Section 6.4

Logarithmic Functions

Recall that a one-to-one function $y = f(x)$ has an inverse function that is defined (implicitly) by the equation $x = f(y)$. In particular, the exponential function $y = f(x) = a^x$, where $a > 0$ and $a \neq 1$, is one-to-one and hence has an inverse function that is defined implicitly by the equation

$$x = a^y, \qquad a > 0, \qquad a \neq 1$$

This inverse function is so important that it is given a name, the *logarithmic function*.

DEFINITION

The **logarithmic function to the base *a***, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm to the base a of x") and is defined by

$$\boxed{y = \log_a x \quad \text{if and only if} \quad x = a^y}$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

EXAMPLE Relating Logarithms to Exponents

(a) If $y = \log_3 x$, then $x = 3^y$.

For example, $4 = \log_3 81$ is equivalent to $81 = 3^4$.

(b) If $y = \log_5 x$, then $x = 5^y$.

For example, $-1 = \log_5\left(\frac{1}{5}\right)$ is equivalent to $\frac{1}{5} = 5^{-1}$.

1 Change Exponential Statements to Logarithmic Statements and Logarithmic Statements to Exponential Statements

EXAMPLE

Changing Exponential Statements to Logarithmic Statements

Change each exponential expression to an equivalent expression involving a logarithm.

(a) $1.2^3 = m$ (b) $e^b = 9$ (c) $a^4 = 24$

(a) If $1.2^3 = m$, then $3 = \log_{1.2} m$.

(b) If $e^b = 9$, then $b = \log_e 9$.

(c) If $a^4 = 24$, then $4 = \log_a 24$.

$y = \log_a x$ and $x = a^y$.

EXAMPLE

Changing Logarithmic Statements to Exponential Statements

Change each logarithmic expression to an equivalent expression involving an exponent.

(a) $\log_a 4 = 5$ (b) $\log_e b = -3$ (c) $\log_3 5 = c$

(a) If $\log_a 4 = 5$, then $a^5 = 4$.

(b) If $\log_e b = -3$, then $e^{-3} = b$.

(c) If $\log_3 5 = c$, then $3^c = 5$.

$y = \log_a x$ and $x = a^y$.

2 Evaluate Logarithmic Expressions

EXAMPLE

Finding the Exact Value of a Logarithmic Expression

$$(a)\ \log_3 81 \qquad (b)\ \log_2 \frac{1}{8}$$

(a) 3 raised to what power yields 81?

$$y = \log_3 81$$
$$3^y = 81$$

$$3^y = 3^4$$
$$y = 4$$

Therefore, $\log_3 81 = 4$

(b) 2 raised to what power yields $\frac{1}{8}$?

$$y = \log_2 \frac{1}{8}$$
$$2^y = \frac{1}{8} = \frac{1}{2^3} = 2^{-3}$$

$$2^y = 2^{-3}$$
$$y = -3$$

Therefore, $\log_2 \frac{1}{8} = -3$

3 Determine the Domain of a Logarithmic Function

Domain of the logarithmic function = Range of the exponential function = $(0, \infty)$

Range of the logarithmic function = Domain of the exponential function = $(-\infty, \infty)$

$y = \log_a x$ (defining equation: $x = a^y$)

Domain: $0 < x < \infty$ Range: $-\infty < y < \infty$

EXAMPLE Finding the Domain of a Logarithmic Function

Find the domain of each logarithmic function.

(a) $f(x)=\log_3(x-2)$ (b) $F(x)=\log_2\left(\dfrac{x+3}{x-1}\right)$

(c) $h(x)=\log_2|x-1|$

(a) The domain of f consists of all x for which $x-2>0$.

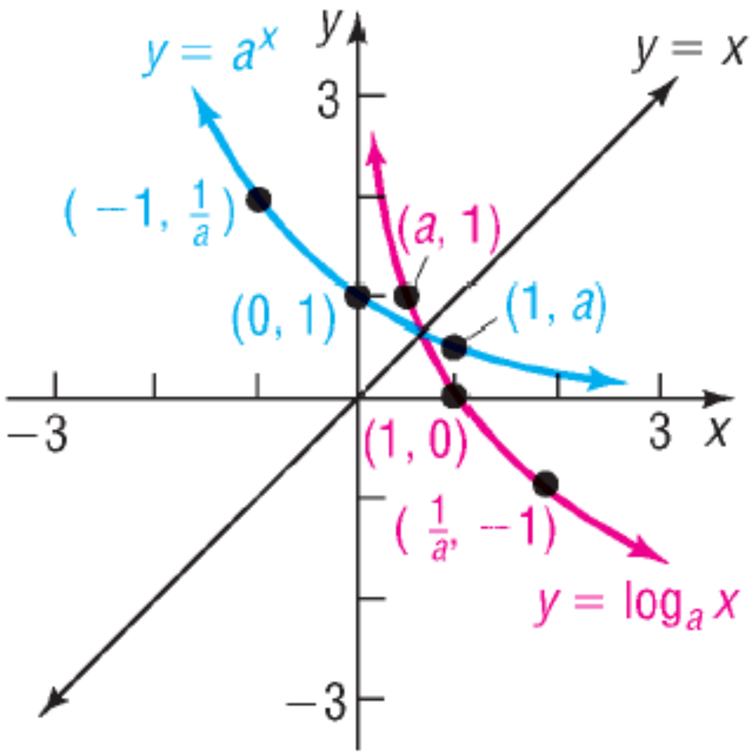
$x>2$ or $(2,\infty)$

(b) The domain of F is restricted to $\left(\dfrac{x+3}{x-1}\right)>0$ $(-\infty,-3)\cup(1,\infty)$.

