = 0 \mid 1$

Default: 0

Type: Integer

0 Symmetry operations won't be saved in "SYMM_INFO".

1 Symmetry operations will be saved in "SYMM_INFO"

PERIODIC-tag = [is_periodic_x, is_periodic_y, is_periodic_z]

Default: 1 1 1

Type: Array of integers

0 Periodic boundary condition will not be used.

1 Periodic boundary condition will be used.

2.3.2 "&interaction"-field

NORDER-tag: The order of force constants to be calculated. With NORDER = m, anharmonic terms up to (m+1)th order will be considered.

Default: None

Type: Integer

Example: NORDER should be 1 for harmonic calculations, and 2 to include cubic force constants.

NBODY-tag: Entry for excluding multiple-body interactions from anharmonic force constants

Default: NBODY = [2, 3, 4, ..., NORDER + 1]

Type: Array of integers

Example: If one wishes to exclude three-body interactions from cubic force constants, one should explicitly give NBODY = 2 2.

2.3.3 "&cutoff"-field

In this entry field, one needs to specify cutoff radii of interaction for each order in units of Bohr. In the current implementation, cutoff radii should be defined for every possible pairs of atomic elements. For example, the cutoff entry for a harmonic calculation (NORDER = 1) of Si (NKD = 1) should be like

```
&cutoff
Si-Si 10.0
/
```

This means that the cutoff radii of $10 a_0$ will be used for harmonic Si-Si terms. Please note that the first column should be two character strings, which are contained in the KD-tag, connected by a hyphen ('-'). When one wishes to consider cubic terms (NORDER = 2), please specify the cutoff radius for cubic terms in the third column as the following.

```
&cutoff
Si-Si 10.0 5.6 # Pair r_{2} r_{3}
/
Instead of giving specific cutoff radii, one can write None as follows
&cutoff
Si-Si None 5.6
/
```

which means that all possible harmonic terms between Si-Si atoms will be included. Please keep in mind that writing None for anharmonic terms can greatly increase the number of parameters, and hereby increase the computational cost.

When there are more than two atomic elements, please specify the cutoff radii between every possible pairs of atomic elements. In the case of MgO (NKD = 2), cutoff should be given like

```
&cutoff
Mg-Mg 8.0
0-0 8.0
Mg-0 10.0
/
which can equivalently be written by using the wild card ('*') as
&cutoff
*-* 8.0
Mg-0 10.0 # Overwrite the cutoff radius for Mg-0 harmonic interactions
/
```

As indicated by the above example, cutoff radii specified by an earlier entry will be overwritten by a new entry that comes later.

Once the cutoff radii are properly given, harmonic force constants $\Phi_{i,j}^{\mu,\nu}$ satisfying $r_{ij} \leq r_c^{\text{KD}[i]-\text{KD}[j]}$ will be searched. In the case of cubic terms, force constants $\Phi_{ijk}^{\mu\nu\lambda}$ satisfying $r_{ij} \leq r_c^{\text{KD}[i]-\text{KD}[j]}$, $r_{ik} \leq r_c^{\text{KD}[i]-\text{KD}[k]}$, and $r_{jk} \leq r_c^{\text{KD}[j]-\text{KD}[k]}$ will be searched and determined by the fitting.

2.3.4 "&cell"-field

Please give the cell parameters in this entry as the following in units of Bohr:

```
&cell
a
a11 a12 a13
a21 a22 a23
a31 a32 a33
/
```

The cell parameters are then given by $\vec{a}_1 = a \times (a_{11}, a_{12}, a_{13}), \ \vec{a}_2 = a \times (a_{21}, a_{22}, a_{23}),$ and $\vec{a}_3 = a \times (a_{31}, a_{32}, a_{33}).$

2.3.5 "&position"-field

In this field, one needs to specify the atomic element and fractional coordinate of atoms in the supercell. Each line should be

ikd xf[1] xf[2] xf[3]

where ikd is an integer specifying the atomic element (ikd = 1, ..., NKD) and xk[i] is the fractional coordinate of an atom. There should be NAT such lines in the &position entry field.

2.3.6 "&fitting"-field (Necessarily when MODE = fitting)

DFILE-tag: File name containing atomic displacements in Cartesian coordinate

Default: None

Type: String

FFILE-tag: File name containing atomic forces in Cartesian coordinate

Default: None

Type: String

NDATA-tag: The number of data contained in the DFILE and FFILE

Default: None

Type: Integer

NSTART, NEND-tag: Specifies the range of data to be used for fitting

Default: NSTART = 1, NEND = NDATA

Type: Integer

Example: If one wishes to use the data in the range of [20:30] out of 50 data, one should set NSTART = 20 and NEND = 30. Please note that $1 \le NSTART \le NEND \le NDATA$ must be satisfied.

 $ICONST-tag = 0 \mid 1$

Default: 1

Type: Integer

- 0 No constraints
- 1 Constraints for translational invariance will be imposed.
- In addition to ICONST = 1, constraints for rotational invariance will be imposed up to (NORDER + 1)th-order.
- 3 In addition to ICONST = 2, constraints for rotational invariance between (NORDER + 1)th-order and (NORDER + 2)th-order, which are zeros, will be imposed.

