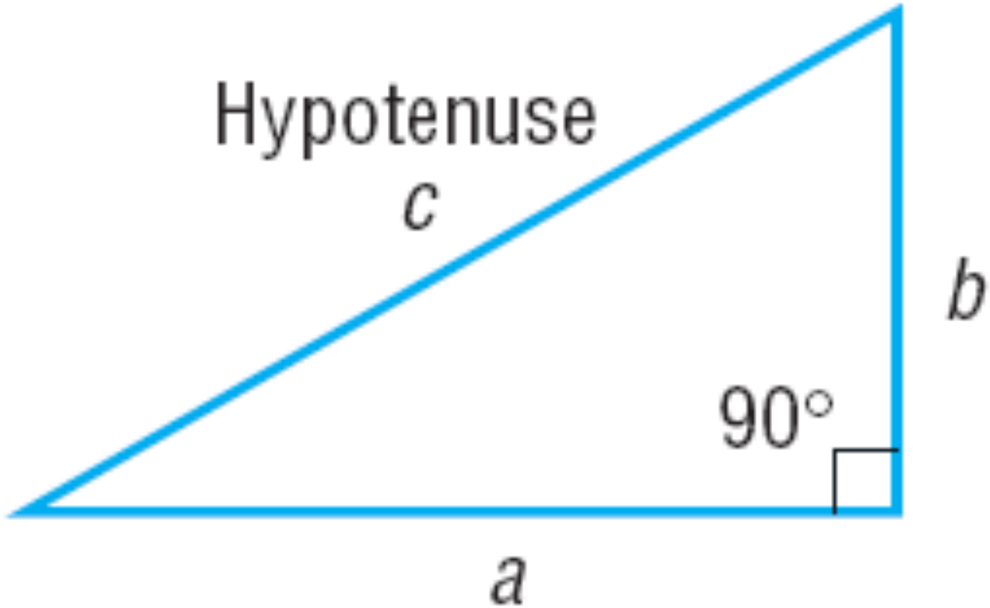
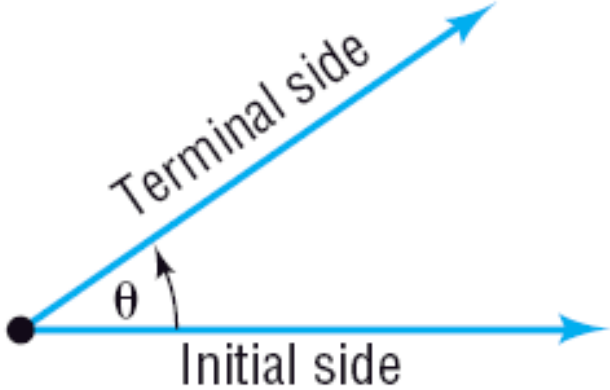
Section 7.2

Right Triangle Trigonometry

1 Find the Values of Trigonometric Functions of Acute Angles

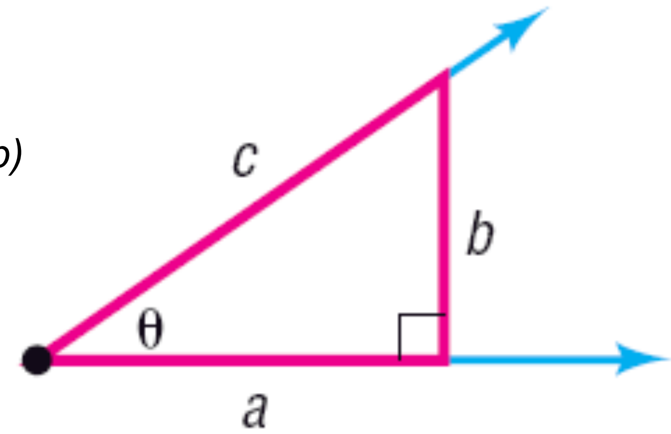


$$\boxed{c^2 = a^2 + b^2}$$



(a) Acute angle

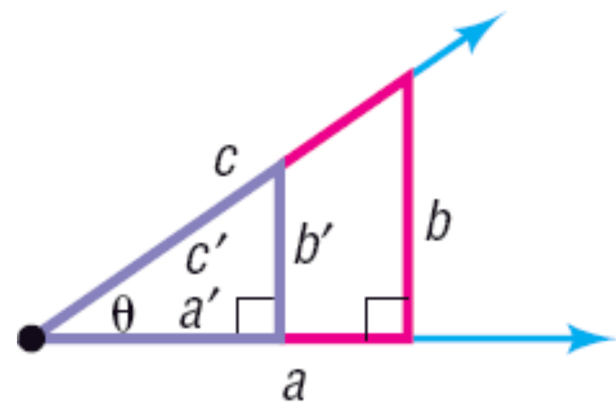
Figure 19 (b)



