Section 4.3

Quadratic Functions and Their Properties

Quadratic Functions

$$F(x) = 3x^2 - 5x + 1 \quad g(x) = -6x^2 + 1 \quad H(x) = \frac{1}{2}x^2 + \frac{2}{3}x$$

DEFINITION

A **quadratic function** is a function of the form

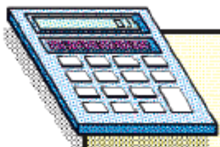
$$\boxed{f(x) = ax^2 + bx + c}$$

where a, b, and c are real numbers and $a \neq 0$. The domain of a quadratic function consists of all real numbers.

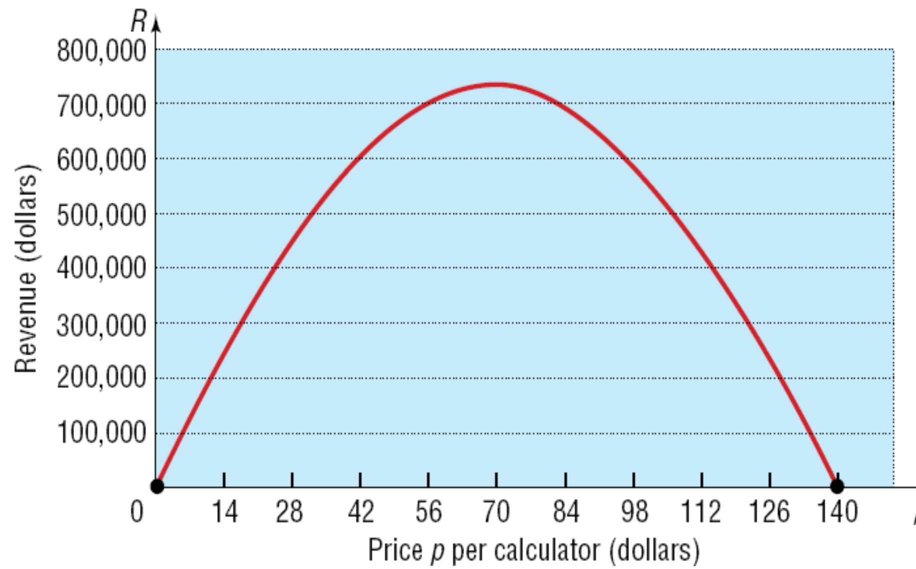
Suppose that Texas Instruments collects the data shown in Table 1, which relate the number of calculators sold at the price p (in dollars) per calculator. Since the price of a product determines the quantity that will be purchased, we treat price as the independent variable. The relationship between the number x of calculators sold and the price p per calculator may be approximated by the linear equation

$$x = 21{,}000 - 150p$$

$$\begin{aligned} R &= xp \\ R(p) &= (21{,}000 - 150p)p \\ &= -150p^2 + 21{,}000p \end{aligned}$$



Price per Calculator, p (Dollars)	Number of Calculators, x
60	11,100
65	10,115
70	9,652
75	8,731
80	8,087
85	7,205
90	6,439



Then the revenue R derived from selling x calculators at the price p per calculator is equal to the unit selling price p of the product times the number x of units actually sold.

A second situation in which a quadratic function appears involves the motion of a projectile. Based on Newton's second law of motion (force equals mass times acceleration, $F = ma$), it can be shown that, ignoring air resistance, the path of a projectile propelled upward at an inclination to the horizontal is the graph of a quadratic function. See Figure 2 for an illustration. Later in this section we shall analyze the path of a projectile.

