# Section 4.1

# Linear Functions and Their Properties

## 1 Graph Linear Functions

## DEFINITION

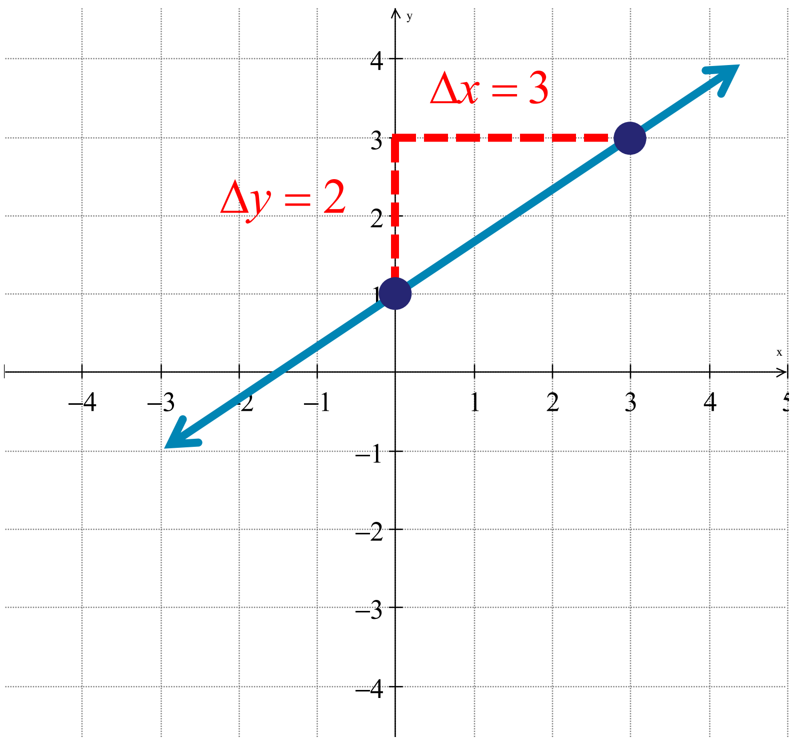
A **linear function** is a function of the form

$$\boxed{f(x) = mx + b}$$

The graph of a linear function is a line with slope $m$ and $y$-intercept $b$.

# EXAMPLE Graphing a Linear Function

Graph the linear function: $f(x) = \frac{2}{3}x + 1$




This linear function has a slope of $\frac{2}{3}$ and a $y$-intercept of 1 so first plot the point (0,1), the $y$-intercept.

Next, the slope $\frac{\Delta y}{\Delta x} = \frac{2}{3}$ so change the $y$-value of the point on the graph by 2 and the $x$-value by 3.

## 2 Use Average Rate of Change to Identify Linear Functions

| $x$ | $y = f(x) = -3x + 7$ | Average Rate of Change $= \dfrac{\Delta y}{\Delta x}$ |
|---|---|---|
| $-2$ | $13$ | |
| | | $\dfrac{10-13}{-1-(-2)} = \dfrac{-3}{1} = -3$ |
| $-1$ | $10$ | |
| | | $\dfrac{7-10}{0-(-1)} = \dfrac{-3}{1} = -3$ |
| $0$ | $7$ | |
| | | $-3$ |
| $1$ | $4$ | |
| | | $-3$ |
| $2$ | $1$ | |
| | | $-3$ |
| $3$ | $-2$ | |

# Theorem

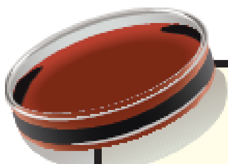
**Average Rate of Change of a Linear Function**

Linear functions have a constant average rate of change. That is, the average rate of change of a linear function $f(x) = mx + b$ is

$$\boxed{\frac{\Delta y}{\Delta x} = m}$$

# EXAMPLE Using the Average Rate of Change to Identify Linear Functions

A strain of E-coli Beu 397-recA441 is placed into a Petri dish at 30° Celsius and allowed to grow. The data shown in Table 2 are collected. The population is measured in grams and the time in hours. Plot the ordered pairs $(x, y)$ in the Cartesian plane and use the average rate of change to determine whether the function is linear.



