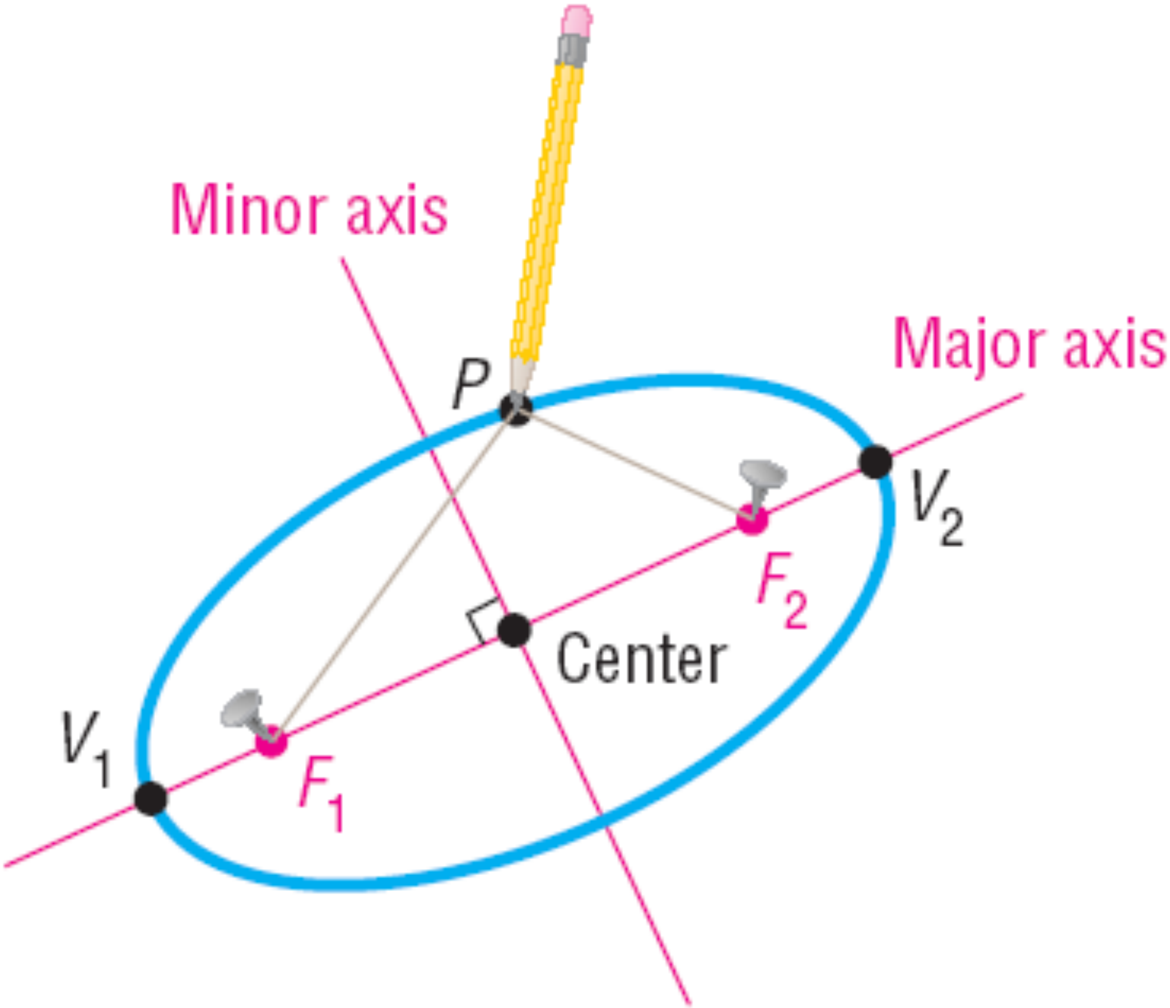
Section 11.3

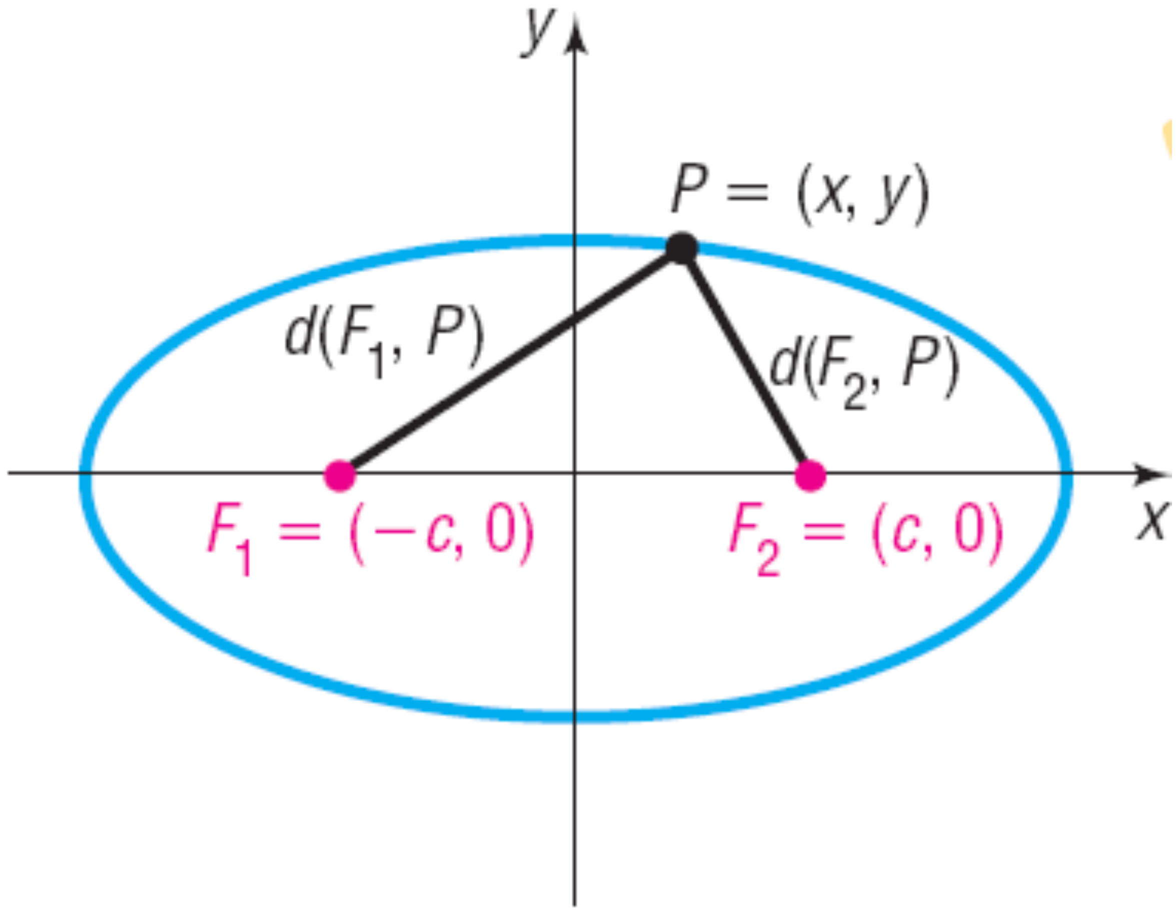
The Ellipse

An **ellipse** is the collection of all points in the plane the sum of whose distances from two fixed points, called the **foci**, is a constant.



1 Analyze Ellipses with Center at the Origin

$$d(F_1, P) + d(F_2, P) = 2a$$

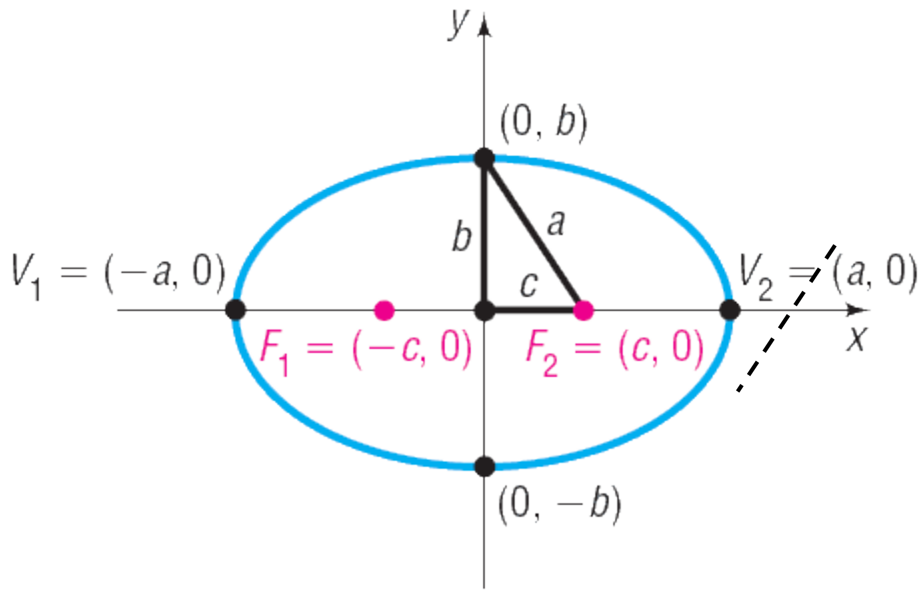


Equation of an Ellipse: Center at (0, 0); Major Axis along the *x*-Axis

An equation of the ellipse with center at $(0, 0)$, foci at $(-c, 0)$ and $(c, 0)$, and vertices at $(-a, 0)$ and $(a, 0)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{where } a > b > 0 \text{ and } b^2 = a^2 - c^2 \tag{2}$$

The major axis is the x-axis. See Figure 19.



