Section 11.7

Plane Curves and Parametric Equations

Let $x = f(t)$ and $y = g(t)$, where f and g are two functions whose common domain is some interval I. The collection of points defined by

$$(x, y) = (f(t), g(t))$$

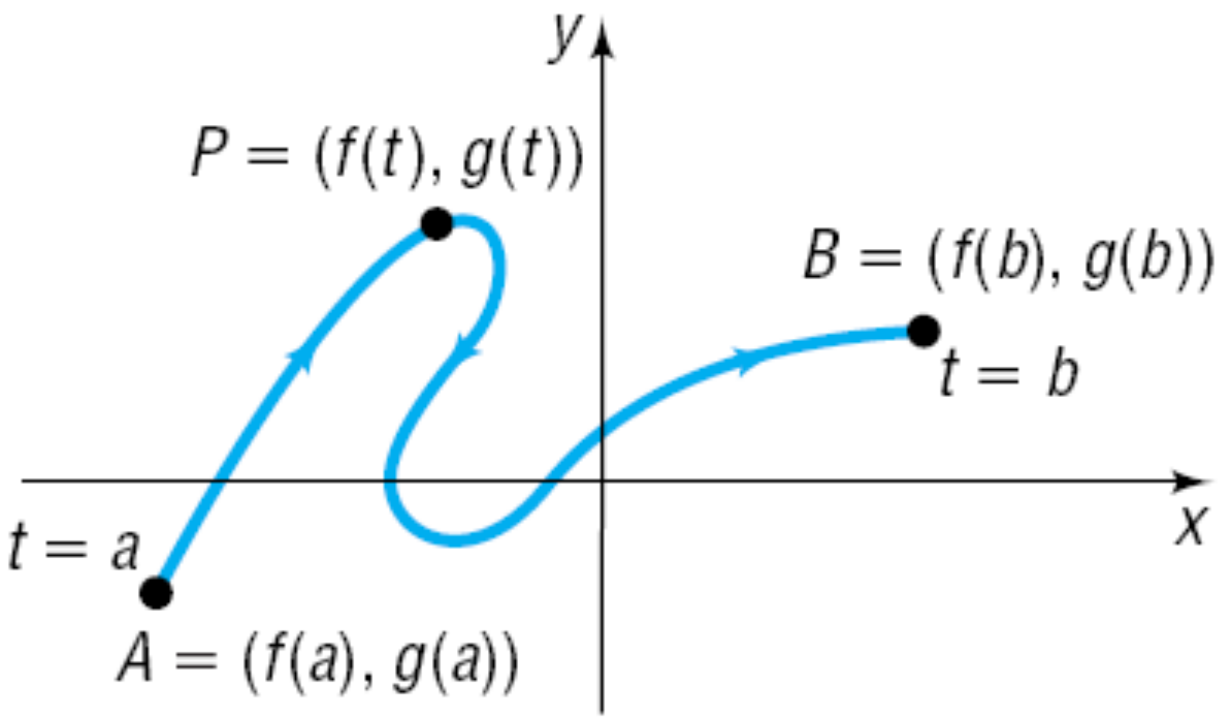
is called a **plane curve.** The equations

$$\boxed{x = f(t) \qquad y = g(t)}$$

where t is in I, are called **parametric equations** of the curve. The variable t is called a **parameter.**

1 Graph Parametric Equations

$$x = f(t), \qquad y = g(t), \qquad a \le t \le b$$



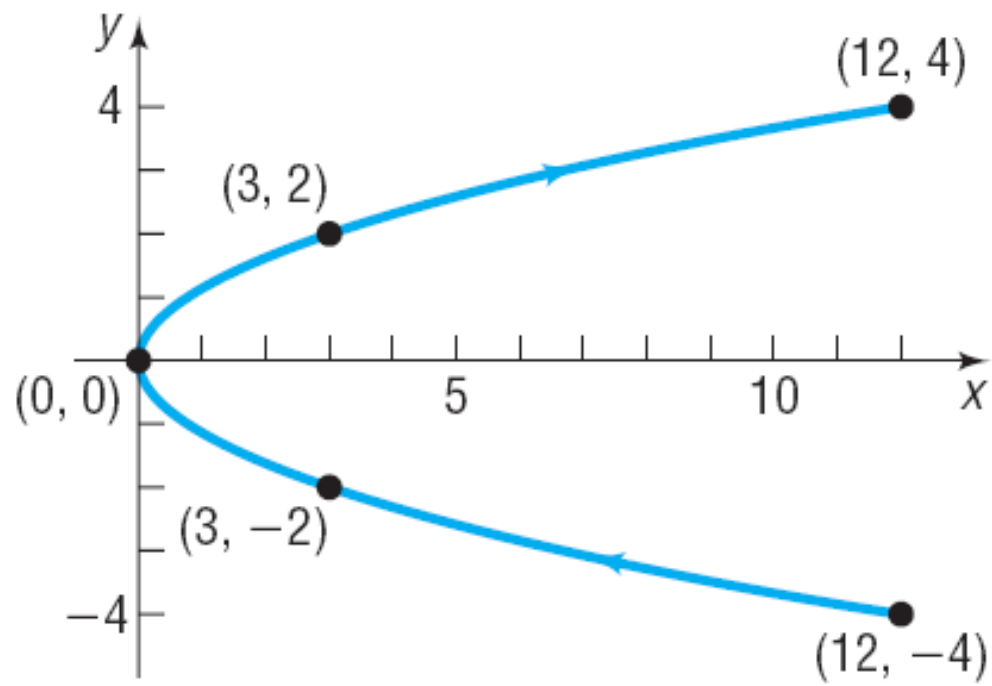
EXAMPLE Graphing a Curve Defined by Parametric Equations

