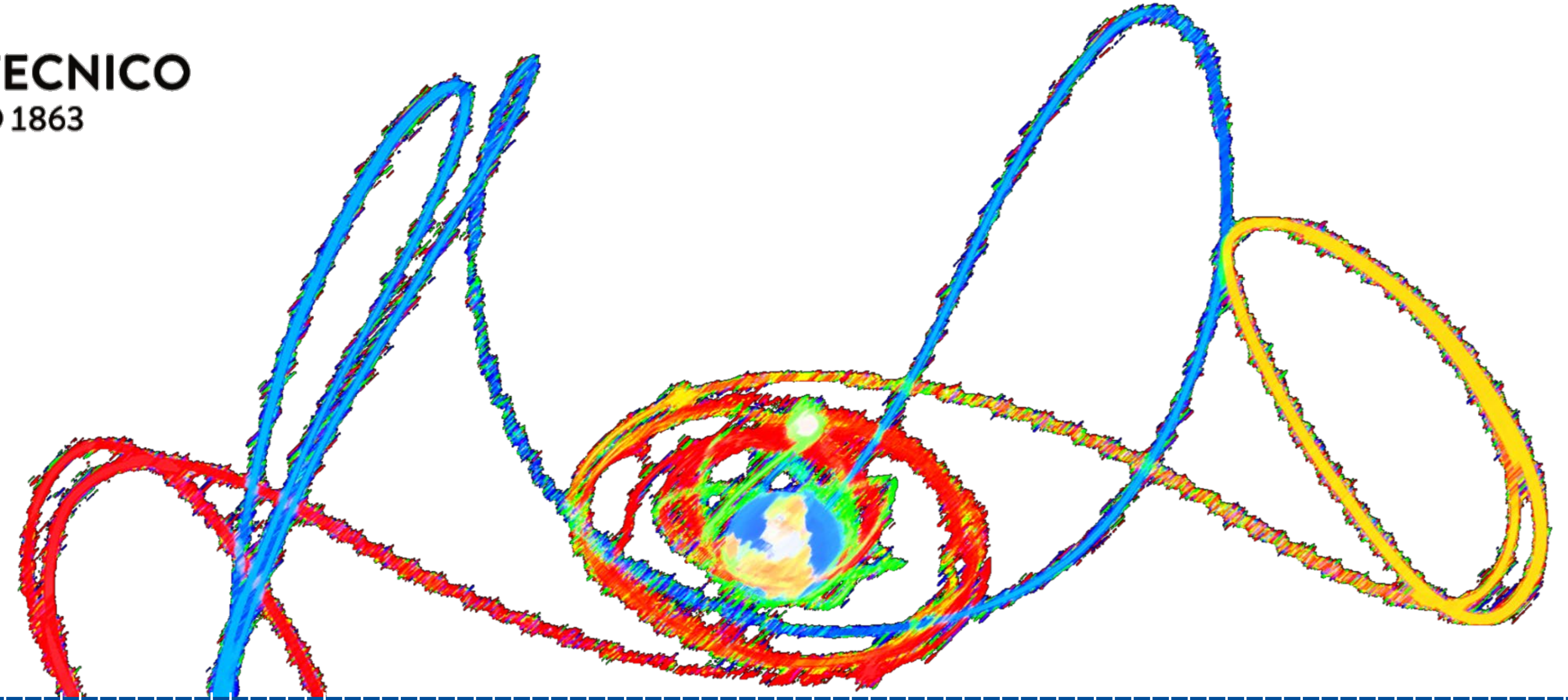

Orbital Mechanics
Module 4: Interplanetary trajectories

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Interplanetary trajectories

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INTERPLANETARY TRANSFERS WITH FLYBYS

Deepening into interplanetary trajectory design

The direct path is not always the cheapest path

The results from the different **Mission Express** proposed in the previous module show that a direct interplanetary transfer using only departure and arrival manoeuvres **may not be technologically feasible** in all cases (**excessive ΔV requirements and/or time of flight**).

Several techniques to enable cost-effective interplanetary travel:

- **Deep space manoeuvres (small orbit corrections).**
- **Planetary flybys (gravity assist).**

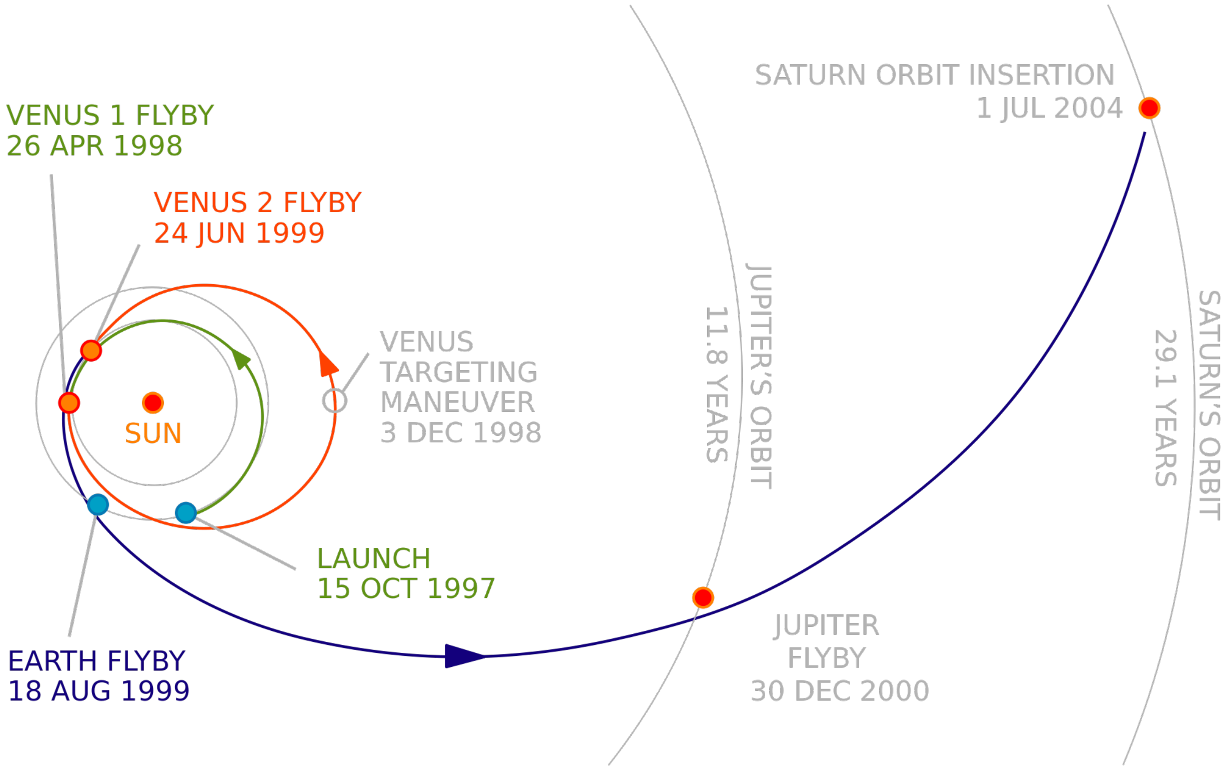
In this module, we will study the preliminary design of interplanetary missions taking advantage of **planetary flybys** (both unpowered and powered) using the **patched conics approach**.

Gravity assist manoeuvres

Harnessing the planet's gravity

Gravity assist manoeuvres are used to modify the momentum of a spacecraft by leveraging the gravitational pull of a planet:

- Provide ΔV values beyond those achievable using current propulsion technology, enabling missions to the furthest reaches of the solar system (e.g., Pioneer 11, Voyager 1, Voyager 2, Cassini,…).
- Reduce time of flight and/or required propellant.
- Higher design complexity.



Cassini's interplanetary trajectory
(source: NASA/JPL)

Method of patched conics

A powerful technique for preliminary interplanetary trajectory design

The **method of patched conics** approximates an interplanetary trajectory as several Keplerian arcs (i.e., **conics**) with different attractive bodies:

- Within a planet's sphere of influence (SOI), unperturbed Keplerian orbit around the planet.
- Outside the SOI, unperturbed heliocentric Keplerian orbit.

Sphere of influence r_{SOI} : *representation of the region around a planet where its gravitational attraction dominates over that of the Sun.*

$$\frac{r_{\text{SOI}}}{r_{\text{P}}} = \left(\frac{m_{\text{P}}}{m_{\odot}}\right)^{\frac{2}{5}}$$

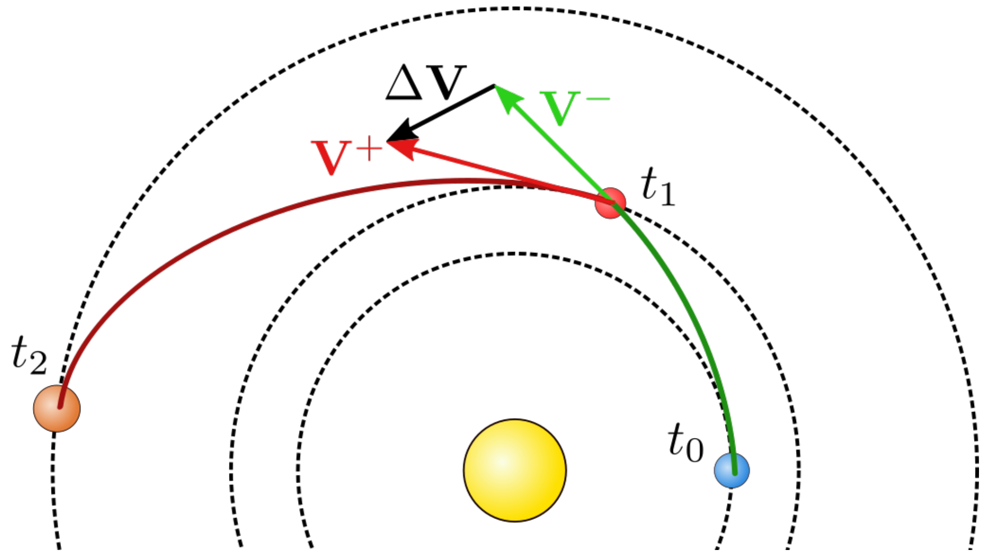
where r_{P} is the mean orbital radius of the planet, m_{P} is the mass of the planet and $m_{\odot}$ is the mass of the Sun.

Method of patched conics

Heliocentric region

The SOIs of all planets are very small compared to their mean distance to the Sun ($r_{SOI}/r_P \ll 1$). Consequently, **in the heliocentric region**:

- Planet's SOI is assumed to be infinitesimal (i.e., a point).
- Flyby is an instantaneous change in velocity, with constant position.
- Each Keplerian arc can be treated as a Lambert problem.



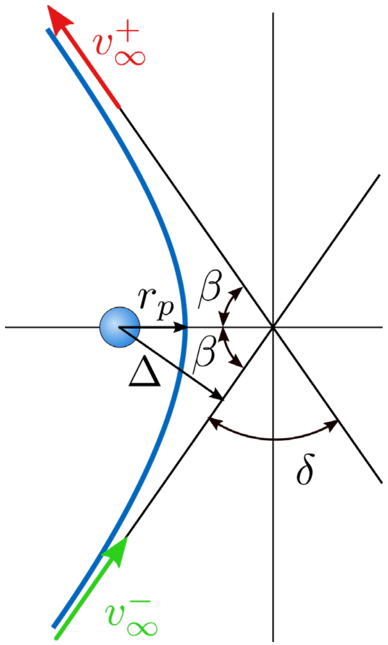
Planet	r_{SOI}/r_P
Mercury	0.0019
Venus	0.0057
Earth	0.0062
Mars	0.0025
Jupiter	0.0620
Saturn	0.0382
Uranus	0.0180
Neptune	0.0192

Method of patched conics

Planetocentric region

The SOI is very large compared to the planet's radius ($r_{\mathrm{SOI}}/R_{\mathrm{P}} \gg 1$). Consequently, **in the planetocentric region**:

- Planet's SOI is assumed to be infinite.
- For a flyby, the planetocentric trajectory is a **hyperbola**.
  - Excess velocities given by the incoming/outcoming heliocentric velocities with respect to the planet.



Planet	$r_{\mathrm{SOI}}/R_{\mathrm{P}}$
Mercury	46.1
Venus	101.7
Earth	145.3
Mars	170.0
Jupiter	675.1
Saturn	906.9
Uranus	2024.6
Neptune	3494.8

PLANETARY FLYBYS

Planetocentric hyperbola

Equations for the hyperbola (Keplerian motion)