(c) Since the absolute value function is never negative, the domain would consist of all real numbers except $x-1=0$.

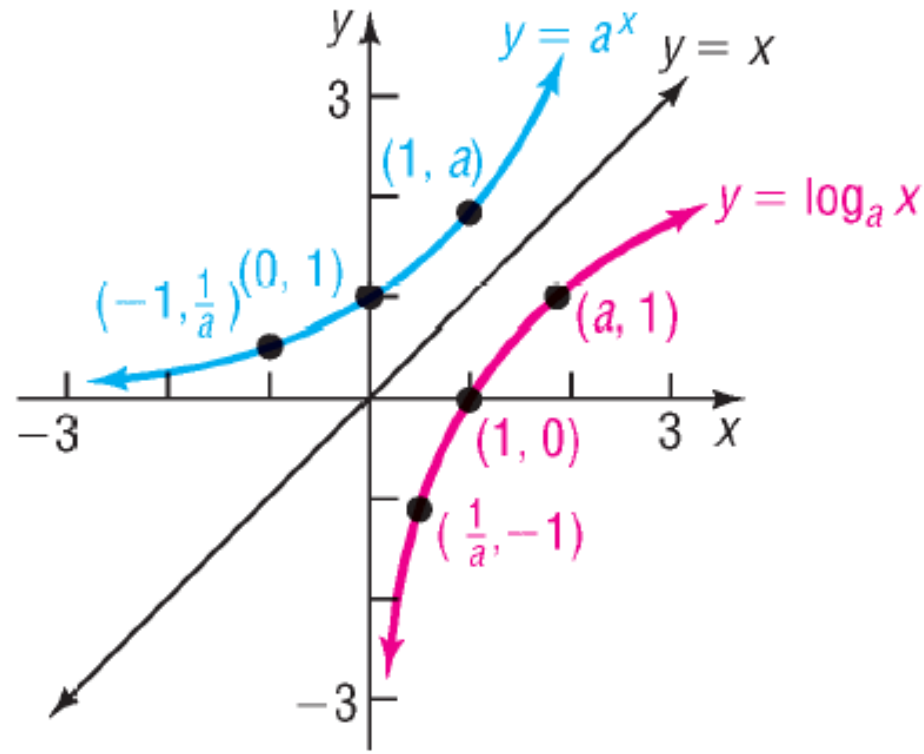
$(-\infty,1)\cup(1,\infty)$

4 Graph Logarithmic Functions



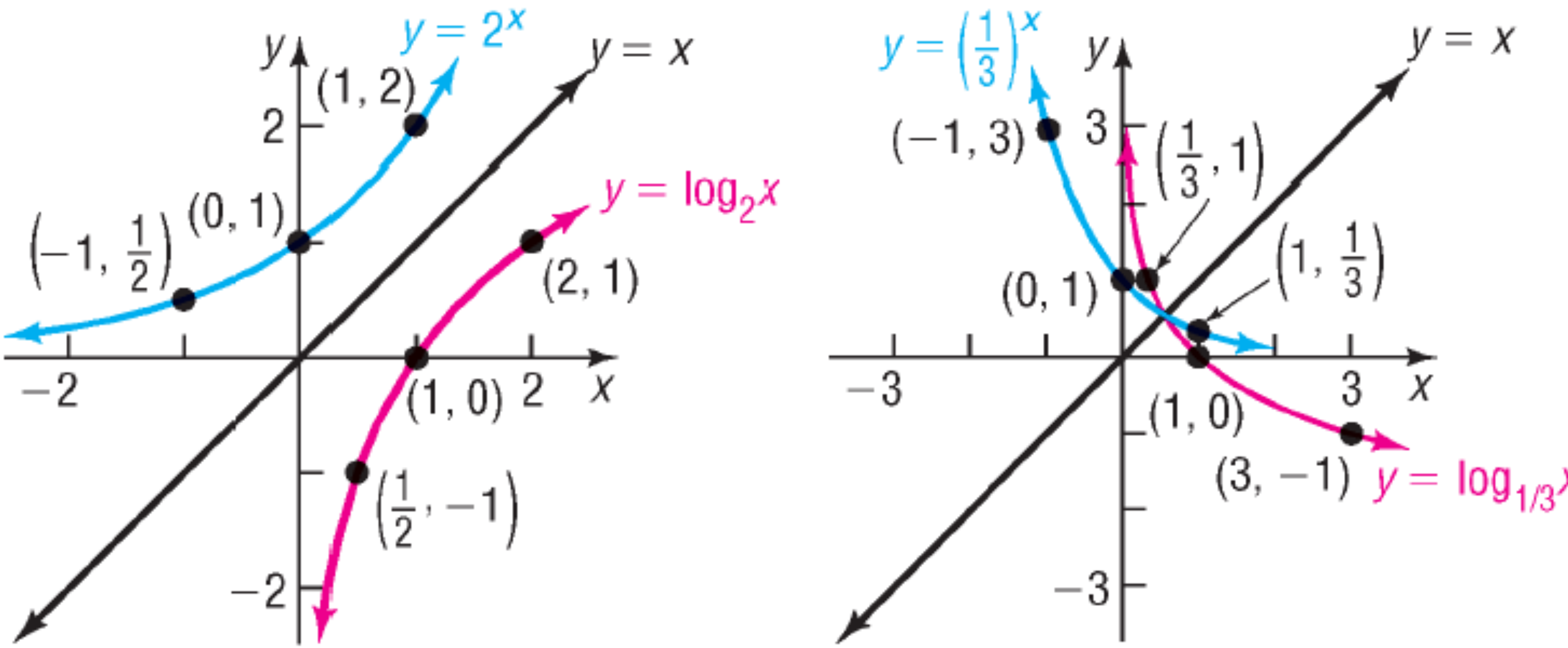
y
y = a^x
y = x
3
(−1, 1/a)
(a, 1)
(0, 1)
(1, a)
−3
3
x
(1, 0)
(1/a, −1)
y = log_a x
−3

