Section 13.3

Geometric Sequences; Geometric Series

1 Determine If a Sequence Is Geometric

Geometric Sequence

A **geometric sequence*** may be defined recursively as $a_1 = a$, $\dfrac{a_n}{a_{n-1}} = r$, or as

$$\boxed{a_1 = a, \qquad a_n = r a_{n-1}} \qquad (1)$$

where $a_1 = a$ and $r \neq 0$ are real numbers. The number a_1 is the first term, and the nonzero number r is called the **common ratio.**

$$a_1, \quad a_1 r, \quad a_1 r^2, \quad a_1 r^3, \ldots$$

EXAMPLE

Determining If a Sequence Is Geometric

Show that the sequence is geometric. List the first term and the common ratio.

(a) 2, 8, 32, 128, . . .

(a) $a_1 = 2 \quad r = 4$

(b) $\{s_n\} = \{3^{n+1}\}$

(b) $a_1 = 3^{1+1} = 9 \quad r = 3$

(c) $\{t_n\} = \{3(2)^n\}$

(c) $a_1 = 3(2)^1 = 6 \quad r = 2$

2 Find a Formula for a Geometric Sequence

*n*th Term of a Geometric Sequence

For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r, the nth term is determined by the formula

$$\boxed{a_n = a_1 r^{n-1} \qquad r \neq 0}$$

EXAMPLE

Finding a Particular Term of a Geometric Sequence

(a) Find the nth term of the geometric sequence:

$3, 2, \frac{4}{3}, \frac{8}{9}, \ldots$

(a) $a_1 = 3 \quad r = \frac{2}{3} \qquad a_n = 3\left(\frac{2}{3}\right)^{n-1}$

(b) Find the ninth term of this sequence.

$$a_9 = 3\left(\frac{2}{3}\right)^{9-1} = \frac{256}{2187}$$

(c) Find a recursive formula for this sequence.

$$a_1 = 3, \;\; a_n = \frac{2}{3}a_{n-1}$$

3 Find the Sum of a Geometric Sequence

Sum of the First *n* Terms of a Geometric Sequence

Let $\{a_n\}$ be a geometric sequence with first term a_1 and common ratio r, where $r \neq 0, r \neq 1$. The sum S_n of the first n terms of $\{a_n\}$ is

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} \\ &= \sum_{k=1}^{n} a_1 r^{k-1} = a_1 \cdot \frac{1 - r^n}{1 - r} \qquad r \neq 0, 1 \end{aligned} \qquad \textbf{(3)}$$

EXAMPLE

Finding the Sum of the First *n* Terms of a Geometric Sequence

Find the sum of the first n terms of the sequence $\left\{3^n\right\}$.

$$a_1 = 3 \quad r = 3$$

$$s_n = 3 \cdot \frac{1-3^n}{1-3} = -\frac{3}{2}\left(1-3^n\right)$$

$$s_n = a_1 \cdot \frac{1-r^n}{1-r}$$

EXAMPLE

Using a Graphing Utility to Find the Sum of a Geometric Sequence

Use a graphing utility to find the sum of the first 15 terms of the sequence $\left\{\left(\frac{1}{3}\right)^n\right\}$; that is, find

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\left(\frac{1}{3}\right)^{15}=\sum_{k=1}^{15}\frac{1}{3}\left(\frac{1}{3}\right)^{k-1}$$

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sum(seq((1/3)^n,
n,1,15,1))
          .4999999652
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4 Determine Whether a Geometric Series Converges or Diverges

DEFINITION

An infinite sum of the form

$$a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots$$

with first term a_1 and common ratio r, is called an **infinite geometric series** and is denoted by

$$\sum_{k=1}^{\infty} a_1 r^{k-1}$$

Convergence of an Infinite Geometric Series

If $|r| < 1$, the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is

$$\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1 - r} \tag{7}$$

EXAMPLE

Determining Whether a Geometric Series Converges or Diverges

Determine if the geometric series

$$\sum_{k=1}^{\infty} 4\left(-\frac{3}{2}\right)^{k-1} = 4-6+9-\ldots$$

converges or diverges. If it converges, find its sum.

$$a_1 = 4 \quad r = \frac{3}{2}$$

Since $|r| > 1$ the series diverges.

$$\boxed{\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}}$$

EXAMPLE Repeating Decimals

Show that the repeating decimal 0.999 … equals 1.

$$0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$a_1 = \frac{9}{10} \quad r = \frac{1}{10}$$

$$S = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$$

$$\boxed{\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}}$$

EXAMPLE Pendulum Swings

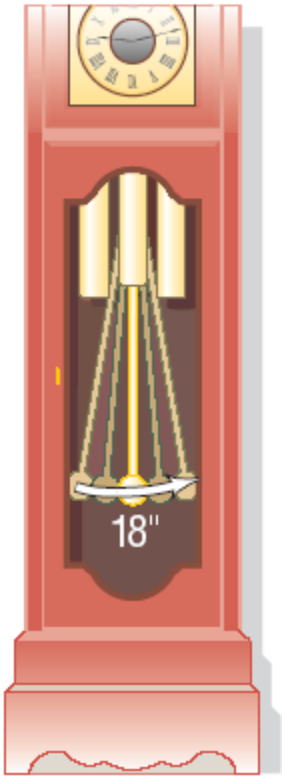
Initially, a pendulum swings through an arc of 18 inches. See Figure 15. On each successive swing, the length of the arc is 0.98 of the previous length.