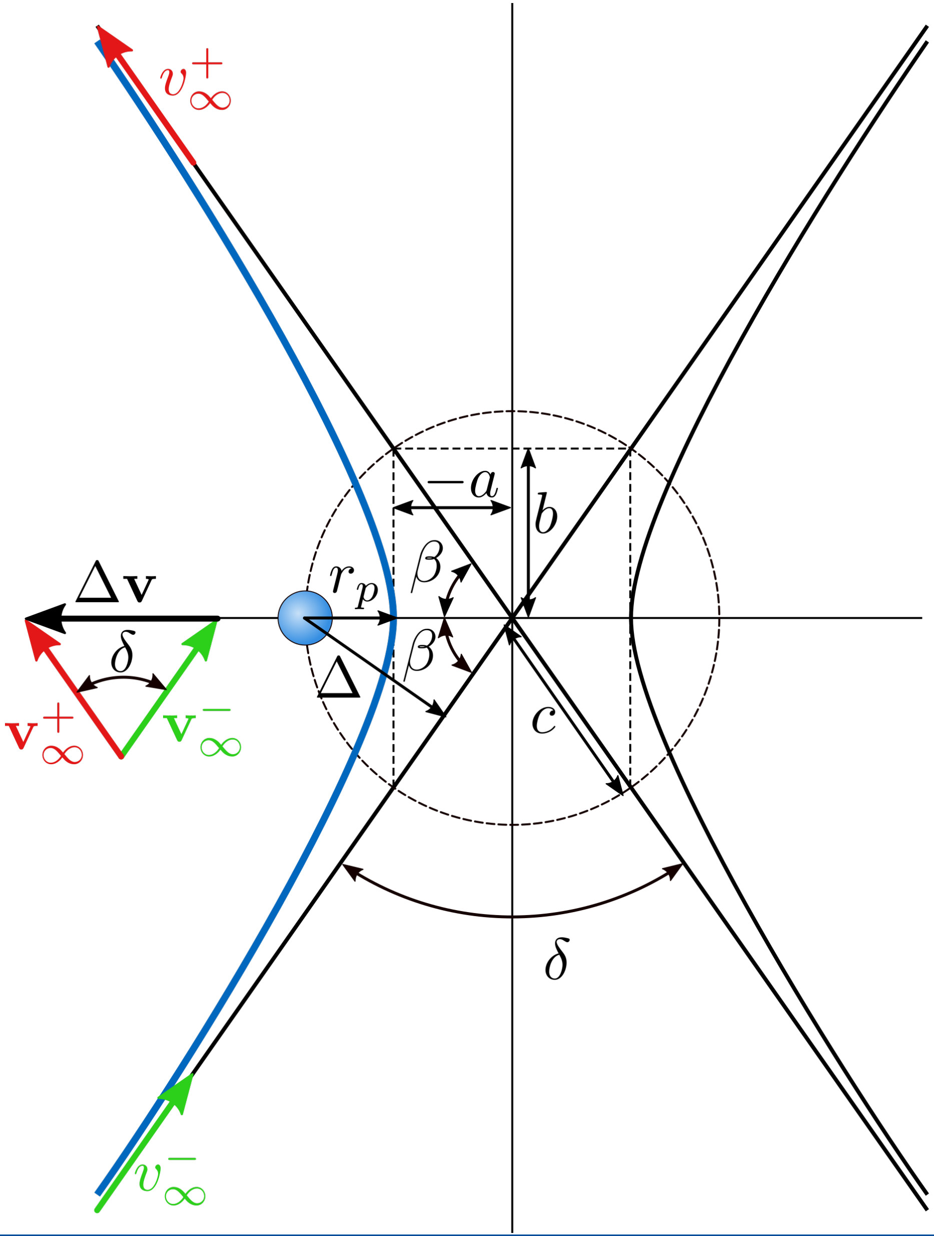
$$v_\infty^- = v_\infty^+ = v_\infty$$

$$v_\infty^2 = -\frac{\mu}{a} \Rightarrow a = -\frac{\mu}{v_\infty^2}$$

$$r_p = a(1-e) = -\frac{\mu}{v_\infty^2}(1-e)$$

$$e = 1 + \frac{r_p v_\infty^2}{\mu} = \frac{1}{\sin\frac{\delta}{2}}$$

$$\Delta = -a\, e \cos\frac{\delta}{2} = \frac{-a}{\tan\frac{\delta}{2}}$$



Planetocentric hyperbola

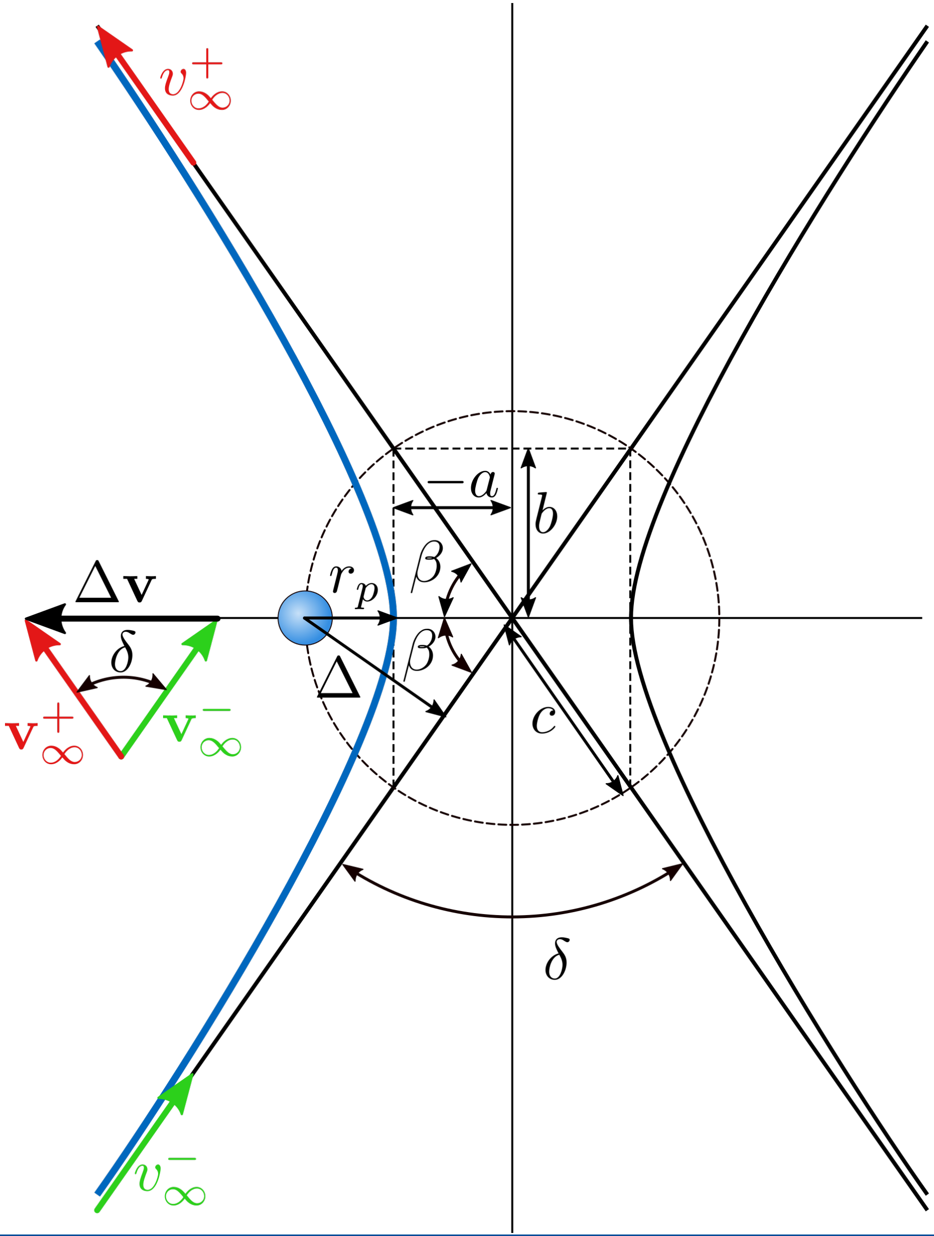
Required data

The geometry of the planetocentric hyperbola is totally defined by setting **two** of the following:

v_∞	Excess velocity
Δ	Impact parameter
δ	Turning angle
r_p	Perigee radius

Turning angle δ: Angle formed by the incoming and outcoming asymptotes.

Impact parameter Δ: minimum distance between asymptote and planet.



Planetocentric hyperbola

Change in velocity

$$\Delta \mathbf{v} = \mathbf{v}_\infty^+ - \mathbf{v}_\infty^- = \mathbf{V}^+ - \mathbf{V}^- = \Delta \mathbf{V}$$

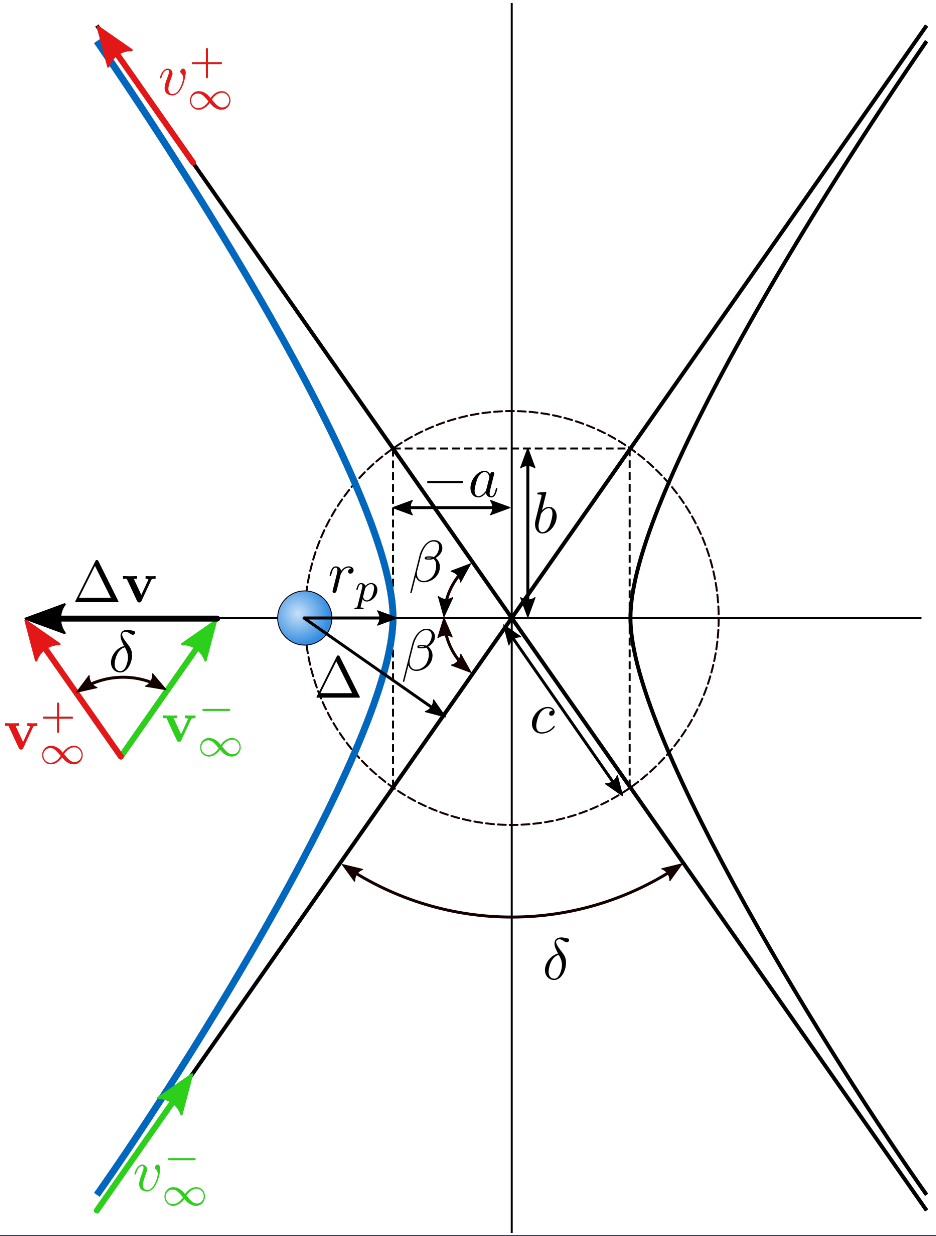
Planetocentric Heliocentric

$$\mathbf{V}^+ = \mathbf{V}_P + \mathbf{v}_\infty^+$$

$$\mathbf{V}^- = \mathbf{V}_P + \mathbf{v}_\infty^-$$

$\Delta\mathbf{v}$ is always oriented along the apse line, and pointing opposite to the pericentre

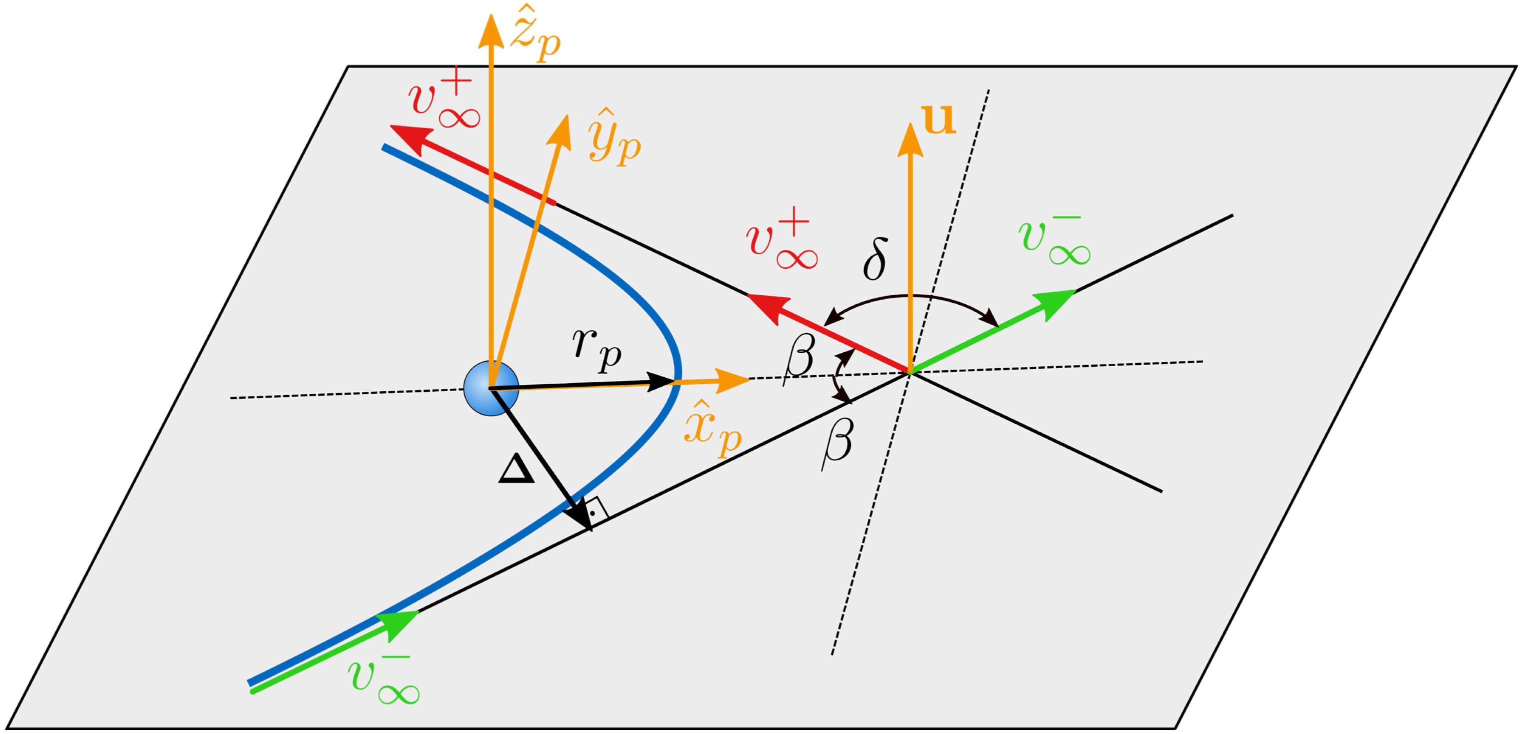
$$\|\Delta \mathbf{v}\| = 2 v_\infty \sin\frac{\delta}{2}$$



Planetocentric hyperbola

Hyperbola orientation in space

- The previous expressions define the hyperbola in its **perifocal frame**.
  - *Perifocal frame*: Frame centred at the planet, with $\hat{x}_p$ aligned with the pericentre, $\hat{z}_p$ normal to the orbital plane in the direction of the angular momentum, and $\hat{y}_p$ completing a right-handed frame.

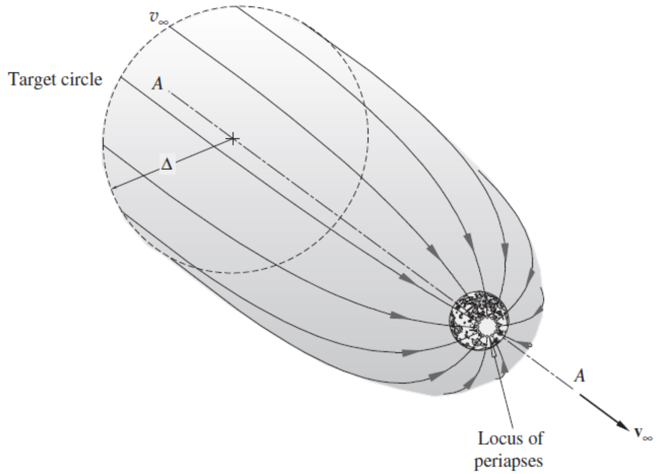


- The flyby rotates the excess velocity an angle δ around the unit vector normal to its plane $\mathbf{u}$ (counter-clockwise).
- The orientation of the flyby hyperbola with respect to another frame (e.g., heliocentric) can be obtained computing its i, Ω, and ω from $\hat{z}_p$ and $\hat{x}_p$.

