Section 11.6

Polar Equations of Conics

1 Analyze and Graph Polar Equations of Conics

Let D denote a fixed line called the **directrix**; let F denote a fixed point called the **focus**, which is not on D; and let e be a fixed positive number called the **eccentricity**. A **conic** is the set of points P in the plane such that the ratio of the distance from F to P to the distance from D to P equals e. That is, a conic is the collection of points P for which

$$\frac{d(F, P)}{d(D, P)} = e \tag{1}$$

If $e = 1$, the conic is a **parabola**.

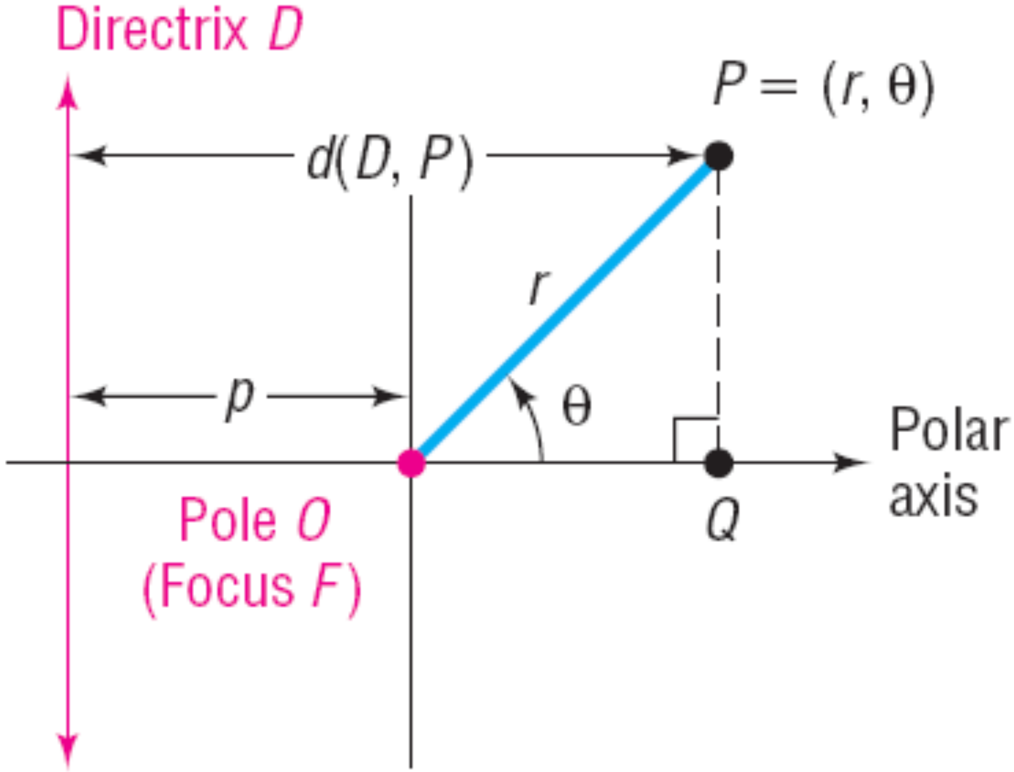
If $e < 1$, the conic is an **ellipse**.

If $e > 1$, the conic is a **hyperbola**.

For both an ellipse and a hyperbola, the eccentricity e satisfies

$$\boxed{e = \frac{c}{a}}$$

where c is the distance from the center to the focus and a is the distance from the center to a vertex.



$$\frac{d(F, P)}{d(D, P)} = e \quad \text{or} \quad d(F, P) = e \cdot d(D, P)$$

THEOREM

Polar Equation of a Conic; Focus at Pole; Directrix Perpendicular to Polar Axis a Distance *p* to the Left of the Pole

The polar equation of a conic with focus at the pole and directrix perpendicular to the polar axis at a distance p to the left of the pole is

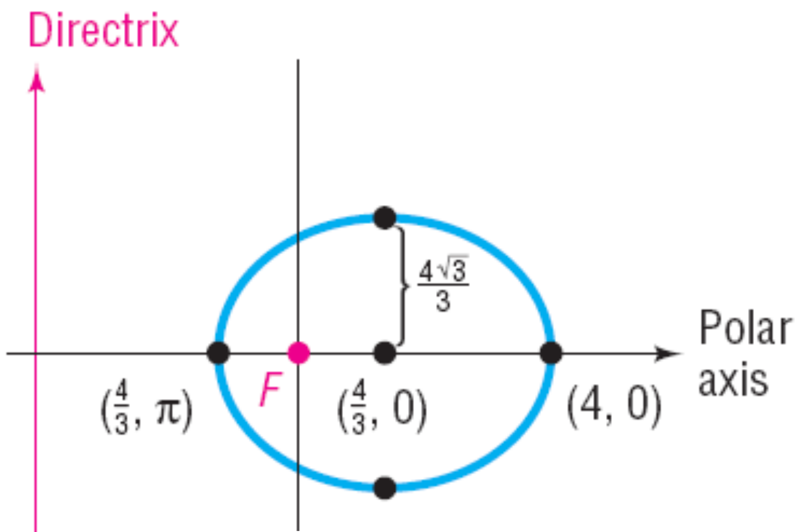
$$r = \frac{ep}{1 - e\cos\theta} \qquad \textbf{(4)}$$

where e is the eccentricity of the conic.

