# J

# SOLVING POLYNOMIAL EQUATIONS

We will assume in this appendix that you know how to divide polynomials using long division and synthetic division. If you need to review those techniques, refer to an algebra book.

### A BRIEF REVIEW OF POLYNOMIALS

Recall that if n is a nonnegative integer, then a **polynomial** of **degree** n is a function that can be written in the following forms, depending on whether you want the powers of x in ascending or descending order:

$$c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n \quad (c_n \neq 0)$$
  
 $c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0 \quad (c_n \neq 0)$ 

The numbers  $c_0, c_1, \ldots, c_n$  are called the *coefficients* of the polynomial. The coefficient  $c_n$  (which multiplies the highest power of x) is called the *leading coefficient*, the term  $c_n x^n$  is called the *leading term*, and the coefficient  $c_0$  is called the *constant term*. Polynomials of degree 1, 2, 3, 4, and 5 are called *linear*, *quadratic*, *cubic*, *quartic*, and *quintic*, respectively. For simplicity, general polynomials of low degree are often written without subscripts on the coefficients:

$$p(x) = a$$
 Constant polynomial  $p(x) = ax + b \quad (a \neq 0)$  Linear polynomial  $p(x) = ax^2 + bx + c \quad (a \neq 0)$  Quadratic polynomial  $p(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$  Cubic polynomial

When you attempt to factor a polynomial completely, one of three things can happen:

• You may be able to decompose the polynomial into distinct linear factors using only real numbers; for example,

$$x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x - 1)(x + 2)$$

• You may be able to decompose the polynomial into linear factors using only real numbers, but some of the factors may be repeated; for example,

$$x^{6} - 3x^{4} + 2x^{3} = x^{3}(x^{3} - 3x + 2) = x^{3}(x - 1)^{2}(x + 2)$$
(1)

You may be able to decompose the polynomial into linear and quadratic factors using only real numbers, but you may not be able to decompose the quadratic factors

into linear factors using only real numbers (such quadratic factors are said to be irreducible over the real numbers); for example,

$$x^{4} - 1 = (x^{2} - 1)(x^{2} + 1) = (x - 1)(x + 1)(x^{2} + 1)$$
$$= (x - 1)(x + 1)(x - i)(x + i)$$

Here, the factor  $x^2 + 1$  is irreducible over the real numbers.

In general, if p(x) is a polynomial of degree n with leading coefficient a, and if complex numbers are allowed, then p(x) can be factored as

$$p(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$$
(2)

where  $r_1, r_2, \dots, r_n$  are called the **zeros** of p(x) or the **roots** of the equation p(x) = 0, and (2) is called the *complete linear factorization* of p(x). If some of the factors in (2) are repeated, then they can be combined; for example, if the first k factors are distinct and the rest are repetitions of the first k, then (2) can be expressed in the form

$$p(x) = a(x - r_1)^{m_1} (x - r_2)^{m_2} \cdots (x - r_k)^{m_k}$$
(3)

where  $r_1, r_2, \dots, r_k$  are the distinct roots of p(x) = 0. The exponents  $m_1, m_2, \dots, m_k$  tell us how many times the various factors occur in the complete linear factorization; for example, in (3) the factor  $(x - r_1)$  occurs  $m_1$  times, the factor  $(x - r_2)$  occurs  $m_2$  times, and so forth. Some techniques for factoring polynomials are discussed later in this appendix. In general, if a factor (x-r) occurs m times in the complete linear factorization of a polynomial, then we say that r is a root or zero of *multiplicity m*, and if (x - r) has no repetitions (i.e., r has multiplicity 1), then we say that r is a *simple* root or zero. For example, it follows from (1) that the equation  $x^6 - 3x^4 + 2x^3 = 0$  can be expressed as

$$x^{3}(x-1)^{2}(x+2) = 0 (4)$$

so this equation has three distinct roots—a root x = 0 of multiplicity 3, a root x = 1 of multiplicity 2, and a simple root x = -2.

Note that in (3) the multiplicities of the roots must add up to n, since p(x) has degree n; that is,  $m_1 + m_2 + \cdots + m_k = n$ 

$$m_1+m_2+\cdots+m_k=n$$

For example, in (4) the multiplicities add up to 6, which is the same as the degree of the polynomial.

It follows from (2) that a polynomial of degree n can have at most n distinct roots; if all of the roots are simple, then there will be exactly n, but if some are repeated, then there will be fewer than n. However, when counting the roots of a polynomial, it is standard practice to count multiplicities, since that convention allows us to say that a polynomial of degree n has n roots. For example, from (1) the six roots of the polynomial  $p(x) = x^6 - 3x^4 + 2x^3$ are

$$r = 0, 0, 0, 1, 1, -2$$

In summary, we have the following important theorem.