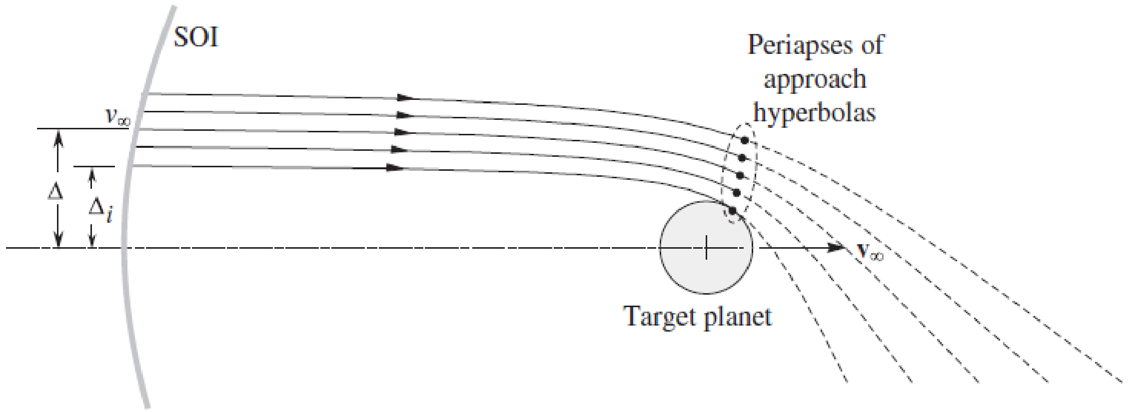
Matching the regions

Impact parameter and location of the incoming asymptote

- **The incoming heliocentric arc alone is not enough to fully define the hyperbola,** because patched conics reduces the SOI to a point:
  - The incoming heliocentric arc actually corresponds to a collision.
  - **The plane of the hyperbola is undefined** (degree of freedom).
  - **Impact parameter Δ is undefined** (degree of freedom).



Family of incoming hyperbolas for a fixed impact parameter Δ and different locations of the asymptote (taken from [1])



Family of incoming hyperbolas for a fixed location of the asymptote and different impact parameter Δ (taken from [1])

[1] Curtis, H. D., *Orbital mechanics for engineering students,* Butterworth-Heinemann , 2014

Matching the regions

Fully-defined cases

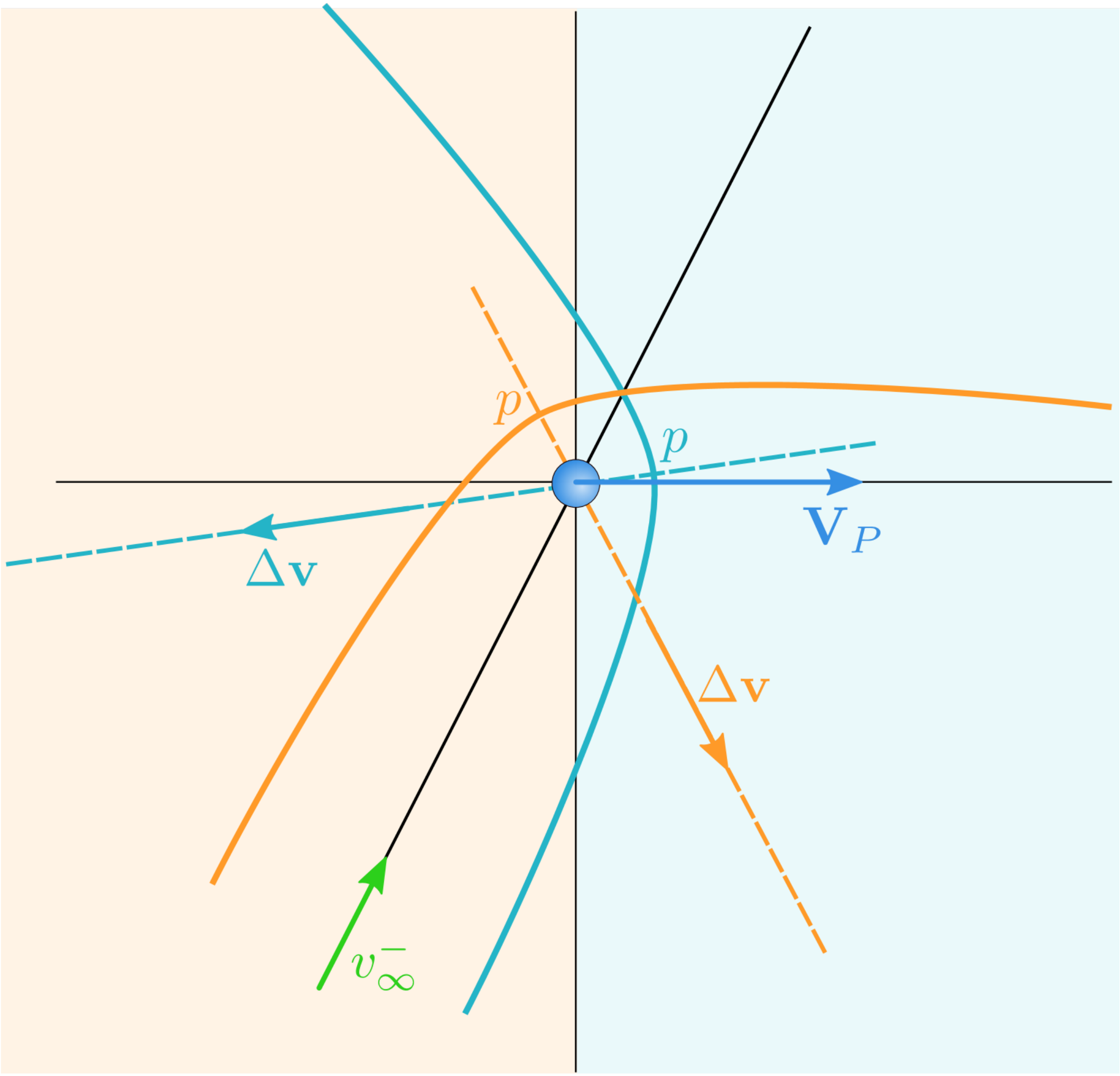
- The flyby can be fully defined providing additional information:
  - Incoming excess velocity vector $\mathbf{v}_\infty^-$ and impact parameter vector $\mathbf{\Delta}$
    - Hyperbola geometry (in plane) given by $v_\infty = \|\mathbf{v}_\infty^-\|$ and $\Delta = \|\mathbf{\Delta}\|$
    - Hyperbola plane given by $\mathbf{u} = \frac{\mathbf{\Delta}\times\mathbf{v}_\infty^-}{\|\mathbf{\Delta}\times\mathbf{v}_\infty^-\|}$
  - Incoming and outcoming heliocentric arcs:
    - Hyperbola geometry (in plane) given by $v_\infty = \|\mathbf{v}_\infty^-\| = \|\mathbf{v}_\infty^+\|$, and $\delta = \operatorname{acos}\frac{\mathbf{v}_\infty^-\cdot\mathbf{v}_\infty^+}{v_\infty^2}$
    - Hyperbola plane given by $\mathbf{u} = \frac{\mathbf{v}_\infty^-\times\mathbf{v}_\infty^+}{\|\mathbf{v}_\infty^-\times\mathbf{v}_\infty^+\|}$
- In all cases, **apse line** is in the direction of $-\Delta\mathbf{v}$. It can also be located through the angle $\beta = \frac{(\pi-\delta)}{2}$ formed by the apse line and the asymptotes.
- **Remember**: The normal to the hyperbola plane $\mathbf{u}$ is also the vector around which the excess velocity is rotated.

Leading- and trailing-side flybys

Reducing or increasing heliocentric velocity

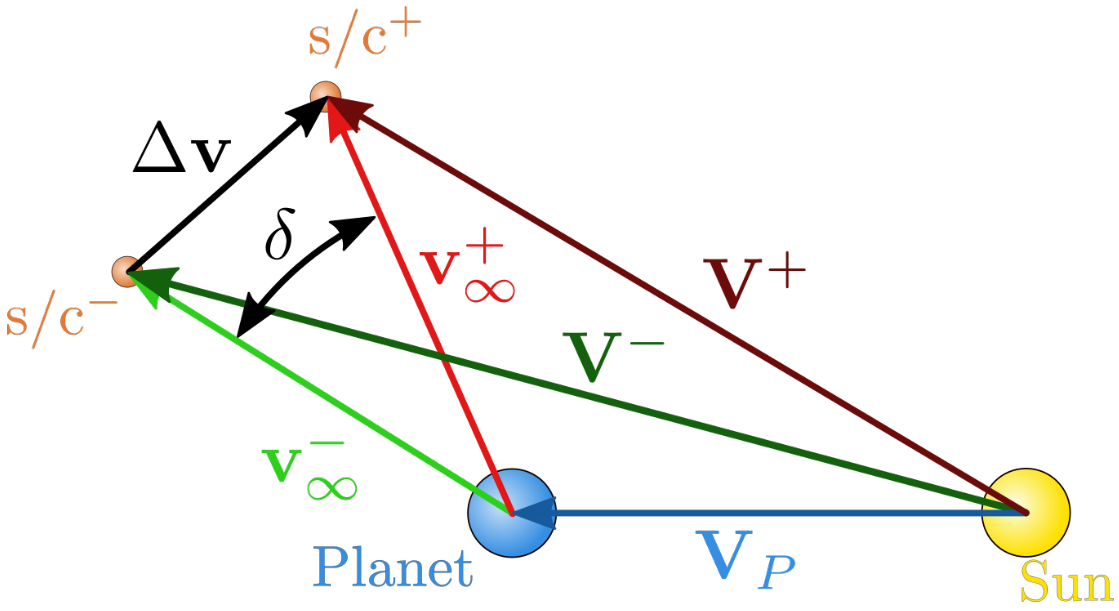
As previously seen, $\Delta\mathbf{v}$ is parallel to the apse line and points away from the pericentre. Two possibilities:

- **Leading-side flyby**: Pericentre lies on the side of the planet facing in the direction of motion. **Heliocentric velocity is reduced.**
- **Trailing-side flyby**: Pericentre lies on the side of the planet facing opposite to the direction of motion. **Heliocentric velocity is increased.**

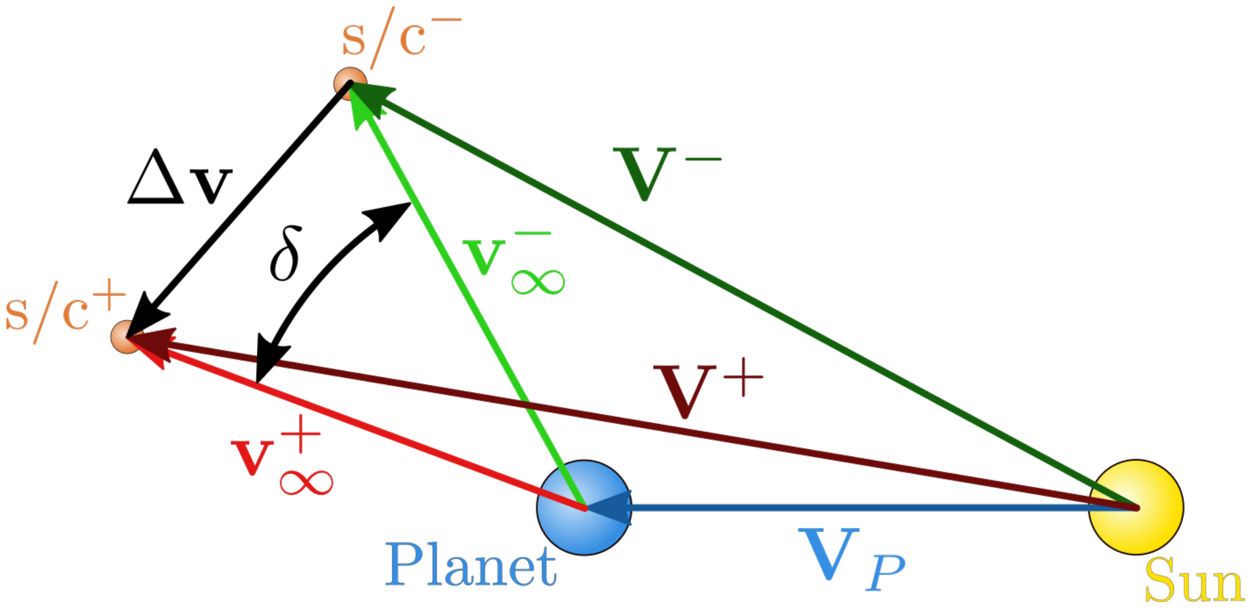


Triangle of velocities

Same-plane flyby



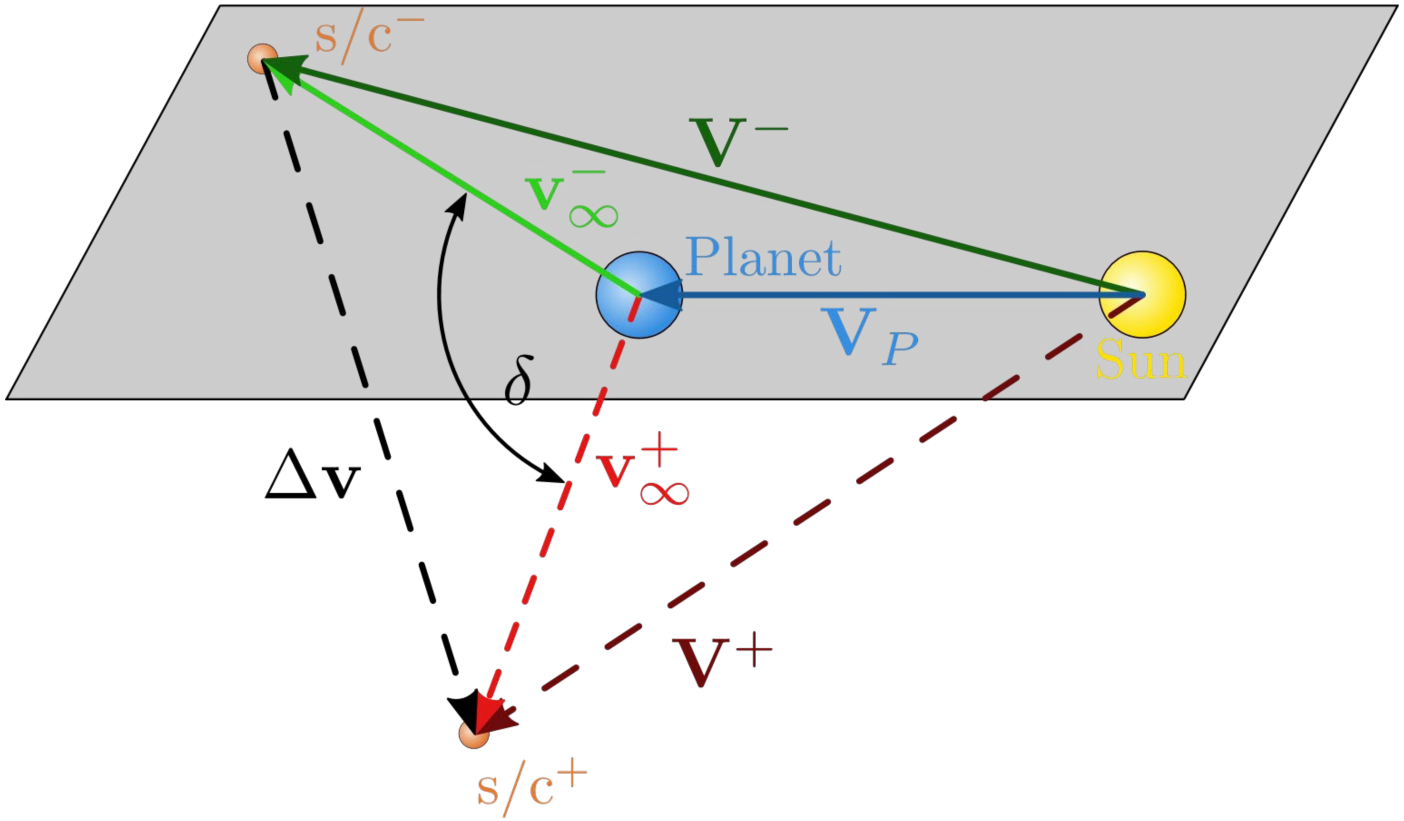
Leading-side flyby



Trailing-side flyby

Triangle of velocities

Change-of-plane flyby



The planetocentric hyperbola lies on a different plane
to the incoming and outcoming heliocentric arcs

Exercise 1a: Flyby

Different locations of the incoming asymptote

Exercise 1a: Design a flyby around the Earth for fixed impact parameter and different locations of the incoming asymptote.

