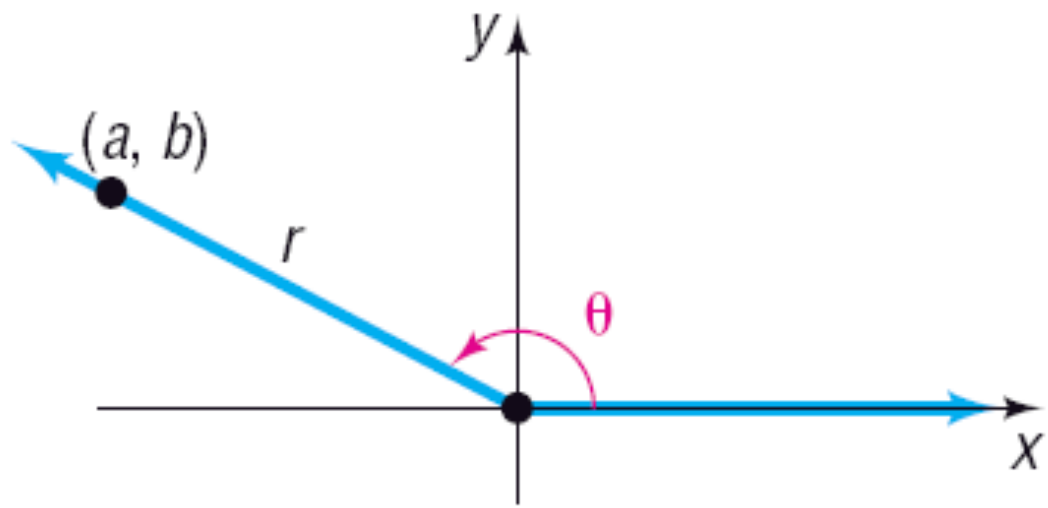
Section 7.4

Trigonometric Functions of General Angles

1 Find the Exact Values of the Trigonometric Functions for Any Angle

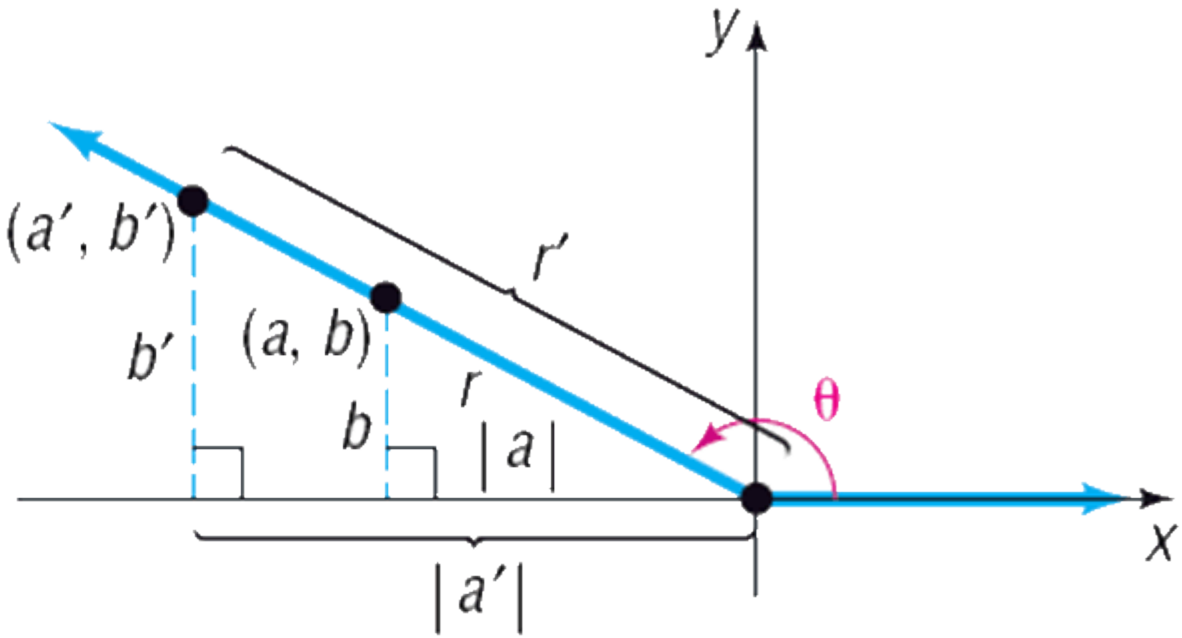


Let θ be any angle in standard position, and let (a, b) denote the coordinates of any point, except the origin $(0, 0)$, on the terminal side of θ. If $r = \sqrt{a^2 + b^2}$ denotes the distance from $(0, 0)$ to (a, b), then the **six trigonometric functions of $\boldsymbol{\theta}$** are defined as the ratios

$$\sin\theta = \frac{b}{r} \qquad \cos\theta = \frac{a}{r} \qquad \tan\theta = \frac{b}{a}$$

$$\csc\theta = \frac{r}{b} \qquad \sec\theta = \frac{r}{a} \qquad \cot\theta = \frac{a}{b}$$

