M

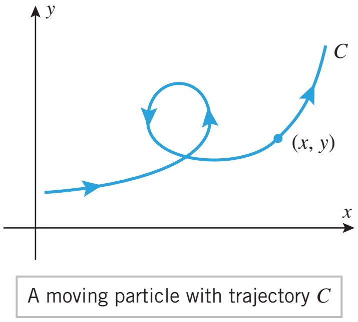
EARLY PARAMETRIC EQUATIONS OPTION

PARAMETRIC EQUATIONS

Suppose that a particle moves along a curve C in the xy-plane in such a way that its x- and y-coordinates, as functions of time, are

$$x = f(t), \quad y = g(t)$$

We call these the ***parametric equations*** of motion for the particle and refer to C as the ***trajectory*** of the particle or the ***graph*** of the equations (Figure M.1). The variable t is called the ***parameter*** for the equations.



▲ **Figure M.1**

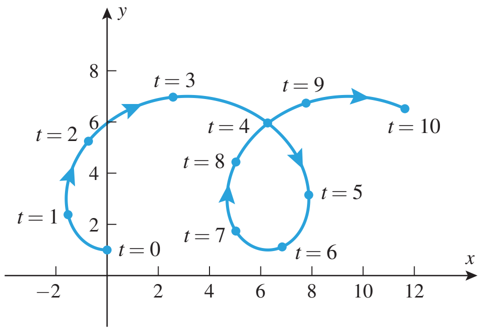
▶ **Example 1** Sketch the trajectory over the time interval $0 \le t \le 10$ of the particle whose parametric equations of motion are

$$x = t - 3\sin t, \quad y = 4 - 3\cos t \tag{1}$$

Solution. One way to sketch the trajectory is to choose a representative succession of times, plot the (x, y) coordinates of points on the trajectory at those times, and connect the points with a smooth curve. The trajectory in Figure M.2 was obtained in this way from the data in Table M.1 in which the approximate coordinates of the particle are given at time increments of 1 unit. Observe that there is no t-axis in the picture; the values of t appear only as labels on the plotted points, and even these are usually omitted unless it is important to emphasize the locations of the particle at specific times. ◀

TECHNOLOGY MASTERY

Read the documentation for your graphing utility to learn how to graph parametric equations, and then generate the trajectory in Example 1. Explore the behavior of the particle beyond time $t = 10$.



▲ **Figure M.2**

Table M.1

t	x	y
0	0.0	1.0
1	–1.5	2.4
2	–0.7	5.2
3	2.6	7.0
4	6.3	6.0
5	7.9	3.1
6	6.8	1.1
7	5.0	1.7
8	5.0	4.4
9	7.8	6.7
10	11.6	6.5

Although parametric equations commonly arise in problems of motion with time as the parameter, they arise in other contexts as well. Thus, unless the problem dictates that the parameter t in the equations

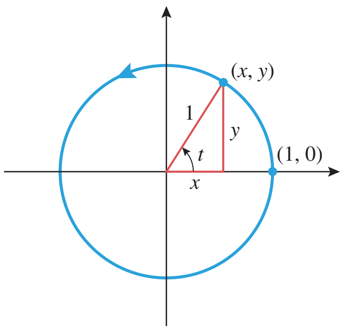
$$x = f(t), \quad y = g(t)$$

represents time, it should be viewed simply as an independent variable that varies over some interval of real numbers. (In fact, there is no need to use the letter t for the parameter; any letter not reserved for another purpose can be used.) If no restrictions on the parameter are stated explicitly or implied by the equations, then it is understood that it varies from $-\infty$ to $+\infty$. To indicate that a parameter t is restricted to an interval $[a, b]$, we will write

$$x = f(t), \quad y = g(t) \qquad (a \le t \le b)$$

▶ Example 2 Find the graph of the parametric equations

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi) \tag{2}$$



$x = \cos t,\ y = \sin t$
$(0 \le t \le 2\pi)$

▲ **Figure M.3**

Solution. One way to find the graph is to eliminate the parameter t by noting that