(b) Right triangle

Figure 19(b), with hypotenuse of length c and legs of lengths a and b. Using the three sides of this triangle, we can form exactly six ratios:

$$\frac{b}{c},\quad \frac{a}{c},\quad \frac{b}{a},\quad \frac{c}{b},\quad \frac{c}{a},\quad \frac{a}{b}$$

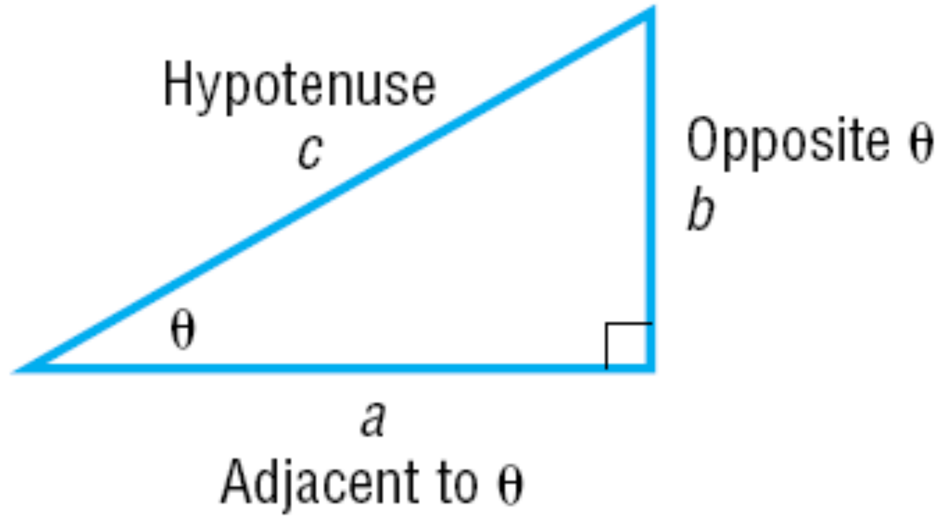


(c) Similar triangles

Trigonometric Functions of Acute Angles

Function Name	Abbreviation	Value
sine of θ	$\sin\theta$	$\frac{b}{c}$
cosine of θ	$\cos\theta$	$\frac{a}{c}$
tangent of θ	$\tan\theta$	$\frac{b}{a}$

Function Name	Abbreviation	Value
cosecant of θ	$\csc\theta$	$\frac{c}{b}$
secant of θ	$\sec\theta$	$\frac{c}{a}$
cotangent of θ	$\cot\theta$	$\frac{a}{b}$

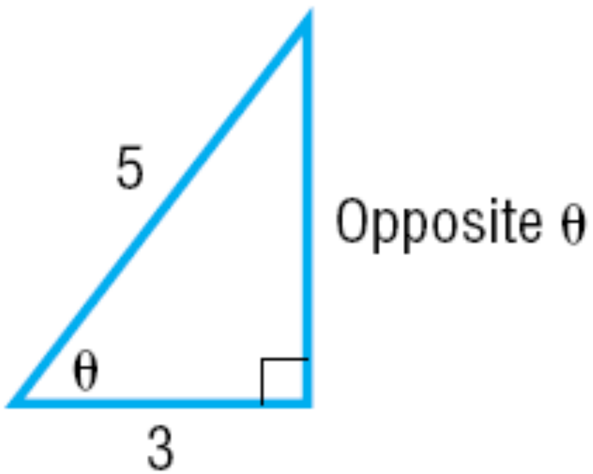


$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}$$

EXAMPLE Finding the Values of Trigonometric Functions

Find the value of each of the six trigonometric functions of the angle θ.



