Name: M. Venkata Gopi Jayaram. Reg-NO- 19BQ1A05E3 section- CSE-C

MII Assignment

1.a Find the laplace transform of sint sin2t sin3t.

Sol sint sin2t sin3t $= \sin t\left[\frac{1}{2}(\cos t - \cos 5t)\right]$

$= \frac{1}{2}\sin t\cos t - \frac{1}{2}\sin t\cos 5t$

$= \frac{1}{4}\sin 2t - \frac{1}{2}\left[\frac{1}{2}(\sin 6t - \sin 4t)\right]$

$= \frac{1}{4}\sin 2t - \frac{1}{4}\sin 6t + \frac{1}{4}\sin 4t$

$L(\sin t\sin 2t\sin 3t) = L\left(\frac{1}{4}\sin 2t - \frac{1}{4}\sin 6t + \frac{1}{4}\sin 4t\right)$

$= \frac{1}{4}\left(L(\sin 2t) - L(\sin 6t) + L(\sin 4t)\right)$

$= \frac{1}{4}\left(\frac{2}{s^2+4} - \frac{6}{s^2+36} + \frac{4}{s^2+16}\right)$

$= \frac{1}{2}\left(\frac{(s^2+36)(s^2+16) - 3(s^2+4)(s^2+16) + 2(s^2+4)(s^2+36)}{(s^2+4)(s^2+36)(s^2+16)}\right)$

$= \frac{1}{2}\left(\frac{72s^2+528}{(s^2+4)(s^2+16)(s^2+36)}\right)$

$L(\sin t\sin 2t\sin 3t) = \frac{36s^2+264}{(s^2+4)(s^2+16)(s^2+36)}$

1.b find the laplace transform of $f(t) = \begin{cases} t^2 & 0<t<2 \\ t-1 & 2<t<3 \\ 7 & t>3 \end{cases}$

Sol w.k.T $L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt = \int_0^2 e^{-st}t^2\,dt + \int_2^3 (t-1)e^{-st}\,dt + \int_3^\infty e^{-st}\cdot 7\,dt$

$L(f(t)) = \left[\frac{-t^2e^{-st}}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3}\right]_0^2 + \left[\frac{e^{-st}}{s}(t-1) - \frac{e^{-st}}{s^2}\right]_2^3 + \left[7\cdot\frac{e^{-st}}{-s}\right]_3^\infty$

$$= \left[\frac{-4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3}\right] - \left[0 - \frac{7}{s}e^{-3s}\right] + \left[\frac{e^{-3s}}{s}(-2) - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2}\right]$$

$$= \frac{2}{s^3} - \frac{2e^{-2s}}{s^3} - \frac{4e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{7}{s}e^{-3s} - \frac{2e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2}$$

$$= \frac{1}{s^3}\left(2 - 2e^{-2s}\right) + \frac{1}{s^2}\left(e^{-2s} - 4e^{-2s} - e^{-3s}\right) + \frac{1}{s}\left(e^{-2s} - 4e^{-2s} + 7e^{-3s} - 2e^{-3s}\right)$$

$$= \frac{2}{s^3} + \frac{e^{-2s}}{s^3}\left(2 + 3s + 3s^2\right) + \frac{e^{-3s}}{s^2}(5s - 1)$$

2(a) Find $L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\}$

Sol: we know that $L(\sin t) = \frac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\,ds = \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s.$$

$$L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}s$$

thus by shifting property

$$L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$$

2(b) show that $\int_0^\infty e^{-2t}\,\frac{\sinh t}{t}\,dt = \frac{1}{2}\log 3$

Sol: $LHS = \int_0^\infty e^{-2t}\left(\frac{\sinh t}{t}\right)dt = L\left(\frac{\sinh t}{t}\right)$ where $s = +2$

$$L(\sinh t) = \frac{1}{s^2-1}$$

$$L\left(\frac{\sinh t}{t}\right) = \int_s^\infty \frac{ds}{s^2-1}\,ds = \frac{1}{2}\int_s^\infty \frac{(s+1)-(s-1)}{(s+1)(s-1)}\,ds$$

$$= \frac{1}{2}\int_{s+0}^\infty \left(\frac{1}{s-1} - \frac{1}{s+1}\right)ds$$

Scanned with CamScanner

$$= \frac{1}{2}\left[\log(s-1) - \log(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

$$= \frac{1}{2}\log 3 = R.H.S$$

L.H.S = R.H.S

Hence proved.