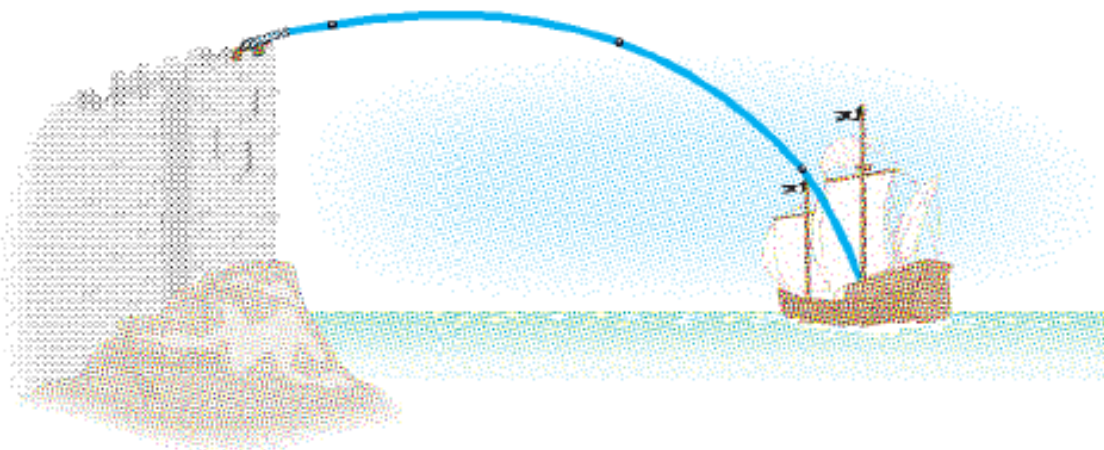
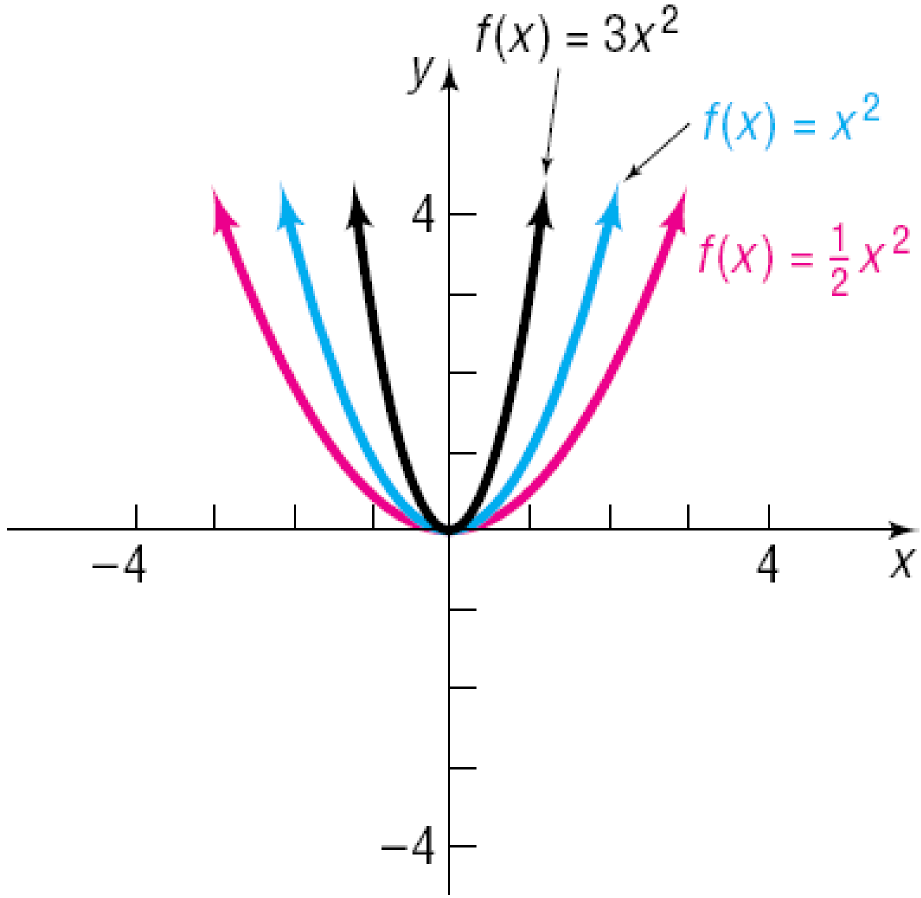
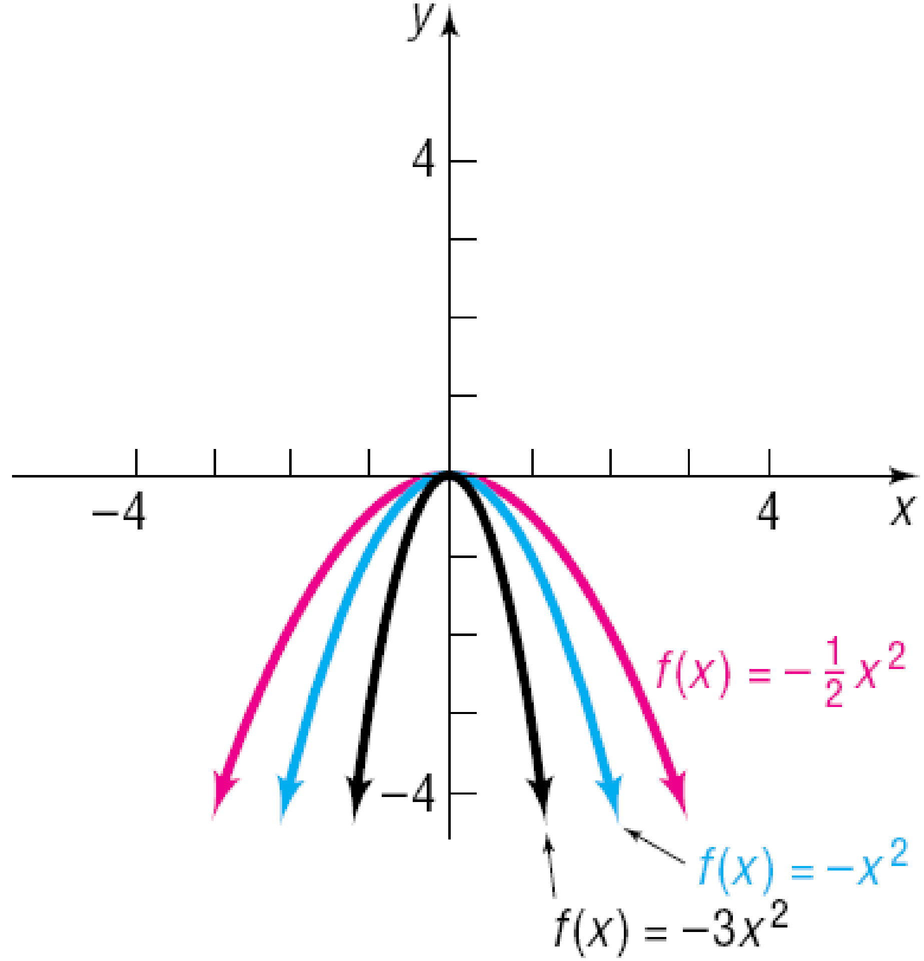


