C

NEW FUNCTIONS FROM OLD

ARITHMETIC OPERATIONS ON FUNCTIONS

Two functions, f and g, can be added, subtracted, multiplied, and divided in a natural way to form new functions $f+g$, $f-g$, fg, and f/g. For example, $f+g$ is defined by the formula

$$(f+g)(x) = f(x) + g(x) \tag{1}$$

which states that for each input the value of $f+g$ is obtained by adding the values of f and g. Equation (1) provides a formula for $f+g$ but does not say anything about the domain of $f+g$. However, for the right side of this equation to be defined, x must lie in the domains of both f and g, so we define the domain of $f+g$ to be the intersection of these two domains. More generally, we make the following definition.

If f is a constant function, that is, $f(x) = c$ for all x, then the product of f and g is cg, so multiplying a function by a constant is a special case of multiplying two functions.

C.1 DEFINITION Given functions f and g, we define

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(f/g)(x) = f(x)/g(x)$$

For the functions $f+g$, $f-g$, and fg we define the domain to be the intersection of the domains of f and g, and for the function f/g we define the domain to be the intersection of the domains of f and g but with the points where $g(x) = 0$ excluded (to avoid division by zero).

Example 1 Let

$$f(x) = 1 + \sqrt{x-2} \quad \text{and} \quad g(x) = x - 3$$

Find the domains and formulas for the functions $f+g$, $f-g$, fg, f/g, and $7f$.

Solution. First, we will find the formulas and then the domains. The formulas are

$$(f+g)(x) = f(x) + g(x) = (1 + \sqrt{x-2}\,) + (x-3) = x - 2 + \sqrt{x-2} \tag{2}$$

$$(f-g)(x) = f(x) - g(x) = (1 + \sqrt{x-2}\,) - (x-3) = 4 - x + \sqrt{x-2} \tag{3}$$

$$(fg)(x) = f(x)g(x) = (1 + \sqrt{x-2}\,)(x-3) \tag{4}$$

$$(f/g)(x) = f(x)/g(x) = \frac{1 + \sqrt{x-2}}{x-3} \tag{5}$$

$$(7f)(x) = 7f(x) = 7 + 7\sqrt{x-2} \tag{6}$$

The domains of f and g are $[2, +\infty)$ and $(-\infty, +\infty)$, respectively (their natural domains). Thus, it follows from Definition C.1 that the domains of $f+g$, $f-g$, and fg are the intersection of these two domains, namely,

$$[2, +\infty) \cap (-\infty, +\infty) = [2, +\infty) \tag{7}$$

Moreover, since $g(x) = 0$ if $x = 3$, the domain of f/g is (7) with $x = 3$ removed, namely,

$$[2, 3) \cup (3, +\infty)$$

Finally, the domain of $7f$ is the same as the domain of f. ◀

We saw in the last example that the domains of the functions $f+g$, $f-g$, fg, and f/g were the natural domains resulting from the formulas obtained for these functions. The following example shows that this will not always be the case.

▶ **Example 2** Show that if $f(x) = \sqrt{x}$, $g(x) = \sqrt{x}$, and $h(x) = x$, then the domain of fg is not the same as the natural domain of h.

Solution. The natural domain of $h(x) = x$ is $(-\infty, +\infty)$. Note that

$$(fg)(x) = \sqrt{x}\sqrt{x} = x = h(x)$$

on the domain of fg. The domains of both f and g are $[0, +\infty)$, so the domain of fg is

$$[0, +\infty) \cap [0, +\infty) = [0, +\infty)$$

by Definition C.1. Since the domains of fg and h are different, it would be misleading to write $(fg)(x) = x$ without including the restriction that this formula holds only for $x \geq 0$. ◀

COMPOSITION OF FUNCTIONS

We now consider an operation on functions, called *composition*, which has no direct analog in ordinary arithmetic. Informally stated, the operation of composition is performed by substituting some function for the independent variable of another function. For example, suppose that

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1$$

If we substitute $g(x)$ for x in the formula for f, we obtain a new function

$$f(g(x)) = (g(x))^2 = (x+1)^2$$

which we denote by $f \circ g$. Thus,

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (x+1)^2$$

In general, we make the following definition.

Although the domain of $f \circ g$ may seem complicated at first glance, it makes sense intuitively: To compute $f(g(x))$ one needs x in the domain of g to compute $g(x)$, and one needs $g(x)$ in the domain of f to compute $f(g(x))$.

C.2 DEFINITION Given functions f and g, the ***composition*** of f with g, denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is defined to consist of all x in the domain of g for which $g(x)$ is in the domain of f.

