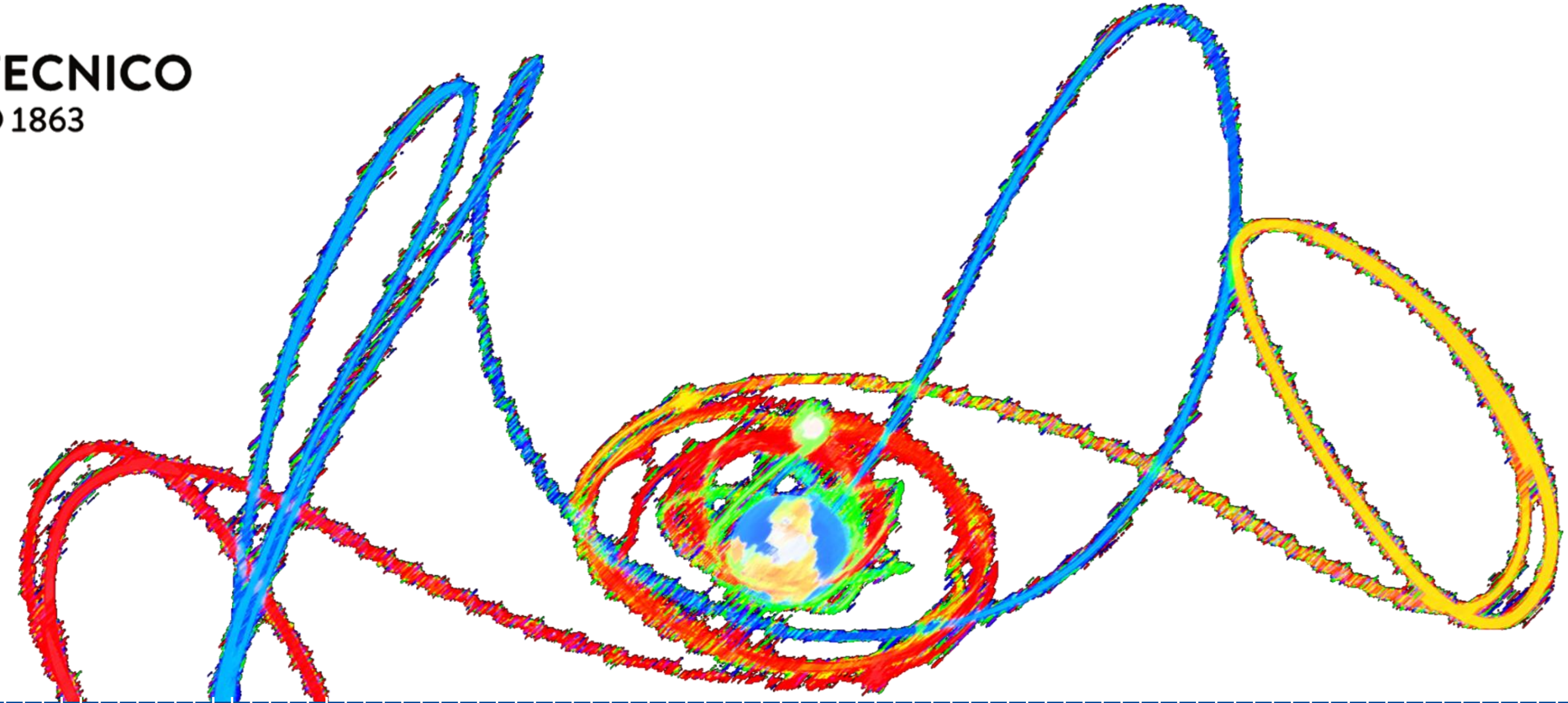

Orbital Mechanics
Module 3: Orbital manoeuvres

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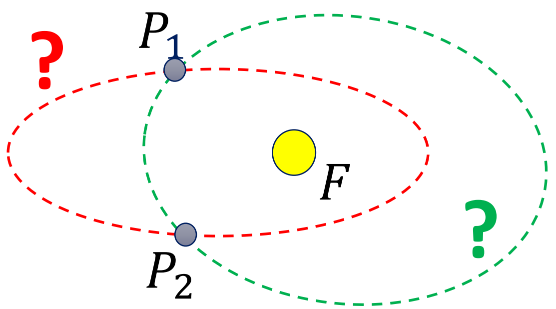
LAMBERT’S PROBLEM

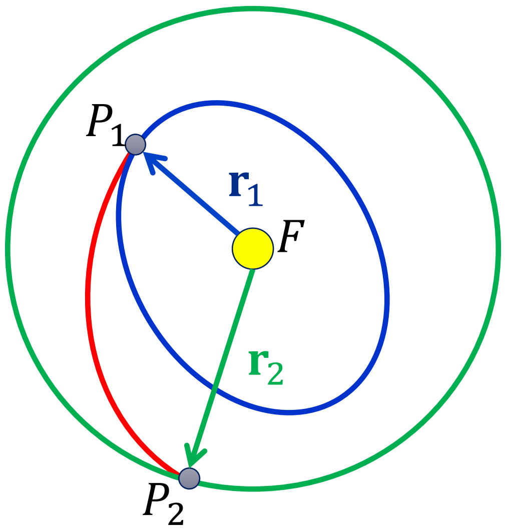
Motivation

Two-body orbital boundary value problem

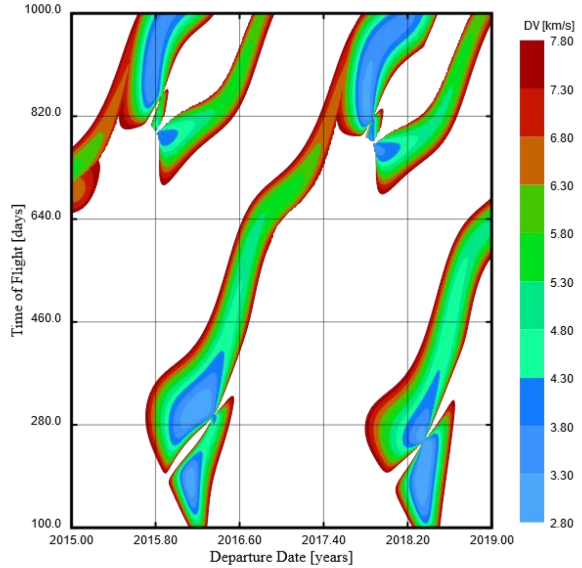
There are many practical situations where we are interested in constructing a **two-body problem orbit** (i.e., a conic) **that passes through given points $\boldsymbol{P_1}$** and $\boldsymbol{P_2}$, with the primary at the focus $\boldsymbol{F}$.



Orbit reconstruction



Orbital transfer



Mission design

This is a **boundary value problem** (state **partially specified at more than one point**), compared to the **initial value problems** (state **completely defined at one point**) we have considered so far.

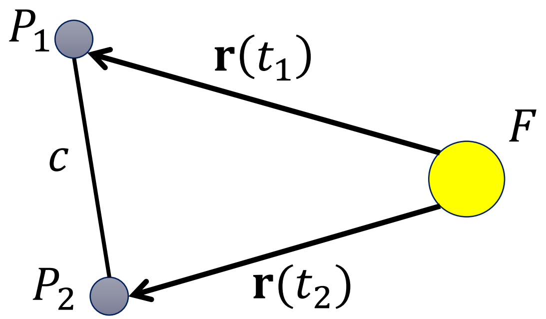
Lambert's Theorem

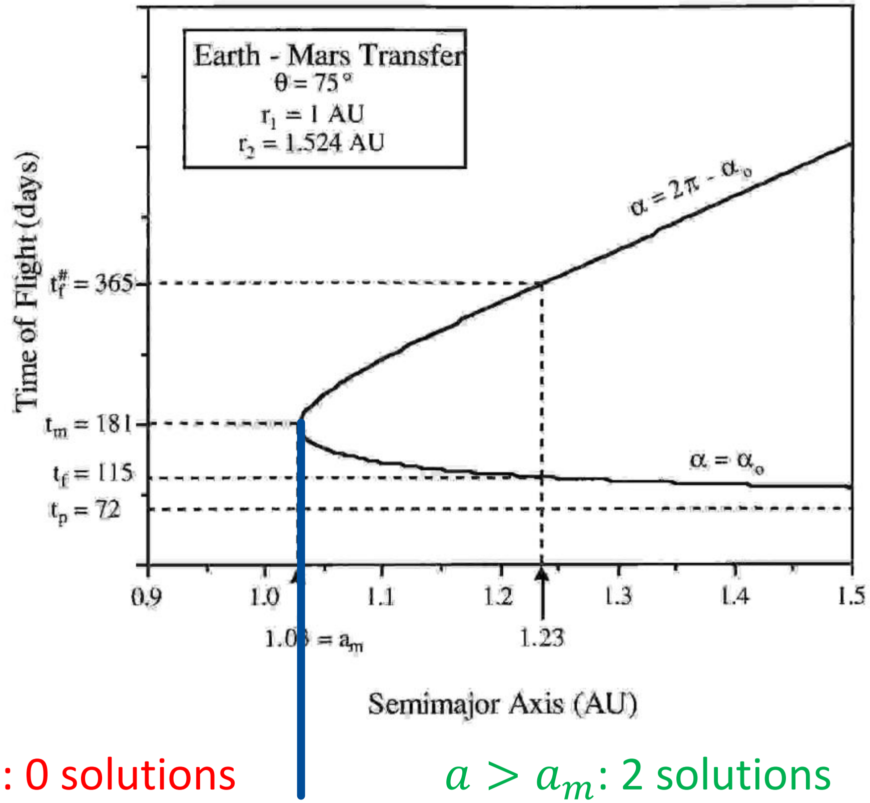
Semi-major axis and time of flight

The focus F and two points P_1 and P_2 are **not enough to univocally define an ellipse**. We have a **family of solutions with infinite possible orbits**. We will choose one using **time information**.