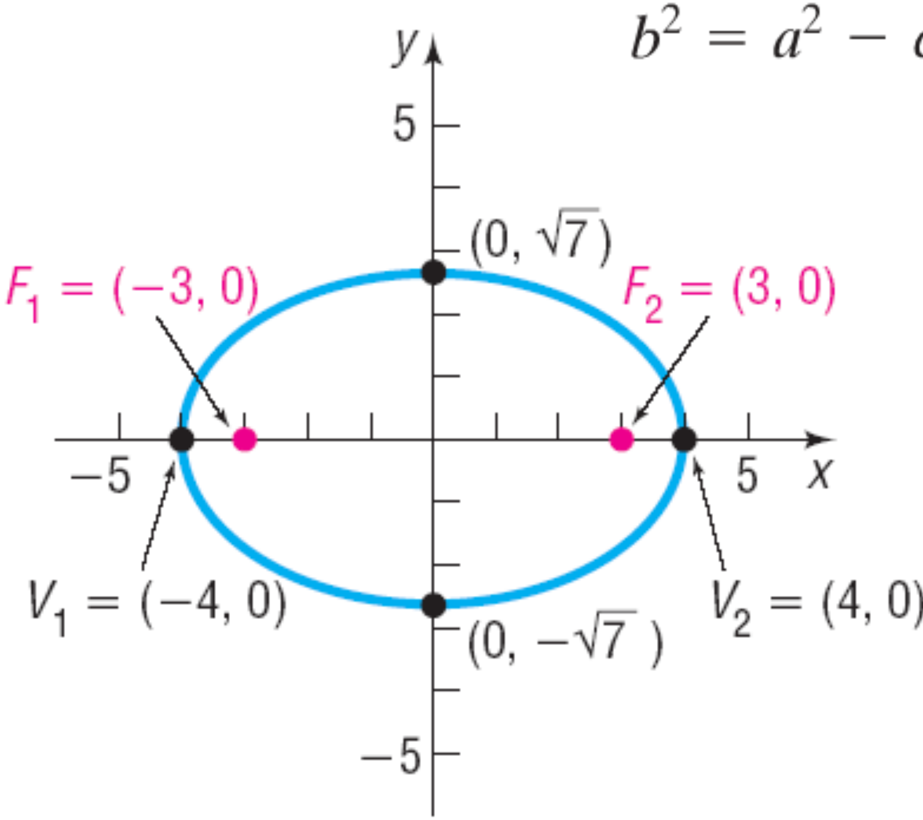
EXAMPLE **Finding an Equation of an Ellipse**

Find an equation of the ellipse with center at the origin, one focus at (3, 0) and a vertex at (– 4, 0). Graph the equation.

From the center to one vertex is $a = 4$. From the center to one focus is $c = 3$.

$$b^2 = a^2 - c^2 = 16 - 9 = 7$$

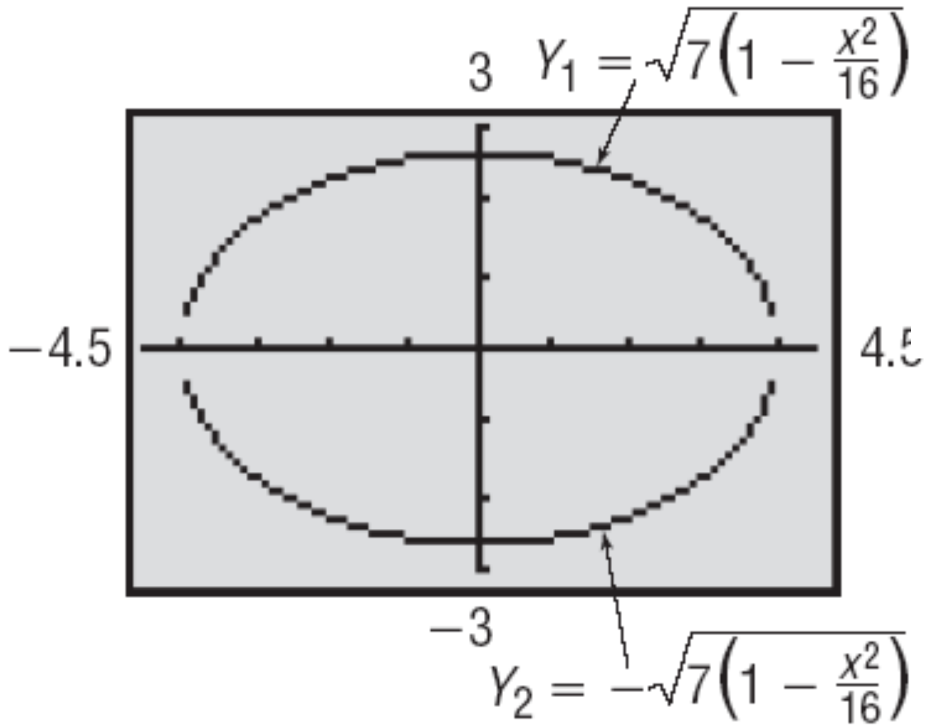
$$\frac{x^2}{16} + \frac{y^2}{7} = 1$$



$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$\boxed{b^2 = a^2 - c^2}$$

Use a graphing utility to graph the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1.$