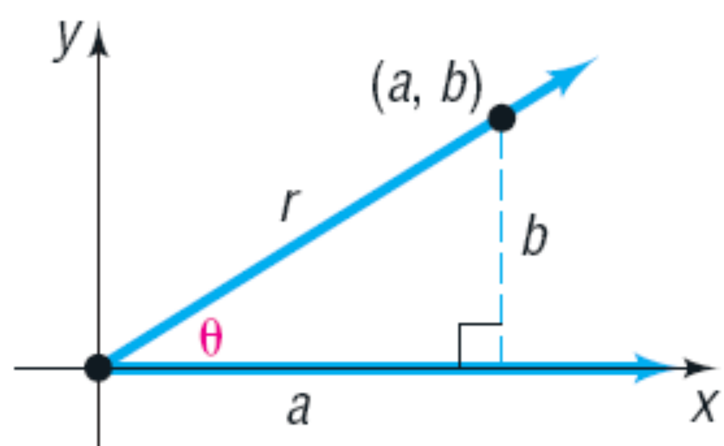
provided no denominator equals 0. If a denominator equals 0, that trigonometric function of the angle θ is not defined.



$$\sin\theta = \frac{b}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{a}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

and so on.



EXAMPLE

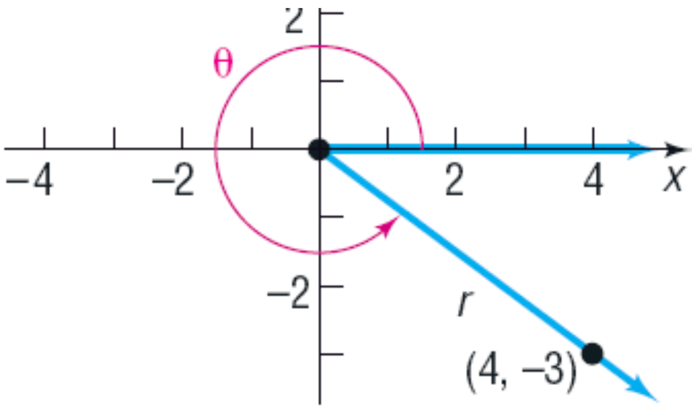
Finding the Exact Values of the Six Trigonometric Functions of θ, Given a Point on the Terminal Side

Find the exact value of each of the six trigonometric functions of a positive angle θ if $(4, -3)$ is a point on its terminal side.

$$a = 4 \text{ and } b = -3 \text{ so } r = \sqrt{a^2 + b^2} = \sqrt{16+9} = 5$$

$$\sin\theta = \frac{b}{r} = -\frac{3}{5} \qquad \cos\theta = \frac{a}{r} = \frac{4}{5} \qquad \tan\theta = \frac{b}{a} = -\frac{3}{4}$$

$$\csc\theta = \frac{r}{b} = -\frac{5}{3} \qquad \sec\theta = \frac{r}{a} = \frac{5}{4} \qquad \cot\theta = \frac{a}{b} = -\frac{4}{3}$$



EXAMPLE **Finding the Exact Values of the Six Trigonometric Functions of Quadrantal Angles**

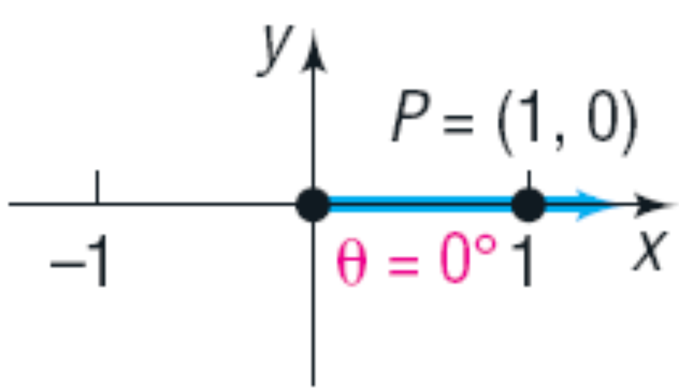
Find the exact values of each of the six trigonometric functions of