Lambert's Theorem: *The orbital transfer time depends only upon the **semi-major axis**, the sum of the distances of the initial and final points of the arc from the center of force, and the length of the cord joining these points* [1]

$$\sqrt{\mu}(t_2 - t_1) = F(a, r_1 + r_2, c)$$





$a < a_m$: 0 solutions

$a > a_m$: 2 solutions

$a = a_m$: 1 solution

[1] Battin, R., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Education Series, 1999

Solving Lambert's Problem

Lambert's problem: *Definition of an orbit, having a specified transfer time and connecting two position vectors.*

- *For a given Δt and less than one revolution, we have a single solution for semi-major axis a*
- *If we consider orbits with multiple revolutions, there are two solutions: one for large a and one for small a.*

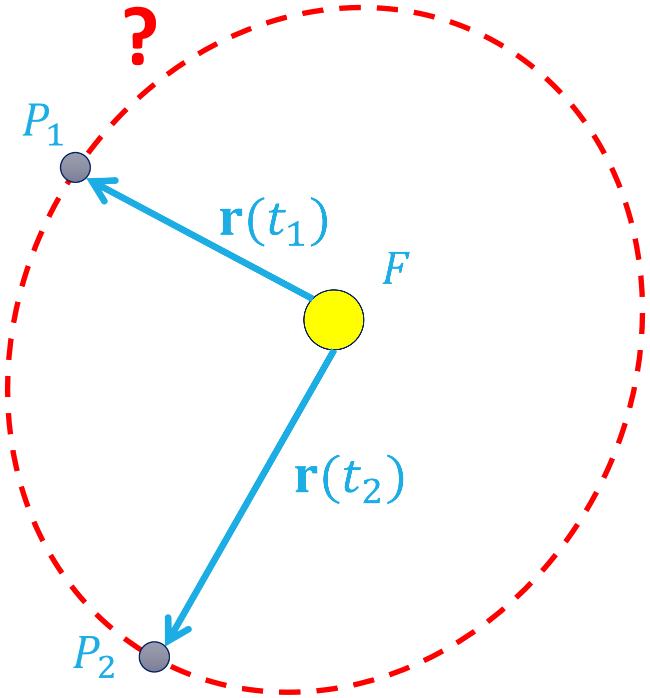
- Many algorithms have been developed to tackle this problem:
  - First one by C.F. Gauss (in book *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium*)
  - More recent results:
    - Battin, R. H., and Vaughan, R. M., "An elegant Lambert algorithm," *Journal of Guidance, Control, and Dynamics*, *7*(6):662-670, 1984.
    - Avanzini, G., "A simple Lambert algorithm," *Journal of Guidance, Control, and Dynamics*, *31*(6):1587-1594, 2008.
    - Arora, N., and Russell, R., "A Fast and Robust Multiple Revolution Lambert Algorithm Using a Cosine Transformation," *Astrodynamics 2013, Advances in the Astronautical Sciences*, 150, AAS Paper 13-728, 2013.
    - Gooding, R., "A Procedure for the Solution of Lambert's Orbital Boundary-Value Problem," *Celestial Mechanics and Dynamical Astronomy*, 48(2):145–165, 1990.
    - Izzo, D., "Revisiting Lambert's Problem," *Celestial Mechanics and Dynamical Astronomy*, 121(1):1–15, 2015.
    - Bombardelli, C., Gonzalo, J. L., and Roa, J., "Approximate analytical solution of the multiple revolution Lambert's targeting problem," *Journal of Guidance, Control, and Dynamics*, 41(3):792-801, 2018. Online app available.
- Typical output are the velocities at the initial and final points, $\mathbf{v}(t_1)$ and $\mathbf{v}(t_2)$
- **For the lab, you are provided a Lambert solver (download it from WeBeep)**

Exercise 1: State reconstruction problem

Where will we be?

We have two position vectors evaluated at different times t_1 and t_2 (for instance, coming from telescope or radar observations)

Can we reconstruct the orbit?



$\mathbf{r}(t_1)$
$\mathbf{r}(t_2)$

Position at two specific times

? →

$\mathbf{r}(t)$
$\mathbf{v}(t)$

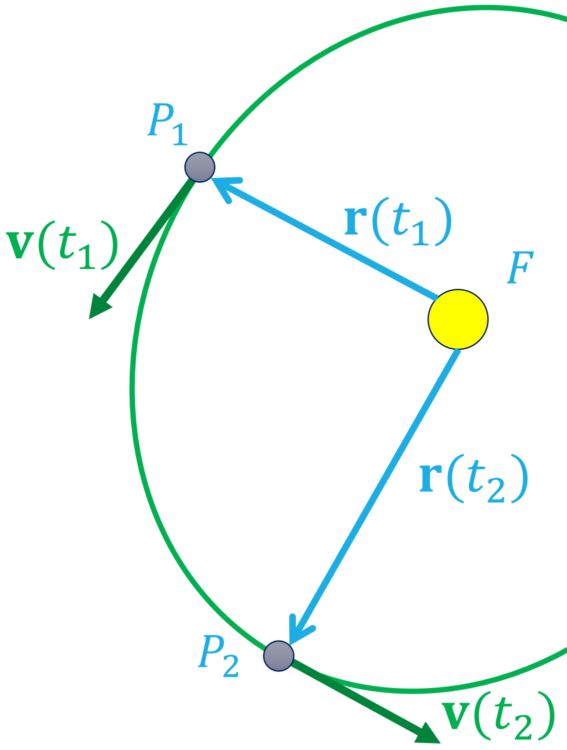
Position and velocity at a generic time

Exercise 1: State reconstruction problem

Where will we be?

We have two position vectors evaluated at different times t_1 and t_2 (for instance, coming from telescope or radar observations)

We can reconstruct the orbit



In many Lambert solvers, outputs are the velocity vectors at P_1 and P_2 (either one can be used to define the orbit)

Exercise 1: State reconstruction problem

Workflow

INPUTS
$\mathbf{r}(t_1)$
$\mathbf{r}(t_2)$
Δt
prograde/retrograde

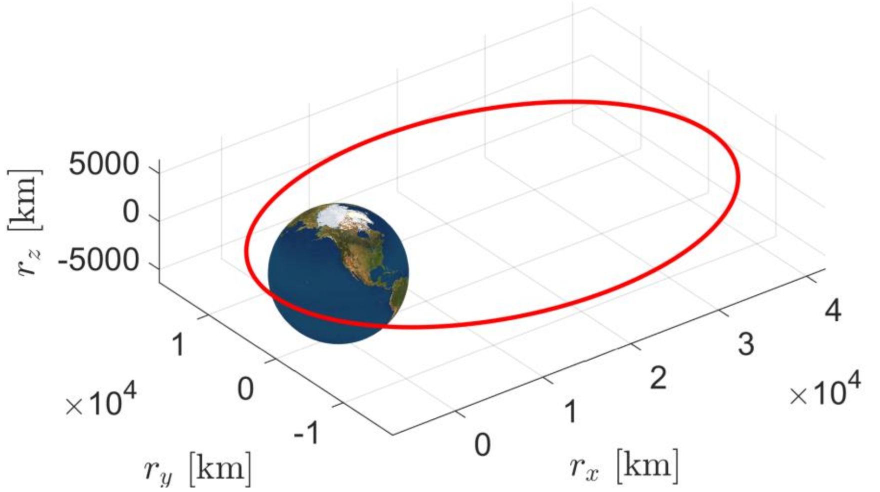
Lambert Solver

OUTPUTS
$\mathbf{v}(t_1)$
$\mathbf{v}(t_2)$

Initial conditions
$\mathbf{r}(t_0), \mathbf{v}(t_0)$
[either $\mathbf{r}(t_1), \mathbf{v}(t_1)$ or $\mathbf{r}(t_2), \mathbf{v}(t_2)$]

Orbit propagation

Orbit propagation: prediction of the orbital characteristics of a body at some future date given the current orbital characteristics.