EXAMPLE Analyzing and Graphing the Polar Equation of a Conic

Analyze and graph the equation: $r = \dfrac{4}{2 - \cos\theta} = \dfrac{2}{1 - \frac{1}{2}\cos\theta}$

Divide numerator and denominator by 2



$$e = \frac{1}{2} \quad \text{and} \quad ep = 2$$

$$\frac{1}{2}p = 2, \quad \text{so} \quad p = 4$$

We conclude that the conic is an ellipse, since $e = \dfrac{1}{2} < 1$.

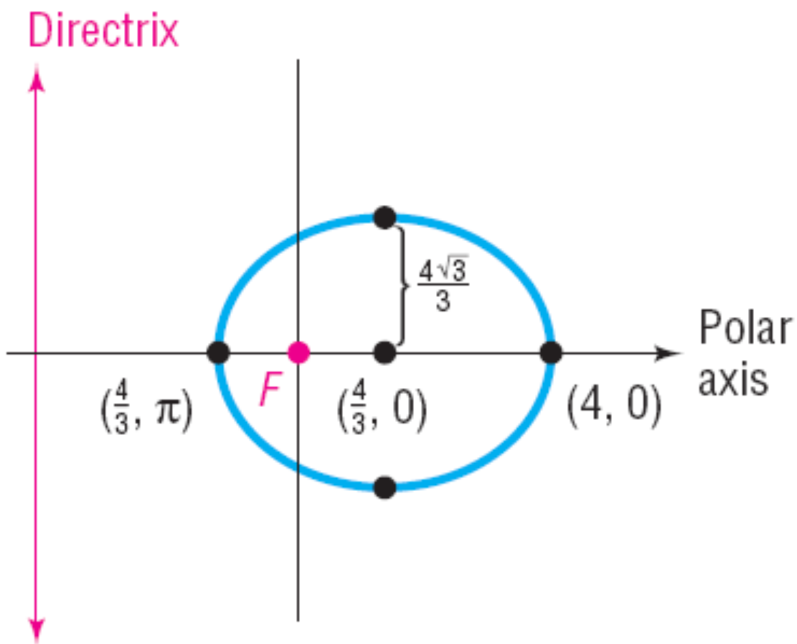
One focus is at the pole and the directrix is perpendicular to the polar axis 4 units to the left of the pole.

Let $\theta = 0$ and π to find vertices which are $(4,0)$ and $\left(\dfrac{4\pi}{3}, \pi\right)$

$$\boxed{r = \frac{ep}{1 - e\cos\theta}}$$

EXAMPLE Analyzing and Graphing the Polar Equation of a Conic

Analyze and graph the equation: $r = \dfrac{4}{2 - \cos\theta} = \dfrac{2}{1 - \frac{1}{2}\cos\theta}$



$$e = \frac{c}{a} \text{ so } c = ea = \left(\frac{1}{2}\right)\left(\frac{8}{3}\right) = \frac{4}{3}$$

$$b^2 = a^2 - c^2 = \frac{64}{9} - \frac{16}{9} = \frac{48}{9}$$

$$b = \frac{4\sqrt{3}}{3}$$

Center is at $\left(\dfrac{4}{3}, 0\right)$ and distance from center to vertex is $a = \dfrac{8}{3}$.

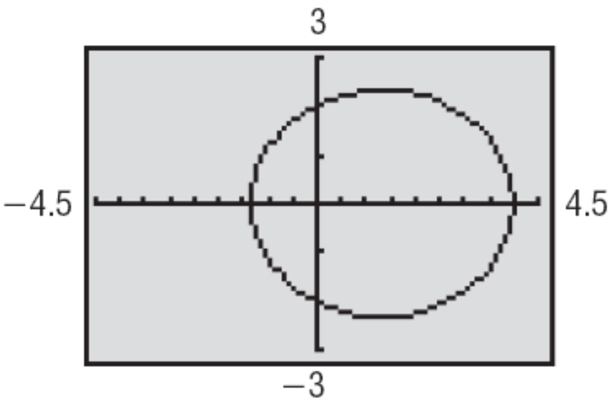
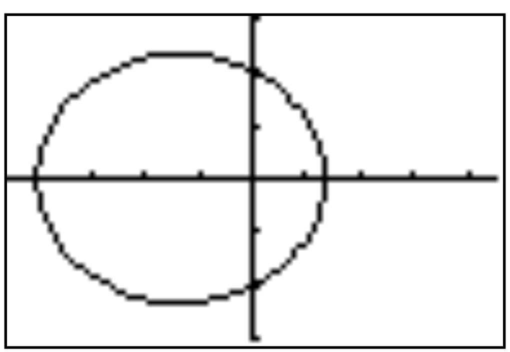