(a) $\theta = 0 = 0°$

$$\sin 0 = \sin 0° = \frac{b}{r} = \frac{0}{1} = 0 \qquad \cos 0 = \cos 0° = \frac{a}{r} = \frac{1}{1} = 1$$

$$\tan 0 = \tan 0° = \frac{b}{a} = \frac{0}{1} = 0 \qquad \sec 0 = \sec 0° = \frac{r}{a} = \frac{1}{1} = 1$$

Since the y-coordinate of P is 0, csc 0 and cot 0 are not defined.



EXAMPLE

Finding the Exact Values of the Six Trigonometric Functions of Quadrantal Angles

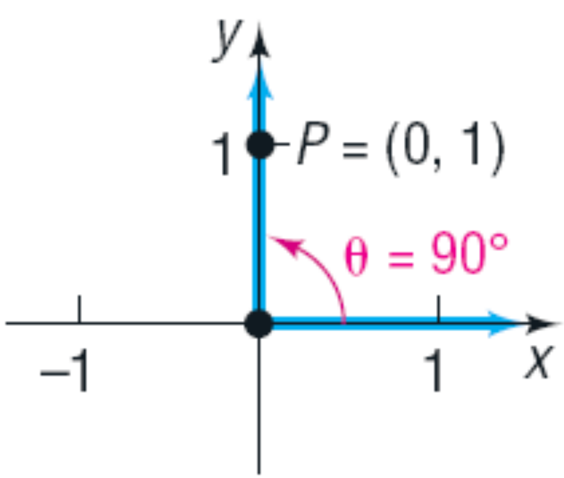
Find the exact values of each of the six trigonometric functions of

(b) $\theta = \dfrac{\pi}{2} = 90^\circ$

$$\sin\frac{\pi}{2} = \sin 90^\circ = \frac{b}{r} = \frac{1}{1} = 1 \qquad \cos\frac{\pi}{2} = \cos 90^\circ = \frac{a}{r} = \frac{0}{1} = 0$$

$$\csc\frac{\pi}{2} = \csc 90^\circ = \frac{r}{b} = \frac{1}{1} = 1 \qquad \cot\frac{\pi}{2} = \cot 90^\circ = \frac{a}{b} = \frac{0}{1} = 0$$

Since the x-coordinate of P is 0, $\tan\dfrac{\pi}{2}$ and $\sec\dfrac{\pi}{2}$ are not defined.



EXAMPLE Finding the Exact Values of the Six Trigonometric Functions of Quadrantal Angles

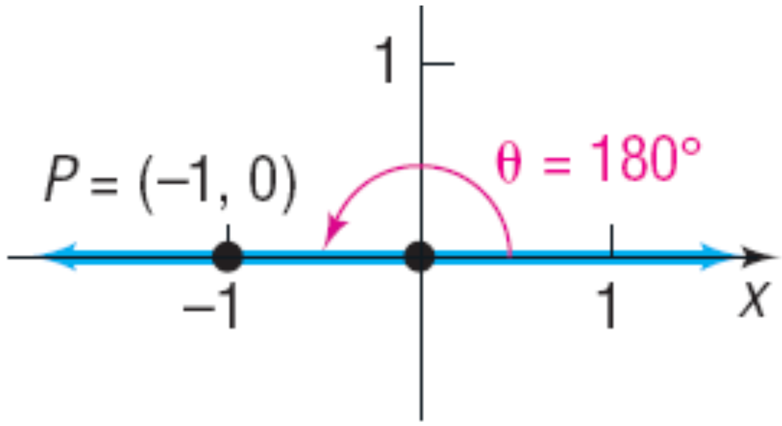
Find the exact values of each of the six trigonometric functions of

(c) $\theta = \pi = 180°$

$$\sin \pi = \sin 180° = \frac{0}{1} = 0 \qquad \cos \pi = \cos 180° = \frac{-1}{1} = -1$$

$$\tan \pi = \tan 180° = \frac{0}{-1} = 0 \qquad \sec \pi = \sec 180° = \frac{1}{-1} = -1$$

Since the y-coordinate of P is 0, $\csc \pi$ and $\cot \pi$ are not defined.