Graph the curve defined by the parametric equations

$$x = 3t^2, \qquad y = 2t, \qquad -2 \le t \le 2$$

For each number t, $-2 \le t \le 2$, there corresponds a number x and a number y. For example, when $t = -2$, then $x = 3(-2)^2 = 12$ and $y = 2(-2) = -4$. When $t = 0$, then $x = 0$ and $y = 0$. Set up a table listing various choices of the parameter t and the corresponding values for x and y, as shown in Table 6. Plotting these points and connecting them with a smooth curve leads to Figure 56. The arrows in Figure 56 are used to indicate the orientation.

t	x	y	(x, y)
−2	12	−4	(12, −4)
−1	3	−2	(3, −2)
0	0	0	(0, 0)
1	3	2	(3, 2)
2	12	4	(12, 4)



Exploration

Graph the following parametric equations using a graphing utility with $X\text{min} = 0, X\text{max} = 15$, $Y\text{min} = -5, Y\text{max} = 5$, and $T\text{step} = 0.1$:

1. $x = \dfrac{3t^2}{4}, y = t, -4 \le t \le 4$
2. $x = 3t^2 + 12t + 12, y = 2t + 4, -4 \le t \le 0$
3. $x = 3t^{\frac{2}{3}}, y = 2\sqrt[3]{t}, -8 \le t \le 8$

Compare these graphs to the graph in Figure 56. Conclude that parametric equations defining a curve are not unique; that is, different parametric equations can represent the same graph.

2 Find a Rectangular Equation for a Curve Defined Parametrically

EXAMPLE

Finding the Rectangular Equation of a Curve Defined Parametrically

Find the rectangular equation of the curve whose parametric equations are

$$x = a\cos t \qquad y = a\sin t$$

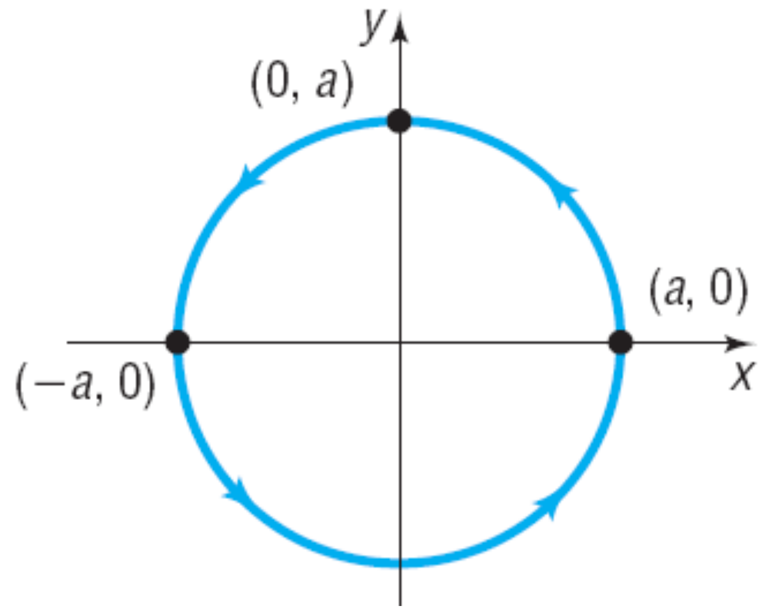
where $a > 0$ is a constant. By hand, graph this curve, indicating its orientation.

$$\cos t = \frac{x}{a} \qquad \sin t = \frac{y}{a}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

$$x^2 + y^2 = a^2$$



The curve is a circle with center at (0, 0) and radius a. As the parameter t increases, say from $t = 0$ [the point $(a, 0)$] to $t = \frac{\pi}{2}$ [the point $(0, a)$] to $t = \pi$ [the point $(-a, 0)$], we see that the corresponding points are traced in a counterclockwise direction around the circle. The orientation is as indicated in Figure 57.