Exercise 1: State reconstruction problem

Exercise 1: State reconstruction problem

1. Write a script to solve Lambert's problem for the values of $\mathbf{r}(t_1)$, $\mathbf{r}(t_2)$ and $\Delta t = t_2 - t_1$ given below
   - Use the provided Lambert solver `lambertMR.m`. Check sample script `call_lambertMR.m` to learn how to use it
2. Propagate and plot the resulting orbit
   - Reuse the orbit propagation functions from **Module 1**
   - Use as initial conditions for orbit propagation either $[\mathbf{r}(t_1), \mathbf{v}(t_1)]$ (propagation forward in time) or $[\mathbf{r}(t_2), \mathbf{v}(t_2)]$ (propagation backward in time)

Data

Prograde orbit

$\mu_\oplus$ from `astroConstants.m` (identifier `13`)

$\mathbf{r}(t_1) = [-21800\,;\;\; 37900\,;\;\; 0\;]\ \mathrm{km}$

$\mathbf{r}(t_2) = [\,27300\,;\;\; 27700\,;\;\; 0\;]\ \mathrm{km}$

$\Delta t = 15\ \mathrm{h}, 6\ \mathrm{min}, 40\ \mathrm{s}$

Exercise 1: State reconstruction problem

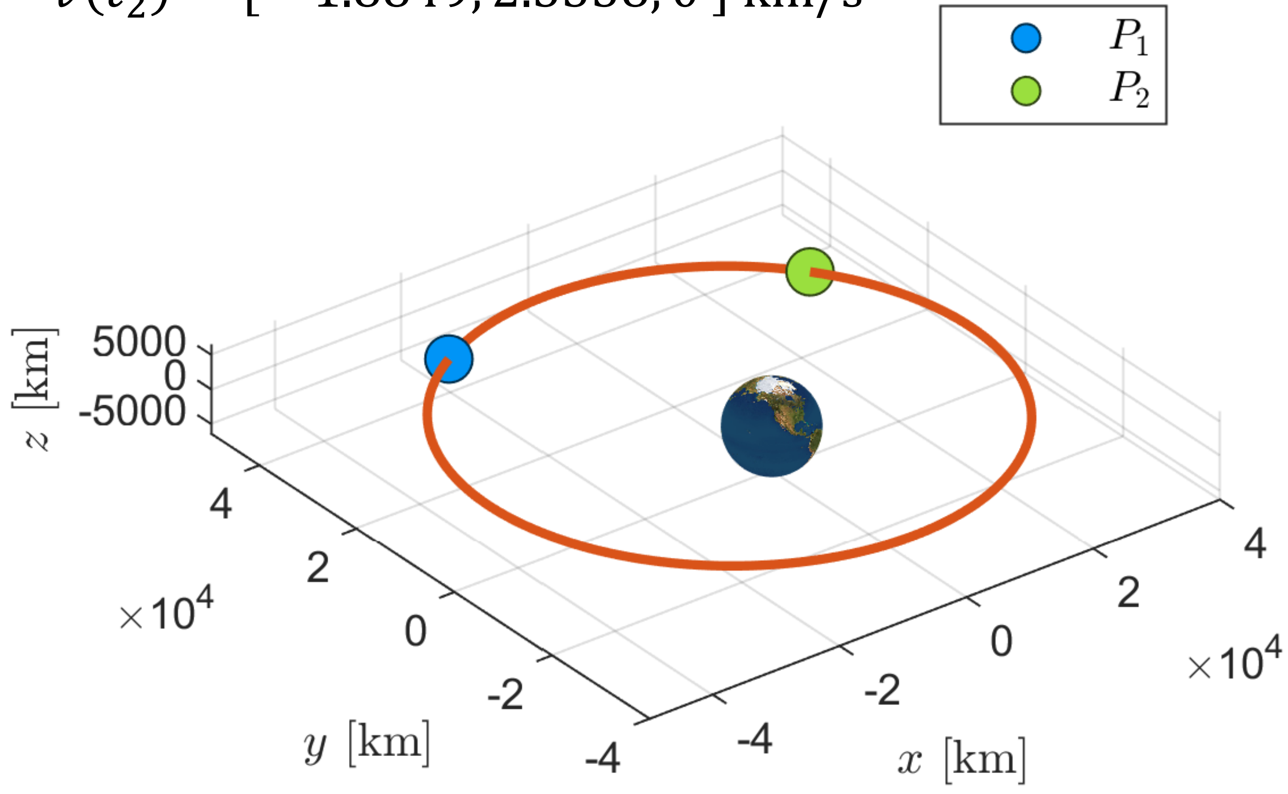
Sample solution

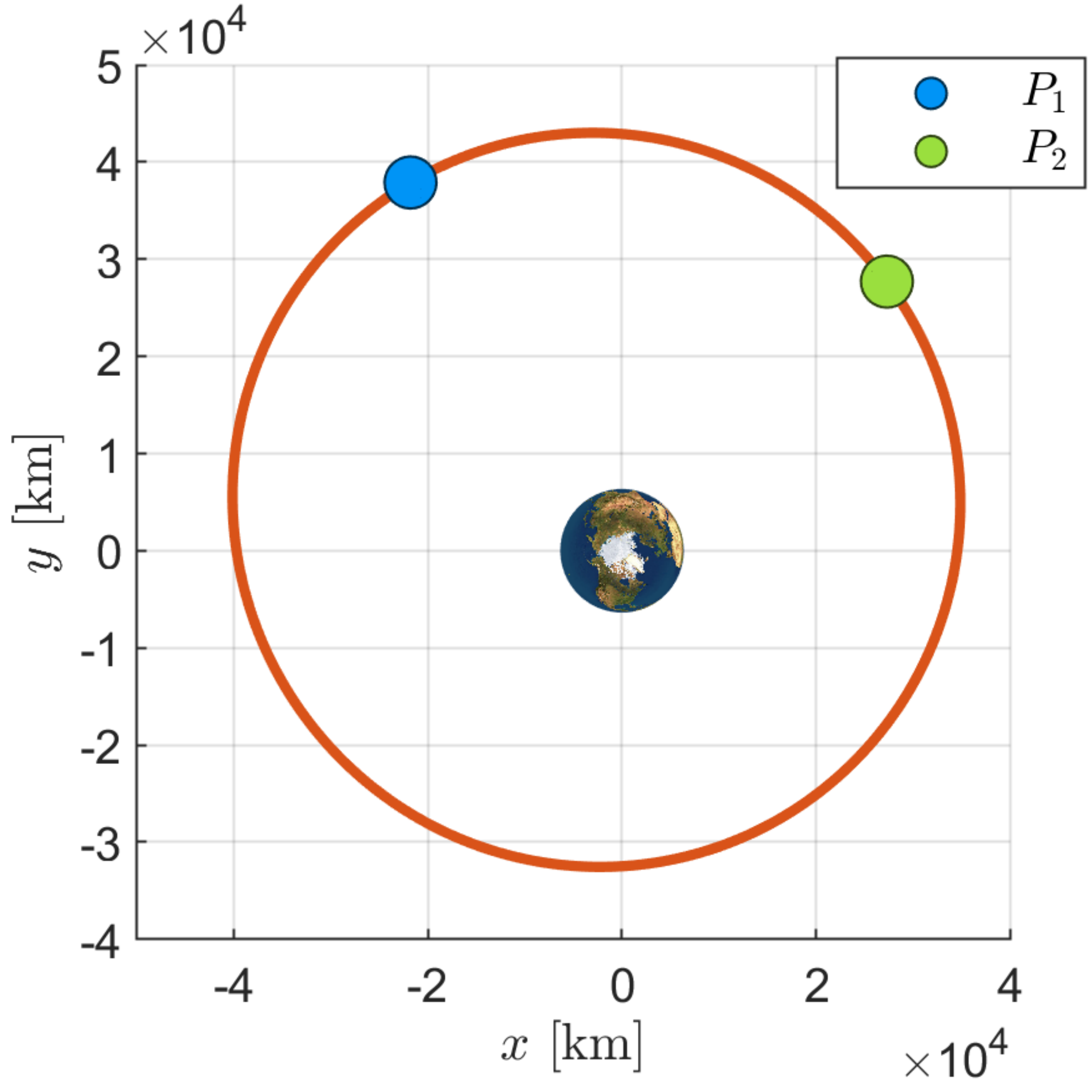
Solution for Lambert's problem:

$a = 3787.1212\ \text{km}$

$v(t_1) = [-2.3925, -1.4086, 0]\ \text{km/s}$

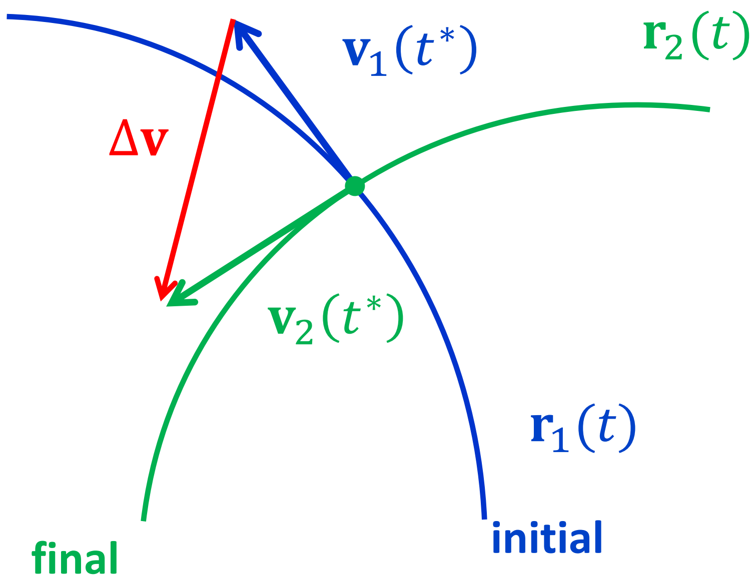
$v(t_2) = [-1.8849, 2.5338, 0]\ \text{km/s}$





Exercise 2: Orbit transfer problem

Transfer problem with intersection



For intersecting orbits, orbit transfer can be performed through a single manoeuvre at intersection time t^*

At intersection time t^*:

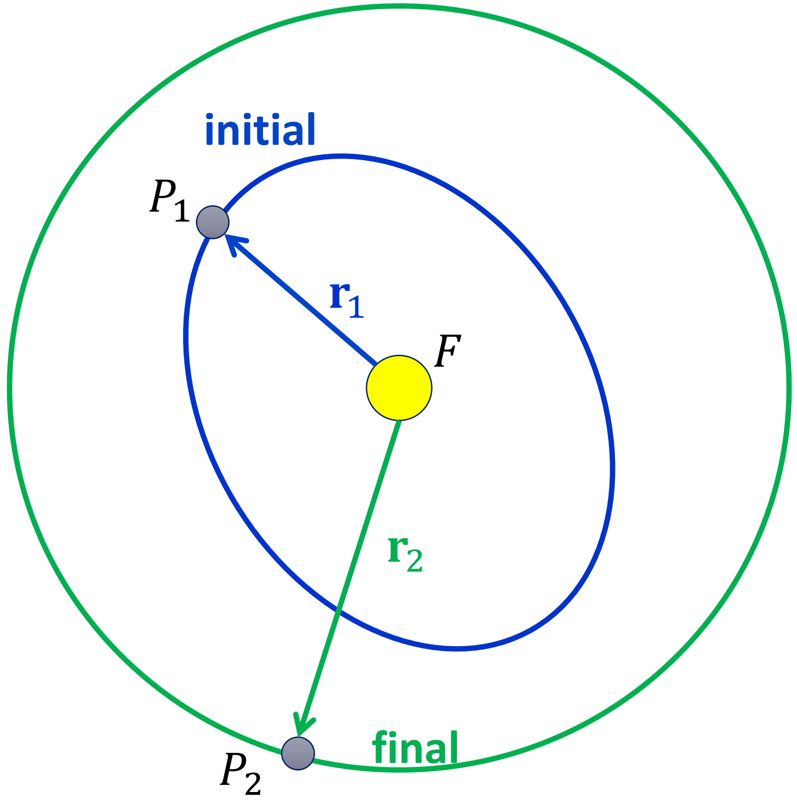
- ☑ $\mathbf{r}_1(t^*) = \mathbf{r}_2(t^*)$
- ☒ $\mathbf{v}_1(t^*) \neq \mathbf{v}_2(t^*)$

We want to move from **initial** to **final** orbit (that is, we have to **change our state**) → $\mathbf{v}_1(t^*) + \Delta\mathbf{v} = \mathbf{v}_2(t^*)$

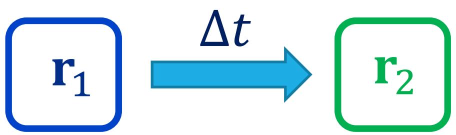
Cost of the manoeuvre: $\|\Delta\mathbf{v}\|$

Exercise 2: Orbit transfer problem

Transfer problem with no intersections

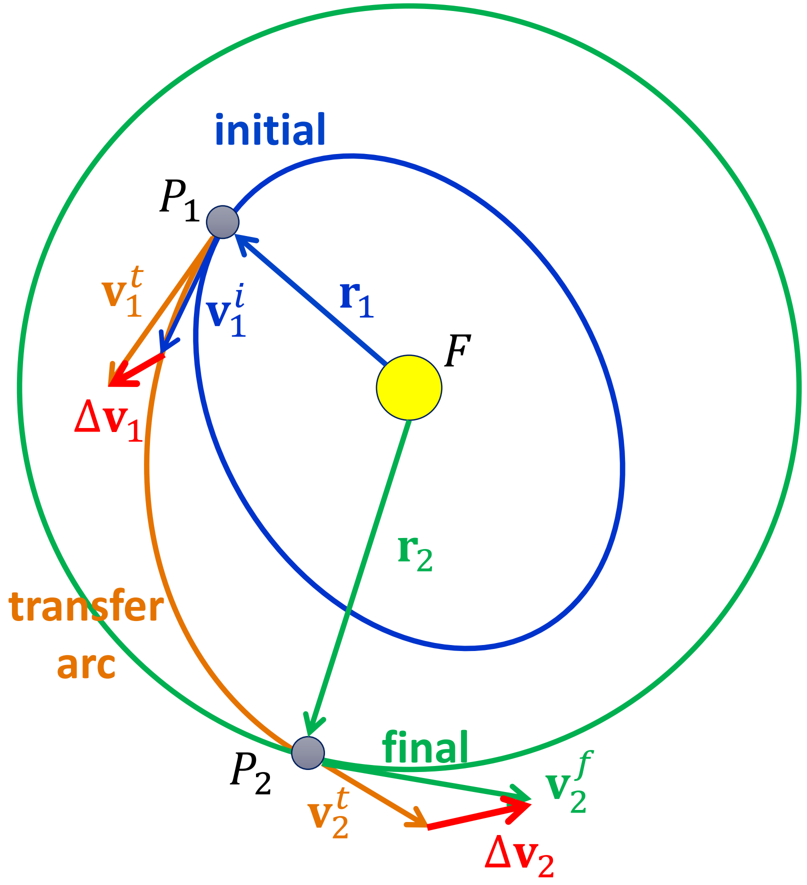


At least two manoeuvers are required



Exercise 2: Orbit transfer problem

Transfer problem with no intersections



$\mathbf{r}_1$
$\mathbf{r}_2$
Δt

Lambert Problem →

Transfer arc

Injection manoeuvre (from initial orbit to transfer arc):

$$\Delta\mathbf{v}_1 = \mathbf{v}_1^t - \mathbf{v}_1^i$$

Arrival manoeuvre (from transfer arc to final orbit):

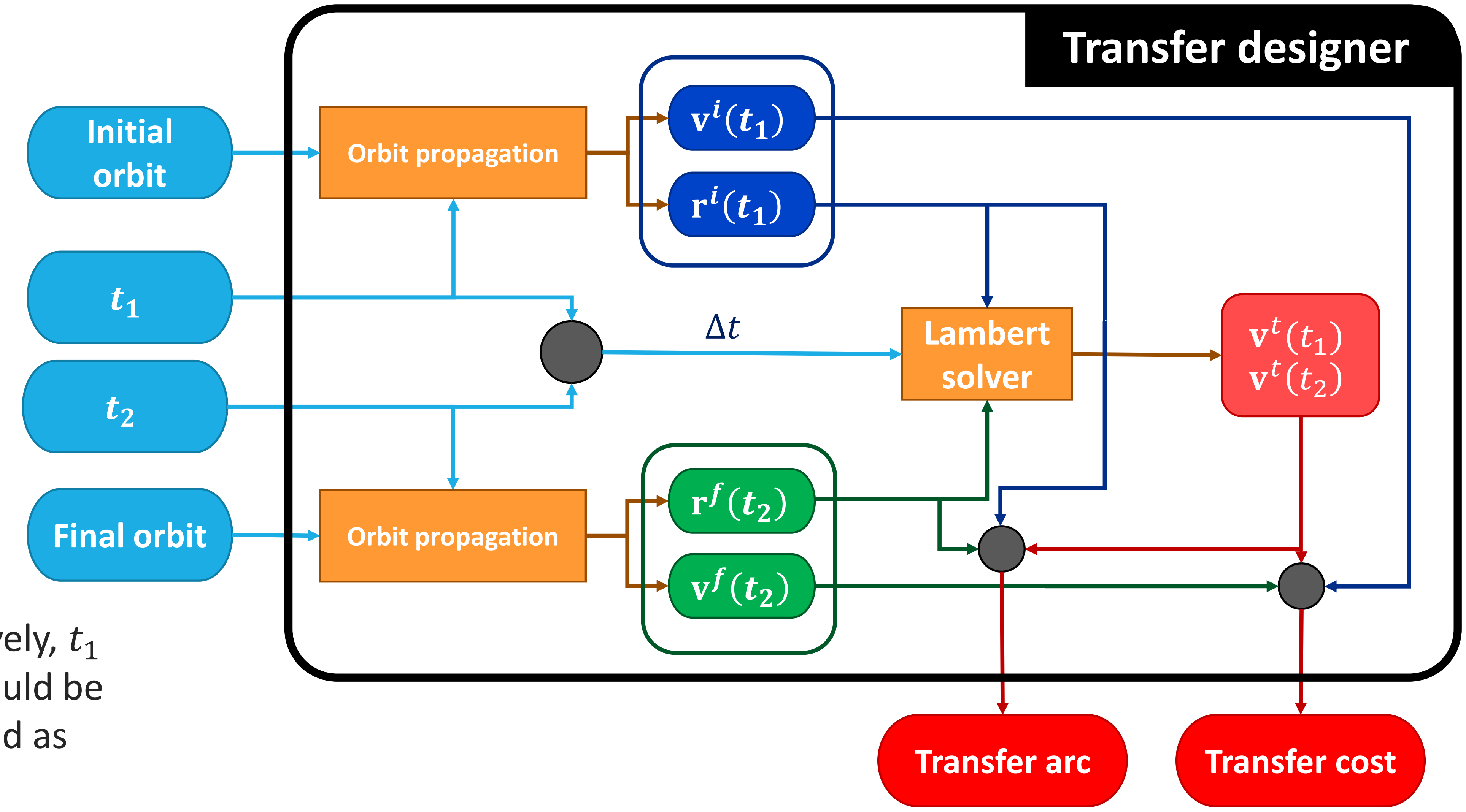
$$\Delta\mathbf{v}_2 = \mathbf{v}_2^f - \mathbf{v}_2^t$$

The problem has 0 degrees of freedom for given P_1, P_2, and Δt

Total cost of the mission: $\Delta v_{tot} = \|\Delta\mathbf{v}_1\| + \|\Delta\mathbf{v}_2\|$

Exercise 2: Orbit transfer problem

Workflow for a fixed-time transfer



Alternatively, t_1 and Δt could be considered as inputs

Exercise 2: Orbit transfer problem

Exercise 2: Orbit transfer problem

1. Compute the initial and final states in Cartesian coordinates
2. Solve Lambert's problem for the transfer arc
3. Compute the total cost of the manoeuvre $||\Delta\mathbf{v}_1|| + ||\Delta\mathbf{v}_2||$
4. Propagate the transfer arc, from t_1 to t_2
5. Plot the initial and final orbits, and the transfer arc

Data

Earth-bound orbits, $\mu_\oplus$ from `astroConstants.m` (identifier `13`)