1. Solve the 2D hyperbola (common to all orientations).
2. Compute $\mathbf{v}_\infty^+$ for three locations of the incoming asymptote:
   - In front of the planet,
   - Behind the planet,
   - Under the planet.
3. Compute $\mathbf{V}^-$, $\mathbf{V}^+$, and the incoming and outcoming heliocentric arcs.
4. Plot the heliocentric trajectory before and after the flyby, and the planetocentric flyby hyperbola.
   - For the flyby, it is enough to plot the hyperbola in its perifocal frame.

Data:

Vectors in the heliocentric ecliptic frame; assume circular Earth orbit around the Sun

$\mu_\oplus$, $\mu_\odot$, and AU from `astroConstants` (identifiers `13`, `4`, and `2`, respectively)

$\mathbf{v}_\infty^- = [\,15.1;\;\; 0;\;\; 0\,]\ \mathrm{km/s}$ $\quad\Delta = 9200\ \mathrm{km}$ $\quad\mathbf{r}_\oplus = [\,1;\;\; 0;\;\; 0\,]\ \mathrm{AU}$

Exercise 1a: Flyby

Location of the incoming asymptote

- The location of the incoming asymptote does not affect:
  - The geometry of the hyperbola (δ, r_p), which depends only on Δ and v_∞.
  - The incoming heliocentric arc, due to the patched conics approximation.

- The location of the incoming asymptote determines the plane of the hyperbola, and so the direction $\mathbf{u}$ around which the velocity vector gets rotated by the flyby, affecting:
  - The direction of $\mathbf{v}_\infty^+$ (the angle formed by $\mathbf{v}_\infty^-$ and $\mathbf{v}_\infty^+$ is always δ).
  - The heliocentric arc after the flyby.

- You can use Rodrigues's formula to rotate a vector $\mathbf{v}$ an angle δ around unit vector $\mathbf{u}$ (counter-clockwise):

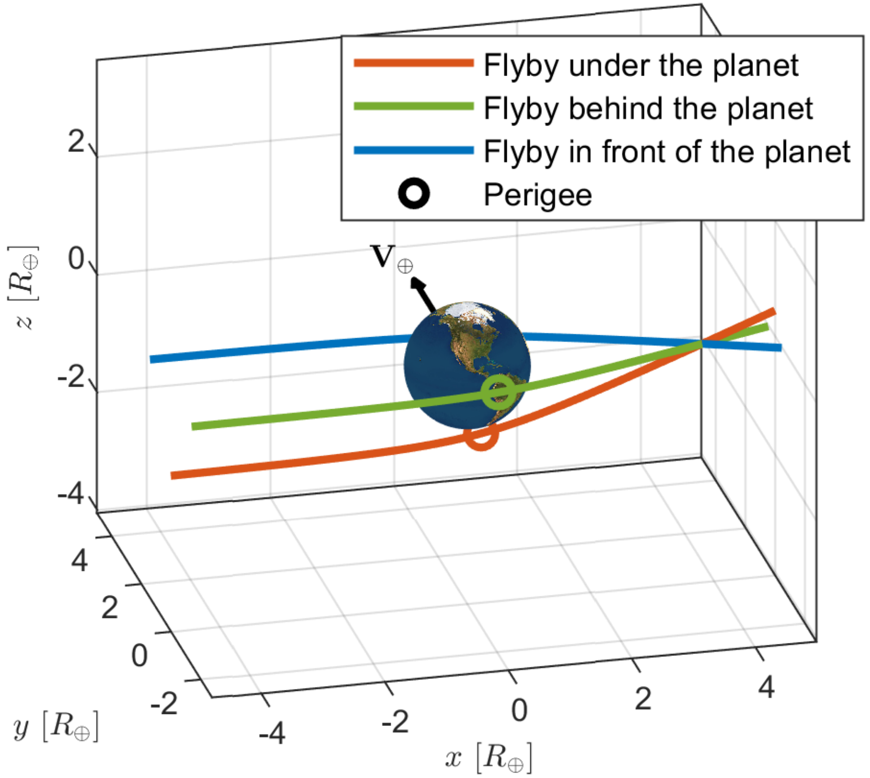
$$\mathbf{v}^{\text{rotated}} = \mathbf{v}\cos\delta + (\mathbf{u}\times\mathbf{v})\sin\delta + \mathbf{u}(\mathbf{u}\cdot\mathbf{v})(1-\cos\delta)$$

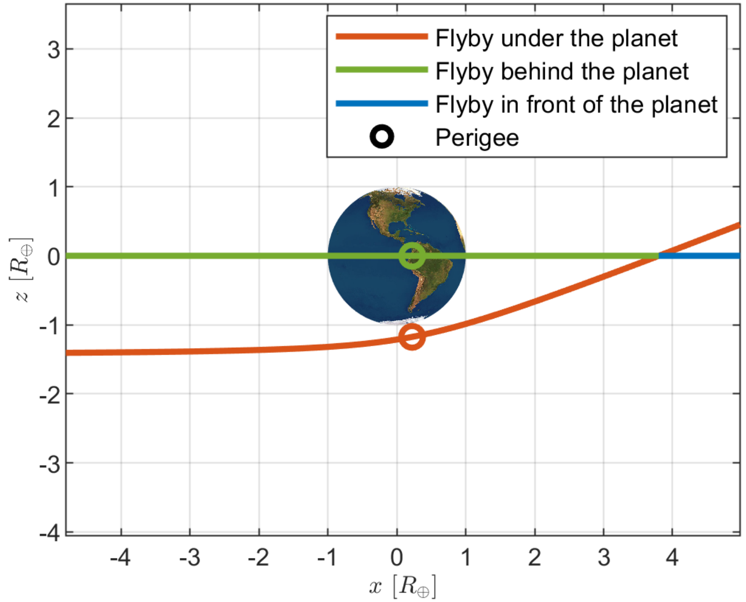
  - You can implement it as a Matlab function so you can reuse it later

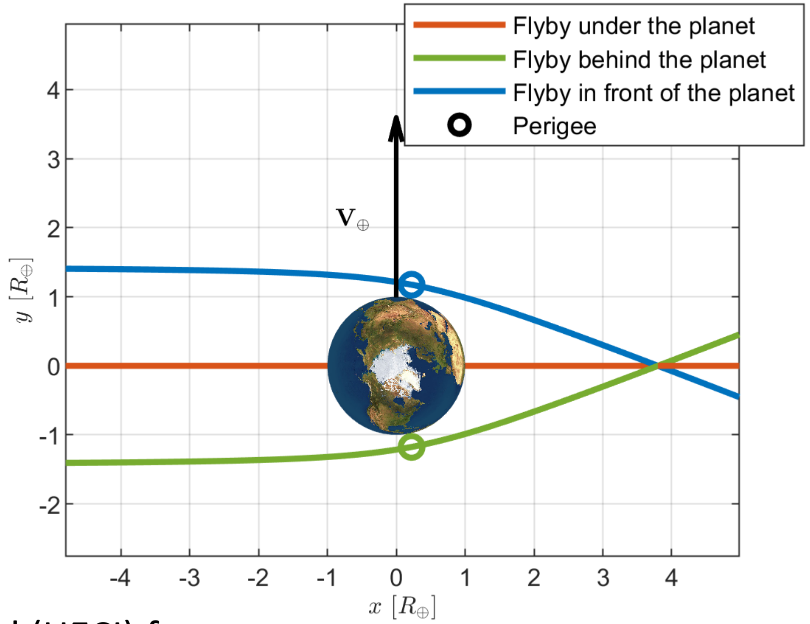
Exercise 1a: Flyby

Location of the incoming asymptote

You can use these figures to help you identify the position of the plane and direction of $\mathbf{u}$ for each asymptote location



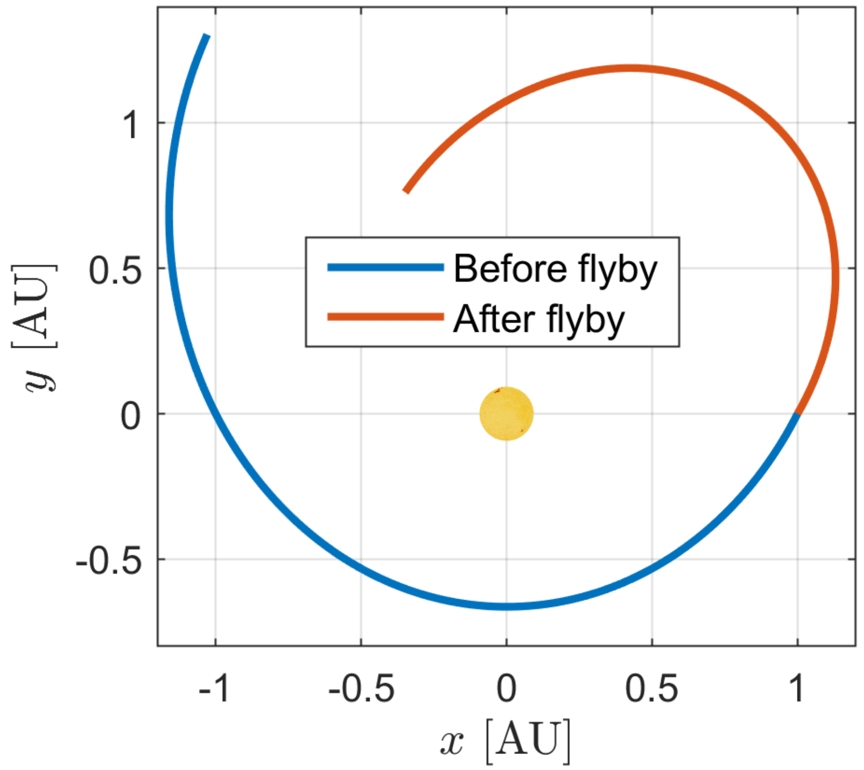




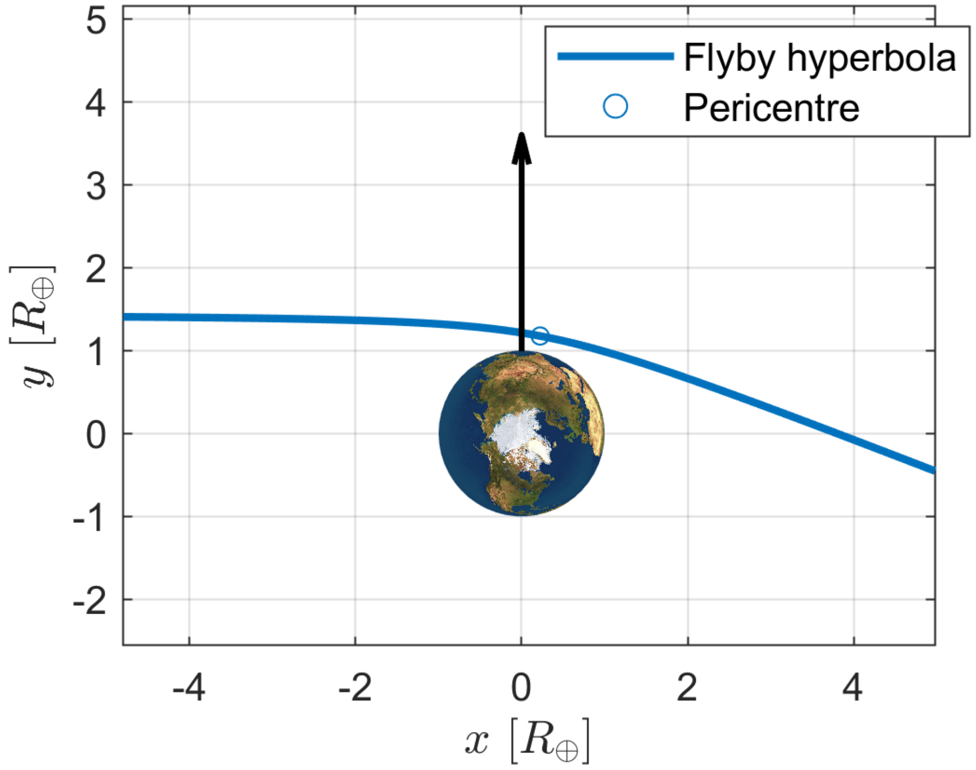
Flybys in Earth-centred frame parallel to the heliocentric ecliptic inertial (HECI) frame

Exercise 1a: Flyby

Results – In front of the planet (leading-side flyby)



Trajectory in heliocentric ecliptic (HECI) frame



Flyby in Earth-centred frame parallel to HECI

$\delta = 21.5180$ deg

$r_p = 7616.4488$ km

$\Delta v = 5.6377$ km/s

$a = -1748.1708$ km

$e = 5.3568$

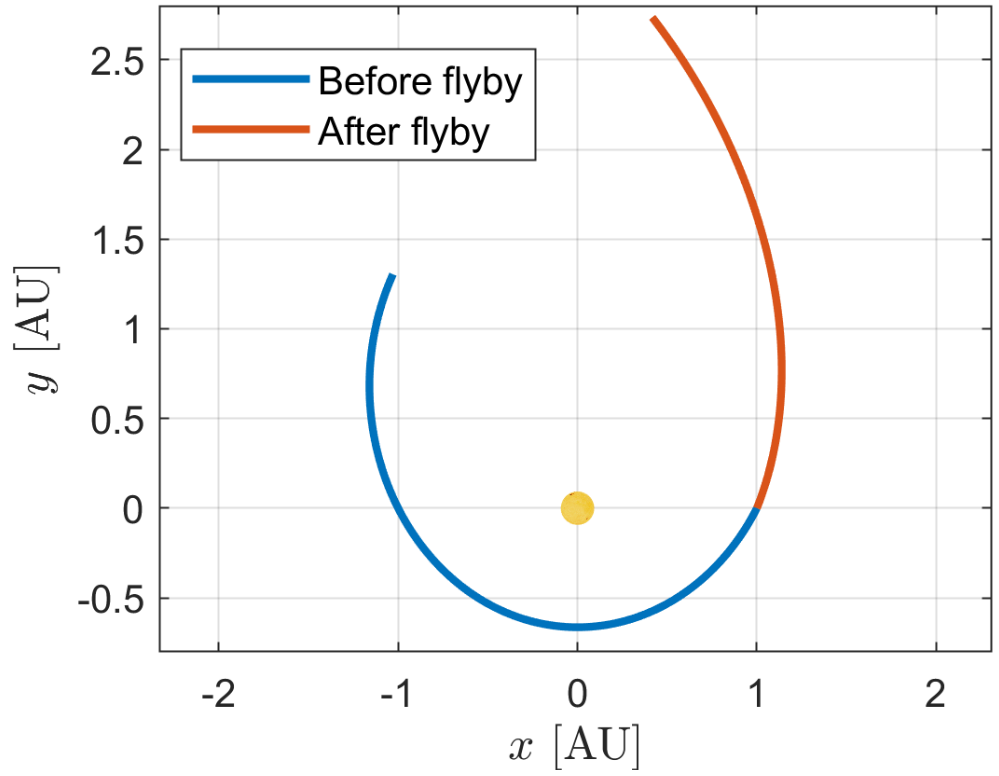
$\mathbf{V}^- = [15.1000;\ 29.7847;\ 0]$ km/s

