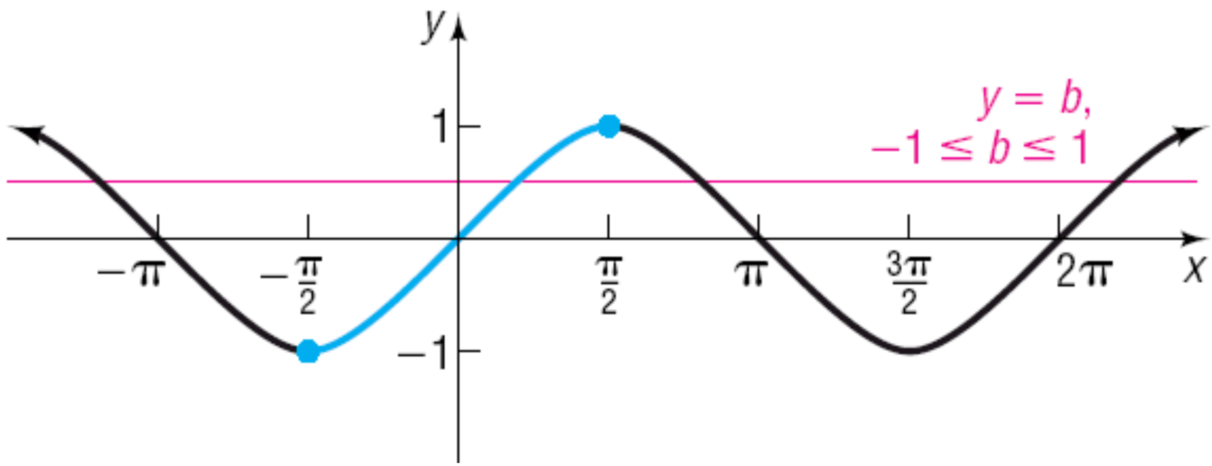
Section 8.1

The Inverse Sine, Cosine, and Tangent Functions

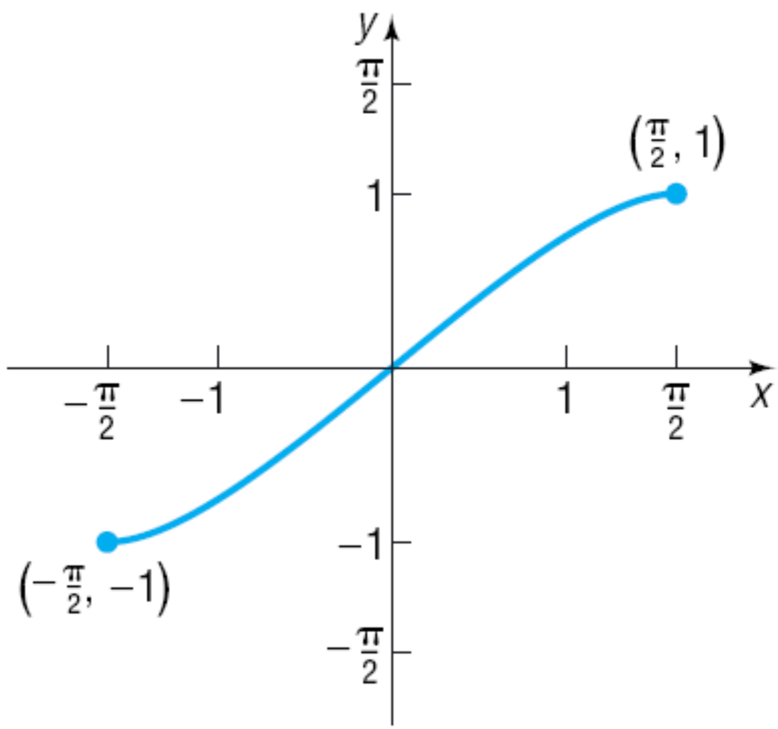
Review of Properties of Functions and Their Inverses

1. $f^{-1}(f(x)) = x$ for every x in the domain of f and $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.
2. Domain of f = range of f^{-1}, and range of f = domain of f^{-1}.
3. The graph of f and the graph of f^{-1} are symmetric with respect to the line $y = x$.
4. If a function $y = f(x)$ has an inverse function, the equation of the inverse function is $x = f(y)$. The solution of this equation is $y = f^{-1}(x)$.