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

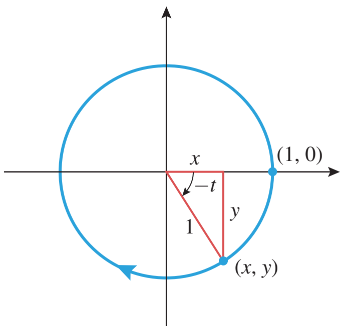
Thus, the graph is contained in the unit circle $x^2 + y^2 = 1$. Geometrically, the parameter t can be interpreted as the angle swept out by the radial line from the origin to the point $(x, y) = (\cos t, \sin t)$ on the unit circle (Figure M.3). As t increases from 0 to 2π, the point traces the circle counterclockwise, starting at $(1, 0)$ when $t = 0$ and completing one full revolution when $t = 2\pi$. One can obtain different portions of the circle by varying the interval over which the parameter varies. For example,

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le \pi) \tag{3}$$

represents just the upper semicircle in Figure M.3. ◀

ORIENTATION

The direction in which the graph of a pair of parametric equations is traced as the parameter increases is called the ***direction of increasing parameter*** or sometimes the ***orientation*** imposed on the curve by the equations. Thus, we make a distinction between a ***curve***, which is a set of points, and a ***parametric curve***, which is a curve with an orientation imposed on it by a set of parametric equations. For example, we saw in Example 2 that the circle represented parametrically by (2) is traced counterclockwise as t increases and hence has *counterclockwise orientation*. As shown in Figures M.2 and M.3, the orientation of a parametric curve can be indicated by arrowheads.



$x = \cos(-t),\ y = \sin(-t)$
$(0 \le t \le 2\pi)$

▲ **Figure M.4**

To obtain parametric equations for the unit circle with *clockwise orientation*, we can replace t by $-t$ in (2) and use the identities $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$. This yields

$$x = \cos t, \quad y = -\sin t \qquad (0 \le t \le 2\pi)$$

Here, the circle is traced clockwise by a point that starts at $(1, 0)$ when $t = 0$ and completes one full revolution when $t = 2\pi$ (Figure M.4).

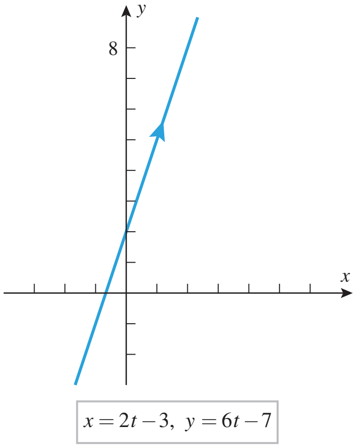
TECHNOLOGY MASTERY

When parametric equations are graphed using a calculator, the orientation can often be determined by watching the direction in which the graph is traced on the screen. However, many computers graph so fast that it is often hard to discern the orientation. See if you can use your graphing utility to confirm that (3) has a counterclockwise orientation.

▶ Example 3 Graph the parametric curve

$$x = 2t - 3, \quad y = 6t - 7$$

by eliminating the parameter, and indicate the orientation on the graph.



▲ Figure M.5

Solution. To eliminate the parameter we will solve the first equation for t as a function of x, and then substitute this expression for t into the second equation:

$$t = \left(\tfrac{1}{2}\right)(x+3)$$

$$y = 6\left(\tfrac{1}{2}\right)(x+3) - 7$$

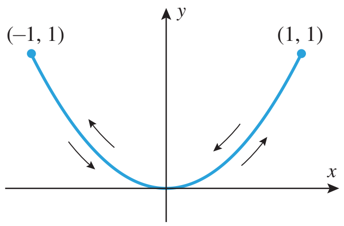
$$y = 3x + 2$$

Thus, the graph is a line of slope 3 and y-intercept 2. To find the orientation we must look to the original equations; the direction of increasing t can be deduced by observing that x increases as t increases *or* by observing that y increases as t increases. Either piece of information tells us that the line is traced left to right as shown in Figure M.5. ◀

REMARK Not all parametric equations produce curves with definite orientations; if the equations are badly behaved, then the point tracing the curve may leap around sporadically or move back and forth, failing to determine a definite direction. For example, if

$$x = \sin t, \quad y = \sin^2 t$$

then the point (x, y) moves along the parabola $y = x^2$. However, the value of x varies periodically between -1 and 1, so the point (x, y) moves periodically back and forth along the parabola between the points $(-1, 1)$ and $(1, 1)$ (as shown in Figure M.6).



