Section R.1

Real Numbers

1 Work with Sets

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$D = \{ \quad x \quad | \quad x \text{ is a digit}\}$$

Read as "D is the set of all x such that x is a digit."

EXAMPLE

Using Set-builder Notation and the Roster Method

(a) $E = \{x | x \text{ is an even digit}\} = \{0, 2, 4, 6, 8\}$

(b) $O = \{x | x \text{ is an odd digit}\} = \{1, 3, 5, 7, 9\}$

If A and B are sets, the **intersection** of A with B, denoted $A \cap B$, is the set consisting of elements that belong to both A and B. The **union** of A with B, denoted $A \cup B$, is the set consisting of elements that belong to either A or B, or both.

EXAMPLE Finding the Intersection and Union of Sets

Let $A = \{1, 3, 5, 8\}$, $B = \{3, 5, 7\}$, and $C = \{2, 4, 6, 8\}$. Find:

(a) $A \cap B$ (b) $A \cup B$ (c) $B \cap (A \cup C)$

(a) $A \cap B = \{1, 3, 5, 8\} \cap \{3, 5, 7\} = \{3, 5\}$

(b) $A \cup B = \{1, 3, 5, 8\} \cup \{3, 5, 7\} = \{1, 3, 5, 7, 8\}$

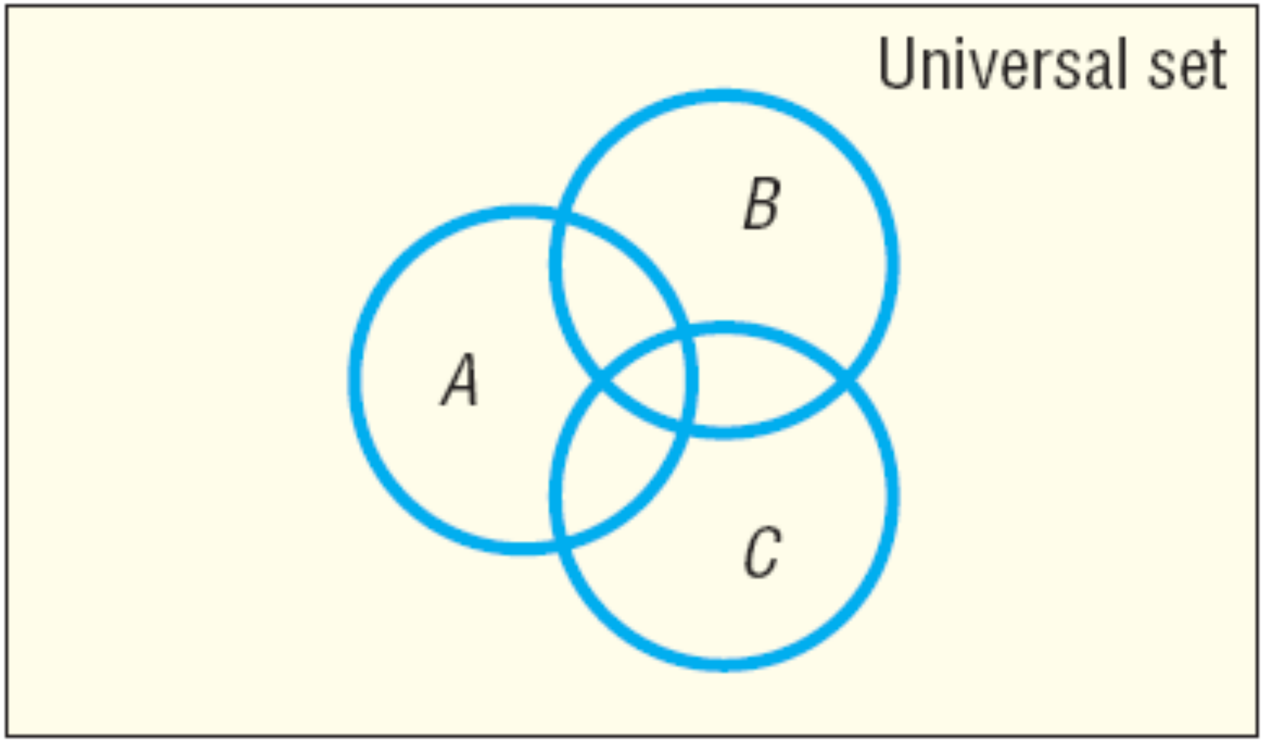
(c) $$\begin{aligned} B \cap (A \cup C) &= \{3, 5, 7\} \cap [\{1, 3, 5, 8\} \cup \{2, 4, 6, 8\}] \\ &= \{3, 5, 7\} \cap \{1, 2, 3, 4, 5, 6, 8\} = \{3, 5\} \end{aligned}$$

If A is a set, the **complement** of A, denoted $\overline{A}$, is the set consisting of all the elements in the universal set that are not in A.*

