Section 12.5

Partial Fraction Decomposition

Consider the problem of adding two rational expressions:

$$\frac{3}{x+4} \quad \text{and} \quad \frac{2}{x-3}$$

The result is

$$\frac{3}{x+4} + \frac{2}{x-3} = \frac{3(x-3) + 2(x+4)}{(x+4)(x-3)} = \frac{5x-1}{x^2+x-12}$$

The reverse procedure is referred to as **partial fraction decomposition**

1 Decompose $\frac{P}{Q}$, Where Q Has Only Nonrepeated Linear Factors

Case 1: Q has only nonrepeated linear factors.

Under the assumption that Q has only nonrepeated linear factors, the polynomial Q has the form

$$\boxed{Q(x) = (x - a_1)(x - a_2) \cdot \cdots \cdot (x - a_n)}$$

where none of the numbers $a_1, a_2, \ldots, a_n$ is equal. In this case, the partial fraction decomposition of $\frac{P}{Q}$ is of the form

$$\boxed{\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n} \qquad \textbf{(1)}}$$

where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

EXAMPLE **Nonrepeated Linear Factors**

Write the partial fraction decomposition of $\dfrac{7x+1}{x^2+x-6}$

$$\frac{7x+1}{x^2+x-6}=\frac{A}{x+3}+\frac{B}{x-2}$$

$$(x+3)(x-2)$$

$$\frac{(7x+1)\cancel{(x+3)(x-2)}}{\cancel{x^2+x-6}}=\frac{A\cancel{(x+3)}(x-2)}{\overset{B(x+3)}{\cancel{x+3}}}+\frac{B(x+3)\cancel{(x-2)}}{\cancel{x-2}}$$

$$7x+1=A(x-2)+B(x+3)=\left(A+B\right)x+\left(-2A+3B\right)$$

Equate coefficients of like powers of *x* to get: $\begin{cases}7=A+B\\1=-2A+3B\end{cases}$ $\quad A=4 \quad B=3$

$$\frac{7x+1}{x^2+x-6}=\frac{4}{x+3}+\frac{3}{x-2}$$

2 Decompose $\dfrac{P}{Q}$, Where Q Has Repeated Linear Factors

Case 2: Q has repeated linear factors.

If the polynomial Q has a repeated linear factor, say $(x-a)^n$, $n \geq 2$ an integer, then, in the partial fraction decomposition of $\dfrac{P}{Q}$, we allow for the terms

$$\boxed{\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}}$$

where the numbers $A_1, A_2, \ldots, A_n$ are to be determined.

EXAMPLE

Repeated Linear Factors

Write the partial fraction decomposition of $\dfrac{x^2+2}{(x-1)(x+2)^2}$

$$\frac{x^2+2}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}=\frac{\frac{1}{3}}{x-1}+\frac{\frac{2}{3}}{x+2}+\frac{-2}{(x+2)^2}$$

Clear of fractions by multiplying all terms by $(x-1)(x+2)^2$

$$x^2+2=A(x+2)^2+B(x-1)(x+2)+C(x-1)$$

$$x^2+2=Ax^2+4Ax+4A+Bx^2+Bx-2B+Cx-C$$

$$x^2+2=(A+B)x^2+(4A+B+C)x+(4A-2B-C)$$

$$\begin{aligned}1&=A+B\\0&=4A+B+C\\2&=4A-2B-C\end{aligned}\qquad A=\frac{1}{3},B=\frac{2}{3},C=-2$$

EXAMPLE Repeated Linear Factors

Write the partial fraction decomposition of $\dfrac{x^2-4}{x^3(x-1)^2}$

$$\frac{x^2-4}{x^3(x-1)^2}=\frac{-11}{x}+\frac{-8}{x^2}+\frac{-4}{x^3}+\frac{11}{x-1}+\frac{-3}{(x-1)^2}$$

$$x^2-4=Ax^2(x-1)^2+Bx(x-1)^2+C(x-1)^2+Dx^3(x-1)+Ex^3$$

Let $x=0$: $-4=C(0-1)^2$ $\quad C=-4$ $\quad$ Let $x=1$: $1^2-4=E(1)^3$ $\quad E=-3$

$$x^2-4+4(x-1)^2+3x^3=Ax^2(x-1)^2+Bx(x-1)^2+Dx^3(x-1)$$

$$x(3x+8)(x-1)=x(x-1)\left(Ax(x-1)+B(x-1)+Dx^2\right)$$

$3x+8=Ax(x-1)+B(x-1)+Dx^2$ $\quad$ Let $x=0:8=B(0-1)$ $\quad B=-8$

Let $x=1:3(1)+8=D(1)^2$ $\quad D=11$

$$3x+8=Ax(x-1)-8(x-1)+11x^2$$

Let $x=2:3(2)+8=A(2)(2-1)-8(2-1)+11(2)^2$ $\quad A=-11$

3 Decompose $\frac{P}{Q}$, Where Q Has a Nonrepeated Irreducible Quadratic Factor

Case 3: Q contains a nonrepeated irreducible quadratic factor.

If Q contains a nonrepeated irreducible quadratic factor of the form $ax^2 + bx + c$, then, in the partial fraction decomposition of $\frac{P}{Q}$, allow for the term

$$\boxed{\frac{Ax + B}{ax^2 + bx + c}}$$

where the numbers A and B are to be determined.

EXAMPLE Nonrepeated Irreducible Quadratic Factor

Write the partial fraction decomposition of $\dfrac{1}{(x+1)(x^2+4)}$

$$\frac{1}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4} = \frac{\frac{1}{5}}{x+1} + \frac{-\frac{1}{5}x+\frac{1}{5}}{x^2+4}$$

$$1 = A(x^2+4) + (Bx+C)(x+1)$$

$$1 = Ax^2 + 4A + Bx^2 + Bx + Cx + C$$

$$1 = (A+B)x^2 + (B+C)x + (4A+C)$$

$$0 = A + B$$
$$0 = B + C$$
$$1 = 4A + C$$

$$A = \frac{1}{5}$$

$$B = -\frac{1}{5}$$

$$C = \frac{1}{5}$$

4 Decompose $\frac{P}{Q}$, Where Q Has a Repeated Irreducible Quadratic Factor

Case 4: Q contains repeated irreducible quadratic factors.

If the polynomial Q contains a repeated irreducible quadratic factor $(ax^2 + bx + c)^n$, $n \geq 2$, n an integer, then, in the partial fraction decomposition of $\frac{P}{Q}$, allow for the terms

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where the numbers $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ are to be determined.

EXAMPLE Repeated Irreducible Quadratic Factor

Write the partial fraction decomposition of $\dfrac{x^2+1}{\left(x^2+4\right)^2}$

$$\frac{x^2+1}{\left(x^2+4\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{\left(x^2+4\right)^2}=\frac{1}{x^2+4}+\frac{-3}{\left(x^2+4\right)^2}$$

$$x^2+1=\left(Ax+B\right)\left(x^2+4\right)+Cx+D$$

$$x^2+1=Ax^3+Bx^2+4Ax+4B+Cx+D$$

$$x^2+1=Ax^3+Bx^2+\left(4A+C\right)x+\left(4B+D\right)$$

$0=A$	$A=0$
$1=B$	$B=1$
$0=4A+C$	$C=0$
$1=4B+D$	$D=-3$