3(a) Evaluate $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\,dt$

Sol:- $\int_0^\infty e^{0t}\left(\frac{e^{-t}-e^{-2t}}{t}\right)dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]_{s=0} \longrightarrow (1)$

$$L\left[\frac{e^{-t}-e^{-2t}}{t}\right] = \int_s^\infty L(e^{-t}-e^{-2t})\,ds$$

$$= \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+2}\right)ds$$

$$= \left(\log(s+1) - \log(s+2)\right)_s^\infty = \log\left(\frac{s+2}{s+1}\right)$$

From (1)

$$\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]_{s=0} = \log 2.$$

3(b) Find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

Sol:- since $F(t) = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$ and

$$g(t) = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \frac{1}{a}\sin at$$

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \cos au \cdot \frac{\sin a(t-u)}{a}\,du$$

Scanned with CamScanner

$$= \frac{1}{2a}\int_0^t \left[\sin at - \sin(2au - at)\right] dt \qquad \left[\begin{array}{l} \because f(u) = \cos au \\ g(t-u) = \frac{1}{a}\sin a(t-u) \end{array}\right]$$

$$= \frac{1}{2a}\left[u\sin at + \frac{1}{2a}\cos(2au - at)\right]_0^t$$

$$= \frac{1}{2a} t\sin at$$

Hence $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t\sin at$

4a) Find the inverse Laplace transform of $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$ using convolution theorem

Sol: Since $L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$, $L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{\sin 3t}{3}$

By convolution theorem, we get

$$L^{-1}\left[\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right] = \int_0^t \sin u \cdot \frac{\sin 3(t-u)}{3}\, du$$

$$= \frac{1}{6}\int_0^t \left[\cos(4u-3t) - \cos(3t-2u)\right] du$$

$$= \frac{1}{6}\left[\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{-2}\right]_0^t$$

$$= \frac{1}{6}\left\{\frac{1}{4}(\sin t + \sin 3t) + \frac{1}{2}(\sin t - \sin 3t)\right\}$$

$$= \frac{1}{6\times 4}\left\{\sin t + \sin 3t + 2\sin t - 2\sin 3t\right\}$$

$$= \frac{1}{24}\left\{3\sin t - \sin 3t\right\}$$

$$L^{-1}\left(\frac{1}{(s^2+1)(s^2+9)}\right) = \frac{1}{8}\left(\sin t - \frac{1}{3}\sin 3t\right)$$

Scanned with CamScanner

4(b) Find $L\left\{\frac{\sin t \sin 5t}{t}\right\}$

Sol: $L\left\{\frac{\sin t \sin 5t}{t}\right\} = \int_s^\infty L(\sin t \sin 5t)\, ds$

$$L(\sin t \sin 5t) = L\left(\frac{1}{2}(\cos(4t) - \cos(6t))\right)$$

$$= \frac{1}{2}L(\cos 4t) - \frac{1}{2}L(\cos 6t)$$

$$= \frac{1}{2}\left[\frac{s}{s^2+16}\right] - \frac{1}{2}\left[\frac{s}{s^2+36}\right]$$

$$= \frac{s}{2}\left[\frac{\cancel{s^2}+36-\cancel{s^2}-16}{(s^2+16)(s^2+36)}\right]$$

$$= \frac{10s}{(s^2+16)(s^2+36)}$$

$$L\left(\frac{\sin t \sin 5t}{t}\right) = \int_s^\infty \frac{10s}{(s^2+16)(s^2+36)}\, ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{s}{s^2+16} - \frac{s}{s^2+36}\right) ds$$

$$= \frac{1}{4}\left[\log(s^2+16) - \log(s^2+36)\right]_s^\infty$$

$$\therefore L\left(\frac{\sin(t)\sin(5t)}{t}\right) = \frac{1}{4}\log\left(\frac{s^2+36}{s^2+16}\right)$$

5(a) Find $L\{t^2 e^{-t}\cos t\}$

Sol: $L\{t^2 e^{-t}\cos t\} = \frac{d^2}{ds^2}F(s)$ $\quad F(s) = L(e^{-t}\cos t)$

$L(\cos t) = \frac{s^2}{s^2+a^2} = \frac{s}{s^2+1}$ $\quad L(e^{-t}\cos t) = \frac{s+1}{(s+1)^2+1^2}$