EXAMPLE Describing Parametric Equations

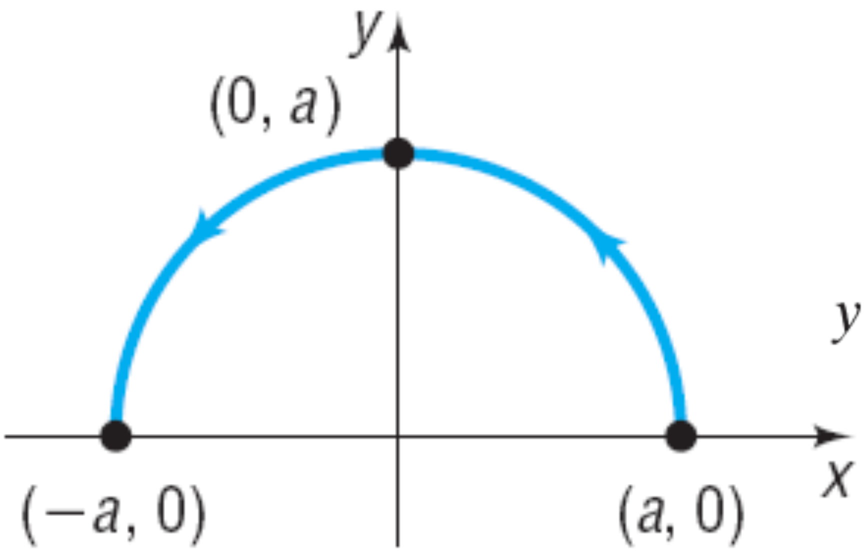
Find rectangular equations for the following curves defined by parametric equations. Graph each curve.

(a) $x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le \pi, \quad a > 0$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

$$x^2 + y^2 = a^2$$



$$y = \sqrt{a^2 - x^2} \qquad -a \le x \le a$$

The curve defined by these parametric equations lies on a circle, with radius a and center at $(0, 0)$. The curve begins at the point $(a, 0)$, $t = 0$; passes through the point $(0, a)$, $t = \frac{\pi}{2}$; and ends at the point $(-a, 0)$, $t = \pi$.

EXAMPLE Describing Parametric Equations

Find rectangular equations for the following curves defined by parametric equations. Graph each curve.

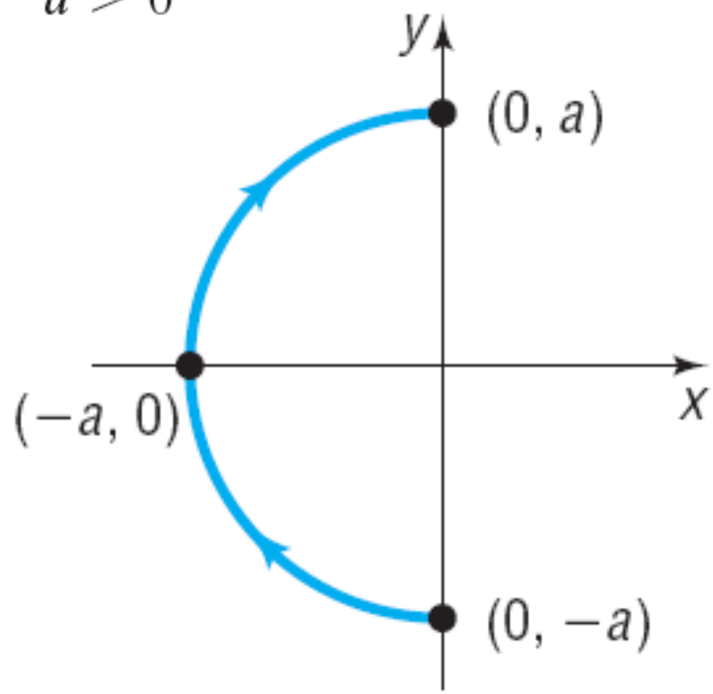
(b) $x = -a \sin t, \quad y = -a \cos t, \quad 0 \le t \le \pi, \quad a > 0$

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{x}{-a}\right)^2 + \left(\frac{y}{-a}\right)^2 = 1$$

