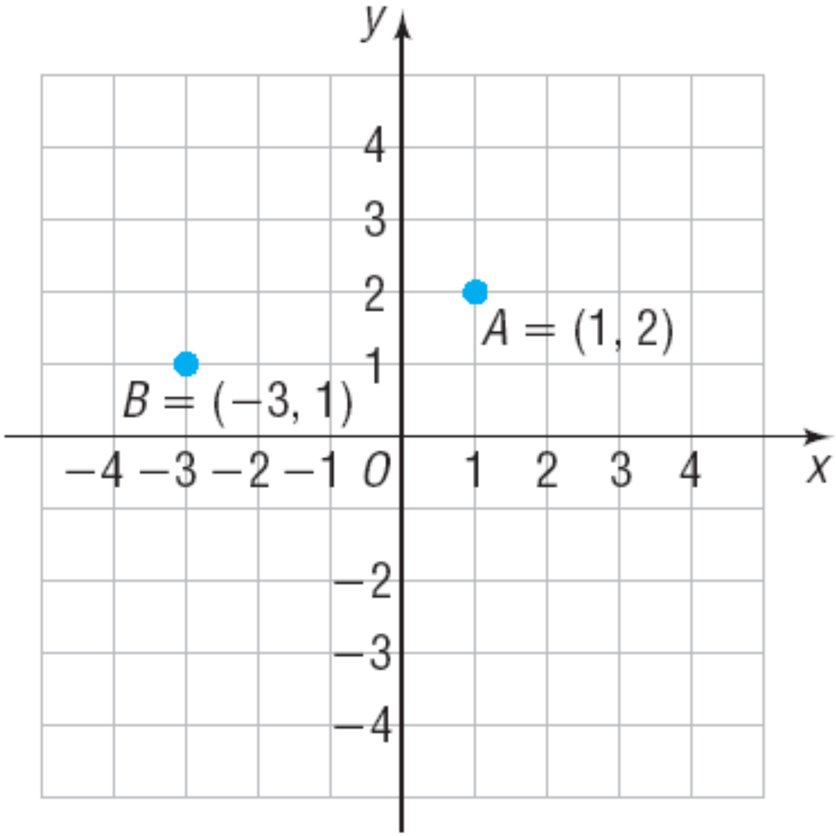
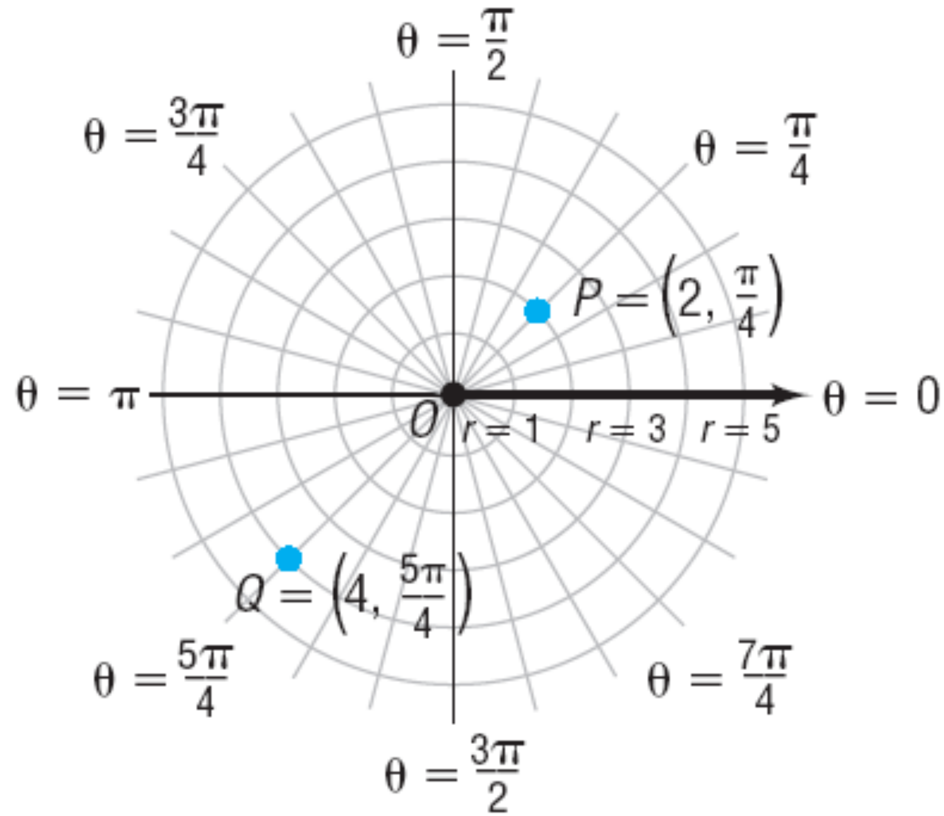
Section 10.2

Polar Equations and Graphs



Rectangular grid



Polar grid

An equation whose variables are polar coordinates is called a **polar equation**. The **graph of a polar equation** consists of all points whose polar coordinates satisfy the equation.

1 Identify and Graph Polar Equations by Converting to Rectangular Equations

EXAMPLE **Identifying and Graphing a Polar Equation (Circle)**

Identify and graph the equation: $r = 3$

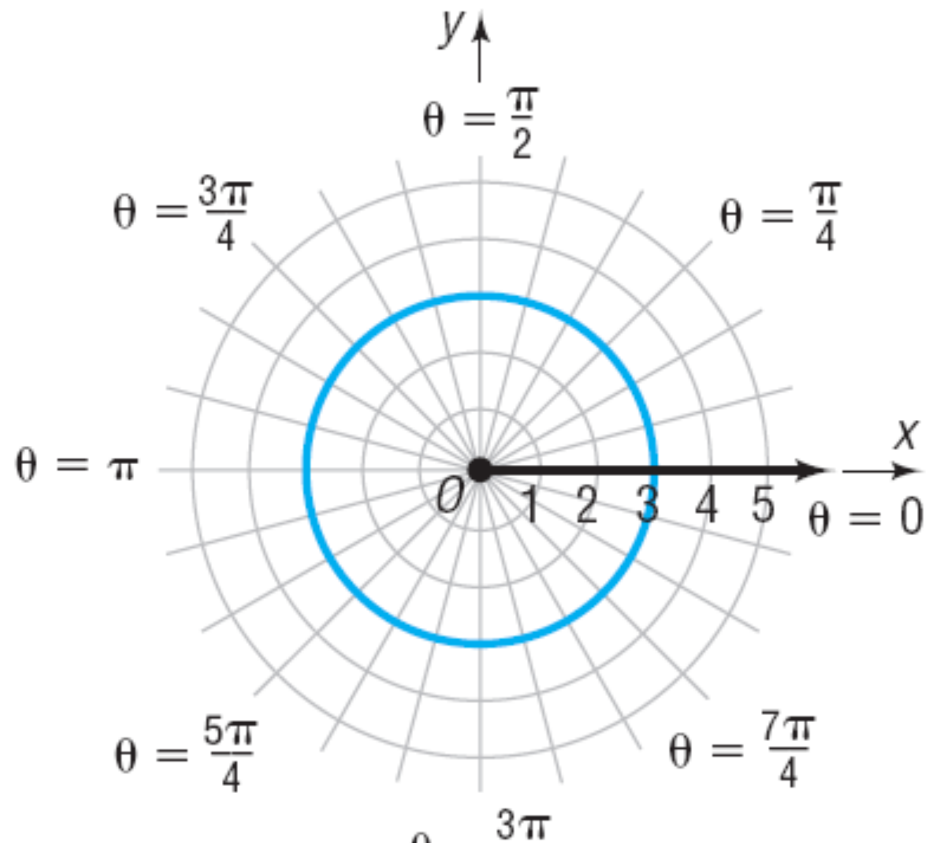
Convert the polar equation to a rectangular equation.

$r = 3$

$r^2 = 9$ Square both sides.

$x^2 + y^2 = 9$ $r^2 = x^2 + y^2$

$r = 3$ or $x^2 + y^2 = 9$



The graph of $r = 3$ is a circle, with center at the pole and radius 3.

EXAMPLE

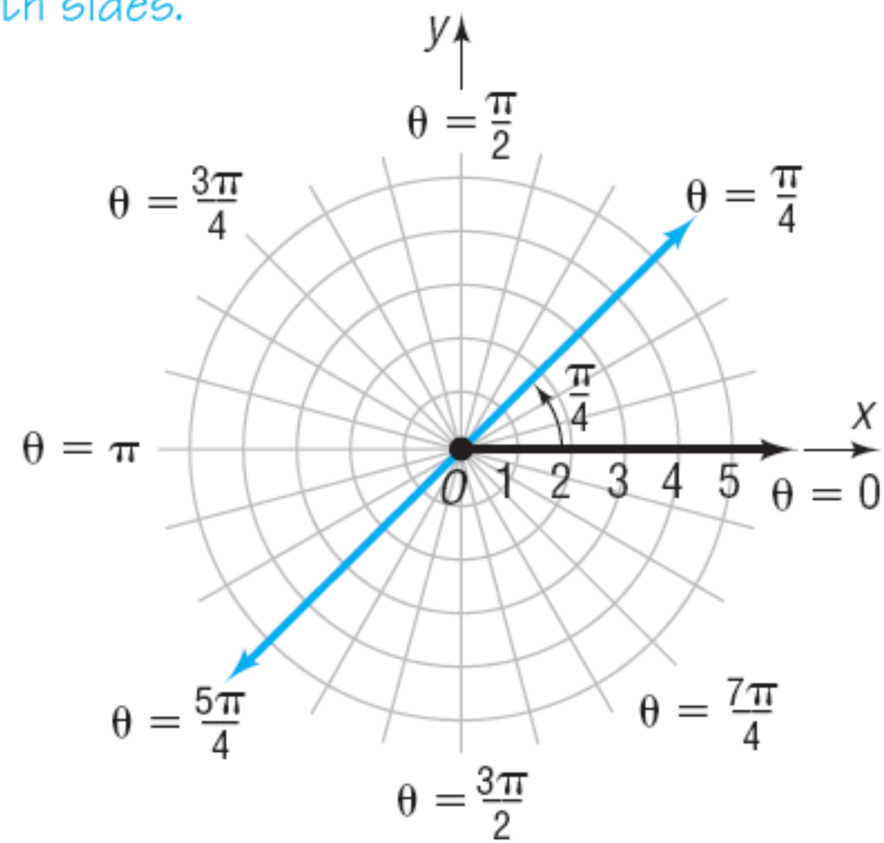
Identifying and Graphing a Polar Equation (Line)

Identify and graph the equation: $\theta = \frac{\pi}{4}$

$\tan\theta = \tan\frac{\pi}{4}$ Take the tangent of both sides.