▶ **Example 3** Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Find

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution (a). The formula for $f(g(x))$ is

$$f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x}\,)^2 + 3 = x + 3$$

Since the domain of g is $[0, +\infty)$ and the domain of f is $(-\infty, +\infty)$, the domain of $f \circ g$ consists of all x in $[0, +\infty)$ such that $g(x) = \sqrt{x}$ lies in $(-\infty, +\infty)$; thus, the domain of $f \circ g$ is $[0, +\infty)$. Therefore,

$$(f \circ g)(x) = x + 3, \quad x \geq 0$$

Solution (b). The formula for $g(f(x))$ is

$$g(f(x)) = \sqrt{f(x)} = \sqrt{x^2 + 3}$$

Since the domain of f is $(-\infty, +\infty)$ and the domain of g is $[0, +\infty)$, the domain of $g \circ f$ consists of all x in $(-\infty, +\infty)$ such that $f(x) = x^2 + 3$ lies in $[0, +\infty)$. Thus, the domain of $g \circ f$ is $(-\infty, +\infty)$. Therefore,

$$(g \circ f)(x) = \sqrt{x^2 + 3}$$

There is no need to indicate that the domain is $(-\infty, +\infty)$, since this is the natural domain of $\sqrt{x^2 + 3}$. ◀

Note that the functions $f \circ g$ and $g \circ f$ in Example 3 are not the same. Thus, the order in which functions are composed can (and usually will) make a difference in the end result.

Compositions can also be defined for three or more functions; for example, $(f \circ g \circ h)(x)$ is computed as

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

In other words, first find $h(x)$, then find $g(h(x))$, and then find $f(g(h(x)))$.

▶ **Example 4** Find $(f \circ g \circ h)(x)$ if

$$f(x) = \sqrt{x}, \quad g(x) = 1/x, \quad h(x) = x^3$$

Solution.

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3)) = f(1/x^3) = \sqrt{1/x^3} = 1/x^{3/2}$$ ◀

EXPRESSING A FUNCTION AS A COMPOSITION

Many problems in mathematics are solved by "decomposing" functions into compositions of simpler functions. For example, consider the function h given by

$$h(x) = (x + 1)^2$$

To evaluate $h(x)$ for a given value of x, we would first compute $x + 1$ and then square the result. These two operations are performed by the functions

$$g(x) = x + 1 \quad \text{and} \quad f(x) = x^2$$

We can express h in terms of f and g by writing

$$h(x) = (x + 1)^2 = [g(x)]^2 = f(g(x))$$

so we have succeeded in expressing h as the composition $h = f \circ g$.

The thought process in this example suggests a general procedure for decomposing a function h into a composition $h = f \circ g$:

- Think about how you would evaluate $h(x)$ for a specific value of x, trying to break the evaluation into two steps performed in succession.

- The first operation in the evaluation will determine a function g and the second a function f.
- The formula for h can then be written as $h(x) = f(g(x))$.

For descriptive purposes, we will refer to g as the "inside function" and f as the "outside function" in the expression $f(g(x))$. The inside function performs the first operation and the outside function performs the second.

▶ Example 5 Express $\sin(x^3)$ as a composition of two functions.

Solution. To evaluate $\sin(x^3)$, we would first compute x^3 and then take the sine, so $g(x) = x^3$ is the inside function and $f(x) = \sin x$ the outside function. Therefore,

$$\sin(x^3) = f(g(x)) \qquad \boxed{g(x) = x^3 \text{ and } f(x) = \sin x}$$ ◀

Table C.1 gives some more examples of decomposing functions into compositions.

Table C.1

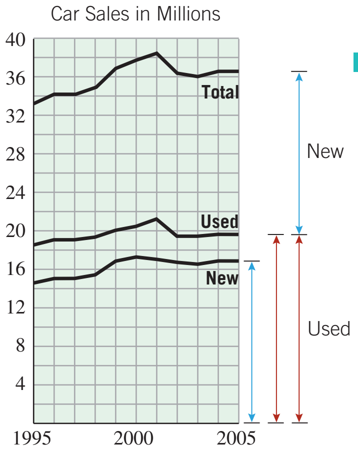
COMPOSING FUNCTIONS

FUNCTION	$g(x)$ INSIDE	$f(x)$ OUTSIDE	COMPOSITION
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10} = f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x = f(g(x))$
$\tan(x^5)$	x^5	$\tan x$	$\tan(x^5) = f(g(x))$
$\sqrt{4-3x}$	$4-3x$	$\sqrt{x}$	$\sqrt{4-3x} = f(g(x))$
$8+\sqrt{x}$	$\sqrt{x}$	$8+x$	$8+\sqrt{x} = f(g(x))$
$\dfrac{1}{x+1}$	$x+1$	$\dfrac{1}{x}$	$\dfrac{1}{x+1} = f(g(x))$

REMARK There is always more than one way to express a function as a composition. For example, here are two ways to express $(x^2+1)^{10}$ as a composition that differ from that in Table C.1:

$$(x^2+1)^{10} = [(x^2+1)^2]^5 = f(g(x)) \qquad \boxed{g(x) = (x^2+1)^2 \text{ and } f(x) = x^5}$$

$$(x^2+1)^{10} = [(x^2+1)^3]^{10/3} = f(g(x)) \qquad \boxed{g(x) = (x^2+1)^3 \text{ and } f(x) = x^{10/3}}$$



Source: NADA.