| Time (hours), $x$ | Population (grams), $y$ | $(x, y)$ |
|---|---|---|
| 0 | 0.09 | (0, 0.09) |
| 1 | 0.12 | (1, 0.12) |
| 2 | 0.16 | (2, 0.16) |
| 3 | 0.22 | (3, 0.22) |
| 4 | 0.29 | (4, 0.29) |
| 5 | 0.39 | (5, 0.39) |

| Time (hours), *x* | Population (grams), *y* | (*x*, *y*) |
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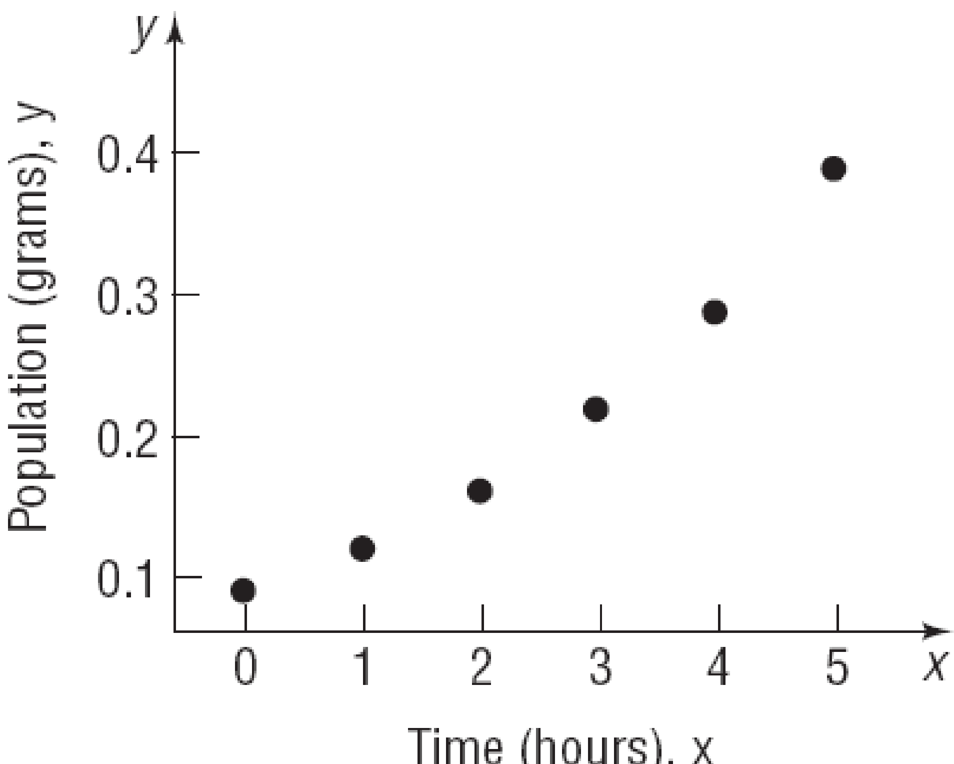
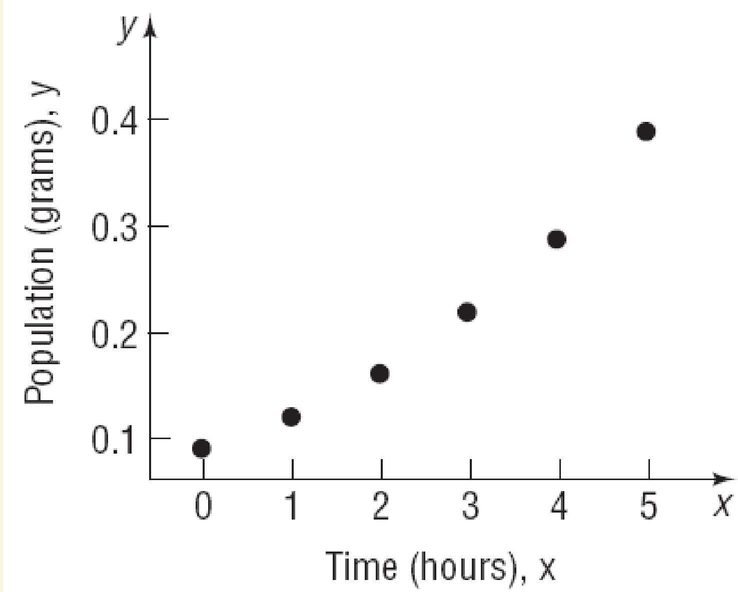
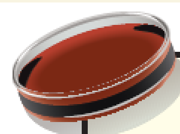