$$x^2 + y^2 = a^2$$

$$x = -\sqrt{a^2 - y^2} \qquad -a \le y \le a$$

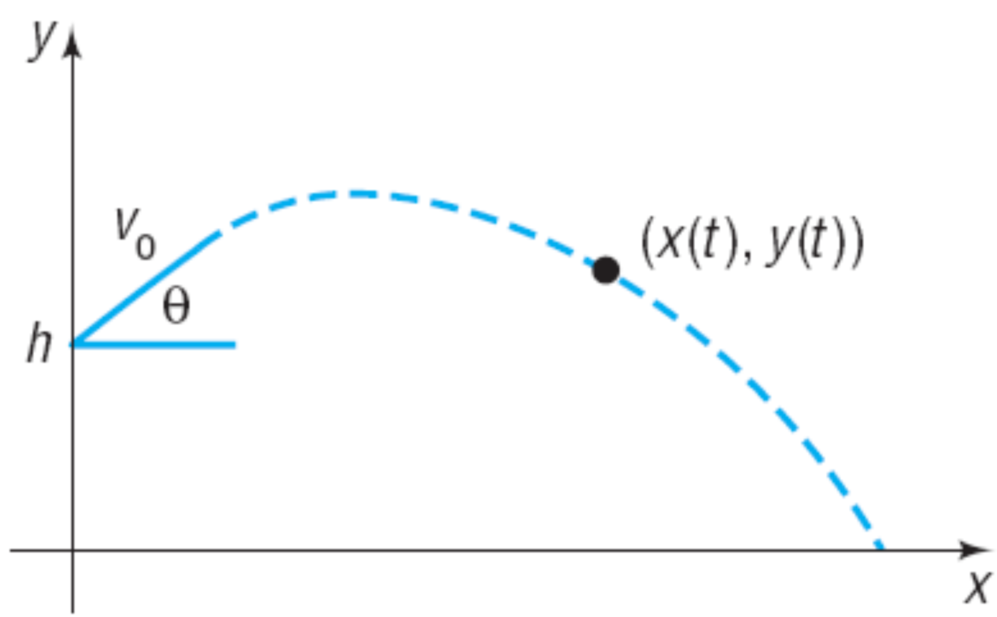


The curve defined by these parametric equations lies on a circle, with radius a and center at $(0, 0)$. The curve begins at the point $(0, -a)$, $t = 0$; passes through the point $(-a, 0)$, $t = \frac{\pi}{2}$; and ends at the point $(0, a)$, $t = \pi$.

3 Use Time as a Parameter in Parametric Equations

projectile motion

$$x = (v_0 \cos \theta)t \qquad y = -\frac{1}{2}gt^2 + (v_0 \sin \theta)t + h$$



EXAMPLE

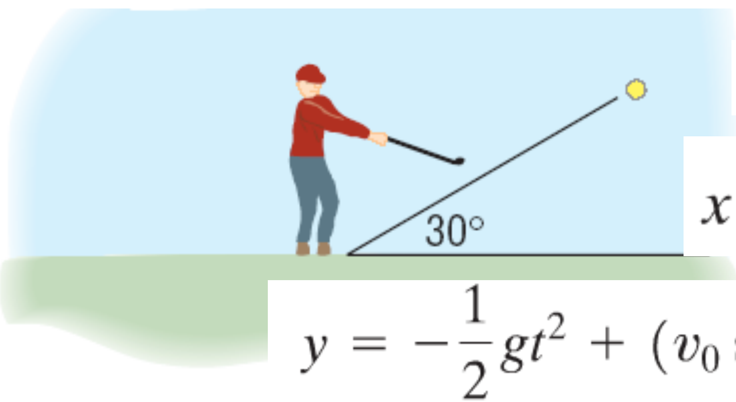
Projectile Motion

Suppose that Jim hit a golf ball with an initial velocity of 150 feet per second at an angle of 30° to the horizontal. See Figure 66.

(a) Find parametric equations that describe the position of the ball as a function of time.

(b) To determine the length of time that the ball was in the air, solve the equation $y = 0$.

$$-16t^2 + 75t = 0$$
$$t(-16t + 75) = 0$$
$$t = 0 \text{ sec} \quad \text{or} \quad t = \frac{75}{16} = 4.6875 \text{ sec}$$



The ball struck the ground after 4.6875 seconds.

$$x = (v_0 \cos\theta)t = (150 \cos 30°)t = 75\sqrt{3}\,t$$

$$y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + h = -\frac{1}{2}(32)t^2 + (150 \sin 30°)t + 0$$
$$= -16t^2 + 75t$$

EXAMPLE

Projectile Motion

Suppose that Jim hit a golf ball with an initial velocity of 150 feet per second at an angle of 30° to the horizontal. See Figure 66.

(a) Find parametric equations that describe the position of the ball as a function of time.

(b) How long is the golf ball in the air?

(c) Notice that the height y of the ball is a quadratic function of t, so the maximum height of the ball can be found by determining the vertex of $y = -16t^2 + 75t$. The value of t at the vertex is

$$t = \frac{-b}{2a} = \frac{-75}{-32} = 2.34375 \text{ sec}$$

The ball was at its maximum height after 2.34375 seconds. The maximum height of the ball is found by evaluating the function y at $t = 2.34375$ seconds.

$$\text{Maximum height} = -16(2.34375)^2 + 75(2.34375) \approx 87.89 \text{ feet}$$

$$x = 75\sqrt{3}\,t \qquad y = -16t^2 + 75t$$

EXAMPLE

Projectile Motion

Suppose that Jim hit a golf ball with an initial velocity of 150 feet per second at an angle of 30° to the horizontal. See Figure 66.

(a) Find parametric equations that describe the position of the ball as a function of time.

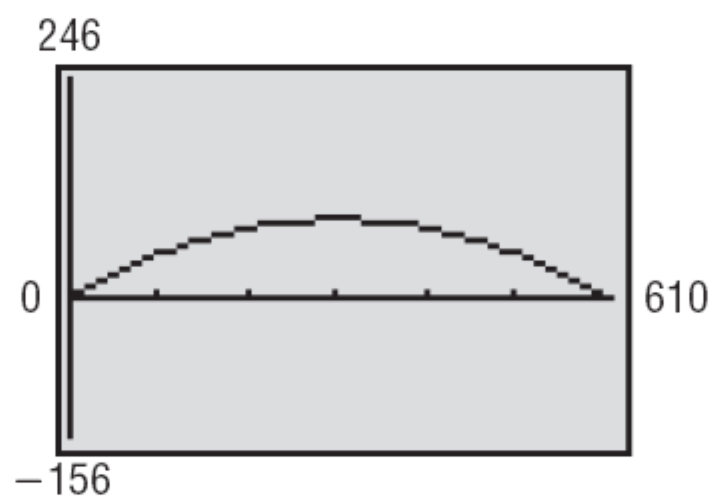
(b) How long is the golf ball in the air?