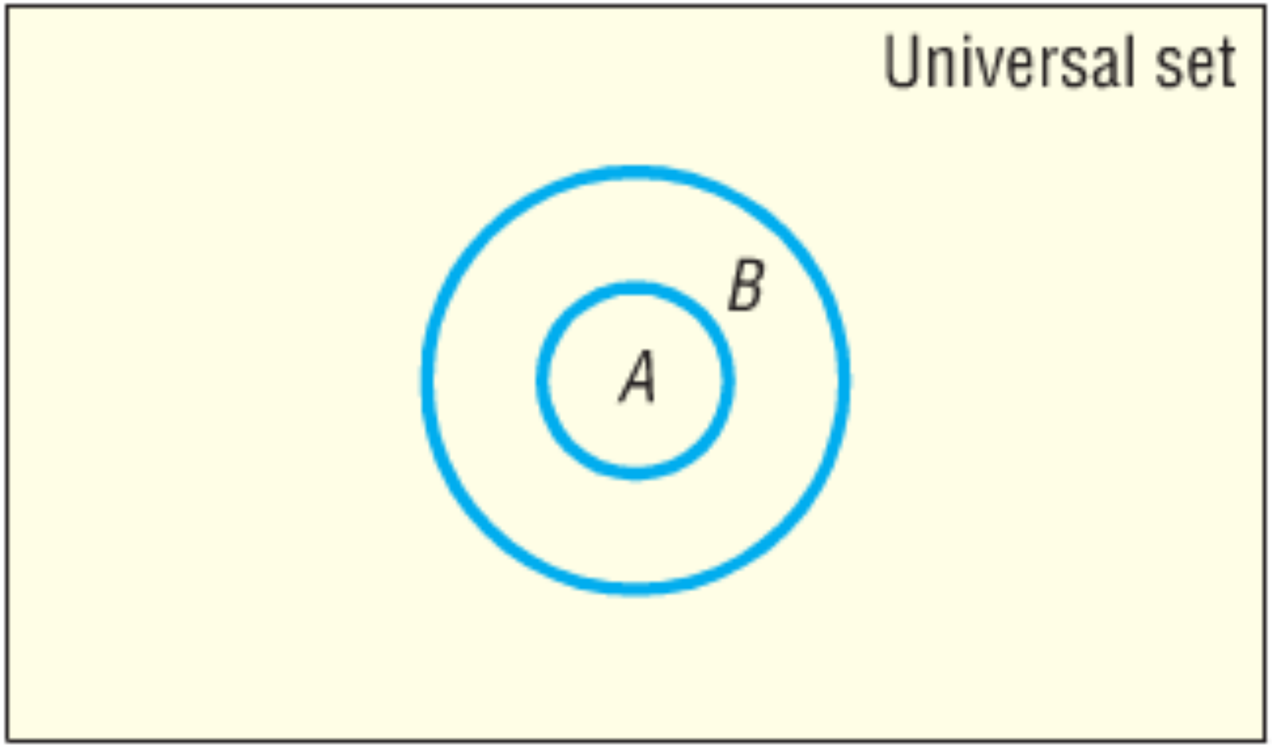
*Some books use the notation A' for the complement of A.

EXAMPLE Finding the Complement of a Set

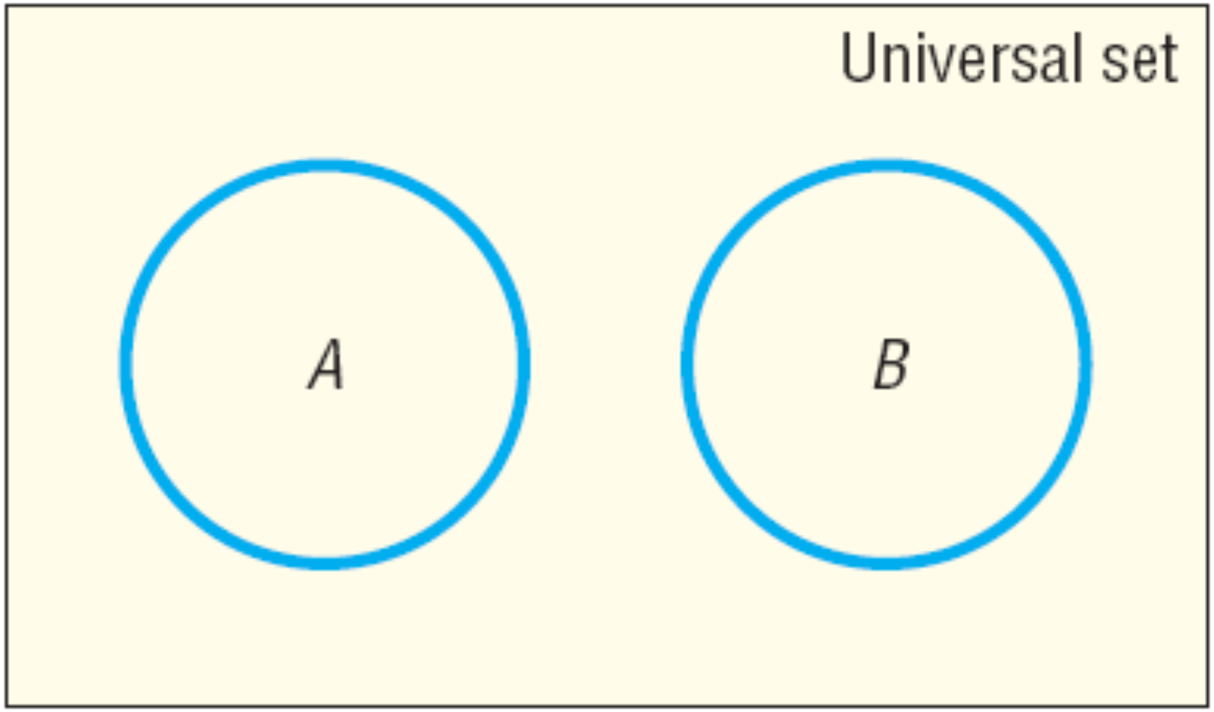
If the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and if $A = \{1, 3, 5, 7, 9\}$, then $\overline{A} = \{2, 4, 6, 8\}$.



