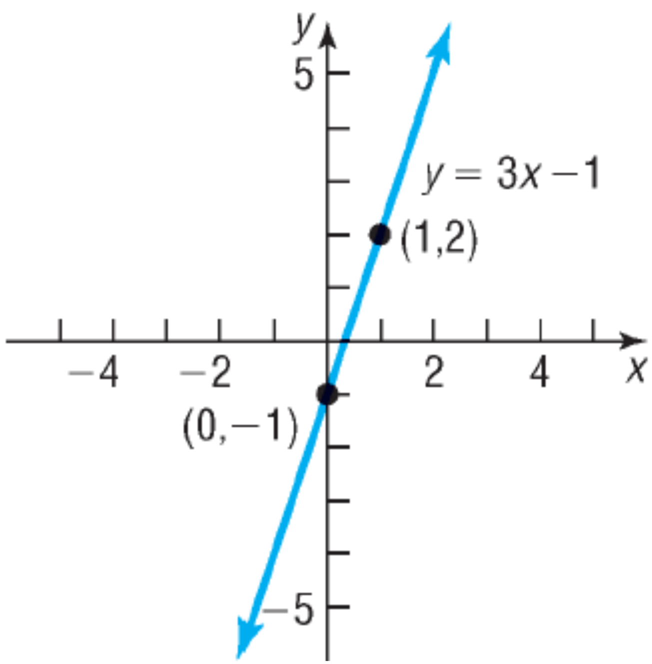
Section 3.1

Functions

1 Determine Whether a Relation Represents a Function

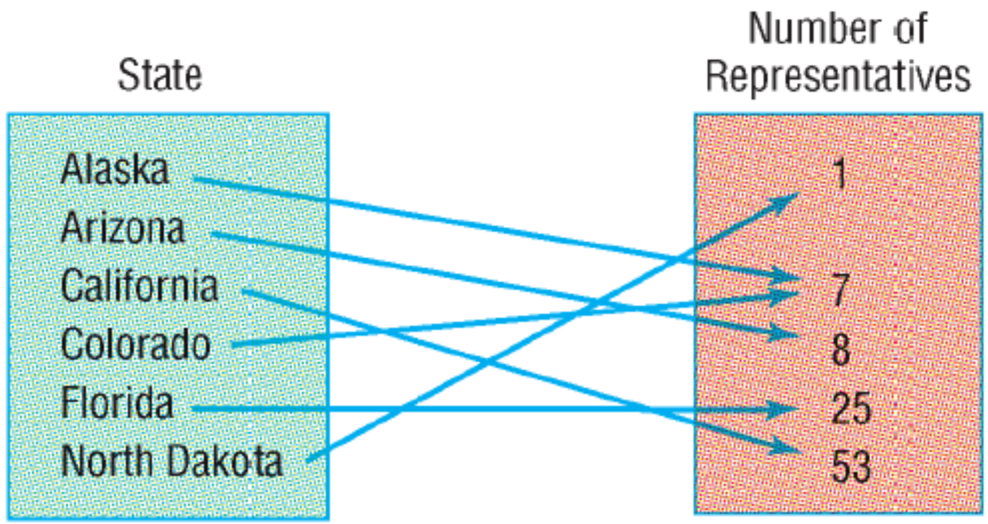
A **relation** is a correspondence between two sets.

If x and y are two elements in these sets and if a relation exists between x and y, then we say that x corresponds to y or that y depends on x, and we write $x \rightarrow y$.