(a) $0 < a < 1$



y
y = a^x
y = x
3
(1, a)
y = log_a x
(−1, 1/a)
(0, 1)
(a, 1)
−3
(1, 0)
3
x
(1/a, −1)
−3

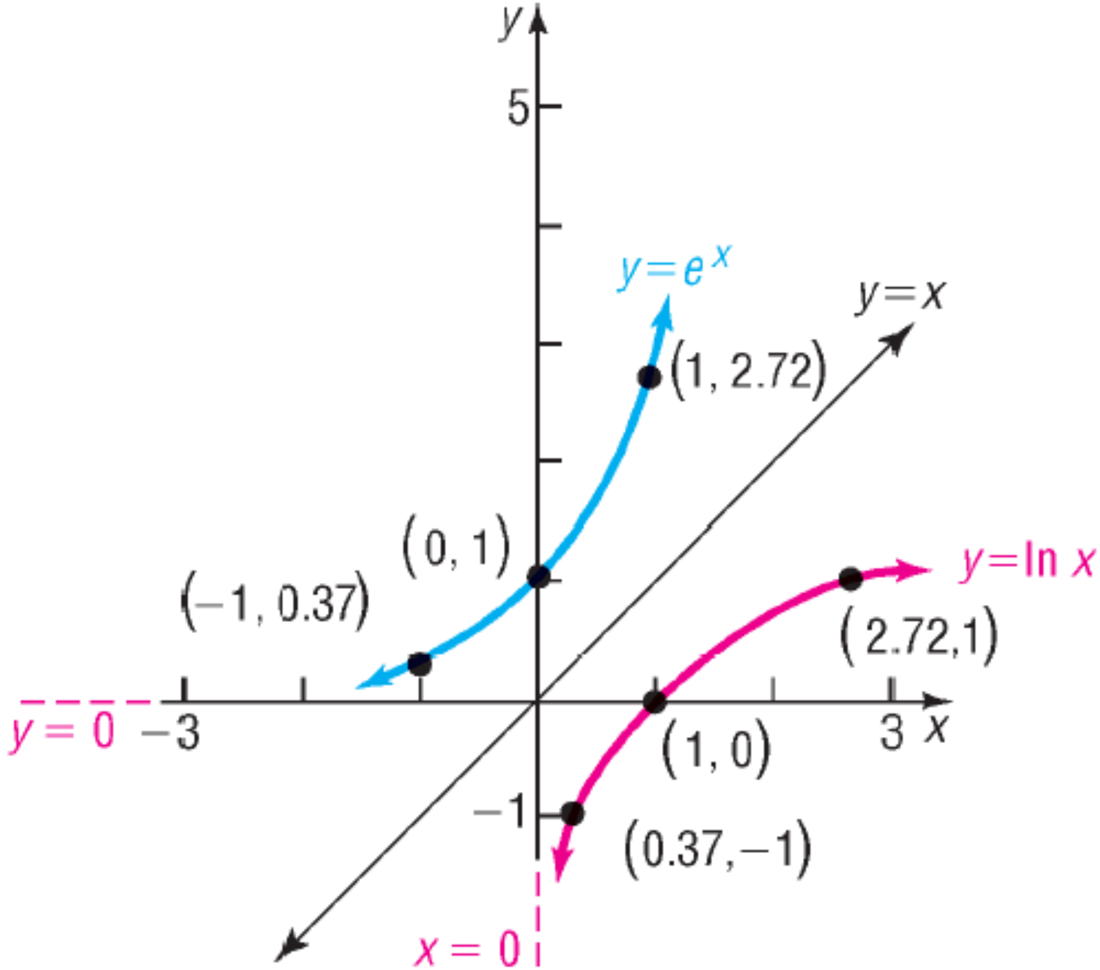
(b) $a > 1$



$y = 2^x$
$y = x$
y
$(1, 2)$
2
$y = \log_2 x$
$(0, 1)$
$\left(-1, \frac{1}{2}\right)$
$(2, 1)$
-2
$(1, 0)$
2
x
$\left(\frac{1}{2}, -1\right)$
-2
$y = \left(\frac{1}{3}\right)^x$
y
$y = x$
$(-1, 3)$
3
$\left(\frac{1}{3}, 1\right)$
$\left(1, \frac{1}{3}\right)$
$(0, 1)$
-3
3
x
$(1, 0)$
$(3, -1)$
$y = \log_{1/3} x$
-3