FC2XML-tag: XML file to which the harmonic terms will be fixed upon fitting.

Default: None

Type: String

Example: When FC2XML-tag is given, harmonic force constants will be fixed to the values written in the FC2XML file. This may be used to optimize cubic and higher-order terms without changing the harmonic terms. Please make sure that the number of harmonic terms in the new computational condition be the same as that in the FC2XML file.

2.4 Formats of DFILE and FFILE

The displacement-force data sets obtained by first-principles (or classical force-field) calculations have to be saved to DFILE and FFILE to estimate IFCS with MODE = fitting.

3 Input variables of anphon

This page describes the input variables for **anphon** program that calculates phonon frequencies, phonon lifetimes, and lattice thermal conductivity.

3.1 Structure of the input file

The input file consists of 4 different entry fields: &general, &cell, &analysis, and &kpoint. The format of input file is the same as that of alm which can be found here.

3.2 List of input variables

3.2.1 "&general"-field

PREFIX-tag: Prefix of the files to be created

Default: None
Type: String

MODE-tag = phonons | RTA

Default: None

Type: String

phonons Calculating phonon dispersion relations, phonon DOS, Grüneisen pa-

rameters etc.

RTA Calculating phonon lifetimes and lattice thermal conductivity based on

the Boltzmann transport equation (BTE) with the relaxation time ap-

proximation (RTA).

NKD-tag: Number of atomic species

Default: None
Type: Integer

 \mathbf{KD} -tag = Name[1],...,Name[NKD]

Default: None

Type: String

Example: In the case of GaAs with NKD = 2, KD should be KD = Ga As.

MASS-tag = Mass[1], ..., Mass[NKD]

Default:None

Type: Double

Example: In the case of Bi_2Te_3 with NKD = 2, MASS should be MASS = 208.98 127.60.

FCSXML-tag: XML file containing force constants generated by the program alm

Default: None

Type: String

 $\texttt{NSYM-} tag = 0 \mid 1 \mid nsym$

Default: 0

Type: Integer

- The program automatically generates the crystal symmetry operations (rotational and translational parts). When PRINTSYM = 1, symmetry operations will be saved in the file "SYMM_INFO_PRIM".
- 1 Only the identity operation will be considered.

nsym "nsym" symmetry operations will be read from "SYMM_INFO_PRIM"

file.

TOLERANCE-tag: Tolerance to estimate symmetry operations

Default: 1.0e-8

Type: Double

 $PRINTSYM-tag = 0 \mid 1$

Default: 0

Type: Integer

- 0 Symmetry operations won't be saved in "SYMM_INFO_PRIM".
- 1 Symmetry operations will be saved in "SYMM_INFO_PRIM"

$\mathtt{NONANALYTIC} = 0 \mid 1$

Default: 0

Type: Integer

0 Non-analytic correction is not considered.

1 Non-analytic correction will be considered. Appropriate NA_SIGMA and BORNINFO should be given.

NA_SIGMA-tag: Damping factor for non-analytic term in units of kayser

Default: None
Type: Double

 ${\tt BORNINFO-tag: File\ containing\ dielectric\ tensor\ and\ Born\ effective\ charges\ for\ the\ non-analytic}$

correction

Default: None

Type: String

The details of file format can be found here.

TMIN, TMAX, DT-tags: Temperature range and stride to be calculated in units of Kelvin

Default: TMIN = 0, TMAX = 1000, DT = 10

Type: Double

EMIN, EMAX, DELTA_E-tags: Energy range and stride to be calculated in units of kayser

 $Default: EMIN = 0, EMAX = 1000, DELTA_E = 10$

Type: Double

ISMEAR-tag = $-1 \mid 0 \mid 1$

Default: -1

Type: Integer

ISMEAR specifies the method for Brillouin zone integration as follows:

-1 Tetrahedron method

0 Lorentzian smearing with width of EPSILON

1 Gaussian smearing with width of EPSILON

TRISYM-tag: Flag to use symmetry operations to reduce the number of triples of k points necessarily for calculating phonon self-energies

Default: 1

Type: Integer

- 0 Symmetry will not be used
- 1 Use symmetry to find irreducible triplets

This variable is used only when MODE = RTA. TRISYM = 1 can reduce the computational cost, but phonon linewidth stored to the file "PREFIX.result" needs to be averaged at points of degeneracy. For that purpose, a subsidiary program **analyze_phonons.py** may be used.

RESTART-tag: Flag to restart the calculation when MODE = RTA

Default: 1 if there is a file named "PREFIX.result"; otherwise 0

Type: Integer

- 0 Calculation will be performed from scratch
- 1 Restart from the existing file

3.2.2 "&cell"-field

Please specify the cell parameters of the **primitive cell** in this field.

```
&cell
a
a11 a12 a13
a21 a22 a23
a31 a32 a33
/
```

The cell parameters are then given by $\vec{a}_1 = a \times (a_{11}, a_{12}, a_{13})$, $\vec{a}_2 = a \times (a_{21}, a_{22}, a_{23})$, and $\vec{a}_3 = a \times (a_{31}, a_{32}, a_{33})$. Please note that the lattice constant a must be correspond to the value used for **alm**. For example, if one used a = 20.4 a_0 for a $2 \times 2 \times 2$ supercell of Si, one has to use a = 10.2 a_0 here for the primitive cell.

