10. Schur decomposition

- eigenvalues of nonsymmetric matrix
- Schur decomposition

Eigenvalues and eigenvectors

a nonzero vector x is an *eigenvector* of the $n \times n$ matrix A, with *eigenvalue* λ, if

$$Ax = \lambda x$$

- the eigenvalues are the roots of the characteristic polynomial

$$\det(\lambda I - A) = 0$$

- eigenvectors are nonzero vectors in the nullspace of $\lambda I - A$

for most of the lecture, we assume that A is a complex $n \times n$ matrix

Linear indepence of eigenvectors

suppose $x_1, \ldots, x_k$ are eigenvectors for k different eigenvalues:

$$Ax_1 = \lambda_1 x_1, \qquad \ldots, \qquad Ax_k = \lambda_k x_k$$

then $x_1, \ldots, x_k$ are linearly independent

- the result holds for $k = 1$ because eigenvectors are nonzero
- suppose it holds for $k - 1$, and assume $\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$; then

$$0 = A(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k) = \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_2 x_2 + \cdots + \alpha_k \lambda_k x_k$$

- subtracting $\lambda_1(\alpha_1 x_1 + \cdots + \alpha_k x_k) = 0$ shows that

$$\alpha_2(\lambda_2 - \lambda_1)x_2 + \cdots + \alpha_k(\lambda_k - \lambda_1)x_k = 0$$

- since $x_2, \ldots, x_k$ are linearly independent, $\alpha_2 = \cdots = \alpha_k = 0$
- hence $\alpha_1 = \cdots = \alpha_k = 0$, so $x_1, \ldots, x_k$ are linearly independent

Multiplicity of eigenvalues

Algebraic multiplicity

- the multiplicity of the eigenvalue as a root of the characteristic polynomial
- the sum of the algebraic multiplicities of the eigenvalues of an $n \times n$ matrix is n

Geometric multiplicity

- the geometric multiplicity is the dimension of $\mathrm{null}(\lambda I - A)$
- the maximum number of linearly independent eigenvectors with eigenvalue λ
- sum is the maximum number of linearly independent eigenvectors of the matrix

Defective eigenvalue

- geometric multipicity never exceeds algebraic multiplicity (proof on page 10.7)
- eigenvalue is *defective* if geometric muliplicity is less than algebraic multiplicity
- a matrix is *defective* if some of its eigenvalues are defective

Example

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \qquad \det(\lambda I - A) = (\lambda - 1)^2(\lambda - 2)^2$$

- eigenvalue $\lambda = 1$ has algebraic multiplicity two and geometric multiplicity one
- eigenvectors with eigenvalue 1 are the nonzero multiples of $(1, 0, 0, 0)$
- eigenvalue $\lambda = 2$ has algebraic and geometric multiplicity two
- eigenvectors with eigenvalue 2 are nonzero vectors in the subspace

$$\mathrm{null}(2I - A) = \mathrm{span}\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\}$$

- maximum number of linearly independent eigenvectors is three; for example,

$$(1, 0, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1)$$

Similarity transformation

two matrices A and B are *similar* if

$$B = X^{-1}AX$$

for some nonsingular matrix X

- similarity transformation preserves eigenvalues and algebraic multiplicities:

$$\det(\lambda I - B) = \det(\lambda I - X^{-1}AX) = \det(X^{-1}(\lambda I - A)X) = \det(\lambda I - A)$$

- if x is an eigenvector of A then $y = X^{-1}x$ is an eigenvector of B:

$$By = (X^{-1}AX)(X^{-1}x) = X^{-1}Ax = X^{-1}(\lambda x) = \lambda y$$

- similarity transformation preserves geometric multiplicities:

$$\dim \operatorname{null}(\lambda I - B) = \dim \operatorname{null}(\lambda I - A)$$