$\frac{y}{x} = 1$ $\tan\theta = \frac{y}{x}; \tan\frac{\pi}{4} = 1$

$y = x$



$$\theta = \frac{\pi}{4} \text{ or } y = x$$

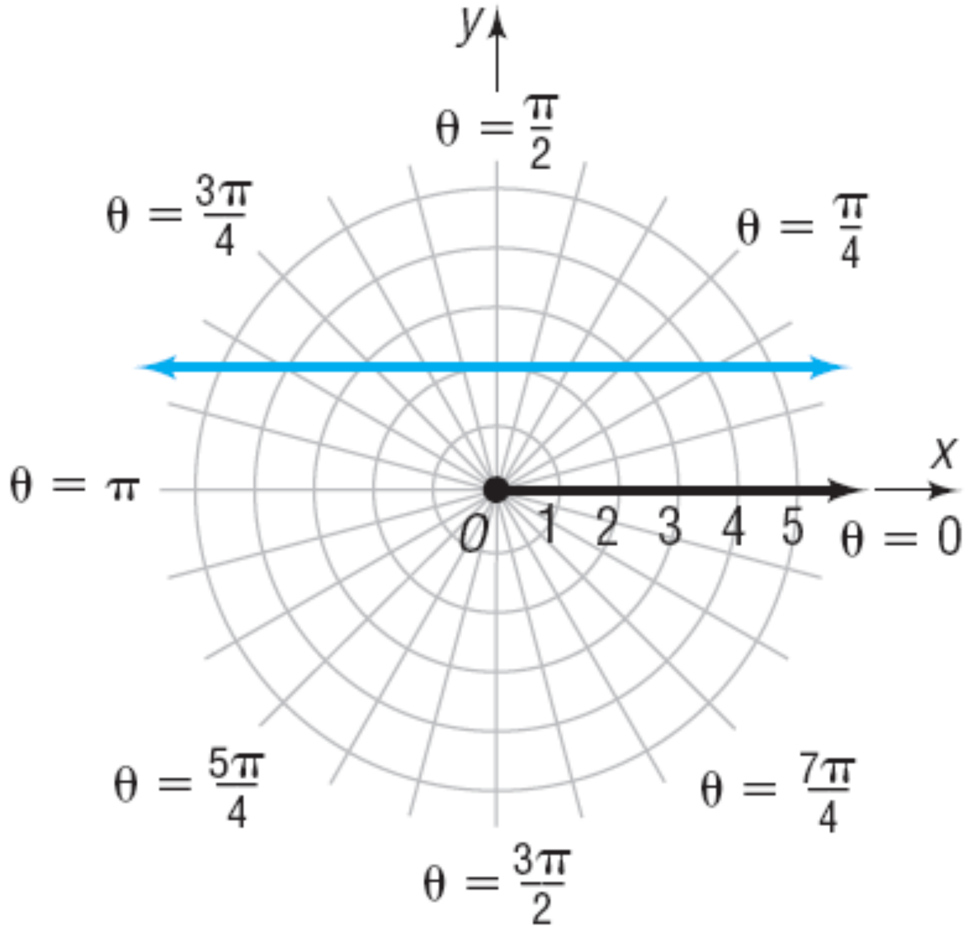
EXAMPLE

Identifying and Graphing a Polar Equation (Horizontal Line)

Identify and graph the equation: $r \sin \theta = 2$

Since $y = r \sin \theta$, we can write the equation as

$$y = 2$$



$r \sin \theta = 2$ or $y = 2$

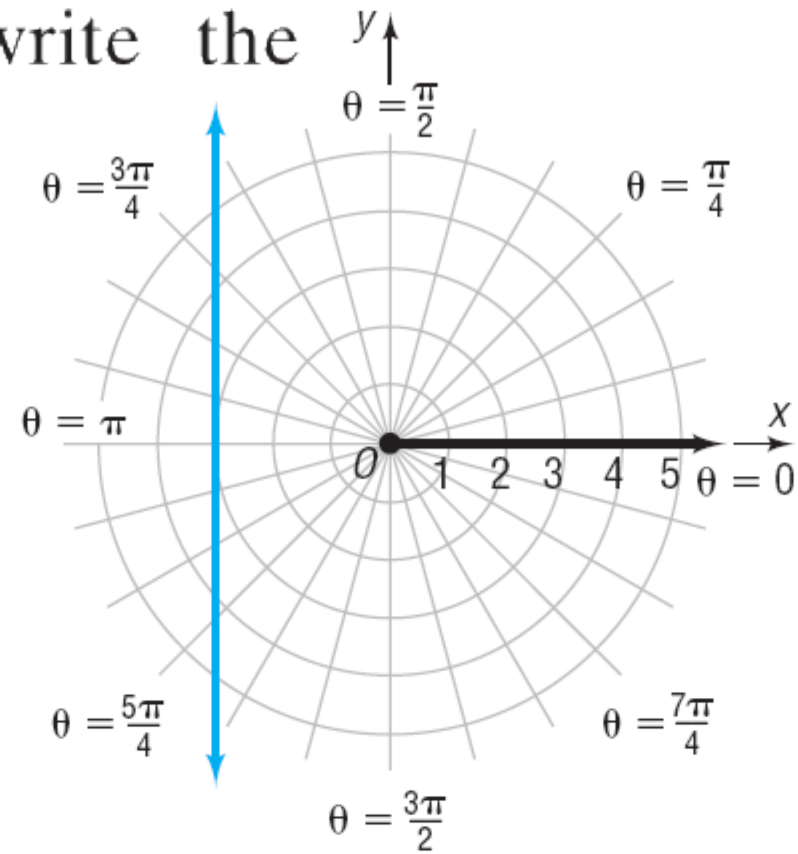
EXAMPLE

Identifying and Graphing a Polar Equation (Vertical Line)

Identify and graph the equation: $r \cos\theta = -3$

Since $x = r\cos\theta$, we can write the equation as

$$x = -3$$



$r\cos\theta = -3$ or $x = -3$

THEOREM

Let a be a nonzero real number. Then the graph of the equation

$$\boxed{r \sin \theta = a}$$

is a horizontal line a units above the pole if $a > 0$ and $|a|$ units below the pole if $a < 0$.

The graph of the equation

$$\boxed{r \cos \theta = a}$$

is a vertical line a units to the right of the pole if $a > 0$ and $|a|$ units to the left of the pole if $a < 0$.

EXAMPLE

Identifying and Graphing a Polar Equation (Circle)

Identify and graph the equation: $r = 4\sin\theta$

$r^2 = 4r\sin\theta$ multiply by sides by *r*

Now use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$.

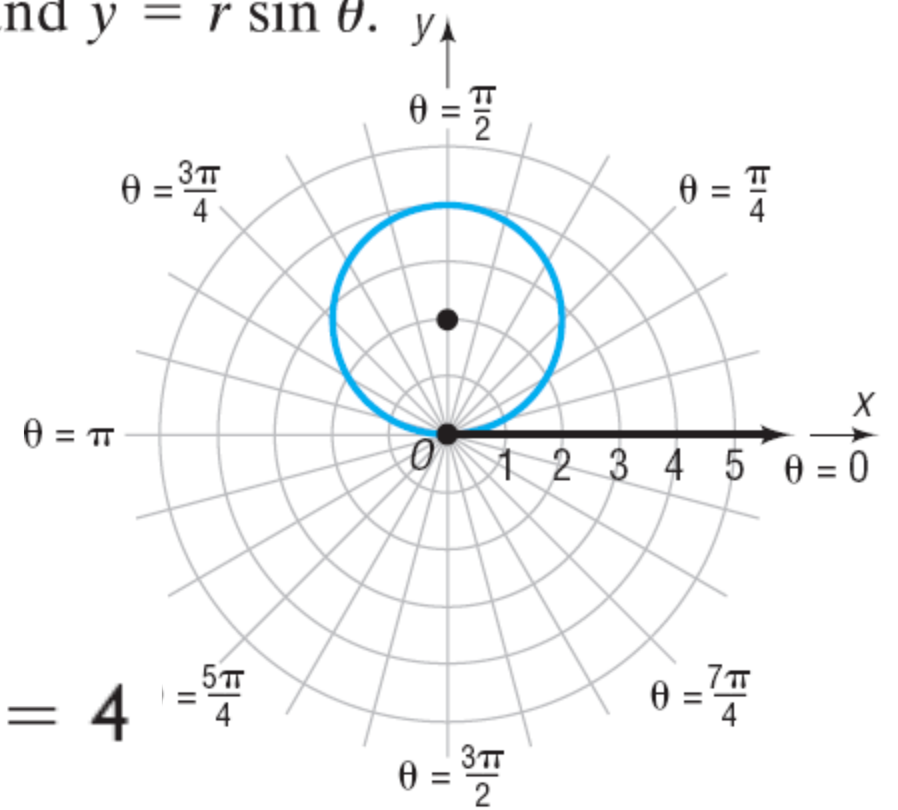
$x^2 + y^2 = 4y$

$x^2 + (y^2 - 4y) = 0$

$x^2 + (y^2 - 4y + 4) = 4$

$x^2 + (y - 2)^2 = 4$

$r = 4\sin\theta$ or $x^2 + (y - 2)^2 = 4$



EXAMPLE

Identifying and Graphing a Polar Equation (Circle)

