Section 6.7

Financial Models

1 Determine the Future Value of a Lump Sum of Money

THEOREM

Simple Interest Formula

If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is

$$\boxed{I = Prt \qquad \textbf{(1)}}$$

Interest charged according to formula (1) is called **simple interest.**

Annually:	Once per year
Semiannually:	Twice per year
Quarterly:	Four times per year
Monthly:	12 times per year
Daily:	365 times per year

EXAMPLE Computing Compound Interest

$$I = Prt$$

A credit union pays interest of 4% per annum compounded quarterly on a certain savings plan. If \$2000 is deposited in such a plan and the interest is left to accumulate, how much is in the account after 1 year?

Interest after first quarter: $I = 2000(0.04)\left(\frac{1}{4}\right) = \$20$

New Principal $= 2000 + 20 = \$2020$

Interest after second quarter: $I = 2020(0.04)\left(\frac{1}{4}\right) = \$20.20$

New Principal $= 2020 + 20.20 = \$2040.20$

Interest after third quarter: $I = 2040.20(0.04)\left(\frac{1}{4}\right) = \$20.40$

New Principal $= 2040.20 + 20.40 = \$2060.60$

Interest after fourth quarter: $I = 2060.60(0.04)\left(\frac{1}{4}\right) = \$20.61$

New Principal $= 2060.60 + 20.61 = \$2081.21$

$$\text{Interest} = \text{principal} \times \text{rate} \times \text{time} = P \cdot r \cdot \frac{1}{n} = P \cdot \left(\frac{r}{n}\right)$$

The amount A after one compounding period is

$$A = P + I = P + P \cdot \left(\frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)$$

After two compounding periods,

$$A = P \cdot \underbrace{\left(1 + \frac{r}{n}\right)}_{\text{New principal}} + \underbrace{P \cdot \left(1 + \frac{r}{n}\right)\left(\frac{r}{n}\right)}_{\text{Interest on new principal}} = P \cdot \left(1 + \frac{r}{n}\right)\left(1 + \frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)^2$$

After three compounding periods, the amount A is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^2 + P \cdot \left(1 + \frac{r}{n}\right)^2\left(\frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)^2 \cdot \left(1 + \frac{r}{n}\right) = P \cdot \left(1 + \frac{r}{n}\right)^3$$

Because t years will contain $n \cdot t$ compounding periods, after t years we have

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

THEOREM

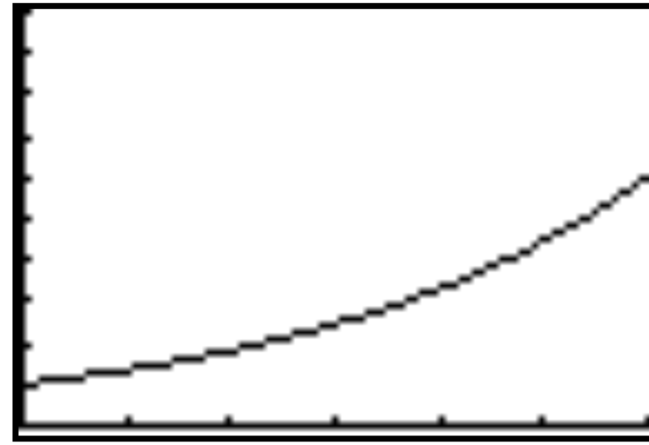
Compound Interest Formula

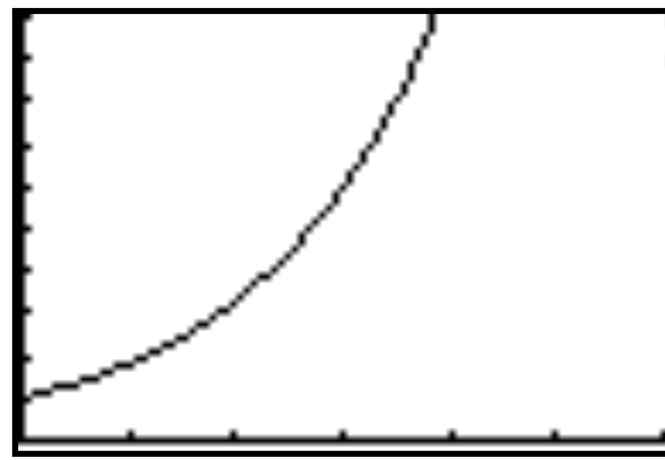
The amount A after t years due to a principal P invested at an annual interest rate r compounded n times per year is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt} \qquad \textbf{(2)}$$

Exploration

To see the effects of compounding interest monthly on an initial deposit of \$1, graph $Y_1 = \left(1 + \frac{r}{12}\right)^{12x}$ with $r = 0.06$ and $r = 0.12$ for $0 \leq x \leq 30$. What is the future value of \$1 in 30 years when the interest rate per annum is $r = 0.06$ (6%)? What is the future value of \$1 in 30 years when the interest rate per annum is $r = 0.12$ (12%)? Does doubling the interest rate double the future value?