$\mathbf{V}^+ = [14.0476;\ 24.2461;\ 0]$ km/s

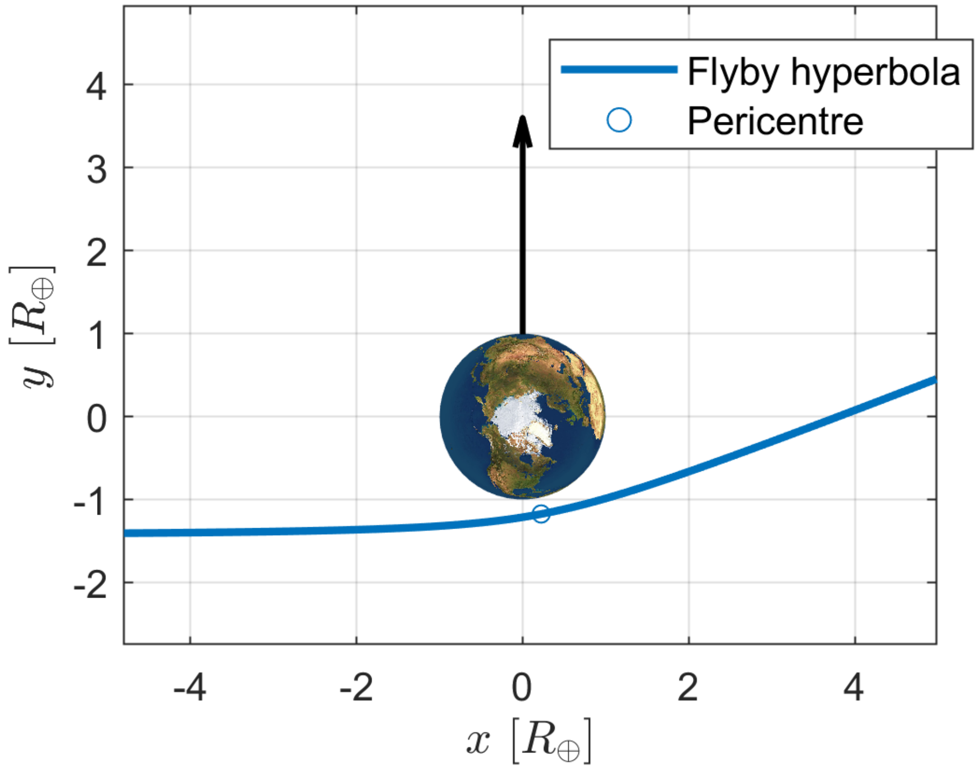
$\mathbf{v}_\infty^+ = [14.0476;\ -5.5386;\ 0]$ km/s

Exercise 1a: Flyby

Results – Behind the planet (trailing-side flyby)



Trajectory in heliocentric ecliptic (HECI) frame



Flyby in Earth-centred frame parallel to HECI

$\delta = 21.5180$ deg

$r_p = 7616.4488$ km

$\Delta v = 5.6377$ km/s

$a = -1748.1708$ km

$e = 5.3568$

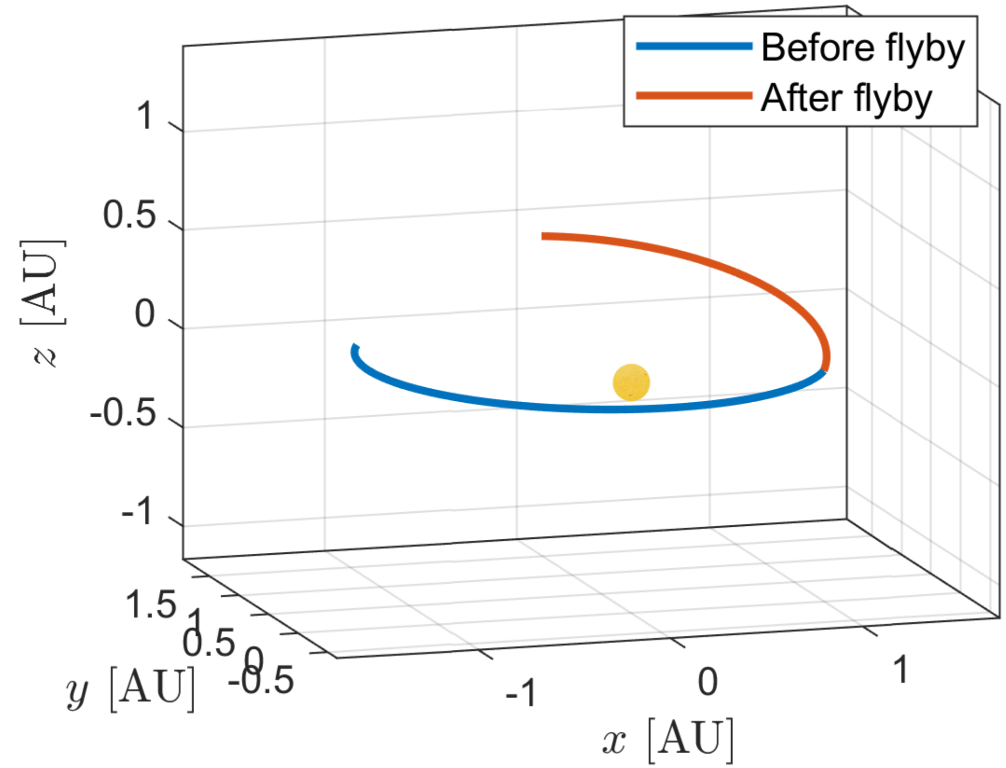
$\mathbf{V}^- = [15.1000;\ 29.7847;\ 0]$ km/s

$\mathbf{V}^+ = [14.0476;\ 35.3233;\ 0]$ km/s

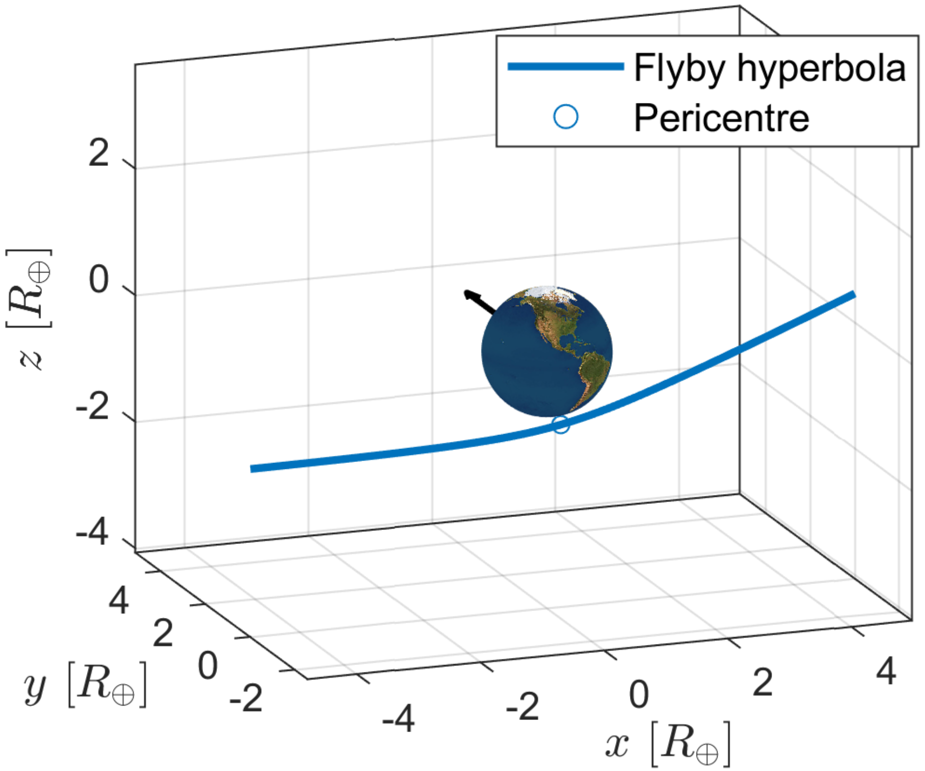
$\mathbf{v}_\infty^+ = [14.0476;\ 5.5386;\ 0]$ km/s

Exercise 1a: Flyby

Results – Under the planet



Trajectory in heliocentric ecliptic (HECI) frame



Flyby in Earth-centred frame parallel to HECI

$\delta = 21.5180$ deg
$r_p = 7616.4488$ km
$\Delta v = 5.6377$ km/s

$a = -1748.1708$ km
$e = 5.3568$

$\mathbf{V}^- = [15.1000;\ 29.7847;\ 0]$ km/s
$\mathbf{V}^+ = [14.0476;\ 29.7847;\ 5.5386]$ km/s
$\mathbf{v}_\infty^+ = [14.0476;\ 0;\ 5.5386]$ km/s

Exercise 1b: Flyby

Different impact parameters

Exercise 1b: Design a flyby around the Earth for fixed location of the incoming asymptote and different impact parameters.

1. Choose a location for the incoming asymptote.
2. Solve and plot the 2D hyperbola for different values of Δ.
3. Compute $\mathbf{V}^-$ and the incoming heliocentric arc (common to all Δ).
4. Compute $\mathbf{v}_\infty^+$, $\mathbf{V}^+$, and the outcoming heliocentric arc for each Δ.
5. Plot the heliocentric trajectory before and after the flyby, for the different values of Δ.

Data:

Vectors in the heliocentric ecliptic frame; assume circular Earth orbit around the Sun

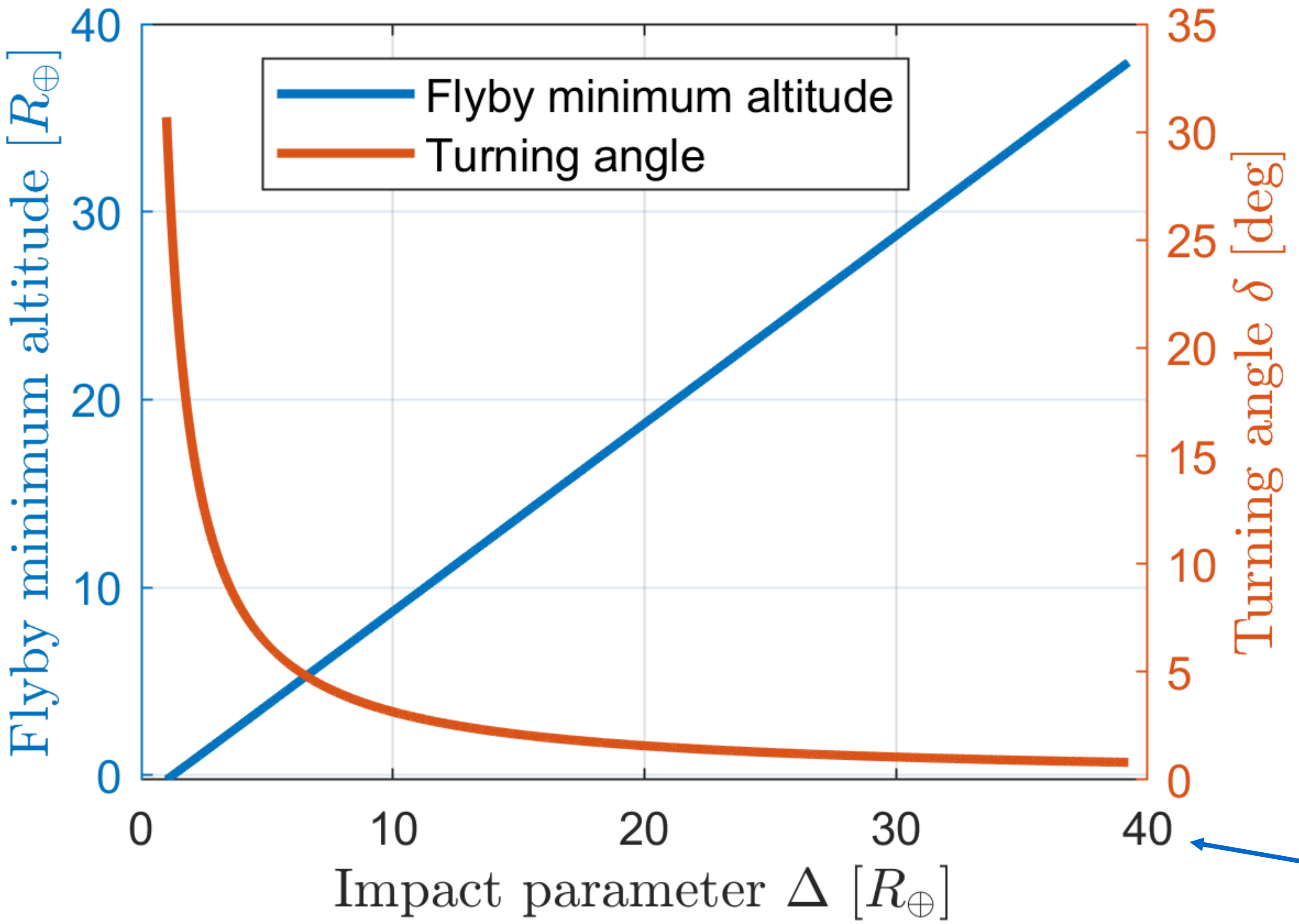
$\mu_\oplus$, $\mu_\odot$, and AU from `astroConstants` (identifiers `13`, `4`, and `2`, respectively)