Identify and graph the equation: $r = -2\cos\theta$

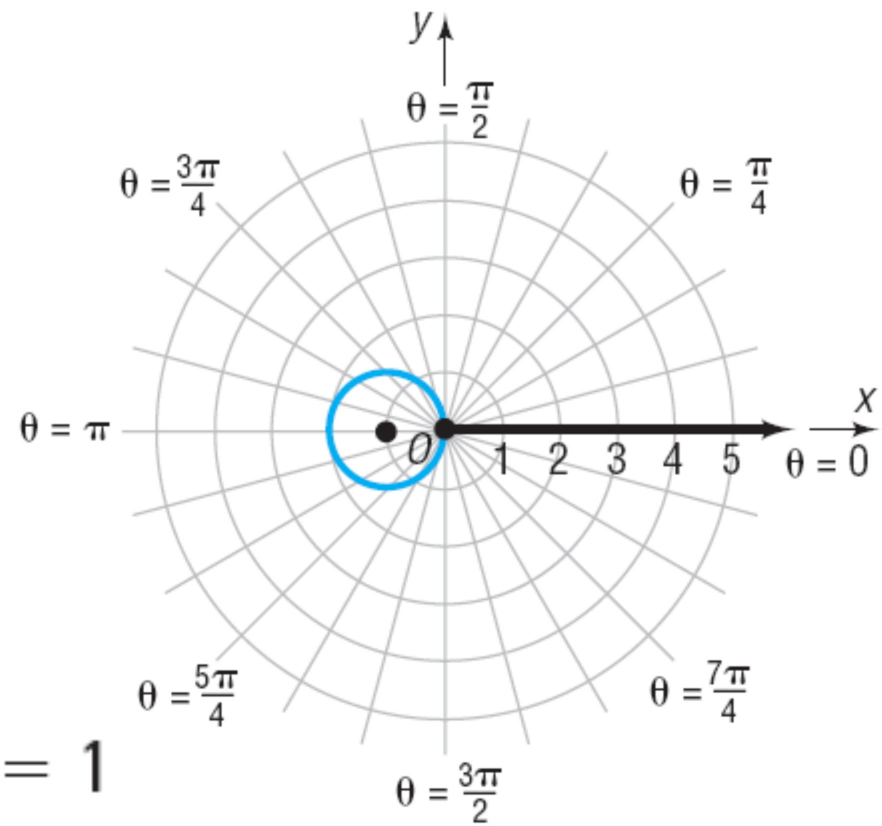
$$r^2 = -2r\cos\theta$$

$$x^2 + y^2 = -2x$$

$$x^2 + 2x + y^2 = 0$$

$$(x^2 + 2x + 1) + y^2 = 1$$

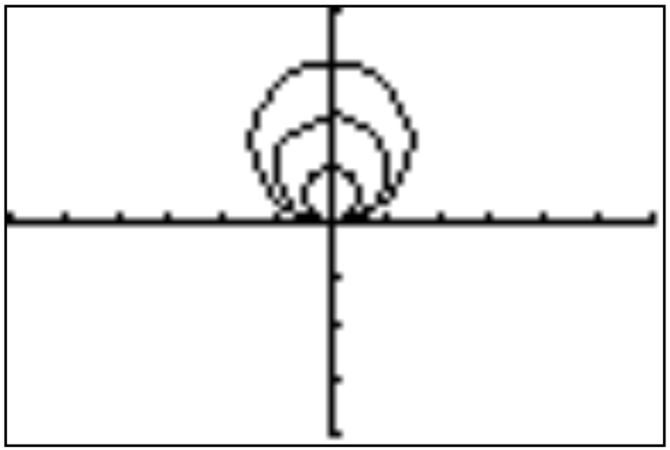
$$(x + 1)^2 + y^2 = 1$$

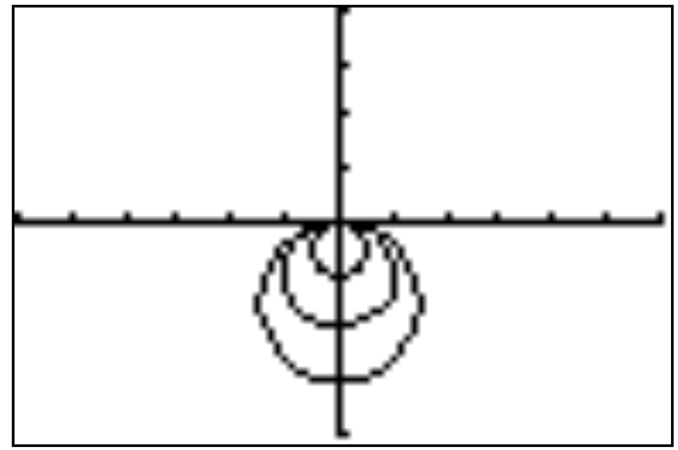


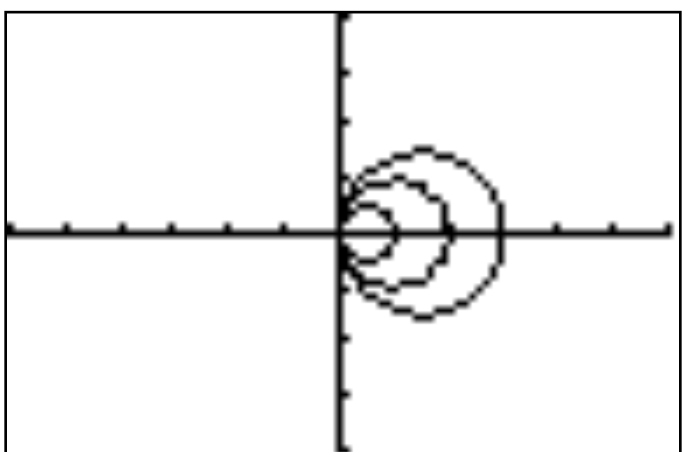
$r = -2\cos\theta$ or $(x + 1)^2 + y^2 = 1$

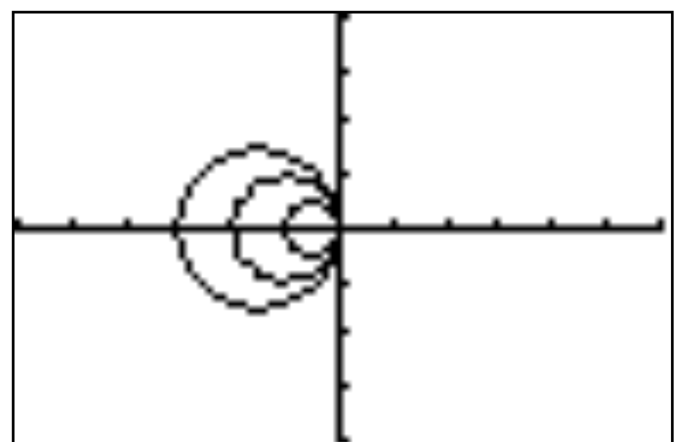
Exploration

Using a square screen, graph $r_1 = \sin\theta$, $r_2 = 2\sin\theta$, and $r_3 = 3\sin\theta$. Do you see the pattern? Clear the screen and graph $r_1 = -\sin\theta$, $r_2 = -2\sin\theta$, and $r_3 = -3\sin\theta$. Do you see the pattern? Clear the screen and graph $r_1 = \cos\theta$, $r_2 = 2\cos\theta$, and $r_3 = 3\cos\theta$. Do you see the pattern? Clear the screen and graph $r_1 = -\cos\theta$, $r_2 = -2\cos\theta$, and $r_3 = -3\cos\theta$. Do you see the pattern?









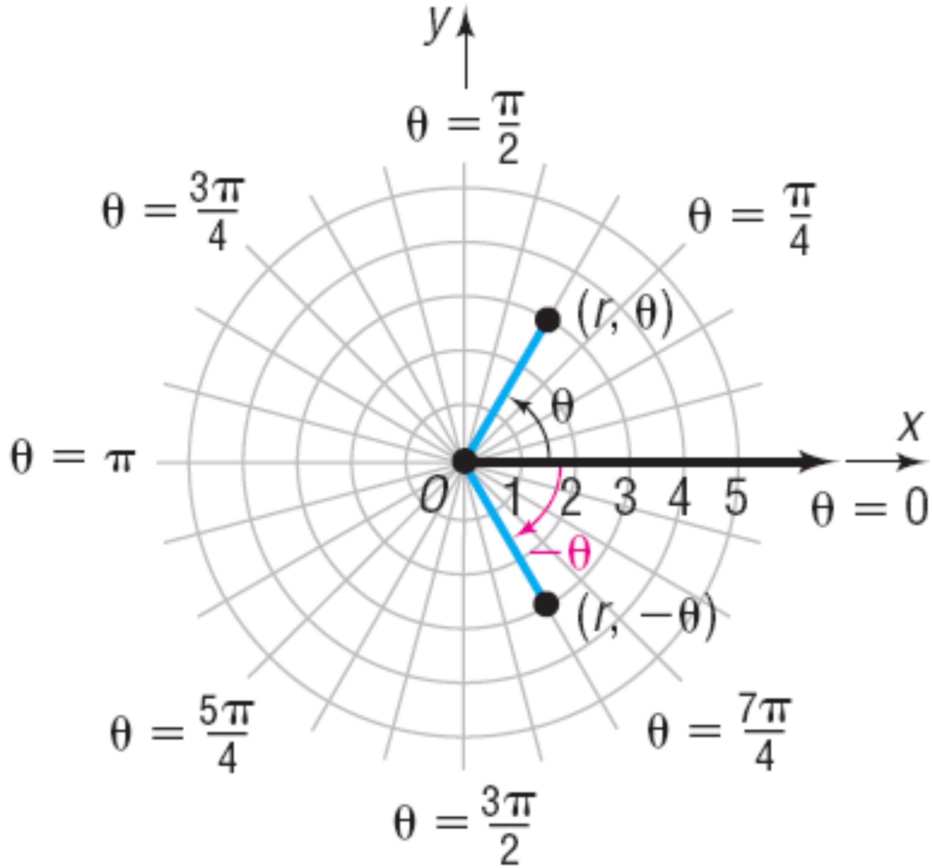
THEOREM

Let a be a positive real number. Then,

Equation	**Description**
(a) $r = 2a \sin \theta$	Circle: radius a; center at $(0, a)$ in rectangular coordinates
(b) $r = -2a \sin \theta$	Circle: radius a; center at $(0, -a)$ in rectangular coordinates
(c) $r = 2a \cos \theta$	Circle: radius a; center at $(a, 0)$ in rectangular coordinates
(d) $r = -2a \cos \theta$	Circle: radius a; center at $(-a, 0)$ in rectangular coordinates

Each circle passes through the pole.