$$y = \pm\sqrt{7\left(1 - \frac{x^2}{16}\right)}$$

EXAMPLE

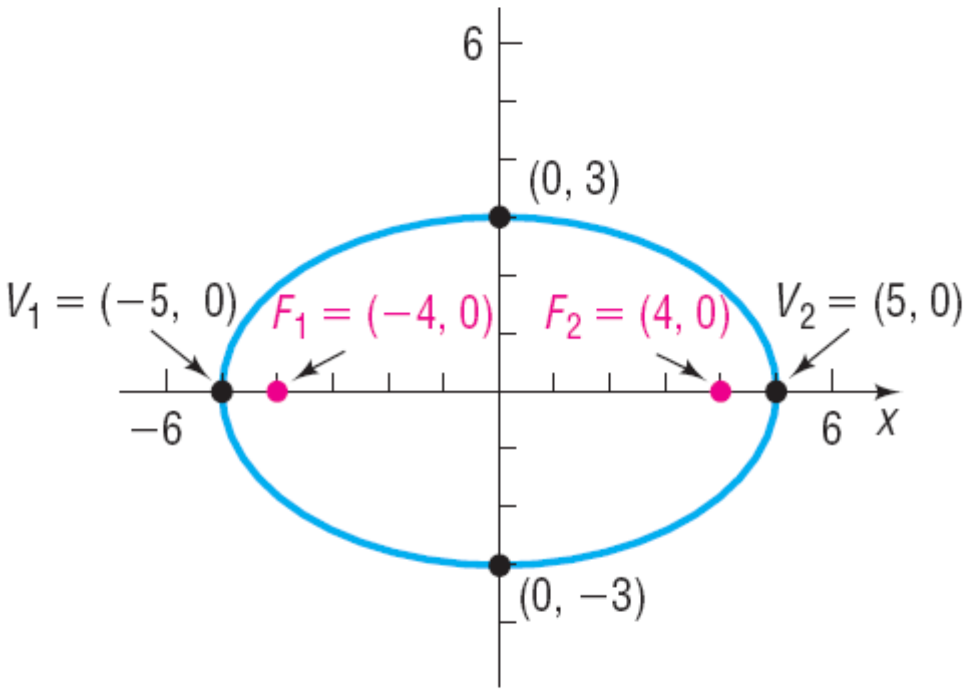
Analyzing the Equation of an Ellipse

Analyze the equation: $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

The given equation is of the form of equation (2), with $a^2 = 25$ and $b^2 = 9$. The equation is that of an ellipse with center (0, 0) and major axis along the x-axis. The vertices are at $(\pm a, 0) = (\pm 5, 0)$. Because $b^2 = a^2 - c^2$, we find that

$$c^2 = a^2 - b^2 = 25 - 9 = 16$$

The foci are at $(\pm c, 0) = (\pm 4, 0)$. Figure 22 shows the graph.



$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$\boxed{c^2 = a^2 - b^2}$$

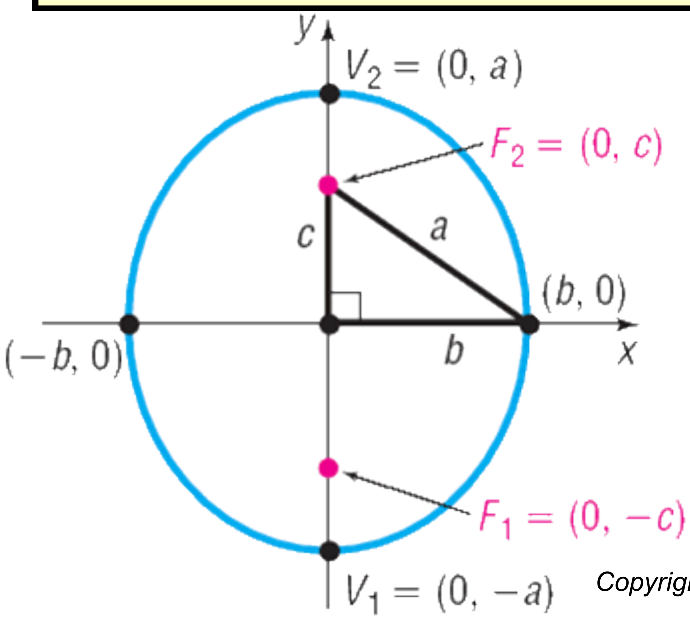
Equation of an Ellipse

Center at (0, 0); Major Axis along the *y*-Axis

An equation of the ellipse with center at $(0, 0)$, foci at $(0, -c)$ and $(0, c)$, and vertices at $(0, -a)$ and $(0, a)$ is

$$\boxed{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \qquad \text{where } a > b > 0 \text{ and } b^2 = a^2 - c^2}$$

The major axis is the y-axis.