The Inverse Sine Function



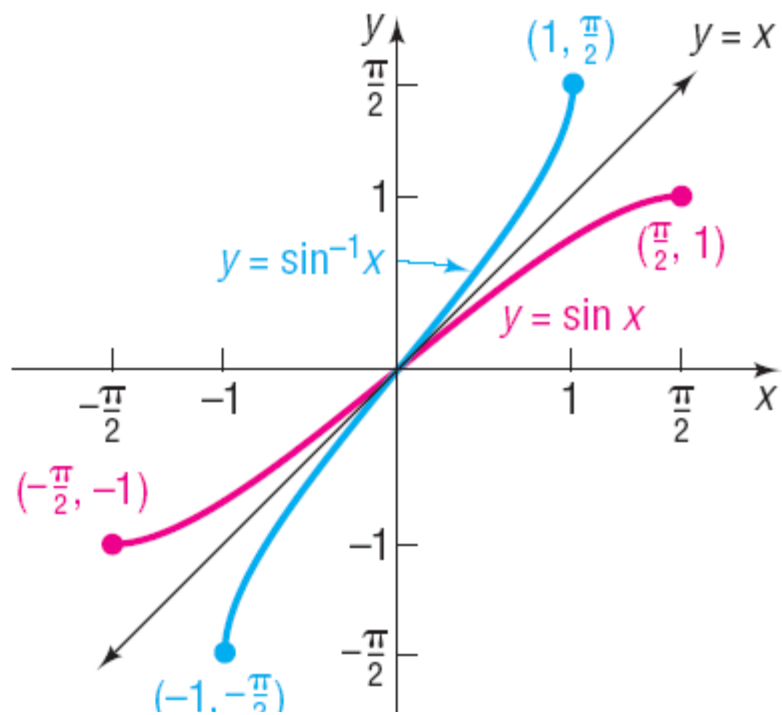
$$y = \sin x, -\infty < x < \infty, -1 \le y \le 1$$



$$y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}, -1 \le y \le 1$$

$$y = \sin^{-1} x \quad \text{means} \quad x = \sin y$$

$$\text{where} \quad -1 \le x \le 1 \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$



$$y = \sin^{-1} x, -1 \le x \le 1, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

1 Find the Exact Value of an Inverse Sine Function

EXAMPLE Finding the Exact Value of an Inverse Sine Function

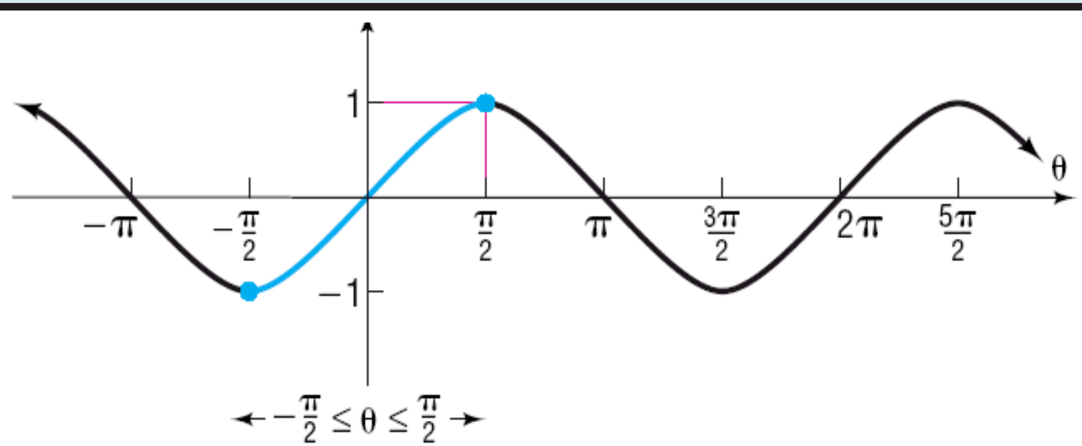
Find the exact value of: $\sin^{-1} 1$

We seek the angle $\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals 1.

We see that the only angle θ within the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is 1 is $\frac{\pi}{2}$.

(Note that $\sin \frac{5\pi}{2}$ also equals 1, but $\frac{5\pi}{2}$ lies outside the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1