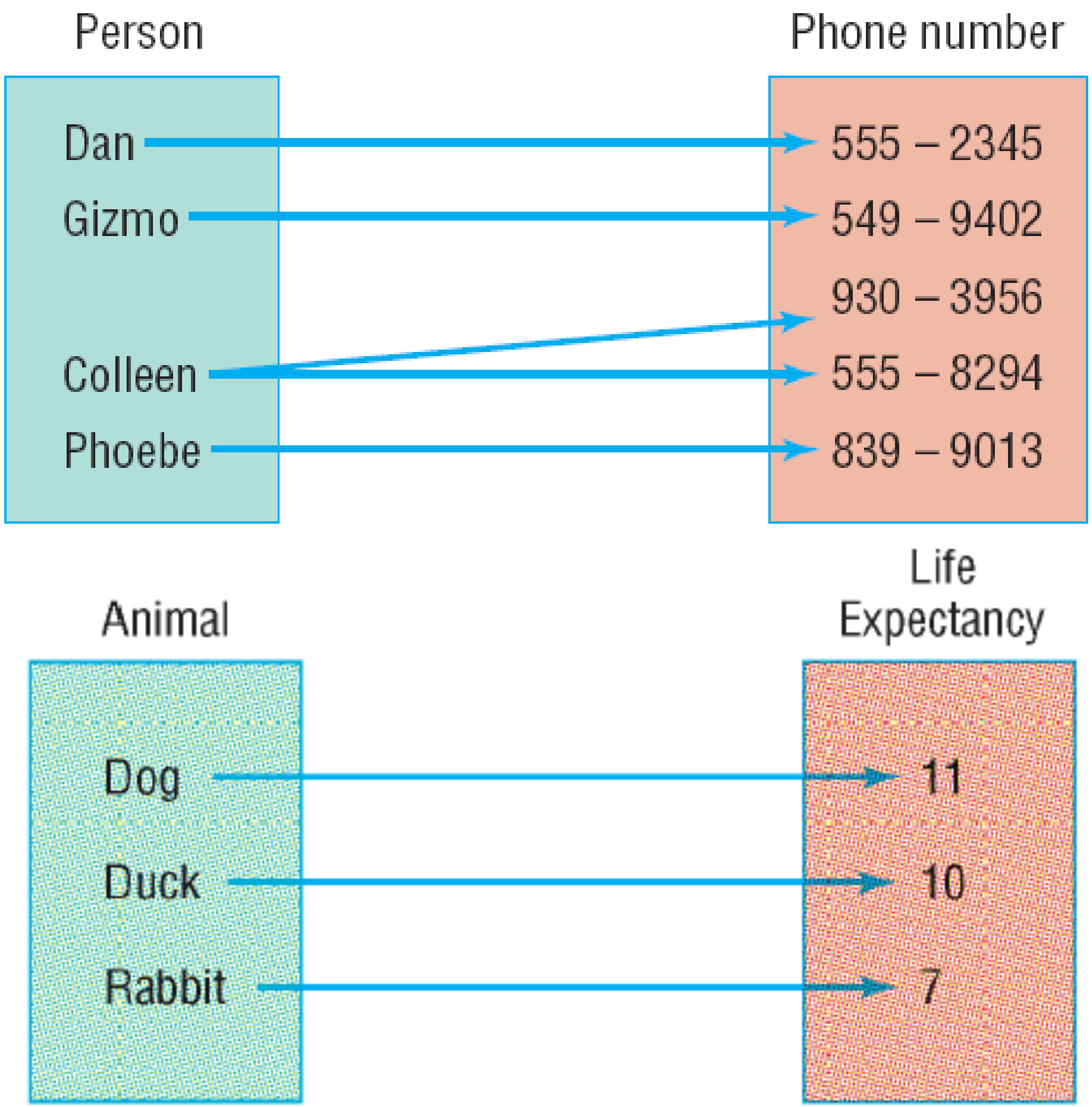
EXAMPLE

Maps and Ordered Pairs as Relations



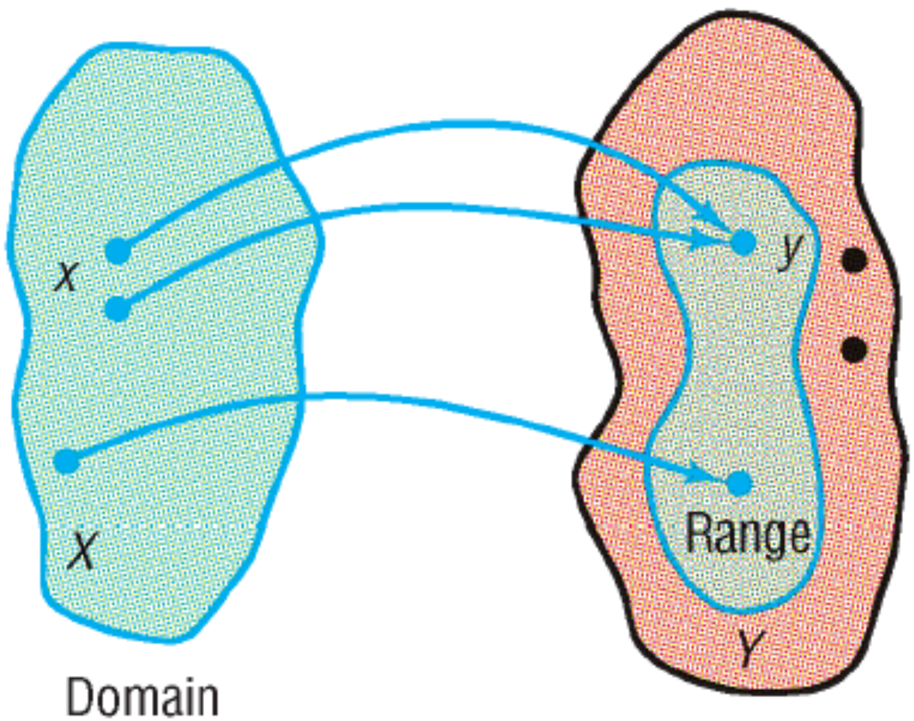
{(Alaska, 7), (Arizona, 8), (California, 53), (Colorado, 7), (Florida, 25), (North Dakota, 1)}

FUNCTION


Person
Phone number
Dan
Gizmo
Colleen
Phoebe
555 – 2345
549 – 9402
930 – 3956
555 – 8294
839 – 9013
Animal
Life Expectancy
Dog
Duck
Rabbit
11
10
7