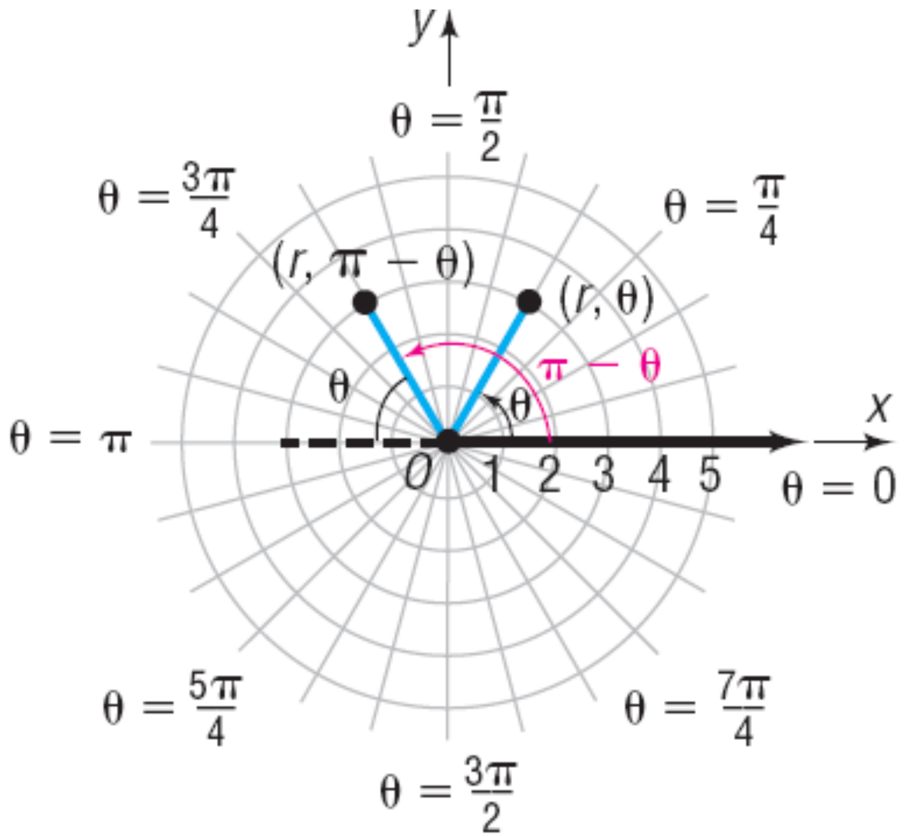
2 Test Polar Equations for Symmetry



(a) Points symmetric with respect to the polar axis

Symmetry with Respect to the Polar Axis (x-Axis)

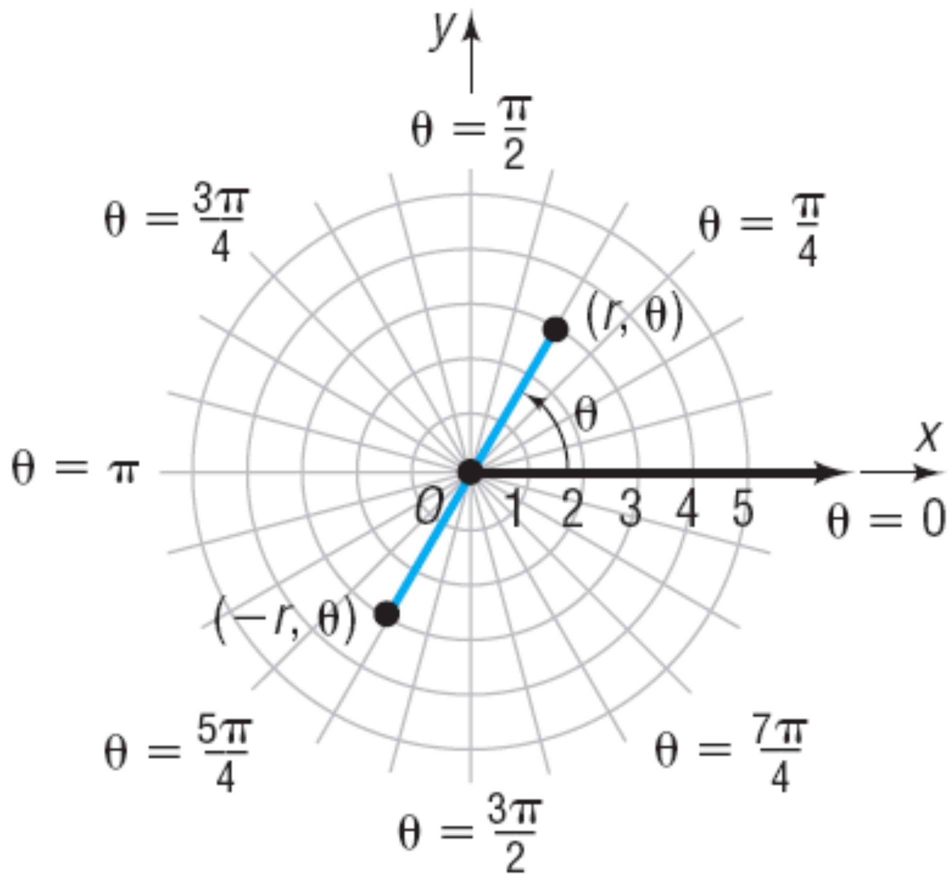
In a polar equation, replace θ by $-\theta$. If an equivalent equation results, the graph is symmetric with respect to the polar axis.



(b) Points symmetric with respect to the line $\theta = \frac{\pi}{2}$

Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$ (y-Axis)

In a polar equation, replace θ by $\pi - \theta$. If an equivalent equation results, the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.



(c) Points symmetric with respect to the pole

Symmetry with Respect to the Pole (Origin)

In a polar equation, replace r by $-r$. If an equivalent equation results, the graph is symmetric with respect to the pole.

3 Graph Polar Equations by Plotting Points

EXAMPLE

Graphing a Polar Equation (Cardioid)

Graph the equation: $r = 1 - \sin\theta$

Check for symmetry first.

Polar Axis: Replace θ by $-\theta$. The result is

$$r = 1 - \sin(-\theta) = 1 + \sin\theta \qquad \sin(-\theta) = -\sin\theta$$

The test fails, so the graph may or may not be symmetric with respect to the polar axis.

The Line $\boldsymbol{\theta = \frac{\pi}{2}}$: Replace θ by $\pi - \theta$. The result is

$$\begin{aligned} r = 1 - \sin(\pi - \theta) &= 1 - (\sin\pi\cos\theta - \cos\pi\sin\theta) \\ &= 1 - [0\cdot\cos\theta - (-1)\sin\theta] = 1 - \sin\theta \end{aligned}$$

The test is satisfied, so the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

EXAMPLE Graphing a Polar Equation (Cardioid)

Graph the equation: $r = 1 - \sin\theta$

The Pole: Replace r by $-r$. Then the result is $-r = 1 - \sin\theta$, so $r = -1 + \sin\theta$. The test fails. Replace θ by $\theta + \pi$. The result is

$$\begin{aligned} r &= 1 - \sin(\theta + \pi) \\ &= 1 - [\sin\theta\cos\pi + \cos\theta\sin\pi] \\ &= 1 - [\sin\theta\cdot(-1) + \cos\theta\cdot 0] \\ &= 1 + \sin\theta \end{aligned}$$

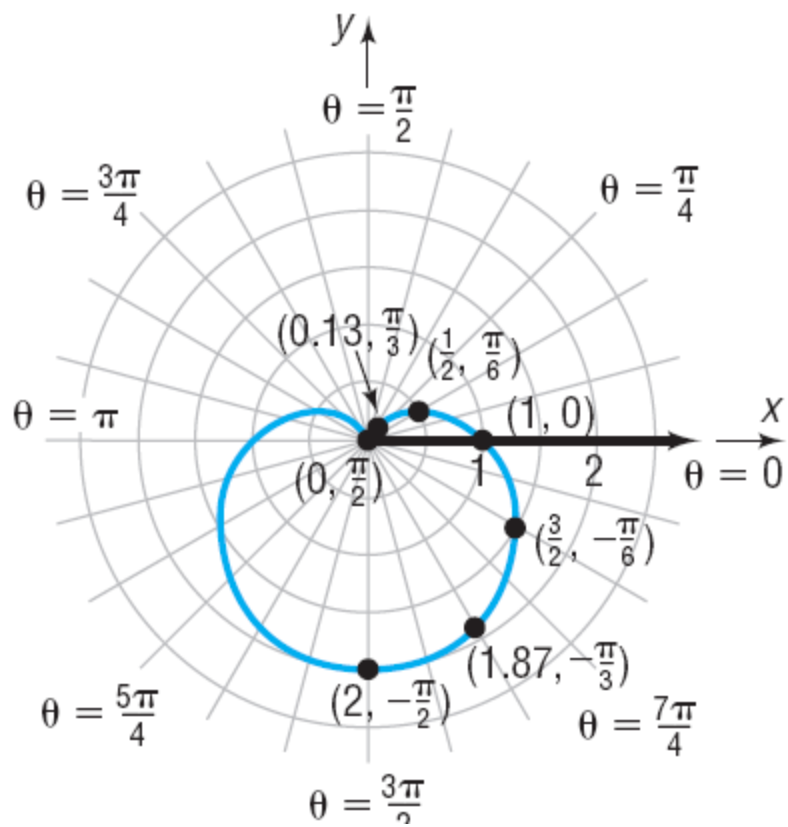
This test also fails. So the graph may or may not be symmetric with respect to the pole.

