Section 1.3

Complex Numbers

Quadratic Equations in the Complex Number System

imaginary unit

$$\boxed{i^2 = -1}$$

Complex numbers are numbers of the form $\boldsymbol{a + bi}$, where a and b are real numbers. The real number a is called the **real part** of the number $a + bi$; the real number b is called the **imaginary part** of $a + bi$.

1 Add, Subtract, Multiply, and Divide Complex Numbers

Equality of Complex Numbers

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

Sum of Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Difference of Complex Numbers

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

EXAMPLE

Adding and Subtracting Complex Numbers

a) $(3 - 2i) + (5 - 4i)$

$(3+5) + [(-2) +(- 4)]\ i$

$8-6\ i$

b) $(3 - 2i) - (5 - 4i)$

$(3 - 5) + [(-2) -(- 4)]\ i$

$-2+2\ i$

EXAMPLE **Multiplying Complex Numbers**

$$(3-2i)\,(5-4i)$$

$$3(5)+3(-4i)+(-2i)(5)+(-2i)(-4i)$$

$$15-12i-10i+8i^2 = 15-22i+8(-1) = 7-22i$$

Product of Complex Numbers

$$\boxed{(a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i}$$

Conjugates

If $z = a + bi$ is a complex number, then its **conjugate**, denoted by $\overline{z}$, is defined as

$$\boxed{\overline{z} = \overline{a + bi} = a - bi}$$

EXAMPLE

Multiplying a Complex Number by Its Conjugate

$$z = 3 - 2i \qquad \overline{z} = 3 + 2i \qquad z\overline{z} = (3-2i)(3+2i)$$

$$z\overline{z} = 9 + 6i - 6i - 4i^2 = 9 - 4(-1) = 13$$

Theorem

The product of a complex number and its conjugate is a nonnegative real number.

That is, if $z = a + bi$, then

$$\boxed{z\overline{z} = a^2 + b^2}$$

EXAMPLE

Writing the Reciprocal of a Complex Number in Standard Form

Write $\dfrac{1}{3-2i}$ in standard form $a+bi$;

That is, find the reciprocal of $3-2i$.

$$\frac{1}{3-2i}\cdot\frac{3+2i}{3+2i}=\frac{3+2i}{9+4}=\frac{3}{13}+\frac{2}{13}i$$

EXAMPLE

Writing the Quotient of Two Complex Numbers in Standard Form

Write in standard form: $\dfrac{3-2i}{5-4i}$

$$\frac{3-2i}{5-4i}\cdot\frac{5+4i}{5+4i}=\frac{15+12i-10i-8i^2}{25+16}=\frac{23+2i}{41}$$

$$=\frac{23}{41}+\frac{2}{41}i$$

EXAMPLE

Writing Other Expressions in Standard Form

If $z = 3 - i$ and $w = 2 - 5i$, write each of the following in standard form:

$$a)\quad \frac{z}{w} = \frac{3-i}{2-5i}\cdot\frac{2+5i}{2+5i} = \frac{6+15i-2i-5i^2}{4+25} = \frac{11}{29}+\frac{13}{29}i$$

$$b)\quad \overline{z+w} = \overline{(3-i)+(2-5i)} = \overline{5-6i} = 5+6i$$

$$c)\quad z+\overline{z} = (3-i)+(3+i) = 6$$

Theorem

The conjugate of a real number is the real number itself.

Theorem

The conjugate of the conjugate of a complex number is the complex number itself.

$$\boxed{(\overline{\overline{z}}) = z}$$

The conjugate of the sum of two complex numbers equals the sum of their conjugates.

$$\overline{z + w} = \overline{z} + \overline{w}$$

The conjugate of the product of two complex numbers equals the product of their conjugates.

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}$$

Powers of i

The powers of i follow a pattern that is useful to know.

$$i^1 = i \qquad i^5 = i^4 \cdot i = 1 \cdot i = i$$
$$i^2 = -1 \qquad i^6 = i^4 \cdot i^2 = -1$$
$$i^3 = i^2 \cdot i = -i \qquad i^7 = i^4 \cdot i^3 = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 \qquad i^8 = i^4 \cdot i^4 = 1$$

And so on. The powers of i repeat with every fourth power.

EXAMPLE **Evaluating Powers of i**

a) $i^{33} = i^{32} \cdot i$

$= \left(i^4\right)^8 \cdot i = (1)^8 \cdot i = i$

b) $i^{82} = i^{80} \cdot i^2$

$= \left(i^4\right)^{20} \cdot i^2 = (1)^{20} \cdot (-1) = -1$

EXAMPLE

Writing the Power of a Complex Number in Standard Form

Write $(3-2i)^3$ in standard form.

We use the special product formula for $(x + a)^3$.

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$

$$(3-2i)^3 = (3)^3 + 3(-2i)(3)^2 + 3(-2i)^2(3) + (-2i)^3$$

$$= 27 - 54i + 36i^2 - 8i^3$$

$$= 27 - 54i + 36(-1) - 8(-i)$$

$$= -9 - 46i$$

2 Solve Quadratic Equations in the Complex Number System

If N is a positive real number, we define the **principal square root of $-N$**, denoted by $\sqrt{-N}$, as

$$\boxed{\sqrt{-N} = \sqrt{N}i}$$

EXAMPLE

Evaluating the Square Root of a Negative Number

$$a)\ \sqrt{-1} = \sqrt{1}\, i = i$$

$$b)\ \sqrt{-9} = \sqrt{9}\, i = 3i$$

$$c)\ \sqrt{-18} = \sqrt{18}\, i = 3\sqrt{2}i$$

EXAMPLE **Solving Equations**

Solve the equation in the complex number system

$a)\quad x^2 = 9 \qquad x = \pm\sqrt{9} = \pm 3$

$b)\quad x^2 + 25 = 0 \qquad x^2 = -25$

$$x = \pm\sqrt{-25}$$

$$x = \pm\sqrt{25}\, i = \pm 5i$$

Theorem

In the complex number system, the solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textbf{(9)}$$

EXAMPLE

Solving a Quadratic Equation in the Complex Number System

Solve the equation $x^2 - 6x + 13 = 0$ in the complex number system.

$$b^2 - 4ac = (-6)^2 - 4(1)(13) = 36 - 52 = -16$$

$$x = \frac{-(-6) \pm \sqrt{-16}}{2(1)} = \frac{6 \pm \sqrt{16}i}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

Character of the Solutions of a Quadratic Equation

In the complex number system, consider a quadratic equation $ax^2 + bx + c = 0$ with real coefficients.

1. If $b^2 - 4ac > 0$, the equation has two unequal real solutions.
2. If $b^2 - 4ac = 0$, the equation has a repeated real solution, a double root.
3. If $b^2 - 4ac < 0$, the equation has two complex solutions that are not real. The solutions are conjugates of each other.

EXAMPLE

Determining the Character of the Solutions of a Quadratic Equation

Without solving, determine the character of the solutions of each equation.

$a)\quad 2x^2 + 3x + 1 = 0 \qquad b^2 - 4ac = (3)^2 - 4(2)(1) = 1$

The solutions are two unequal real numbers.

$b)\quad 5x^2 - 2x + 4 = 0 \qquad b^2 - 4ac = (-2)^2 - 4(5)(4) = -76$

The solutions are two non-real complex numbers that are conjugates of each other.

$c)\quad 4x^2 - 4x + 1 = 0 \qquad b^2 - 4ac = (-4)^2 - 4(4)(1) = 0$

The solution is a repeated real number, that is, a double root.