$$
\begin{aligned}
(\text{adjacent})^2 + (\text{opposite})^2 &= (\text{hypotenuse})^2 \\
3^2 + (\text{opposite})^2 &= 5^2 \\
(\text{opposite})^2 &= 25 - 9 = 16 \\
\text{opposite} &= 4
\end{aligned}
$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4}$$

2 Use the Fundamental Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

EXAMPLE

Finding the Values of the Remaining Trigonometric Functions, Given $\sin\theta$ and $\cos\theta$

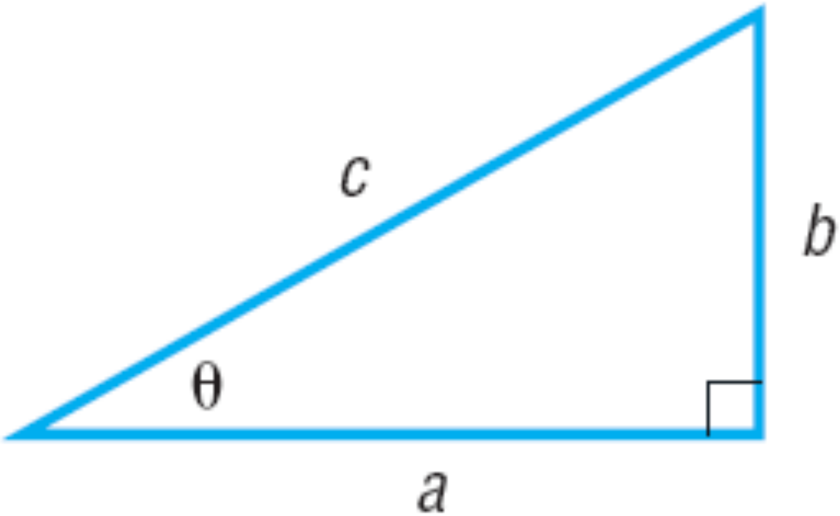
Given $\sin\theta = \dfrac{\sqrt{10}}{10}$ and $\cos\theta = \dfrac{3\sqrt{10}}{10}$, find the value of each of the four remaining trigonometric functions of θ.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{10}}{10}}{\frac{3\sqrt{10}}{10}} = \frac{\sqrt{10}}{10}\cdot\frac{10}{3\sqrt{10}} = \frac{1}{3}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{1}{3}} = 3$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{\sqrt{10}}{10}} = \frac{10}{\sqrt{10}} = \sqrt{10}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{3\sqrt{10}}{10}} = \frac{10}{3\sqrt{10}} = \frac{\sqrt{10}}{3}$$



$$b^2 + a^2 = c^2$$

Dividing each side by c^2, we get

$$\frac{b^2}{c^2} + \frac{a^2}{c^2} = 1 \quad \text{or} \quad \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 = 1$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

$$\boxed{\cot^2\theta + 1 = \csc^2\theta}$$

Fundamental Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

EXAMPLE

Finding the Exact Value of a Trigonometric Expression Using Identities

Find the exact value of each expression. Do not use a calculator.

(a) $\dfrac{1}{\csc^2 35°} + \cos^2 35°$ (b) $\dfrac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} - \cot\dfrac{\pi}{3}$

(a) $$\frac{1}{\csc^2 35°} + \cos^2 35° = \sin^2 35° + \cos^2 35° = 1$$

$\csc\theta = \dfrac{1}{\sin\theta}$

$\sin^2\theta + \cos^2\theta = 1$

(b) $$\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} - \cot\frac{\pi}{3} = \cot\frac{\pi}{3} - \cot\frac{\pi}{3} = 0$$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

3 Find the Values of the Remaining Trigonometric Functions, Given the Value of One of Them

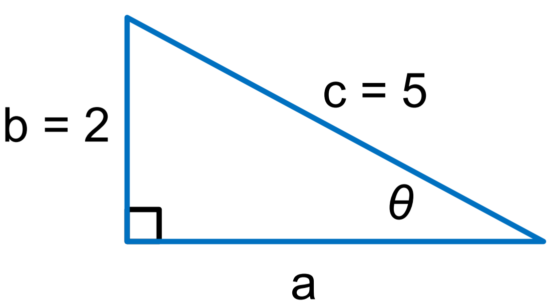
EXAMPLE

Solution 1
Using the Definition

Finding the Values of the Remaining Trigonometric Functions, Given sin θ, θ Acute

Given that $\sin\theta = \dfrac{2}{5}$ and θ is an acute angle, find the exact value of each of the remaining five trigonometric functions of θ.