Figure 2
Path of a cannonball

1 Graph a Quadratic Function Using Transformations

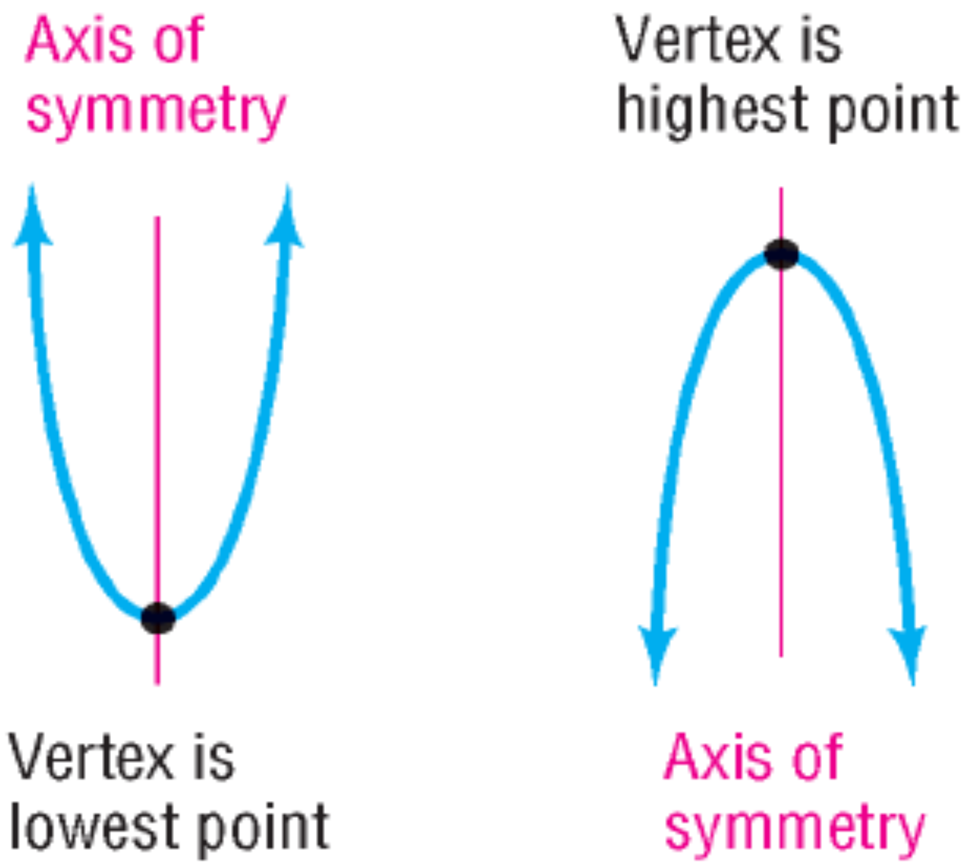


$$f(x) = ax^2, a > 0, \text{ for } a = 1, a = \frac{1}{2}, \text{ and } a = 3.$$



$$f(x) = ax^2 \text{ for } a < 0.$$

Graphs of a quadratic function,
$f(x) = ax^2 + bx + c, a \neq 0$



(a) Opens up

$a > 0$

(b) Opens down

$a < 0$

EXAMPLE

Graphing a Quadratic Function Using Transformations

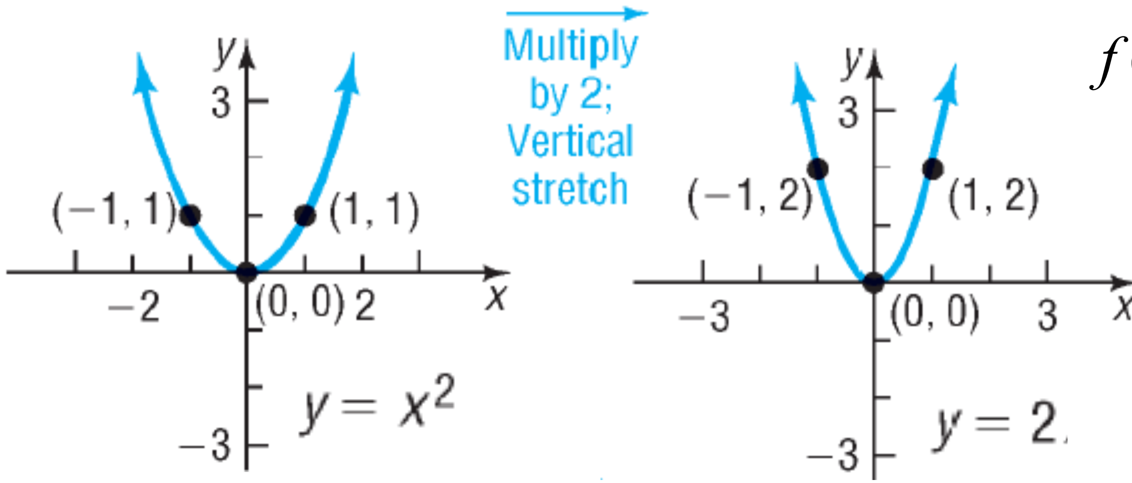
Graph the function $f(x) = 2x^2 + 8x + 5$.