▲ **Figure C.1**

NEW FUNCTIONS FROM OLD

The remainder of this section will be devoted to considering the geometric effect of performing basic operations on functions. This will enable us to use known graphs of functions to visualize or sketch graphs of related functions. For example, Figure C.1 shows the graphs of yearly new car sales $N(t)$ and used car sales $U(t)$ over a certain time period. Those graphs can be used to construct the graph of the total car sales

$$T(t) = N(t) + U(t)$$

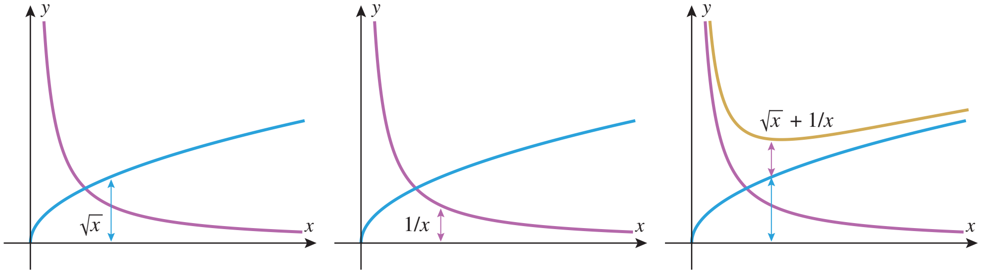
by adding the values of $N(t)$ and $U(t)$ for each value of t. In general, the graph of $y = f(x) + g(x)$ can be constructed from the graphs of $y = f(x)$ and $y = g(x)$ by adding corresponding y-values for each x.

▶ Example 6 Referring to Figure C.2 for the graphs of $y = \sqrt{x}$ and $y = 1/x$, make a sketch that shows the general shape of the graph of $y = \sqrt{x} + 1/x$ for $x \geq 0$.

Solution. To add the corresponding y-values of $y = \sqrt{x}$ and $y = 1/x$ graphically, just imagine them to be "stacked" on top of one another. This yields the sketch in Figure C.2. ◀

Use the technique in Example 6 to sketch the graph of the function

$$\sqrt{x} - \frac{1}{x}$$

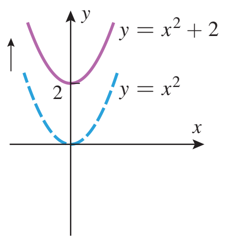
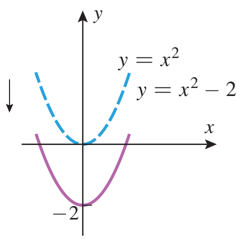
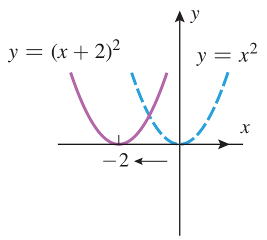
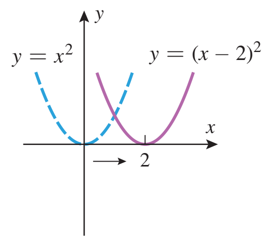


▶ **Figure C.2**
Add the y-coordinates of $\sqrt{x}$ and $1/x$ to obtain the y-coordinate of $\sqrt{x} + 1/x$.

TRANSLATIONS

Table C.2 illustrates the geometric effect on the graph of $y = f(x)$ of adding or subtracting a *positive* constant c to f or to its independent variable x. For example, the first result in the table illustrates that adding a positive constant c to a function f adds c to each y-coordinate of its graph, thereby shifting the graph of f up by c units. Similarly, subtracting c from f shifts the graph down by c units. On the other hand, if a positive constant c is added to x, then the value of $y = f(x + c)$ at $x - c$ is $f(x)$; and since the point $x - c$ is c units to the left of x on the x-axis, the graph of $y = f(x + c)$ must be the graph of $y = f(x)$ shifted left by c units. Similarly, subtracting c from x shifts the graph of $y = f(x)$ right by c units.

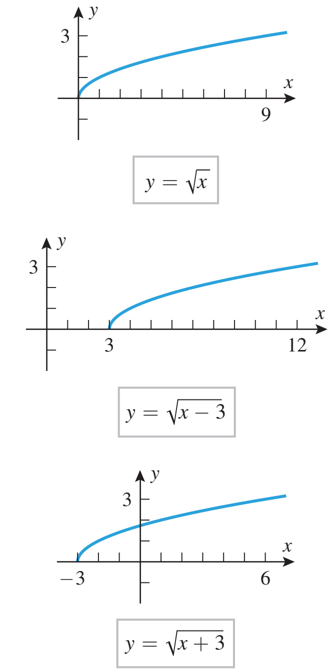
Table C.2
TRANSLATION PRINCIPLES

OPERATION ON $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
NEW EQUATION	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
GEOMETRIC EFFECT	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units
EXAMPLE				

▶ **Example 7** Sketch the graph of

$$\text{(a)}\ \ y = \sqrt{x - 3} \qquad \text{(b)}\ \ y = \sqrt{x + 3}$$

Solution. Using the translation principles given in Table C.2, the graph of the equation $y = \sqrt{x - 3}$ can be obtained by translating the graph of $y = \sqrt{x}$ right 3 units. The graph of $y = \sqrt{x + 3}$ can be obtained by translating the graph of $y = \sqrt{x}$ left 3 units (Figure C.3). ◀