EXAMPLE

Comparing Investments Using Different Compounding Periods

Investing \$1000 at an annual rate of 10% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

Annual compounding ($n = 1$):
$$A = P \cdot (1 + r) = (\$1000)(1 + 0.10) = \$1100.00$$

Semiannual compounding ($n = 2$):
$$A = P \cdot \left(1 + \frac{r}{2}\right)^2 = (\$1000)(1 + 0.05)^2 = \$1102.50$$

Quarterly compounding ($n = 4$):
$$A = P \cdot \left(1 + \frac{r}{4}\right)^4 = (\$1000)(1 + 0.025)^4 = \$1103.81$$

Monthly compounding ($n = 12$):
$$A = P \cdot \left(1 + \frac{r}{12}\right)^{12} = (\$1000)(1 + 0.00833)^{12} = \$1104.71$$

Daily compounding ($n = 365$):
$$A = P \cdot \left(1 + \frac{r}{365}\right)^{365} = (\$1000)(1 + 0.000274)^{365} = \$1105.16$$

Comparing Investments Using Different Compounding Periods

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{n} = P \cdot \left(1 + \frac{1}{\frac{n}{r}}\right)^{n} = P \cdot \left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{n/r}\right]^{r} \underset{h = \frac{n}{r}}{\uparrow}{=} P \cdot \left[\left(1 + \frac{1}{h}\right)^{h}\right]^{r}$$

	$\left(1 + \frac{r}{n}\right)^{n}$			
	$n = 100$	$n = 1000$	$n = 10{,}000$	e^r
$r = 0.05$	1.0512580	1.0512698	1.051271	1.0512711
$r = 0.10$	1.1051157	1.1051654	1.1051704	1.1051709
$r = 0.15$	1.1617037	1.1618212	1.1618329	1.1618342
$r = 1$	2.7048138	2.7169239	2.7181459	2.7182818

THEOREM

Continuous Compounding

The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is

$$\boxed{A = Pe^{rt} \qquad \textbf{(4)}}$$

EXAMPLE Using Continuous Compounding

Find the amount A that results from investing a principal P of \$2000 at an annual rate r of 8% compounded continuously for a time t of 1 year.

$$A = 2000\,e^{(0.08)(1)} = 2000\,e^{(0.08)(1)} = \$2166.57$$

$$\boxed{A = Pe^{rt}}$$

2 Calculate Effective Rates of Return

Suppose that you have $1000 and a bank offers to pay you 3% annual interest on a savings account with interest compounded monthly. In one year:

$$A = \$1000\left(1 + \frac{0.03}{12}\right)^{12} \qquad \text{Use } A = P\left(1 + \frac{r}{n}\right)^{n} \text{ with } P = \$1000, r = 0.03, n = 12.$$

$$= \$1030.42$$

So the interest earned is \$30.42. Using $I = Prt$ with $t = 1$, $I = \$30.42$, and $P = \$1000$, we find the annual simple interest rate is $0.03042 = 3.042\%$. This interest rate is known as the *effective rate of interest.*

The **effective rate** of interest is the equivalent annual simple rate of interest that would yield the same amount as compounding after 1 year.

Effective Rate of Interest

The effective rate of interest r_e of an investment earning an annual interest rate r is given by

$$\text{Compounding } n \text{ times per year: } r_e = \left(1 + \frac{r}{n}\right)^n - 1$$

$$\text{Continuous compounding: } r_e = e^r - 1$$

EXAMPLE

Computing the Effective Rate of Interest —Which Is the Best Deal?

Suppose you want to open a money market account. You visit three banks to determine their money market rates. Bank A offers you 4% annual interest compounded daily, Bank B offers you 4.1% compounded monthly, and Bank C offers 3.95% compounded continuously. Determine which bank is offering the best deal.

Bank A	Bank B	Bank C
$r_e = \left(1 + \frac{0.04}{365}\right)^{365} - 1$	$r_e = \left(1 + \frac{0.041}{12}\right)^{12} - 1$	$r_e = e^{0.0395} - 1$
$r_e = 0.0408084$	$r_e = 0.0417793$	$r_e = 0.0402905$
$r_e = 4.081\%$	$\boxed{r_e = 4.178\%}$	$r_e = 4.029\%$

Bank B is offering the best deal

3 Determine the Present Value of a Lump Sum of Money

THEOREM

Present Value Formulas

The present value P of A dollars to be received after t years, assuming a per annum interest rate r compounded n times per year, is

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt} \quad \textbf{(5)}$$

If the interest is compounded continuously,

$$P = Ae^{-rt} \quad \textbf{(6)}$$

EXAMPLE

Computing the Value of a Zero-coupon Bond

A zero-coupon (noninterest-bearing) bond can be redeemed in 10 years for \$1000. How much should you be willing to pay for it now if you want a return of

(a) 7% compounded monthly? (b) 6% compounded continuously?

$$\text{(a)}\quad p = 1000\left(1+\frac{0.07}{12}\right)^{-12(10)} = \$497.60$$

$$\text{(b)}\quad p = 1000e^{-0.06(10)} = \$548.81$$

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt}$$

$$P = Ae^{-rt}$$

4 Determine the Rate of Interest or Time Required to Double a Lump Sum of Money

EXAMPLE

Rate of Interest Required to Double an Investment

What annual rate of interest compounded quarterly should you seek if you want to double your investment in 6 years?

$$2P = P\left(1+\frac{r}{4}\right)^{4(6)} \qquad 2 = \left(1+\frac{r}{4}\right)^{24} \qquad \sqrt[24]{2} = 1+\frac{r}{4}$$

$$r = 4\left(\sqrt[24]{2}-1\right) = 0.1172089466$$

$$A = P\cdot\left(1+\frac{r}{n}\right)^{nt}$$

The annual rate of interest needed to double the principal in 6 years is 11.72%.

EXAMPLE

Time Required to Double or Triple an Investment

(a) How long will it take for an investment to double in value if it earns 6% compounded continuously?

(b) How long will it take to triple at this rate?

(a) $2P = Pe^{0.06t}$

$$2 = e^{0.06t}$$

$$\ln 2 = \ln e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years}$$

(b) $3P = Pe^{0.06t}$

$$3 = e^{0.06t}$$

$$\ln 3 = \ln e^{0.06t}$$

$$\ln 3 = 0.06t$$

$$t = \frac{\ln 3}{0.06} \approx 18.31 \text{ years}$$

$$\boxed{A = Pe^{rt}}$$