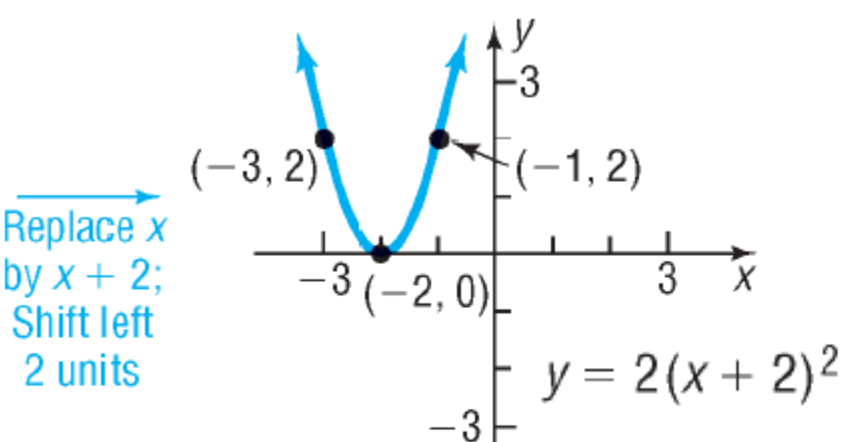
Find the vertex and axis of symmetry.

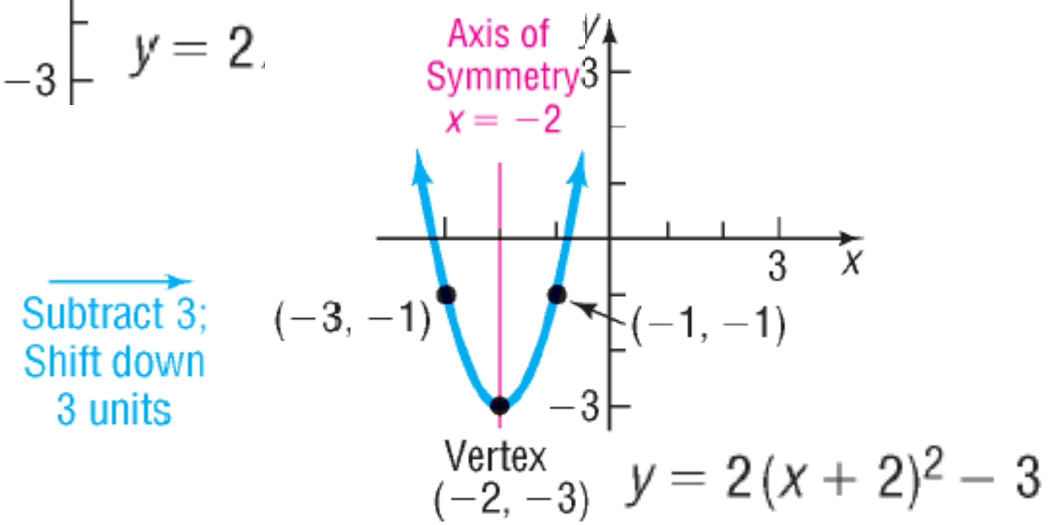
$$f(x) = 2\left(x^2 + 4x + \_\_\right) + 5 - 2(\_\_)$$

$$f(x) = 2\left(x^2 + 4x + 4\right) + 5 - 2(4)$$

$$f(x) = 2\left(x + 2\right)^2 - 3$$







$$f(x) = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c$$ Factor out a from $ax^2 + bx$.

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right)$$ Complete the square by adding $\frac{b^2}{4a^2}$. Look closely at this step!

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$ Factor.

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$ $c - \frac{b^2}{4a} = c \cdot \frac{4a}{4a} - \frac{b^2}{4a} = \frac{4ac - b^2}{4a}$

If $h = -\frac{b}{2a}$ and $k = \frac{4ac - b^2}{4a}$, then

$$\boxed{f(x) = ax^2 + bx + c = a(x - h)^2 + k}$$

2 Identify the Vertex and Axis of Symmetry of a Quadratic Function

Properties of the Graph of a Quadratic Function

$$f(x) = ax^2 + bx + c \qquad a \neq 0$$

Vertex $= \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$ Axis of symmetry: the line $x = -\dfrac{b}{2a}$

Parabola opens up if $a > 0$; the vertex is a minimum point.

Parabola opens down if $a < 0$; the vertex is a maximum point.

EXAMPLE Locating the Vertex without Graphing

Without graphing, locate the vertex and axis of symmetry of the parabola defined by $f(x)=2x^2-3x+2$. Does it open up or down?

$$-\frac{b}{2a}=\frac{-(-3)}{2(2)}=\frac{3}{4}$$

$$\text{Vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

$$f\left(\frac{3}{4}\right)=2\left(\frac{3}{4}\right)^2-3\left(\frac{3}{4}\right)+2=\frac{7}{8}$$

Vertex is $\left(\frac{3}{4},\frac{7}{8}\right)$.

Axis of symmetry is $x=\frac{3}{4}$.