**J.1 THEOREM** If complex roots are allowed, and if roots are counted according to their multiplicities, then a polynomial of degree n has exactly n roots.

### THE REMAINDER THEOREM

When two positive integers are divided, the numerator can be expressed as the quotient plus the remainder over the divisor, where the remainder is less than the divisor. For example,

$$\frac{17}{5} = 3 + \frac{2}{5}$$

If we multiply this equation through by 5, we obtain

$$17 = 5 \cdot 3 + 2$$

which states that the *numerator* is the divisor times the quotient plus the remainder.

**J.2 THEOREM** If p(x) and s(x) are polynomials, and if s(x) is not the zero polynomial, then p(x) can be expressed as

$$p(x) = s(x)q(x) + r(x)$$

where q(x) and r(x) are the quotient and remainder that result when p(x) is divided by s(x), and either r(x) is the zero polynomial or the degree of r(x) is less than the degree of s(x).

In the special case where p(x) is divided by a first-degree polynomial of the form x - c, the remainder must be some constant r, since it is either zero or has degree less than 1. Thus, Theorem J.2 implies that

$$p(x) = (x - c)q(x) + r$$

and this in turn implies that p(c) = r. In summary, we have the following theorem.

**J.3 THEOREM** (*Remainder Theorem*) If a polynomial p(x) is divided by x - c, then the remainder is p(c).

**Example 1** According to the Remainder Theorem, the remainder on dividing

$$p(x) = 2x^3 + 3x^2 - 4x - 3$$

by x + 4 should be

$$p(-4) = 2(-4)^3 + 3(-4)^2 - 4(-4) - 3 = -67$$

Show that this is so.

**Solution.** By long division

$$\begin{array}{r}
2x^2 - 5x + 16 \\
x + 4 \overline{\smash)2x^3 + 3x^2 - 4x - 3} \\
\underline{2x^3 + 8x^2} \\
-5x^2 - 4x \\
\underline{-5x^2 - 20x} \\
16x - 3 \\
\underline{16x + 64} \\
-67
\end{array}$$

which shows that the remainder is -67.

**Alternative Solution.** Because we are dividing by an expression of the form x - c (where c = -4), we can use synthetic division rather than long division. The computations are

which again shows that the remainder is -67.

### ■ THE FACTOR THEOREM

To *factor* a polynomial p(x) is to write it as a product of lower-degree polynomials, called *factors* of p(x). For s(x) to be a factor of p(x) there must be no remainder when p(x) is divided by s(x). For example, if p(x) can be factored as

$$p(x) = s(x)q(x) \tag{5}$$

then

$$\frac{p(x)}{s(x)} = q(x) \tag{6}$$

so dividing p(x) by s(x) produces a quotient q(x) with no remainder. Conversely, (6) implies (5), so s(x) is a factor of p(x) if there is no remainder when p(x) is divided by s(x).

In the special case where x - c is a factor of p(x), the polynomial p(x) can be expressed as

$$p(x) = (x - c)q(x)$$

which implies that p(c) = 0. Conversely, if p(c) = 0, then the Remainder Theorem implies that x - c is a factor of p(x), since the remainder is 0 when p(x) is divided by x - c. These results are summarized in the following theorem.

**J.4 THEOREM** (Factor Theorem) A polynomial p(x) has a factor x - c if and only if p(c) = 0.

It follows from this theorem that the statements below say the same thing in different ways:

- x c is a factor of p(x).
- p(c) = 0.
- c is a zero of p(x).
- c is a root of the equation p(x) = 0.
- c is a solution of the equation p(x) = 0.
- c is an x-intercept of y = p(x).

# **Example 2** Confirm that x - 1 is a factor of

$$p(x) = x^3 - 3x^2 - 13x + 15$$

by dividing x - 1 into p(x) and checking that the remainder is zero.

**Solution.** By long division

$$\begin{array}{r}
x^2 - 2x - 15 \\
x - 1 \overline{\smash)x^3 - 3x^2 - 13x + 15} \\
\underline{x^3 - x^2} \\
-2x^2 - 13x \\
\underline{-2x^2 + 2x} \\
-15x + 15 \\
\underline{-15x + 15} \\
0
\end{array}$$

which shows that the remainder is zero.

**Alternative Solution.** Because we are dividing by an expression of the form x - c, we can use synthetic division rather than long division. The computations are

which again confirms that the remainder is zero.

# USING ONE FACTOR TO FIND OTHER FACTORS

If x - c is a factor of p(x), and if q(x) = p(x)/(x - c), then

$$p(x) = (x - c)q(x) \tag{7}$$

so that additional linear factors of p(x) can be obtained by factoring the quotient q(x).