$\sin\theta = \dfrac{2}{5} = \dfrac{b}{c}$ so draw a triangle with these sides.



$$a^2 + 2^2 = 5^2$$

$$a^2 = 25 - 4 = 21$$

$$a = \sqrt{21}$$

$$\csc\theta = \frac{c}{b} = \frac{5}{2}$$

$$\sec\theta = \frac{c}{a} = \frac{5}{\sqrt{21}} = \frac{5\sqrt{21}}{21}$$

$$\cos\theta = \frac{a}{c} = \frac{\sqrt{21}}{5}$$

$$\tan\theta = \frac{b}{a} = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

$$\cot\theta = \frac{a}{b} = \frac{\sqrt{21}}{2}$$

EXAMPLE

Solution 2
Using Identities

Finding the Values of the Remaining Trigonometric Functions, Given sin θ, θ Acute

Given that $\sin\theta = \dfrac{2}{5}$ and θ is an acute angle, find the exact value of each of the remaining five trigonometric functions of θ.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{2}{5}\right)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{4}{25} = \frac{21}{25}$$

$$\cos\theta = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}$$

Since θ is acute the trig functions are all positive.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{2}{5}}{\frac{\sqrt{21}}{5}} = \frac{2}{5}\cdot\frac{5}{\sqrt{21}} = \frac{2\sqrt{21}}{21}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{2}{5}} = \frac{5}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{21}}{5}} = \frac{5}{\sqrt{21}} = \frac{5\sqrt{21}}{21}$$

Finding the Values of the Trigonometric Functions When One Is Known

Given the value of one trigonometric function of an acute angle θ, the exact value of each of the remaining five trigonometric functions of θ can be found in either of two ways.

Method 1 Using the Definition

STEP 1: Draw a right triangle showing the acute angle θ.

STEP 2: Two of the sides can then be assigned values based on the value of the given trigonometric function.

STEP 3: Find the length of the third side by using the Pythagorean Theorem.

STEP 4: Use the definitions in equation (1) to find the value of each of the remaining trigonometric functions.

Method 2 Using Identities

Use appropriately selected identities to find the value of each of the remaining trigonometric functions.

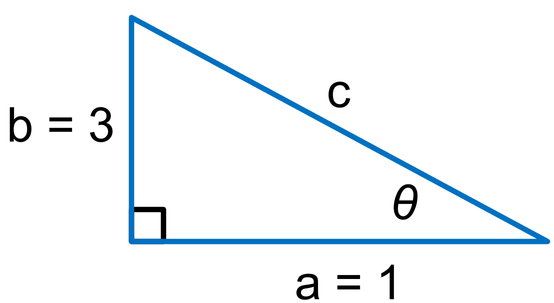
EXAMPLE

Solution 1
Using the Definition

Given One Value of a Trigonometric Function, Find the Values of the Remaining Ones

Given that $\cot\theta = \dfrac{1}{3}$ and θ is an acute angle, find the exact value of each of the remaining five trigonometric functions of θ.

$\cot\theta = \dfrac{1}{3} = \dfrac{a}{b}$ so draw a triangle with these sides.



$$1^2 + 3^2 = c^2$$

$$c^2 = 10$$

$$c = \sqrt{10}$$

$$\csc\theta = \frac{c}{b} = \frac{\sqrt{10}}{3}$$

$$\sec\theta = \frac{c}{a} = \frac{\sqrt{10}}{1} = \sqrt{10}$$

$$\cos\theta = \frac{a}{c} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$\tan\theta = \frac{b}{a} = \frac{3}{1} = 3$$

$$\sin\theta = \frac{b}{c} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

EXAMPLE

Solution 2 Using Identities

Given One Value of a Trigonometric Function, Find the Values of the Remaining Ones

Given that $\cot\theta = \frac{1}{3}$ and θ is an acute angle, find the exact value of each of the remaining five trigonometric functions of θ.

$$\cot^2\theta + 1 = \csc^2\theta$$

$$\left(\frac{1}{3}\right)^2 + 1 = \csc^2\theta$$

$$\csc^2\theta = \frac{1}{9} + 1 = \frac{10}{9}$$

$$\csc\theta = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3}$$

Since θ is acute the trig functions are all positive.