DEFINITION

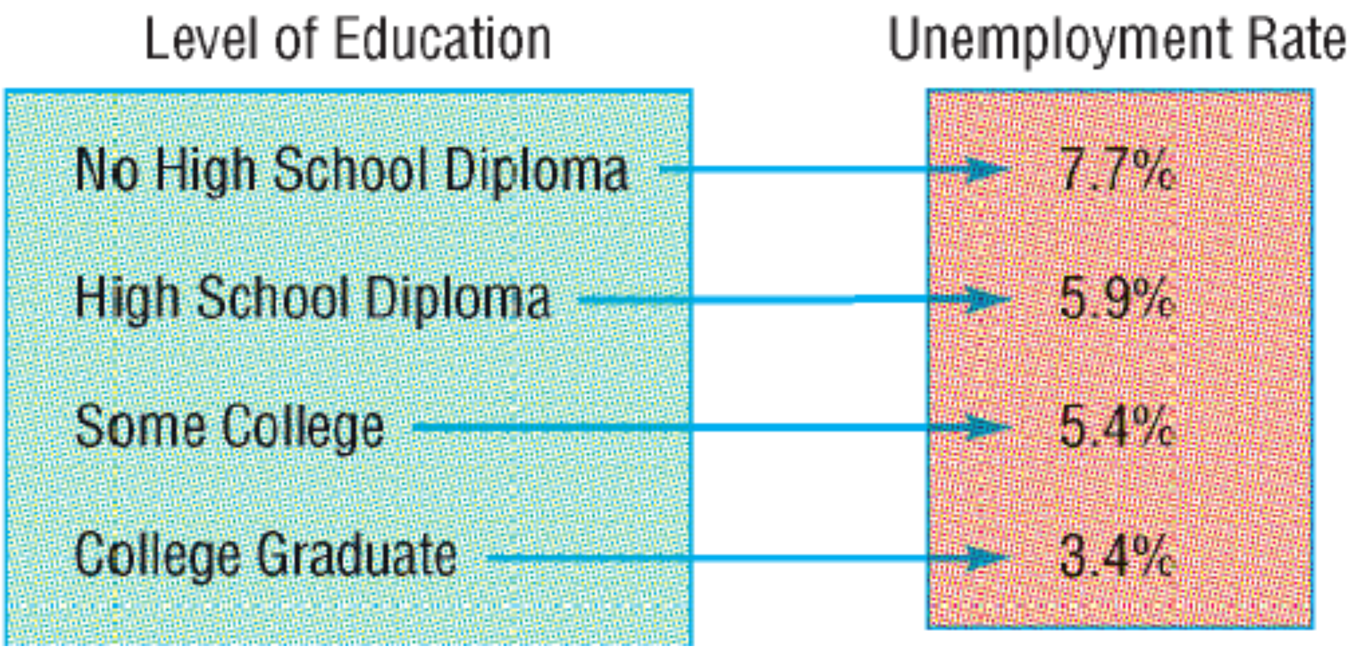
Let X and Y be two nonempty sets.* A **function** from X into Y is a relation that associates with each element of X exactly one element of Y.



EXAMPLE

Determining Whether a Relation Represents a Function

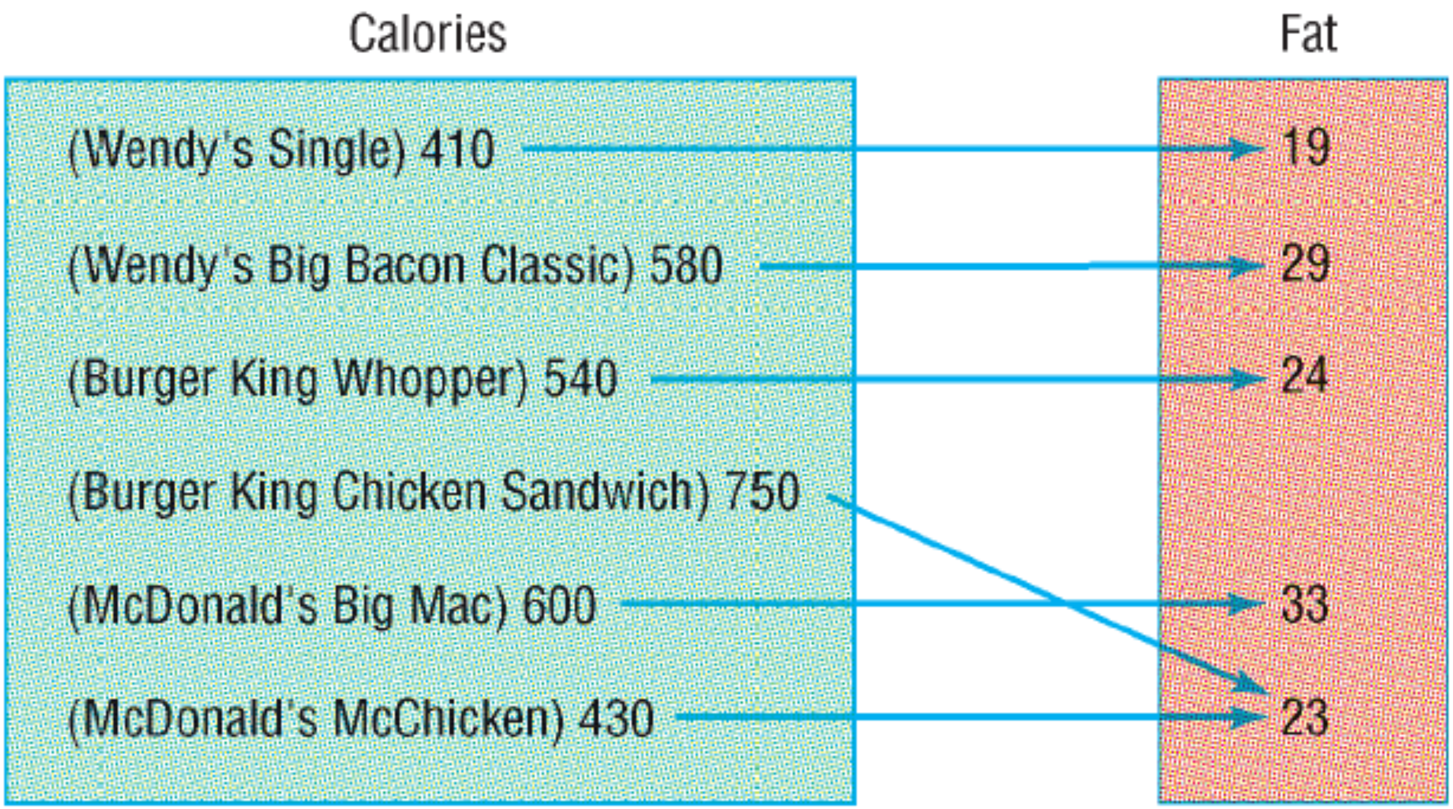
Determine if the following relations represent functions. If the relation is a function, then state its domain and range.