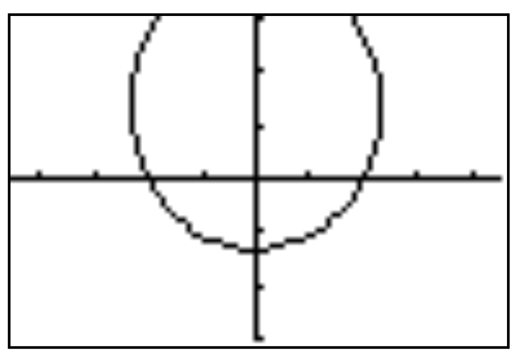
Figure 56

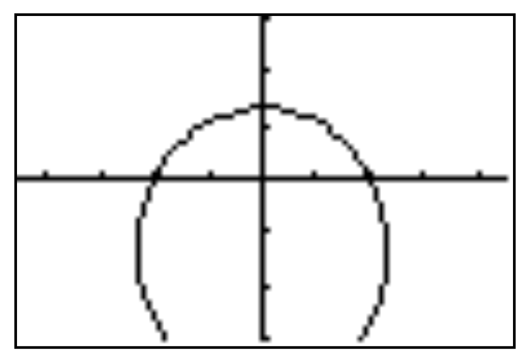
$$r = \frac{4}{2 - \cos\theta}$$

Exploration

Graph $r_1 = \dfrac{4}{2 + \cos\theta}$ and compare the result with Figure 56. What do you conclude? Clear the screen and graph $r_1 = \dfrac{4}{2 - \sin\theta}$ and then $r_1 = \dfrac{4}{2 + \sin\theta}$. Compare each of these graphs with Figure 56. What do you conclude?







The polar equation of a conic with focus at the pole and directrix perpendicular to the polar axis at a distance p units to the right of the pole is

$$\boxed{r = \frac{ep}{1 + e\cos\theta}}$$

where e is the eccentricity.

POLAR EQUATIONS OF CONICS (FOCUS AT THE POLE, ECCENTRICITY e)

Equation	Description
(a) $r = \dfrac{ep}{1 - e\cos\theta}$	Directrix is perpendicular to the polar axis at a distance p units to the left of the pole.
(b) $r = \dfrac{ep}{1 + e\cos\theta}$	Directrix is perpendicular to the polar axis at a distance p units to the right of the pole.
(c) $r = \dfrac{ep}{1 + e\sin\theta}$	Directrix is parallel to the polar axis at a distance p units above the pole.
(d) $r = \dfrac{ep}{1 - e\sin\theta}$	Directrix is parallel to the polar axis at a distance p units below the pole.

Eccentricity

If $e = 1$, the conic is a parabola; the axis of symmetry is perpendicular to the directrix.

If $e < 1$, the conic is an ellipse; the major axis is perpendicular to the directrix.

If $e > 1$, the conic is a hyperbola; the transverse axis is perpendicular to the directrix.