EXAMPLE **Finding the Exact Values of the Six Trigonometric Functions of Quadrantal Angles**

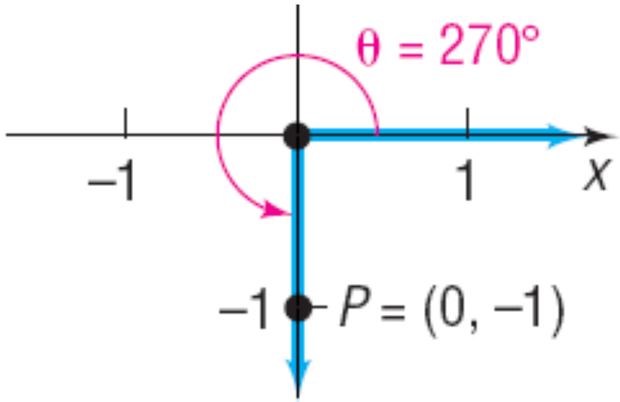
Find the exact values of each of the six trigonometric functions of

$$\text{(d)}\ \theta = \frac{3\pi}{2} = 270^\circ$$

$$\sin\frac{3\pi}{2} = \sin 270^\circ = \frac{-1}{1} = -1 \qquad \cos\frac{3\pi}{2} = \cos 270^\circ = \frac{0}{1} = 0$$

$$\csc\frac{3\pi}{2} = \csc 270^\circ = \frac{1}{-1} = -1 \qquad \cot\frac{3\pi}{2} = \cot 270^\circ = \frac{0}{-1} = 0$$

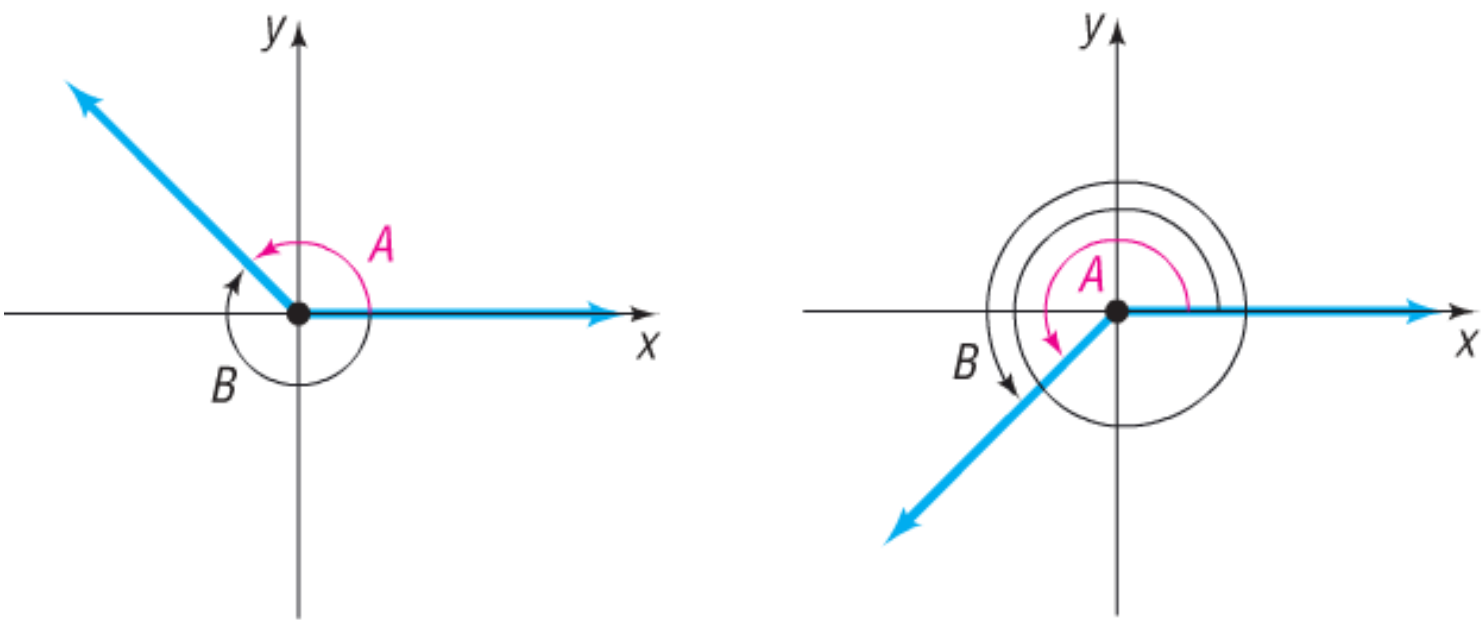
Since the x-coordinate of P is 0, $\tan\frac{3\pi}{2}$ and $\sec\frac{3\pi}{2}$ are not defined.