EXAMPLE

Graphing a Polar Equation (Cardioid)

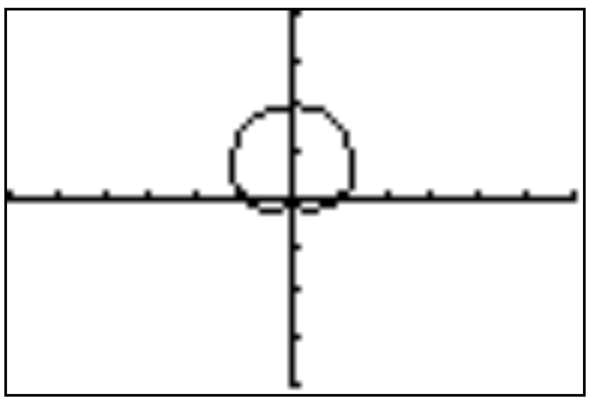
Graph the equation: $r = 1 - \sin\theta$

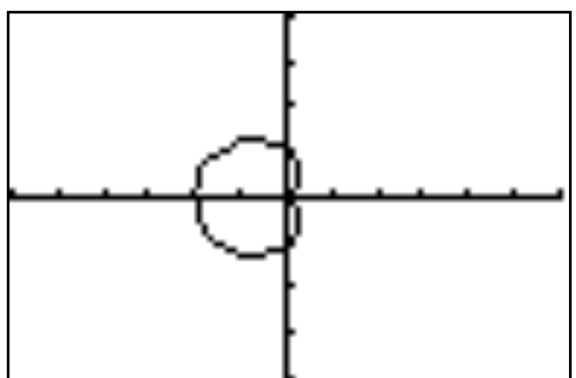
θ	$r = 1 - \sin\theta$
$-\frac{\pi}{2}$	$1 - (-1) = 2$
$-\frac{\pi}{3}$	$1 - \left(-\frac{\sqrt{3}}{2}\right) \approx 1.87$
$-\frac{\pi}{6}$	$1 - \left(-\frac{1}{2}\right) = \frac{3}{2}$
0	$1 - 0 = 1$
$\frac{\pi}{6}$	$1 - \frac{1}{2} = \frac{1}{2}$
$\frac{\pi}{3}$	$1 - \frac{\sqrt{3}}{2} \approx 0.13$
$\frac{\pi}{2}$	$1 - 1 = 0$

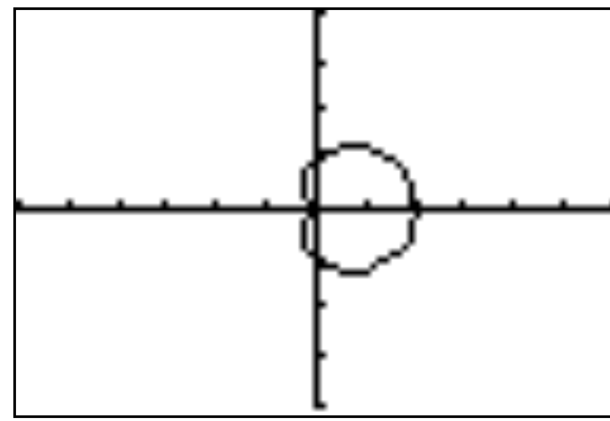


Exploration

Graph $r_1 = 1 + \sin\theta$. Clear the screen and graph $r_1 = 1 - \cos\theta$. Clear the screen and graph $r_1 = 1 + \cos\theta$. Do you see a pattern?







Cardioids are characterized by equations of the form

$$r = a(1 + \cos\theta) \qquad r = a(1 + \sin\theta)$$
$$r = a(1 - \cos\theta) \qquad r = a(1 - \sin\theta)$$

where $a > 0$. The graph of a cardioid passes through the pole.

EXAMPLE

Graphing a Polar Equation (Limaçon without an Inner Loop)

Graph the equation: $r = 3 + 2\cos\theta$

Check for symmetry first.

Polar Axis: Replace θ by $-\theta$. The result is

$$r = 3 + 2\cos(-\theta) = 3 + 2\cos\theta \qquad \cos(-\theta) = \cos\theta$$

The test is satisfied, so the graph is symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: Replace θ by $\pi - \theta$. The result is

$$\begin{aligned} r = 3 + 2\cos(\pi - \theta) &= 3 + 2(\cos\pi\cos\theta + \sin\pi\sin\theta) \\ &= 3 - 2\cos\theta \end{aligned}$$

The test fails, so the graph may or may not be symmetric with respect to the line $\theta = \frac{\pi}{2}$.

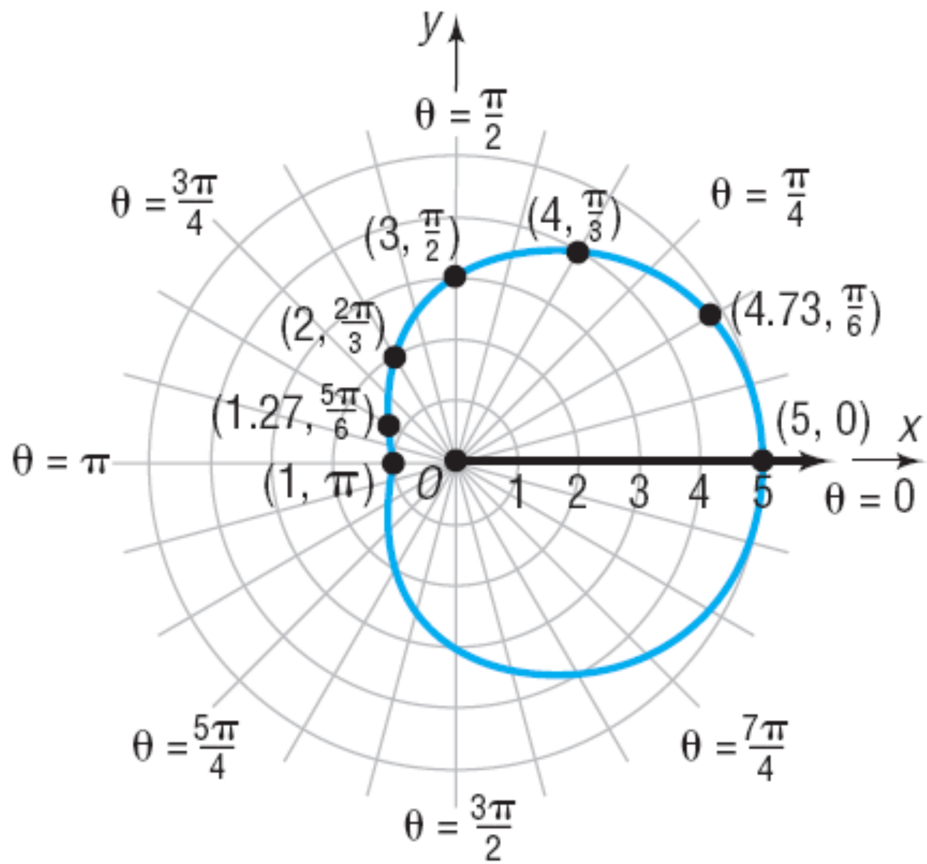
The Pole: Replace r by $-r$. The test fails, so the graph may or may not be symmetric with respect to the pole. Replace θ by $\theta + \pi$. The test fails, so the graph may or may not be symmetric with respect to the pole.

EXAMPLE

Graphing a Polar Equation (Limaçon without an Inner Loop)

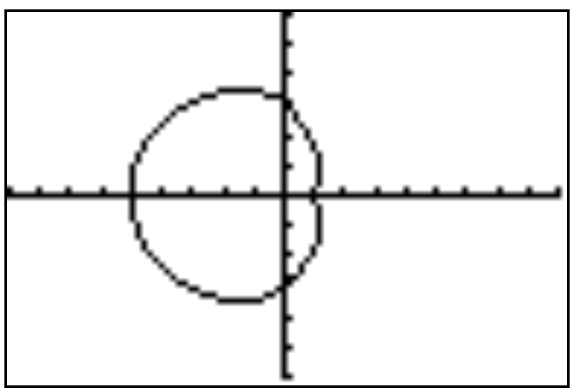
Graph the equation: $r = 3 + 2\cos\theta$

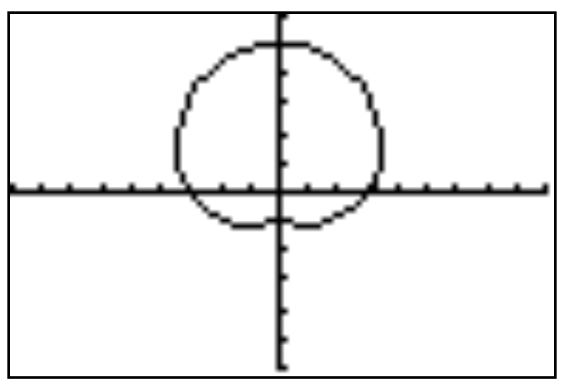
θ	$r = 3 + 2\cos\theta$
0	$3 + 2(1) = 5$
$\frac{\pi}{6}$	$3 + 2\left(\frac{\sqrt{3}}{2}\right) \approx 4.73$
$\frac{\pi}{3}$	$3 + 2\left(\frac{1}{2}\right) = 4$
$\frac{\pi}{2}$	$3 + 2(0) = 3$
$\frac{2\pi}{3}$	$3 + 2\left(-\frac{1}{2}\right) = 2$
$\frac{5\pi}{6}$	$3 + 2\left(-\frac{\sqrt{3}}{2}\right) \approx 1.27$
π	$3 + 2(-1) = 1$

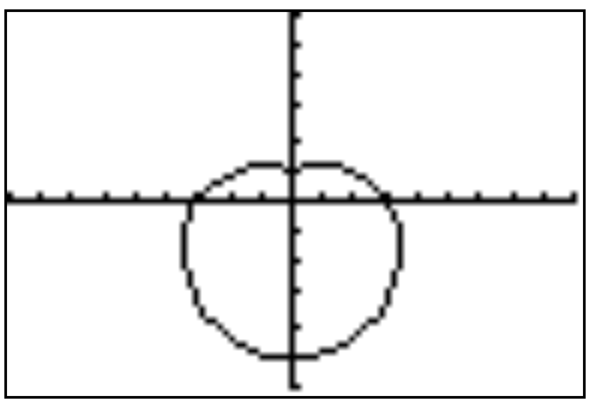


Exploration

Graph $r_1 = 3 - 2\cos\theta$. Clear the screen and graph $r_1 = 3 + 2\sin\theta$. Clear the screen and graph $r_1 = 3 - 2\sin\theta$. Do you see a pattern?







Limaçons without an inner loop are characterized by equations of the form

$$r = a + b\cos\theta \qquad r = a + b\sin\theta$$
$$r = a - b\cos\theta \qquad r = a - b\sin\theta$$

where $a > 0$, $b > 0$, and $a > b$. The graph of a limaçon without an inner loop does not pass through the pole.

EXAMPLE

Graphing a Polar Equation (Limaçon with an Inner Loop)

Graph the equation: $r = 1 + 2\cos\theta$

First, check for symmetry.