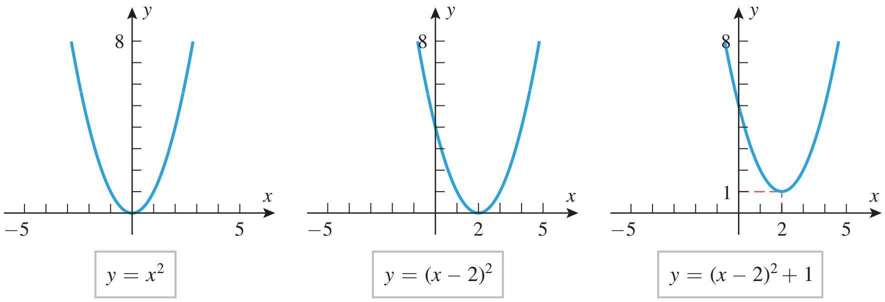
▲ **Figure C.3**

▶ **Example 8** Sketch the graph of $y = x^2 - 4x + 5$.

Solution. Completing the square on the first two terms yields

$$y = (x^2 - 4x + 4) - 4 + 5 = (x - 2)^2 + 1$$

(see Web Appendix I for a review of this technique). In this form we see that the graph can be obtained by translating the graph of $y = x^2$ right 2 units because of the $x - 2$, and up 1 unit because of the $+1$ (Figure C.4). ◀



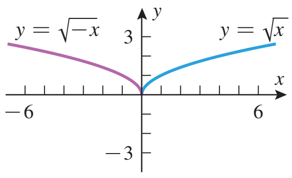
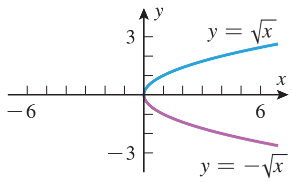
▲ **Figure C.4**

REFLECTIONS

The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(-x, y)$. Similarly, the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(x, -y)$ [the equation $y = -f(x)$ is equivalent to $-y = f(x)$]. This is summarized in Table C.3.

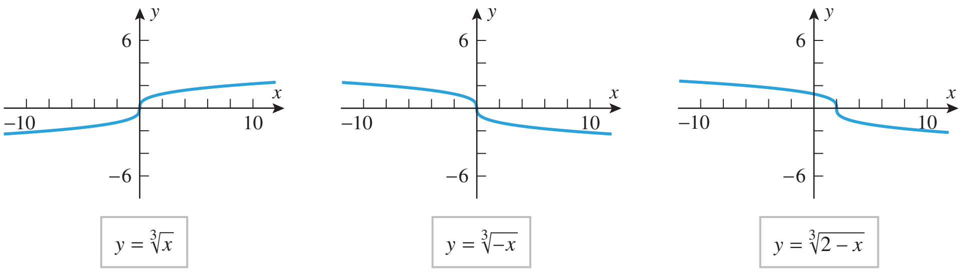
Table C.3

REFLECTION PRINCIPLES

OPERATION ON $y = f(x)$	Replace x by $-x$	Multiply $f(x)$ by -1
NEW EQUATION	$y = f(-x)$	$y = -f(x)$
GEOMETRIC EFFECT	Reflects the graph of $y = f(x)$ about the y-axis	Reflects the graph of $y = f(x)$ about the x-axis
EXAMPLE		

▶ **Example 9** Sketch the graph of $y = \sqrt[3]{2 - x}$.

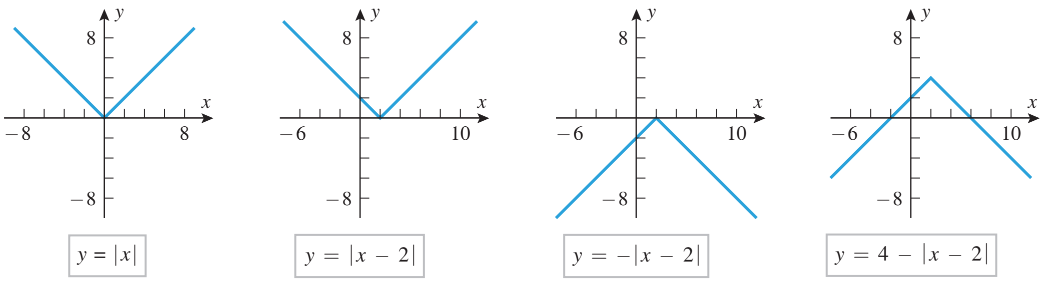
Solution. Using the translation and reflection principles in Tables C.2 and C.3, we can obtain the graph by a reflection followed by a translation as follows: First reflect the graph of $y = \sqrt[3]{x}$ about the y-axis to obtain the graph of $y = \sqrt[3]{-x}$, then translate this graph right 2 units to obtain the graph of the equation $y = \sqrt[3]{-(x - 2)} = \sqrt[3]{2 - x}$ (Figure C.5). ◀



▶ Figure C.5

▶ **Example 10** Sketch the graph of $y = 4 - |x - 2|$.