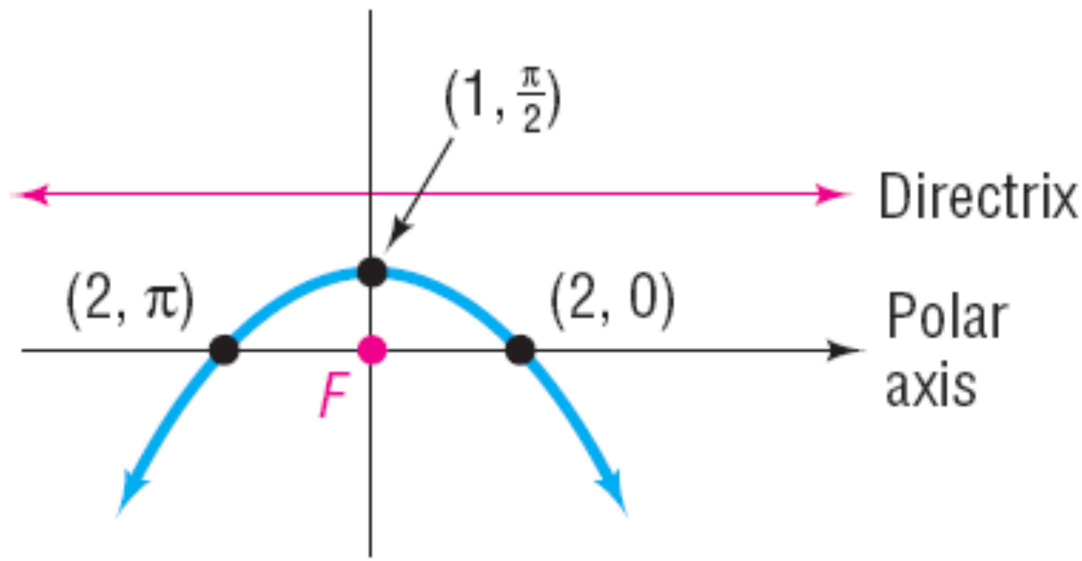
EXAMPLE

Analyzing and Graphing the Polar Equation of a Conic

Analyze and graph the equation: $r = \dfrac{6}{3 + 3\sin\theta} = \dfrac{2}{1 + \sin\theta}$

$$e = 1 \quad \text{and} \quad ep = 2 \qquad p = 2$$

The conic is a parabola with focus at the pole. The directrix is parallel to the polar axis at a distance 2 units above the pole; the axis of symmetry is perpendicular to the polar axis. The vertex of the parabola is at $\left(1, \dfrac{\pi}{2}\right)$. (Do you see why?) See Figure 53 for the graph. Notice that we plotted two additional points, $(2, 0)$ and $(2, \pi)$, to assist in graphing.



$$\boxed{r = \frac{ep}{1 + e\sin\theta}}$$

EXAMPLE Analyzing and Graphing the Polar Equation of a Conic

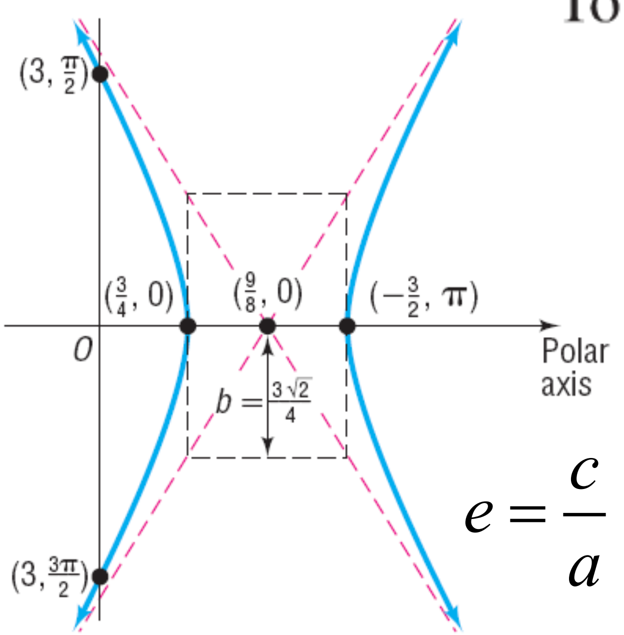
Analyze and graph the equation: $r = \dfrac{3}{1 + 3\cos\theta}$

$$b^2 = c^2 - a^2 = \frac{81}{64} - \frac{9}{64} = \frac{72}{64} = \frac{9}{8} \qquad b = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

This is the equation of a hyperbola with a focus at the pole.

The directrix is perpendicular to the polar axis on unit to the right of the pole and the transverse axis is along the polar axis.

To find the vertices, we let $\theta = 0$ and $\theta = \pi$.



Vertices: $\left(\frac{3}{4}, 0\right), \left(-\frac{3}{2}, \pi\right)$

Center: $\left(\frac{9}{8}, 0\right)$

$e = \dfrac{c}{a}$ so $a = \dfrac{c}{e} = \left(\dfrac{9}{8}\right)\left(\dfrac{1}{3}\right) = \dfrac{3}{8}$

$$\boxed{r = \frac{ep}{1 + e\cos\theta}}$$

2 Convert the Polar Equation of a Conic to a Rectangular Equation

EXAMPLE **Converting a Polar Equation to a Rectangular Equation**

Convert the polar equation to a rectangular equation.

$$r = \frac{8}{4 + 3\cos\theta}$$

$$r(4 + 3\cos\theta) = 8$$

$$4r + 3r\cos\theta = 8$$

$$4r = 8 - 3r\cos\theta$$

$$16r^2 = (8 - 3r\cos\theta)^2$$

$$16(x^2 + y^2) = (8 - 3x)^2$$

$$16x^2 + 16y^2 = 64 - 48x + 9x^2$$

$$7x^2 + 16y^2 + 48x - 64 = 0$$

$x^2 + y^2 = r^2; x = r\cos\theta$