Polar Axis: Replace θ by $-\theta$. The result is

$$r = 1 + 2\cos(-\theta) = 1 + 2\cos\theta$$

The test is satisfied, so the graph is symmetric with respect to the polar axis.

The Line $\boldsymbol{\theta = \frac{\pi}{2}}$: Replace θ by $\pi - \theta$. The result is

$$\begin{aligned} r &= 1 + 2\cos(\pi - \theta) = 1 + 2(\cos\pi\cos\theta + \sin\pi\sin\theta) \\ &= 1 - 2\cos\theta \end{aligned}$$

The test fails, so the graph may or may not be symmetric with respect to the line $\theta = \frac{\pi}{2}$.

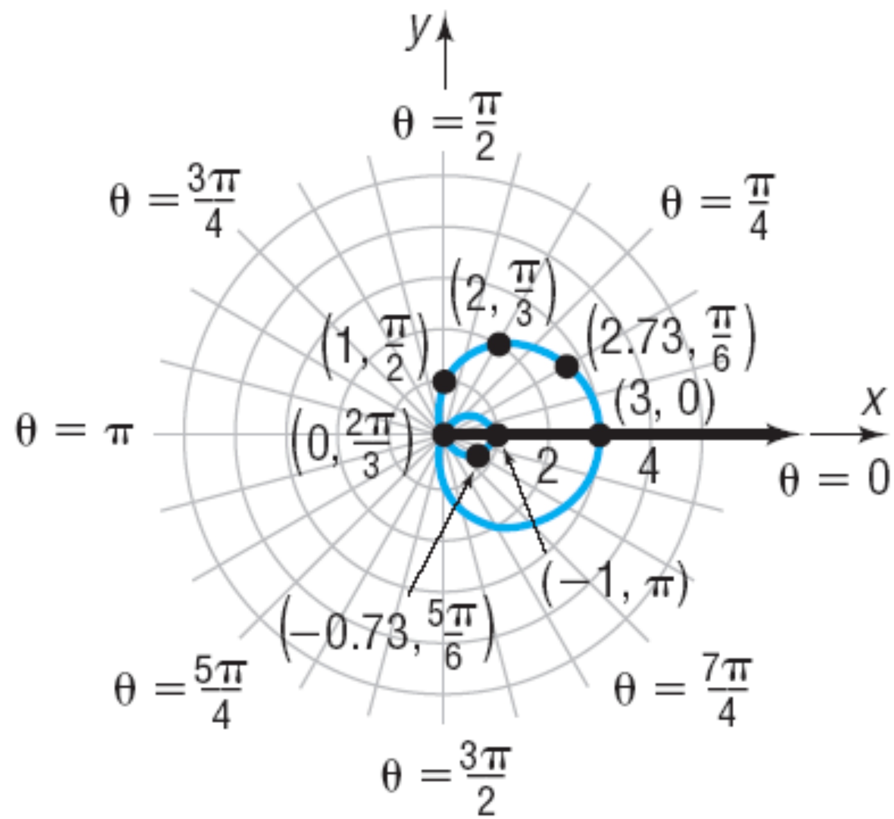
The Pole: Replace r by $-r$. The test fails, so the graph may or may not be symmetric with respect to the pole. Replace θ by $\theta + \pi$. The test fails, so the graph may or may not be symmetric with respect to the pole.

EXAMPLE

Graphing a Polar Equation (Limaçon with an Inner Loop)

Graph the equation: $r = 1 + 2\cos\theta$

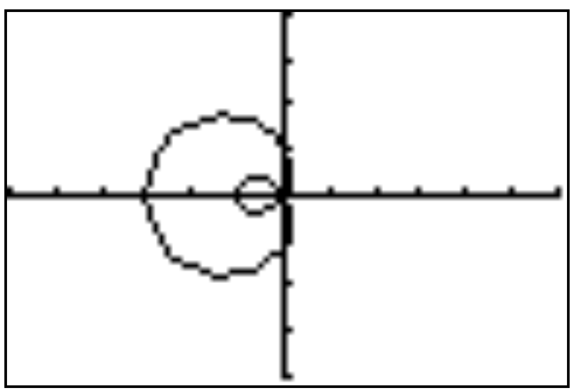
θ	$r = 1 + 2\cos\theta$
0	$1 + 2(1) = 3$
$\frac{\pi}{6}$	$1 + 2\left(\frac{\sqrt{3}}{2}\right) \approx 2.73$
$\frac{\pi}{3}$	$1 + 2\left(\frac{1}{2}\right) = 2$
$\frac{\pi}{2}$	$1 + 2(0) = 1$
$\frac{2\pi}{3}$	$1 + 2\left(-\frac{1}{2}\right) = 0$
$\frac{5\pi}{6}$	$1 + 2\left(-\frac{\sqrt{3}}{2}\right) \approx -0.73$
π	$1 + 2(-1) = -1$

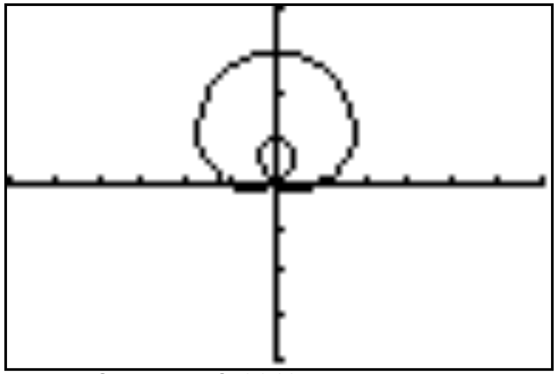


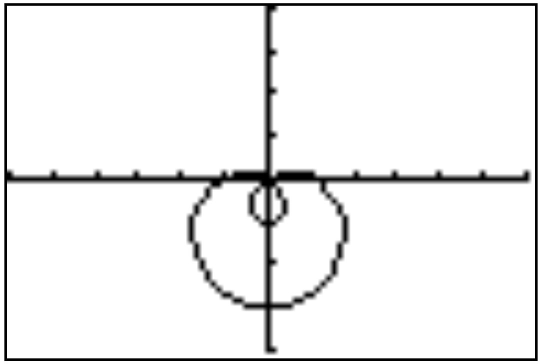
$r = 1 + 2\cos\theta$

Exploration

Graph $r_1 = 1 - 2\cos\theta$. Clear the screen and graph $r_1 = 1 + 2\sin\theta$. Clear the screen and graph $r_1 = 1 - 2\sin\theta$. Do you see a pattern?







Limaçons with an inner loop are characterized by equations of the form

$$r = a + b\cos\theta \qquad r = a + b\sin\theta$$
$$r = a - b\cos\theta \qquad r = a - b\sin\theta$$

where $a > 0$, $b > 0$, and $a < b$. The graph of a limaçon with an inner loop will pass through the pole twice.

EXAMPLE Graphing a Polar Equation (Rose)

Graph the equation: $r = 2\cos(2\theta)$

Check for symmetry.

Polar Axis: If we replace θ by $-\theta$, the result is

$$r = 2\cos[2(-\theta)] = 2\cos(2\theta)$$

The test is satisfied, so the graph is symmetric with respect to the polar axis.

The Line $\boldsymbol{\theta = \frac{\pi}{2}}$: If we replace θ by $\pi - \theta$, we obtain

$$r = 2\cos[2(\pi - \theta)] = 2\cos(2\pi - 2\theta) = 2\cos(2\theta)$$

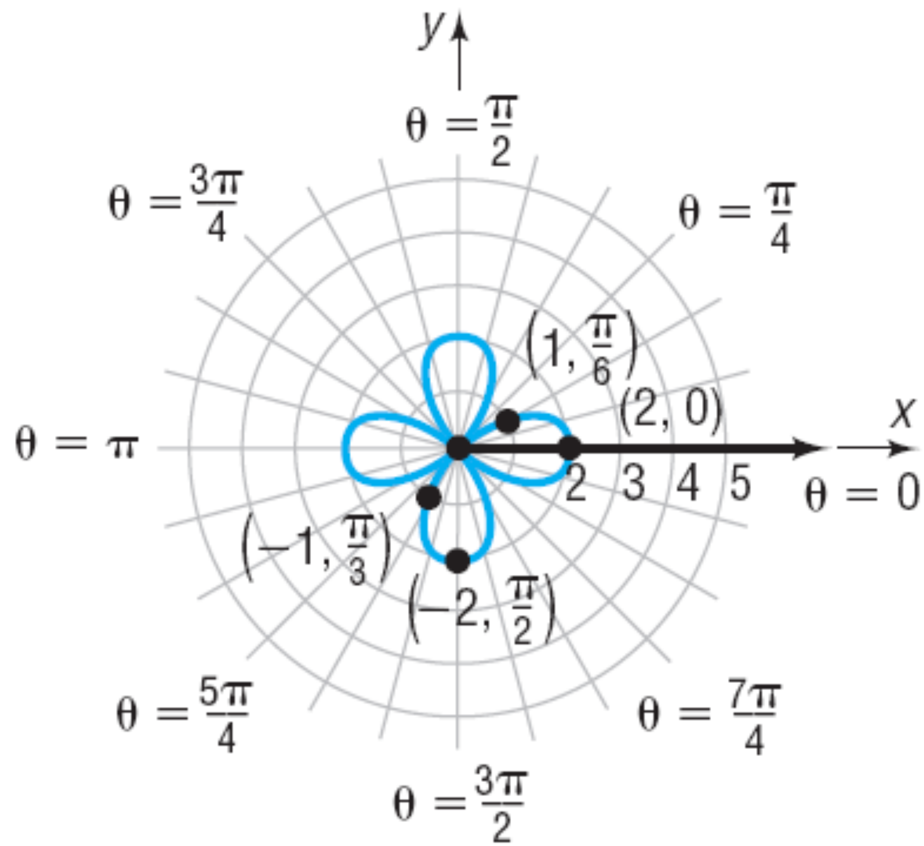
The test is satisfied, so the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Since the graph is symmetric with respect to both the polar axis and the line $\theta = \frac{\pi}{2}$, it must be symmetric with respect to the pole.