Solution. The graph can be obtained by a reflection and two translations: First translate the graph of $y = |x|$ right 2 units to obtain the graph of $y = |x - 2|$; then reflect this graph about the x-axis to obtain the graph of $y = -|x - 2|$; and then translate this graph up 4 units to obtain the graph of the equation $y = -|x - 2| + 4 = 4 - |x - 2|$ (Figure C.6). ◀



▲ Figure C.6

STRETCHES AND COMPRESSIONS

Multiplying $f(x)$ by a *positive* constant c has the geometric effect of stretching the graph of $y = f(x)$ in the y-direction by a factor of c if $c > 1$ and compressing it in the y-direction by a factor of $1/c$ if $0 < c < 1$. For example, multiplying $f(x)$ by 2 doubles each y-coordinate, thereby stretching the graph vertically by a factor of 2, and multiplying by $\frac{1}{2}$ cuts each y-coordinate in half, thereby compressing the graph vertically by a factor of 2. Similarly, multiplying x by a *positive* constant c has the geometric effect of compressing the graph of $y = f(x)$ by a factor of c in the x-direction if $c > 1$ and stretching it by a factor of $1/c$ if $0 < c < 1$. [If this seems backwards to you, then think of it this way: The value of $2x$ changes twice as fast as x, so a point moving along the x-axis from the origin will only have to move half as far for $y = f(2x)$ to have the same value as $y = f(x)$, thereby creating a horizontal compression of the graph.] All of this is summarized in Table C.4.

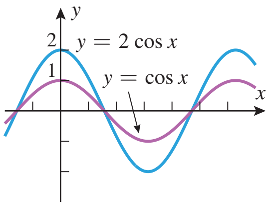
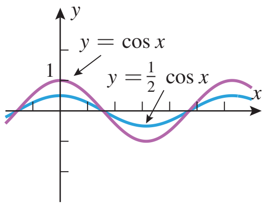
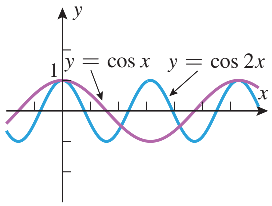
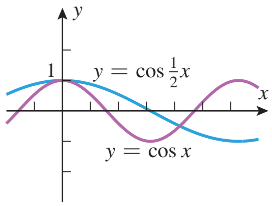
Describe the geometric effect of multiplying a function f by a *negative* constant in terms of reflection and stretching or compressing. What is the geometric effect of multiplying the independent variable of a function f by a *negative* constant?

SYMMETRY

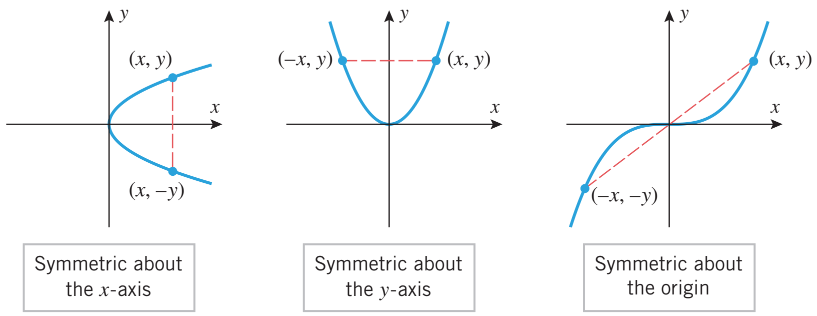
Figure C.7 illustrates three types of symmetries: ***symmetry about the x-axis***, ***symmetry about the y-axis***, and ***symmetry about the origin***. As illustrated in the figure, a curve is symmetric about the x-axis if for each point (x, y) on the graph the point $(x, -y)$ is also on the graph, and it is symmetric about the y-axis if for each point (x, y) on the graph the point $(-x, y)$ is also on the graph. A curve is symmetric about the origin if for each point (x, y)

Table C.4

STRETCHING AND COMPRESSING PRINCIPLES

OPERATION ON $y = f(x)$	Multiply $f(x)$ by c $(c > 1)$	Multiply $f(x)$ by c $(0 < c < 1)$	Multiply x by c $(c > 1)$	Multiply x by c $(0 < c < 1)$
NEW EQUATION	$y = cf(x)$	$y = cf(x)$	$y = f(cx)$	$y = f(cx)$
GEOMETRIC EFFECT	Stretches the graph of $y = f(x)$ vertically by a factor of c	Compresses the graph of $y = f(x)$ vertically by a factor of $1/c$	Compresses the graph of $y = f(x)$ horizontally by a factor of c	Stretches the graph of $y = f(x)$ horizontally by a factor of $1/c$
EXAMPLE				

Explain why the graph of a nonzero function cannot be symmetric about the x-axis.



▶ **Figure C.7**

on the graph, the point $(-x, -y)$ is also on the graph. (Equivalently, a graph is symmetric about the origin if rotating the graph 180° about the origin leaves it unchanged.) This suggests the following symmetry tests.

C.3 THEOREM (*Symmetry Tests*)

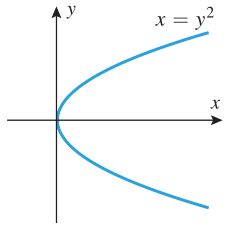
(*a*) *A plane curve is symmetric about the y-axis if and only if replacing x by $-x$ in its equation produces an equivalent equation.*

(*b*) *A plane curve is symmetric about the x-axis if and only if replacing y by $-y$ in its equation produces an equivalent equation.*

(*c*) *A plane curve is symmetric about the origin if and only if replacing both x by $-x$ and y by $-y$ in its equation produces an equivalent equation.*

▶ **Example 11** Use Theorem C.3 to identify symmetries in the graph of $x = y^2$.