EXAMPLE

Finding the Exact Value of an Inverse Sine Function

Find the exact value of: $\sin^{-1}\left(-\frac{1}{2}\right)$

We seek the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

2 Find an Approximate Value of an Inverse Sine Function

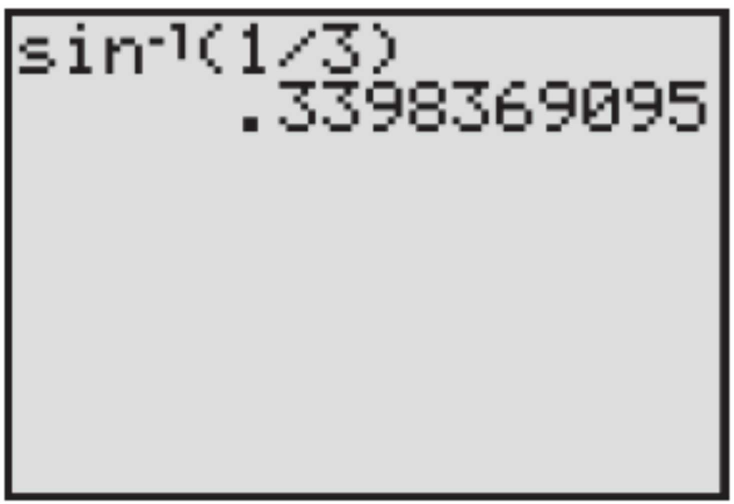
EXAMPLE

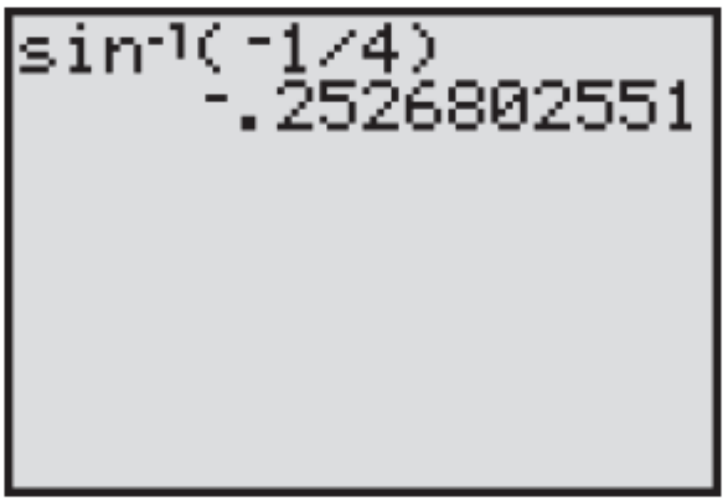
Finding an Approximate Value of an Inverse Sine Function

Find an approximate value of:

(a) $\sin^{-1} \dfrac{1}{3}$ (b) $\sin^{-1}\left(-\dfrac{1}{4}\right)$

Express the answer in radians rounded to two decimal places.





3 Use Properties of Inverse Functions to Find Exact Values of Certain Composite Functions

Properties of Inverse Functions

$$f^{-1}(f(x)) = \sin^{-1}(\sin x) = x, \qquad \text{where } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$f(f^{-1}(x)) = \sin(\sin^{-1} x) = x, \qquad \text{where } -1 \le x \le 1$$

EXAMPLE Finding the Exact Value of Certain Composite Functions

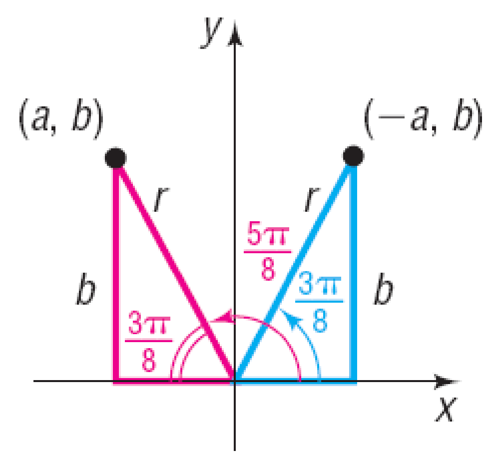
Find the exact value of each of the following composite functions:

(a) $\sin^{-1}\left(\sin\frac{\pi}{8}\right)$

Because $\frac{\pi}{8}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$,

$$\sin^{-1}\left(\sin\frac{\pi}{8}\right)=\frac{\pi}{8}$$

(b) $\sin^{-1}\left(\sin\frac{5\pi}{8}\right)$



$\frac{5\pi}{8}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ so we need an angle in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\sin\theta=\sin\frac{5\pi}{8}$.

$$\sin^{-1}\left(\sin\frac{5\pi}{8}\right)=\sin^{-1}\left(\sin\frac{3\pi}{8}\right)=\frac{3\pi}{8}$$

EXAMPLE

Finding the Exact Value of Certain Composite Functions

Find the exact value, if any, of each composite function.