Yes, it is a function. The domain is {No High School Diploma, High School Diploma, Some College, College Graduate}. The range is {3.4%, 5.4%, 5.9%, 7.7%}.

EXAMPLE

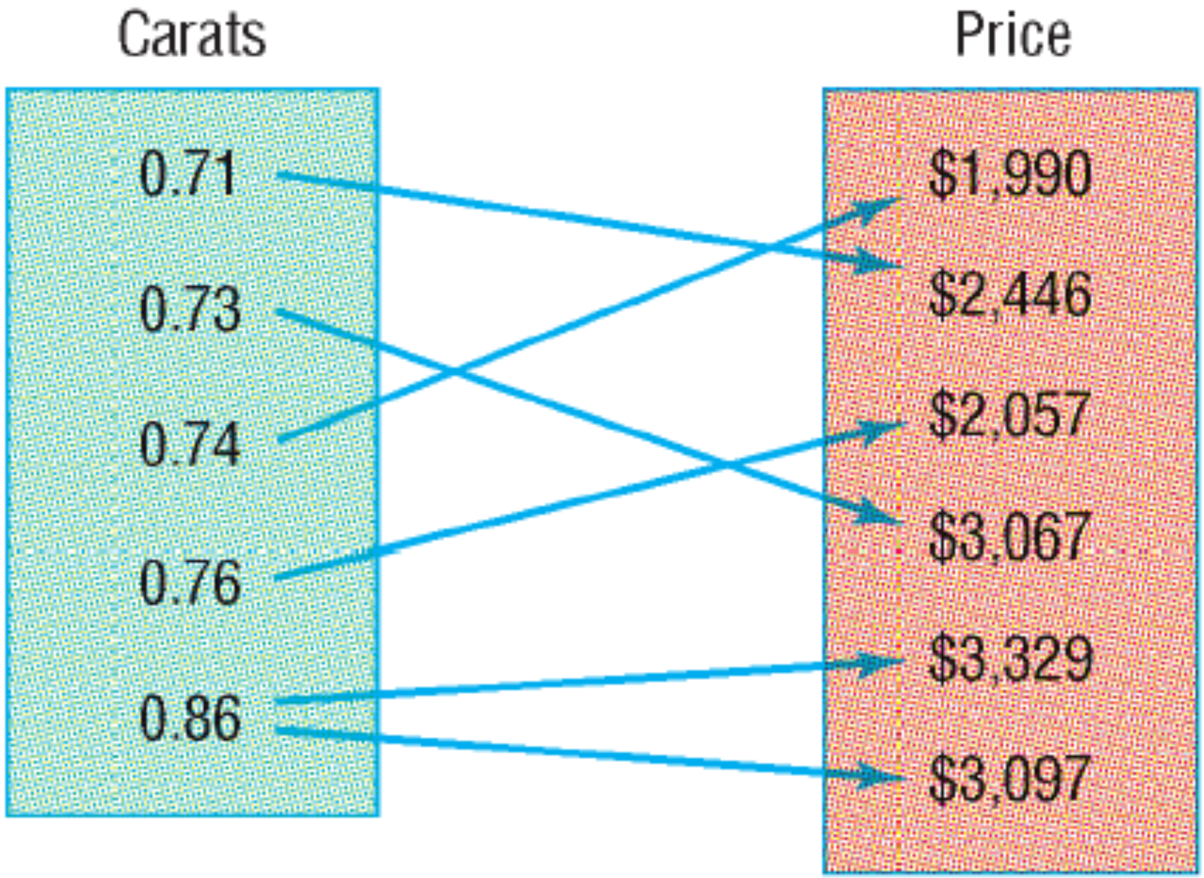
Determining Whether a Relation Represents a Function



Yes, it is a function. The domain is {410, 430, 540, 580, 600, 750}. The range is {19, 23, 24, 29, 33}. Note that it is okay for more than one element in the domain to correspond to the same element in the range.

EXAMPLE

Determining Whether a Relation Represents a Function



No, not a function. Each element in the domain does not correspond to exactly one element in the range (0.86 has two prices assigned to it).