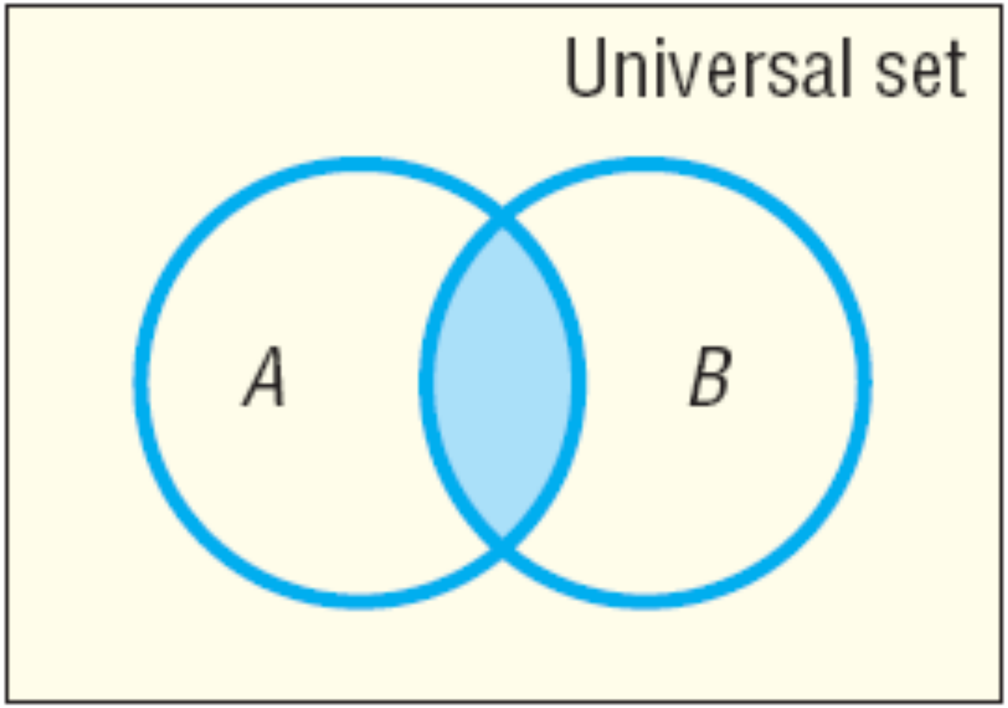
Venn diagram



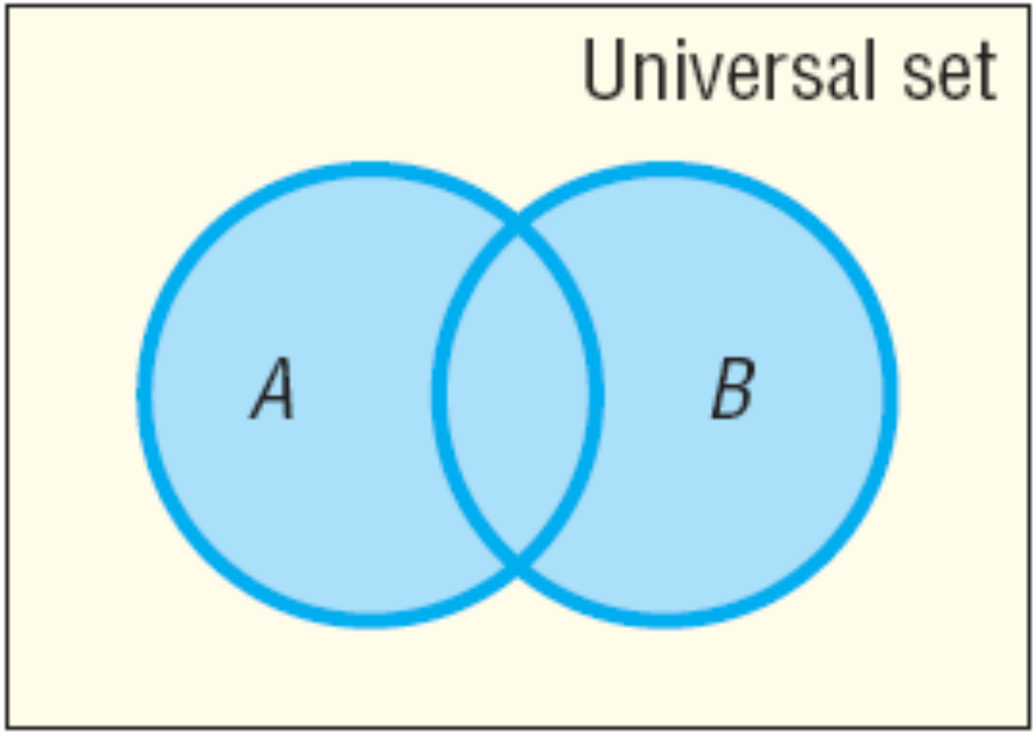
(a) $A \subseteq B$
subset



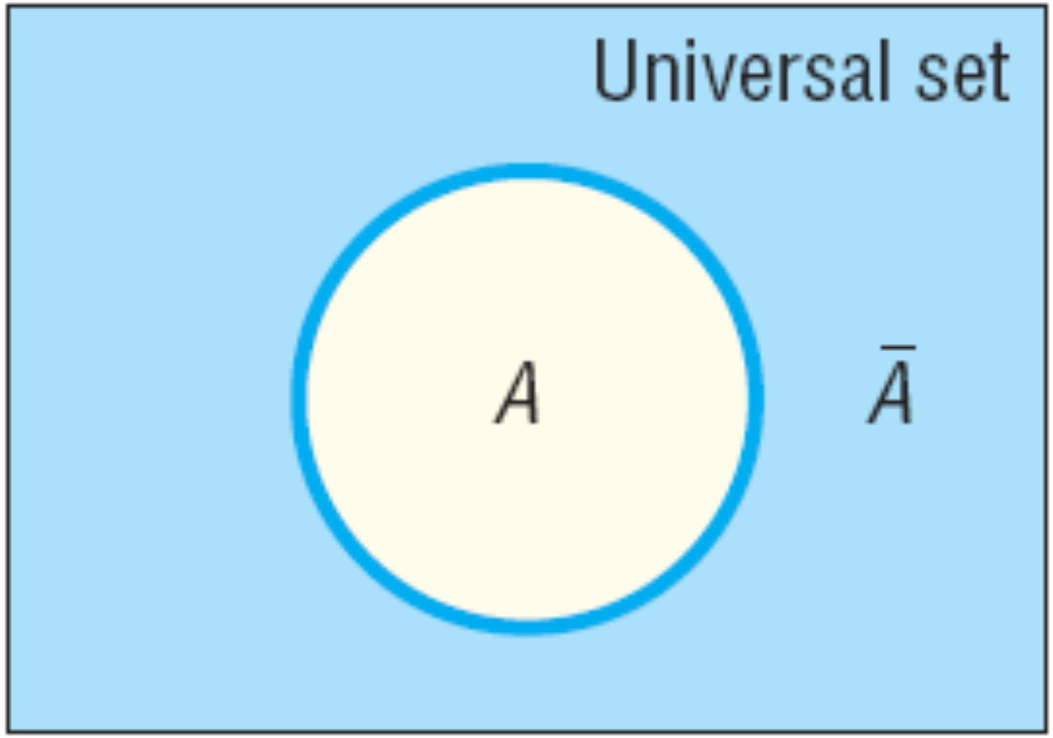
(b) $A \cap B = \varnothing$
disjoint sets



(a) $A \cap B$
intersection



(b) $A \cup B$
union



(c) $\bar{A}$
complement

2 Classify Numbers

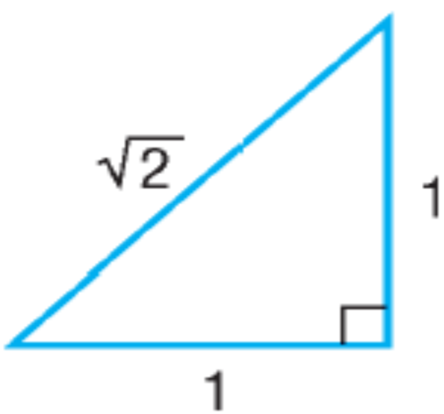
The **integers** are the set of numbers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

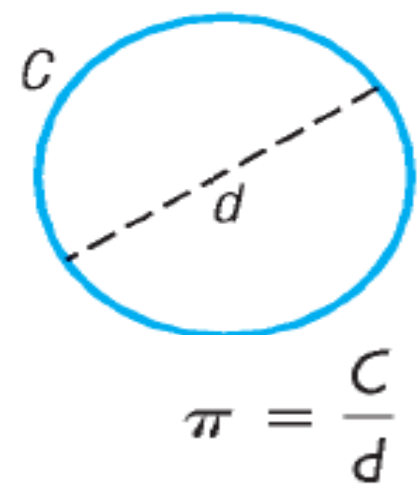
A **rational number** is a number that can be expressed as a quotient $\frac{a}{b}$ of two integers. The integer a is called the **numerator,** and the integer b, which cannot be 0, is called the **denominator.** The rational numbers are the numbers in the set $\left\{x \,\middle|\, x = \frac{a}{b}, \text{ where } a, b \text{ are integers and } b \neq 0\right\}$.