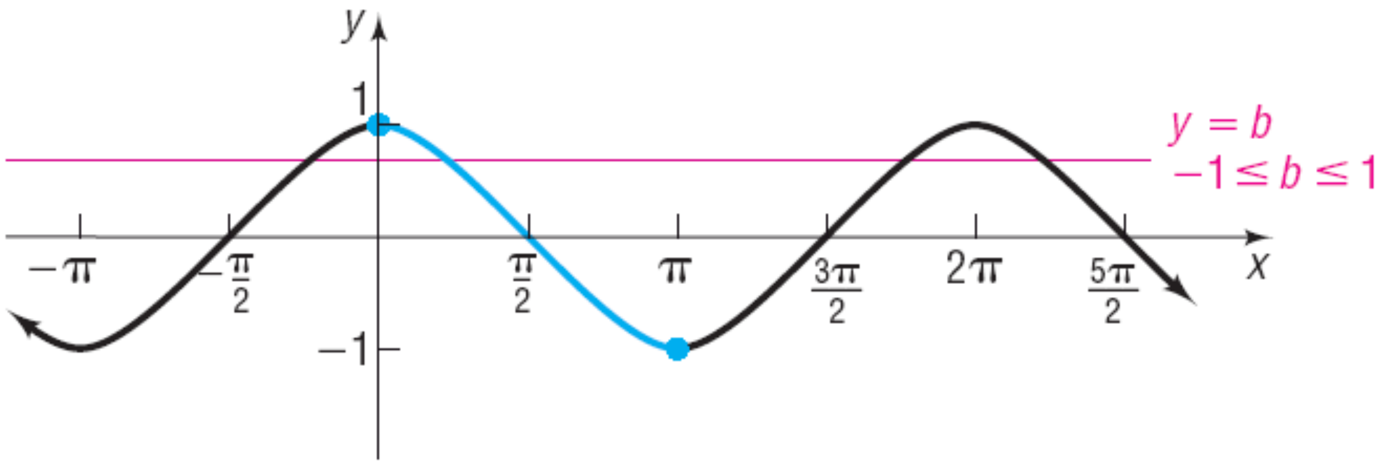
(a) $\sin(\sin^{-1} 0.5)$ (b) $\sin(\sin^{-1} 1.8)$

(a) The composite function $\sin(\sin^{-1} 0.5)$ follows the form of equation (2b) and 0.5 is in the interval $[-1, 1]$. So we use (2b):

$$\sin(\sin^{-1} 0.5) = 0.5$$

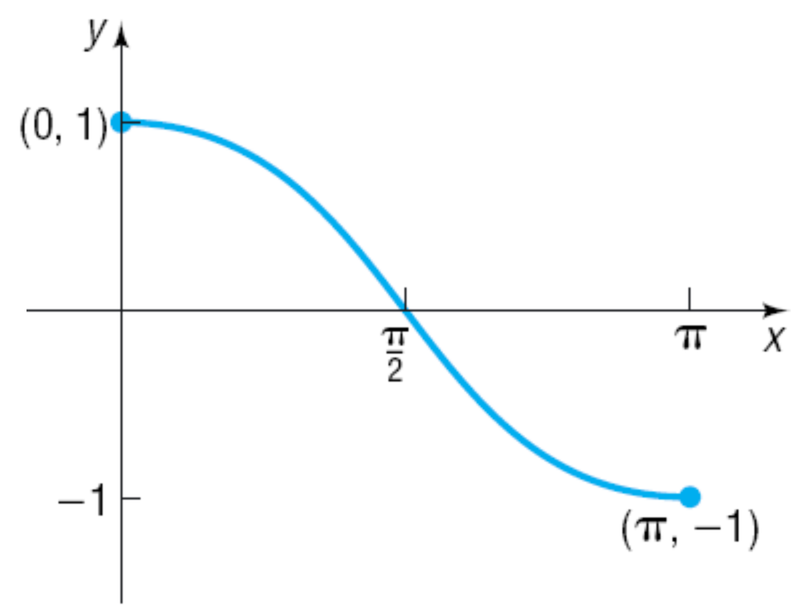
(b) The composite function $\sin(\sin^{-1} 1.8)$ follows the form of equation (2b), but 1.8 is not in the domain of the inverse sine function. This composite function is not defined.