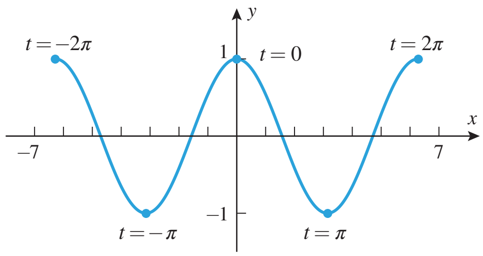
▲ Figure M.6

EXPRESSING ORDINARY FUNCTIONS PARAMETRICALLY

An equation $y = f(x)$ can be expressed in parametric form by introducing the parameter $t = x$; this yields the parametric equations $x = t$, $y = f(t)$. For example, the portion of the curve $y = \cos x$ over the interval $[-2\pi, 2\pi]$ can be expressed parametrically as

$$x = t, \quad y = \cos t \qquad (-2\pi \le t \le 2\pi)$$

(Figure M.7).



▶ Figure M.7

GENERATING PARAMETRIC CURVES WITH GRAPHING UTILITIES

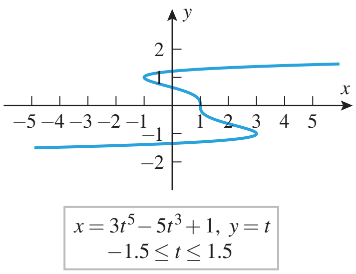
Many graphing utilities allow you to graph equations of the form $y = f(x)$ but not equations of the form $x = g(y)$. Sometimes you will be able to rewrite $x = g(y)$ in the form $y = f(x)$; however, if this is inconvenient or impossible, then you can graph $x = g(y)$ by introducing a parameter $t = y$ and expressing the equation in the parametric form $x = g(t)$, $y = t$. (You may have to experiment with various intervals for t to produce a complete graph.)

▶ **Example 4** Use a graphing utility to graph the equation $x = 3y^5 - 5y^3 + 1$.

Solution. If we let $t = y$ be the parameter, then the equation can be written in parametric form as

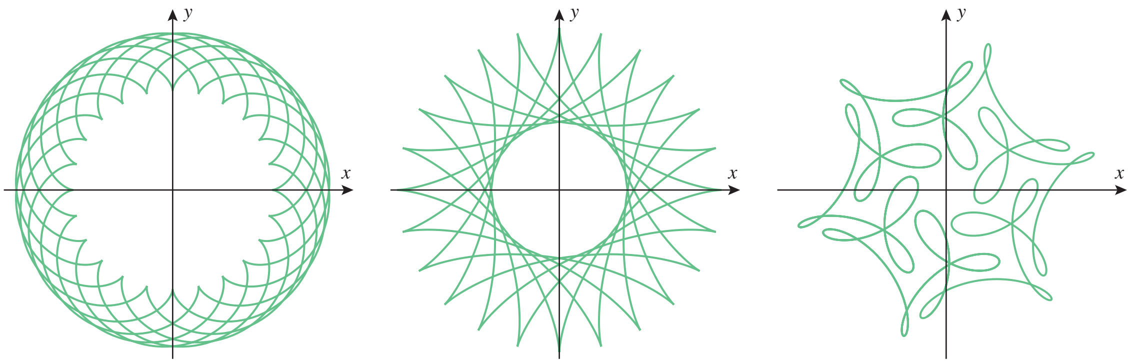
$$x = 3t^5 - 5t^3 + 1, \quad y = t$$

Figure M.8 shows the graph of these equations for $-1.5 \le t \le 1.5$. ◀



▲ Figure M.8

Some parametric curves are so complex that it is virtually impossible to visualize them without using some kind of graphing utility. Figure M.9 shows three such curves.