Prograde transfer arc

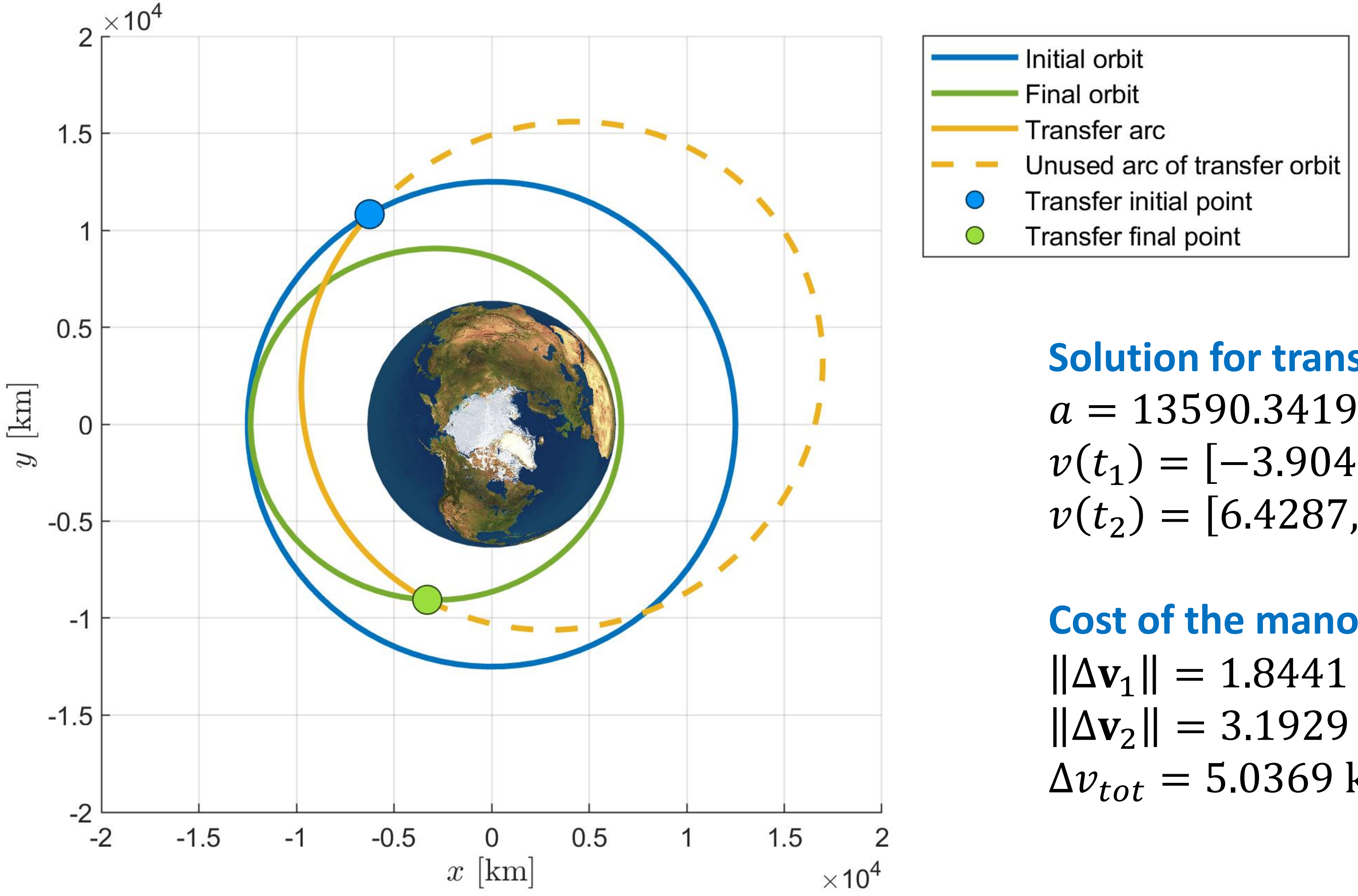
$kep_1 = [a_1; e_1; i_1; \Omega_1; \omega_1; f_1] = [\ 12500 \text{ km};\ 0\ ; 0 \text{ deg}; 0 \text{ deg}; 0 \text{ deg};\ 120 \text{ deg}\ ]$

$kep_2 = [a_2; e_2; i_2; \Omega_2; \omega_2; f_2] = [\ 9500 \text{ km};\ 0.3\ ; 0 \text{ deg}; 0 \text{ deg}; 0 \text{ deg}; 250 \text{ deg}]$

$tof\ = \Delta t = 3300 \text{ s}$

Exercise 2: Orbit transfer problem

Sample solution



Solution for transfer arc:

$a = 13590.3419$ km

$v(t_1) = [-3.9045, -4.3819, 0]$ km/s

$v(t_2) = [6.4287, -3.4778, 0\,]$ km/s

Cost of the manoeuvre:

$\|\Delta \mathbf{v}_1\| = 1.8441$ km/s

$\|\Delta \mathbf{v}_2\| = 3.1929$ km/s

$\Delta v_{tot} = 5.0369$ km/s

TRANSFER DESIGN

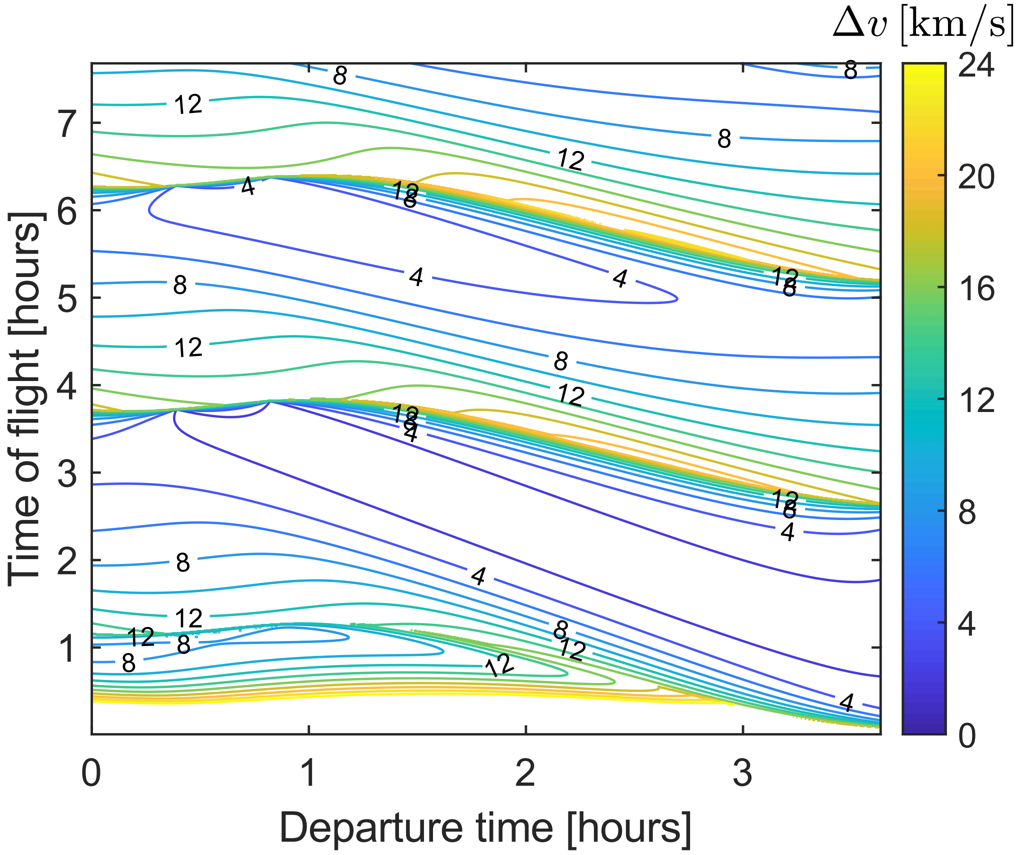
Transfer design

A parametric optimization problem

In **Exercise 2**, initial time t_1 and final time t_2 were fixed, leading to a **single possible transfer arc**.

What happens if instead we want to design a transfer between two celestial bodies, without *a priori* fixed values for departure and arrival time?

- **Departure and arrival time are free parameters**, leading to a **family of possible transfer arcs**, each one with different Δv
- State (position and velocity) at the initial and final orbits is a known function of time. Therefore, we have just **2 degrees of freedom**
- $\Delta v(t_1, t_2)$ can be plotted as a contour plot known as **porkchop plot**
- This is a **powerful tool for mission design**