3.2.3 "&kpoint"-field

This entry field specifies the list of k points to be calculated. The first entry **KPMODE** specifies the types of calculation which is followed by detailed entries.

KPMODE = $\mathbf{0}$: Calculate phonon frequencies at given k points

For example, if one wishes to calculate phonon frequencies at Γ (0, 0, 0) and X (0, 1/2, 1/2) of a FCC crystal, the &kpoint entry can be written as

```
&kpoint
0
0.000 0.000 0.000
0.000 0.500 0.500
/
```

$\mathbf{KPMODE} = \mathbf{1}$: Band dispersion calculation

For example, if one wishes to calculate phonon dispersion relations along $\Gamma - K - X - \Gamma - L$ of a FCC crystal, the &kpoint entry can be written as follows:

```
&kpoint
1
G 0.000 0.000 0.000 K 0.375 0.375 0.750 51
K 0.375 0.375 0.750 X 0.500 0.500 1.000 51
X 0.000 0.500 0.500 G 0.000 0.500 0.500 51
G 0.000 0.000 0.000 L 0.500 0.500 51
/
```

KPMODE = 2: Uniform k grid for phonon DOS and thermal conductivity

When one wishes to perform a calculation with $20 \times 20 \times 20$ k-grid, the &kpoint entry should be

```
&kpoint
2
20 20 20
/
```

3.2.4 "&analysis"-field (Optional)

```
GRUNEISEN-tag = 0 \mid 1
```

Default: 0

Type: Integer

GRUNEISEN specifies whether or not to compute Grüneisen parameters.

- 0 Grüneisen parameters won't be calculated
- Grüneisen parameters will be stored in "PREFIX.phvel" (when $\mathbf{KP-MODE}=1$) or "PREFIX.phvel_all" (when $\mathbf{KPMODE}=2$)

This flag is available only when MODE = phonons. Please note that cubic force constants should be included in the FCSXML file when GRUNEISEN = 1.

```
PRINTEVEC-tag = 0 \mid 1
```

Default: 0

Type: Integer

PRINTEVEC determines whether or not print phonon eigenvectors.

- 0 Phonon eigenvectors won't be printed
- 1 Print phonon eigenvectors to "PREFIX.evec"

$PRINTXSF-tag = 0 \mid 1$

Default: 0

Type: Integer

PRINTXSF determines whether or not print a AXSF file for visualization of phonon modes.

- 0 AXSF file won't be created
- 1 AXSF file "PREFIX.axsf" will be created

The created AXSF file may be visualized by XCrysDen.

$PRINTVEL-tag = 0 \mid 1$

Default: 0

Type: Integer

PRINTVEL determines whether or not to print group velocities of phonons.

- 0 Group velocities won't be printed
- 1 Phonon velocities will be stored in "PREFIX.phvel" (when KPMODE
 - = 1) or "PREFIX.phvel_all" (when $\mathbf{KPMODE} = 2$) in units of m/s.

This flag is available only when MODE = phonons.

$\mathtt{PRINTRMSD-tag} = 0 \mid 1$

Default: 0

Type: Integer

PRINTRMSD determines whether or not to print root-mean-square-displacements (RMSD) of atoms.

- 0 RMSD won't be printed
- 1 RMSD will be stored in "PREFIX.rmsd"

This flag is available only when MODE = phonons and KPMODE = 2.

$$PDOS-tag = 0 \mid 1$$

Default: 0

Type: Integer

PDOS determines whether or not to print atom-projected phonon DOS.

- 0 Atom-projected phonon DOS won't be printed
- 1 Atom-projected phonon DOS will be stored in "PREFIX.dos"

This flag is available only when MODE = phonons and KPMODE = 2.

 $TDOS-tag = 0 \mid 1$

Default: 0

Type: Integer

TDOS determines whether or not to print two-phonon DOS.

0 Two-phonon DOS won't be printed

1 Two-phonon DOS will be stored in "PREFIX.tdos"

This flag is available only when MODE = phonons and KPMODE = 2.

 $ISOTOPE-tag = 0 \mid 1$

Default: 0

Type: Integer

ISOTOPE determines whether or not to include isotope-phonon scatterings in thermal conductivity calculations

0 Isotope effect won't be considered.

1 Isotope-phonon scattering will be considered by Tamura's formula. ISOFACT should be properly given.

This flag is available only when MODE = RTA.

ISOFACT-tag = isofact[1], ..., isofact[NKD]

Default: None

Type: Array of doubles

Isotope factor is a dimensionless value which is given by $\sum_i f_i (1 - m_i/\bar{m})^2$. Here, f_i is the fraction of i th isotope of an element having mass m_i , and $\bar{m} = \sum_i f_i m_i$ is the average mass, respectively. This quantity is equivalent to g_2 appearing in the original paper by S. Tamura [Phys. Rev. B, 27, 858.].