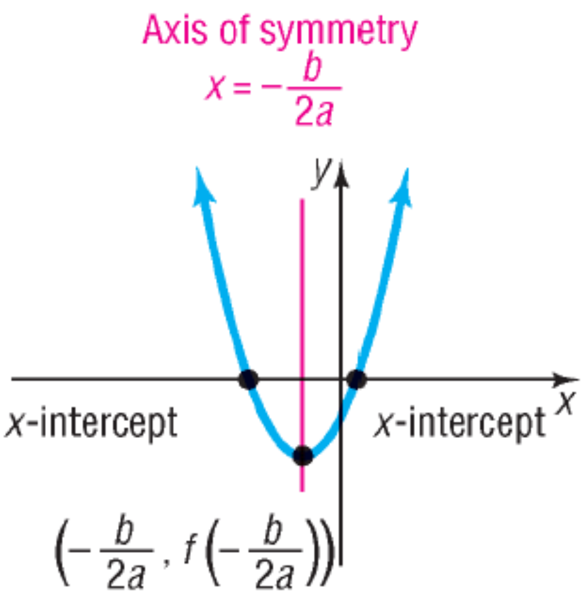
Because $a = 2 > 0$, the parabola opens up.

3 Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts

The *x*-Intercepts of a Quadratic Function

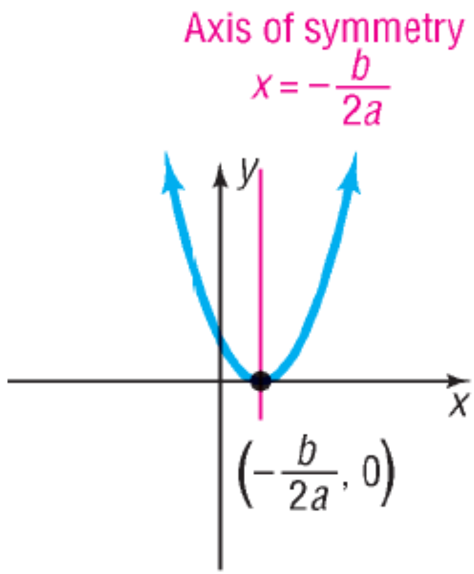
1. If the discriminant $b^2 - 4ac > 0$, the graph of $f(x) = ax^2 + bx + c$ has two distinct x-intercepts so it crosses the x-axis in two places.
2. If the discriminant $b^2 - 4ac = 0$, the graph of $f(x) = ax^2 + bx + c$ has one x-intercept so it touches the x-axis at its vertex.
3. If the discriminant $b^2 - 4ac < 0$, the graph of $f(x) = ax^2 + bx + c$ has no x-intercept so it does not cross or touch the x-axis.

$$f(x) = ax^2 + bx + c, a > 0$$



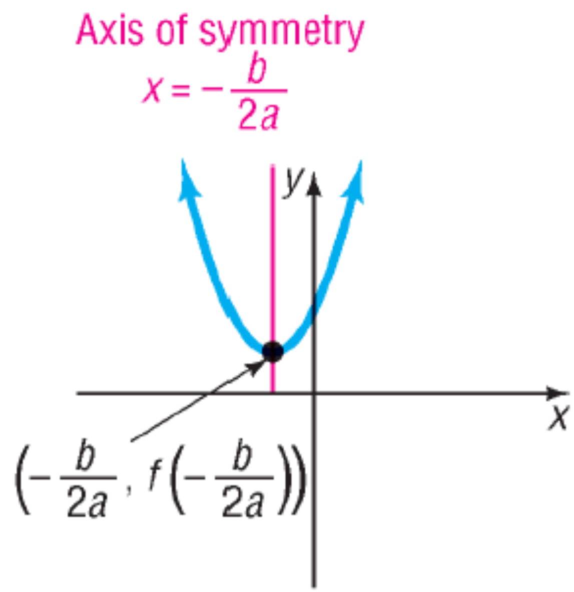
(a) $b^2 - 4ac > 0$

Two x-intercepts



(b) $b^2 - 4ac = 0$

One x-intercept



(c) $b^2 - 4ac < 0$

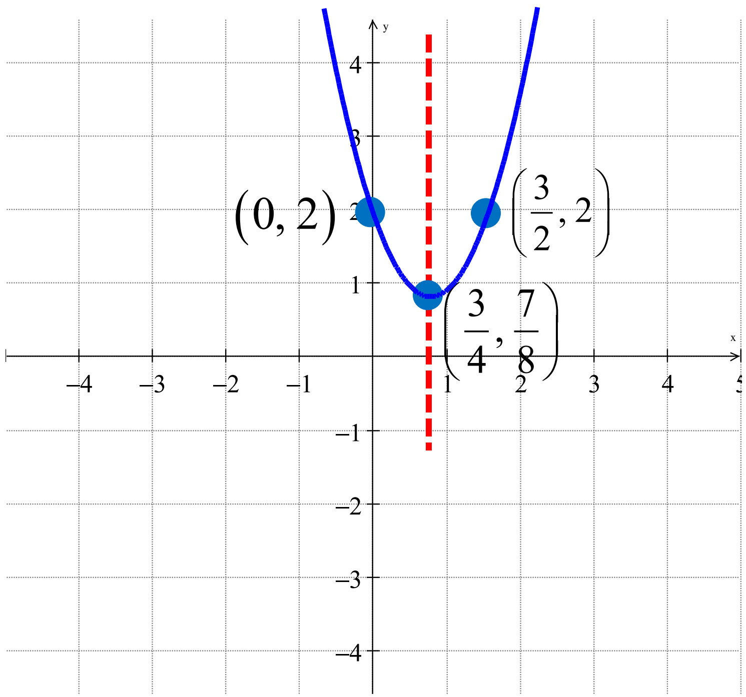
No x-intercepts

EXAMPLE **Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts**

(a) Use the information from the previous example and the locations of the intercepts to graph $f(x) = 2x^2 - 3x + 2$.