Properties of the Logarithmic Function $f(x) = \log_a x$

1. The domain is the set of positive real numbers; the range is the set of all real numbers.
2. The x-intercept of the graph is 1. There is no y-intercept.
3. The y-axis $(x = 0)$ is a vertical asymptote of the graph.
4. A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.
5. The graph of f contains the points $(1, 0)$, $(a, 1)$, and $\left(\frac{1}{a}, -1\right)$.
6. The graph is smooth and continuous, with no corners or gaps.



x	$\ln x$
$\frac{1}{2}$	−0.69
2	0.69
3	1.10

Natural Logarithm Function

$$y = \ln x \quad \text{if and only if} \quad x = e^y$$

EXAMPLE

Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = -\ln(x - 2)$.

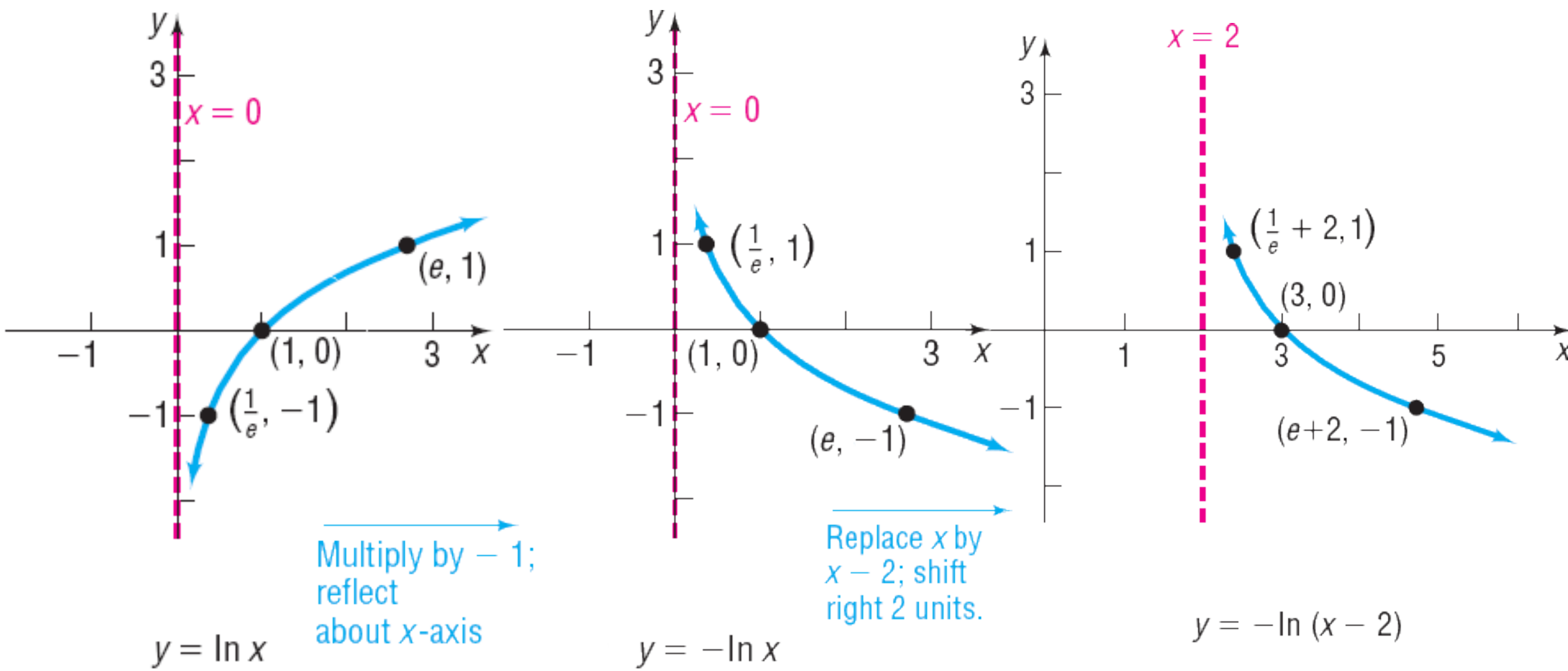
$$x - 2 > 3 \text{ so } x > 5.$$

$$\text{The domain of } f \text{ is } (2, \infty).$$

EXAMPLE

Graphing a Logarithmic Function and Its Inverse

(b) Graph f.

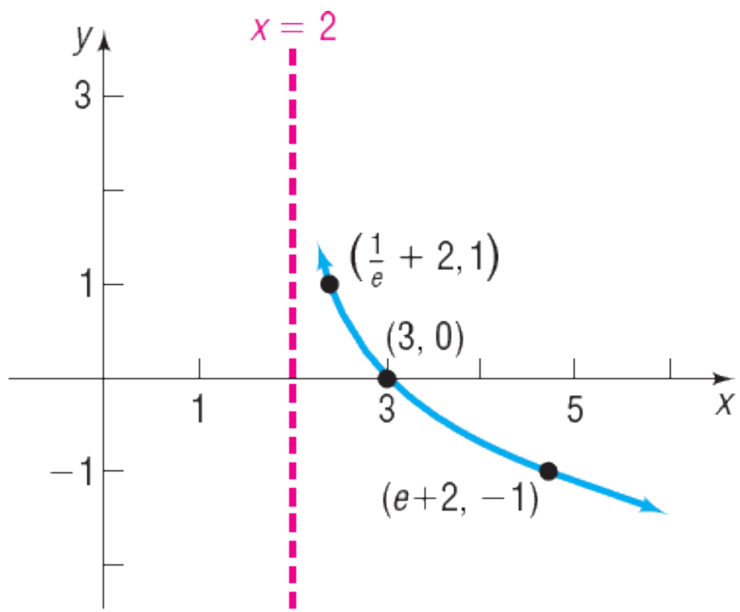


EXAMPLE

Graphing a Logarithmic Function and Its Inverse

(c) From the graph, determine the range and vertical asymptote of f.