$\mathbf{v}_\infty^- = [\,15.1;\ \ 0;\ \ 0\ ]\,\mathrm{km/s}$

$\mathbf{r}_\oplus = [\,1;\ \ 0;\ \ 0\ ]\,\mathrm{AU}$

Exercise 1b: Flyby

Results

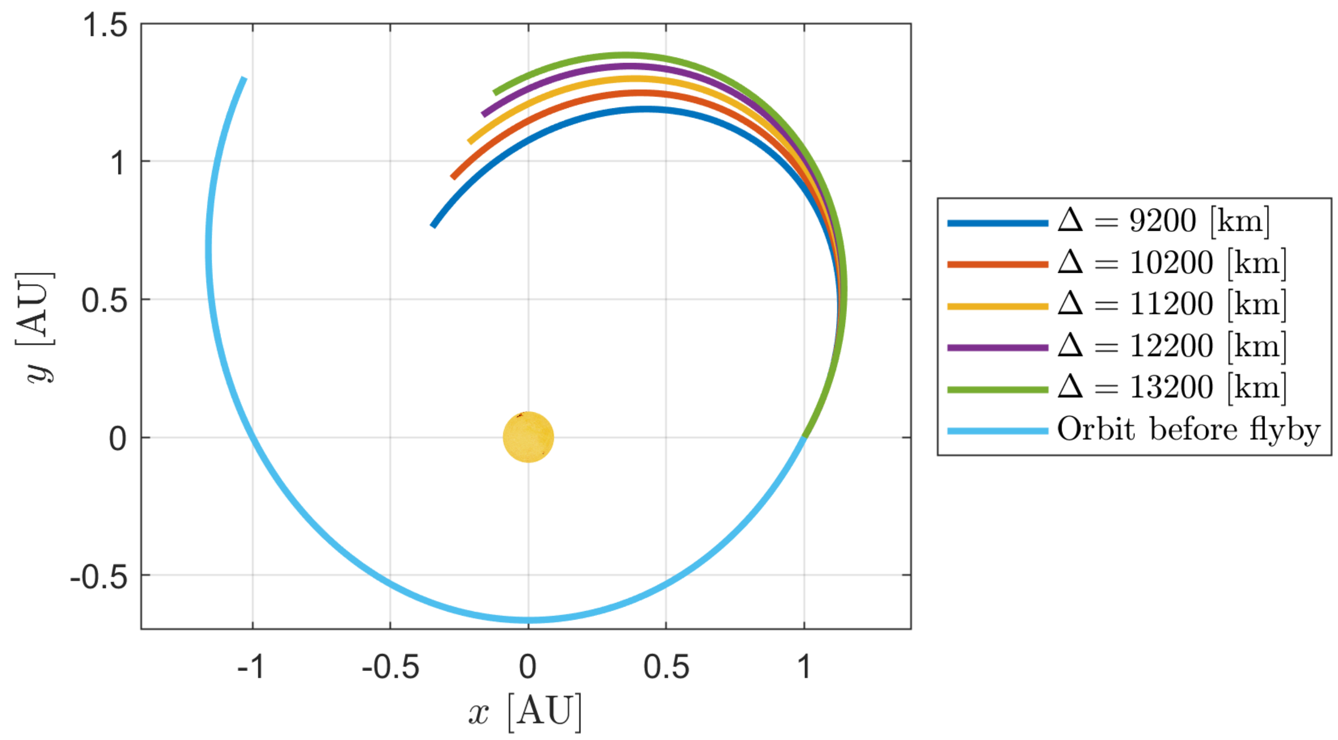


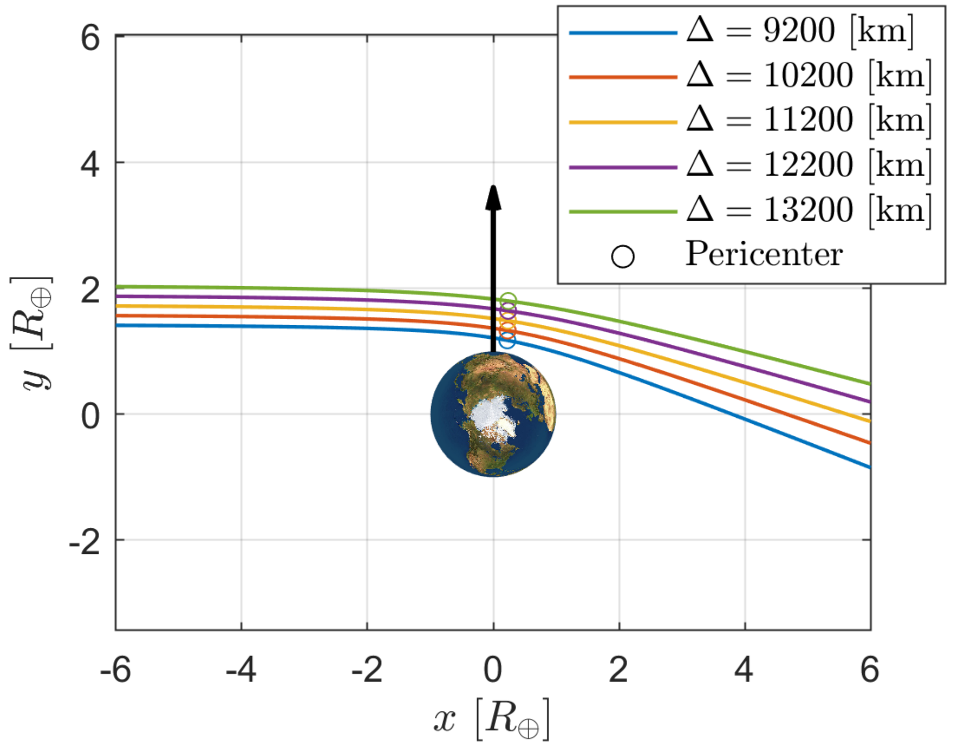
Evolution of flyby altitude and turning angle with impact parameter

Remember,
$r_{SOI} = 145.3 R_\oplus$

Exercise 1b: Flyby

Results





Heliocentric trajectories in HECI frame, for flybys in front of the planet with same $\mathbf{v}_\infty^-$ and different Δ

Planetocentric trajectories in Earth-centred frame parallel to HECI, for flybys in front of the planet with same $\mathbf{v}_\infty^-$ and different Δ

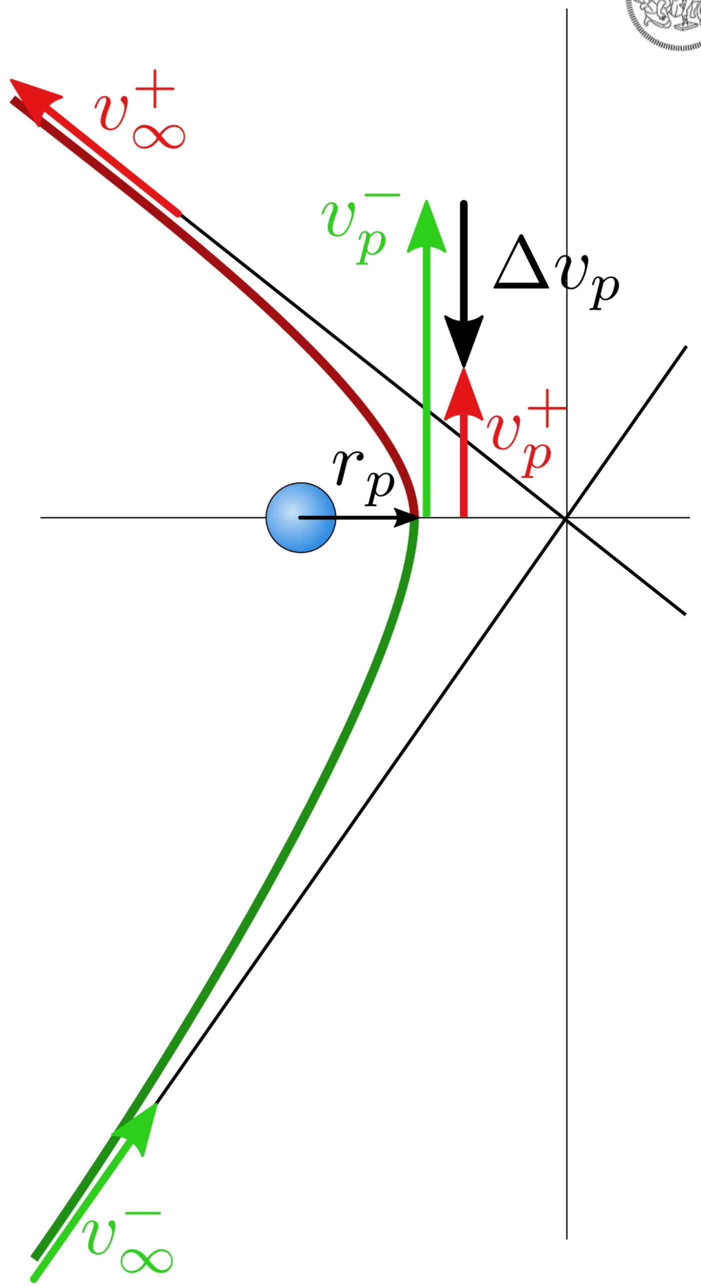
Powered gravity assist

Motivation

Previous flybys were **limited to $\boldsymbol{v_\infty^- = v_\infty^+}$** (constant excess velocity).

However, in a general case the incoming and outcoming heliocentric arcs will need to have $v_\infty^- \neq v_\infty^+$:

- For instance, to match two interplanetary arcs obtained from Lambert's problem.
- This gravity assist cannot be performed with a single hyperbolic arc.

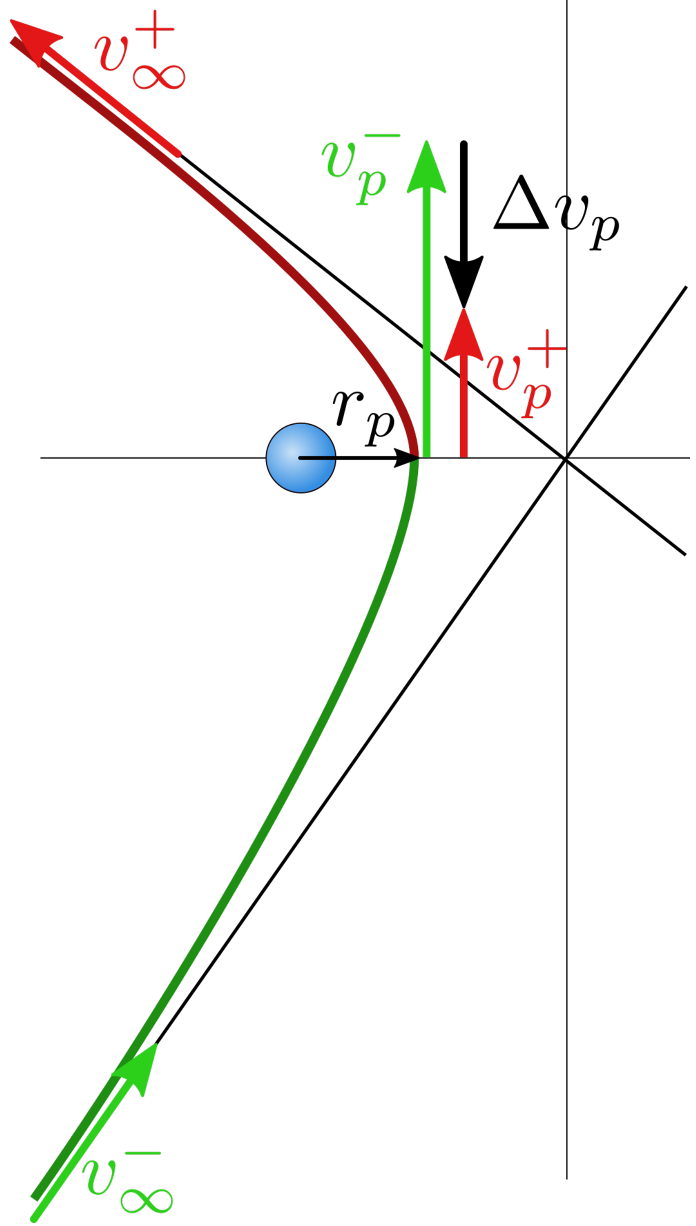


Powered gravity assist

Definition

We can combine two different hyperbolic arcs (with different excess velocities) by performing a **tangential impulsive manoeuvre at their common pericentre.**

- The magnitude of the manoeuvre will be the difference of the velocities of each arc at the common pericentre, $\Delta v_p = \left| v_p^- - v_p^+ \right|$.
- The total turning angle δ is still the angle formed by $\mathbf{v}_\infty^-$ and $\mathbf{v}_\infty^+$.
- Other strategies could be applied (e.g., deep space manoeuvres before and after the flyby, multiple impulses inside the SOI, etc.), but we will not consider them in this lab.
- Note that $\Delta \mathbf{v}$ is no longer oriented along the apse line. The orientation of the apse line can still be computed from the angles formed with it by the asymptotes.



Powered gravity assist

Design strategy

Inputs $\mathbf{v}_\infty^-$ $\mathbf{v}_\infty^+$ → **Compute** $\delta, v_\infty^-, v_\infty^+$ → **Solve for** r_p for given $\delta, v_\infty^-, v_\infty^+$ → **Check validity** $r_p > r_p^*$ → **Compute** Δv_p

$$e^- = 1 + \frac{r_p \, (v_\infty^-)^2}{\mu}, \quad \delta^- = 2 \operatorname{asin} \frac{1}{e^-} \quad \Rightarrow \quad \delta^-(r_p; v_\infty^-)$$

$$e^+ = 1 + \frac{r_p \, (v_\infty^+)^2}{\mu}, \quad \delta^+ = 2 \operatorname{asin} \frac{1}{e^+} \quad \Rightarrow \quad \delta^+(r_p; v_\infty^+)$$

$$\delta = f_\delta(r_p; v_\infty^-, v_\infty^+) = \frac{\delta^-}{2} + \frac{\delta^+}{2}$$

Hint: solve for r_p using `fzero` or `fsolve`

Remember to check if the radius of pericentre obtained is physically feasible:

$$r_p > r_p^* = R_P + h_{atm}$$

Exercise 2: Powered gravity assist

Exercise 2: Design a powered GA around the Earth given the heliocentric velocities before and after the flyby, and Earth's position.

1. Compute the velocities relative to the planet before and after the flyby, $\mathbf{v}_\infty^-$ and $\mathbf{v}_\infty^+$.
2. Compute the turning angle δ.
3. Solve the non-linear system for r_p and check its validity.
4. Compute the velocities of the two hyperbolic arcs at pericentre and the required Δv_p.
5. Plot the two planetocentric hyperbolic arcs.