Geometric and algebraic multiplicities

suppose α is an eigenvalue with geometric multiplicity r:

$$\dim \mathrm{null}(\alpha I - A) = r$$

- define an $n \times r$ matrix U with orthonormal columns that span $\mathrm{null}(\alpha I - A)$
- complete U to define a unitary matrix $X = \begin{bmatrix} U & V \end{bmatrix}$ and define $B = X^H A X$:

$$B = X^H A X = \begin{bmatrix} U^H A U & U^H A V \\ V^H A U & V^H A V \end{bmatrix} = \begin{bmatrix} \alpha I & U^H A V \\ 0 & V^H A V \end{bmatrix}$$

- the characteristic polynomial of B is

$$\det(\lambda I - B) = (\lambda - \alpha)^r \det(\lambda I - V^H A V)$$

this shows that the algebraic multiplicity of eigenvalue α of B is at least r

Diagonalizable matrices

the following properties are equivalent

1. A is diagonalizable by a similarity: there exists nonsingular X, diagonal Λ s.t.

$$X^{-1}AX = \Lambda$$

2. A has a set of n linearly independent eigenvectors (the columns of X):

$$AX = X\Lambda$$

3. all eigenvalues of A are nondefective:

algebraic multiplicity = geometric multiplicity

not all matrices are diagonalizable (as are real symmetric matrices)

Outline

- eigenvalues of nonsymmetric matrix
- **Schur decomposition**

Schur decomposition

every $A \in \mathbf{C}^{n\times n}$ can be factored as

$$A = UTU^H \tag{1}$$

- U is unitary: $U^H U = UU^H = I$
- T is upper triangular, with the eigenvalues of A on its diagonal
- the eigenvalues can be chosen to appear in any order on the diagonal of T

complexity of computing the factorization is order n^3

Proof by induction

- the decomposition (1) obviously exists if $n = 1$
- suppose it exists if $n = m$ and A is an $(m+1) \times (m+1)$ matrix
- let λ be any eigenvalue and u a corresponding eigenvector, with $\|u\| = 1$
- let V be an $(m+1) \times m$ matrix that makes the matrix $\begin{bmatrix} u & V \end{bmatrix}$ unitary:

$$\begin{bmatrix} u^H \\ V^H \end{bmatrix} A \begin{bmatrix} u & V \end{bmatrix} = \begin{bmatrix} u^H A u & u^H A V \\ V^H A u & V^H A V \end{bmatrix} = \begin{bmatrix} \lambda u^H u & u^H A V \\ \lambda V^H u & V^H A V \end{bmatrix} = \begin{bmatrix} \lambda & u^H A V \\ 0 & V^H A V \end{bmatrix}$$

- $V^H A V$ is an $m \times m$ matrix, so by the induction hypothesis,

$$V^T A V = \tilde{U} \tilde{T} \tilde{U}^H \quad \text{for some unitary } \tilde{U} \text{ and upper triangular } \tilde{T}$$

- the matrix $U = \begin{bmatrix} u & V\tilde{U} \end{bmatrix}$ is unitary and defines a similarity that triangularizes A:

$$U^H A U = \begin{bmatrix} u^H \\ \tilde{U}^H V^H \end{bmatrix} A \begin{bmatrix} u & V\tilde{U} \end{bmatrix} = \begin{bmatrix} \lambda & u^H A V \tilde{U} \\ 0 & \tilde{Q}^H V^H A V \tilde{Q} \end{bmatrix} = \begin{bmatrix} \lambda & u^H A V \tilde{U} \\ 0 & \tilde{T} \end{bmatrix}$$

Real Schur decomposition

if A is real, a factorization with real matrices exists:

$$A = UTU^T$$

- U is orthogonal: $U^T U = UU^T = I$
- T is *quasi-triangular:*

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1m} \\ 0 & T_{22} & \cdots & T_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{mm} \end{bmatrix}$$

  the diagonal blocks T_{ii} are 1×1 or 2×2

- the scalar diagonal blocks are real eigenvalues of A
- the eigenvalues of the 2×2 diagonal blocks are complex eigenvalues of A