Scanned with CamScanner

$$L\{t^2e^{-t}\cos t\} = \frac{d^2}{ds^2}\left(\frac{s+1}{(s+1)^2+1}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{(s+1)}{(s+1)^2+1}\right)\right)$$

$$= \frac{1}{2}\,\frac{d}{ds}\left(\log\left((s+1)^2+1\right)\right)$$

$$= \frac{1}{2}\cdot\frac{1}{1+(s+1)^2}\,2(s+1)$$

$$= \frac{s+1}{(s+1)^2+1}$$

$$\therefore\ L\{t^2e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1}$$

5(b) Solve $(D^3+D)x = 2$ if $x(0)=3$, $x'(0)=1$, $x''(0)=-2$ using Laplace transform method

Sol: Given equation is $x''' + x' = 2 \rightarrow ①$

By applying Laplace transform to equ①, we get

$$L[x'''+x'] = L[2]$$

$$L[x'''] + L[x'] = L[2]$$

$$\Rightarrow s^3[L(x)] - s^2x(0) - s(x'(0)) - x''(0) + sL[x] - x[0] = \frac{2}{s}$$

$$s^3L[x] + sL[x] - 3s^2 - s - 1 - \frac{2}{s} = 0$$

$$L[x](s^3+s) = \frac{3s^3+s^2+s+2}{s}$$

$$L[x] = \frac{3s^3+s^2+s+2}{s^2(s^2+1)}$$

$$L[x] = \frac{3s^3+s^2+s+2}{s^2(s^2+1)}$$

$$L[x] = \frac{s^2(3s+1)}{s^2(s^2+1)} + \frac{s+2}{s^2(s^2+1)}$$

$$x = L^{-1}\left[\frac{3s+1}{s^2+1} + \frac{s+2}{s^2(s^2+1)}\right]$$

$$= L^{-1}\left[3\cdot\frac{s}{s^2+1} + \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} + \frac{2}{s^2(s^2+1)}\right]$$

$$= 3\cos t + \sin t + L^{-1}\left[\frac{s+2}{s^2(s^2+1)}\right]$$

$$= 3\cos t + \sin t + L^{-1}\left[\frac{s+2}{s^4+s^2}\right] = 3\cos t + \sin t + L^{-1}\left[\frac{1}{s(s^2+1)} + \frac{2}{s^2(s^2+1)}\right]$$

$$x = 3\cos t + \sin t + \frac{1-\cos t}{1} + L^{-1}\left[\frac{2}{s^2(s^2+1)}\right]$$

$$L^{-1}\left[\frac{1}{s^2}\right] = \frac{t^{2-1}}{(2-1)(1)} = t \qquad L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

By convolution theorem

$$L^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right] = \int_0^t u.\sin(t-u)\,du$$

$$= \left[u\cos(t-u) + \sin(t-u)\right]_0^t = t + \sin t.$$

$$\therefore x = 3\cos t + \sin t + 1 - \cos t + 2t + 2\sin t$$

$$x = 2\cos t + 3\sin t + 2t + 1.$$

Scanned with CamScanner

6(a) show that $\int_0^\infty \frac{e^{-2t}\sinh t}{t}\,dt = \frac{1}{2}\log 3$

Sol: $LHS = \int_0^\infty \frac{e^{-2t}\sinh t}{t}\,dt = L\left(\frac{\sinh t}{t}\right)\Big|_{s=2}$

$$L\left(\frac{\sinh t}{t}\right) = \int_s^\infty L(\sinh t)\,ds$$

$$= \int_s^\infty \frac{ds}{s^2-1} = \frac{1}{2}\int_s^\infty \frac{(s+1)-(s-1)}{(s+1)(s-1)}\,ds$$

$$= \frac{1}{2}\left[\log(s-1) - \log(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

$$\int_0^\infty e^{-2t}\frac{\sinh t}{t} = \frac{1}{2}\log 3 \quad [\because s=2]$$

Hence proved.

6(b) show that D.E $y''+2y'+5y = 8\sin t + 4\cos t$, $y(0)=1$ and $y(\pi/4)=\sqrt{2}$ using Laplace transform method.