### **Example 3** Factor $p(x) = x^3 - 3x^2 - 13x + 15$ (8)

completely into linear factors.

**Solution.** We showed in Example 2 that x-1 is a factor of p(x) and we also showed that  $p(x)/(x-1) = x^2 - 2x - 15$ . Thus,

$$x^3 - 3x^2 - 13x + 15 = (x - 1)(x^2 - 2x - 15)$$

Factoring  $x^2 - 2x - 15$  by inspection yields

$$x^3 - 3x^2 - 13x + 15 = (x - 1)(x - 5)(x + 3)$$

which is the complete linear factorization of p(x).

### METHODS FOR FINDING ROOTS

A general quadratic equation  $ax^2 + bx + c = 0$  can be solved by using the quadratic formula to express the solutions of the equation in terms of the coefficients. Versions of this formula were known since Babylonian times, and by the seventeenth century formulas had been obtained for solving general cubic and quartic equations. However, attempts to find formulas for the solutions of general fifth-degree equations and higher proved fruitless. The reason for this became clear in 1829 when the French mathematician Evariste Galois (1811–1832) proved that it is impossible to express the solutions of a general fifth-degree equation or higher in terms of its coefficients using algebraic operations.

Today, we have powerful computer programs for finding the zeros of specific polynomials. For example, it takes only seconds for a computer algebra system, such as Mathematica or Maple, to show that the zeros of the polynomial

$$p(x) = 10x^4 - 23x^3 - 10x^2 + 29x + 6 (9)$$

are

$$x = -1$$
,  $x = -\frac{1}{5}$ ,  $x = \frac{3}{2}$ , and  $x = 2$  (10)

The algorithms that these programs use to find the integer and rational zeros of a polynomial, if any, are based on the following theorem, which is proved in advanced algebra courses.

### J.5 THEOREM Suppose that

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$$

is a polynomial with integer coefficients.

- (a) If r is an integer zero of p(x), then r must be a divisor of the constant term  $c_0$ .
- (b) If r = a/b is a rational zero of p(x) in which all common factors of a and b have been canceled, then a must be a divisor of the constant term  $c_0$ , and b must be a divisor of the leading coefficient  $c_n$ .

For example, in (9) the constant term is 6 (which has divisors  $\pm 1$ ,  $\pm 2$ ,  $\pm 3$ , and  $\pm 6$ ) and the leading coefficient is 10 (which has divisors  $\pm 1$ ,  $\pm 2$ ,  $\pm 5$ , and  $\pm 10$ ). Thus, the only possible integer zeros of p(x) are

$$\pm 1, \pm 2, \pm 3, \pm 6$$

and the only possible noninteger rational zeros are

$$\pm \frac{1}{2}$$
,  $\pm \frac{1}{5}$ ,  $\pm \frac{1}{10}$ ,  $\pm \frac{2}{5}$ ,  $\pm \frac{3}{2}$ ,  $\pm \frac{3}{5}$ ,  $\pm \frac{3}{10}$ ,  $\pm \frac{6}{5}$ 

Using a computer, it is a simple matter to evaluate p(x) at each of the numbers in these lists to show that its only rational zeros are the numbers in (10).

# **Example 4** Solve the equation $x^3 + 3x^2 - 7x - 21 = 0$ .

**Solution.** The solutions of the equation are the zeros of the polynomial

$$p(x) = x^3 + 3x^2 - 7x - 21$$

We will look for integer zeros first. All such zeros must divide the constant term, so the only possibilities are  $\pm 1, \pm 3, \pm 7$ , and  $\pm 21$ . Substituting these values into p(x) (or using the method of Exercise 6) shows that x = -3 is an integer zero. This tells us that x + 3 is a factor of p(x) and that p(x) can be written as

$$x^3 + 3x^2 - 7x - 21 = (x+3)q(x)$$

where q(x) is the quotient that results when  $x^3 + 3x^2 - 7x - 21$  is divided by x + 3. We leave it for you to perform the division and show that  $q(x) = x^2 - 7$ ; hence,

$$x^{3} + 3x^{2} - 7x - 21 = (x+3)(x^{2} - 7) = (x+3)(x+\sqrt{7})(x-\sqrt{7})$$

which tells us that the solutions of the given equation are x = 3,  $x = \sqrt{7} \approx 2.65$ , and  $x = -\sqrt{7} \approx -2.65$ .