Data:

Vectors in the heliocentric ecliptic frame; assume circular Earth orbit around the Sun

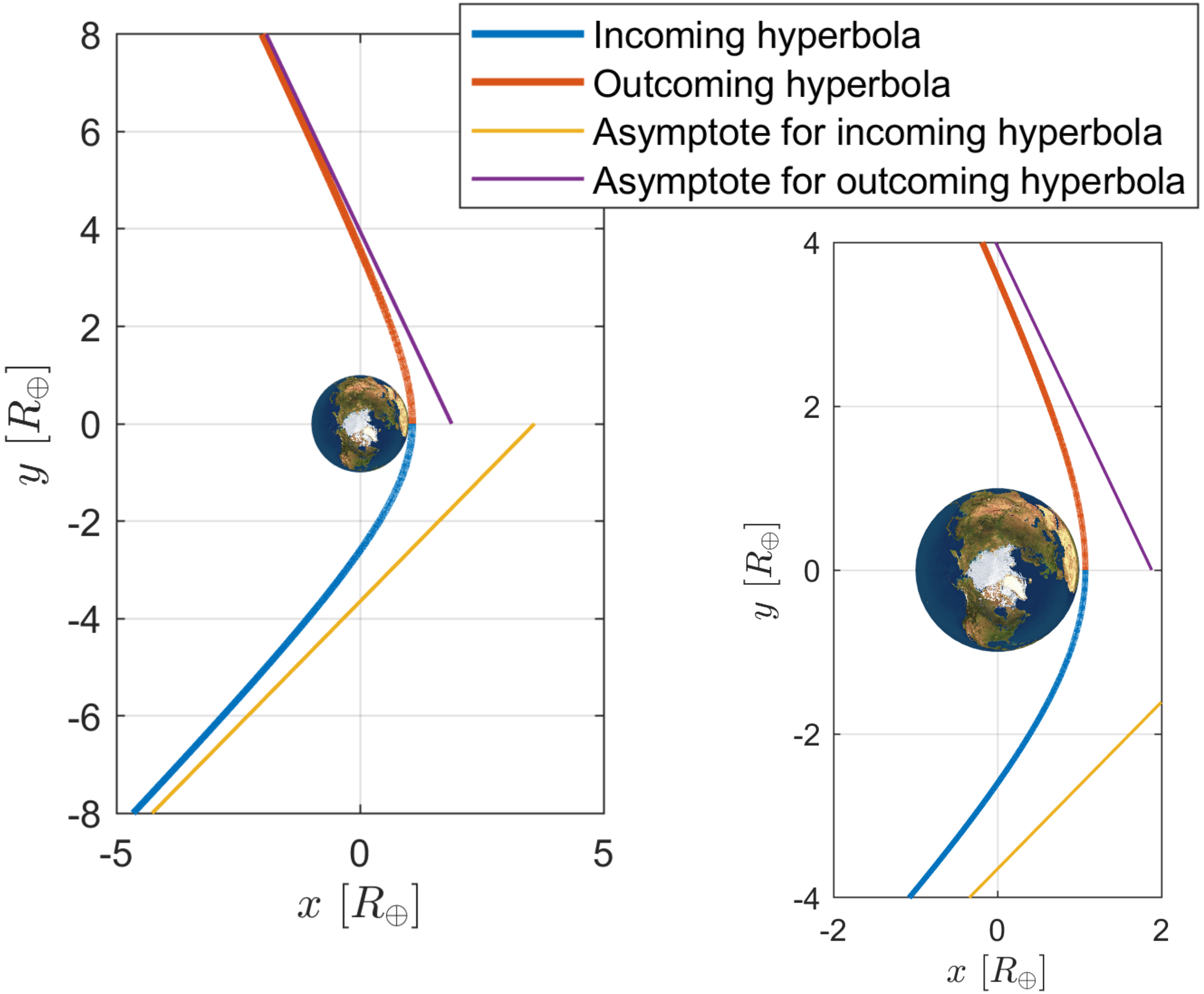
$\mu_\oplus$, $\mu_\odot$, and AU from `astroConstants` (identifiers `13`, `4`, and `2`, respectively)

$\mathbf{V}^- = [\,31.5;\;\; 4.69;\;\; 0\;\,]$ km/s $\qquad$ $\mathbf{V}^+ = [\,38.58;\;\; 0;\;\; 0\;\,]$ km/s

$\mathbf{r}_\oplus = [\,0;\;\; -1;\;\; 0\,]$ AU

Exercise 2: Powered gravity assist

Results



$\delta = 69.9106\ \mathrm{deg}$

$r_p = 6837.1763\ \mathrm{km}$

$h_{ga} = 466.1663\ \mathrm{km}$

$\Delta v_p = 2.0299\ \mathrm{km/s}$

$\Delta v_{\mathrm{tot}} = 8.4925\ \mathrm{km/s}$

$e^- = 1.4278$

$a^- = -15983.4119\ \mathrm{km}$

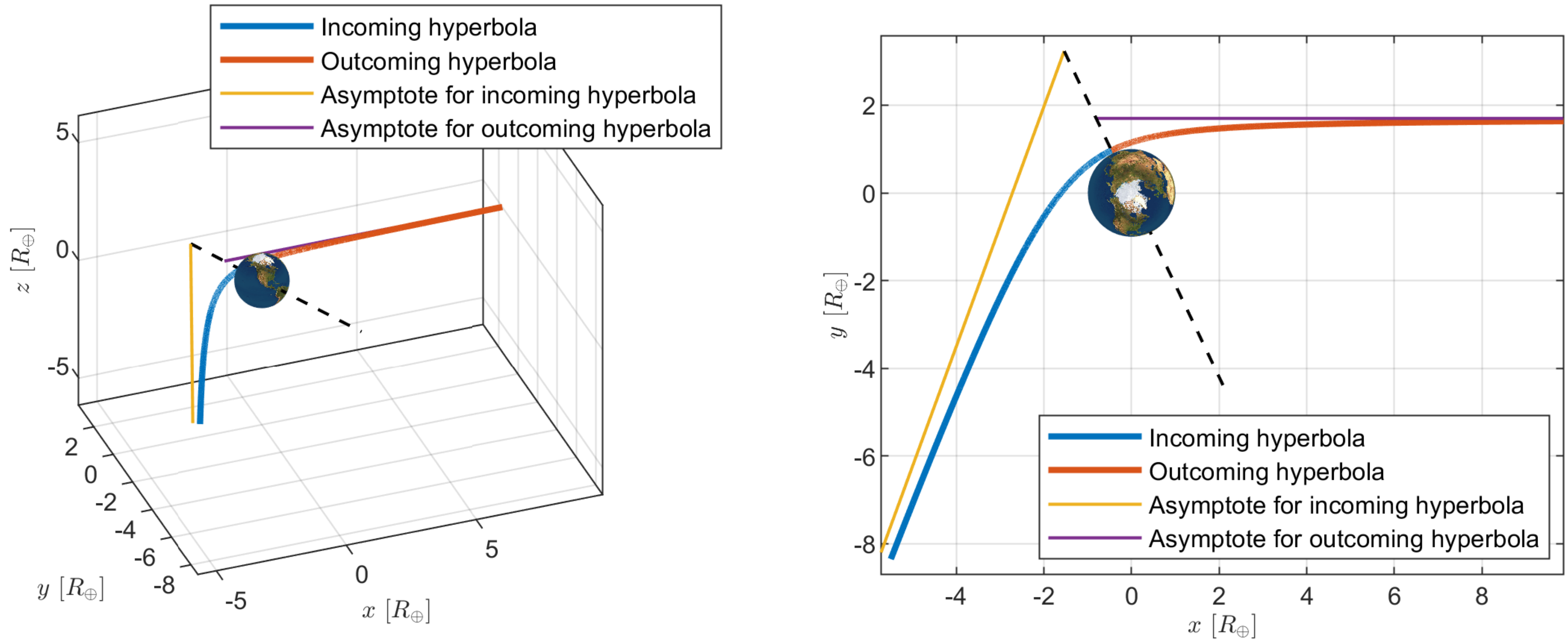
$e^+ = 2.3269$

$a^+ = -5152.7093\ \mathrm{km}$

Plots in flyby perifocal frame

Exercise 2: Powered gravity assist

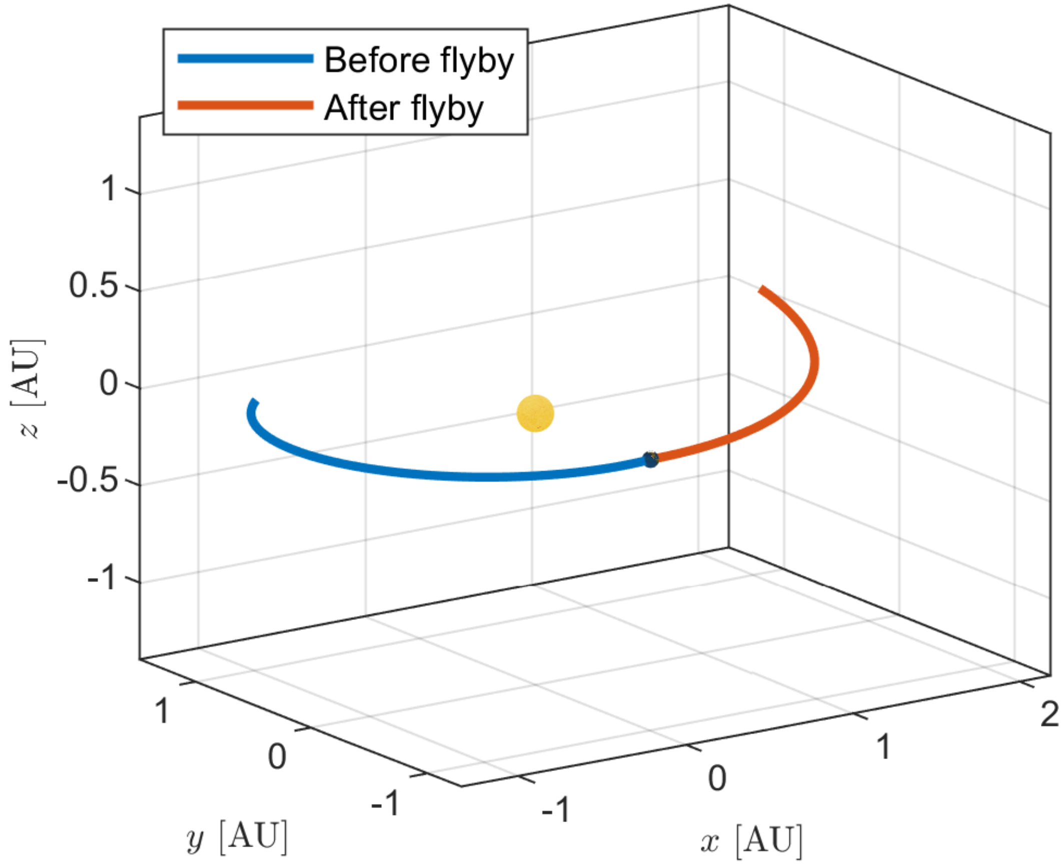
Results

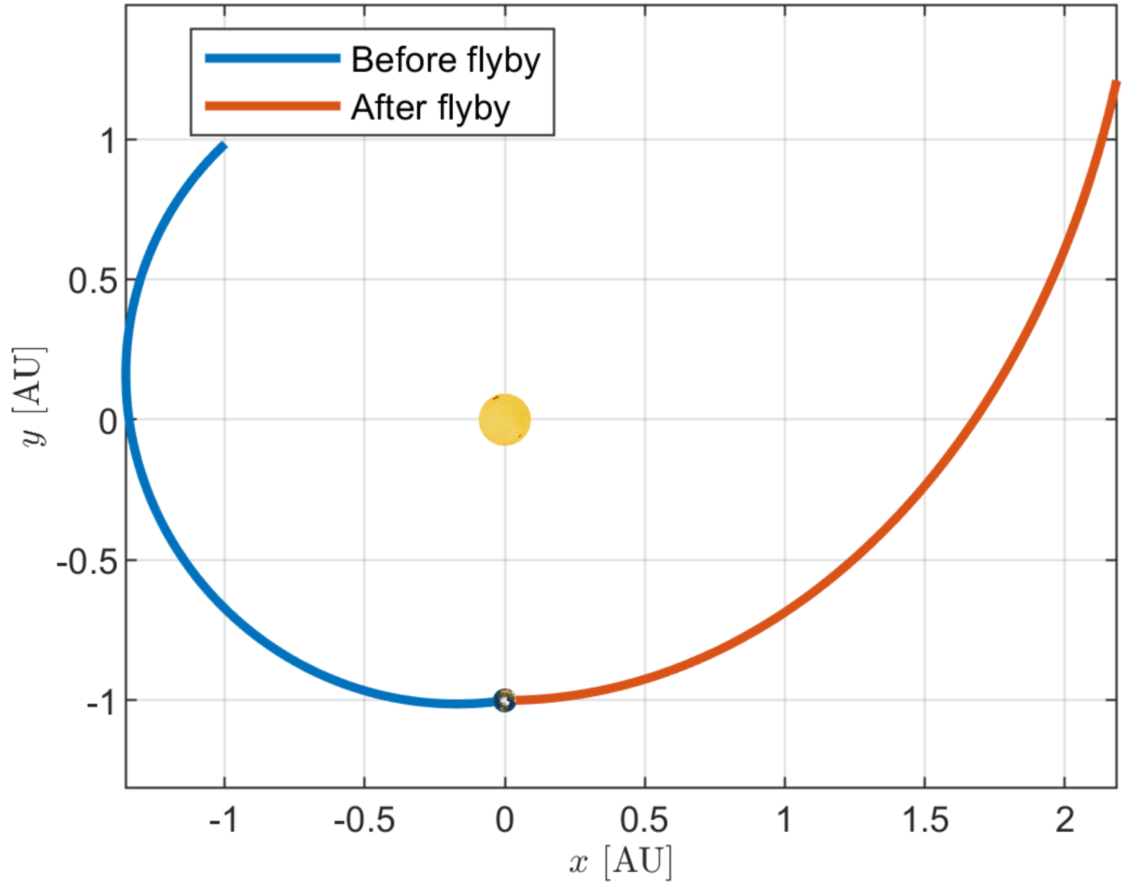


Flyby in Earth-centred frame parallel to HECI

Exercise 2: Powered gravity assist

Results





Heliocentric trajectories in HECI frame

DESIGN OF GRAVITY-ASSISTED INTERPLANETARY TRANSFERS

Design of gravity-assisted interplanetary transfers

More than the sum of the parts

Designing an **interplanetary transfer with intermediate GAs** is more complex than just sequentially computing the optimum Lambert arc for each leg of the travel:

- The optimum arrival time for the incoming arc may be a suboptimum (*or even terrible*) departure time for the outcoming arc.
- The required powered GA to connect the incoming and outcoming arcs may require too high Δv (*or just pierce through the planet*).

We need a compromise solution: a sequence of suboptimal transfer arcs leading to a globally optimal (and feasible) solution.

- Porkchop plots for each arc are still a useful design tool, as they provide information about the minimum possible Δv, or high-Δv regions to be avoided.