The Inverse Cosine Function

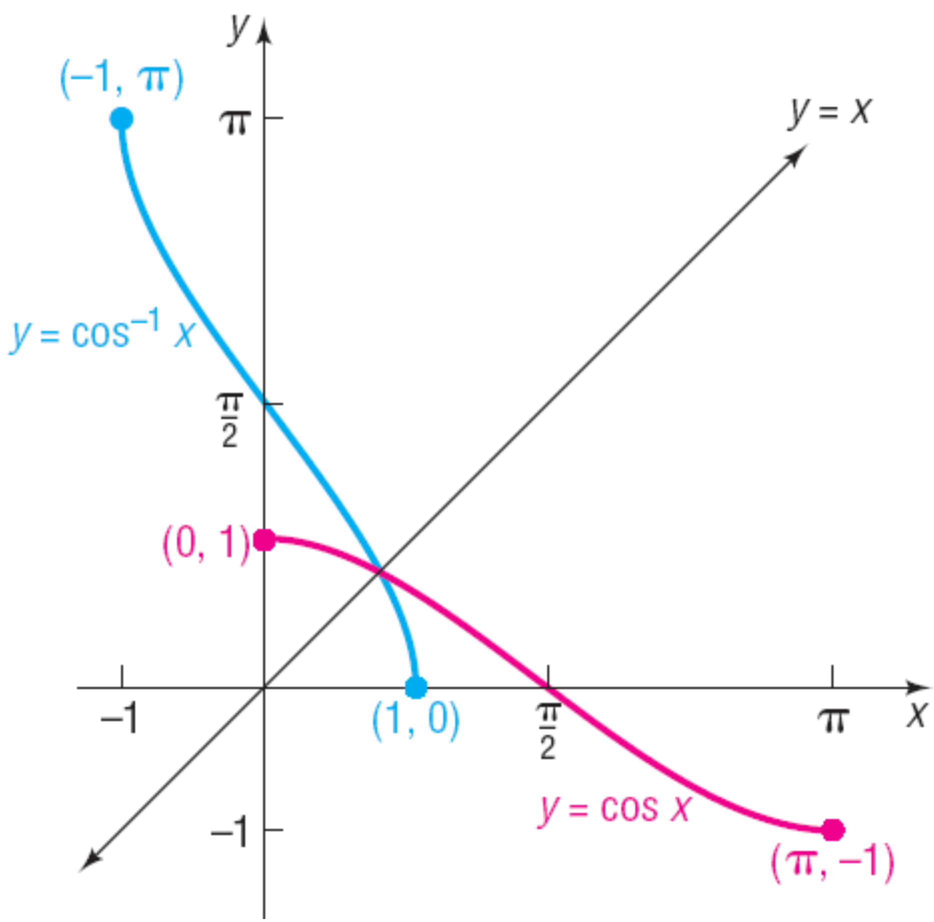


$$y = \cos x, -\infty < x < \infty, \quad -1 \le y \le 1$$

$$y = \cos x, 0 \le x \le \pi, -1 \le y \le 1$$



$$y = \cos^{-1} x \quad \text{means} \quad x = \cos y$$
$$\text{where} \quad -1 \le x \le 1 \quad \text{and} \quad 0 \le y \le \pi$$



$$y = \cos^{-1} x, -1 \le x \le 1, 0 \le y \le \pi$$

EXAMPLE

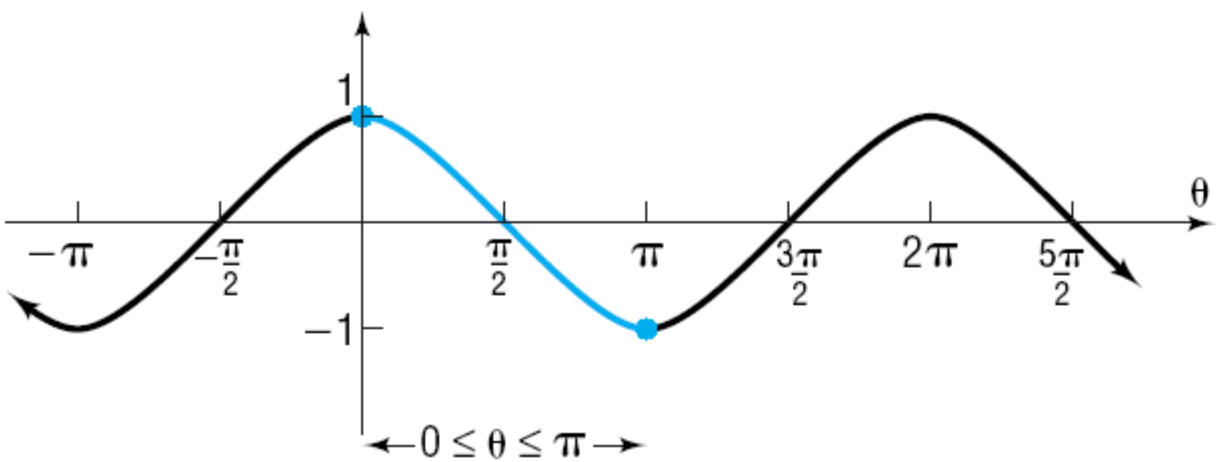
Finding the Exact Value of an Inverse Cosine Function

θ	$\cos\theta$
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$
π	-1

Find the exact value of: $\cos^{-1} 0$

We seek the angle θ, $0 \le \theta \le \pi$, whose cosine equals 0.

