3.3 Differentiation Rules

① Derivative of a constant function ($f(x) = c$)	$\frac{d}{dx}(c) = 0$
② Derivative of $f(x) = x$	$\frac{d}{dx}(x) = 1$
③ Derivative of $f(x) = x^2$	$\frac{d}{dx}(x^2) = 2x$
④ Derivative of $2x$	$\frac{d}{dx}(2x) = 2$
⑤ Derivative of x^3	$\frac{d}{dx}(x^3) = 3x^2$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

For any real number n

Examples: Differentiate

1) $\frac{d}{dx}(x^6) =$; $\frac{d}{dx}(x^8) =$; $\frac{d}{dt} t^{15}$

2) $\frac{d}{dx}\left(\frac{1}{x^2}\right) =$

3) $f(x) = \sqrt[3]{x^2}$

4) $\frac{d}{ds}(s^{-3}) =$

The Constant Multiple Rule: c constant and f differentiable

$$\frac{d}{dx}[c\,f(x)] = c\frac{d}{dx}f(x)$$

Examples: Differentiate

a) $f(x) = 3x^4$

b) $f(x) = -x^{-2}$

c) $\frac{d}{dx}\left(\frac{\pi}{x^2}\right) =$

The sum and difference Rule: If f and g are differentiable

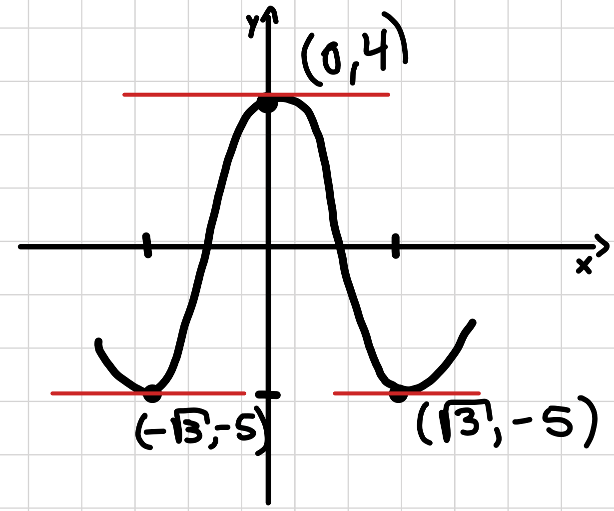
$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

Examples: Differentiate

a) $f(x) = x^5 - 2x^4 + 2x - 3$

b) $\frac{d}{dx}(3x^9 - x^{-3}) =$

Example: Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.



Exponential Functions: Let $f(x) = b^x$ be an exponential function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x b^h - b^x}{h}$$

$$= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} = b^x \lim_{h \to 0} \frac{b^h - 1}{h} = f'(0)$$

Then if $f(x) = b^x$ is differentiable at 0,

then is differentiable everywhere and $f'(x) = f'(0)b^x$

The number e: Is the unique number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

Among all possible exponential $y = b^x$, e is the base for which

the slope of the tangent line is 1.

Then, $\frac{d}{dx}(e^x) = e^x$ $\left(\frac{d}{dx} e^x = f'(0) e^x \quad f(x) = e^x \right)$

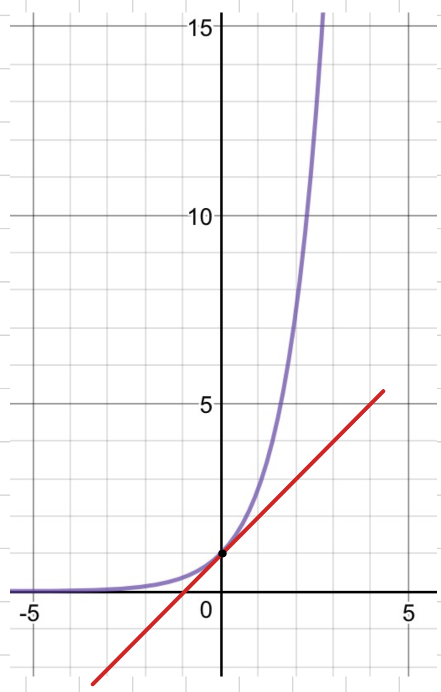
Example: If $f(x) = e^x - x$, find $f'(x)$ and $f''(x)$

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

Example: At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$.

(because we want the slope of the tangent line to be the same as $y=2x$)



Higher Derivatives

Example: Find all higher derivatives of $f(x) = 2x^4 - 3x^3 + 5x^2 - 6x + 18$

What is the derivative of the product of two functions?

Let $f(x) = x^2$ and $g(x) = x^3$ then $(f \cdot g)' = ?$ $\frac{d}{dx}(f \cdot g)(x) = ?$

The Product Rule

Suppose f and g are two differentiable functions, then

$$(f \cdot g)' = fg' + f'g$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + \frac{d}{dx}[f(x)] \cdot g(x)$$

Returning to the above example

$\frac{d}{dx}(x^2 \cdot x^3)$

Examples: Find the derivative of the following functions:

1) $y = (x^2 - 1)(3x^4 + 2x)$

2) $f(x) = (1 + x^2)\sqrt{x}$

3) $g(x) = (x^2 - 1)f(x)$ and $f(2) = 3$, $f'(2) = -1$. Find $g'(2)$

$f(x) = xe^x$ $\quad f'(x) =$

$=$

$f''(x) =$

$f'''(x) =$

$\vdots$

$f^{(n)}(x) =$

The Quotient Rule

Suppose f and g are two differentiable functions, then

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg}{g^2}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Again $\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$ For example, let $f(x) = x^3$ and $g(x) = x^6$

Examples: Find the derivative of the following functions:

1) $y = \dfrac{x^3 - 3x^2 - 5}{2x + 5}$

2) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

3) Find an equation of the tangent line to the curve $y = \frac{e^x}{1+x^2}$ at the point $(1, \frac{1}{2}e)$.

4) $f(x) = (x^2+1)\left(\frac{x+1}{x+2}\right)$

Practice:

① Find the first and second derivatives

(a) $y = -x^2 + 3$

(b) $y = 6x^2 - 10x - 5x^{-2}$

② Find the derivatives of the functions

(a) $h(t) = (6t^3 - t)(10 - 20t)$

(b) $f(x) = \sqrt[3]{x^2}\,(2x - x^2)$

(c) $g(x) = \sqrt{x}\,(a + bx)$ a,b ctt.

(d) $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$

(e) $y = 2e^{-x} + e^{3x}$

(f) $h(x) = x^3 e^x$

(g) $y = \dfrac{e^x}{s}$

(h) $y = x^{-3/5} + \pi^{3/2}$

Normal line to a curve at a: Is the line perpendicular to the tangent line at a.

③ Find an equation for the normal line to the curve $y = x^3 - 4x + 1$ at $(2,1)$

④ Find equations for the horizontal tangent lines to the curve $y = x^3 - 3x - 2$

What is the smallest slope on the curve? At what points?