Section 11.5

Rotation of Axes;

General Form of a Conic

1 Identify a Conic

THEOREM

Identifying Conics without Completing the Squares

Excluding degenerate cases, the equation

$$\boxed{Ax^2 + Cy^2 + Dx + Ey + F = 0}$$

where A and C cannot both equal zero:

(a) Defines a parabola if $AC = 0$.

(b) Defines an ellipse (or a circle) if $AC > 0$.

(c) Defines a hyperbola if $AC < 0$.

EXAMPLE **Identifying a Conic without Completing the Squares**

Identify each equation without completing the squares.

$$2x^2 - y^2 - 8x - 4y + 2 = 0$$

$AC < 0$ Hyperbola

$$-2y^2 + 3y - 3x = 0$$

$AC = 0$ Parabola

$$4x^2 + 3y^2 - 8x - 6y + 1 = 0$$

$AC > 0$ and $A \neq C$ Ellipse

$$4x^2 + 4y^2 - 2x + 7y = 0$$

$AC > 0$ and $A = C$ Circle

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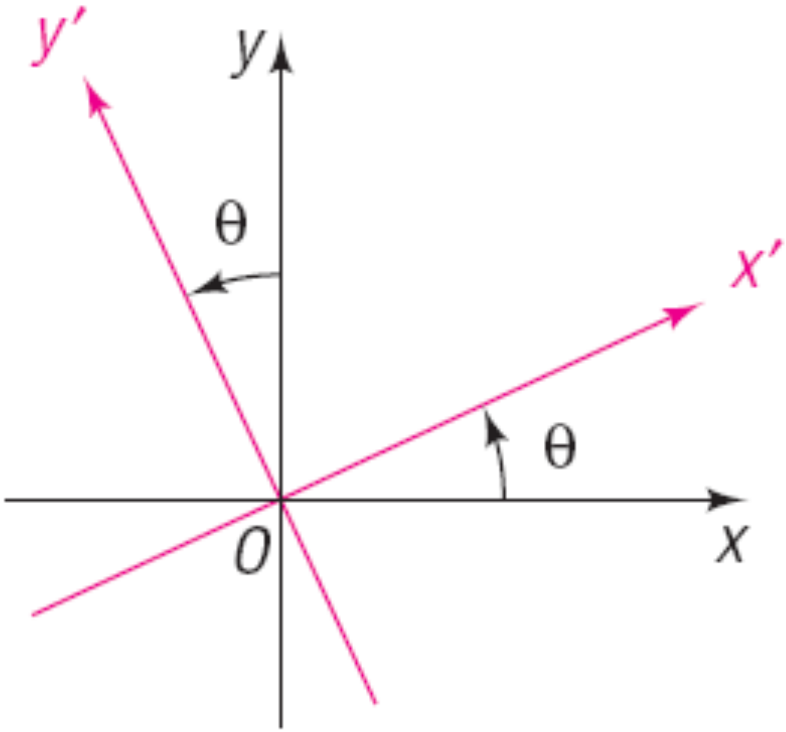
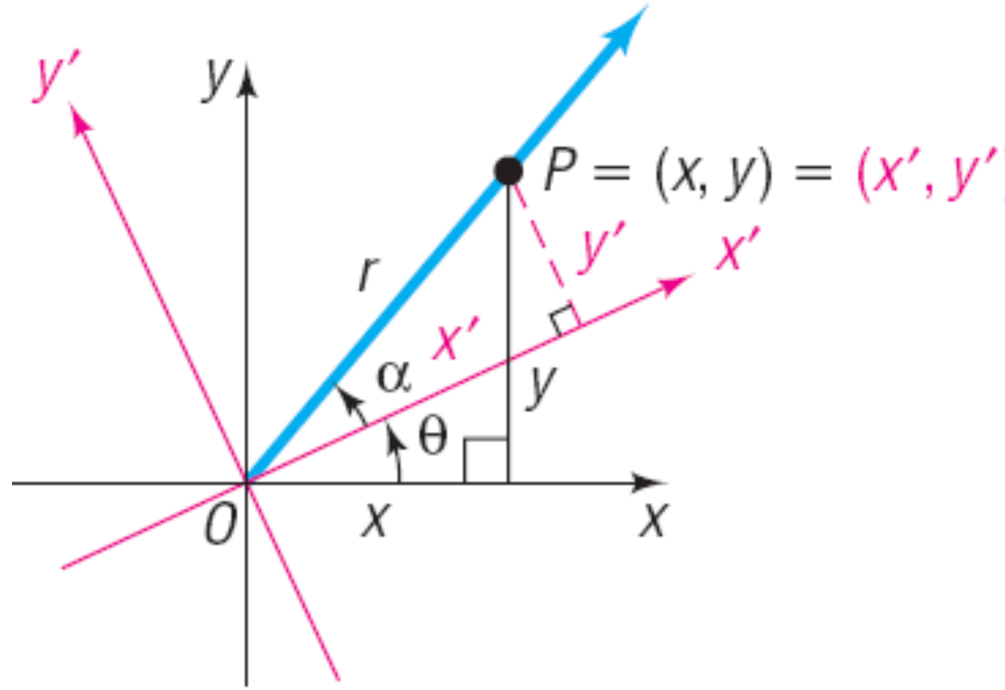
2 Use a Rotation of Axes to Transform Equations





