Section 12.4

Matrix Algebra

A **matrix** is defined as a rectangular array of numbers:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \text{Column } 1 & \text{Column } 2 & & \text{Column } j & & \text{Column } n \end{array} \\
\begin{array}{c} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } i \\ \vdots \\ \text{Row } m \end{array} &
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}
\end{array}
$$

EXAMPLE Arranging Data in a Matrix

In a survey of 900 people, the following information was obtained:

200 males	Thought federal defense spending was too high
150 males	Thought federal defense spending was too low
45 males	Had no opinion
315 females	Thought federal defense spending was too high
125 females	Thought federal defense spending was too low
65 females	Had no opinion

We can arrange these data in a rectangular array as follows:

	Too High	Too Low	No Opinion
Male	200	150	45
Female	315	125	65

or as the matrix

$$\begin{bmatrix} 200 & 150 & 45 \\ 315 & 125 & 65 \end{bmatrix}$$

This matrix has two rows (representing males and females) and three columns (representing "too high," "too low," and "no opinion").

EXAMPLE

Examples of Matrices

(a) $\begin{bmatrix} 5 & 0 \\ -6 & 1 \end{bmatrix}$ A 2 by 2 square matrix

(b) $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$ A 1 by 3 matrix

(c) $\begin{bmatrix} 6 & -2 & 4 \\ 4 & 3 & 5 \\ 8 & 0 & 1 \end{bmatrix}$ A 3 by 3 square matrix

1 Find the Sum and Difference of Two Matrices

DEFINITION

Two matrices A and B are said to be **equal,** written as

$$\boxed{A = B}$$

provided that A and B have the same number of rows and the same number of columns and each entry a_{ij} in A is equal to the corresponding entry b_{ij} in B.

If A and B are both $m \times n$ matrices then the **sum** of A and B, denoted $A + B$, is a matrix obtained by adding ***corresponding entries*** of A and B. The **difference** of A and B, denoted $A - B$, is obtained by subtracting ***corresponding entries*** of A and B.

EXAMPLE Adding and Subtracting Matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 0 & 4 \\ 2 & 1 & -4 \end{bmatrix}$$

Find: (a) $A + B$ (b) $A - B$

$$A + B = \begin{bmatrix} -2 & -2 & 6 \\ 2 & 0 & -1 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 4 & -2 & -2 \\ -2 & -2 & 7 \end{bmatrix}$$

Commutative Property of Matrix Addition

$$A + B = B + A$$

Associative Property of Matrix Addition

$$(A + B) + C = A + (B + C)$$

EXAMPLE

Demonstrating the Commutative Property

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 5 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 2 + (-1) & 3 + 2 & -1 + 1 \\ 4 + 5 & 0 + (-3) & 7 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 + 2 & 2 + 3 & 1 + (-1) \\ 5 + 4 & -3 + 0 & 4 + 7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 1 \\ 5 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 7 \end{bmatrix}$$

The Zero Matrix

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 2 by 2 square zero matrix

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 2 by 3 zero matrix

$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ 1 by 3 zero matrix

$$\boxed{A + 0 = 0 + A = A}$$

2 Find Scalar Multiples of a Matrix

If A is an $m \times n$ matrix and s is a scalar, then we let kA denote the matrix obtained by multiplying every element of A by k. This procedure is called **scalar multiplication.**

EXAMPLE Operations Using Matrices

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 1 \\ 0 & -4 \end{bmatrix}$$

Find: (a) $4A$ (b) $\frac{1}{3}C$ (c) $3A - 2B$

(a) $$4A = 4\begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -8 & 8 \\ 0 & -4 & 12 \end{bmatrix}$$

(b) $$\frac{1}{3}C = \frac{1}{3}\begin{bmatrix} 3 & 1 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{4}{3} \end{bmatrix}$$

(c) $$3A - 2B = 3\begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} - 2\begin{bmatrix} 0 & -2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \\ -2 & -7 & 7 \end{bmatrix}$$

Properties of Scalar Multiplication

$$k(hA) = (kh)A$$
$$(k + h)A = kA + hA$$
$$k(A + B) = kA + kB$$

3 Find the Product of Two Matrices

A **row vector** $\boldsymbol{R}$ is a 1 by n matrix

$$R = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

A **column vector** $\boldsymbol{C}$ is an n by 1 matrix

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The **product** $\boldsymbol{RC}$ of R times C is defined as the number

$$\boxed{RC = \begin{bmatrix} r_1 & r_2 \cdots r_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n}$$

EXAMPLE

The Product of a Row Vector and a Column Vector

Find RC if $R = \begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$$RC = (1)(2) + (-2)(-1) + (4)(3) = 2 + 2 + 12 = 16$$

EXAMPLE Using Matrices to Compute Revenue

A clothing store sells men's shirts for \$40, silk ties for \$20, and wool suits for \$400. Last month, the store had sales consisting of 100 shirts, 200 ties, and 50 suits. What was the total revenue due to these sales?