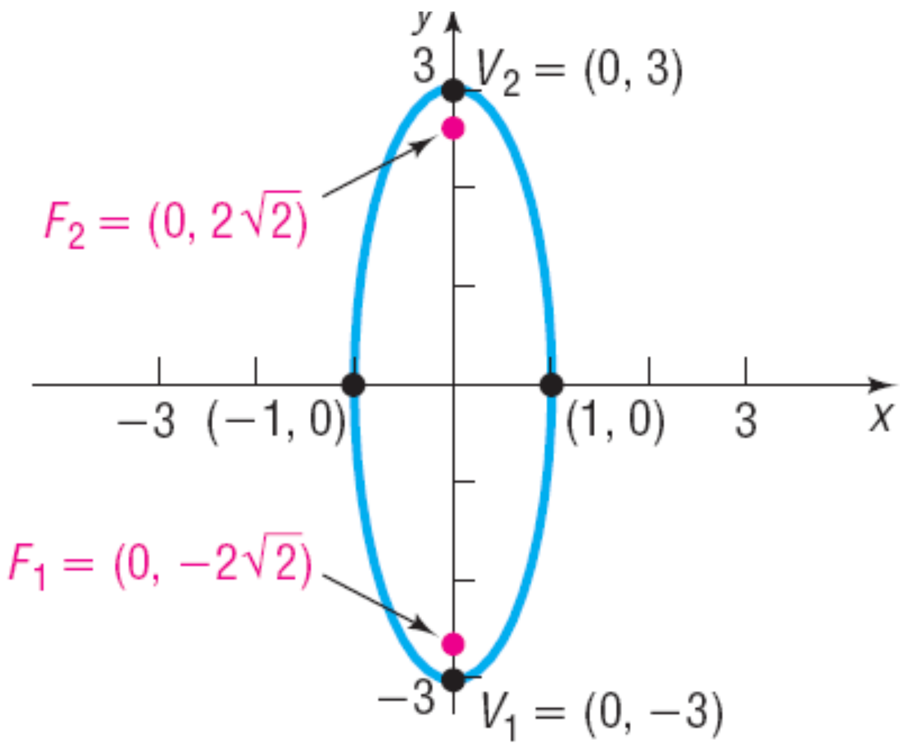
EXAMPLE

Analyzing the Equation of an Ellipse

Analyze the equation: $9x^2 + y^2 = 9$

$$x^2 + \frac{y^2}{9} = 1$$

The larger denominator, 9, is in the y^2-term so, based on equation (3), this is the equation of an ellipse with center at the origin and major axis along the y-axis. Also, we conclude that $a^2 = 9$, $b^2 = 1$, and $c^2 = a^2 - b^2 = 9 - 1 = 8$. The vertices are at $(0, \pm a) = (0, \pm 3)$, and the foci are at $(0, \pm c) = (0, \pm 2\sqrt{2})$. Figure 24 shows the graph.



EXAMPLE

Finding an Equation of an Ellipse

Find an equation of the ellipse having one focus at (0, 2) and vertices at (0, –3) and (0, 3). Graph the equation by hand.

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

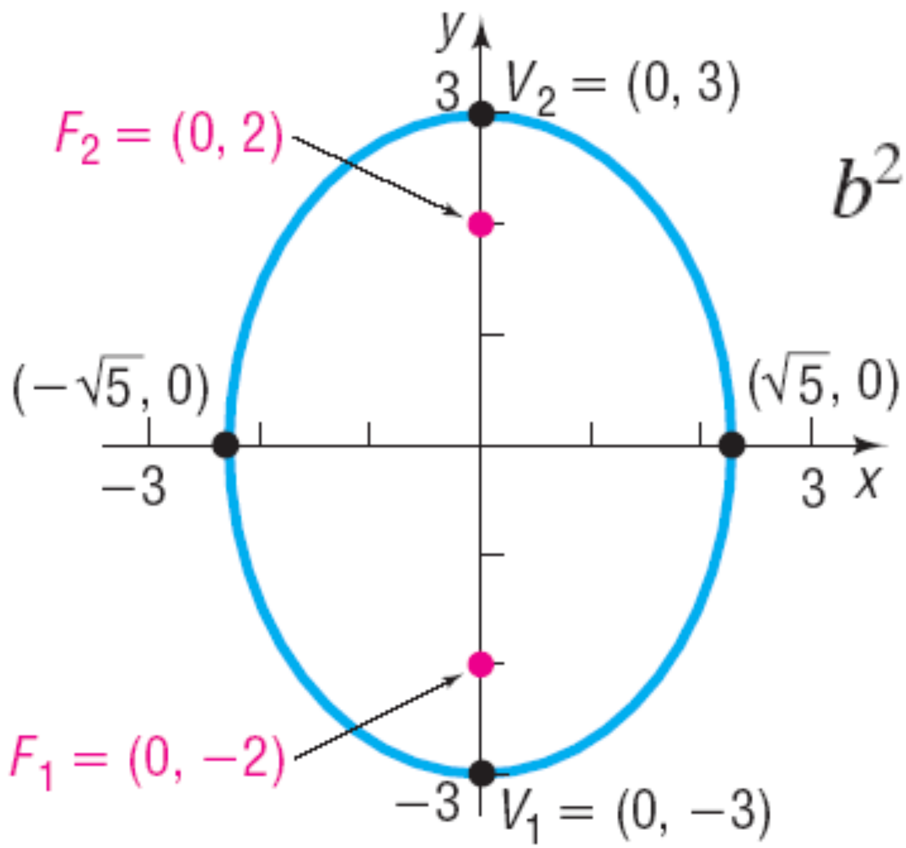
The distance from the center to a focus is $c = 2$