θ (Radians)	θ (Degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	$0°$	0	1	0	Not defined	1	Not defined
$\frac{\pi}{2}$	$90°$	1	0	Not defined	1	Not defined	0
π	$180°$	0	-1	0	Not defined	-1	Not defined
$\frac{3\pi}{2}$	$270°$	-1	0	Not defined	-1	Not defined	0

2 Use Coterminal Angles to Find the Exact Value of a Trigonometric Function

Two angles in standard position are said to be **coterminal** if they have the same terminal side.



(a) *A* and *B* are coterminal

(b) *A* and *B* are coterminal

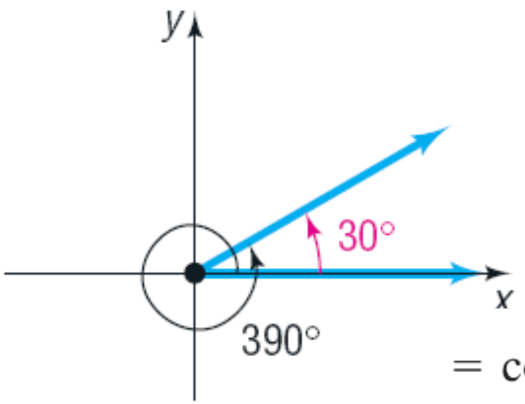
EXAMPLE

Using a Coterminal Angle to Find the Exact Value of a Trigonometric Function

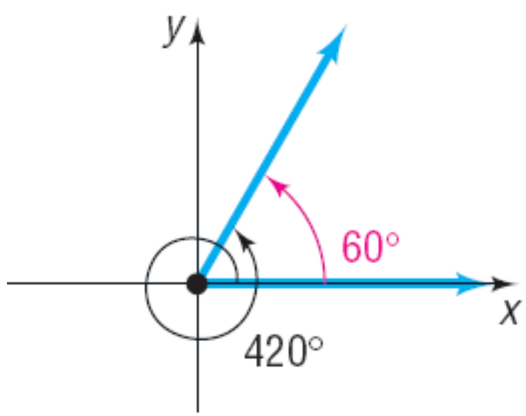
Find the exact value of each of the following:

(a) $\sin 390°$ (b) $\cos 420°$ (c) $\tan \dfrac{9\pi}{4}$ (d) $\sec\left(-\dfrac{7\pi}{4}\right)$ (e) $\csc(-270°)$