$x = 31\cos t - 7\cos(31/7)t$
$y = 31\sin t - 7\sin(31/7)t$
$(0 \le t \le 14\pi)$

$x = 17\cos t + 7\cos(17/7)t$
$y = 17\sin t - 7\sin(17/7)t$
$(0 \le t \le 14\pi)$

$x = \cos t + (1/2)\cos 7t + (1/3)\sin 17t$
$y = \sin t + (1/2)\sin 7t + (1/3)\cos 17t$
$(0 \le t \le 2\pi)$

▲ **Figure M.9**

TECHNOLOGY MASTERY

Try your hand at using a graphing utility to generate some parametric curves that you think are interesting or beautiful.

GRAPHING INVERSE FUNCTIONS WITH GRAPHING UTILITIES

Most graphing utilities cannot graph inverse functions directly. However, there is a way of graphing inverse functions by expressing the graphs parametrically. To see how this can be done, suppose that we are interested in graphing the inverse of a one-to-one function f. We know that the equation $y = f(x)$ can be expressed parametrically as

$$x = t, \quad y = f(t) \tag{4}$$

and we know that the graph of f^{-1} can be obtained by interchanging x and y, since this reflects the graph of f about the line $y = x$. Thus, from (4) the graph of f^{-1} can be represented parametrically as

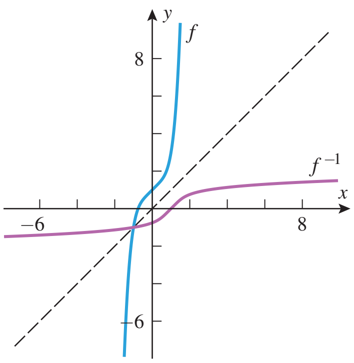
$$x = f(t), \quad y = t \tag{5}$$

For example, Figure M.10 shows the graph of $f(x) = x^5 + x + 1$ and its inverse generated with a graphing utility. The graph of f was generated from the parametric equations

$$x = t, \quad y = t^5 + t + 1$$

and the graph of f^{-1} was generated from the parametric equations

$$x = t^5 + t + 1, \quad y = t$$



▲ **Figure M.10**

TRANSLATION

If a parametric curve C is given by the equations $x = f(t)$, $y = g(t)$, then adding a constant to $f(t)$ translates the curve C in the x-direction, and adding a constant to $g(t)$ translates it in the y-direction. Thus, a circle of radius r centered at (x_0, y_0) can be represented parametrically as

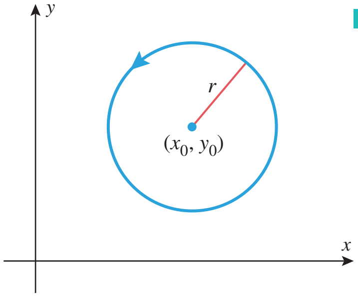
$$x = x_0 + r\cos t, \quad y = y_0 + r\sin t \quad (0 \le t \le 2\pi) \tag{6}$$

(Figure M.11). If desired, we can eliminate the parameter from these equations by noting that

$$(x - x_0)^2 + (y - y_0)^2 = (r\cos t)^2 + (r\sin t)^2 = r^2$$

Thus, we have obtained the familiar equation in rectangular coordinates for a circle of radius r centered at (x_0, y_0):

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{7}$$



$x = x_0 + r\cos t$
$y = y_0 + r\sin t$
$(0 \le t \le 2\pi)$

▲ **Figure M.11**

SCALING

If a parametric curve C is given by the equations $x = f(t)$, $y = g(t)$, then multiplying $f(t)$ by a constant stretches or compresses C in the x-direction, and multiplying $g(t)$ by a constant stretches or compresses C in the y-direction. For example, we would expect the parametric equations

$$x = 3\cos t, \quad y = 2\sin t \qquad (0 \le t \le 2\pi)$$

to represent an ellipse, centered at the origin, since the graph of these equations results from stretching the unit circle