EXAMPLE

Determining Whether a Relation Represents a Function

Determine whether each relation represents a function. If it is a function, state the domain and range.

a) $\{(2, 3), (4, 1), (3, -2), (2, -1)\}$

No, it is not a function. The element 2 is assigned to both 3 and –1.

b) $\{(-2, 3), (4, 1), (3, -2), (2, -1)\}$

Yes, it is a function because no ordered pairs have the same first element and different second elements.

c) $\{(4, 3), (3, 3), (4, -3), (2, 1)\}$

No, it is not a function. The element 4 is assigned to both 3 and –3.

EXAMPLE

Determining Whether a Relation Represents a Function

Determine if the equation $y = -\frac{1}{2}x - 3$ defines y as a function of x.

Yes, this is a function since for any input x, when you multiply by –1/2 and then subtract 3, you would only get one output y.

EXAMPLE

Determining Whether a Relation Represents a Function

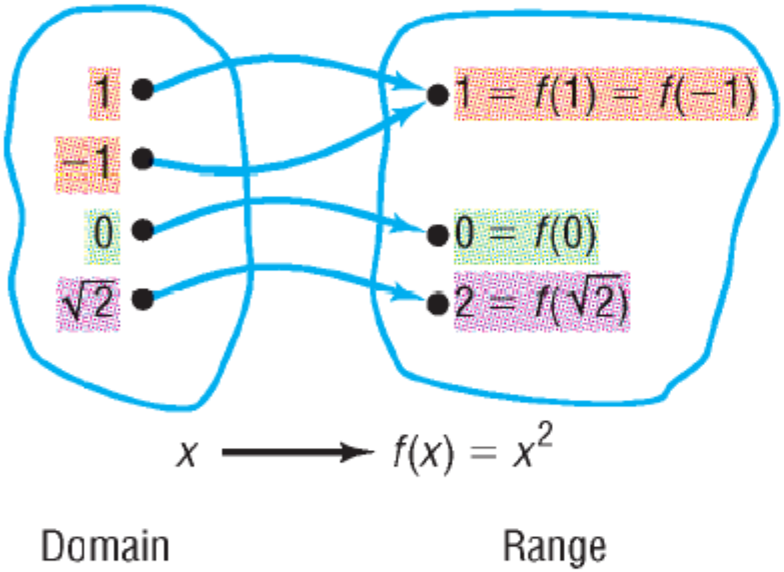
Determine if the equation $x = y^2 + 1$ defines y as a function of x.

$$x = y^2 + 1 \qquad \text{Solve for } y$$

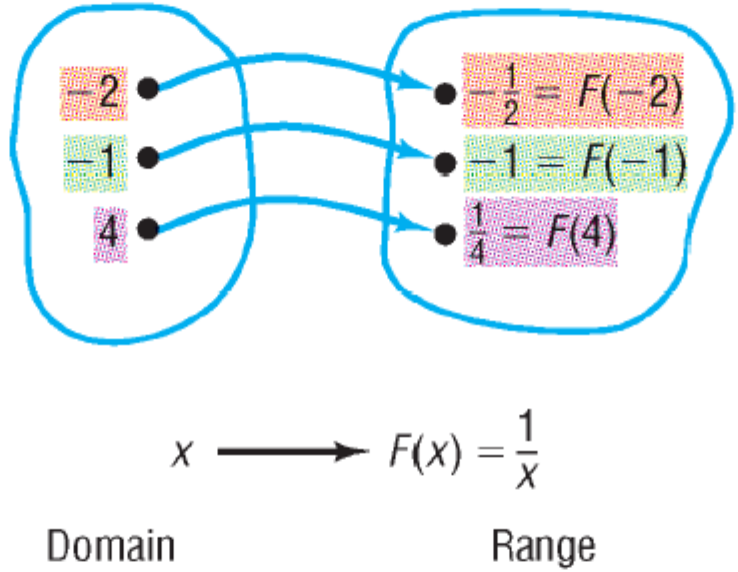
$$y^2 = x - 1 \qquad y = \pm\sqrt{x-1}$$

No, this it not a function since for values of x greater than 1, you would have two outputs for y.