(c) When is the ball at its maximum height? Determine the maximum height of the ball.

(d) Since the ball was in the air for 4.6875 seconds, the horizontal distance that the ball traveled is

$$x = \left(75\sqrt{3}\right)4.6875 \approx 608.92 \text{ feet}$$

$$x = 75\sqrt{3}\,t \qquad y = -16t^2 + 75t$$

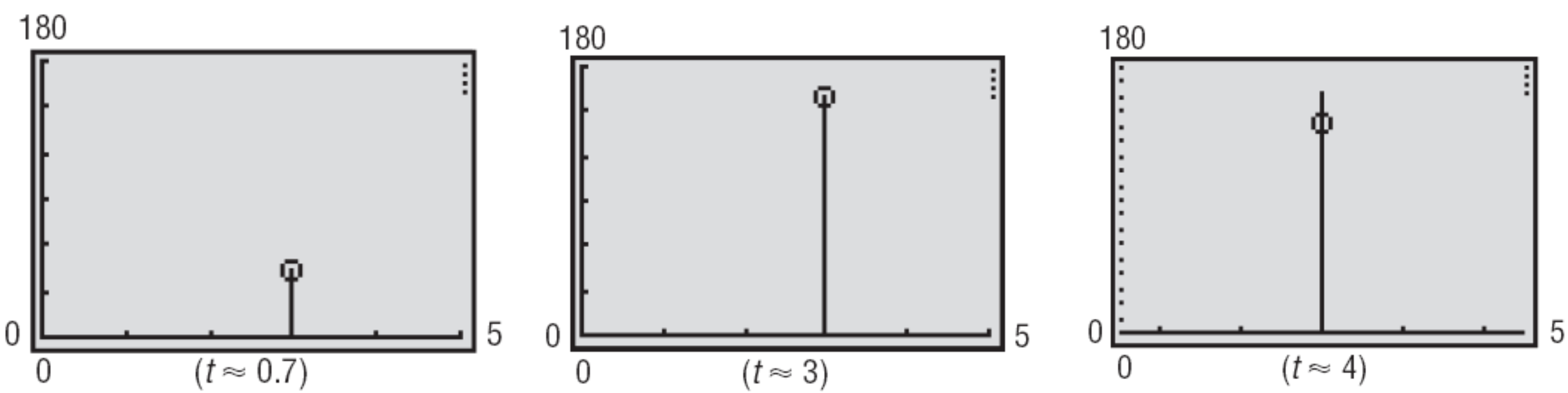


Exploration

Simulate the motion of a ball thrown straight up with an initial speed of 100 feet per second from a height of 5 feet above the ground. Use PARametric mode with Tmin = 0, Tmax = 6.5, Tstep = 0.1, Xmin = 0, Xmax = 5, Ymin = 0, and Ymax = 180. What happens to the speed with which the graph is drawn as the ball goes up and then comes back down? How do you interpret this physically? Repeat the experiment using other values for Tstep. How does this affect the experiment?

[**Hint:** In the projectile motion equations, let $\theta = 90°$, $v_0 = 100$, $h = 5$, and $g = 32$. We use $x = 3$ instead of $x = 0$ to see the vertical motion better.]

Result See Figure 68. In Figure 68(a) the ball is going up. In Figure 68(b) the ball is near its highest point. Finally, in Figure 68(c) the ball is coming back down.

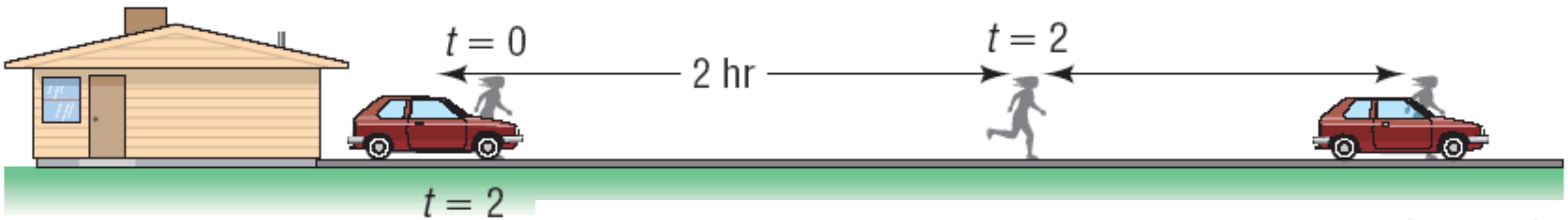


(a) (b) (c)

EXAMPLE Simulating Motion

Tanya, who is a long distance runner, runs at an average velocity of 8 miles per hour. Two hours after Tanya leaves your house, you leave in your Honda and follow the same route. If your average velocity is 40 miles per hour, how long will it be before you catch up to Tanya? See Figure 69. Use a simulation of the two motions to verify the answer.