The distance from the center to a vertex is $a = 3$

$$b^2 = a^2 - c^2$$

$$= 9 - 4 = 5.$$

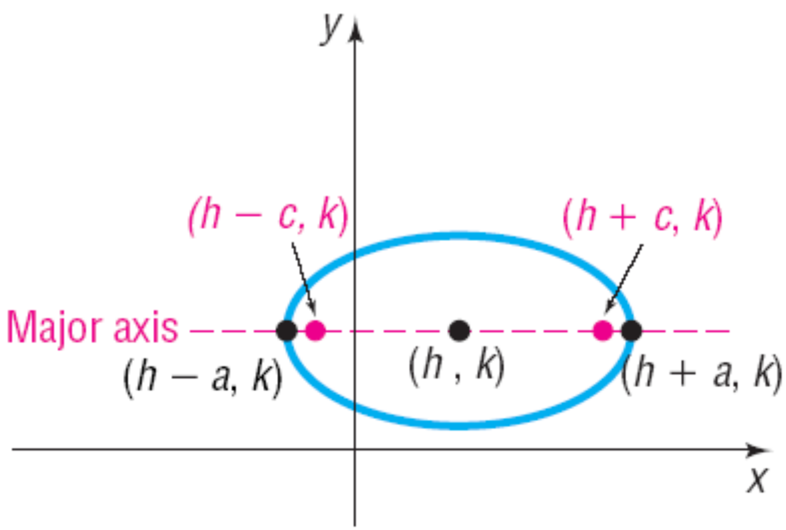
$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$



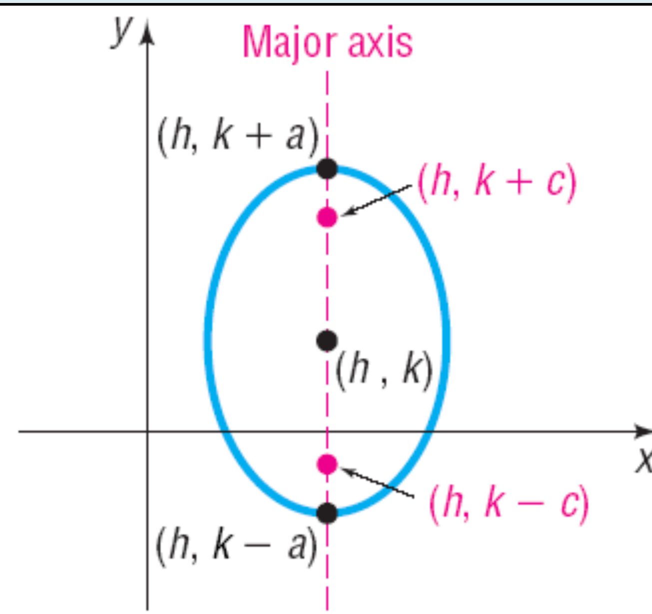
2 Analyze Ellipses with Center at (h, k)

Equations of an Ellipse: Center at (h, k); Major Axis Parallel to a Coordinate Axis

Center	Major Axis	Foci	Vertices	Equation
(h, k)	Parallel to the x-axis	$(h + c, k)$ $(h - c, k)$	$(h + a, k)$ $(h - a, k)$	$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1,$ $a > b > 0$ and $b^2 = a^2 - c^2$
(h, k)	Parallel to the y-axis	$(h, k + c)$ $(h, k - c)$	$(h, k + a)$ $(h, k - a)$	$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1,$ $a > b > 0$ and $b^2 = a^2 - c^2$



(a) $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$



(b) $\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$

EXAMPLE Finding an Equation of an Ellipse, Center Not at the Origin

Find an equation for the ellipse with center at (2, –3), one focus at (3, –3), and one vertex at (5, –3). Graph the equation by hand.

The center is at $(h, k) = (2, -3)$, so $h = 2$ and $k = -3$.