Sol: Since $y'(0)$ is not given, we assume $y'(0)=a$

Given D.E is $y''+2y'+5y = 8\sin t + 4\cos t$

Taking the Laplace transform on both sides of equation

$$L[y''] + 2L[y'] + 5L[y] = 8L[\sin t] + 4L[\cos t]$$

$$s^2 L[y] - s\,y(0) - y'(0) + 2[s[L(y)] - y[0]] + 5L[y] = 8\left(\frac{1}{s^2+1}\right) + 4\left(\frac{s}{s^2+1}\right)$$

$$L[y](s^2+2s+6) - s(1) - a - 2 = \frac{4s+8}{s^2+1}$$

$$L[y] = \frac{\frac{4s+8}{s^2+1} + s + a + 2}{s^2+2s+5} = \frac{4s+8+s^3+s+as^2+a+2s^2+2}{(s^2+1)(s^2+2s+5)}$$

$$= \frac{s^3+5s+2s^2+10+as^2+a}{(s^2+1)(s^2+2s+5)}$$

$$= \frac{s(s^2+2s+5)+as^2+a+10}{(s^2+1)(s^2+2s+5)}$$

$$L[y] = \frac{s}{s^2+1} + \frac{a}{s^2+2s+5} + \frac{10}{(s^2+1)(s^2+2s+5)}$$

on inversion

$$y = \cos t + a\left(\tfrac{1}{2}\right)e^{-t}\sin 2t + L^{-1}\left(\frac{10}{(s^2+1)((s+1)^2+2^2)}\right)$$

$$\frac{1}{(s^2+1)(s^2+2s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}$$

$$1 = (As+B)(s^2+2s+5) + (s^2+1)(Cs+D)$$

$$1 = As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + Ds^2 + Cs + D$$

$$1 = s^3(A+C) + s^2(2A+B+D) + s(5A+2B+C) + (5B+D)$$

on equating coefficients

$A+C=0 \rightarrow (1)$ $\quad 2A+B+D=0 \rightarrow (2)$ $\quad 5A+2B+C=0 \rightarrow (3)$ $\quad 5B+D=1 \rightarrow (4)$

From (1) & (3)

$$4A+2B+A+C=0$$
$$4A+2B=0$$
$$2A+B=0$$

From (2)

$$D=0$$

From (4)

$$B = 1/5.$$

From (2)

$$2A+\frac{1}{5}=0 \Rightarrow 2A = -1/5 \Rightarrow A = -1/10$$

From (3)

$$-\frac{1}{2}+\frac{2}{5}+C=0 \Rightarrow C = \frac{1}{2}-\frac{2}{5} = \frac{1}{10}$$

$$\frac{1}{(s^2+1)(s^2+2s+5)} = \frac{-\frac{1}{10}s+\frac{1}{5}}{s^2+1} + \frac{\frac{1}{10}s+0}{s^2+2s+5}$$

$$L^{-1}\left(\frac{10}{(s^2+1)(s^2+2s+5)}\right) = L^{-1}\left(\frac{-s+2}{s^2+1} + \frac{s}{(s+1)^2+2^2}\right)$$

$$= -L^{-1}\left(\frac{s}{s^2+1}\right) + 2L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left(\frac{s}{(s+1)^2+2^2}\right)$$

Scanned with CamScanner

$$= -\cos t + 2\sin t + e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

$$\therefore y = \cos t + \frac{a}{2}e^{-t}\sin 2t + 2\sin t - \cos t + e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

$$\therefore y = \frac{a}{2}e^{-t}\sin 2t + 2\sin t + e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

given $y(\pi/4) = \sqrt{2}$

$$\sqrt{2} = \frac{a}{2}e^{-\pi/4}(1) + \frac{2}{\sqrt{2}} + e^{-\pi/4}(0) - \frac{1}{2}e^{-\pi/4}(1)$$

$$\sqrt{2} - \sqrt{2} = e^{-\pi/4}\left(\frac{a}{2} - \frac{1}{2}\right)$$

$$0 = e^{-\pi/4}\left(\frac{a}{2} - \frac{1}{2}\right)$$

$$\therefore \frac{a}{2} - \frac{1}{2} = 0$$

$$\therefore \boxed{a = 1}$$

$$\therefore y = \cos t + \frac{1}{2}e^{-t}\sin 2t + 2\sin t - \cos t + e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

$$\therefore y = \frac{1}{2}e^{-t}\sin 2t + 2\sin t + e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t$$

$$\therefore y = 2\sin t + e^{-t}\cos 2t$$

Scanned with CamScanner