Rational numbers may be represented as **decimals.** For example, the rational numbers $\frac{3}{4}, \frac{5}{2}, -\frac{2}{3}$, and $\frac{7}{66}$ may be represented as decimals by merely carrying out the indicated division:

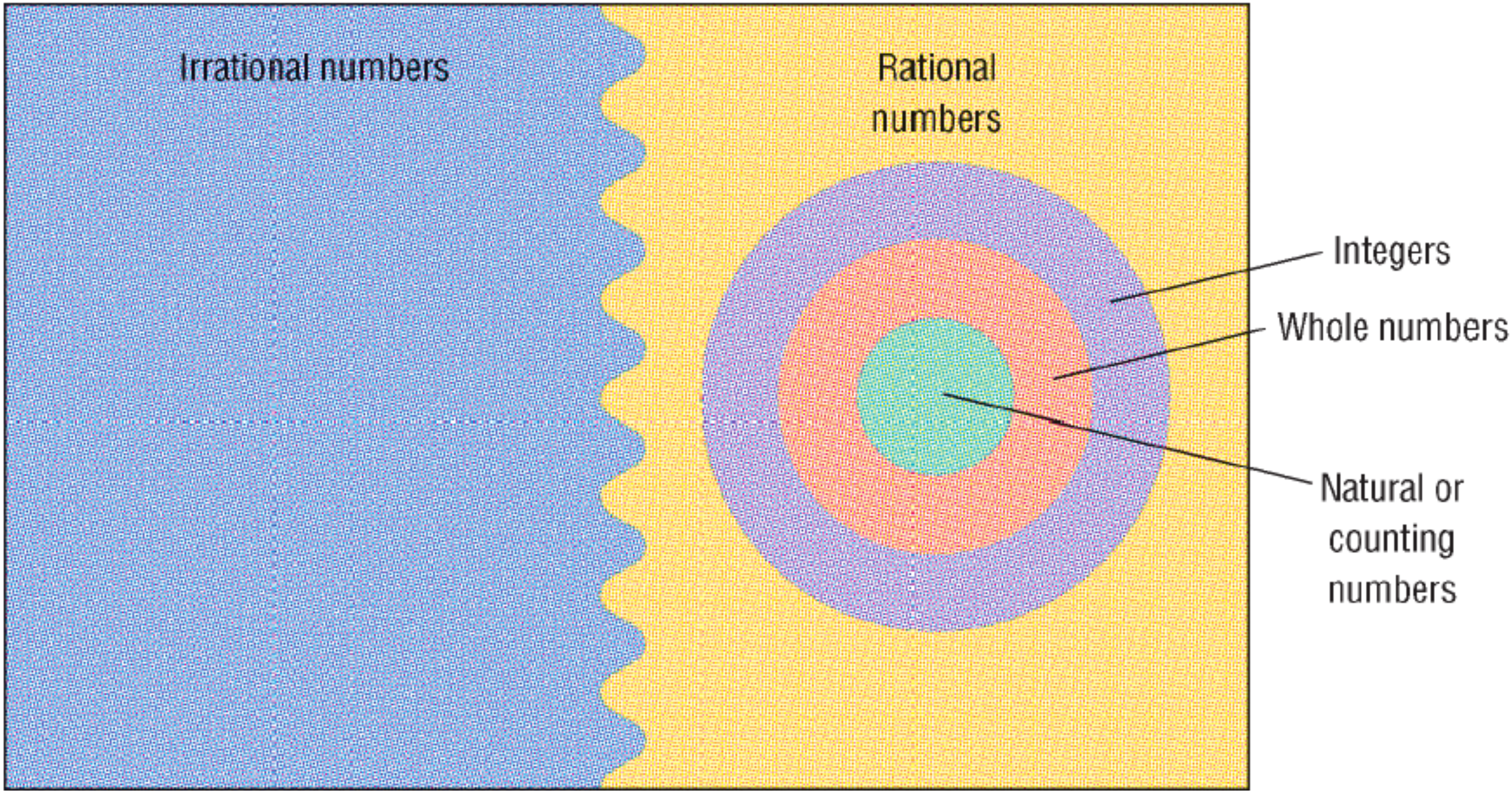
$$\frac{3}{4} = 0.75 \qquad \frac{5}{2} = 2.5 \qquad -\frac{2}{3} = -0.666\ldots = -0.\overline{6} \qquad \frac{7}{66} = 0.1060606\ldots = 0.1\overline{06}$$

Some decimals do not terminate or end in a pattern. Such decimals represent **irrational numbers**. Every irrational number may be represented by a decimal that neither repeats nor terminates. In other words, irrational numbers cannot be written in the form $\frac{a}{b}$, where a, b are integers and $b \neq 0$.