The range of f is all real numbers and the vertical asymptote is $x = 2$.



$y = -\ln(x - 2)$

EXAMPLE

Graphing a Logarithmic Function and Its Inverse

(d) Find f^{-1}, the inverse of f.

$$f(x) = -\ln(x-2)$$

The inverse implicitly is $x = -\ln(y-2)$

$$-x = \ln(y-2)$$

$$e^{-x} = y-2$$

$$e^{-x} + 2 = y = f^{-1}(x)$$

EXAMPLE

Graphing a Logarithmic Function and Its Inverse

(e) Use f^{-1} to find the range of f.

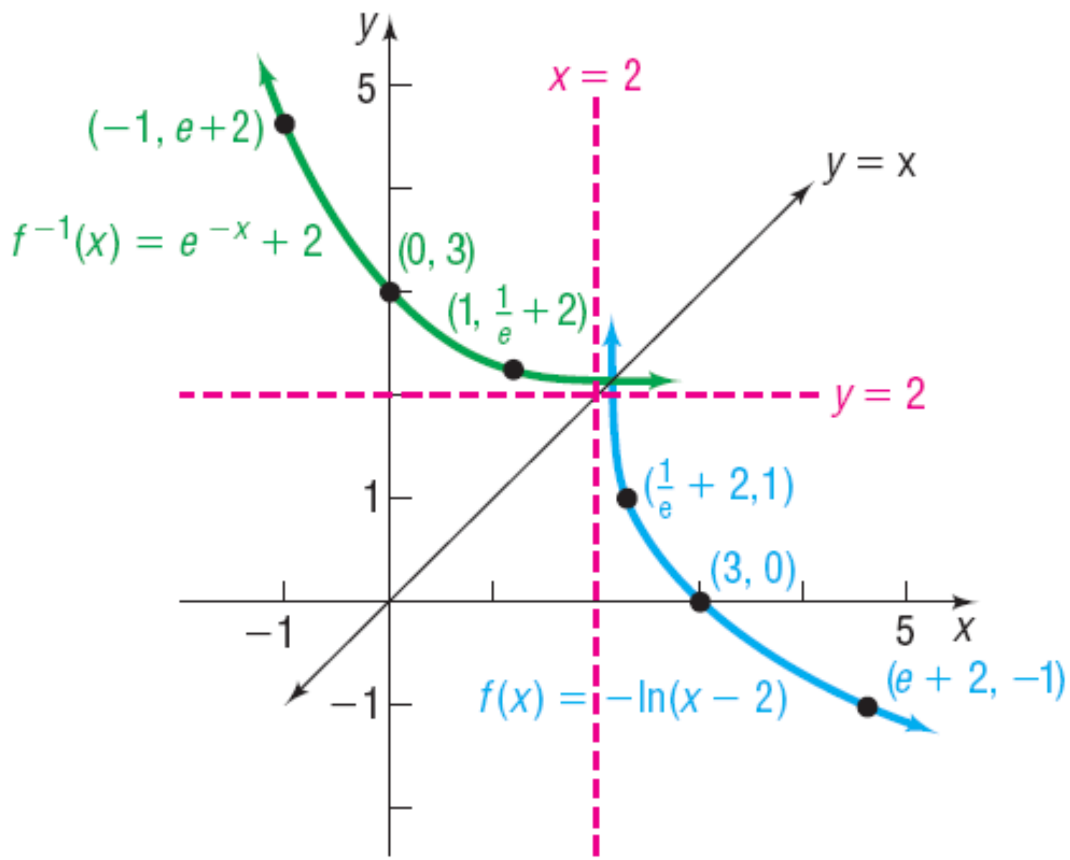
$$f^{-1}(x) = e^{-x} + 2$$

Since the range of f equals the domain of f^{-1}, the range of f is all real numbers.