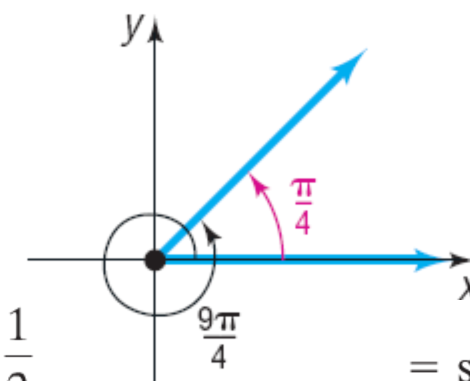
(a) $= \sin 30° = \dfrac{1}{2}$



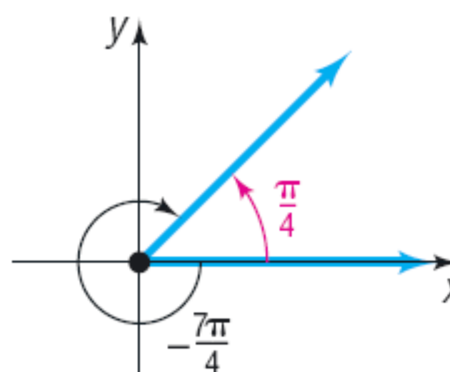
(b) $= \cos 60° = \dfrac{1}{2}$



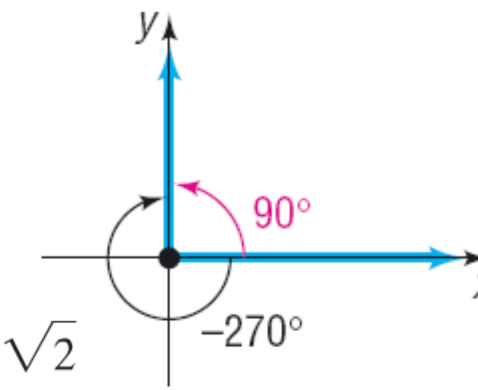
(c) $= \tan \dfrac{\pi}{4} = 1$



(d) $= \sec \dfrac{\pi}{4} = \sqrt{2}$



(e) $= \csc 90° = 1$

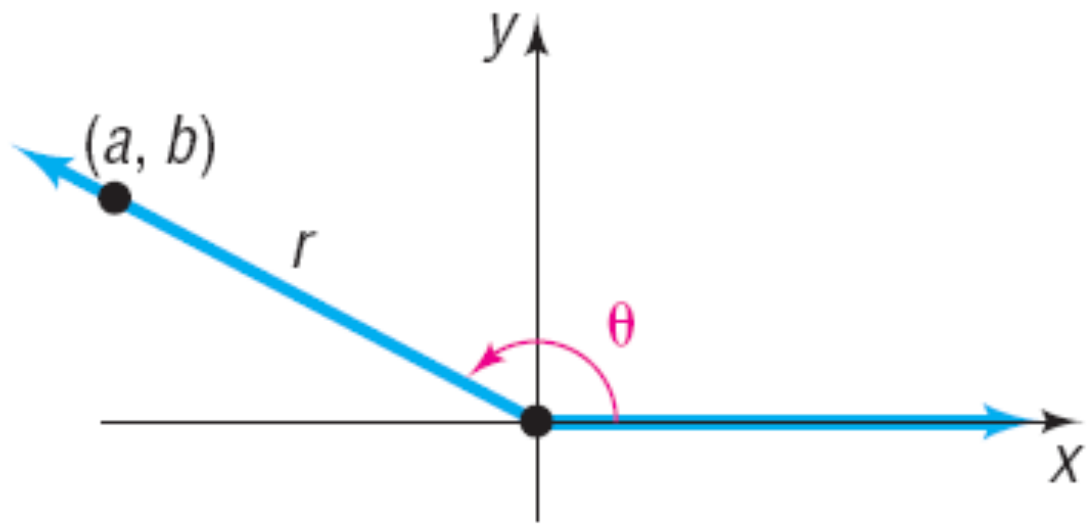


$\boldsymbol{\theta}$ **degrees**	$\boldsymbol{\theta}$ **radians**
$\sin(\theta + 360°k) = \sin\theta$	$\sin(\theta + 2\pi k) = \sin\theta$
$\cos(\theta + 360°k) = \cos\theta$	$\cos(\theta + 2\pi k) = \cos\theta$
$\tan(\theta + 360°k) = \tan\theta$	$\tan(\theta + 2\pi k) = \tan\theta$
$\csc(\theta + 360°k) = \csc\theta$	$\csc(\theta + 2\pi k) = \csc\theta$
$\sec(\theta + 360°k) = \sec\theta$	$\sec(\theta + 2\pi k) = \sec\theta$
$\cot(\theta + 360°k) = \cot\theta$	$\cot(\theta + 2\pi k) = \cot\theta$

where k is any integer.

3 Determine the Signs of the Trigonometric Functions of an Angle in a Given Quadrant

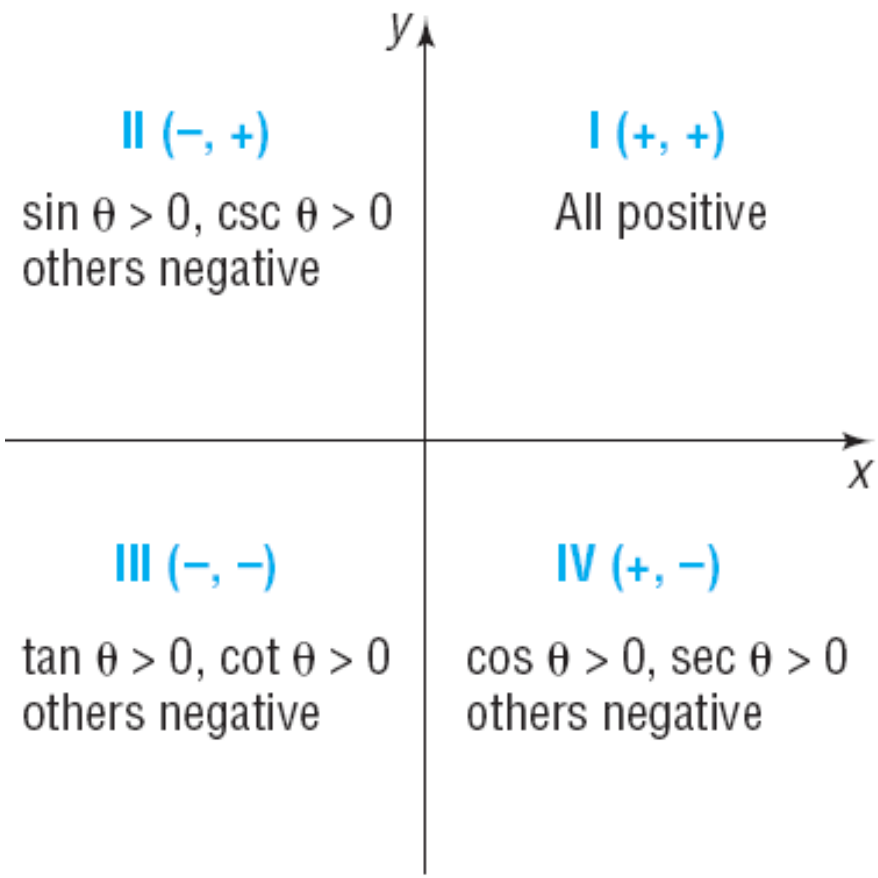
θ in quadrant II, $a < 0$, $b > 0$, $r > 0$