Rotation Formulas

If the x- and y-axes are rotated through an angle θ, the coordinates (x, y) of a point P relative to the xy-plane and the coordinates (x', y') of the same point relative to the new x'- and y'-axes are related by the formulas

$$x = x' \cos\theta - y' \sin\theta \qquad y = x' \sin\theta + y' \cos\theta \tag{5}$$

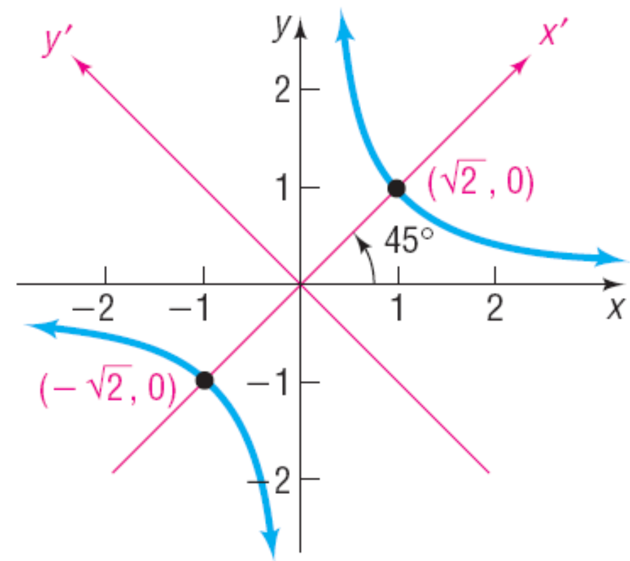
EXAMPLE Rotating Axes

Express the equation $xy = 1$ in terms of new $x'y'$-coordinates by rotating the axes through a 45° angle. Discuss the new equation.

$$x = x' \cos 45^\circ - y' \sin 45^\circ = x' \frac{\sqrt{2}}{2} - y' \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x' - y')$$

$$y = x' \sin 45^\circ + y' \cos 45^\circ = x' \frac{\sqrt{2}}{2} + y' \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x' + y')$$

Substituting these expressions for x and y in $xy = 1$ gives



$$\left[\frac{\sqrt{2}}{2}(x' - y')\right]\left[\frac{\sqrt{2}}{2}(x' + y')\right] = 1$$

$$\frac{1}{2}(x'^2 - y'^2) = 1 \qquad \frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

$$\boxed{x = x' \cos\theta - y' \sin\theta \qquad y = x' \sin\theta + y' \cos\theta}$$

THEOREM

To transform the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad B \neq 0$$

into an equation in x' and y' without an $x'y'$-term, rotate the axes through an angle θ that satisfies the equation

$$\boxed{\cot(2\theta) = \frac{A - C}{B}}$$

3 Analyze an Equation Using a Rotation of Axes

EXAMPLE Analyzing an Equation Using a Rotation of Axes

Analyze the equation: $x^2 + \sqrt{3}xy + 2y^2 - 10 = 0$

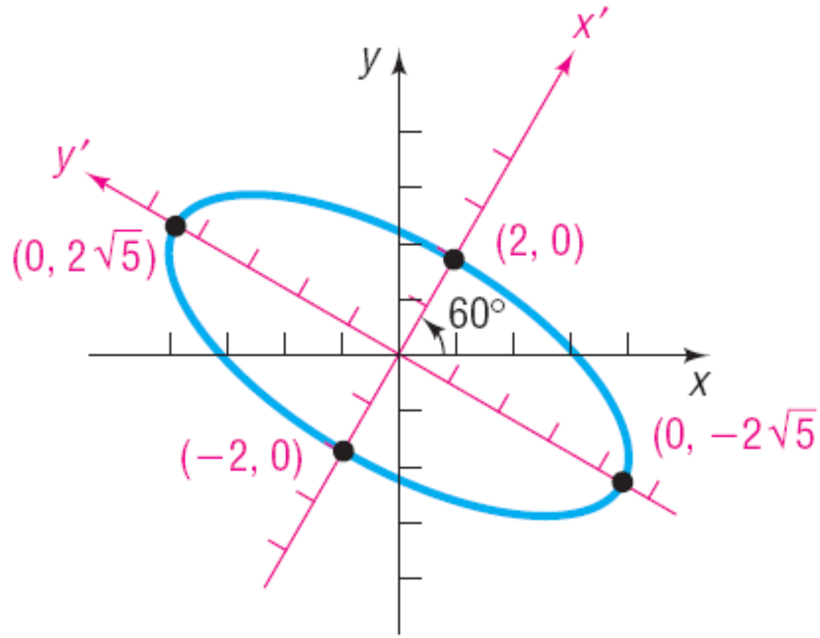
Since an xy-term is present, we must rotate the axes.

