Section 5.5

The Real Zeros of a Polynomial Function

1 Use the Remainder and Factor Theorems

Theorem

Division Algorithm for Polynomials

If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is a polynomial whose degree is greater than zero, then there are unique polynomial functions $q(x)$ and $r(x)$ such that

$$\boxed{\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad \underset{\text{dividend}}{f(x)} = \underset{\text{quotient}}{q(x)}\,\underset{\text{divisor}}{g(x)} + \underset{\text{remainder}}{r(x)} \qquad \textbf{(1)}}$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

Remainder Theorem

Let f be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

EXAMPLE Using the Remainder Theorem

Find the remainder if $f(x) = x^3 + 3x^2 + 2x - 1$ is divided by

(a) $x + 2$ (b) $x - 1$

(a) $f(-2) = (-2)^3 + 3(-2)^2 + 2(-2) - 1 = -1$

The remainder is –1.

(b) $f(1) = (1)^3 + 3(1)^2 + 2(1) - 1 = 5$

The remainder is 5.

Factor Theorem

Let f be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$.

1. If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
2. If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

EXAMPLE Using the Factor Theorem

Use the Factor Theorem to determine whether the function $f(x) = -2x^3 - x^2 + 4x + 3$ has the factor

(a) $x+1$ (b) $x-1$

(a) $f(-1) = -2(-1)^3 - (-1)^2 + 4(-1) + 3 = 0$

By the factor theorem, $x + 1$ is a factor of $f(x)$.

(b) $f(1) = -2(1)^3 - (1)^2 + 4(1) + 3 = 4$

By the factor theorem, $x - 1$ is not a factor of $f(x)$.

Theorem

Number of Real Zeros

A polynomial function cannot have more real zeros than its degree.

2 Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function

Theorem

Rational Zeros Theorem

Let f be a polynomial function of degree 1 or higher of the form

$$\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0, a_0 \neq 0}$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of f, then p must be a factor of a_0, and q must be a factor of a_n.

EXAMPLE Listing Potential Rational Zeros

List the potential rational zeros of

$$f(x) = 3x^3 + 8x^2 - 7x - 12$$

$$\frac{\text{Factors of the constant}}{\text{Factors of the leading coefficient}}$$

$$p: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$$

$$q: \pm 1, \ \pm 3$$

$$\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{3}, \pm \frac{2}{3}, \ \pm \frac{4}{3}$$

3 Find the Real Zeros of a Polynomial Function

EXAMPLE

How to Find the Real Zeros of a Polynomial Function

Find the real zeros of the polynomial function $f(x) = 3x^3 + 8x^2 - 7x - 12$. Write f in factored form.

Step 1: Use the degree of the polynomial to determine the maximum number of zeros.

Since f is a polynomial of degree 3, there are at most three real zeros.

Step 2: If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros. Use the Factor Theorem to determine if each potential rational zero is a zero. If it is, use synthetic division or long division to factor the polynomial function. Repeat Step 2 until all the zeros of the polynomial function have been identified and the polynomial function is completely factored.

$$\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$$

$$\begin{array}{r|rrrr} -1 & 3 & 8 & -7 & -12 \\ & & -3 & -5 & 12 \\ \hline & 3 & 5 & -12 & 0 \end{array}$$

$$\begin{aligned} f(x) &= (x+1)\left(3x^2 + 5x - 12\right) \\ &= (x+1)(x+3)(3x-4) \end{aligned}$$

$$(x+1)(x+3)(3x-4) = 0$$

$$x = -1 \text{ or } x = -3 \text{ or } x = \frac{4}{3}$$

SUMMARY Steps for Finding the Real Zeros of a Polynomial Function

STEP 1: Use the degree of the polynomial to determine the maximum number of real zeros.

STEP 2: (a) If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially could be zeros.

(b) Use substitution, synthetic division, or long division to test each potential rational zero. Each time that a zero (and thus a factor) is found, repeat Step 2 on the depressed equation.

In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

EXAMPLE Finding the Real Zeros of a Polynomial Function

Find the real zeros of $f(x) = 2x^4 + 13x^3 + 29x^2 + 27x + 9.$
Write f in factored form.

Step 1: There are at most 4 real zeros.

Step 2: The potential rational zeros are $\frac{p}{q}: \pm 1,\ \pm\frac{1}{2},\ \pm 3,\ \pm\frac{3}{2},\ \pm 9,\ \pm\frac{9}{2}.$

$$\begin{array}{r|rrrrr} -1 & 2 & 13 & 29 & 27 & 9 \\ & & -2 & -11 & -18 & -9 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 2 & 11 & 18 & 9 \\ & & -2 & -9 & -9 \\ \hline & 2 & 9 & 9 & 0 \end{array}$$

$$f(x) = (x+1)\left(2x^3 + 11x^2 + 18x + 9\right)$$

$$f(x) = (x+1)^2\left(2x^2 + 9x + 9\right)$$

$$f(x) = (x+1)^2(x+3)(2x+3)$$

$$(x+1)^2(x+3)(2x+3) = 0$$

$$x = -1 \text{ or } x = -3 \text{ or } x = -\frac{3}{2}$$

4 Solve Polynomial Equations

EXAMPLE Solving a Polynomial Equation

Solve the equation: $2x^4+13x^3+29x^2+27x+9=0$

$$(x+1)^2(2x+3)(x+3)=0$$

$$\left\{-1,-\frac{3}{2},-3\right\}$$

THEOREM

Every polynomial function with real coefficients can be uniquely factored into a product of linear factors and/or irreducible (prime) quadratic factors.

THEOREM

A polynomial function of odd degree that has real coefficients has at least one real zero.