Solution. Replacing y by $-y$ yields $x = (-y)^2$, which simplifies to the original equation $x = y^2$. Thus, the graph is symmetric about the x-axis. The graph is not symmetric about the y-axis because replacing x by $-x$ yields $-x = y^2$, which is not equivalent to the original equation $x = y^2$. Similarly, the graph is not symmetric about the origin because replacing x by $-x$ and y by $-y$ yields $-x = (-y)^2$, which simplifies to $-x = y^2$, and this is again not



▲ **Figure C.8**

equivalent to the original equation. These results are consistent with the graph of $x = y^2$ shown in Figure C.8. ◀

EVEN AND ODD FUNCTIONS

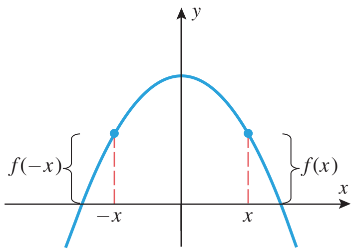
A function f is said to be an ***even function*** if

$$f(-x) = f(x) \tag{8}$$

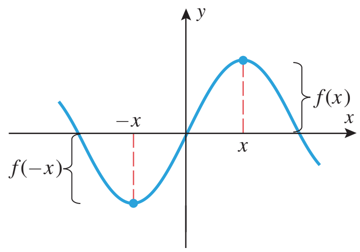
and is said to be an ***odd function*** if

$$f(-x) = -f(x) \tag{9}$$

Geometrically, the graphs of even functions are symmetric about the y-axis because replacing x by $-x$ in the equation $y = f(x)$ yields $y = f(-x)$, which is equivalent to the original equation $y = f(x)$ by (8) (see Figure C.9). Similarly, it follows from (9) that graphs of odd functions are symmetric about the origin (see Figure C.10). Some examples of even functions are x^2, x^4, x^6, and $\cos x$; and some examples of odd functions are x^3, x^5, x^7, and $\sin x$.



▲ **Figure C.9** This is the graph of an even function since $f(-x) = f(x)$.



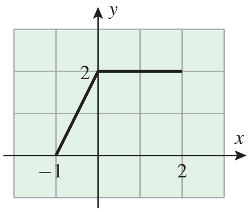
▲ **Figure C.10** This is the graph of an odd function since $f(-x) = -f(x)$.

EXERCISE SET C

Graphing Utility

FOCUS ON CONCEPTS

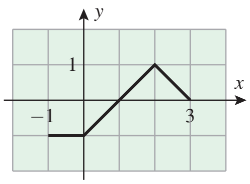
1. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.
(a) $y = f(x) - 1$ (b) $y = f(x-1)$
(c) $y = \frac{1}{2}f(x)$ (d) $y = f\left(-\frac{1}{2}x\right)$



◀ **Figure Ex-1**

2. Use the graph in Exercise 1 to sketch the graphs of the following equations.
(a) $y = -f(-x)$ (b) $y = f(2-x)$
(c) $y = 1 - f(2-x)$ (d) $y = \frac{1}{2}f(2x)$

3. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.
(a) $y = f(x+1)$ (b) $y = f(2x)$
(c) $y = |f(x)|$ (d) $y = 1 - |f(x)|$



◀ **Figure Ex-3**

4. Use the graph in Exercise 3 to sketch the graph of the equation $y = f(|x|)$.

5–24 Sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = x^2$, $y = \sqrt{x}$, $y = 1/x$, $y = |x|$, or $y = \sqrt[3]{x}$ appropriately. Then use a graphing utility to confirm that your sketch is correct.

5. $y = -2(x+1)^2 - 3$
6. $y = \frac{1}{2}(x-3)^2 + 2$
7. $y = x^2 + 6x$
8. $y = \frac{1}{2}(x^2 - 2x + 3)$
9. $y = 3 - \sqrt{x+1}$
10. $y = 1 + \sqrt{x-4}$
11. $y = \frac{1}{2}\sqrt{x} + 1$
12. $y = -\sqrt{3x}$
13. $y = \dfrac{1}{x-3}$
14. $y = \dfrac{1}{1-x}$

15. $y = 2 - \dfrac{1}{x+1}$ **16.** $y = \dfrac{x-1}{x}$

17. $y = |x+2| - 2$ **18.** $y = 1 - |x-3|$

19. $y = |2x-1| + 1$ **20.** $y = \sqrt{x^2 - 4x + 4}$

21. $y = 1 - 2\sqrt[3]{x}$ **22.** $y = \sqrt[3]{x-2} - 3$

23. $y = 2 + \sqrt[3]{x+1}$ **24.** $y + \sqrt[3]{x-2} = 0$

25. (a) Sketch the graph of $y = x + |x|$ by adding the corresponding y-coordinates on the graphs of $y = x$ and $y = |x|$.
(b) Express the equation $y = x + |x|$ in piecewise form with no absolute values, and confirm that the graph you obtained in part (a) is consistent with this equation.