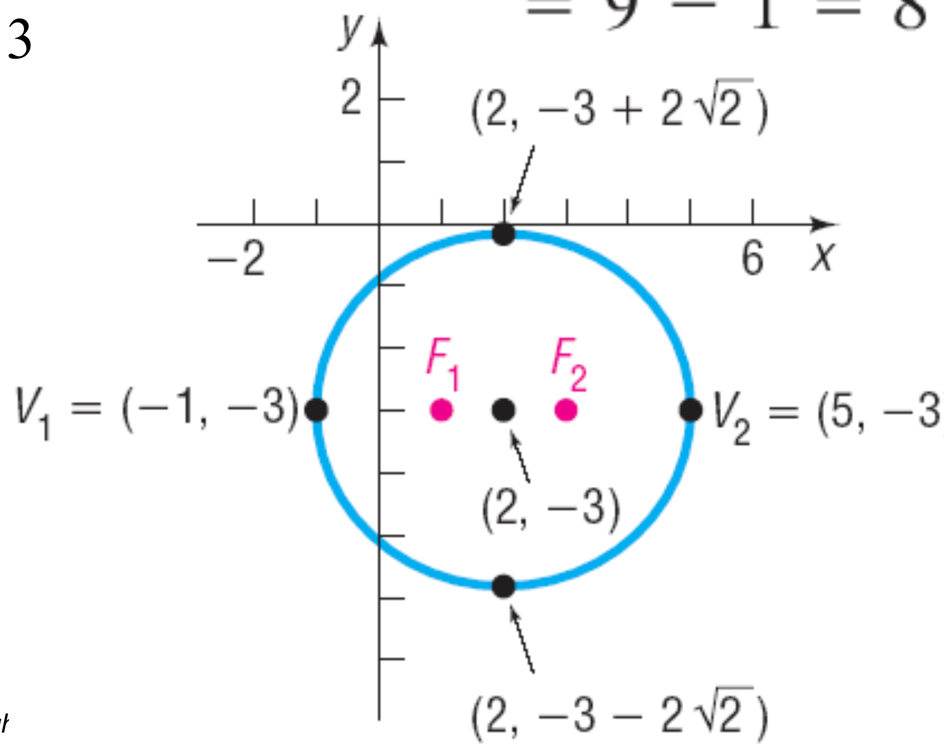
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$b^2 = a^2 - c^2$$
$$= 9 - 1 = 8$$

Center and vertices lie on line $y = -3$ so major axis is parallel to x-axis.

The distance from the center to a focus is $c = 1$;The distance from the center to a vertex is $a = 3$