| Time | Population (grams), y | Average Rate of Change |
|---|---|---|
| 0 | 0.09 | |
| | | $\frac{0.12 - 0.09}{1 - 0} = 0.03$ |
| 1 | 0.12 | |
| | | 0.04 |
| 2 | 0.16 | |
| | | 0.06 |
| 3 | 0.22 | |
| | | 0.07 |
| 4 | 0.29 | |
| | | 0.10 |
| 5 | 0.39 | |



| Time (hours), $x$ | Population (grams), $y$ | $(x, y)$ |
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| 0 | 0.09 | (0, 0.09) |
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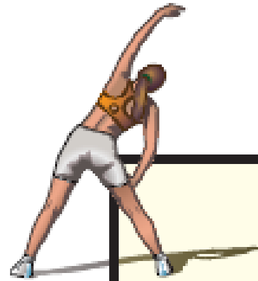
| 5 | 0.39 | (5, 0.39) |

Since the average rate of change is not constant, the function is not linear.

## EXAMPLE Using the Average Rate of Change to Identify Linear Functions

The data in Table 3 represent the maximum number of heartbeats that a healthy individual should have during a 15-second interval of time while exercising for different ages. Plot the ordered pairs $(x, y)$ in the Cartesian plane and use the average rate of change to determine whether the function is linear.

| Age, $x$ | Maximum Number of Heart Beats, $y$ | $(x, y)$ |
|---|---|---|
| 20 | 50 | (20, 50) |
| 30 | 47.5 | (30, 47.5) |
| 40 | 45 | (40, 45) |
| 50 | 42.5 | (50, 42.5) |
| 60 | 40 | (60, 40) |
| 70 | 37.5 | (70, 37.5) |



| Age, $x$ | Maximum Number of Heart Beats, $y$ | $(x, y)$ |
|---|---|---|
| 20 | 50 | (20, 50) |
| 30 | 47.5 | (30, 47.5) |
| 40 | 45 | (40, 45) |
| 50 | 42.5 | (50, 42.5) |
| 60 | 40 | (60, 40) |
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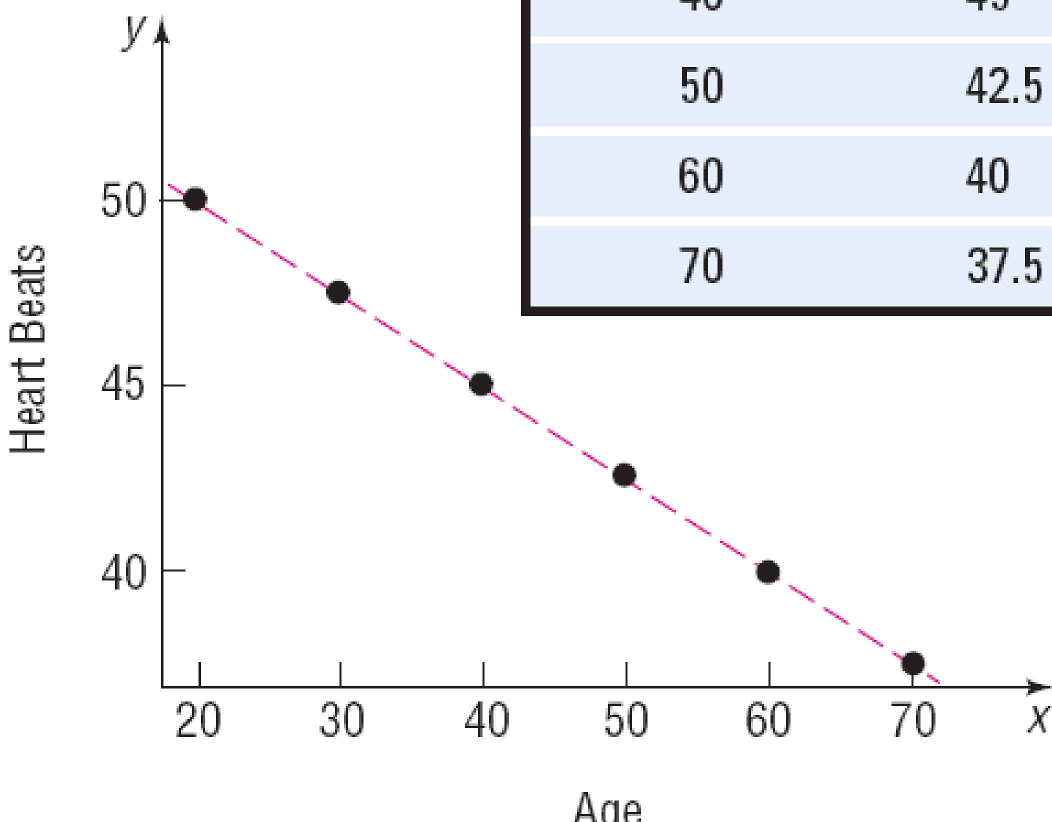