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

by a factor of 3 in the x-direction and a factor of 2 in the y-direction. In general, if a and b are positive constants, then the parametric equations

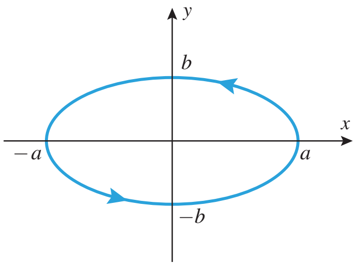
$$x = a\cos t, \quad y = b\sin t \qquad (0 \le t \le 2\pi) \tag{8}$$

represent an ellipse, centered at the origin, and extending between $-a$ and a on the x-axis and between $-b$ and b on the y-axis (Figure M.12). The numbers a and b are called the ***semiaxes*** of the ellipse. If desired, we can eliminate the parameter t in (8) and rewrite the equations in rectangular coordinates as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{9}$$

TECHNOLOGY MASTERY

Use the parametric capability of your graphing utility to generate a circle of radius 5 that is centered at $(3, -2)$.



$x = a\cos t,\ y = b\sin t$
$(0 \le t \le 2\pi)$

Figure M.12

TECHNOLOGY MASTERY

Use the parametric capability of your graphing utility to generate an ellipse that is centered at the origin and that extends between -4 and 4 in the x-direction and between -3 and 3 in the y-direction. Generate an ellipse with the same dimensions, but translated so that its center is at the point $(2, 3)$.

EXERCISE SET M

Graphing Utility

1. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \le t \le 5$ of the particle whose parametric equations of motion are
$$x = t - 1, \quad y = t + 1$$
(b) Indicate the direction of motion on your sketch.
(c) Make a table of x- and y-coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.
(d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of t.

2. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \le t \le 1$ of the particle whose parametric equations of motion are
$$x = \cos(\pi t), \quad y = \sin(\pi t)$$
(b) Indicate the direction of motion on your sketch.
(c) Make a table of x- and y-coordinates of the particle at times $t = 0, 0.25, 0.5, 0.75, 1$.
(d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of t.

3–12 Sketch the curve by eliminating the parameter, and indicate the direction of increasing t.

3. $x = 3t - 4,\ y = 6t + 2$

4. $x = t - 3,\ y = 3t - 7 \quad (0 \le t \le 3)$

5. $x = 2\cos t,\ y = 5\sin t \quad (0 \le t \le 2\pi)$

6. $x = \sqrt{t},\ y = 2t + 4$

7. $x = 3 + 2\cos t,\ y = 2 + 4\sin t \quad (0 \le t \le 2\pi)$

8. $x = \sec t,\ y = \tan t \quad (\pi \le t < 3\pi/2)$

9. $x = \cos 2t,\ y = \sin t \quad (-\pi/2 \le t \le \pi/2)$

10. $x = 4t + 3,\ y = 16t^2 - 9$

11. $x = 2\sin^2 t,\ y = 3\cos^2 t \quad (0 \le t \le \pi/2)$

12. $x = \sec^2 t,\ y = \tan^2 t \quad (0 \le t < \pi/2)$

13–18 Find parametric equations for the curve, and check your work by generating the curve with a graphing utility.

13. A circle of radius 5, centered at the origin, oriented clockwise.

14. The portion of the circle $x^2 + y^2 = 1$ that lies in the third quadrant, oriented counterclockwise.

15. A vertical line intersecting the x-axis at $x = 2$, oriented upward.

16. The ellipse $x^2/4 + y^2/9 = 1$, oriented counterclockwise.

17. The portion of the parabola $x = y^2$ joining $(1, -1)$ and $(1, 1)$, oriented down to up.

18. The circle of radius 4, centered at $(1, -3)$, oriented counterclockwise.

19. (a) Use a graphing utility to generate the trajectory of a particle whose equations of motion over the time interval $0 \le t \le 5$ are
$$x = 6t - \tfrac{1}{2}t^3, \quad y = 1 + \tfrac{1}{2}t^2$$

(cont.)