EXAMPLE **Graphing a Polar Equation (Rose)**

Graph the equation: $r = 2\cos(2\theta)$

θ	$r = 2\cos(2\theta)$
0	$2(1) = 2$
$\frac{\pi}{6}$	$2\left(\frac{1}{2}\right) = 1$
$\frac{\pi}{4}$	$2(0) = 0$
$\frac{\pi}{3}$	$2\left(-\frac{1}{2}\right) = -1$
$\frac{\pi}{2}$	$2(-1) = -2$

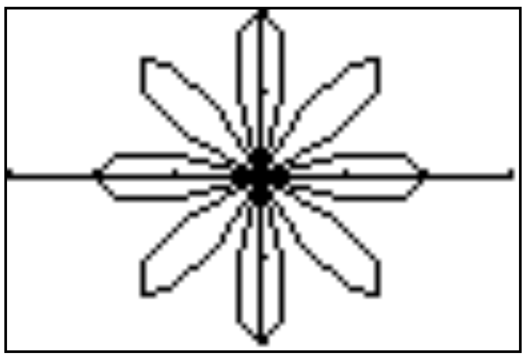
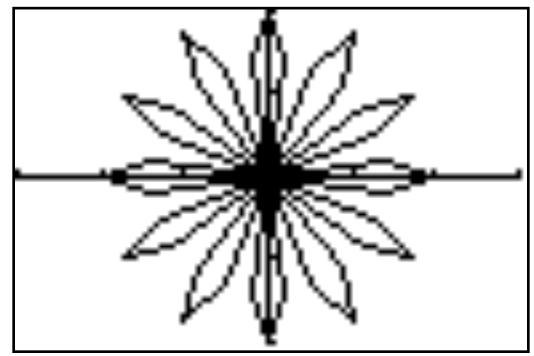


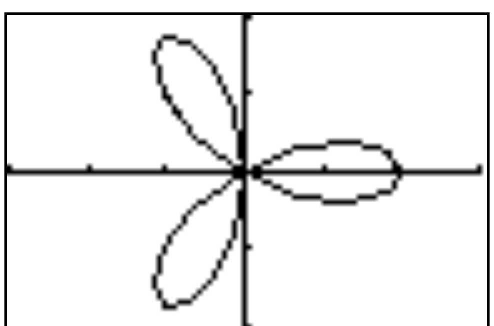
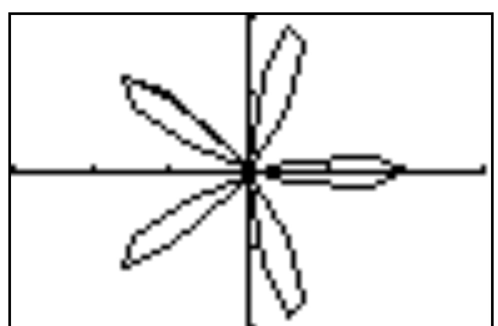
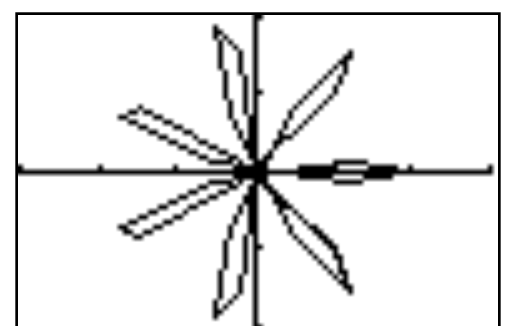
$r = 2\cos(2\theta)$

Exploration

Graph $r_1 = 2\cos(4\theta)$; clear the screen and graph $r_1 = 2\cos(6\theta)$. How many petals did each of these graphs have?

Clear the screen and graph, in order, each on a clear screen, $r_1 = 2\cos(3\theta)$, $r_1 = 2\cos(5\theta)$, and $r_1 = 2\cos(7\theta)$. What do you notice about the number of petals?

Rose curves are characterized by equations of the form

$$\boxed{r = a\cos(n\theta), \qquad r = a\sin(n\theta), \qquad a \neq 0}$$

and have graphs that are rose shaped. If $n \neq 0$ is even, the rose has $2n$ petals; if $n \neq \pm 1$ is odd, the rose has n petals.

EXAMPLE

Graphing a Polar Equation (Lemniscate)

Graph the equation: $r^2 = 4\sin(2\theta)$

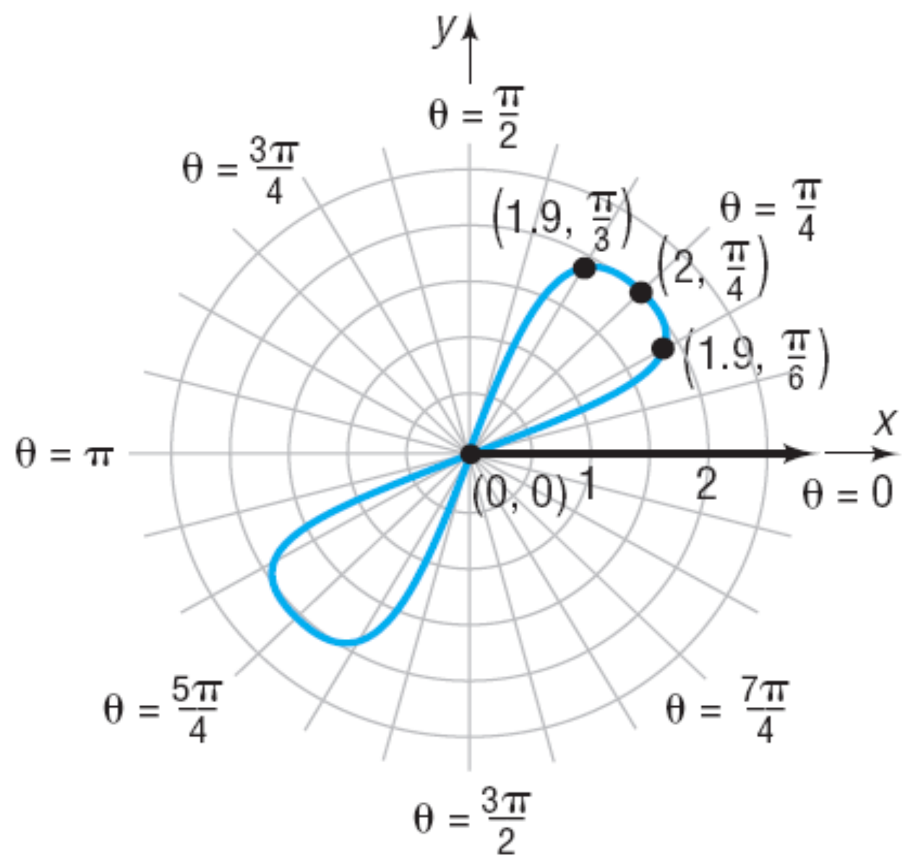
We leave it to you to verify that the graph is symmetric with respect to the pole. Because of the symmetry with respect to the pole, we only need to consider values of θ between $\theta = 0$ and $\theta = \pi$. Note that there are no points on the graph for $\frac{\pi}{2} < \theta < \pi$ (quadrant II), since $r^2 < 0$ for such values.

EXAMPLE

Graphing a Polar Equation (Lemniscate)

Graph the equation: $r^2 = 4\sin(2\theta)$

θ	$r^2 = 4\sin(2\theta)$	r
0	$4(0) = 0$	0
$\frac{\pi}{6}$	$4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$	± 1.9
$\frac{\pi}{4}$	$4(1) = 4$	± 2
$\frac{\pi}{3}$	$4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$	± 1.9
$\frac{\pi}{2}$	$4(0) = 0$	0



$r^2 = 4\sin(2\theta)$

Lemniscates are characterized by equations of the form

$$r^2 = a^2 \sin(2\theta) \qquad r^2 = a^2 \cos(2\theta)$$

where $a \neq 0$, and have graphs that are propeller shaped.

EXAMPLE

Graphing a Polar Equation (Spiral)

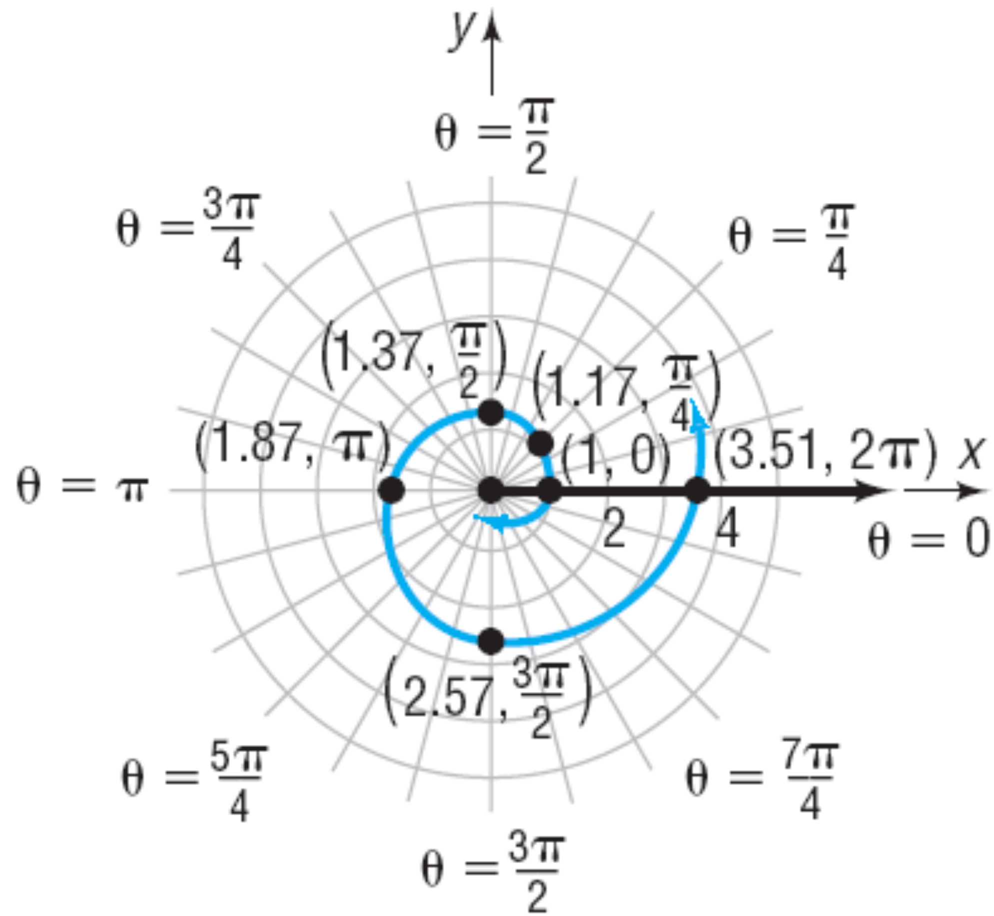
Graph the equation: $r = e^{\theta/5}$

The tests for symmetry with respect to the pole, the polar axis, and the line $\theta = \frac{\pi}{2}$ fail. Furthermore, there is no number θ for which $r = 0$, so the graph does not pass through the pole. Observe that r is positive for all θ, r increases as θ increases, $r \to 0$ as $\theta \to -\infty$, and $r \to \infty$ as $\theta \to \infty$.