$$\cos^{-1} 0 = \frac{\pi}{2}$$



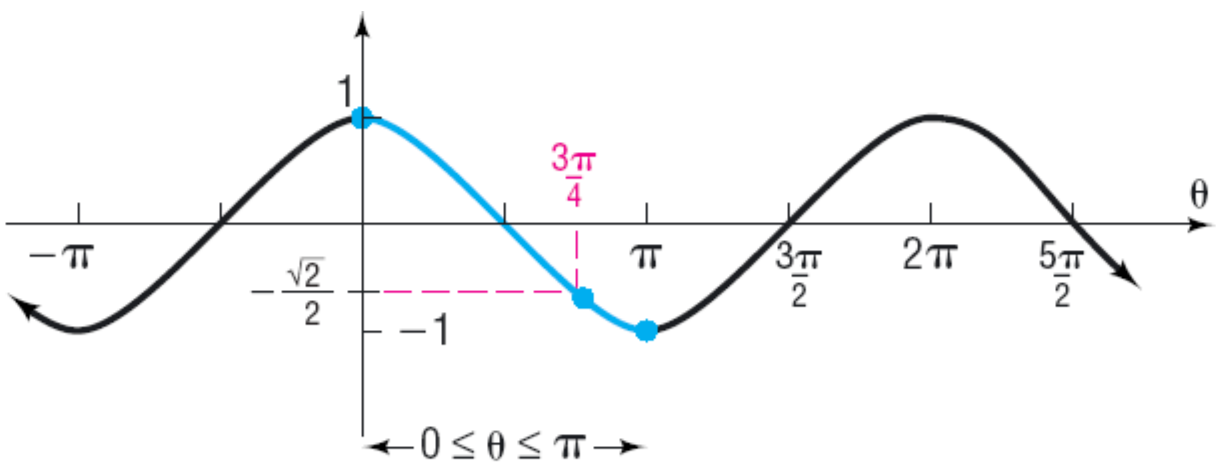
EXAMPLE

Finding the Exact Value of an Inverse Cosine Function

θ	$\cos\theta$
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$
π	-1

Find the exact value of: $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$



Properties of Inverse Functions

$$f^{-1}(f(x)) = \cos^{-1}(\cos x) = x, \qquad \text{where } 0 \le x \le \pi$$
$$f(f^{-1}(x)) = \cos(\cos^{-1} x) = x, \qquad \text{where } -1 \le x \le 1$$

EXAMPLE

Using Properties of Inverse Functions to Find the Exact Value of Certain Composite Functions

Find the exact value of:

(a) $\cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right] = \frac{5\pi}{6}$ since $\frac{5\pi}{6}$ is in the interval $\left[0,\pi\right)$.

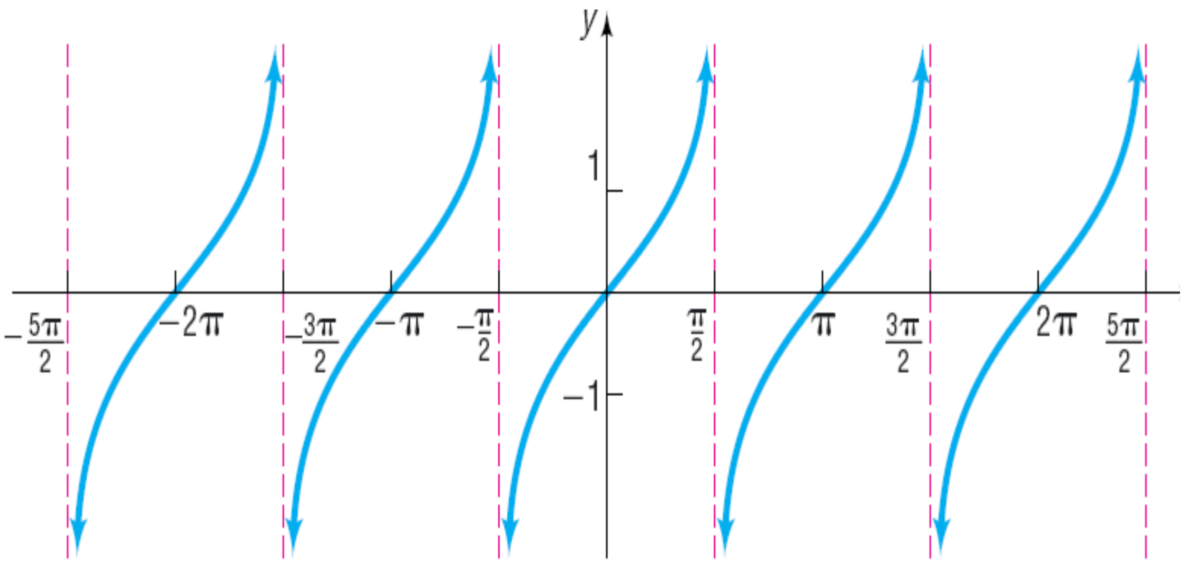
(b) $\cos\left(\cos^{-1}0.2\right) = 0.2$ since 0.2 is in the interval $\left[-1,1\right]$.