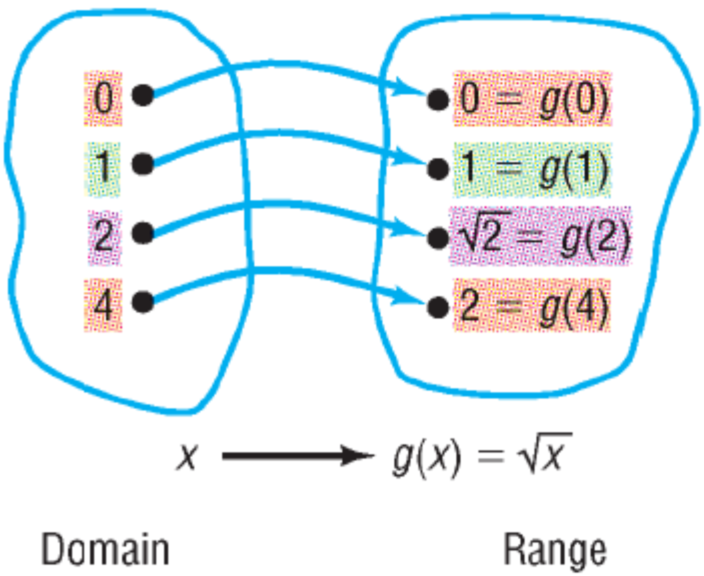
2 Find the Value of a Function



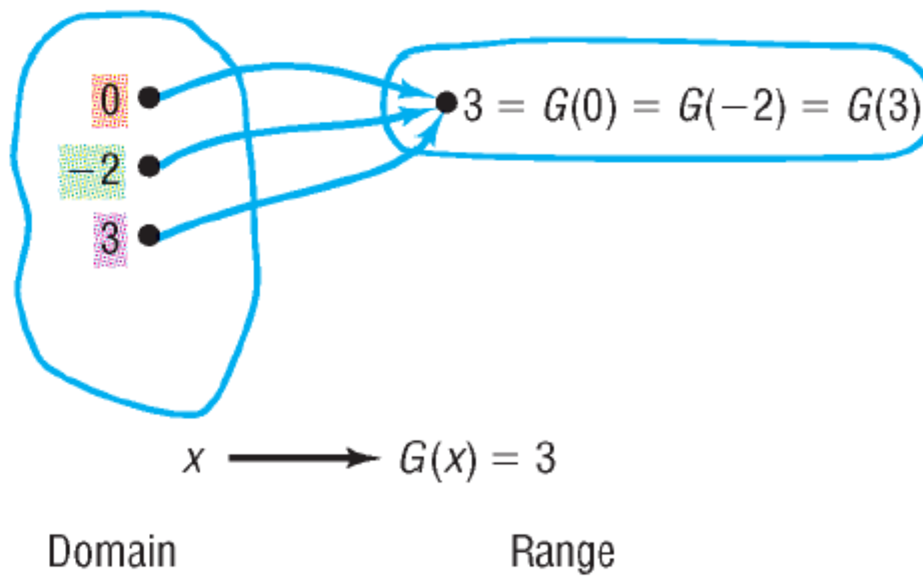
(a) $f(x) = x^2$



(b) $F(x) = \frac{1}{x}$

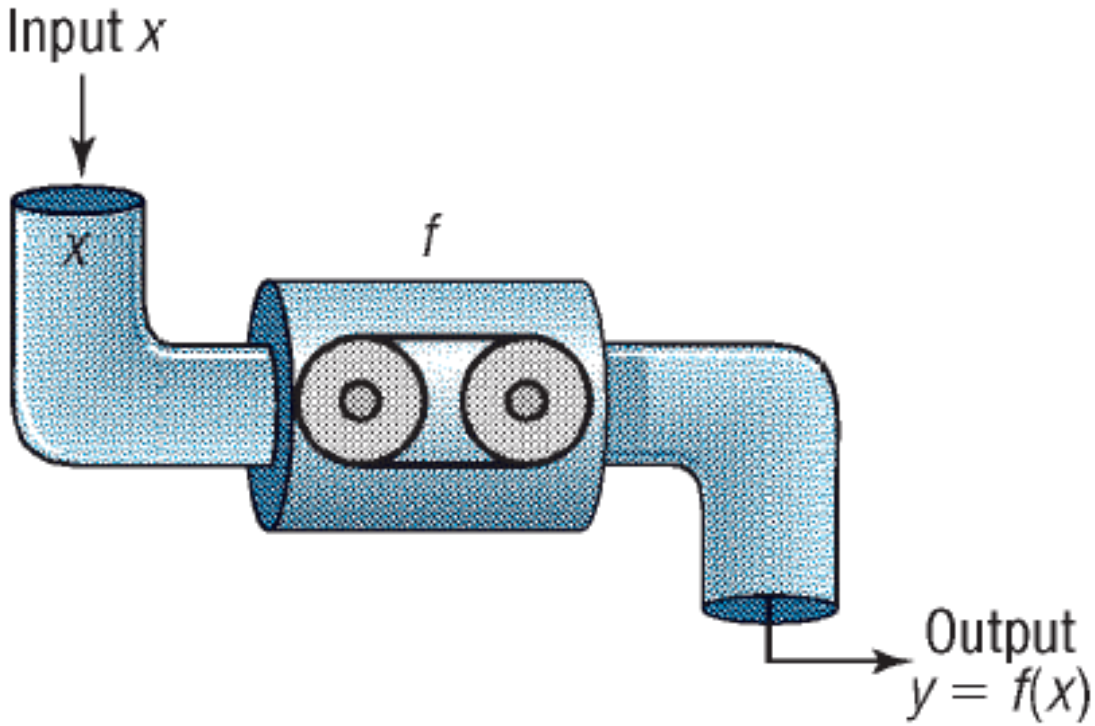


(c) $g(x) = \sqrt{x}$



(d) $G(x) = 3$

FUNCTION MACHINE



1. It only accepts numbers from the domain of the function.
2. For each input, there is exactly one output (which may be repeated for different inputs).