### C CAS **EXERCISE SET J**

**1–2** Find the quotient q(x) and the remainder r(x) that result when p(x) is divided by s(x).

- **1.** (a)  $p(x) = x^4 + 3x^3 5x + 10$ ;  $s(x) = x^2 x + 2$ (b)  $p(x) = 6x^4 + 10x^2 + 5$ ;  $s(x) = 3x^2 1$ 

  - (c)  $p(x) = x^5 + x^3 + 1$ ;  $s(x) = x^2 + x$
- **2.** (a)  $p(x) = 2x^4 3x^3 + 5x^2 + 2x + 7$ ;  $s(x) = x^2 x + 1$ (b)  $p(x) = 2x^5 + 5x^4 4x^3 + 8x^2 + 1$ ;  $s(x) = 2x^2 x + 1$ (c)  $p(x) = 5x^6 + 4x^2 + 5$ ;  $s(x) = x^3 + 1$

- **3–4** Use synthetic division to find the quotient q(x) and the remainder r(x) that result when p(x) is divided by s(x).
- **3.** (a)  $p(x) = 3x^3 4x 1$ ; s(x) = x 2
  - (b)  $p(x) = x^4 5x^2 + 4$ ; s(x) = x + 5(c)  $p(x) = x^5 1$ ; s(x) = x 1
- **4.** (a)  $p(x) = 2x^3 x^2 2x + 1$ ; s(x) = x 1
  - (b)  $p(x) = 2x^4 + 3x^3 17x^2 27x 9$ ; s(x) = x + 4
  - (c)  $p(x) = x^7 + 1$ ; s(x) = x 1

- **5.** Let  $p(x) = 2x^4 + x^3 3x^2 + x 4$ . Use synthetic division and the Remainder Theorem to find p(0), p(1), p(-3), and p(7).
- **6.** Let p(x) be the polynomial in Example 4. Use synthetic division and the Remainder Theorem to evaluate p(x) at  $x = \pm 1, \pm 3, \pm 7, \text{ and } \pm 21.$
- 7. Let  $p(x) = x^3 + 4x^2 + x 6$ . Find a polynomial q(x) and a constant r such that
  - (a) p(x) = (x 2)q(x) + r
  - (b) p(x) = (x+1)q(x) + r.
- **8.** Let  $p(x) = x^5 1$ . Find a polynomial q(x) and a constant r
  - (a) p(x) = (x+1)q(x) + r
  - (b) p(x) = (x 1)q(x) + r.
- 9. In each part, make a list of all possible candidates for the rational zeros of p(x).
  - (a)  $p(x) = x^7 + 3x^3 x + 24$
  - (b)  $p(x) = 3x^4 2x^2 + 7x 10$ (c)  $p(x) = x^{35} 17$
- 10. Find all integer zeros of

$$p(x) = x^6 + 5x^5 - 16x^4 - 15x^3 - 12x^2 - 38x - 21$$

- 11–15 Factor the polynomials completely.
- **11.**  $p(x) = x^3 2x^2 x + 2$
- **12.**  $p(x) = 3x^3 + x^2 12x 4$
- **13.**  $p(x) = x^4 + 10x^3 + 36x^2 + 54x + 27$
- **14.**  $p(x) = 2x^4 + x^3 + 3x^2 + 3x 9$
- **15.**  $p(x) = x^5 + 4x^4 4x^3 34x^2 45x 18$

- **16.** For each of the factorizations that you obtained in Exercises 11–15, check your answer using a CAS.
  - 17–21 Find all real solutions of the equations.
  - **17.**  $x^3 + 3x^2 + 4x + 12 = 0$
  - **18.**  $2x^3 5x^2 10x + 3 = 0$
  - **19.**  $3x^4 + 14x^3 + 14x^2 8x 8 = 0$
  - **20.**  $2x^4 x^3 14x^2 5x + 6 = 0$
  - **21.**  $x^5 2x^4 6x^3 + 5x^2 + 8x + 12 = 0$
- **22.** For each of the equations you solved in Exercises 17–21, check your answer using a CAS.
  - **23.** Find all values of k for which x 1 is a factor of the polynomial  $p(x) = k^2 x^3 - 7kx + 10$ .
  - **24.** Is x + 3 a factor of  $x^7 + 2187$ ? Justify your answer.
- **25.** A 3 cm thick slice is cut from a cube, leaving a volume of 196 cm<sup>3</sup>. Use a CAS to find the length of a side of the original cube.
  - 26. (a) Show that there is no positive rational number that exceeds its cube by 1.
    - (b) Does there exist a real number that exceeds its cube by 1? Justify your answer.
  - 27. Use the Factor Theorem to show each of the following.
    - (a) x y is a factor of  $x^n y^n$  for all positive integer values of n.
    - (b) x + y is a factor of  $x^n y^n$  for all positive even integer values of n.
    - (c) x + y is a factor of  $x^n + y^n$  for all positive odd integer values of n.