The Honda catches up to Tanya 2.5 hours after Tanya leaves the house.



Tanya: $x_1 = 8t$ Honda: $x_2 = 40(t - 2)$

$y_1 = 2$ $y_2 = 4$

The Honda catches up to Tanya when $x_1 = x_2$.

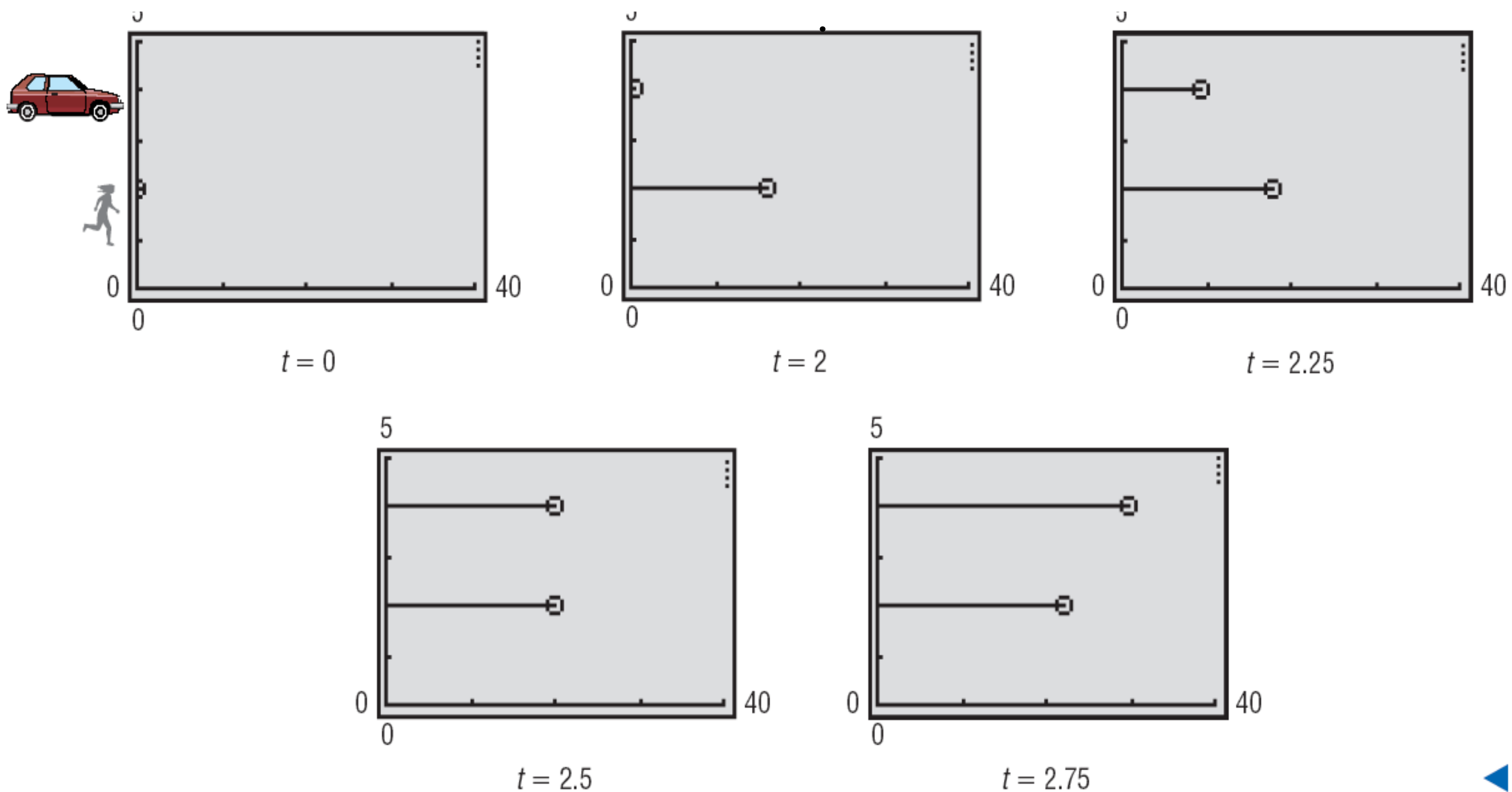
$$\begin{aligned}
8t &= 40(t - 2) \\
8t &= 40t - 80 \\
-32t &= -80 \\
t &= \frac{-80}{-32} = 2.5
\end{aligned}$$

In PARametric mode with Tstep $= 0.01$, simultaneously graph

$$\text{Tanya:} \quad x_1 = 8t \qquad \text{Honda:} \quad x_2 = 40(t-2)$$
$$y_1 = 2 \qquad\qquad\qquad y_2 = 4$$

for $0 \le t \le 3$.

Figure 65 shows the relative position of Tanya and the Honda for $t = 0$, $t = 2$, $t = 2.25$, $t = 2.5$, and $t = 2.75$.



