Section 13.5

The Binomial Theorem

$$(x + a)^1 = x + a$$

$$(x + a)^2 = x^2 + 2ax + a^2$$

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$

$$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$$

$$(x + a)^n = x^n + \underline{\qquad}\,ax^{n-1} + \underline{\qquad}\,a^2x^{n-2} + \cdots + \underline{\qquad}\,a^{n-1}x + a^n$$

1 Evaluate $\binom{n}{j}$

DEFINITION

If j and n are integers with $0 \leq j \leq n$, the symbol $\binom{n}{j}$ is defined as

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} \tag{1}$$

EXAMPLE Evaluating $\binom{n}{j}$

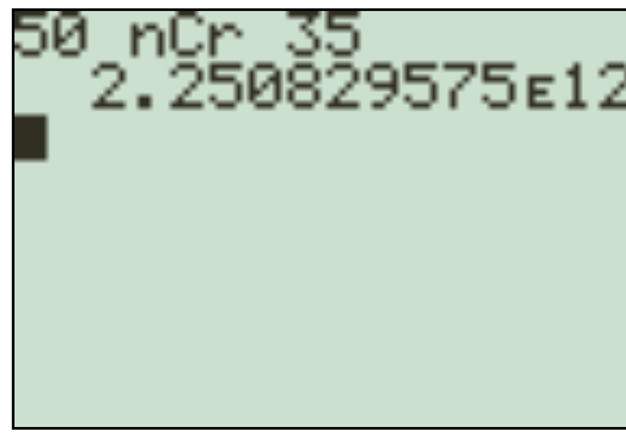
$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

Find:

(a) $\binom{4}{1} = \frac{4!}{1!(4-1)!} = \frac{4!}{1!3!} = \frac{4\cdot 3\cdot 2\cdot 1}{1(3\cdot 2\cdot 1)} = 4$

(b) $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1(2\cdot 1)} = 10$

(c) $\binom{7}{5} = \frac{7!}{5!(7-5)!} = \frac{7!}{5!2!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1(2\cdot 1)} = 21$



(d) $\binom{50}{35} = \frac{50!}{35!(50-35)!} = \frac{50!}{35!15!}$ Use a calculator.

Four Useful Formulas

$$\binom{n}{0} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{n-1} = n \qquad \binom{n}{n} = 1$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$j = 0$ $j = 1$ $j = 2$ $j = 3$ $j = 4$ $j = 5$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Pascal triangle

$n = 5 \rightarrow$ 1 5 10 10 5 1

$n = 6 \rightarrow$ 1 6 15 20 15 6 1

2 Use the Binomial Theorem

Binomial Theorem

Let x and a be real numbers. For any positive integer n, we have

$$(x+a)^n = \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n$$
$$= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} a^j \qquad \textbf{(2)}$$

EXAMPLE **Expanding a Binomial**

Use the Binomial Theorem to expand $(3y+2)^4$

$$=\binom{4}{0}(3y)^4+\binom{4}{1}(2)(3y)^3+\binom{4}{2}(2)^2(3y)^2+\binom{4}{3}(2)^3(3y)+\binom{4}{4}(2)^4$$

$$=81y^4+4(2)(27y^3)+6(4)(9y^2)+4(8)(3y)+16$$

$$=81y^4+216y^3+216y^2+96y+16$$

$$(x+a)^n=\binom{n}{0}x^n+\binom{n}{1}ax^{n-1}+\cdots+\binom{n}{j}a^jx^{n-j}+\cdots+\binom{n}{n}a^n$$

EXAMPLE

Finding a Particular Coefficient in a Binomial Expansion

Find the coefficient of x^7 in the expansion of $(3x-1)^{10}$

The x^7 term would be $\binom{10}{3}(3x)^7(-1)^3 = \dfrac{10!}{3!7!}(2187x^7)(-1)$

$$= \frac{10\cdot 9\cdot 8\cdot 7!}{6\cdot 7!}(2187x^7)(-1) = -262{,}440x^7$$

Based on the expansion of $(x+a)^n$, the term containing x^j is

$$\binom{n}{n-j}a^{n-j}x^j$$

EXAMPLE

Finding a Particular Term in a Binomial Expansion

Find the seventh term in the expansion of $(y+3)^8$

The 7th term would be $\binom{8}{6}(y)^2(3)^6 = \frac{8!}{6!2!}(y^2)(3)^6$

$$= \frac{8 \cdot 7 \cdot 6!}{6!2!}(729y^2) = 28(729y^2) = 20{,}412y^2$$

Based on the expansion of $(x+a)^n$, the term containing x^j is

$$\binom{n}{n-j} a^{n-j} x^j$$

THEOREM

If n and j are integers with $1 \leq j \leq n$, then

$$\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j} \tag{4}$$

EXAMPLE Expanding a Binomial

Use the Binomial Theorem to expand $(x-3)^6$

$$=\binom{6}{0}x^6+\binom{6}{1}(-3)x^5+\binom{6}{2}(-3)^2x^4+\binom{6}{3}(-3)^3x^3+\binom{6}{4}(-3)^4x^2+\binom{6}{5}(-3)^5x+\binom{6}{6}(-3)^6$$

$$=x^6+6\cdot(-3)x^5+15(9)x^4+20(-27)x^3+15(81)x^2+6(-243)x+729$$

$$=x^6-18x^5+135x^4-540x^3+1215x^2-1458x+729$$

$$(x+a)^n=\binom{n}{0}x^n+\binom{n}{1}ax^{n-1}+\cdots+\binom{n}{j}a^jx^{n-j}+\cdots+\binom{n}{n}a^n$$