(c) $\cos^{-1}\left[\cos\left(\frac{5\pi}{4}\right)\right] = \cos^{-1}\left[\cos\left(\frac{3\pi}{4}\right)\right] = \frac{3\pi}{4}$

Since $\frac{5\pi}{4}$ is not in the interval $\left[0,\pi\right)$ we find an angle that has the same cosine value that is in that interval. $\cos\left(\frac{5\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$

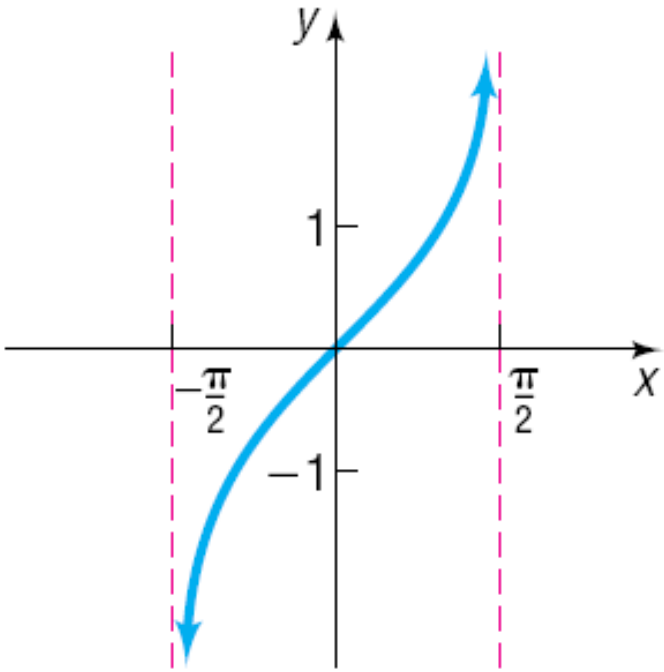
(d) $\cos\left(\cos^{-1}2\right)$ This is undefined since 2 is not the interval $\left[-1,1\right]$.

The Inverse Tangent Function



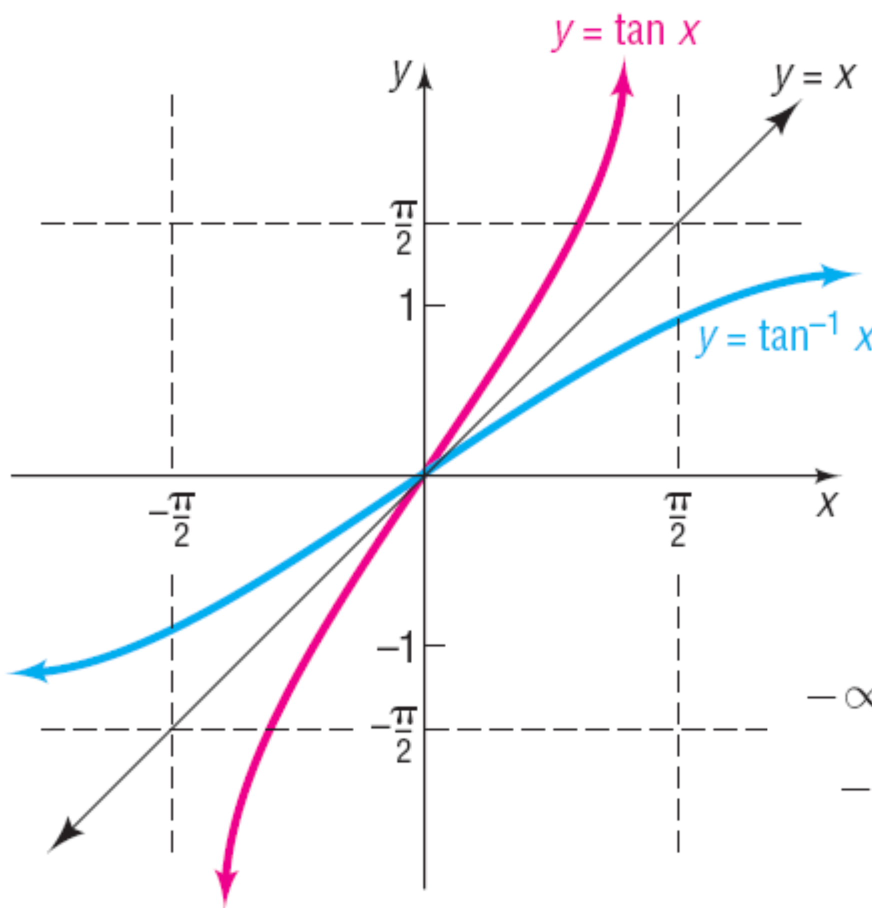
$y = \tan x, -\infty < x < \infty$, x not equal to odd multiples of $\frac{\pi}{2}$, $-\infty < y < \infty$