EXAMPLE **Graphing a Polar Equation (Spiral)**

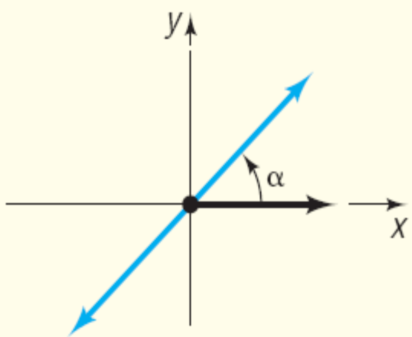
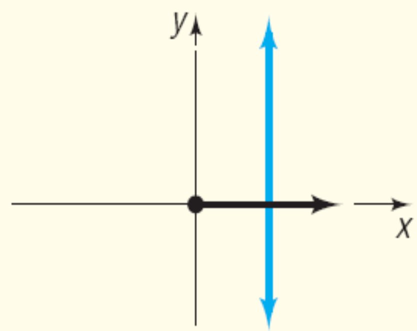
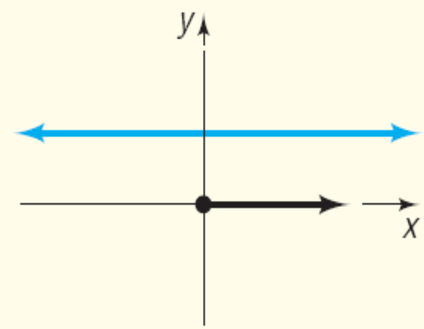
Graph the equation: $r = e^{\theta/5}$

θ	$r = e^{\theta/5}$
$-\frac{3\pi}{2}$	0.39
$-\pi$	0.53
$-\frac{\pi}{2}$	0.73
$-\frac{\pi}{4}$	0.85
0	1
$\frac{\pi}{4}$	1.17
$\frac{\pi}{2}$	1.37
π	1.87
$\frac{3\pi}{2}$	2.57
2π	3.51

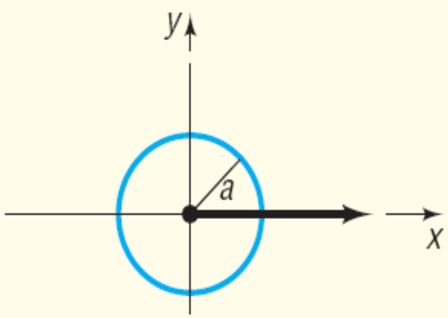
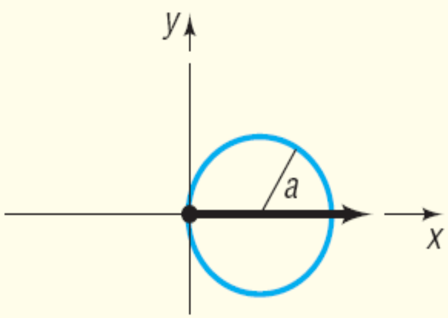
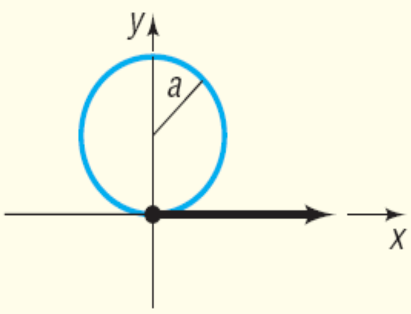


The curve in Figure 32 is called a **logarithmic spiral,** since its equation may be written as $\theta = 5 \ln r$ and it spirals infinitely both toward the pole and away from it.

Classification of Polar Equations

Lines			
Description	Line passing through the pole making an angle α with the polar axis	Vertical line	Horizontal line
Rectangular equation	$y = (\tan \alpha)x$	$x = a$	$y = b$
Polar equation	$\theta = \alpha$	$r \cos \theta = a$	$r \sin \theta = b$
Typical graph			

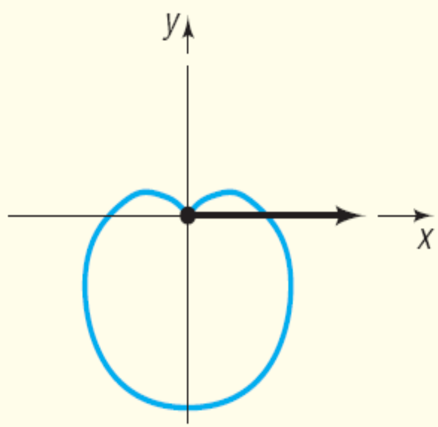
Circles

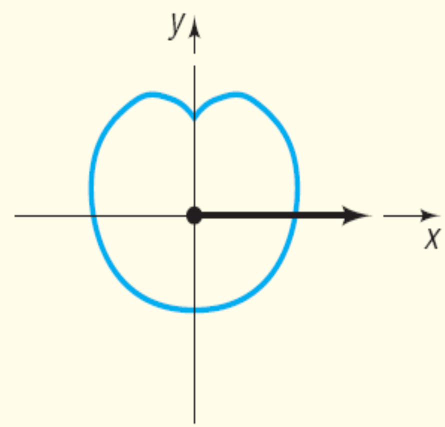
Description	Center at the pole, radius a	Passing through the pole, tangent to the line $\theta = \frac{\pi}{2}$, center on the polar axis, radius a	Passing through the pole, tangent to the polar axis, center on the line $\theta = \frac{\pi}{2}$, radius a
Rectangular equation	$x^2 + y^2 = a^2, \quad a > 0$	$x^2 + y^2 = \pm 2ax, \quad a > 0$	$x^2 + y^2 = \pm 2ay, \quad a > 0$
Polar equation	$r = a, \quad a > 0$	$r = \pm 2a \cos \theta, \quad a > 0$	$r = \pm 2a \sin \theta, \quad a > 0$
Typical graph			

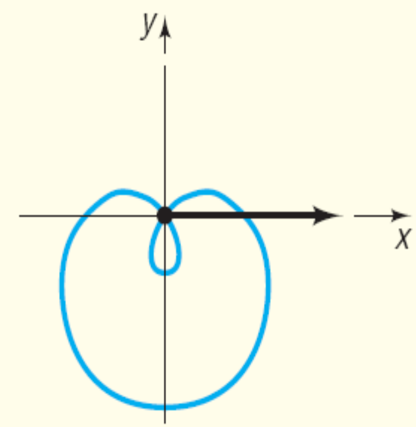
Other Equations

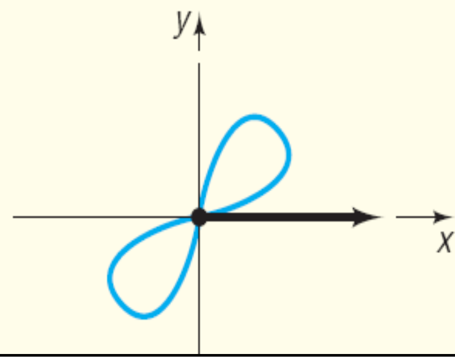
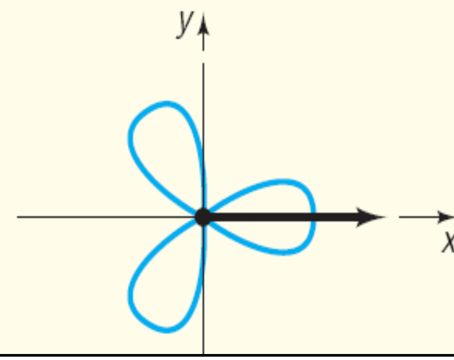
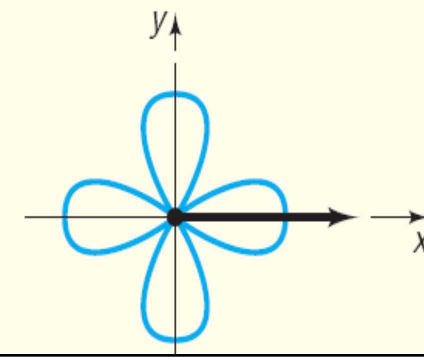
Name	Cardioid	Limaçon without inner loop	Limaçon with inner loop
Polar equations	$r = a \pm a \cos\theta, \quad a > 0$ $r = a \pm a \sin\theta, \quad a > 0$	$r = a \pm b \cos\theta, \quad 0 < b < a$ $r = a \pm b \sin\theta, \quad 0 < b < a$	$r = a \pm b \cos\theta, \quad 0 < a < b$ $r = a \pm b \sin\theta, \quad 0 < a < b$

Typical graph







Name	Lemniscate	Rose with three petals	Rose with four petals
Polar equations	$r^2 = a^2 \cos(2\theta), \quad a > 0$ $r^2 = a^2 \sin(2\theta), \quad a > 0$	$r = a \sin(3\theta), \quad a > 0$ $r = a \cos(3\theta), \quad a > 0$	$r = a \sin(2\theta), \quad a > 0$ $r = a \cos(2\theta), \quad a > 0$
Typical graph			

EXAMPLE

Sketching the Graph of a Polar Equation Quickly

Graph the equation: $r = 2 + 2\sin\theta$

You should recognize the polar equation: Its graph is a cardioid. The period of sin θ is 2π, so form a table using $0 \le \theta \le 2\pi$, compute r, plot the points (r, θ), and sketch the graph of a cardioid as θ varies from 0 to 2π. See Table 8 and Figure 33.

θ	$r = 2 + 2\sin\theta$
0	$2 + 2(0) = 2$
$\frac{\pi}{2}$	$2 + 2(1) = 4$
π	$2 + 2(0) = 2$
$\frac{3\pi}{2}$	$2 + 2(-1) = 0$
2π	$2 + 2(0) = 2$