Vertex is $\left(\frac{3}{4}, \frac{7}{8}\right)$ Axis of symmetry is $x = \frac{3}{4}$

Since $a = 2 > 0$ the parabola opens up and therefore will have no x-intercepts.



$f(0) = 2(0)^2 - 3(0) + 2 = 2$ so the y-intercept $= 2$.

By symmetry, since the point $(0, 2)$ is on the graph, so is the point $\left(\frac{3}{2}, 2\right)$.

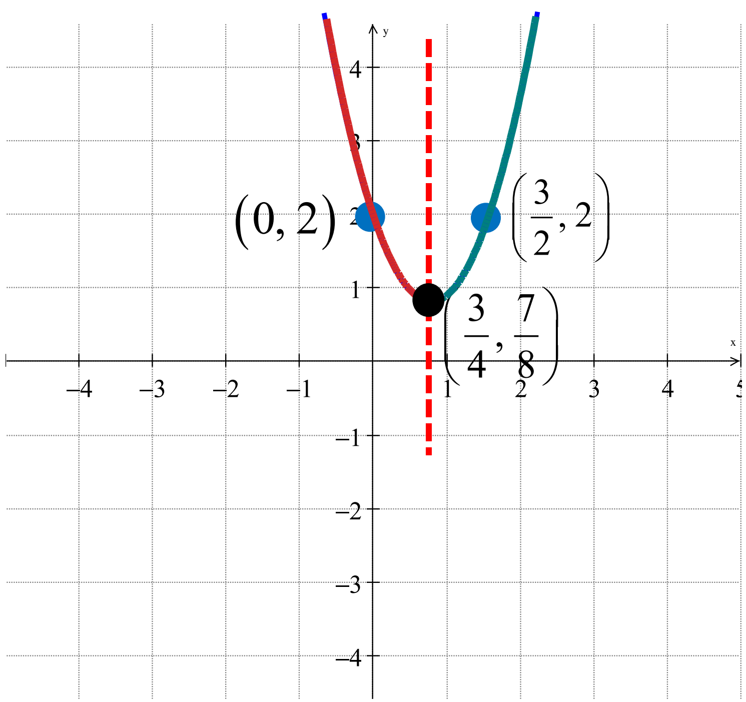
EXAMPLE

Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(b) Determine the domain and the range of f.
(c) Determine where f is increasing and decreasing.

The domain of f is the set of all real numbers.

Based on the graph, the range is the interval $\left[\frac{7}{8},\infty\right)$.



The function is

decreasing from $\left(-\infty,\frac{3}{4}\right)$

and

increasing from $\left(\frac{3}{4},\infty\right)$.

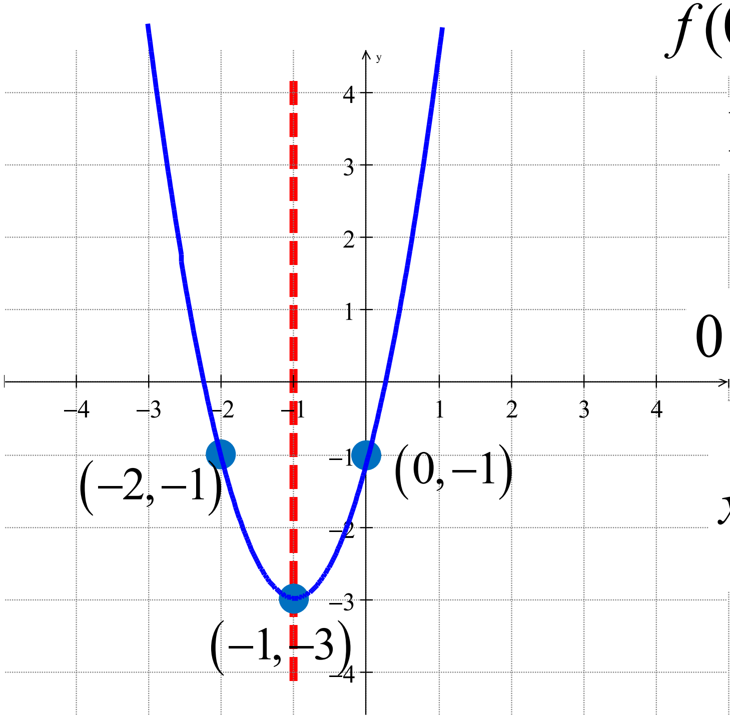
EXAMPLE **Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts**

(a) Graph $f(x)=2x^2+4x-1$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, and x-intercepts and y-intercept if any.

$$h=-\frac{b}{2a}=-\frac{4}{2(2)}=-1 \qquad k=f(-1)=2(-1)^2+4(-1)-1=-3$$

Since $a=2>0$ the parabola opens up.

$\text{Vertex}=(-1,-3)$



Axis of symmetry: $x=-1$

$f(0)=2(0)^2+4(0)-1=-1$ so the y-intercept $=-1$.

By symmetry, the point $(-2,-1)$ is also on the graph.

x-intercepts can be found when $f(x)=0$.