(a) What is the length of the arc of the 10th swing?

(b) On which swing is the length of the arc first less than 12 inches?

(c) After 15 swings, what total distance will the pendulum have swung?

(d) When it stops, what total distance will the pendulum have swung?



(a) The length of the first swing is 18 inches.
The length of the second swing is 0.98(18) inches.
The length of the third swing is $0.98(0.98)(18) = 0.98^2(18)$ inches.
The length of the arc of the 10th swing is

$$(0.98)^9(18) \approx 15.007 \text{ inches}$$

(b) The length of the arc of the nth swing is $(0.98)^{n-1}(18)$. For this to be exactly 12 inches requires that

$$(0.98)^{n-1}(18) = 12$$

$$(0.98)^{n-1} = \frac{12}{18} = \frac{2}{3}$$

$$n - 1 = \log_{0.98}\left(\frac{2}{3}\right)$$

$$n = 1 + \frac{\ln\left(\frac{2}{3}\right)}{\ln 0.98} \approx 1 + 20.07 = 21.07$$

EXAMPLE Pendulum Swings

Initially, a pendulum swings through an arc of 18 inches. See Figure 15. On each successive swing, the length of the arc is 0.98 of the previous length.

(a) What is the length of the arc of the 10th swing?

(b) On which swing is the length of the arc first less than 12 inches?

(c) After 15 swings, what total distance will the pendulum have swung?

(d) When it stops, what total distance will the pendulum have swung?

(c) After 15 swings, the pendulum will have swung the following total distance L:

$$L = \underbrace{18}_{\text{1st}} + \underbrace{0.98(18)}_{\text{2nd}} + \underbrace{(0.98)^2(18)}_{\text{3rd}} + \underbrace{(0.98)^3(18)}_{\text{4th}} + \cdots + \underbrace{(0.98)^{14}(18)}_{\text{15th}}$$

This is the sum of a geometric sequence. The common ratio is 0.98; the first term is 18. The sum has 15 terms, so

$$L = 18 \cdot \frac{1 - 0.98^{15}}{1 - 0.98} \approx 18(13.07) \approx 235.3 \text{ inches}$$

The pendulum will have swung through approximately 235.3 inches after 15 swings.

EXAMPLE Pendulum Swings

Initially, a pendulum swings through an arc of 18 inches. See Figure 15. On each successive swing, the length of the arc is 0.98 of the previous length.

(a) What is the length of the arc of the 10th swing?

(b) On which swing is the length of the arc first less than 12 inches?

(c) After 15 swings, what total distance will the pendulum have swung?

(d) When it stops, what total distance will the pendulum have swung?

(d) When the pendulum stops, it will have swung the following total distance T:

$$T = 18 + 0.98(18) + (0.98)^2(18) + (0.98)^3(18) + \cdots$$

This is the sum of an infinite geometric series. The common ratio is $r = 0.98$; the first term is $a_1 = 18$. Since $|r| < 1$, the series converges. Its sum is

$$T = \frac{a_1}{1 - r} = \frac{18}{1 - 0.98} = 900$$

The pendulum will have swung a total of 900 inches when it finally stops.

5 Solve Annuity Problems

Deposit	1	2	3	...	$n-1$	n
Amount	$P(1+i)^{n-1}$	$P(1+i)^{n-2}$	$P(1+i)^{n-3}$	...	$P(1+i)$	P

Amount of an Annuity

Suppose P is the deposit in dollars made at the end of each payment period for an annuity paying i percent interest per payment period. The amount A of the annuity after n deposits is

$$A = P\frac{(1+i)^n - 1}{i} \qquad \textbf{(8)}$$

EXAMPLE Determining the Amount of an Annuity

To save for retirement, Brett decides to place \$2000 into an Individual Retirement Account (IRA) each year for the next 30 years. What will the value of the IRA be when Brett makes his 30th deposit? Assume that the rate of return of the IRA is 10% per annum compounded annually.

$$A = 2000\frac{(1+0.10)^{30}-1}{0.10} = \$328,988.05$$

$$\boxed{A = P\frac{(1+i)^n - 1}{i}}$$

EXAMPLE Determining the Amount of an Annuity

To save for her daughter's college education, Ms. Miranda decides to put \$50 aside every month in a credit union account paying 10% interest compounded monthly. She begins this savings program when her daughter is 3 years old. How much will she have saved by the time she makes the 180th deposit? How old is her daughter at this time?

$$A = 50\frac{\left(1+\frac{0.10}{12}\right)^{180}-1}{\frac{0.10}{12}} = \$20,723.52$$

$$\boxed{A = P\frac{(1+i)^n - 1}{i}}$$