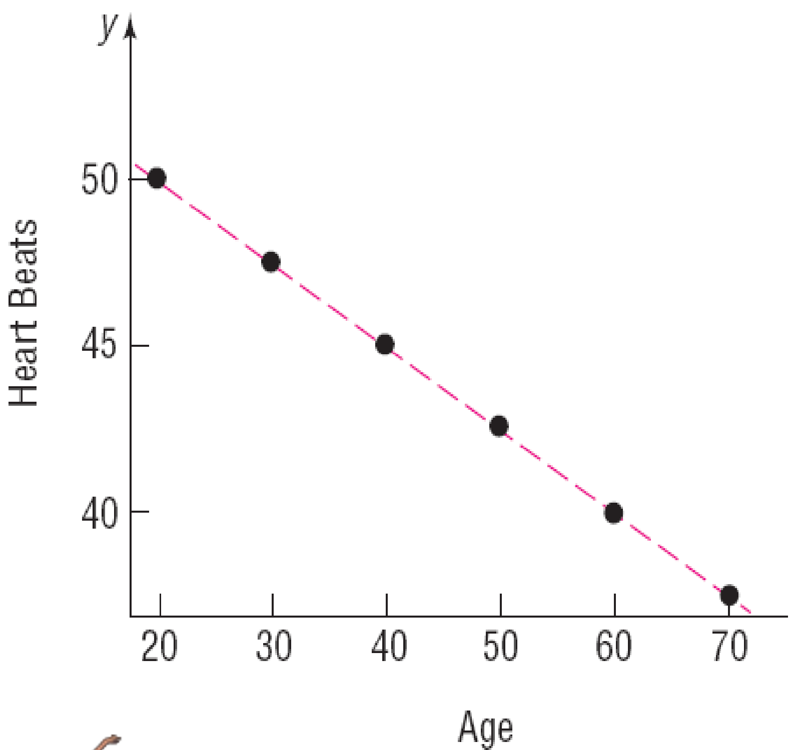


| Age, $x$ | Maximum Number of Heartbeats, $y$ | Average Rate of Change $= \frac{\Delta y}{\Delta x}$ |
|---|---|---|
| 20 | 50 | |
| | | $\frac{47.5 - 50}{30 - 20} = -0.25$ |
| 30 | 47.5 | |
| | | $-0.25$ |
| 40 | 45 | |
| | | $-0.25$ |
| 50 | 42.5 | |
| | | $-0.25$ |
| 60 | 40 | |
| | | $-0.25$ |
| 70 | 37.5 | |

| Age, $x$ | Maximum Number of Heart Beats, $y$ | $(x, y)$ |
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| 20 | 50 | (20, 50) |
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| 50 | 42.5 | (50, 42.5) |
| 60 | 40 | (60, 40) |
| 70 | 37.5 | (70, 37.5) |

Since the average rate of change is a constant –0.25, the function is linear.

## 3 Determine Whether a Linear Function Is Increasing, Decreasing, or Constant

INCREASING

DECREASING

CONSTANT

# Theorem

**Increasing, Decreasing, and Constant Linear Functions**

A linear function $f(x) = mx + b$ is increasing over its domain if its slope, $m$, is positive. It is decreasing over its domain if its slope, $m$, is negative. It is constant over its domain if its slope, $m$, is zero.