$0=2x^2+4x-1$ Use the quadratic formula to solve.

$$x=\frac{-4\pm\sqrt{4^2-4(2)(-1)}}{2(2)}=\frac{-4\pm\sqrt{24}}{4}=\frac{-2\pm\sqrt{6}}{2}$$

x-intercepts ≈ 0.22 and -2.22

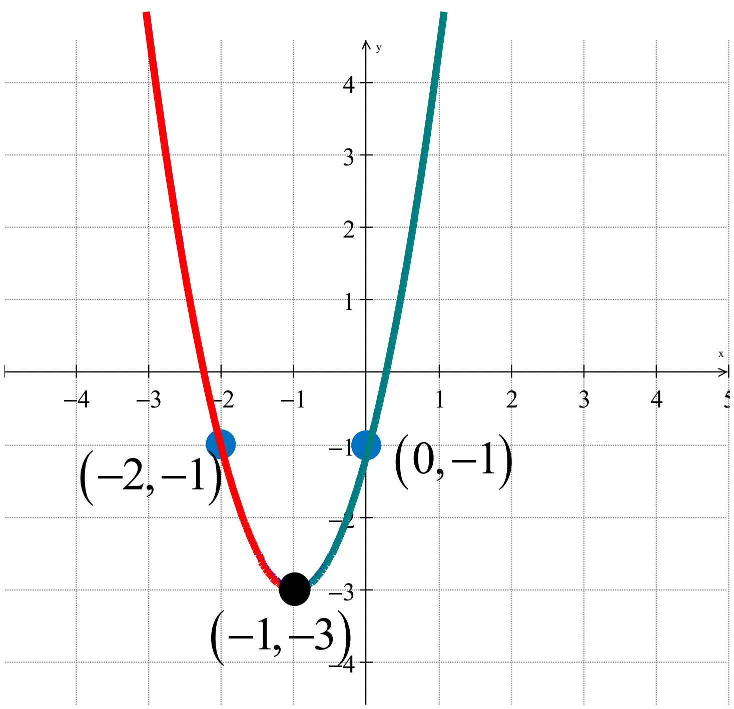
EXAMPLE

Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(b) Determine the domain and the range of f.
(c) Determine where f is increasing and decreasing.

The domain of f is the set of all real numbers.

Based on the graph, the range is the interval $[-3, \infty)$.



The function is

decreasing from $(-\infty, -1)$

and

increasing from $(-1, \infty)$.

EXAMPLE Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(a) Graph $f(x) = -\frac{1}{2}x^2 - 2x - 2$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, and x-intercepts and y-intercept if any.

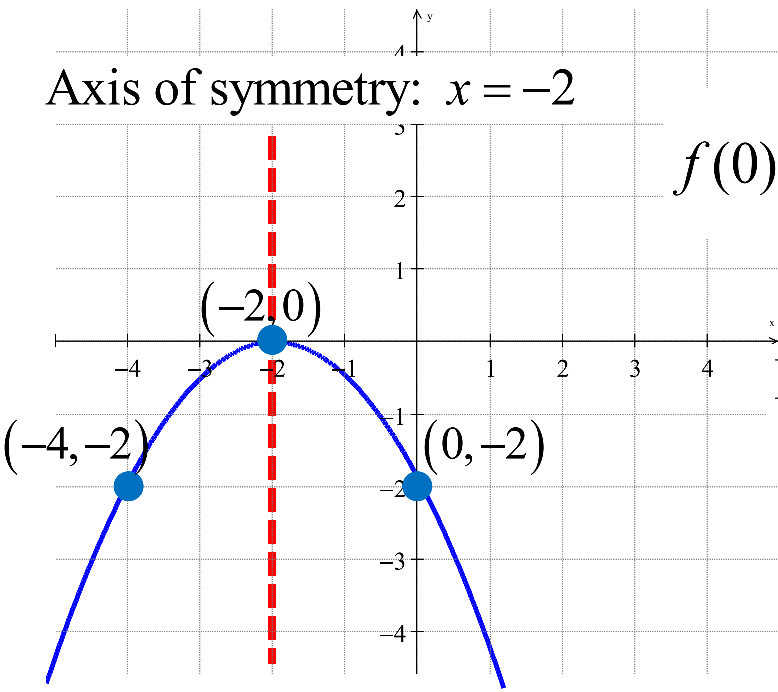
$$h = -\frac{b}{2a} = -\frac{-2}{2\left(-\frac{1}{2}\right)} = -2$$

$$k = f(-2) = -\frac{1}{2}(-2)^2 - 2(-2) - 2 = 0$$

$\text{Vertex} = (-2, 0)$

Since a is negative, the parabola opens down.

Axis of symmetry: $x = -2$



$f(0) = -\frac{1}{2}(0)^2 - 2(0) - 2 = -2$ so the y-intercept $= -2$.

By symmetry, the point $(-4, -2)$ is also on the graph.

As seen on the graph, the x-intercept is -2.