EXAMPLE

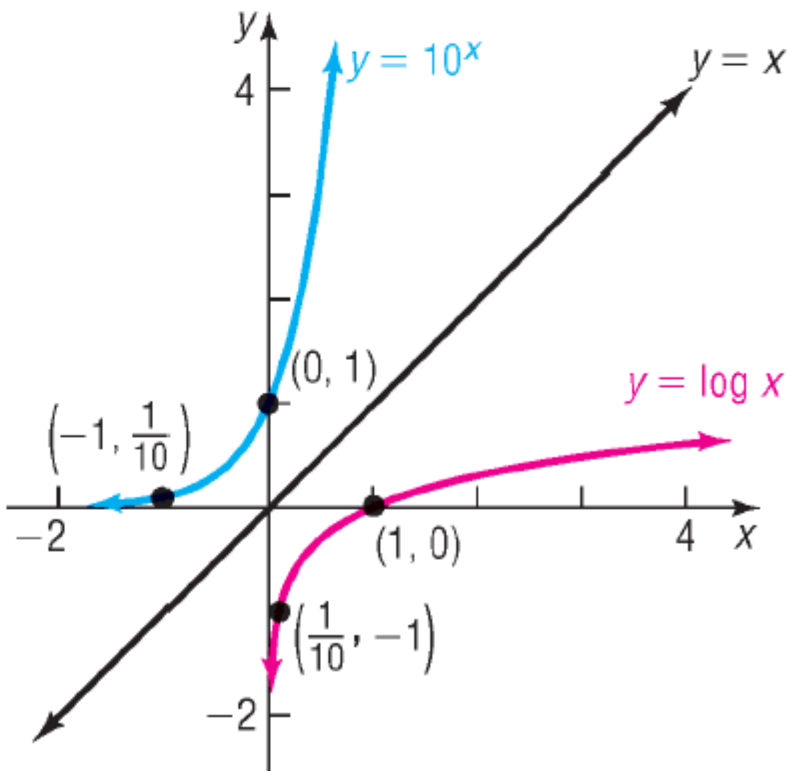
Graphing a Logarithmic Function and Its Inverse

(f) Graph f^{-1}.



Common Logarithm Function

$$y = \log x \quad \text{if and only if} \quad x = 10^y$$



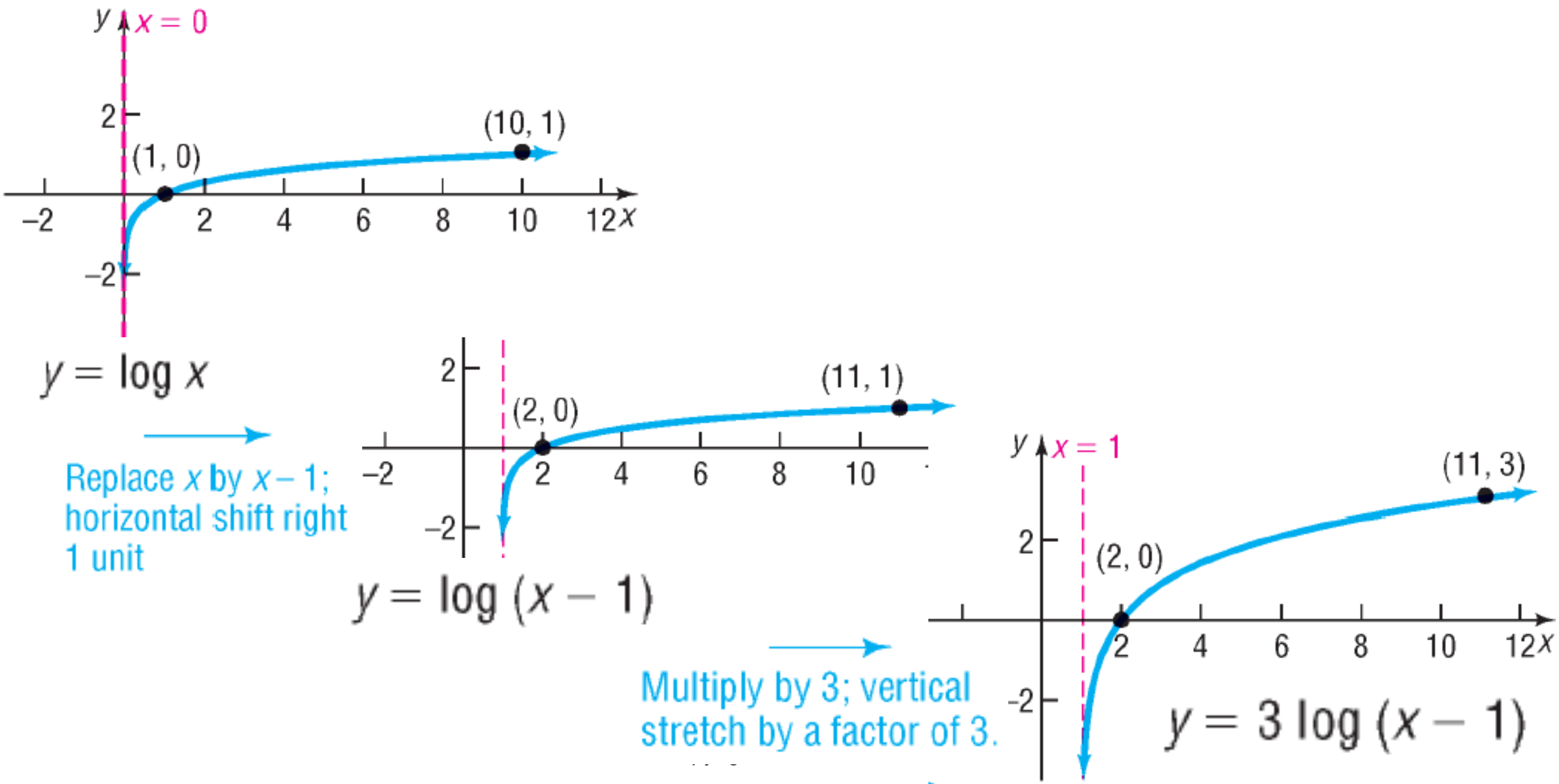
EXAMPLE

Graphing a Logarithmic Function and Its Inverse

(a) Find the domain of the logarithmic function $f(x) = 3\log(x-1)$.