$$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{8} = 1$$



EXAMPLE

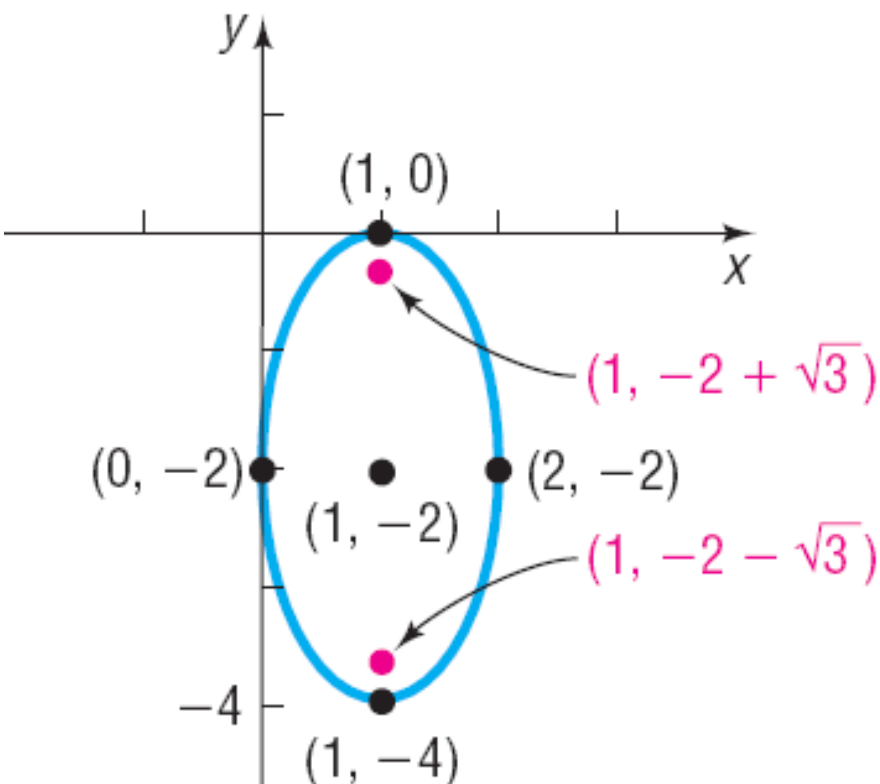
Analyzing the Equation of an Ellipse

Analyze the equation: $4x^2 + y^2 - 8x + 4y + 4 = 0$

$4x^2 - 8x + y^2 + 4y = -4$ $\quad 4(x^2 - 2x) + (y^2 + 4y) = -4$

$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = -4 + 4 + 4$

$4(x - 1)^2 + (y + 2)^2 = 4$ $\quad (x - 1)^2 + \frac{(y + 2)^2}{4} = 1$



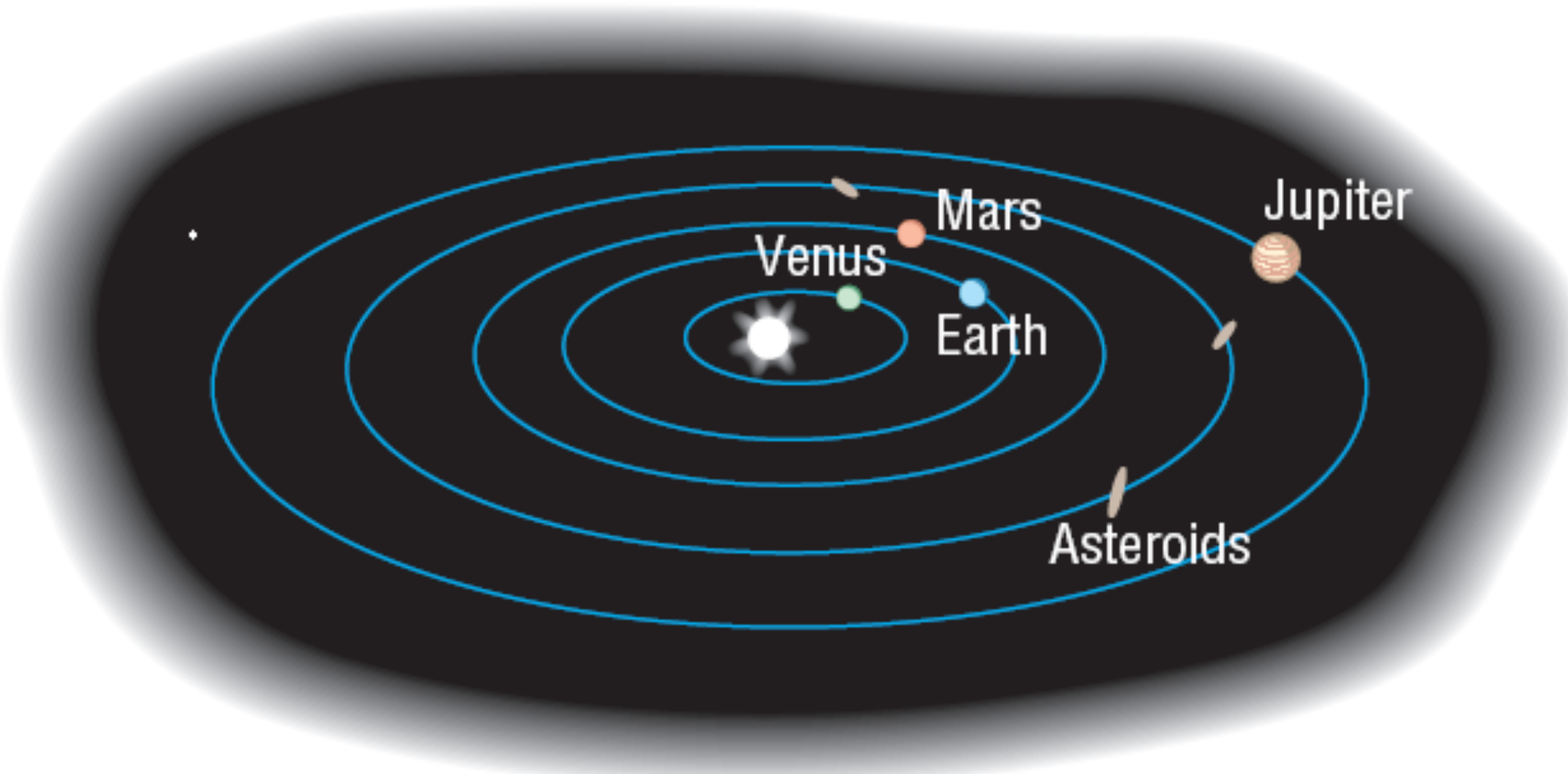
$c^2 = a^2 - b^2 = 4 - 1 = 3$

Center: $(1, -2)$

Vertices: $(1, -2 \pm 2) = (1, 0), (1, -4)$

Foci: $(1, -2 \pm \sqrt{3})$

3 Solve Applied Problems Involving Ellipses


Mars
Jupiter
Venus
Earth
Asteroids

EXAMPLE A Whispering Gallery

The whispering gallery in the Museum of Science and Industry in Chicago is 47.3 feet long. The distance from the center of the room to the foci is 20.3 feet. Find an equation that describes the shape of the room. How high is the room at its center?

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{47.3}{2} = 23.65 \text{ feet; so } a = 23.65 \text{ feet}$$

$$c = 20.3 \text{ feet} \qquad b^2 = 23.65^2 - 20.3^2 = 147.2325$$

The height of the room at its center is $b = \sqrt{147.2325} \approx 12.1$ feet.

$$\frac{x^2}{23.65^2} + \frac{y^2}{147.2325} = 1$$