Section 13.2

Arithmetic Sequences

1 Determine If a Sequence Is Arithmetic

Arithmetic Sequence

An **arithmetic sequence*** may be defined recursively as $a_1 = a, \quad a_n - a_{n-1} = d$, or as

$$\boxed{a_1 = a, \qquad a_n = a_{n-1} + d} \quad \textbf{(1)}$$

where $a_1 = a$ and d are real numbers. The number a is the first term, and the number d is called the **common difference.**

EXAMPLE Determining If a Sequence Is Arithmetic

Show that the sequence is arithmetic. List the first term and the common difference.

(a) 4, 2, 0, –2, . . .

(a) $a_1 = 4 \quad d = -2$

(b) $\{s_n\} = \{4n - 1\}$

(b) $a_1 = 4(1) - 1 = 3 \quad d = S_n - S_{n-1} = 4n - 1 - \big(4(n-1) - 1\big) = 4$

(c) $\{t_n\} = \{2 - 3n\}$

(c) $a_1 = 2 - 3(1) = -1 \quad d = S_n - S_{n-1} = 2 - 3n - \big(2 - 3(n-1)\big) = -3$

2 Find a Formula for an Arithmetic Sequence

*n*th Term of an Arithmetic Sequence

For an arithmetic sequence $\{a_n\}$ whose first term is a and whose common difference is d, the nth term is determined by the formula

$$\boxed{a_n = a + (n - 1)d}$$

EXAMPLE

Finding a Particular Term of an Arithmetic Sequence

Find the twenty fourth term of the arithmetic sequence:

–3, 0, 3, 6, . . .

$$a_1 = -3 \quad d = 3 \qquad a_n = -3 + (n-1)3$$

$$a_{24} = -3 + (24-1)3 = 66$$

$$\boxed{a_n = a + (n-1)d}$$

EXAMPLE

Finding a Recursive Formula for an Arithmetic Sequence

The eighth term of an arithmetic sequence is 75, and the twentieth term is 39.

(a) Find the first term and the common difference.

(b) Give a recursive formula for the sequence.

(c) What is the nth term of the sequence?

$$\begin{cases} a_8 = a_1 + 7d = 75 \\ a_{20} = a_1 + 19d = 39 \end{cases} \qquad \begin{aligned} -12d &= 36 \\ d &= -3 \end{aligned}$$

With $d = -3$, we use $a_1 + 7d = 75$ and find that $a_1 = 75 - 7d = 75 - 7(-3) = 96$. The first term is $a_1 = 96$, and the common difference is $d = -3$.

$$a_1 = 96, \qquad a_n = a_{n-1} - 3$$

$$a_n = a_1 + (n-1)d = 96 + (n-1)(-3) = 99 - 3n$$

3 Find the Sum of an Arithmetic Sequence

Sum of the First *n* Terms of an Arithmetic Sequence

Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and common difference d. The sum S_n of the first n terms of $\{a_n\}$ may be found in two ways:

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= \sum_{k=1}^{n} [a_1 + (k-1)d] = \frac{n}{2}[2a_1 + (n-1)d] \end{aligned} \qquad \textbf{(3)}$$

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= \sum_{k=1}^{n} [a_1 + (k-1)d] = \frac{n}{2}(a_1 + a_n) \end{aligned} \qquad \textbf{(4)}$$

EXAMPLE

Finding the Sum of an Arithmetic Sequence

Find the sum of the first n terms of the sequence $\{4n + 2\}$

$$a_1 = 4(1) + 2 = 6$$

$$S_n = \frac{n}{2}(6 + 4n + 2) = \frac{n}{2}(4n + 8)$$

$$\boxed{S_n = \frac{n}{2}(a_1 + a_n)}$$

EXAMPLE

Finding the Sum of an Arithmetic Sequence

Find the sum: $52 + 57 + 62 + 67 + \ldots + 122$

$$122 = 52 + (n-1)5$$

$$70 = (n-1)5$$

$$14 = n - 1$$

$$15 = n$$

$$S_{15} = \frac{15}{2}(52 + 122)$$

$$= 1305$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

EXAMPLE Creating a Floor Design

A ceramic tile floor is designed in the shape of a trapezoid 20 feet wide at the base and 10 feet wide at the top. See Figure 13. The tiles, 12 inches by 12 inches, are to be placed so that each successive row contains one less tile than the preceding row. How many tiles will be required?

The bottom row requires 20 tiles and the top row, 10 tiles. Since each successive row requires one less tile, the total number of tiles required is

$$S = 20 + 19 + 18 + \cdots + 11 + 10$$

This is the sum of an arithmetic sequence; the common difference is -1. The number of terms to be added is $n = 11$, with the first term $a_1 = 20$ and the last term $a_{11} = 10$. The sum S is

$$S = \frac{n}{2}(a_1 + a_{11}) = \frac{11}{2}(20 + 10) = 165$$