EXAMPLE

## Determining Whether a Linear Function Is Increasing, Decreasing, or Constant

Determine whether the following linear functions are increasing, decreasing, or constant.

(a) $f(x) = -2x + 4$ (b) $g(x) = 5$

(c) $s(t) = \frac{3}{4}t$ (d) $m(z) = z - 3$

(a) The linear function has a slope of $-2$ so the function $f$ is decreasing.

(b) This function could be written $g(x) = 0x + 5$ so the function has a slope of 0 and the function $g$ is constant.

(c) The linear function has a slope of $\frac{3}{4}$ so the function $s$ is increasing.

(d) The linear function has a slope of 1 so the function $m$ is increasing.

## 4 Build Linear Models from Verbal Descriptions

## Modeling with a Linear Function

If the average rate of change of a function is a constant $m$, a linear function $f$ can be used to model the relation between the two variables as follows:

$$\boxed{f(x) = mx + b}$$

where $b$ is the value of $f$ at 0, that is, $b = f(0)$.

## EXAMPLE Straight-line Depreciation

*Book value* is the value of an asset that a company uses to create its balance sheet. Some companies depreciate their assets using straight-line depreciation so that the value of the asset declines by a fixed amount each year. The amount of the decline depends on the useful life that the company places on the asset. Suppose that a company just purchased a fleet of new cars for its sales force at a cost of \$28,000 per car. The company chooses to depreciate each vehicle using the straight-line method over 7 years. This means that each car will depreciate by $\frac{\$28{,}000}{7} = \$4000$ per year.

(a) Write a linear function that expresses the book value $V$ of each car as a function of its age, $x$.

Let $V(x)$ represent the value of each car after $x$ years.

$V(0) = \$28,000$ and the slope is $-4000$ since the car depreciates by that amount per year.

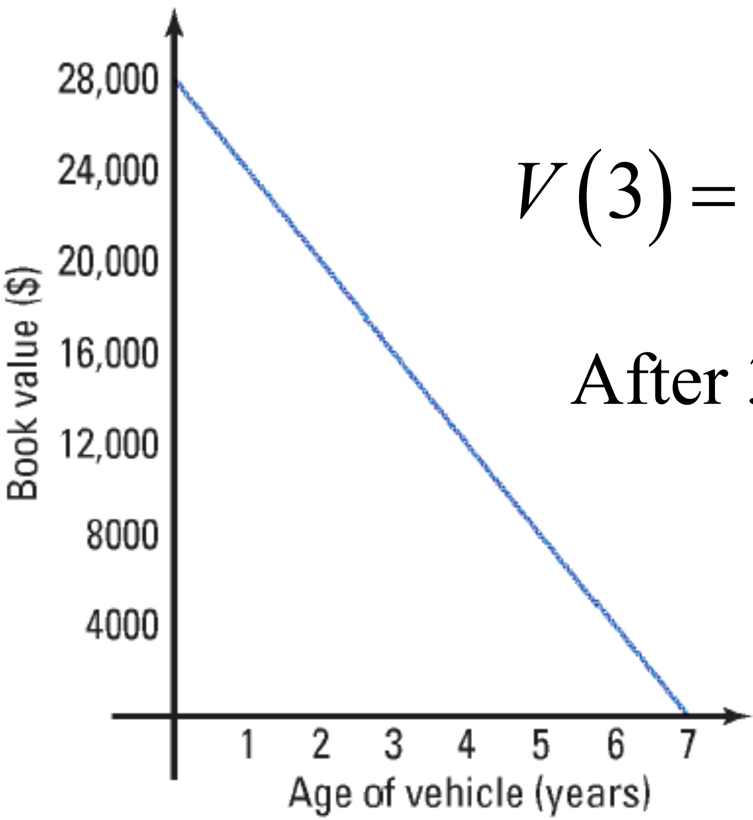
$$V(x) = -4000x + 28000$$

## EXAMPLE Straight-line Depreciation

$$V(x) = -4000x + 28{,}000$$

(b) Graph the linear function.