Real numbers



Together, the rational numbers and irrational numbers form the set of **real numbers**.

EXAMPLE

Classifying the Numbers in a Set

List the numbers in the set

$$\left\{-3, \frac{4}{3}, 0.12, \sqrt{2}, \pi, 2.151515\ldots = 2.\overline{15}, 10\right\}$$

that are

(a) Natural numbers (b) Integers (c) Rational numbers
(d) Irrational numbers (e) Real numbers

(a) 10 is the only natural number.

(b) -3 and 10 are integers.

(c) $-3, 10, \frac{4}{3}, 0.12$, and $2.151515\ldots$ are rational numbers.

(d) $\sqrt{2}$ and π are irrational numbers.

(e) All the numbers listed are real numbers.

Approximations

Truncation: Drop all the digits that follow the specified final digit in the decimal.
Rounding: Identify the specified final digit in the decimal. If the next digit is 5 or more, add 1 to the final digit; if the next digit is 4 or less, leave the final digit as it is. Then truncate following the final digit.

EXAMPLE **Approximating a Decimal to Two Places**

Approximate 20.98752 to two decimal places by

(a) Truncating (b) Rounding

(a) To truncate, we remove all digits following the final digit 8. The truncation of 20.98752 to two decimal places is 20.98.

(b) The digit following the final digit 8 is the digit 7. Since 7 is 5 or more, we add 1 to the final digit 8 and truncate. The rounded form of 20.98752 to two decimal places is 20.99.

EXAMPLE

Approximating a Decimal to Two and Four Places

Number	Rounded to Two Decimal Places	Rounded to Four Decimal Places	Truncated to Two Decimal Places	Truncated to Four Decimal Places
(a) 3.14159	3.14	3.1416	3.14	3.1415
(b) 0.056128	0.06	0.0561	0.05	0.0561
(c) 893.46125	893.46	893.4613	893.46	893.4612

Operations

Operation	Symbol	Words
Addition	$a + b$	Sum: a plus b
Subtraction	$a - b$	Difference: a minus b
Multiplication	$a \cdot b, (a) \cdot b, a \cdot (b), (a) \cdot (b),$	Product: a times b
	$ab, (a)b, a(b), (a)(b)$	
Division	a/b or $\frac{a}{b}$	Quotient: a divided by b

EXAMPLE

Writing Statements Using Symbols

(a) The sum of 2 and 7 equals 9. In symbols, this statement is written as $2 + 7 = 9$.

(b) The product of 3 and 5 is 15. In symbols, this statement is written as $3 \cdot 5 = 15$.

3 Evaluate Numerical Expressions

We agree that whenever the two operations of addition and multiplication separate three numbers the multiplication operation always will be performed first, followed by the addition operation.

EXAMPLE

Finding the Value of an Expression

Evaluate each expression.

(a) $3 + 4 \cdot 5$ (b) $8 \cdot 2 + 1$ (c) $2 + 2 \cdot 2$

(a) $3 + 4 \cdot 5 = 3 + 20 = 23$

Multiply first

(b) $8 \cdot 2 + 1 = 16 + 1 = 17$

Multiply first

(c) $2 + 2 \cdot 2 = 2 + 4 = 6$

EXAMPLE

Finding the Value of an Expression

(a) $(5 + 3) \cdot 4 = 8 \cdot 4 = 32$

(b) $(4 + 5) \cdot (8 - 2) = 9 \cdot 6 = 54$

Rules for the Order of Operations

1. Begin with the innermost parentheses and work outward. Remember that in dividing two expressions the numerator and denominator are treated as if they were enclosed in parentheses.
2. Perform multiplications and divisions, working from left to right.
3. Perform additions and subtractions, working from left to right.

EXAMPLE Finding the Value of an Expression

Evaluate each expression.

(a) $8 \cdot 2 + 3$

(b) $5 \cdot (3 + 4) + 2$

(c) $\dfrac{2 + 5}{2 + 4 \cdot 7}$

(d) $2 + [4 + 2 \cdot (10 + 6)]$

(a) $8 \cdot 2 + 3 = 16 + 3 = 19$

Multiply first

(c) $\dfrac{2 + 5}{2 + 4 \cdot 7} = \dfrac{2 + 5}{2 + 28} = \dfrac{7}{30}$

(b) $5 \cdot (3 + 4) + 2 = 5 \cdot 7 + 2 = 35 + 2 = 37$

Parentheses first   Multiply before adding

(d) $2 + [4 + 2 \cdot (10 + 6)] = 2 + [4 + 2 \cdot (16)]$

$= 2 + [4 + 32] = 2 + [36] = 38$

EXAMPLE Finding the Value of an Expression

Evaluate each expression.