(b) Make a table of x- and y-coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.
(c) At what times is the particle on the y-axis?
(d) During what time interval is $y < 5$?
(e) Use the graph to determine the time when the x-coordinate of the particle reaches a maximum.

20. (a) Use a graphing utility to generate the trajectory of a paper airplane whose equations of motion for $t \geq 0$ are

$$x = t - 2\sin t, \quad y = 3 - 2\cos t$$

(b) Assuming that the plane flies in a room in which the floor is at $y = 0$, explain why the plane will not crash into the floor. [*Note:* For simplicity, ignore the physical size of the plane by treating it as a particle.]
(c) How high must the ceiling be to ensure that the plane does not touch or crash into it?

21–22 Graph the equation using a graphing utility. ■

21. (a) $x = y^2 + 2y + 1$
(b) $x = \sin y, \quad -2\pi \leq y \leq 2\pi$

22. (a) $x = y + 2y^3 - y^5$
(b) $x = \tan y, \quad -\pi/2 < y < \pi/2$

23–26 True–False Determine whether the statement is true or false. Explain your answer. ■

23. The parametric curve given by the equations $x = \cos t$, $y = \sin t\,(0 \leq t \leq \pi)$ is a semicircle.

24. The parametric curve given by the equations $x = \sin t$, $y = \cos^2 t\,(0 \leq t \leq 2\pi)$ is a circle.

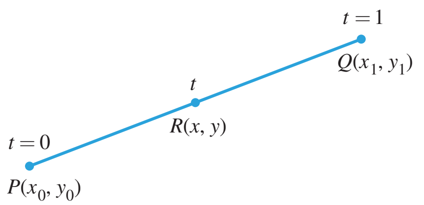
25. The graph of the parametric equations $x = f(t)$, $y = t$ is the reflection of the graph of $y = f(x)$ about the x-axis.

26. If the parametric curve C is described by the equations $x = f(t)$, $y = g(t)$, then the graph of the parametric equations $x = 2 + f(t)$, $y = 3g(t)$ is a horizontal translation of the curve obtained by stretching C in the y-direction.

27. (a) Suppose that the line segment from the point $P(x_0, y_0)$ to $Q(x_1, y_1)$ is represented parametrically by

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t \qquad (0 \leq t \leq 1)$$

and that $R(x, y)$ is the point on the line segment corresponding to a specified value of t (see the accompanying figure). Show that $t = r/q$, where r is the distance from P to R and q is the distance from P to Q.
(b) What value of t produces the midpoint between points P and Q?
(c) What value of t produces the point that is three-fourths of the way from P to Q?



◄ **Figure Ex-27**

28. Find parametric equations for the line segment joining $P(2, -1)$ and $Q(3, 1)$, and use the result in Exercise 27 to find
(a) the midpoint between P and Q
(b) the point that is one-fourth of the way from P to Q
(c) the point that is three-fourths of the way from P to Q.

29. (a) Show that the line segment joining the points (x_0, y_0) and (x_1, y_1) can be represented parametrically as

$$x = x_0 + (x_1 - x_0)\frac{t - t_0}{t_1 - t_0}, \quad y = y_0 + (y_1 - y_0)\frac{t - t_0}{t_1 - t_0} \qquad (t_0 \leq t \leq t_1)$$

(b) Which way is the line segment oriented?
(c) Find parametric equations for the line segment traced from $(3, -1)$ to $(1, 4)$ as t varies from 1 to 2, and check your result with a graphing utility.

30. (a) By eliminating the parameter, show that if a and c are not both zero, then the graph of the parametric equations

$$x = at + b, \quad y = ct + d \qquad (t_0 \leq t \leq t_1)$$

is a line segment.
(b) Sketch the parametric curve

$$x = 2t - 1, \quad y = t + 1 \qquad (1 \leq t \leq 2)$$

and indicate its orientation.
(c) What can you say about the line in part (a) if a or c (but not both) is zero?
(d) What do the equations represent if a and c are both zero?