Design of gravity-assisted interplanetary transfers

A constrained parametric optimisation problem

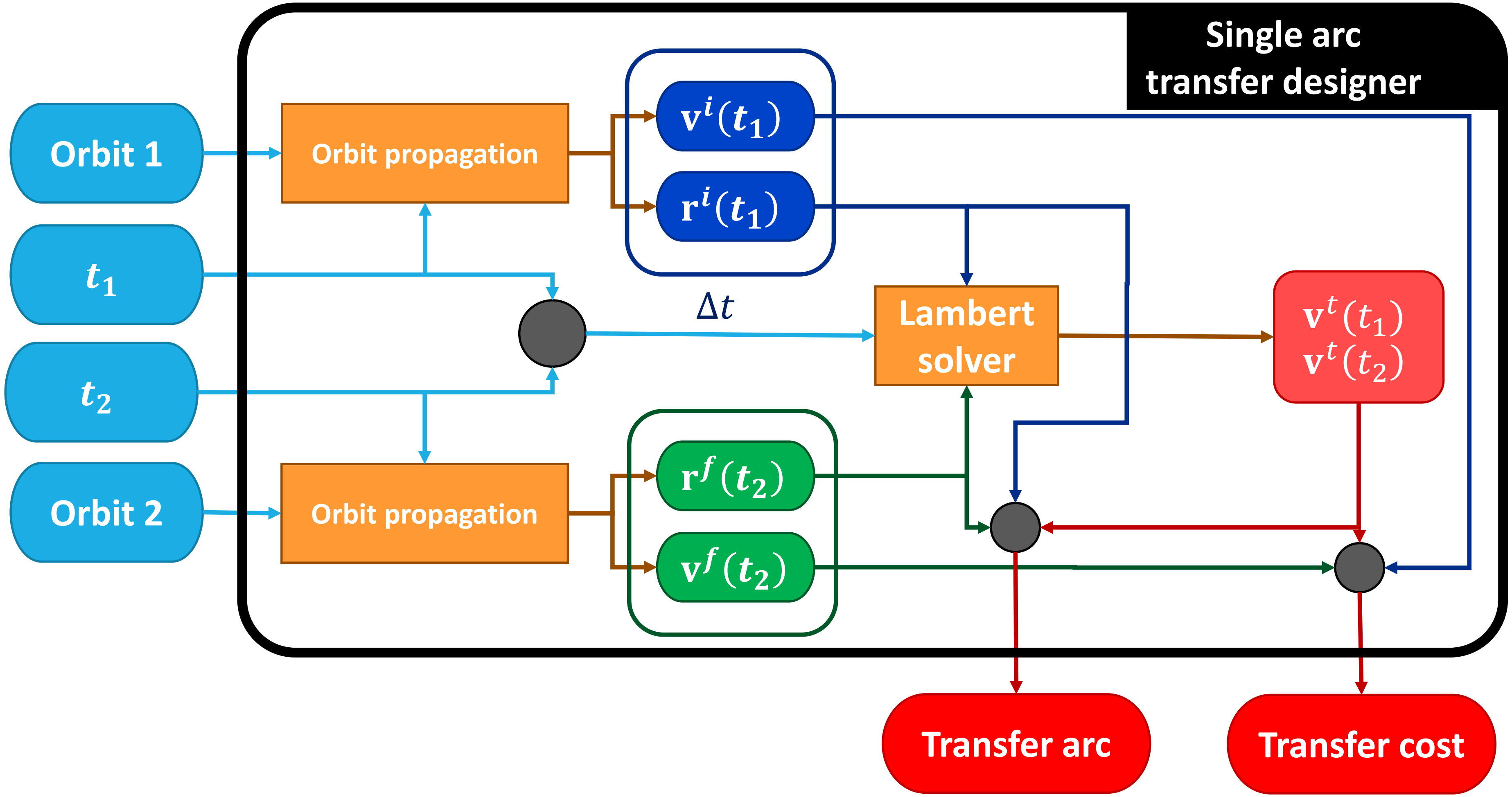
The preliminary design of an interplanetary transfer with one GA can be formulated as a **constrained parametric optimisation problem**

Three degrees of freedom (DoFs)	One figure of merit	Operational constraints
Departure time Flyby time Arrival time (or equivalent parameters, such as the times of flight)	Δv_{tot}	• **Flyby not hitting the planet or its atmosphere** • Time constraints (earliest departure date, latest arrival date). • Etcetera.

Design of gravity-assisted interplanetary transfers

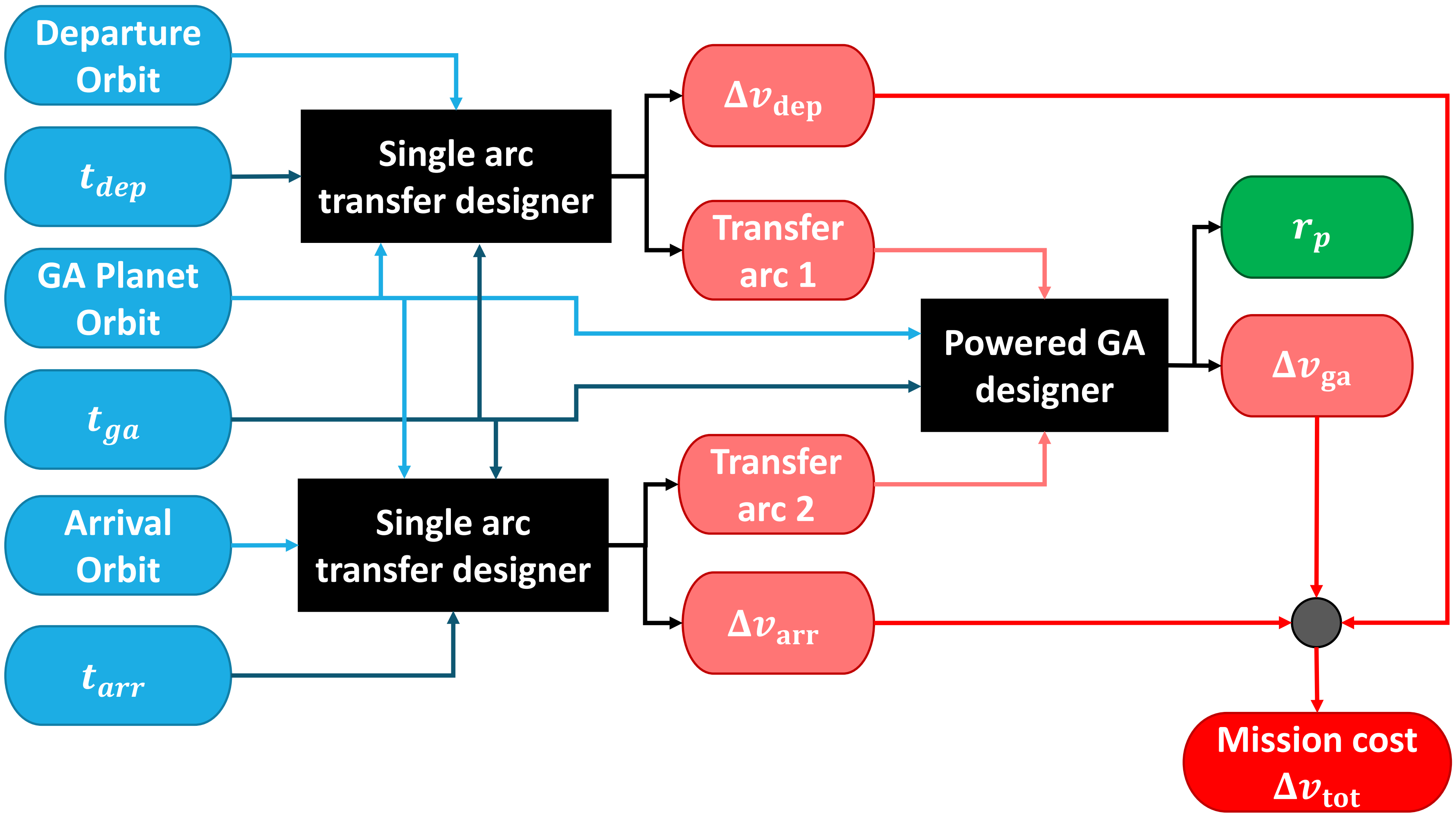
Computing Δv_{tot} for fixed departure, GA, and arrival times

From **Module 3**:



Design of gravity-assisted interplanetary transfers

Computing Δv_{tot} for fixed departure, GA, and arrival times

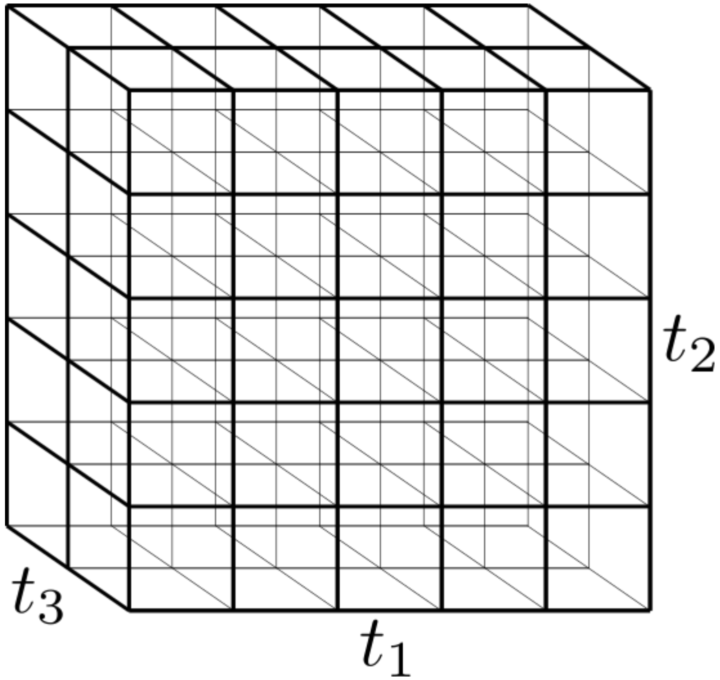


Design of gravity-assisted interplanetary transfers

The search for a solution (the basic way)

The simplest way to search for a solution is to perform a **grid search over the 3 DoFs using three nested loops**:

```
for each departure time
   for each GA time
      for each arrival time
         Compute and store DV(dep,ga,arr)
Find minimum of DV fulfilling the constraints
```



This approach **requires to:**

- Choose adequate ranges for the times.
  - Leverage physical information from the problem, such as the synodic periods, planet orbital periods, time of flight for simplified transfers, etc.
  - Remember that the information relevant for each range is different. E.g., synodic periods are relevant for departure windows, but not for times of flight.
- Choose an adequate number of points to discretise each range.
  - A finer grid improves the accuracy of the results, but also increases the computational cost (both in terms of time and memory).

Design of gravity-assisted interplanetary transfers

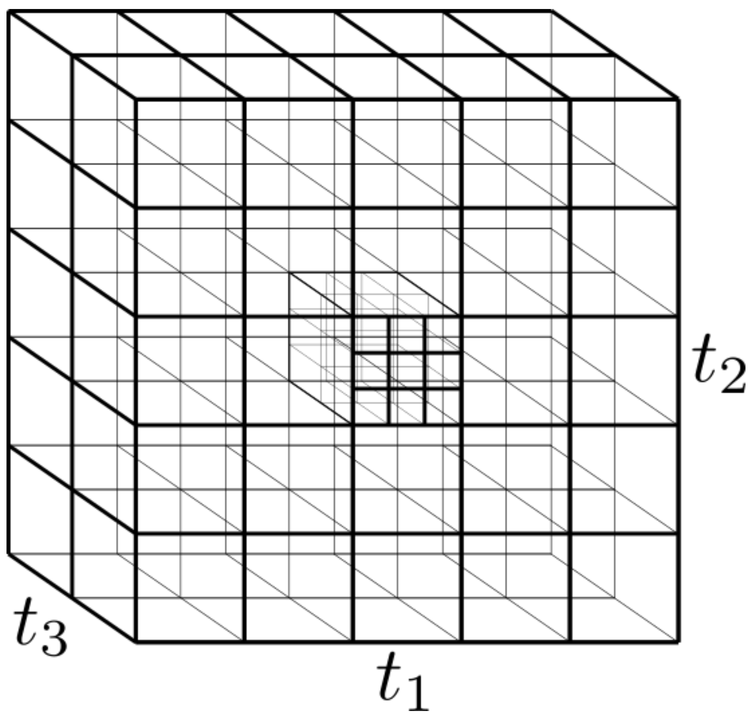
The search for a solution (refined strategy)

The main limitation of the grid search approach is its high computational cost.

- Implies solving $l \times m \times n$ cases, where l, m, and n are the number of points used to discretize each DoF.
  For instance, $l = m = n = 100$ yields a million cases.

This can be improved using a **multi-step approach**:

1. **Coarse grid search** to identify the best regions.
2. **Refined grid searches** for each of the regions identified in the previous step.
3. Repeat the refinement of subregions until the desired accuracy is reached.



Apart from locating minima, **the data from the grid search can also provide valuable insight on the behaviour of the mission** (for instance, making a contour plot for mission cost depending on departure and arrival dates).

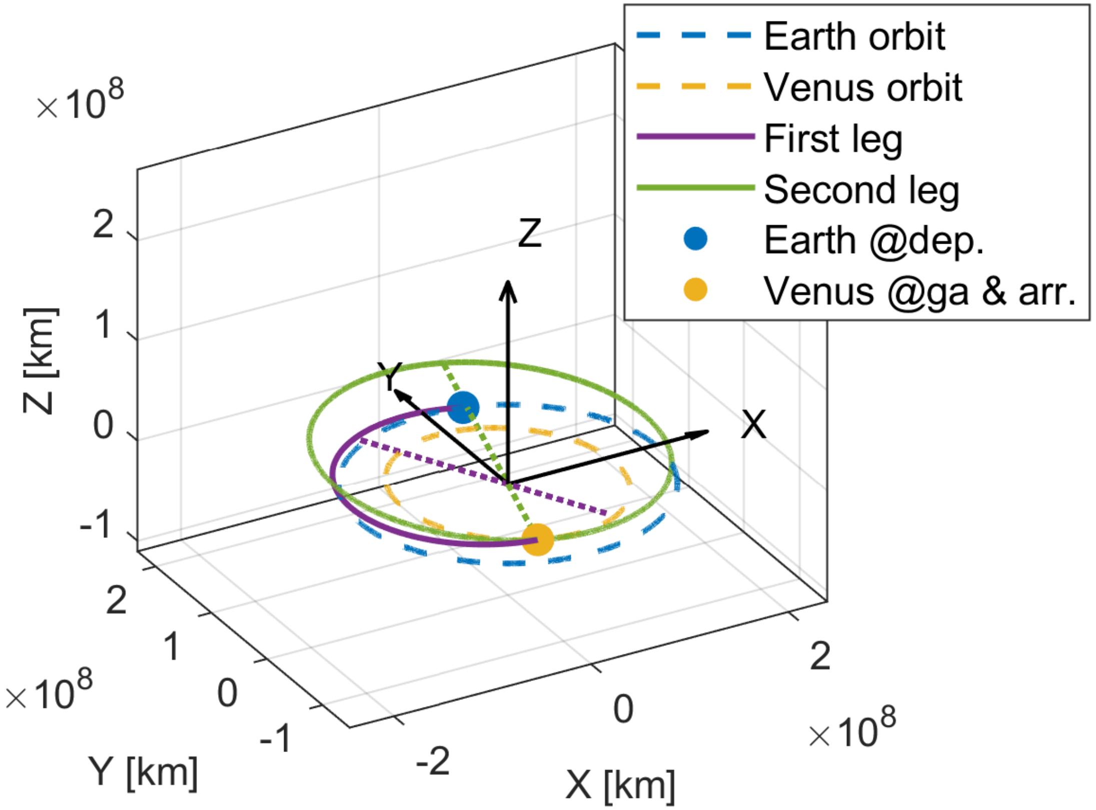
Interplanetary Explorer Mission

First Assignment

The **PoliMi Space Agency** is carrying out a feasibility study for a potential **Interplanetary Explorer Mission** visiting three planets in the Solar System.

As part of the **mission analysis team**, you are requested to perform the **preliminary mission analysis**. You have to study the transfer options from the departure planet to the arrival planet, with a powered gravity assist (flyby) at the intermediate planet, and **propose a solution based on the mission cost** (measured through the total Δv).

The departure, flyby, and arrival planets have been decided by the science team. Constraints on earliest departure and latest arrival have also been set by the launch provider, the systems engineering team, and the Agency's leadership.



Check the slides in WeBeep for all the details on the mission!