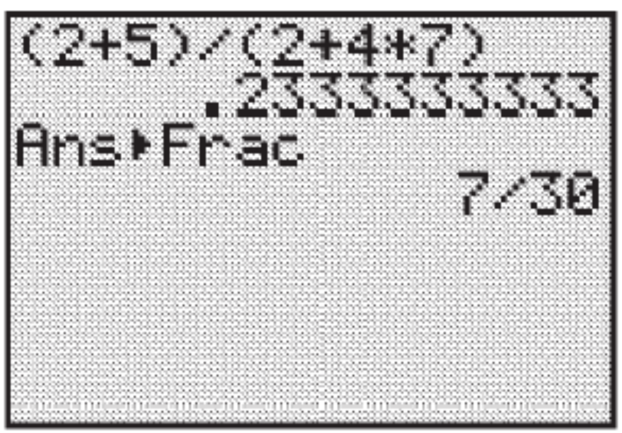
(a) $8 \cdot 2 + 3$

(b) $5 \cdot (3 + 4) + 2$

(c) $\dfrac{2 + 5}{2 + 4 \cdot 7}$

(d) $2 + [4 + 2 \cdot (10 + 6)]$

Be careful if you use a calculator. For example when you compute (c) above, you need to use parentheses (see calculator screenshot below). If you don't, your answer will be incorrect because the calculator will compute the expression



$$2 + \frac{5}{2} + 4 \cdot 7 = 2 + 2.5 + 28 = 32.5$$

4 Work with Properties of Real Numbers

1. The **reflexive property** states that a number always equals itself; that is, $a = a$.
2. The **symmetric property** states that if $a = b$ then $b = a$.
3. The **transitive property** states that if $a = b$ and $b = c$ then $a = c$.
4. The **principle of substitution** states that if $a = b$ then we may substitute b for a in any expression containing a.

EXAMPLE Commutative Properties

(a) $3 + 5 = 8$
$5 + 3 = 8$
$3 + 5 = 5 + 3$

(b) $2 \cdot 3 = 6$
$3 \cdot 2 = 6$
$2 \cdot 3 = 3 \cdot 2$

Commutative Properties

$$a + b = b + a$$
$$a \cdot b = b \cdot a$$

EXAMPLE Associative Properties

(a) $2 + (3 + 4) = 2 + 7 = 9$

$(2 + 3) + 4 = 5 + 4 = 9$

$2 + (3 + 4) = (2 + 3) + 4$

(b) $2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$

$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$

$2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$

Associative Properties

$$a + (b + c) = (a + b) + c = a + b + c$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c = a \cdot b \cdot c$$

Distributive Property

$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$(a + b) \cdot c = a \cdot c + b \cdot c$$

EXAMPLE **Distributive Property**

(a) $2 \cdot (x + 3) = 2 \cdot x + 2 \cdot 3 = 2x + 6$ Use to remove parentheses.

(b) $3x + 5x = (3 + 5)x = 8x$ Use to combine two expressions.

(c) $(x + 2)(x + 3) = x(x + 3) + 2(x + 3) = (x^2 + 3x) + (2x + 6)$
$= x^2 + (3x + 2x) + 6 = x^2 + 5x + 6$

EXAMPLE Identity Properties

(a) $4 + 0 = 0 + 4 = 4$

(b) $3 \cdot 1 = 1 \cdot 3 = 3$

Identity Properties

$$\boxed{\begin{gathered} 0 + a = a + 0 = a \\ a \cdot 1 = 1 \cdot a = a \end{gathered}}$$

Additive Inverse Property

$$\boxed{a + (-a) = -a + a = 0}$$

EXAMPLE **Finding an Additive Inverse**

(a) The additive inverse of 6 is -6, because $6 + (-6) = 0$.

(b) The additive inverse of -8 is $-(-8) = 8$, because $-8 + 8 = 0$.

Multiplicative Inverse Property

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 \qquad \text{if } a \neq 0$$

EXAMPLE **Finding a Reciprocal**

(a) The reciprocal of 6 is $\frac{1}{6}$, because $6 \cdot \frac{1}{6} = 1$.

(b) The reciprocal of -3 is $\frac{1}{-3}$, because $-3 \cdot \frac{1}{-3} = 1$.

(c) The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, because $\frac{2}{3} \cdot \frac{3}{2} = 1$.