$A = 1, B = \sqrt{3}$, and $C = 2$

$$\cot(2\theta) = \frac{A - C}{B} = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \qquad 0° < 2\theta < 180°$$

$2\theta = 120°$, so $\theta = 60°$

$$x = x'\cos 60° - y'\sin 60°$$
$$= \frac{1}{2}x' - \frac{\sqrt{3}}{2}y' = \frac{1}{2}\left(x' - \sqrt{3}y'\right)$$
$$y = x'\sin 60° + y'\cos 60°$$
$$= \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' = \frac{1}{2}\left(\sqrt{3}x' + y'\right)$$



EXAMPLE **Analyzing an Equation Using a Rotation of Axes**

Analyze the equation: $4x^2 - 4xy + y^2 + 5\sqrt{5}x + 5 = 0$

$A = 4, B = -4$, and $C = 1$ $\qquad \cot(2\theta) = \dfrac{A - C}{B} = \dfrac{3}{-4} = -\dfrac{3}{4}$

Now we need to find the value of $\cos(2\theta)$. Since $\cot(2\theta) = -\dfrac{3}{4}$, then $90° < 2\theta < 180°$ (Do you know why?), so $\cos(2\theta) = -\dfrac{3}{5}$. Then

$$\sin\theta = \sqrt{\frac{1 - \cos(2\theta)}{2}} = \sqrt{\frac{1 - \left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos\theta = \sqrt{\frac{1 + \cos(2\theta)}{2}} = \sqrt{\frac{1 + \left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$x = \frac{\sqrt{5}}{5}x' - \frac{2\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}(x' - 2y')$$

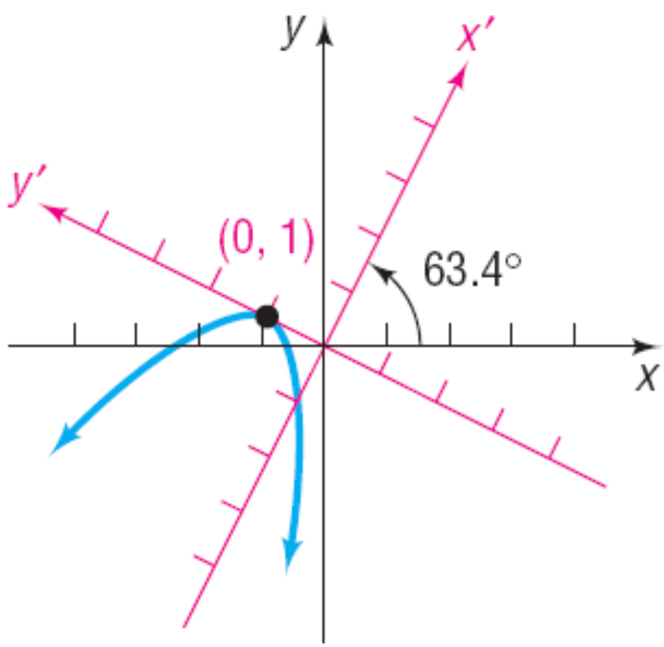
$$y = \frac{2\sqrt{5}}{5}x' + \frac{\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}(2x' + y')$$

EXAMPLE Analyzing an Equation Using a Rotation of Axes

Analyze the equation: $4x^2 - 4xy + y^2 + 5\sqrt{5}x + 5 = 0$

$$4\left[\frac{\sqrt{5}}{5}(x' - 2y')\right]^2 - 4\left[\frac{\sqrt{5}}{5}(x' - 2y')\right]\left[\frac{\sqrt{5}}{5}(2x' + y')\right] + \left[\frac{\sqrt{5}}{5}(2x' + y')\right]^2 + 5\sqrt{5}\left[\frac{\sqrt{5}}{5}(x' - 2y')\right] = -5$$

$$\begin{aligned}
4(x'^2 - 4x'y' + 4y'^2) - 4(2x'^2 - 3x'y' - 2y'^2) \\
+ 4x'^2 + 4x'y' + y'^2 + 25(x' - 2y') &= -25 \\
25y'^2 - 50y' + 25x' &= -25 \\
y'^2 - 2y' + x' &= -1 \\
y'^2 - 2y' + 1 &= -x' \\
(y' - 1)^2 &= -x'
\end{aligned}$$



$' - 2y')$

$x' + y')$

4 Identify Conics without a Rotation of Axes

THEOREM

Identifying Conics without a Rotation of Axes

Except for degenerate cases, the equation

$$\boxed{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0}$$

(a) Defines a parabola if $B^2 - 4AC = 0$.

(b) Defines an ellipse (or a circle) if $B^2 - 4AC < 0$.

(c) Defines a hyperbola if $B^2 - 4AC > 0$.

EXAMPLE

Identifying a Conic without a Rotation of Axes

Identify the equation: $8x^2 - 12xy + 17y^2 - 4\sqrt{5}x - 2\sqrt{5}y - 15 = 0$

$A = 8$, $B = -12$, and $C = 17$, so $B^2 - 4AC = -400$.

Since $B^2 - 4AC < 0$, the equation defines an ellipse.

Identifying Conics without a Rotation of Axes

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