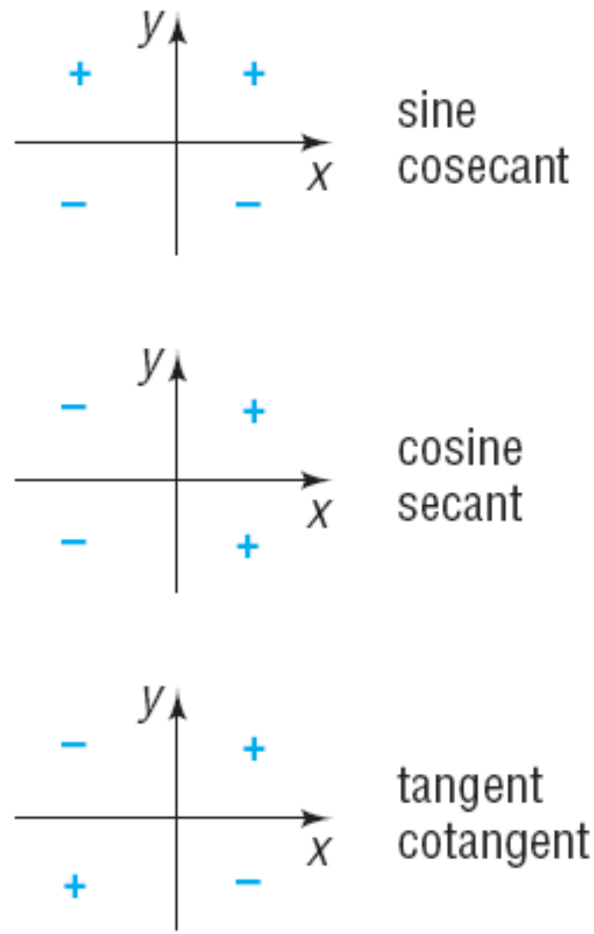


$$\sin\theta = \frac{b}{r} > 0 \qquad \cos\theta = \frac{a}{r} < 0 \qquad \tan\theta = \frac{b}{a} < 0$$

$$\csc\theta = \frac{r}{b} > 0 \qquad \sec\theta = \frac{r}{a} < 0 \qquad \cot\theta = \frac{a}{b} < 0$$

Quadrant of θ	$\sin\theta$, $\csc\theta$	$\cos\theta$, $\sec\theta$	$\tan\theta$, $\cot\theta$
I	Positive	Positive	Positive
II	Positive	Negative	Negative
III	Negative	Negative	Positive
IV	Negative	Positive	Negative





EXAMPLE

Finding the Quadrant in Which an Angle Lies

If $\sin q > 0$ and $\cos q < 0$, name the quadrant in which the angle q lies.