4 Find Parametric Equations for Curves Defined by Rectangular Equations

EXAMPLE

Finding Parametric Equations for a Curve Defined by a Rectangular Equation

Find parametric equations for the equation $y = x^2 - 4$.

$$x = t, \quad y = t^2 - 4, \qquad -\infty < t < \infty$$

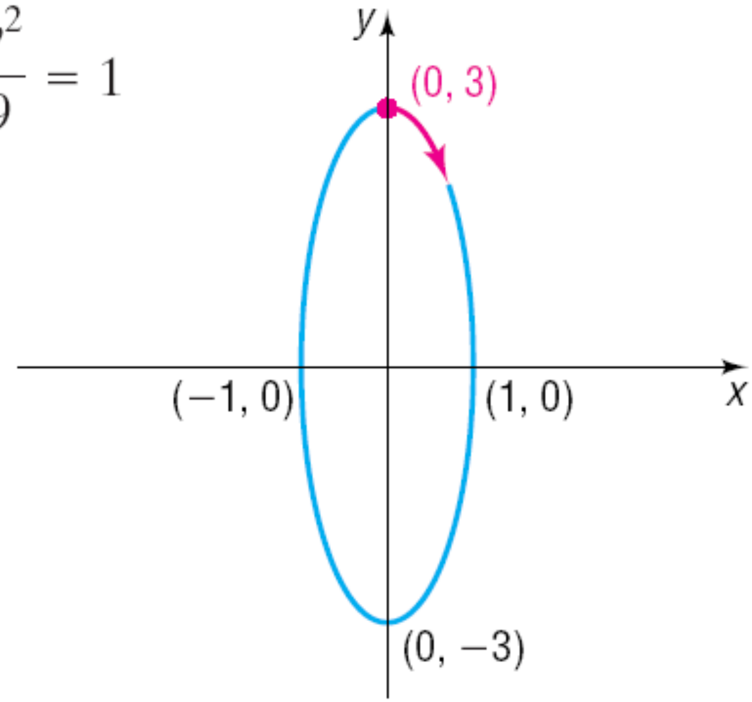
$$x = t^3, \quad y = t^6 - 4, \qquad -\infty < t < \infty$$

EXAMPLE Finding Parametric Equations for an Object in Motion

Find parametric equations for the ellipse $x^2 + \frac{y^2}{9} = 1$

where the parameter t is time (in seconds) and

(a) The motion around the ellipse is clockwise, begins at the point (0,3), and requires 1 second for a complete revolution.



Finally, since 1 revolution requires 1 second, the period $\frac{2\pi}{\omega} = 1$, so $\omega = 2\pi$. Parametric equations that satisfy the conditions stipulated are

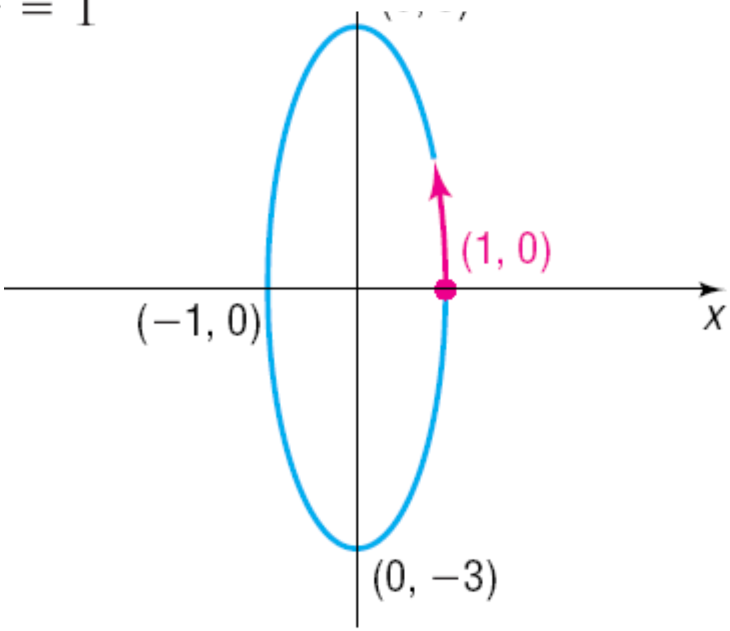
$$x = \sin(2\pi t), \qquad y = 3\cos(2\pi t), \qquad 0 \le t \le 1 \qquad \textbf{(3)}$$