Transfer design

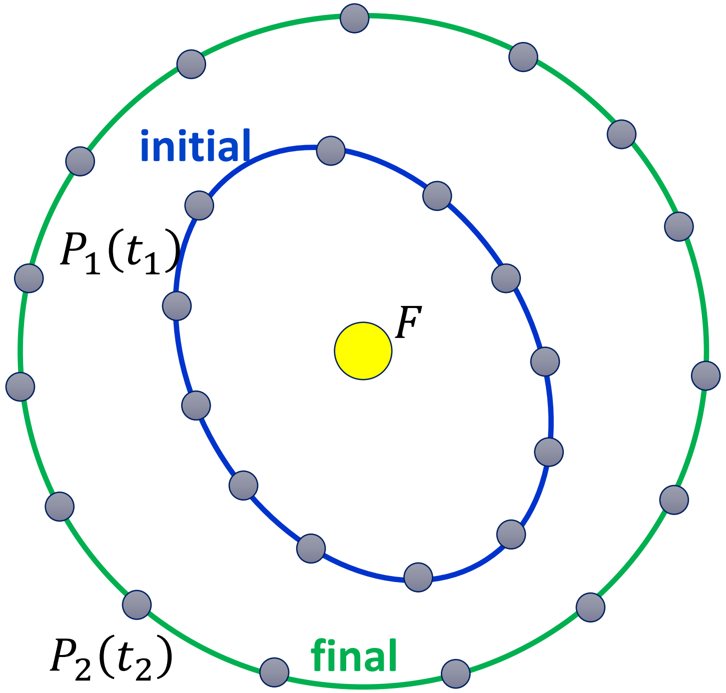
Choice of design variables and ranges

- The **2 degrees of freedom** can be parametrized in different ways. The simplest ones are:
  - **Departure time** t_1 and **arrival time** t_2
  - **Departure time** t_1 and **time of flight** $\Delta t = t_2 - t_1$
    - Note that departure point P_1 changes only with the departure time t_1, whereas arrival point P_2 changes with both departure time t_1 and time of flight Δt because $\mathbf{r}(t_2) = \mathbf{r}(t_1 + \Delta\mathrm{t})$
- To locate the minima, it is important to choose **time windows large enough to capture all possible configurations**:
  - For departure window, try to include all relative positions between both bodies. The synodic period is a useful first estimation
  - For time of flight (ToF), you can make initial estimations from simplified transfers (e.g., assume coplanar, circular orbits and compute the Hohmann transfer; take the parabolic time of flight from Lambert solver; etc.). Synodic period is not a good estimation for the ToF in general (e.g., the synodic period of Mercury and Neptune is very short, but the required ToF is very long)
  - In many cases, operational constraints may limit the feasible size for the time windows (for instance, due to the lifetime of the spacecraft systems)

Time-free transfer between two orbits

A two degrees of freedom problem in time

We want to transfer from a **body in the initial orbit** to another **body in the final orbit**



$\mathbf{r}_1(\boldsymbol{t}_1)$
$\mathbf{r}_2(\boldsymbol{t}_2)$
$\Delta t = t_2 - t_1$

Lambert Problem

$\Delta v_{tot}(t_1, t_2)$

Departure and **arrival** points are **functions of the departure and arrival times** (**within the respective windows**).

The problem has 2 degrees of freedom for given departure and arrival bodies

Not all the transfer arcs will fulfill the launcher constraint $\|\Delta \mathbf{v_1}\| \leq \boldsymbol{v}_\infty$

Workflow

Fixed-time transfer (single solution)

Alternatively, t_1 and Δt could be considered as inputs

Workflow

Time-free transfer (parametric solution)

Initial orbit

Departure window

Arrival window

Final orbit

The **fixed-time transfer** is solved for each feasible combination of t_1 and t_2 (or t_1 and Δt) within the corresponding time windows.

Transfer Designer

Reuse code!!!

$\Delta \mathbf{v}_{tot}(t_1, t_2)$

Output is the mission cost for each (t_1, t_2). Other outputs could also be needed to check mission constraints.

Porkchop plots

Making contour plots

- The required Δv as a function of **departure time** and **arrival time** (or **departure time** and **ToF**) can be represented on a **contour plot known as porkchop plot**.
  - Design tool for the analysis of possible launch opportunities.
  - Named for its resemblance to a pork chop for some missions (e.g., Earth to Mars).

- **Contour plots** can be plotted in Matlab with the `contour` function.
  - Check the **documentation center** to learn how to use `contour`.
  - Remember to add a `colorbar` with ticks and labels.

- $\Delta v(t_1, t_2)$ can also be plotted as a 3D surface using the `surf` function (but keep in mind that this is not a porkchop plot).

- Use enough discretization points for the time windows to get smooth plots.

Ephemerides

Locating objects in space

- *A table of the coordinates of celestial bodies as a function of time is called an **ephemeris*** [1].
  - Refer to **Module 2** for more details

- **Instead of propagating the orbits of the departure and arrival bodies, we will use the analytical ephemerides available in WeBeep**:
  - `uplanet`: Analytical ephemerides of planets of the Solar System

- **Be careful with the units!**
  - The ephemeris functions take as input the date in MJD2000 (i.e., days). Lambert solver takes as input the time of flight in seconds.

- The functions provided in **WeBeep** also include time conversion routines (particularly, to convert from calendar date to MJD2000).

[1] Curtis, H. D.. *Orbital mechanics for engineering students,* Butterworth-Heinemann , 2014

Exercise 3: Mars Express

Mission definition

Mars Express: Design an interplanetary transfer with minimum Δv_{tot} between Earth and Mars, under the following mission requirements:

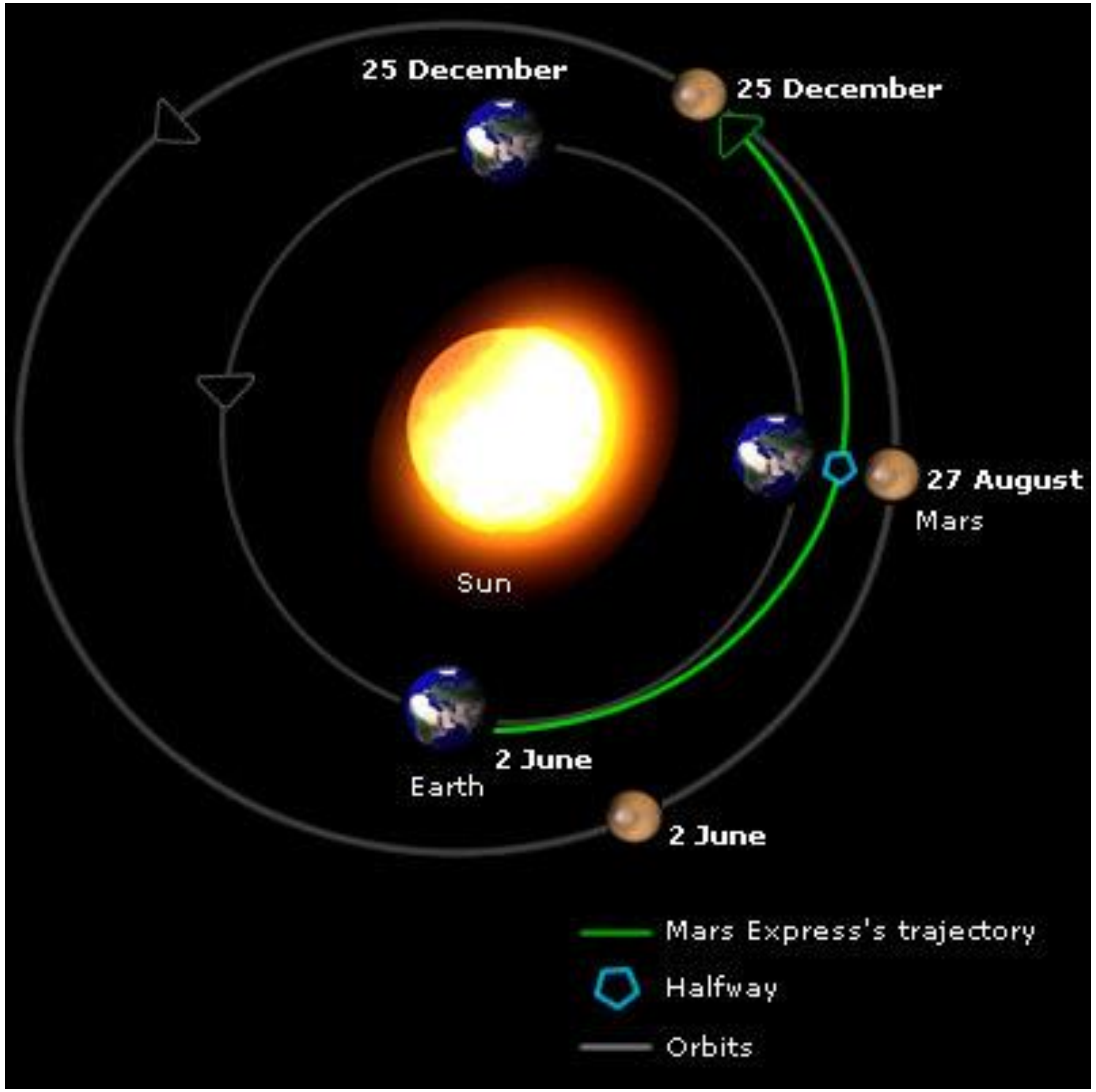
- Departure planet: **Earth**
- Target planet: **Mars**

- Earliest departure requirement: **2003 April 1**
- Latest departure requirement: **2003 August 1**

- Earliest arrival requirement: **2003 September 1**
- Latest arrival requirement: **2004 March 1**

Exercise 3: Mars Express

This is an actual mission!