(b) Graph f.

(c) From the graph, determine the range and vertical asymptote of f.



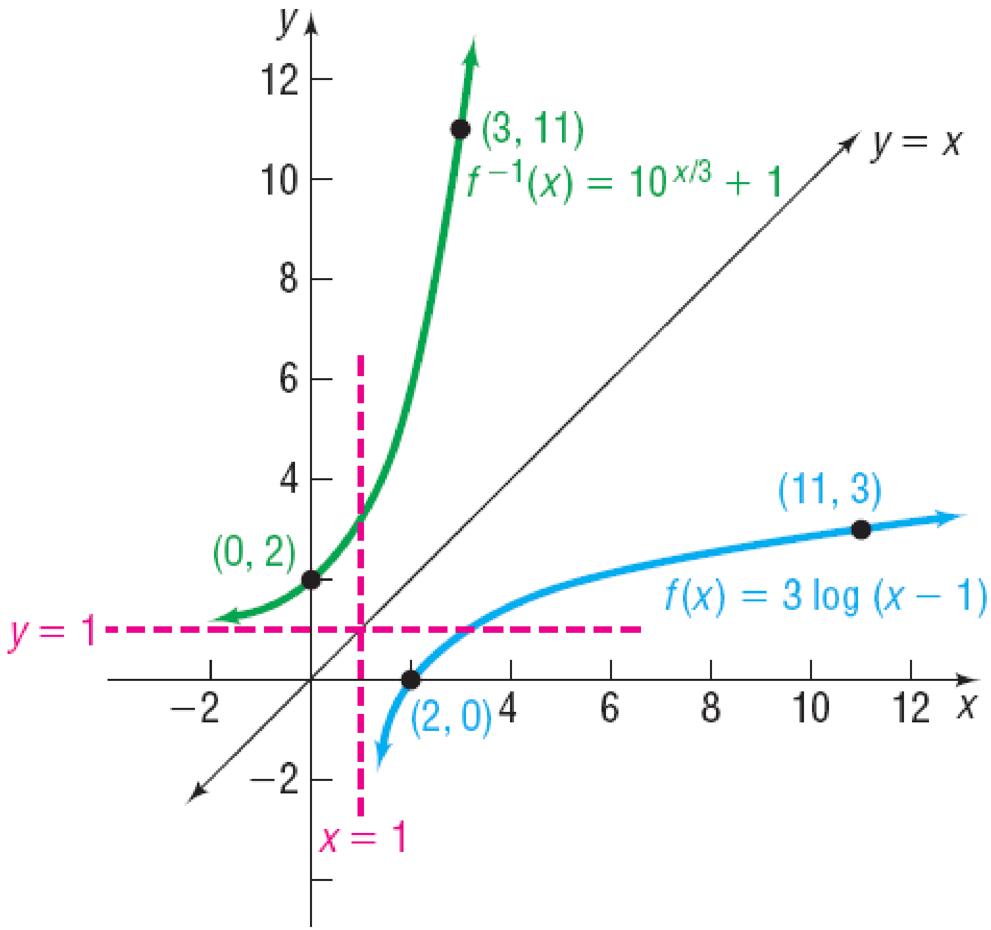
EXAMPLE

Graphing a Logarithmic Function and Its Inverse

(d) Find f^{-1}, the inverse of f.

(e) Use f^{-1} to find the range of f.

(f) Graph f^{-1}.



5 Solve Logarithmic Equations

EXAMPLE

Solving Logarithmic Equations

Solve: $(a)\ \log_2(2x+1)=3$ $(b)\ \log_x 343=3$

(a) Change $\log_2(2x+1)=3$ to exponential form.

$2^3=2x+1$ $8=2x+1$ $x=\dfrac{7}{2}$

✓Check: $\log_2\left(2\left(\dfrac{7}{2}\right)+1\right)=\log_2 8=3$

(b) Change $\log_x 343=3$ to exponential form.

$x^3=343$ $x=7$

✓Check: $\log_7 343=3$

EXAMPLE

Using Logarithms to Solve an Exponential Equation

Solve: $2e^{3x} = 6$

$e^{3x} = 3$ Isolate the exponential.

$\ln 3 = 3x$ Change to logarithmic form.

$x = \dfrac{\ln 3}{3}$ Exact solution

≈ 0.366 Approximate solution

EXAMPLE Alcohol and Driving

Blood alcohol concentration (BAC) is a measure of the amount of alcohol in a person's bloodstream. A BAC of 0.04% means that a person has 4 parts alcohol per 10,000 parts blood in the body. Relative risk is defined as the likelihood of one event occurring divided by the likelihood of a second event occurring. For example, if an individual with a BAC of 0.02% is 1.4 times as likely to have a car accident as an individual that has not been drinking, the relative risk of an accident with a BAC of 0.02% is 1.4. Recent medical research suggests that the relative risk R of having an accident while driving a car can be modeled by an equation of the form

$$R = e^{kx}$$

where x is the percent of concentration of alcohol in the bloodstream and k is a constant.