26. Sketch the graph of $y = x + (1/x)$ by adding corresponding y-coordinates on the graphs of $y = x$ and $y = 1/x$. Use a graphing utility to confirm that your sketch is correct.

27–28 Find formulas for $f+g$, $f-g$, fg, and f/g, and state the domains of the functions.

27. $f(x) = 2\sqrt{x-1},\ g(x) = \sqrt{x-1}$

28. $f(x) = \dfrac{x}{1+x^2},\ g(x) = \dfrac{1}{x}$

29. Let $f(x) = \sqrt{x}$ and $g(x) = x^3 + 1$. Find
(a) $f(g(2))$ (b) $g(f(4))$ (c) $f(f(16))$
(d) $g(g(0))$ (e) $f(2+h)$ (f) $g(3+h)$.

30. Let $g(x) = \sqrt{x}$. Find
(a) $g(5s+2)$ (b) $g(\sqrt{x}+2)$ (c) $3g(5x)$
(d) $\dfrac{1}{g(x)}$ (e) $g(g(x))$ (f) $(g(x))^2 - g(x^2)$
(g) $g(1/\sqrt{x})$ (h) $g((x-1)^2)$ (i) $g(x+h)$.

31–34 Find formulas for $f \circ g$ and $g \circ f$, and state the domains of the compositions.

31. $f(x) = x^2,\ g(x) = \sqrt{1-x}$

32. $f(x) = \sqrt{x-3},\ g(x) = \sqrt{x^2+3}$

33. $f(x) = \dfrac{1+x}{1-x},\ g(x) = \dfrac{x}{1-x}$

34. $f(x) = \dfrac{x}{1+x^2},\ g(x) = \dfrac{1}{x}$

35–36 Find a formula for $f \circ g \circ h$.

35. $f(x) = x^2 + 1,\ g(x) = \dfrac{1}{x},\ h(x) = x^3$

36. $f(x) = \dfrac{1}{1+x},\ g(x) = \sqrt[3]{x},\ h(x) = \dfrac{1}{x^3}$

37–42 Express f as a composition of two functions; that is, find g and h such that $f = g \circ h$. [*Note:* Each exercise has more than one solution.]

37. (a) $f(x) = \sqrt{x+2}$ (b) $f(x) = |x^2 - 3x + 5|$

38. (a) $f(x) = x^2 + 1$ (b) $f(x) = \dfrac{1}{x-3}$

39. (a) $f(x) = \sin^2 x$ (b) $f(x) = \dfrac{3}{5 + \cos x}$

40. (a) $f(x) = 3\sin(x^2)$ (b) $f(x) = 3\sin^2 x + 4\sin x$

41. (a) $f(x) = \left(1 + \sin(x^2)\right)^3$ (b) $f(x) = \sqrt{1 - \sqrt[3]{x}}$

42. (a) $f(x) = \dfrac{1}{1-x^2}$ (b) $f(x) = |5 + 2x|$

43–46 True–False Determine whether the statement is true or false. Explain your answer.

43. The domain of $f+g$ is the intersection of the domains of f and g.

44. The domain of $f \circ g$ consists of all values of x in the domain of g for which $g(x) \neq 0$.

45. The graph of an even function is symmetric about the y-axis.

46. The graph of $y = f(x+2) + 3$ is obtained by translating the graph of $y = f(x)$ right 2 units and up 3 units.

FOCUS ON CONCEPTS

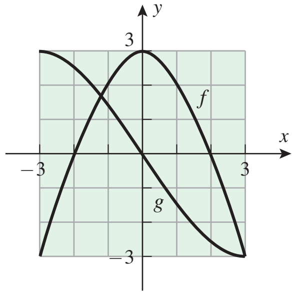
47. Use the data in the accompanying table to make a plot of $y = f(g(x))$.

x	-3	-2	-1	0	1	2	3
$f(x)$	-4	-3	-2	-1	0	1	2
$g(x)$	-1	0	1	2	3	-2	-3

▲ **Table Ex-47**

48. Find the domain of $g \circ f$ for the functions f and g in Exercise 47.

49. Sketch the graph of $y = f(g(x))$ for the functions graphed in the accompanying figure.



◀ **Figure Ex-49**

50. Sketch the graph of $y = g(f(x))$ for the functions graphed in Exercise 49.

51. Use the graphs of f and g in Exercise 49 to estimate the solutions of the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

52. Use the table given in Exercise 47 to solve the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

53–56 Find

$$\frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \frac{f(w) - f(x)}{w - x}$$

Simplify as much as possible.

53. $f(x) = 3x^2 - 5$ **54.** $f(x) = x^2 + 6x$

55. $f(x) = 1/x$ **56.** $f(x) = 1/x^2$

57. Classify the functions whose values are given in the accompanying table as even, odd, or neither.

x	-3	-2	-1	0	1	2	3
$f(x)$	5	3	2	3	1	-3	5
$g(x)$	4	1	-2	0	2	-1	-4
$h(x)$	2	-5	8	-2	8	-5	2

▲ **Table Ex-57**

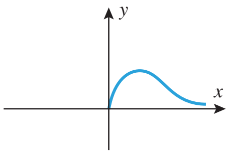
58. Complete the accompanying table so that the graph of $y = f(x)$ is symmetric about
(a) the y-axis (b) the origin.

x	-3	-2	-1	0	1	2	3
$f(x)$	1		-1	0		-5	

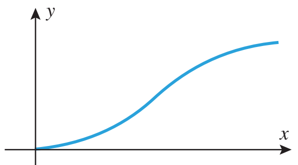
▲ **Table Ex-58**

59. The accompanying figure shows a portion of a graph. Complete the graph so that the entire graph is symmetric about
(a) the x-axis (b) the y-axis (c) the origin.