Results should be very similar to
ESA's Mars Express Mission

- **Departure date:**
  2 June 2003
- **Arrival date:**
  25 December 2003
- $\Delta v_{\text{tot}} = 5.67 \; - 5.7 \text{ km/s}$

Exercise 3: Mars Express

Mission analysis outputs

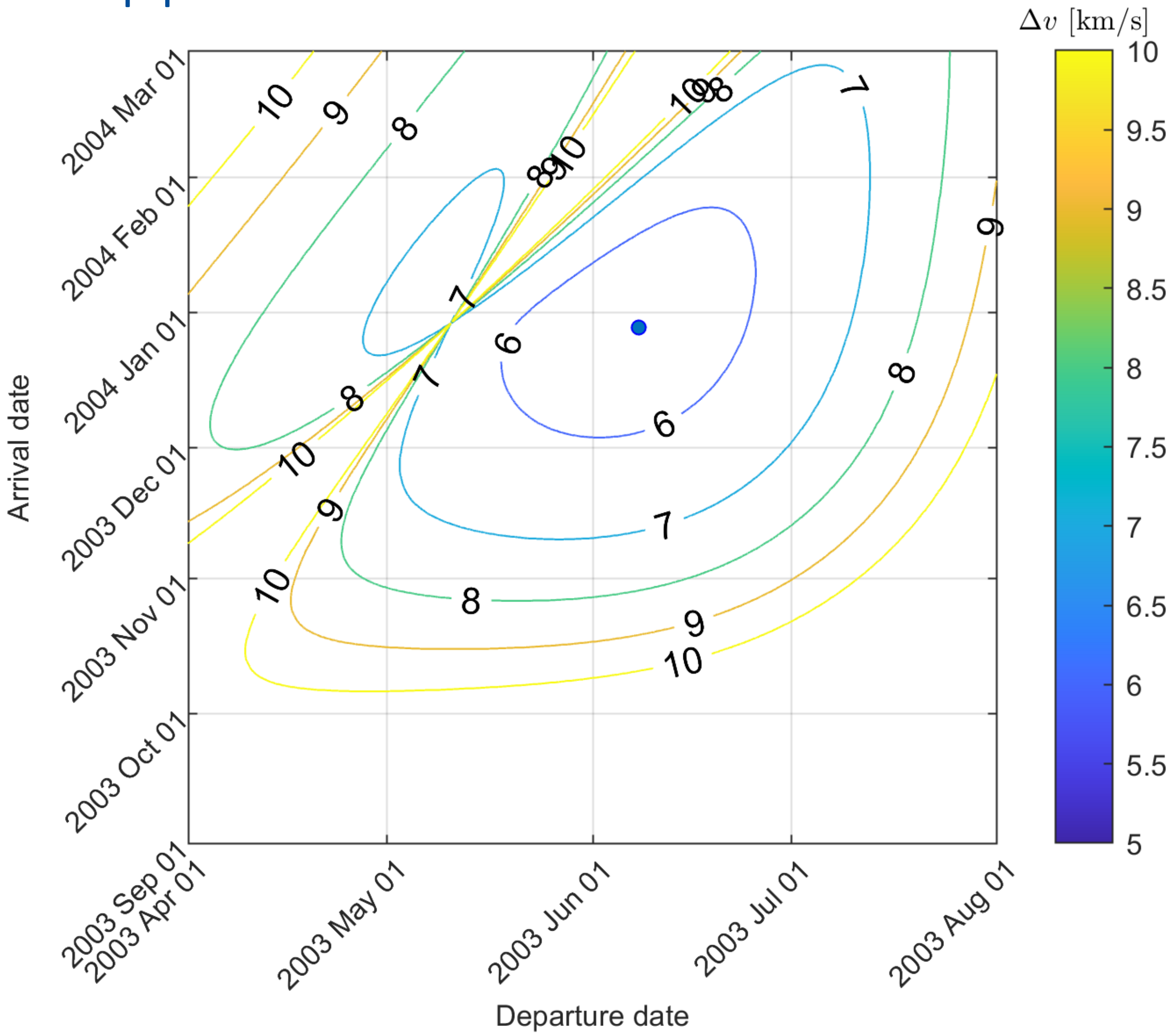
1. Implement a function to compute $\Delta v_{\text{tot}}(t_1, t_2)$.
2. Evaluate Δv_{tot} for a grid of departure and arrival times covering the time windows provided.
3. Draw the **porkchop plot** of the **Mars Express Mission**.
   Plot Δv_{tot} as a function of departure (x-axis) and arrival (y-axis) times, within their respective windows.
   Overlap to the contour plot some lines indicating constant Δt in days.
4. Find the cheapest mission (minimum of Δv_{tot}).
   Use function `min` over the 2D array of Δv_{tot} values.
5. Plot the transfer trajectory for this mission, together with the orbits and initial/final positions of Earth and Mars.
6. **OPTIONAL**: Refine the solution using Matlab's `fminunc` or `fmincon` (unconstrained or constrained gradient-based optimization, respectively), taking the solution in **4.** as initial guess.

Data

Sun's gravitational parameter $\mu_\odot$ from `astroConstants.m` (identifier `4`)

Exercise 3: Mars Express

Porkchop plot



Minimum Δv transfer:

$$\Delta v = 5.6696 \text{ km/s}$$

Departure:
2003/06/07 22:27:34.14

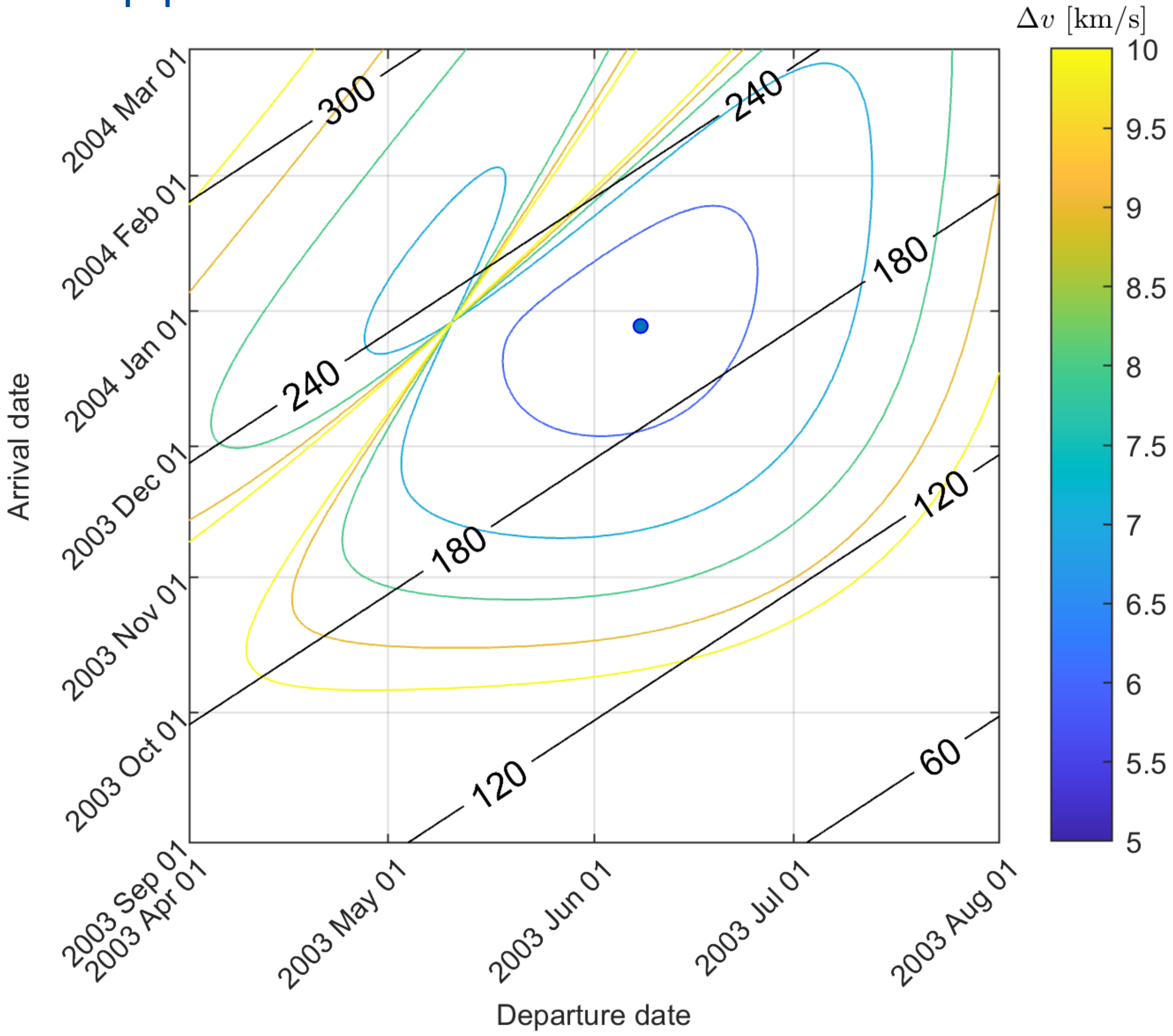
Arrival:
2003/12/28 14:26:08.25

The minimum Δv transfer results given here have been obtained using `fminunc`.

Note that the accuracy of the results from the grid search depends on the number of points used for each time window.