5 Use the Theorem for Bounds on Zeros

BOUND

$$-M \leq \text{any real zero of } f \leq M$$

Theorem

Bounds on Zeros

Let f denote a polynomial function whose leading coefficient is 1.

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

A bound M on the zeros of f is the smaller of the two numbers

$$\boxed{\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}, \quad 1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} \qquad \textbf{(4)}}$$

where Max { } means "choose the largest entry in { }."

EXAMPLE

Using the Theorem for Finding Bounds on Zeros

Find a bound on the real zeros of each polynomial function.

(a) $f(x) = x^5 + 3x^3 - 9x^2 + 5$ (b) $g(x) = 4x^5 - 2x^3 + 2x^2 + 1$

(a) The leading coefficient of f is 1.

$$f(x) = x^5 + 3x^3 - 9x^2 + 5$$

$$\begin{aligned}\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} &= \text{Max}\{1, |5| + |0| + |-9| + |3| + |0|\} \\ &= \text{Max}\{1, 17\} = 17\end{aligned}$$

$$\begin{aligned}1 + \text{Max}\{|a_0|, |a_1|, \cdots, |a_{n-1}|\} &= 1 + \text{Max}\{|5|, |0|, |-9|, |3|, |0|\} \\ &= 1 + 9 = 10\end{aligned}$$

The smaller of the two numbers, 10, is the bound. Every real zero of f lies between -10 and 10.

EXAMPLE **Using the Theorem for Finding Bounds on Zeros**

Find a bound on the real zeros of each polynomial function.

(a) $f(x) = x^5 + 3x^3 - 9x^2 + 5$ (b) $g(x) = 4x^5 - 2x^3 + 2x^2 + 1$

(b) First write g so that it is the product of a constant times a polynomial whose leading coefficient is 1 by factoring out the leading coefficient of g, 4.

$$g(x) = 4x^5 - 2x^3 + 2x^2 + 1 = 4\left(x^5 - \frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{4}\right)$$

$$\begin{aligned}\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} &= \text{Max}\left\{1, \left|\frac{1}{4}\right| + |0| + \left|\frac{1}{2}\right| + \left|-\frac{1}{2}\right| + |0|\right\} \\ &= \text{Max}\left\{1, \frac{5}{4}\right\} = \frac{5}{4}\end{aligned}$$

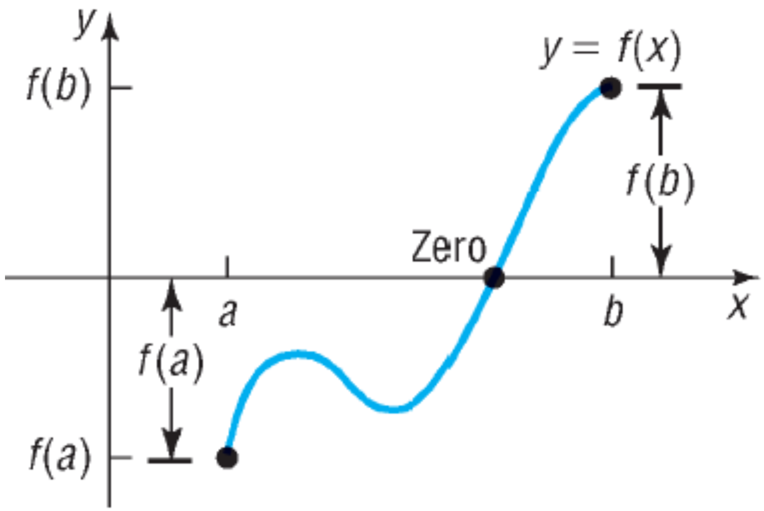
$$\begin{aligned}1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} &= 1 + \text{Max}\left\{\left|\frac{1}{4}\right|, |0|, \left|\frac{1}{2}\right|, \left|-\frac{1}{2}\right|, |0|\right\} \\ &= 1 + \frac{1}{2} = \frac{3}{2}\end{aligned}$$

The smaller of the two numbers, $\frac{5}{4}$, is the bound. Every real zero of g lies between $-\frac{5}{4}$ and $\frac{5}{4}$.

6 Use the Intermediate Value Theorem

Intermediate Value Theorem

Let f denote a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite sign, there is at least one real zero of f between a and b.



If $f(a) < 0$ and $f(b) > 0$ and if f is continuous, there is at least one zero between a and b.

EXAMPLE

Using the Intermediate Value Theorem to Locate a Real Zero

Show that $f(x) = x^5 - x^3 - 1$ has a zero between 1 and 2.

$$f(1) = -1 \quad \text{and} \quad f(2) = 23$$

Because $f(1) < 0$ and $f(2) > 0$, it follows from the Intermediate Value Theorem that the polynomial function f has at least one zero between 1 and 2.

Approximating the Zeros of a Polynomial Function

Step 1: Find two consecutive integers a and $a + 1$ such that f has a zero between them.

Step 2: Divide the interval $[a, a + 1]$ into 10 equal subintervals.

Step 3: Evaluate f at each endpoint of the subintervals until the Intermediate Value Theorem applies; this interval then contains a zero.

Step 4: Repeat the process starting at Step 2 until the desired accuracy is achieved.

EXAMPLE

Approximating a Real Zero of a Polynomial Function

Find the positive zero of $f(x) = x^5 - x^3 - 1$ correct to two decimal places.

$$f(1.0) = -1 \qquad f(1.2) = -0.23968$$
$$f(1.1) = -0.72049 \qquad f(1.3) = 0.51593$$

$$f(1.20) = -0.23968 \qquad f(1.23) \approx -0.0455613$$
$$f(1.21) \approx -0.1778185 \qquad f(1.24) \approx 0.025001$$
$$f(1.22) \approx -0.1131398$$