(a) Research indicates that the relative risk of a person having an accident with a BAC of 0.02% is 1.4. Find the constant k in the equation.

$$1.4 = e^{k(0.02)} \qquad 0.02k = \ln 1.4 \qquad k = \frac{\ln 1.4}{0.02} \approx 16.82$$

EXAMPLE Alcohol and Driving

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$$R = e^{kx} \qquad\qquad k = \frac{\ln 1.4}{0.02} \approx 16.82$$

where x is the percent of concentration of alcohol in the bloodstream and k is a constant.

(b) Using this value of k, what is the relative risk if the concentration is 0.17%?

$$R = e^{kx} = e^{(16.82)(0.17)} \approx 17.5$$

For a concentration of alcohol in the blood of 0.17%, the relative risk of an accident is about 17.5. That is, a person with a BAC of 0.17% is 17.5 times as likely to have a car accident as a person with no alcohol in the bloodstream.

EXAMPLE Alcohol and Driving

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$$R = e^{kx} \qquad k = \frac{\ln 1.4}{0.02} \approx 16.82$$

where x is the percent of concentration of alcohol in the bloodstream and k is a constant.

(c) Using this same value of k, what BAC corresponds to a relative risk of 100?

$$100 = e^{16.82x} \qquad 16.82x = \ln 100 \qquad x = \frac{\ln 100}{16.82} \approx 0.27$$

For a concentration of alcohol in the blood of 0.27%, the relative risk of an accident is 100.

EXAMPLE Alcohol and Driving

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$$R = e^{kx} \qquad\qquad k = \frac{\ln 1.4}{0.02} \approx 16.82$$

where x is the percent of concentration of alcohol in the bloodstream and k is a constant.

(d) If the law asserts that anyone with a relative risk of 5 or more should not have driving privileges, at what concentration of alcohol in the bloodstream should a driver be arrested and charged with a DUI (driving under the influence)?

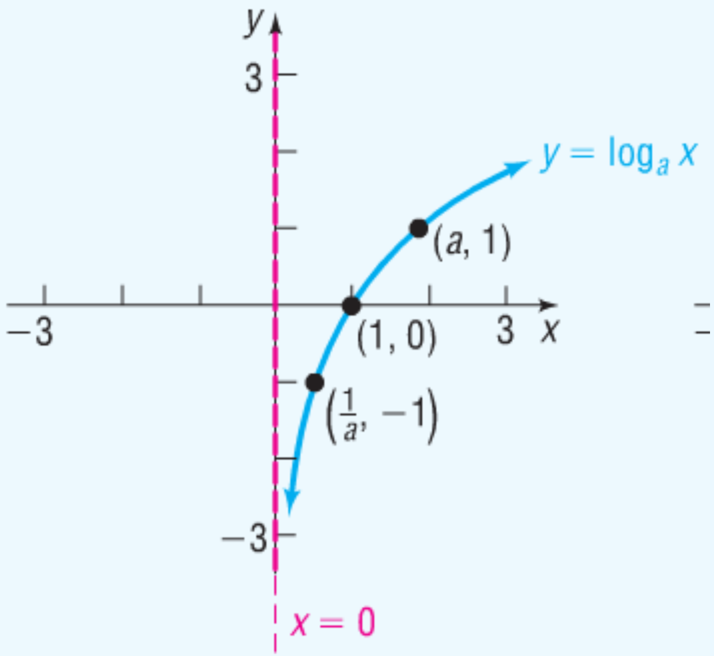
$$5 = e^{16.82x} \qquad 16.82x = \ln 5 \qquad x = \frac{\ln 5}{16.82} \approx 0.096$$

A driver with a BAC of 0.096% or more should be arrested and charged with DUI.

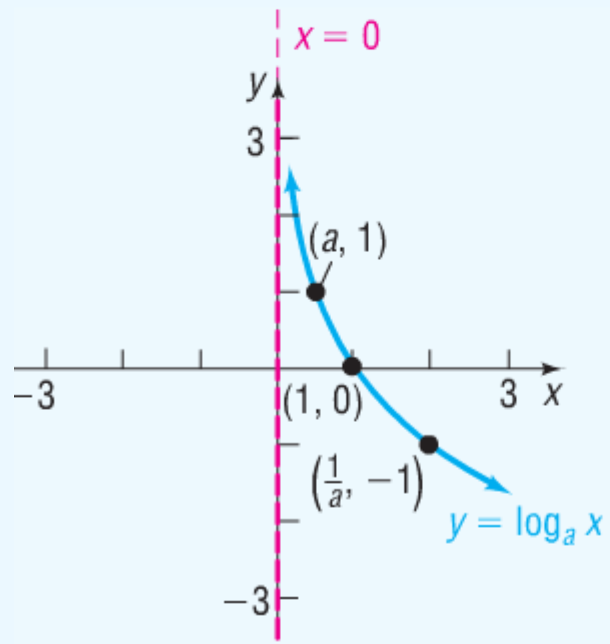
SUMMARY Properties of the Logarithmic Function

$f(x) = \log_a x, \quad a > 1$ Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$

$(y = \log_a x \text{ means } x = a^y)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); increasing;



(a) $a > 1$



(b) $0 < a < 1$

$f(x) = \log_a x, \quad 0 < a < 1$ Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$

$(y = \log_a x \text{ means } x = a^y)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); decreasing; one-to-one