$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$, $-\infty < y < \infty$



$$y = \tan^{-1} x \quad \text{means} \quad x = \tan y$$

$$\text{where} \quad -\infty < x < \infty \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$



$$y = \tan^{-1} x,$$

$$-\infty < x < \infty,$$

$$-\frac{\pi}{2} < y < \frac{\pi}{2}$$

EXAMPLE

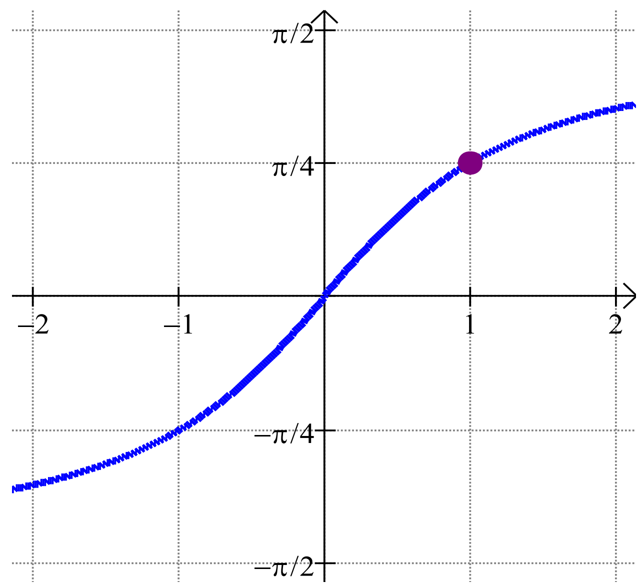
Finding the Exact Value of an Inverse Tangent Function

θ	$\tan\theta$
$-\frac{\pi}{2}$	Undefined
$-\frac{\pi}{3}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3}$
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	Undefined

Find the exact value of: $\tan^{-1} 1$

We seek the angle $\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 1.

$$\tan^{-1} 1 = \frac{\pi}{4}$$



$y = \tan^{-1} x$

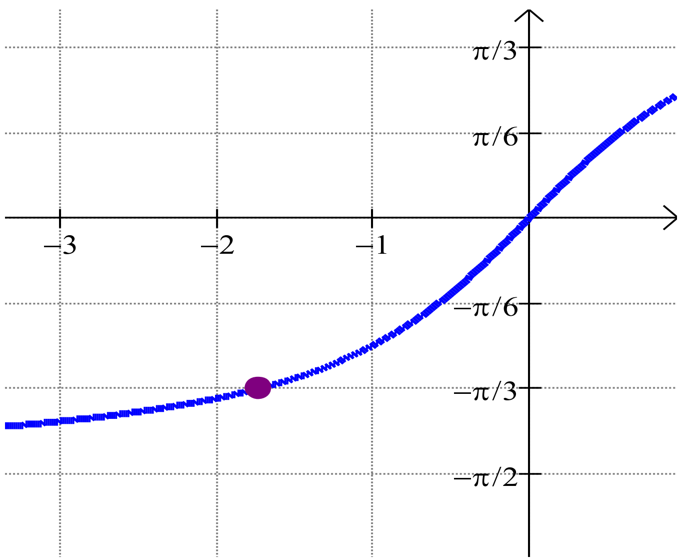
EXAMPLE

Finding the Exact Value of an Inverse Tangent Function

θ	$\tan\theta$
$-\frac{\pi}{2}$	Undefined
$-\frac{\pi}{3}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3}$
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	Undefined

Find the exact value of: $\tan^{-1}\left(-\sqrt{3}\right)$

We seek the angle $\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $-\sqrt{3}$.



$$y = \tan^{-1} x$$

$$\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Properties of Inverse Functions

$$f^{-1}(f(x)) = \tan^{-1}(\tan x) = x \qquad \text{where } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(f^{-1}(x)) = \tan(\tan^{-1} x) = x \qquad \text{where } -\infty < x < \infty$$

4 Find the Inverse Function of a Trigonometric Function

EXAMPLE

Finding the Inverse Function of a Trigonometric Function

Find the inverse function f^{-1} of $f(x) = 3\cos x + 1, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Find the range of f and the domain and range of f^{-1}.

Recall that the domain of *f* is the range of f^{-1} and the range of *f* is the domain of f^{-1}.

$$y = 3\cos x + 1$$

$$x = 3\cos y + 1$$ interchange x and y

$$3\cos y = x - 1$$

So the range of f is $-2 \le y \le 4$

$$\cos y = \frac{x-1}{3}$$

and the range of f^{-1} is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$y = \cos^{-1}\left(\frac{x-1}{3}\right) = f^{-1}(x)$$

Domain of f^{-1} is $-1 \le \frac{x-1}{3} \le 1$

$-3 \le x - 1 \le 3$ $\quad$ $-2 \le x \le 4$

5 Solve Equations Involving Inverse Trigonometric Functions

EXAMPLE

Solving an Equation Involving an Inverse Trigonometric Function

Solve the equation: $2\cos^{-1} x = \dfrac{\pi}{2}$

$$\cos^{-1} x = \frac{\pi}{4}$$

$$x = \cos\frac{\pi}{4}$$

$$x = \frac{\sqrt{2}}{2}$$