(c) What is the book value of each car after 3 years?




$$V(3) = -4000(3) + 28{,}000 = 16{,}000$$

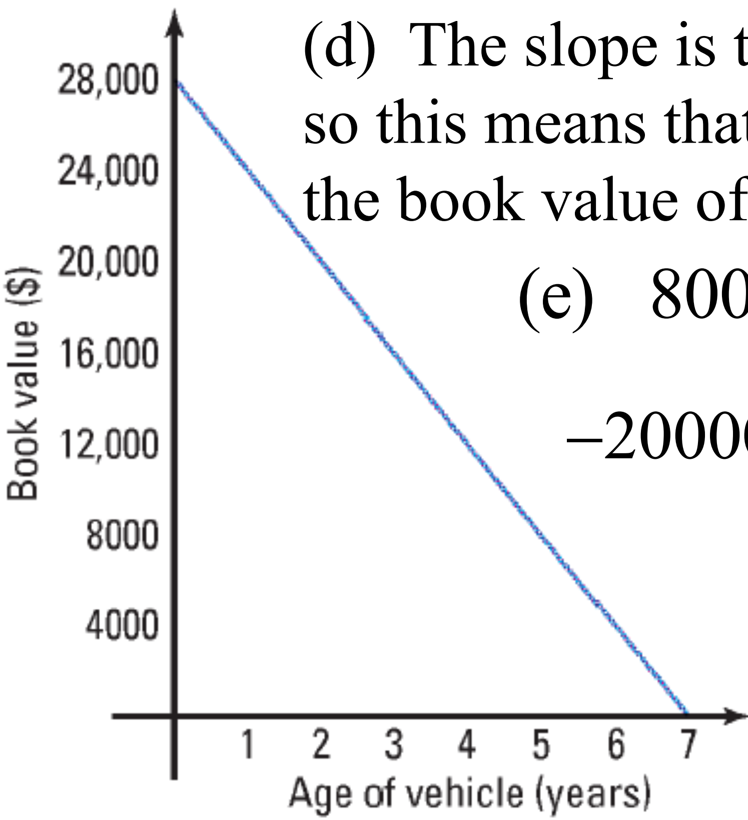
After 3 years the book value is $16,000.

## EXAMPLE Straight-line Depreciation

$$V(x) = -4000x + 28{,}000$$

(d) Interpret the slope.

(e) When will the book value of each car be $8000?




(d) The slope is the average rate of change and is –4000 so this means that for each additional year that passes, the book value of the car decreases by $4000.

(e) $8000 = -4000x + 28000$

$$-20000 = -4000x \qquad x = \frac{-20000}{-4000} = 5$$

The car will have a book value of $8000 when it is 5 years old.

**EXAMPLE** **Supply and Demand**

The quantity supplied of a good is the amount of a product that a company is willing to make available for sale at a given price. The quantity demanded of a good is the amount of a product that consumers are willing to purchase at a given price. Suppose that the quantity supplied, $S$, and quantity demanded, $D$, of cellular telephones each month are given by the following functions:

$$S(p) = 60p - 900$$

$$D(p) = -15p + 2850$$

where $p$ is the price (in dollars) of the telephone.

(a) The **equilibrium price** of a product is defined as the price at which quantity supplied equals quantity demanded. That is, the equilibrium price is the price at which $S(p) = D(p)$. Find the equilibrium price of cellular telephones. What is the equilibrium quantity, the amount demanded (or supplied), at the equilibrium price?

$$60p - 900 = -15p + 2850$$

$$75p = 3750 \qquad p = 50$$

$$s(50) = 60(50) - 900 = 2100$$

So the equilibrium price is \$50 and the equilibrium quantity is 2100 phones.

**EXAMPLE** **Supply and Demand** $S(p) = 60p - 900$

$$D(p) = -15p + 2850$$

(b) Determine the prices for which quantity supplied is greater than quantity demanded. That is, solve the inequality $S(p) > D(p)$.

$$60p - 900 > -15p + 2850$$

$$75p > 3750$$

$$p > 50$$

If the company charges more than \$50 per phone, then quantity supplied will exceed quantity demanded. In this case the company will have excess phones in inventory.

**EXAMPLE** **Supply and Demand**

$$S(p) = 60p - 900$$

$$D(p) = -15p + 2850$$

(c) Graph $S = S(p)$, $D = D(p)$ and label the equilibrium price.