60. The accompanying figure shows a portion of the graph of a function f. Complete the graph assuming that
(a) f is an even function (b) f is an odd function.



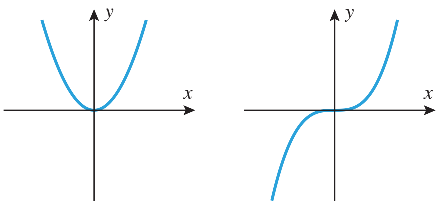
▲ **Figure Ex-59**



▲ **Figure Ex-60**

61–62 Classify the functions graphed in the accompanying figures as even, odd, or neither.

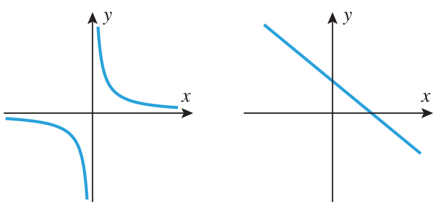
61.



(*a*) (*b*)

▲ **Figure Ex-61**

62.



(*a*) (*b*)

▲ **Figure Ex-62**

63. In each part, classify the function as even, odd, or neither.
(a) $f(x) = x^2$ (b) $f(x) = x^3$
(c) $f(x) = |x|$ (d) $f(x) = x + 1$
(e) $f(x) = \dfrac{x^5 - x}{1 + x^2}$ (f) $f(x) = 2$

64. Suppose that the function f has domain all real numbers. Determine whether each function can be classified as even or odd. Explain.
(a) $g(x) = \dfrac{f(x) + f(-x)}{2}$ (b) $h(x) = \dfrac{f(x) - f(-x)}{2}$

65. Suppose that the function f has domain all real numbers. Show that f can be written as the sum of an even function and an odd function. [*Hint:* See Exercise 64.]

66–67 Use Theorem C.3 to determine whether the graph has symmetries about the x-axis, the y-axis, or the origin.

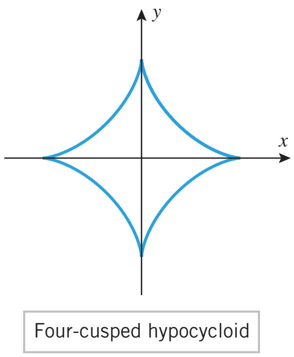
66. (a) $x = 5y^2 + 9$ (b) $x^2 - 2y^2 = 3$
(c) $xy = 5$

67. (a) $x^4 = 2y^3 + y$ (b) $y = \dfrac{x}{3 + x^2}$
(c) $y^2 = |x| - 5$

68–69 (i) Use a graphing utility to graph the equation in the first quadrant. [*Note:* To do this you will have to solve the equation for y in terms of x.] (ii) Use symmetry to make a hand-drawn sketch of the entire graph. (iii) Confirm your work by generating the graph of the equation in the remaining three quadrants.

68. $9x^2 + 4y^2 = 36$ **69.** $4x^2 + 16y^2 = 16$

70. The graph of the equation $x^{2/3} + y^{2/3} = 1$, which is shown in the accompanying figure, is called a ***four-cusped hypocycloid***.
(a) Use Theorem C.3 to confirm that this graph is symmetric about the x-axis, the y-axis, and the origin.
(b) Find a function f whose graph in the first quadrant coincides with the four-cusped hypocycloid, and use a graphing utility to confirm your work.
(c) Repeat part (b) for the remaining three quadrants.



◀ **Figure Ex-70**

71. The equation $y = |f(x)|$ can be written as

$$y = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$$

which shows that the graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by retaining the portion that lies on or above the x-axis and reflecting about the x-axis the portion that lies below the x-axis. Use this method to obtain the graph of $y = |2x - 3|$ from the graph of $y = 2x - 3$.

72–73 Use the method described in Exercise 71. ■

72. Sketch the graph of $y = |1 - x^2|$.

73. Sketch the graph of
(a) $f(x) = |\cos x|$ (b) $f(x) = \cos x + |\cos x|$.

74. The ***greatest integer function***, $\lfloor x \rfloor$, is defined to be the greatest integer that is less than or equal to x. For example, $\lfloor 2.7 \rfloor = 2$, $\lfloor -2.3 \rfloor = -3$, and $\lfloor 4 \rfloor = 4$. In each part, sketch the graph of $y = f(x)$.
(a) $f(x) = \lfloor x \rfloor$ (b) $f(x) = \lfloor x^2 \rfloor$
(c) $f(x) = \lfloor x \rfloor^2$ (d) $f(x) = \lfloor \sin x \rfloor$

75. Is it ever true that $f \circ g = g \circ f$ if f and g are nonconstant functions? If not, prove it; if so, give some examples for which it is true.