31–34 Use a graphing utility and parametric equations to display the graphs of f and f^{-1} on the same screen. ■

31. $f(x) = x^3 + 0.2x - 1, \quad -1 \leq x \leq 2$

32. $f(x) = \sqrt{x^2 + 2} + x, \quad -5 \leq x \leq 5$

33. $f(x) = \cos(\cos 0.5x), \quad 0 \leq x \leq 3$

34. $f(x) = x + \sin x, \quad 0 \leq x \leq 6$

35. Parametric curves can be defined piecewise by using different formulas for different values of the parameter. Sketch the curve that is represented piecewise by the parametric equations

$$\begin{cases} x = 2t, \quad y = 4t^2 & \left(0 \leq t \leq \frac{1}{2}\right) \\ x = 2 - 2t, \quad y = 2t & \left(\frac{1}{2} \leq t \leq 1\right) \end{cases}$$

36. Find parametric equations for the rectangle in the accompanying figure, assuming that the rectangle is traced counterclockwise as t varies from 0 to 1, starting at $(\frac{1}{2}, \frac{1}{2})$ when $t = 0$. [*Hint:* Represent the rectangle piecewise, letting t vary from 0 to $\frac{1}{4}$ for the first edge, from $\frac{1}{4}$ to $\frac{1}{2}$ for the second edge, and so forth.]

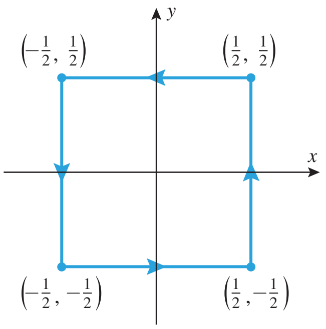


Figure Ex-36

37. (a) Find parametric equations for the ellipse that is centered at the origin and has intercepts $(4, 0)$, $(-4, 0)$, $(0, 3)$, and $(0, -3)$.
(b) Find parametric equations for the ellipse that results by translating the ellipse in part (a) so that its center is at $(-1, 2)$.
(c) Confirm your results in parts (a) and (b) using a graphing utility.

38. If a projectile is fired from ground level with an initial speed of v_0 meters per second at an angle α with the horizontal, and if air resistance is neglected, then its position after t seconds, relative to the coordinate system in the accompanying figure is

$$x = (v_0 \cos \alpha)t, \quad y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2$$

where $g \approx 9.8 \text{ m/s}^2$.
(a) By eliminating the parameter, show that the trajectory is a parabola.
(b) Use a graphing utility to sketch the trajectory if $\alpha = 30^\circ$ and $v_0 = 1000$ m/s.
(c) Using the trajectory of part (b), how high does the shell rise?
(d) Using the trajectory of part (b), how far does the shell travel horizontally?

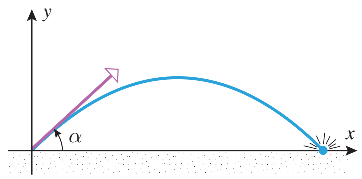


Figure Ex-38

39. Consider the family of curves described by the parametric equations

$$x = a \cos t + h, \quad y = b \cos t + k \qquad (0 \le t \le 2\pi)$$

where $a \neq 0$ and $b \neq 0$. Describe the curves in this family if
(a) h and k are fixed but a and b can vary
(b) a and b are fixed but h and k can vary
(c) $a = 1$ and $b = 1$ but h and k vary so that $h = k + 1$.

40. (a) Use a graphing utility to study how the curves in the family

$$x = 2a \cos^2 t, \quad y = 2a \cos t \sin t \qquad (-2\pi < t < 2\pi)$$

change as a varies from 0 to 5.
(b) Confirm your conclusion algebraically.
(c) Write a brief paragraph that describes your findings.