EXAMPLE

Finding Values of a Function

For the function f defined by $f\left(x\right)=-3x^2+2x$, evaluate:

(a) $f(3)$ (b) $f(x)+f(3)$ (c) $3f(x)$ (d) $f(-x)$

(e) $-f(x)$ (f) $f(3x)$ (g) $f(x+3)$ (h) $\dfrac{f(x+h)-f(x)}{h} \quad h \neq 0$

$$a) \quad f\left(3\right)=-3\left(3\right)^2+2\left(3\right)=-21$$

$$b) \quad f(x)+f\left(3\right)=-3x^2+2x+\left(-3\left(3\right)^2+2\left(3\right)\right)=-3x^2+2x-21$$

$$c) \quad 3f\left(x\right)=3\left(-3x^2+2x\right)=-9x^2+6x$$

$$d) \quad f\left(-x\right)=-3\left(-x\right)^2+2\left(-x\right)=-3x^2-2x$$

EXAMPLE

Finding Values of a Function

For the function f defined by $f(x) = -3x^2 + 2x$, evaluate:

(a) $f(3)$ (b) $f(x) + f(3)$ (c) $3f(x)$ (d) $f(-x)$

(e) $-f(x)$ (f) $f(3x)$ (g) $f(x+3)$ (h) $\dfrac{f(x+h) - f(x)}{h}$ $h \neq 0$

$$e) \quad -f(x) = -\left(-3x^2 + 2x\right) = 3x^2 - 2x$$

$$f) \quad f(3x) = -3(3x)^2 + 2(3x) = -27x^2 + 6x$$

$$g) \quad f(x+3) = -3(x+3)^2 + 2(x+3) = -3\left(x^2 + 6x + 9\right) + 2x + 6$$

$$= -3x^2 - 16x - 21$$

EXAMPLE

Finding Values of a Function

For the function f defined by $f(x) = -3x^2 + 2x$, evaluate:

(a) $f(3)$ (b) $f(x) + f(3)$ (c) $3f(x)$ (d) $f(-x)$

(e) $-f(x)$ (f) $f(3x)$ (g) $f(x+3)$ (h) $\dfrac{f(x+h) - f(x)}{h}$ $h \neq 0$

$$h)\quad \frac{f(x+h)-f(x)}{h} = \frac{\left[-3(x+h)^2 + 2(x+h) - \left(-3x^2 + 2x\right)\right]}{h}$$

$$= \frac{-3\left(x^2 + 2xh + h^2\right) + 2x + 2h + 3x^2 - 2x}{h} = \frac{-3x^2 - 6xh - 3h^2 + 2h + 3x^2}{h}$$