EXAMPLE **Finding Parametric Equations for an Object in Motion**

Find parametric equations for the ellipse $x^2 + \frac{y^2}{9} = 1$

where the parameter t is time (in seconds) and

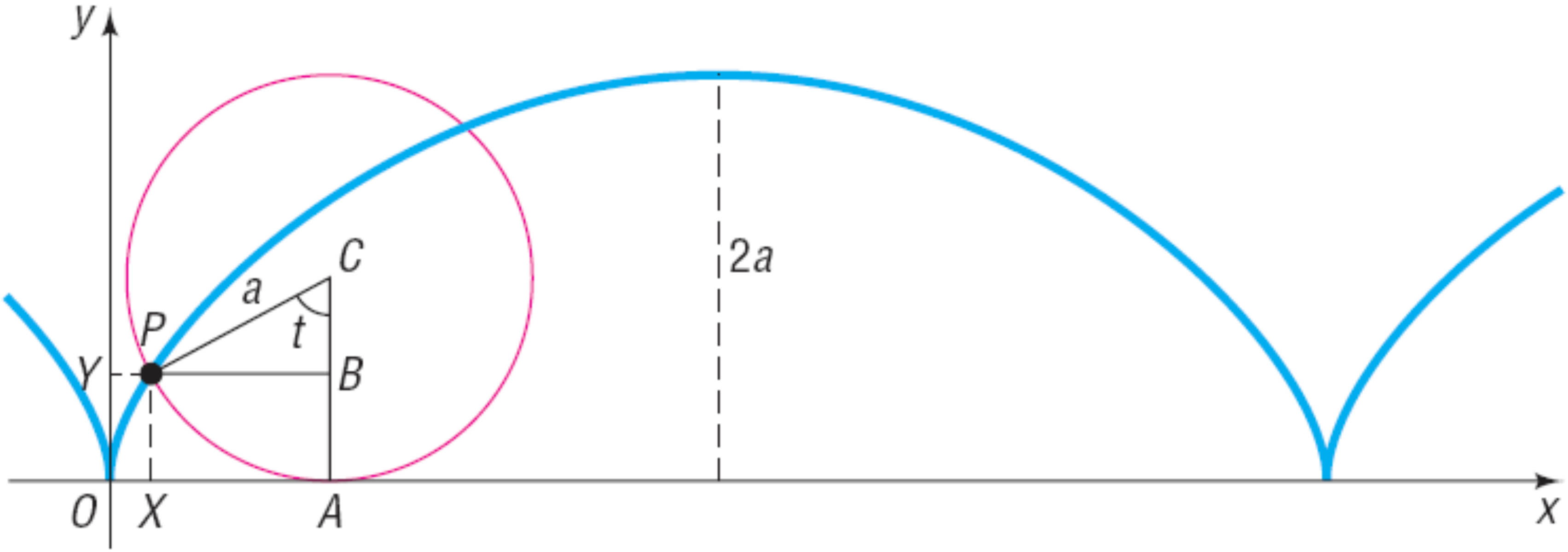
(b) The motion around the ellipse is counterclockwise, begins at the point (1, 0), and requires 2 seconds for a complete revolution.



For the motion to be counterclockwise, the motion will have to begin with the value of x decreasing and y increasing as t increases. This requires that $\omega > 0$. [Do you know why?] Finally, since 1 revolution requires 2 seconds, the period is $\frac{2\pi}{\omega} = 2$, so $\omega = \pi$. The parametric equations that satisfy the conditions stipulated are

$$x = \cos(\pi t), \qquad y = 3\sin(\pi t), \qquad 0 \le t \le 2 \qquad \textbf{(4)}$$

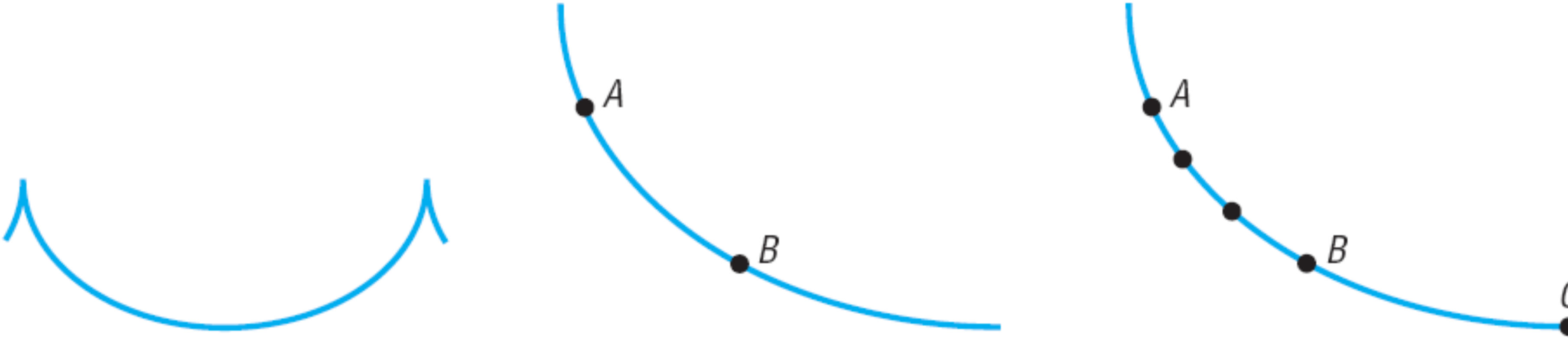
The Cycloid



The parametric equations of the cycloid are

$$\boxed{x = a(t - \sin t) \qquad y = a(1 - \cos t)}$$

Applications to Mechanics



(a) Inverted cycloid

(b) Curve of quickest descent

(c) All reach *Q* at the same time