Section 8.4

Trigonometric Identities

Two functions f and g are said to be **identically equal** if

$$\boxed{f(x) = g(x)}$$

for every value of x for which both functions are defined. Such an equation is referred to as an **identity.** An equation that is not an identity is called a **conditional equation.**

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Even-Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

1 Use Algebra to Simplify Trigonometric Expressions

EXAMPLE Using Algebraic Techniques to Simplify Trigonometric Expressions

(a) Simplify $\dfrac{\tan\theta}{\sec\theta}$ by rewriting each trigonometric function in terms of sine and cosine functions.

$$\text{(a)}\quad \frac{\tan\theta}{\sec\theta} = \frac{\dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\cos}} = \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{1} = \sin\theta$$

(b) Show that $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$ by multiplying the numerator and denominator by $1-\cos\theta$

$$\text{(b)}\quad \frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

EXAMPLE Using Algebraic Techniques to Simplify Trigonometric Expressions

(a) Simplify $\dfrac{1}{1-\sin u}+\dfrac{1}{1+\sin u}$ by rewriting the expression over a common denominator.

$$\text{(a)}\quad \frac{1(1+\sin u)}{(1-\sin u)(1+\sin u)}+\frac{1(1-\sin u)}{(1+\sin u)(1-\sin u)}=\frac{1+\sin u+1-\sin u}{1-\sin^2 u}=\frac{2}{\cos^2 u}=2\sec^2 u$$

(b) Simplify $\dfrac{1-\cos^2 v}{\sin v+\cos v\sin v}$ by factoring.

$$\text{(b)}\quad \frac{1-\cos^2 v}{\sin v+\cos v\sin v}=\frac{(1+\cos v)(1-\cos v)}{\sin v(1+\cos v)}$$

$$=\frac{1-\cos v}{\sin v}=\frac{1}{\sin v}-\frac{\cos v}{\sin v}=\csc v-\cot v$$

2 Establish Identities

EXAMPLE

Establishing an Identity

Establish the identity: $\sin\theta\left(\cot\theta+\tan\theta\right)=\sec\theta$

$$\sin\theta\left(\cot\theta+\tan\theta\right)=\sin\theta\left(\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}\right)$$

$$=\cos\theta+\frac{\sin^2\theta}{\cos\theta}=\frac{\cos^2\theta+\sin^2\theta}{\cos\theta}$$

$$=\frac{1}{\cos\theta}=\sec\theta$$

EXAMPLE Establishing an Identity

Establish the identity: $\csc\theta - \cot\theta = \dfrac{\sin\theta}{1+\cos\theta}$

$$\csc\theta - \cot\theta = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta}$$

EXAMPLE Establishing an Identity

Establish the identity: $$\frac{\sin^2\theta-\tan\theta}{\cos^2\theta-\cot\theta}=\tan^2\theta$$

$$\frac{\sin^2\theta-\tan\theta}{\cos^2\theta-\cot\theta}=\frac{\sin^2\theta-\dfrac{\sin\theta}{\cos\theta}}{\cos^2\theta-\dfrac{\cos\theta}{\sin\theta}}=\frac{\dfrac{\sin^2\theta\cos\theta-\sin\theta}{\cos\theta}}{\dfrac{\sin\theta\cos^2\theta-\cos\theta}{\sin\theta}}$$

$$=\frac{\sin^2\theta\cos\theta-\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta\cos^2\theta-\cos\theta}$$

$$=\frac{\sin^2\theta(\sin\theta\cos\theta-1)}{\cos^2\theta(\sin\theta\cos\theta-1)}=\frac{\sin^2\theta}{\cos^2\theta}=\tan^2\theta$$

EXAMPLE **Establishing an Identity**

Establish the identity: $$\frac{\cos\theta}{1-\sin\theta}=\frac{1+\sin\theta}{\cos\theta}$$

$$\frac{\cos\theta}{1-\sin\theta}=\frac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}=\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$

$$=\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}=\frac{1+\sin\theta}{\cos\theta}$$

EXAMPLE **Establishing an Identity**

Establish the identity: $\cot^2\theta = \dfrac{\csc\theta - \sin\theta}{\sin\theta}$

$$\frac{\csc\theta - \sin\theta}{\sin\theta} = \left(\frac{\dfrac{1}{\sin\theta} - \sin\theta}{\sin\theta}\right)\left(\frac{\sin\theta}{\sin\theta}\right)$$

$$= \frac{1 - \sin^2\theta}{\sin^2\theta} = \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$$

EXAMPLE Establishing an Identity

Establish the identity: $1-\csc\theta\sin^3\theta=\cos^2\theta$

$$1-\csc\theta\sin^3\theta=1-\frac{\sin^3\theta}{\sin\theta}$$

$$=1-\sin^2\theta=\cos^2\theta$$

Guidelines for Establishing Identities

1. It is almost always preferable to start with the side containing the more complicated expression.
2. Rewrite sums or differences of quotients as a single quotient.
3. Sometimes rewriting one side in terms of sine and cosine functions only will help.
4. Always keep your goal in mind. As you manipulate one side of the expression, you must keep in mind the form of the expression on the other side.