# Section 13.4

# Mathematical Induction

## 1 Prove Statements Using Mathematical Induction

## THEOREM

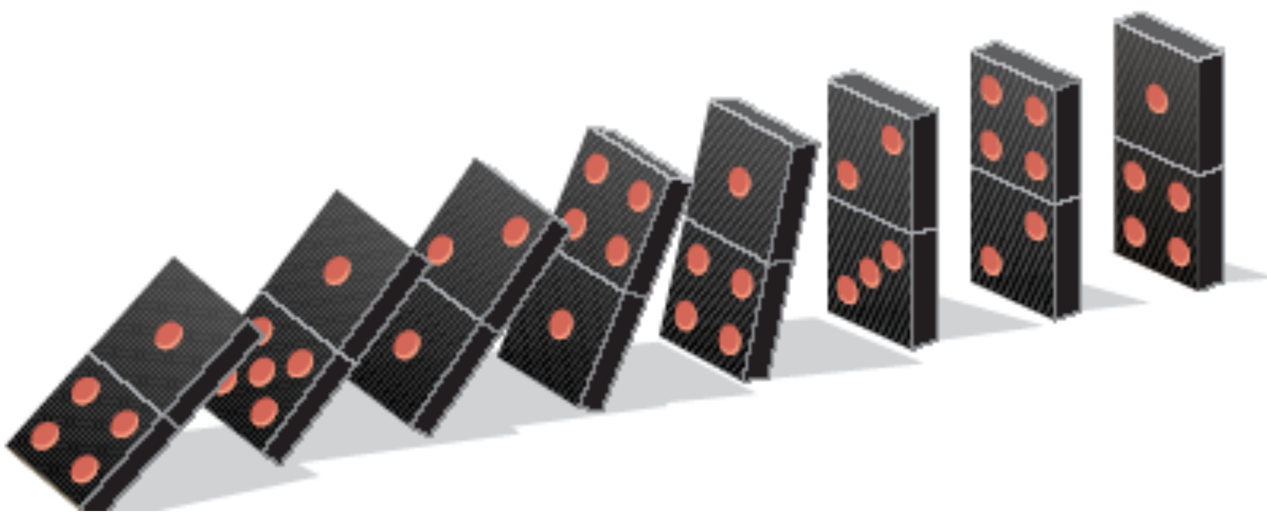
**The Principle of Mathematical Induction**

Suppose that the following two conditions are satisfied with regard to a statement about natural numbers:

CONDITION I: The statement is true for the natural number 1.

CONDITION II: If the statement is true for some natural number $k$, it is also true for the next natural number $k + 1$.

Then the statement is true for all natural numbers.



**EXAMPLE** **Using Mathematical Induction**

Show that the following statement is true for all natural numbers $n$.

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

CONDITION I: The statement is true for the natural number 1.

$$1 = 1^2$$

CONDITION II: If the statement is true for some natural number $k$, it is also true for the next natural number $k + 1$.

Assume: $$1 + 3 + \cdots + (2k - 1) = k^2$$

is true for some natural number $k$.

$$1 + 3 + \cdots + (2k - 1) + [2(k + 1) - 1] = \underbrace{[1 + 3 + \cdots + (2k - 1)]}_{= k^2 \text{ by equation above}} + (2k + 1)$$

$$= k^2 + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$$

Conditions I and II are satisfied; by the Principle of Mathematical Induction, statement (2) is true for all natural numbers $n$.

EXAMPLE

# Using Mathematical Induction

Show that the following statement is true for all natural numbers $n$.

$$2^n > n$$

CONDITION I: The statement is true for the natural number 1.

$$2^1 = 2 > 1$$

CONDITION II: If the statement is true for some natural number $k$, it is also true for the next natural number $k + 1$.

Assume: $2^k > k$

$$2^{k+1} = 2 \cdot 2^k \underset{\substack{\uparrow \\ \text{We know that} \\ 2^k > k.}}{>} 2 \cdot k = k + k \underset{\substack{\uparrow \\ k \geq 1}}{\geq} k + 1$$

EXAMPLE **Using Mathematical Induction**

Show that the following formula is true for all natural numbers $n$.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

CONDITION I: The statement is true for the natural number 1.

$$\frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$$

CONDITION II: If the statement is true for some natural number $k$, it is also true for the next natural number $k + 1$.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \quad \text{for some } k \qquad \textbf{(5)}$$

$$1 + 2 + 3 + \cdots + k + (k+1) = \underbrace{[1 + 2 + 3 + \cdots + k]} + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k^2 + k + 2k + 2}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$$

**EXAMPLE** **Using Mathematical Induction**

Show that $3^n - 1$ is divisible by 2 for all natural numbers $n$.

CONDITION I: The statement is true for the natural number 1.

$$3^1 - 1 = 3 - 1 = 2 \quad \text{is divisible by 2}$$

CONDITION II: If the statement is true for some natural number $k$, it is also true for the next natural number $k + 1$.

Next, we assume that the statement holds for some $k$, and we determine whether the statement then holds for $k + 1$. We assume that $3^k - 1$ is divisible by 2 for some $k$. We need to show that $3^{k+1} - 1$ is divisible by 2. Now

$$\begin{aligned} 3^{k+1} - 1 &= 3^{k+1} - 3^k + 3^k - 1 \\ &= 3^k(3 - 1) + (3^k - 1) = 3^k \cdot 2 + (3^k - 1) \end{aligned}$$

Because $3^k \cdot 2$ is divisible by 2 and $3^k - 1$ is divisible by 2, it follows that $3^k \cdot 2 + (3^k - 1) = 3^{k+1} - 1$ is divisible by 2. Condition II is also satisfied. As a result, the statement "$3^n - 1$ is divisible by 2" is true for all natural numbers $n$.