We set up a row vector R to represent the prices of each item and a column vector C to represent the corresponding number of items sold. Then

$$\begin{array}{c} \text{Prices} \\ \text{Shirts Ties Suits} \\ R = \begin{bmatrix} 40 & 20 & 400 \end{bmatrix} \end{array} \qquad \begin{array}{c} \text{Number} \\ \text{sold} \\ C = \begin{bmatrix} 100 \\ 200 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \text{Shirts} \\ \text{Ties} \\ \text{Suits} \end{array}$$

The total revenue obtained is the product RC.

$$RC = (40)(100) + (20)(200) + (400)(50) = 4000 + 4000 + 20000 = 28000$$

Let A denote an m by r matrix, and let B denote an r by n matrix. The **product** AB is defined as the m by n matrix whose entry in row i, column j is the product of the ith row of A and the jth column of B.

A	B
m by r	r by n

Must be same for AB to be defined

AB is m by n.

EXAMPLE Multiplying Two Matrices

Find the product AB if

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \text{Row 1 of } A \text{ times column 1 of } B & \text{Row 1 of } A \text{ times column 2 of } B \\ \text{Row 2 of } A \text{ times column 1 of } B & \text{Row 2 of } A \text{ times column 2 of } B \end{bmatrix}$$

$$AB = \begin{bmatrix} 3(2)-2(-1)+1(-3) & 3(4)-2(3)+1(1) \\ 0(2)+4(-1)-1(-3) & 0(4)+4(3)-1(1) \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 7 \\ -1 & 11 \end{bmatrix}$$

EXAMPLE Multiplying Two Matrices

Find the product BA if $A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 6 & 12 & -2 \\ -3 & 14 & -4 \\ -9 & 10 & -4 \end{bmatrix}$$

Recall from last example:

$$AB = \begin{bmatrix} 5 & 7 \\ -1 & 11 \end{bmatrix}$$

EXAMPLE **Multiplying Two Square Matrices**

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 0 \\ 3 & 4 \end{bmatrix}$$

find: (a) AB (b) BA

$$AB = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ -17 & -28 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 0 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ -5 & -19 \end{bmatrix}$$

$$\boldsymbol{AB \neq BA}$$

THEOREM

Matrix multiplication is not commutative.

Associative Property of Matrix Multiplication

$$A(BC) = (AB)C$$

Distributive Property

$$A(B + C) = AB + AC$$

For an n by n square matrix, the entries located in row i, column i, $1 \leq i \leq n$, are called the **diagonal entries**. An n by n square matrix whose diagonal entries are 1's, while all other entries are 0's, is called the **identity matrix** $\boldsymbol{I_n}$. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and so on.

EXAMPLE Multiplication with an Identity Matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

Find: (a) AI_3 (b) I_2A (c) BI_2

$$\text{(a) } AI_3 = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

$$\text{(b) } AI_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

$$\text{(c) } BI_2 = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

Identity Property

If A is an m by n matrix, then

$$\boxed{I_m A = A \quad \text{and} \quad A I_n = A}$$

If A is an n by n square matrix, then

$$\boxed{A I_n = I_n A = A}$$

4 Find the Inverse of a Matrix

DEFINITION

Let A be a square n by n matrix. If there exists an n by n matrix A^{-1}, read "A inverse," for which

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

then A^{-1} is called the **inverse** of the matrix A.

EXAMPLE

Multiplying a Matrix by Its Inverse

Show that the inverse of $A = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$.

$$AA^{-1} = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Procedure for Finding the Inverse of a Nonsingular Matrix

To find the inverse of an n by n nonsingular matrix A, proceed as follows:

Step 1: Form the matrix $[A|I_n]$.

Step 2: Transform the matrix $[A|I_n]$ into reduced row echelon form.

Step 3: The reduced row echelon form of $[A|I_n]$ will contain the identity matrix I_n on the left of the vertical bar; the n by n matrix on the right of the vertical bar is the inverse of A.

EXAMPLE **Finding the Inverse of a Matrix**

The matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix}$ is nonsingular. Find its inverse.

$$A|I_3 = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = -2r_1 + r_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 = -r_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 4 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = -4r_2 + r_3]{R_1 = r_1 + r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 9 & -2 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = 1/9r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{9} & -\frac{5}{9} & \frac{1}{9} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{7}{9} & -\frac{5}{9} & \frac{1}{9} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{bmatrix}$$

EXAMPLE Showing That a Matrix Has No Inverse

Show that the matrix $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ has no inverse.

$$\left[\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 4 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = -1/2 r_1} \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 4 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = -4r_1 + r_2} \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

We can see that we cannot get the identity on the left side of the vertical bar. We conclude that A is singular and has no inverse.

5 Solve a System of Linear Equations Using an Inverse Matrix

EXAMPLE

Using the Inverse Matrix to Solve a System of Linear Equations

Solve the system of equations: $\begin{cases} x - y + 2z = 1 \\ \quad -y + 3z = -2 \\ 2x + 2y + z = -1 \end{cases}$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \qquad AX = B$$

$$X = A^{-1}B$$

$$A^{-1}B = \begin{bmatrix} \frac{7}{9} & -\frac{5}{9} & \frac{1}{9} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{16}{9} \\ -\frac{5}{3} \\ -\frac{11}{9} \end{bmatrix}$$