The **difference** $a - b$, also read "a less b" or "a minus b," is defined as

$$a - b = a + (-b) \tag{6}$$

If b is a nonzero real number, the **quotient** $\frac{a}{b}$, also read as "a divided by b" or "the ratio of a to b," is defined as

$$\frac{a}{b} = a \cdot \frac{1}{b} \qquad \text{if } b \neq 0 \tag{7}$$

EXAMPLE

Working with Differences and Quotients

(a) $8 - 5 = 8 + (-5) = 3$

(b) $4 - 9 = 4 + (-9) = -5$

(c) $\dfrac{5}{8} = 5 \cdot \dfrac{1}{8}$

Multiplication by Zero

$$\boxed{a \cdot 0 = 0}$$

Division Properties

$$\boxed{\frac{0}{a} = 0 \qquad \frac{a}{a} = 1 \qquad \text{if } a \neq 0}$$

Rules of Signs

$$a(-b) = -(ab) \qquad (-a)b = -(ab) \qquad (-a)(-b) = ab$$

$$-(-a) = a \qquad \frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b} \qquad \frac{-a}{-b} = \frac{a}{b}$$

EXAMPLE **Applying the Rules of Signs**

(a) $2(-3) = -(2 \cdot 3) = -6$ (b) $(-3)(-5) = 3 \cdot 5 = 15$

(c) $\dfrac{3}{-2} = \dfrac{-3}{2} = -\dfrac{3}{2}$ (d) $\dfrac{-4}{-9} = \dfrac{4}{9}$ (e) $\dfrac{x}{-2} = \dfrac{1}{-2} \cdot x = -\dfrac{1}{2}x$

Cancellation Properties

$$ac = bc \quad \text{implies} \quad a = b \quad \text{if } c \neq 0$$

$$\frac{ac}{bc} = \frac{a}{b} \quad \text{if } b \neq 0, c \neq 0$$

EXAMPLE **Using the Cancellation Properties**

(a) If $2x = 6$, then

$$\begin{aligned} 2x &= 6 \\ 2x &= 2 \cdot 3 && \text{Factor 6.} \\ x &= 3 && \text{Cancel the 2's.} \end{aligned}$$

(b) $\dfrac{18}{12} = \dfrac{3 \cdot \cancel{6}}{2 \cdot \cancel{6}} = \dfrac{3}{2}$

Cancel the 6's.

Zero-Product Property

If $ab = 0$, then $a = 0$, or $b = 0$, or both.

EXAMPLE **Using the Zero-Product Property**

If $2x = 0$, then either $2 = 0$ or $x = 0$.

Since $2 \neq 0$, it follows that $x = 0$.

Arithmetic of Quotients

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd} \qquad \text{if } b \neq 0, d \neq 0$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{if } b \neq 0, d \neq 0$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad \text{if } b \neq 0, c \neq 0, d \neq 0$$

EXAMPLE

Adding, Subtracting, Multiplying, and Dividing Quotients

(a) $$\frac{2}{3}+\frac{5}{2}=\frac{2\cdot 2}{3\cdot 2}+\frac{3\cdot 5}{3\cdot 2}=\frac{2\cdot 2+3\cdot 5}{3\cdot 2}=\frac{4+15}{6}=\frac{19}{6}$$

(b) $$\frac{3}{5}-\frac{2}{3}=\frac{3}{5}+\left(-\frac{2}{3}\right)=\frac{3}{5}+\frac{-2}{3}$$

$$=\frac{3\cdot 3+5\cdot(-2)}{5\cdot 3}=\frac{9+(-10)}{15}=\frac{-1}{15}=-\frac{1}{15}$$

EXAMPLE

Adding, Subtracting, Multiplying, and Dividing Quotients

(c) $\dfrac{8}{3} \cdot \dfrac{15}{4} = \dfrac{8 \cdot 15}{3 \cdot 4} = \dfrac{2 \cdot \cancel{4} \cdot \cancel{3} \cdot 5}{\cancel{3} \cdot \cancel{4} \cdot 1} = \dfrac{2 \cdot 5}{1} = 10$

(d) $\dfrac{\frac{3}{5}}{\frac{7}{9}} = \dfrac{3}{5} \cdot \dfrac{9}{7} = \dfrac{3 \cdot 9}{5 \cdot 7} = \dfrac{27}{35}$

EXAMPLE

Finding the Least Common Multiple of Two Numbers

Find the least common multiple of 15 and 12.

To find the LCM of 15 and 12, we look at multiples of 15 and 12.

$$15, \quad 30, \quad 45, \quad 60, \quad 75, \quad 90, \quad 105, \quad 120, \ldots$$

$$12, \quad 24, \quad 36, \quad 48, \quad 60, \quad 72, \quad 84, \quad 96, \quad 108, \quad 120, \ldots$$

The *common* multiples are in blue. The *least* common multiple is 60.

EXAMPLE

Using the Least Common Multiple to Add Two Fractions

Find: $\dfrac{8}{15} + \dfrac{5}{12}$

We use the LCM of the denominators of the fractions and rewrite each fraction using the LCM as a common denominator. The LCM of the denominators (12 and 15) is 60. Rewrite each fraction using 60 as the denominator.

$$\frac{8}{15} + \frac{5}{12} = \frac{8}{15} \cdot \frac{4}{4} + \frac{5}{12} \cdot \frac{5}{5} = \frac{32}{60} + \frac{25}{60} = \frac{32 + 25}{60} = \frac{57}{60} = \frac{19}{20}$$