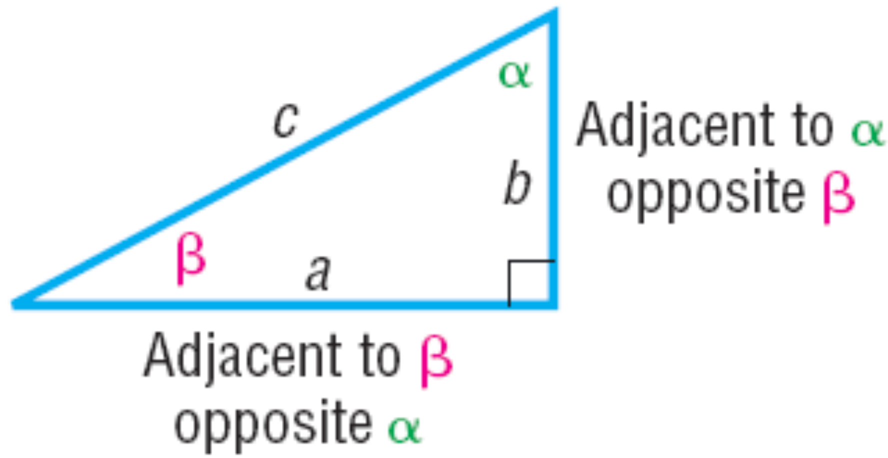
$$\sin\theta = \frac{1}{\csc\theta} = \frac{1}{\frac{\sqrt{10}}{3}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

$$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{\frac{1}{3}} = 3$$

$$\cos\theta = \frac{\sin\theta}{\tan\theta} = \frac{\frac{3\sqrt{10}}{10}}{3} = \frac{3\sqrt{10}}{10}\cdot\frac{1}{3} = \frac{\sqrt{10}}{10}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{10}}{10}} = \frac{10}{\sqrt{10}} = \sqrt{10}$$

4 Use the Complementary Angle Theorem



$$\sin\beta = \frac{b}{c} = \cos\alpha \qquad \cos\beta = \frac{a}{c} = \sin\alpha \qquad \tan\beta = \frac{b}{a} = \cot\alpha$$

$$\csc\beta = \frac{c}{b} = \sec\alpha \qquad \sec\beta = \frac{c}{a} = \csc\alpha \qquad \cot\beta = \frac{a}{b} = \tan\alpha$$

THEOREM

Complementary Angle Theorem

Cofunctions of complementary angles are equal.

Complementary angles

$$\sin 30^\circ = \cos 60^\circ$$

Cofunctions

Complementary angles

$$\tan 40^\circ = \cot 50^\circ$$

Cofunctions

Complementary angles

$$\sec 80^\circ = \csc 10^\circ$$

Cofunctions

θ **(Degrees)**	θ **(Radians)**
$\sin\theta = \cos(90^\circ - \theta)$	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
$\cos\theta = \sin(90^\circ - \theta)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot(90^\circ - \theta)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$
$\csc\theta = \sec(90^\circ - \theta)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc(90^\circ - \theta)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$
$\cot\theta = \tan(90^\circ - \theta)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

EXAMPLE

Using the Complementary Angle Theorem

(a) $\sin 62^\circ = \cos(90^\circ - 62^\circ) = \cos 28^\circ$

(b) $\tan \frac{\pi}{12} = \cot\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = \cot \frac{5\pi}{12}$

(c) $\cos \frac{\pi}{4} = \sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \sin \frac{\pi}{4}$

(d) $\csc \frac{\pi}{6} = \sec\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \sec \frac{\pi}{3}$

EXAMPLE

Using the Complementary Angle Theorem

Find the exact value of each expression. Do not use a calculator.

(a) $\dfrac{\tan 75^\circ}{\cot 15^\circ}$ (b) $\cos 38^\circ - \sin 52^\circ$

(a) $\dfrac{\tan 75^\circ}{\cot 15^\circ} = \dfrac{\tan 75^\circ}{\tan 75^\circ} = 1$

(b) $\cos 38^\circ - \sin 52^\circ = \cos 38^\circ - \cos 38^\circ = 0$