Exercise 3: Mars Express

Porkchop plot



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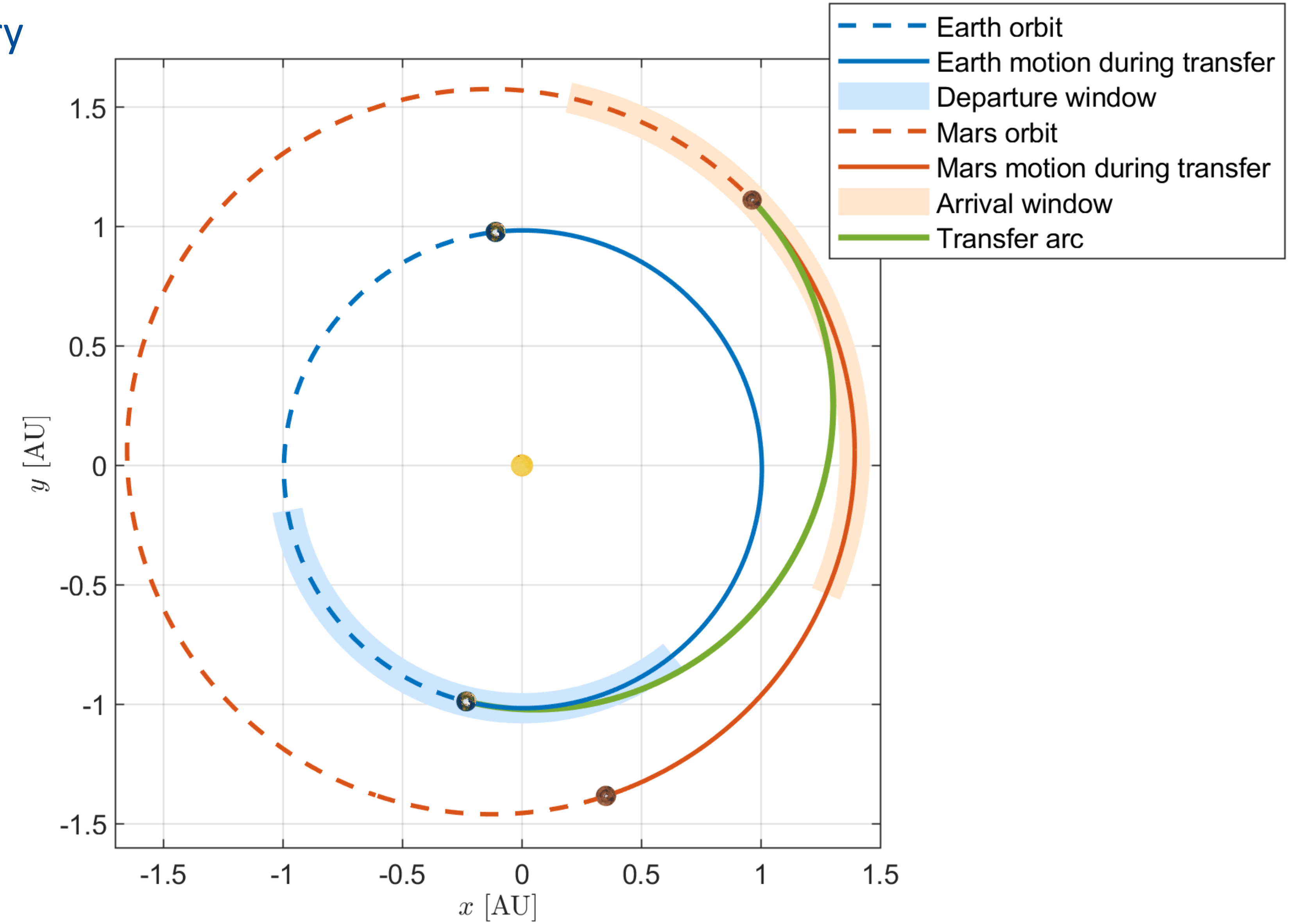
The minimum Δv transfer results given here have been obtained using `fminunc`.

Note that the accuracy of the results from the grid search depends on the number of points used for each time window.

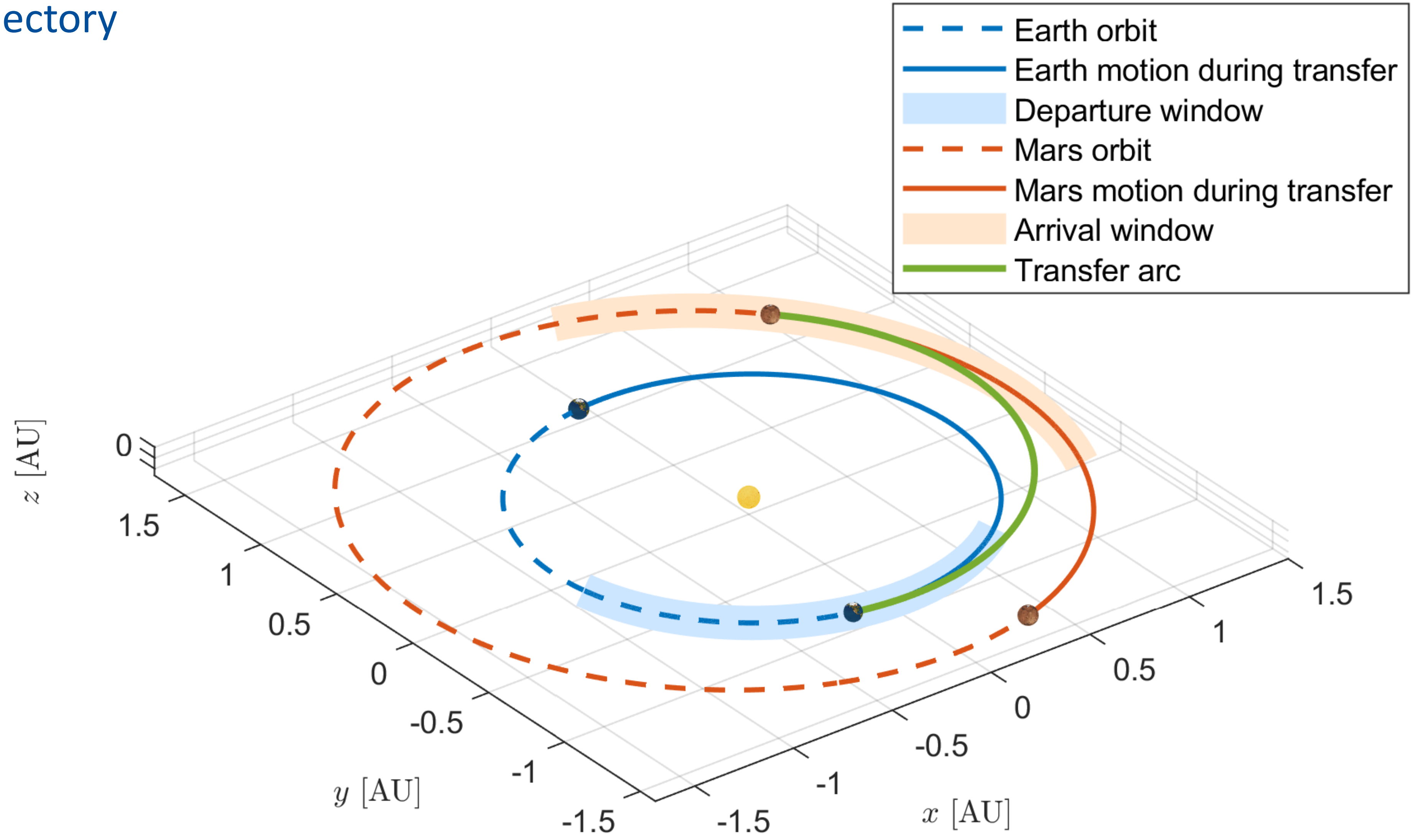
Exercise 3: Mars Express

Transfer trajectory



Exercise 3: Mars Express

Transfer trajectory



Mission Express

We have a mission!

As part of the mission analysis team of the **PoliMi Space Agency**, you are requested to perform the **preliminary mission analysis of an Express Mission** to rendezvous with a planet of the Solar System.

The launcher will inject the spacecraft directly into the interplanetary heliocentric transfer orbit. The maximum excess velocity and the available launch window are design constraints given by our launch provider.

The target planet and the arrival window are set by our science team.

Calculate the transfer options from Earth to the target planet/asteroid within the launch and arrival windows of your **Express Mission**, and select the one with minimum cost in terms of Δv.

Exercise 4: Mission Express

Mission definition

Mission Express: Design a direct transfer from Earth to a planet, with restricted launcher excess velocity.

Requirements for several missions are provided in the slides, with the following data:

▪ Target planet	**n**	Integer (ephemeris ID)
▪ Launch window	$\boldsymbol{t_1 \in [t_{1\,min}\,, t_{1\,max}\,]}$	Date format [yyyy, mm, dd]
▪ Arrival window	$\boldsymbol{t_2 \in [t_{2\,min}, t_{2\,max}]}$	Date format [yyyy, mm, dd]
▪ Maximum excess velocity from launcher	$\boldsymbol{v_\infty}$	[km/s]

Exercise 4: Mission Express

Mission analysis outputs

1. Evaluate Δv_{tot} for a grid of departure and arrival times within the given time windows.
2. Draw the porkchop plot of the **Mission Express**.
3. Find the minimum Δv_{tot}, without considering the launcher constraint.
   Although $||\Delta\mathbf{v}_1||$ is given by the launcher, we want to include it in Δv_{tot} because it gives a measure of the mission cost.
4. Find the cheapest mission (minimum Δv_{tot}) fulfilling the launcher constraint.
5. Plot the transfer trajectory from **4.**, together with the orbits and initial/final positions of Earth and the target planet.
6. **OPTIONAL**: Refine the solution using Matlab's `fmincon`.

Data

Sun's gravitational parameter $\mu_{\odot}$ from `astroConstants.m` (identifier `4`)

Exercise 4: Mission Express

Mission data

Planet (ID)	Departure window	Arrival window	v_∞ [km/s]
Mercury (1)	2023/11/01 - 2025/01/01	2024/04/01 - 2025/03/01	7.0
Venus (2)	2024/06/01 - 2026/11/01	2024/12/01 - 2027/06/01	3.0
Mars (4)	2025/08/01 - 2031/01/01	2026/01/01 - 2032/01/01	3.5
Jupiter (5)	2026/06/01 - 2028/06/01	2028/06/01 - 2034/01/01	9.1
Saturn (6)	2027/09/01 - 2029/10/01	2030/04/01 - 2036/03/01	11.5
Uranus (7)	2027/01/01 - 2029/01/01	2031/04/01 - 2045/12/01	12.1
Neptune (8)	2025/01/01 - 2026/10/01	2036/01/01 - 2055/06/01	12.5