$$= \frac{h(-6x - 3h + 2)}{h} = -6x - 3h + 2$$

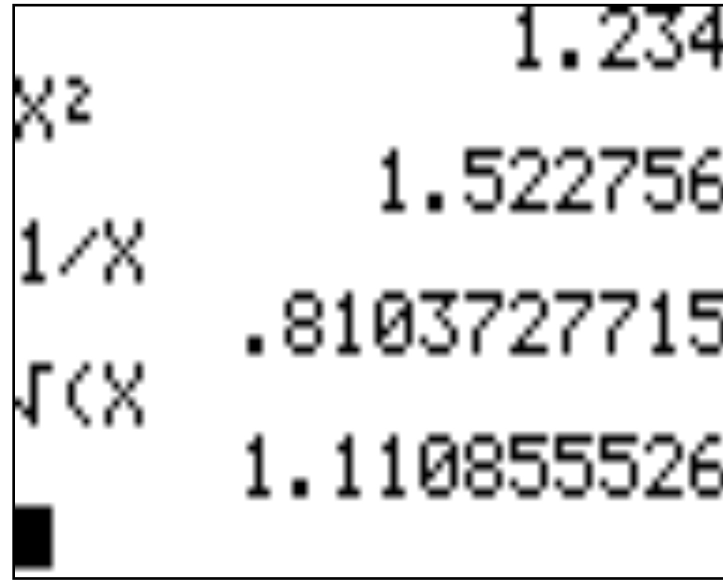
EXAMPLE

Finding Values of a Function on a Calculator

(a) $f(x) = x^2;\quad f(1.234) =$

(b) $F(x) = \dfrac{1}{x};\quad F(1.234) =$

(c) $g(x) = \sqrt{x};\quad g(1.234) =$

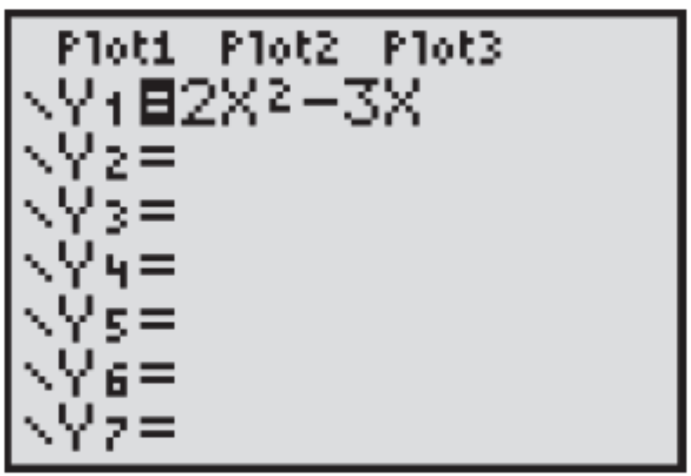


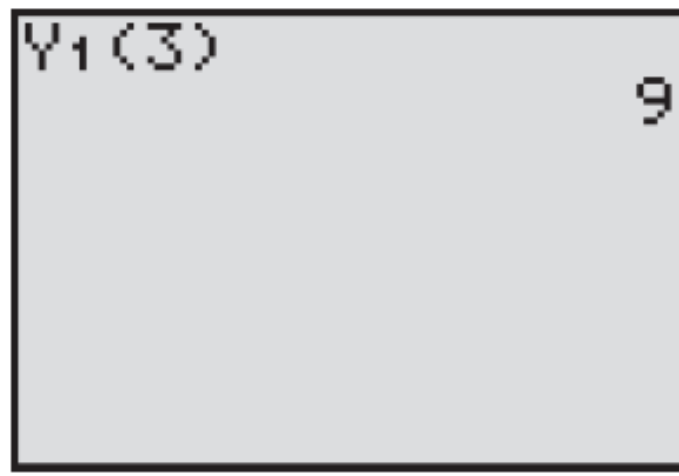
COMMENT

Graphing calculators can be used to evaluate any function that you wish

Figure below shows the result obtained on a TI-84 graphing calculator with

The function to be evaluated, $f(x) = 2x^2 - 3x$, in Y_1.





Implicit Form of a Function

Implicit Form	Explicit Form
$3x + y = 5$	$y = f(x) = -3x + 5$
$x^2 - y = 6$	$y = f(x) = x^2 - 6$
$xy = 4$	$y = f(x) = \dfrac{4}{x}$

Summary

Important Facts About Functions

(a) For each x in the domain of f, there is exactly one image $f(x)$ in the range; however, an element in the range can result from more than one x in the domain.

(b) f is the symbol that we use to denote the function. It is symbolic of the equation that we use to get from an x in the domain to $f(x)$ in the range.

(c) If $y = f(x)$, then x is called the independent variable or argument of f, and y is called the dependent variable or the value of f at x.

3 Find the Domain of a Function Defined by an Equation

EXAMPLE

Finding the Domain of a Function

Find the domain of each of the following functions:

(a) $f(x) = \dfrac{x+4}{x^2 - 2x - 3}$

The denominator $\neq 0$ so find values where $x^2 - 2x - 3 = 0$.

$(x-3)(x+1) = 0$

$\{x \mid x \neq 3, x \neq -1\}$

(b) $g(x) = x^2 - 9$

The set of all real numbers

(c) $h(x) = \sqrt{3 - 2x}$

Only nonnegative numbers have real square roots so $3 - 2x \geq 0$.

$\left\{x \,\middle|\, x \leq \dfrac{3}{2}\right\}$ or $\left(-\infty, \dfrac{3}{2}\right]$

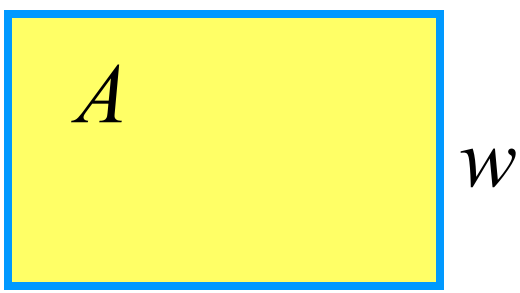
Finding the Domain of a Function Defined by an Equation

1. Start with the domain as the set of real numbers.
2. If the equation has a denominator, exclude any numbers that give a zero denominator.
3. If the equation has a radical of even index, exclude any numbers that cause the expression inside the radical to be negative.

EXAMPLE

Finding the Domain in an Application

A rectangular garden has a perimeter of 100 feet. Express the area A of the garden as a function of the width w. Find the domain.



$$A(w) = w(50 - w)$$

Domain: $0 < w < 50$

4 Form the Sum, Difference, Product, and Quotient of Two Functions

If f and g are functions:

The **sum** $\boldsymbol{f + g}$ is the function defined by

$$\boxed{(f + g)(x) = f(x) + g(x)}$$

The **difference** $\boldsymbol{f - g}$ is the function defined by

$$\boxed{(f - g)(x) = f(x) - g(x)}$$

The product $\boldsymbol{f \cdot g}$ is the function defined by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

The **quotient** $\dfrac{\boldsymbol{f}}{\boldsymbol{g}}$ is the function defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \neq 0$$

EXAMPLE

Operations on Functions

For the functions $f(x) = 2x^2 + 3 \quad g(x) = 4x^3 + 1$

find the following:

(a) $(f + g)(x) = 2x^2 + 3 + 4x^3 + 1 = 4x^3 + 2x^2 + 4$

(b) $(f - g)(x) = 2x^2 + 3 - \left(4x^3 + 1\right) = -4x^3 + 2x^2 + 2$

(c) $(f \cdot g)(x) = (2x^2 + 3)(4x^3 + 1) = 8x^5 + 2x^2 + 12x^3 + 3$

(d) $\left(\frac{f}{g}\right)(x) = \dfrac{2x^2 + 3}{4x^3 + 1}$

SUMMARY

Function	A relation between two sets of real numbers so that each number x in the first set, the domain, has corresponding to it exactly one number y in the second set. A set of ordered pairs (x, y) or $(x, f(x))$ in which no first element is paired with two different second elements. The range is the set of y values of the function that are the images of the x values in the domain. A function f may be defined implicitly by an equation involving x and y or explicitly by writing $y = f(x)$.
Unspecified domain	If a function f is defined by an equation and no domain is specified, then the domain will be taken to be the largest set of real numbers for which the equation defines a real number.
Function notation	$y = f(x)$ f is a symbol for the function. x is the independent variable or argument. y is the dependent variable. $f(x)$ is the value of the function at x, or the image of x.