EXAMPLE

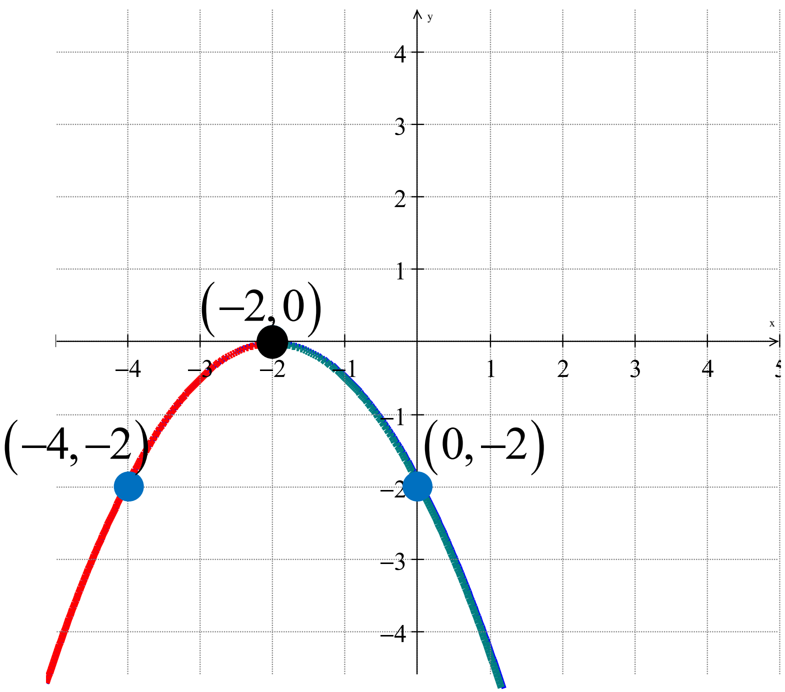
Graphing a Quadratic Function Using Its Vertex, Axis, and Intercepts

(b) Determine the domain and the range of f.

(c) Determine where f is increasing and decreasing.

The domain of f is the set of all real numbers.

Based on the graph, the range is the interval $(-\infty, 0]$.



The function is **increasing** from $(-\infty, -2)$ and **decreasing** from $(-2, \infty)$.

4 Find a Quadratic Function Given Its Vertex and One Other Point

Given the vertex (h, k) and one additional point on the graph of a quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, we can use

$$f(x) = a(x - h)^2 + k \qquad \textbf{(3)}$$

to obtain the quadratic function.

EXAMPLE

Finding the Quadratic Function Given Its Vertex and One Other Point

Determine the quadratic function whose vertex is (–2, 3) and whose y-intercept is 1.

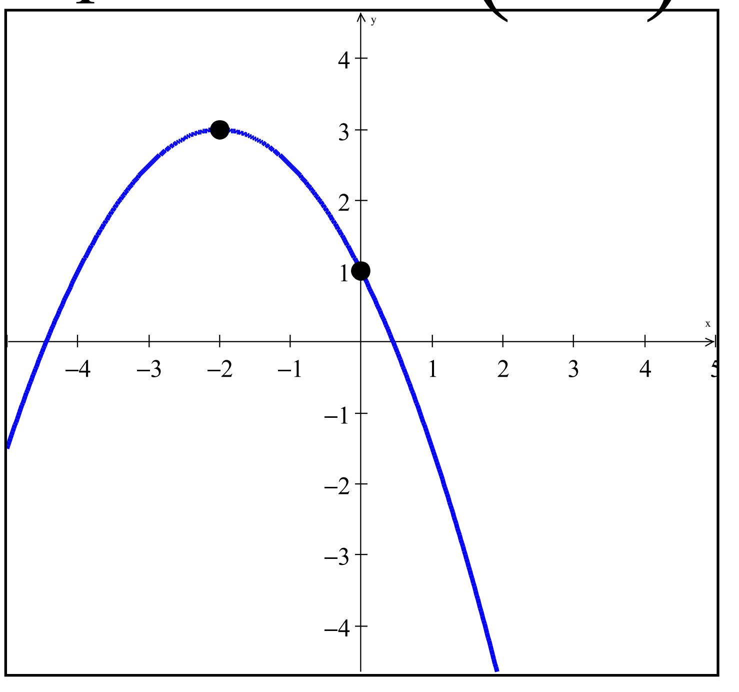
$$f(x) = a(x-h)^2 + k = a(x+2)^2 + 3$$

Using the fact that the y-intercept is 1: $1 = a(0+2)^2 + 3$

$$1 = 4a + 3 \qquad a = -\frac{1}{2}$$

$$f(x) = -\frac{1}{2}(x+2)^2 + 3$$

$$f(x) = -\frac{1}{2}x^2 - 2x + 1$$



5 Find the Maximum or Minimum Value of a Quadratic Function

EXAMPLE

Finding the Maximum or Minimum Value of a Quadratic Function

Determine whether the quadratic function

$$f(x) = -x^2 + 4x + 5$$

has a maximum or minimum value.

Then find the maximum or minimum value.

Since a is negative, the graph of f opens down so the function will have a maximum value.

$$x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$$

So the maximum value is $f(2) = -(2)^2 + 4(2) + 5 = 9$

SUMMARY Steps for Graphing a Quadratic Function $f(x) = ax^2 + bx + c, a \neq 0$

Option 1

STEP 1: Complete the square in x to write the quadratic function in the form $f(x) = a(x - h)^2 + k$.

STEP 2: Graph the function in stages using transformations.

Option 2

STEP 1: Determine whether the parabola opens up ($a > 0$) or down ($a < 0$).

STEP 2: Determine the vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

STEP 3: Determine the axis of symmetry, $x = -\frac{b}{2a}$.

STEP 4: Determine the y-intercept, $f(0)$, and the x-intercepts, if any.

(a) If $b^2 - 4ac > 0$, the graph of the quadratic function has two x-intercepts, which are found by solving the equation $ax^2 + bx + c = 0$.

(b) If $b^2 - 4ac = 0$, the vertex is the x-intercept.

(c) If $b^2 - 4ac < 0$, there are no x-intercepts.

STEP 5: Determine an additional point by using the y-intercept and the axis of symmetry.

STEP 6: Plot the points and draw the graph.