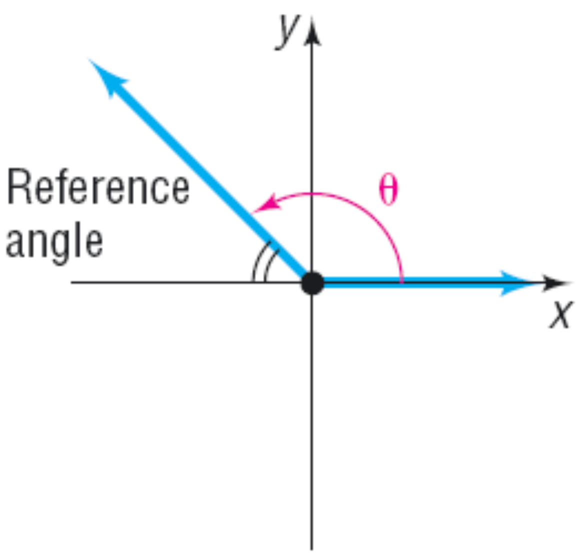
For $\sin q > 0$ an angle must be in quadrant I or II.

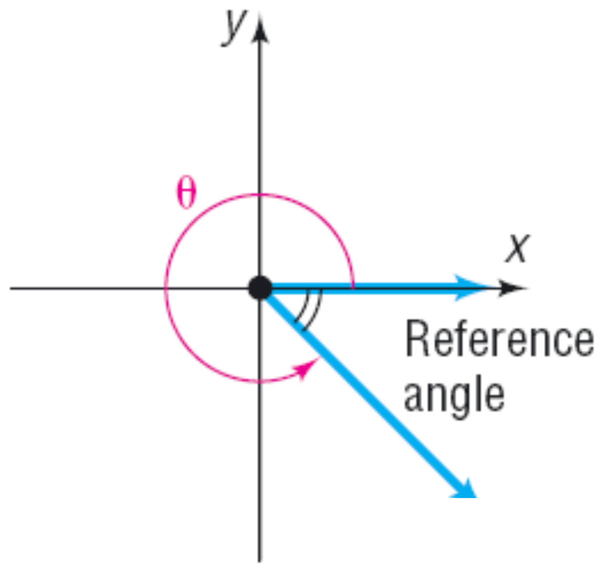
For $\cos q < 0$ an angle must be in quadrant II or III.

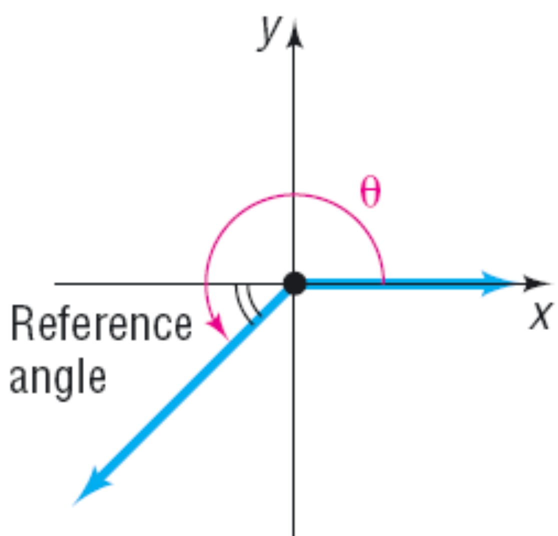
Therefore, this angle must lie in quadrant II.

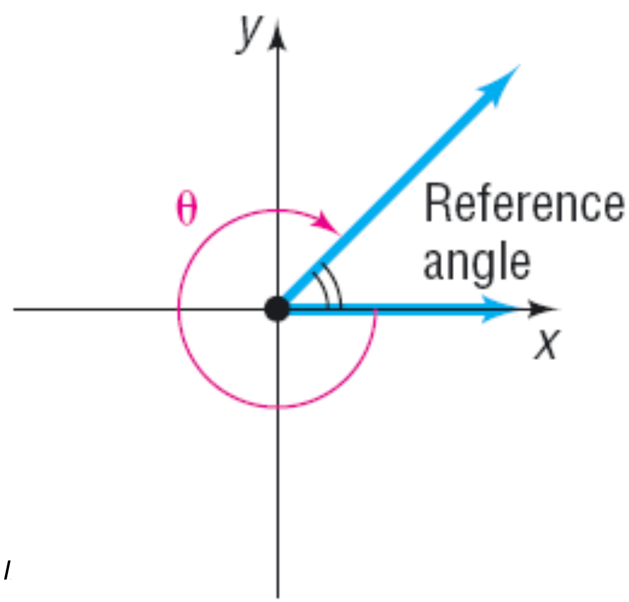
4 Find the Reference Angle of an Angle

Let θ denote a nonacute angle that lies in a quadrant. The acute angle formed by the terminal side of θ and either the positive x-axis or the negative x-axis is called the **reference angle** for θ.





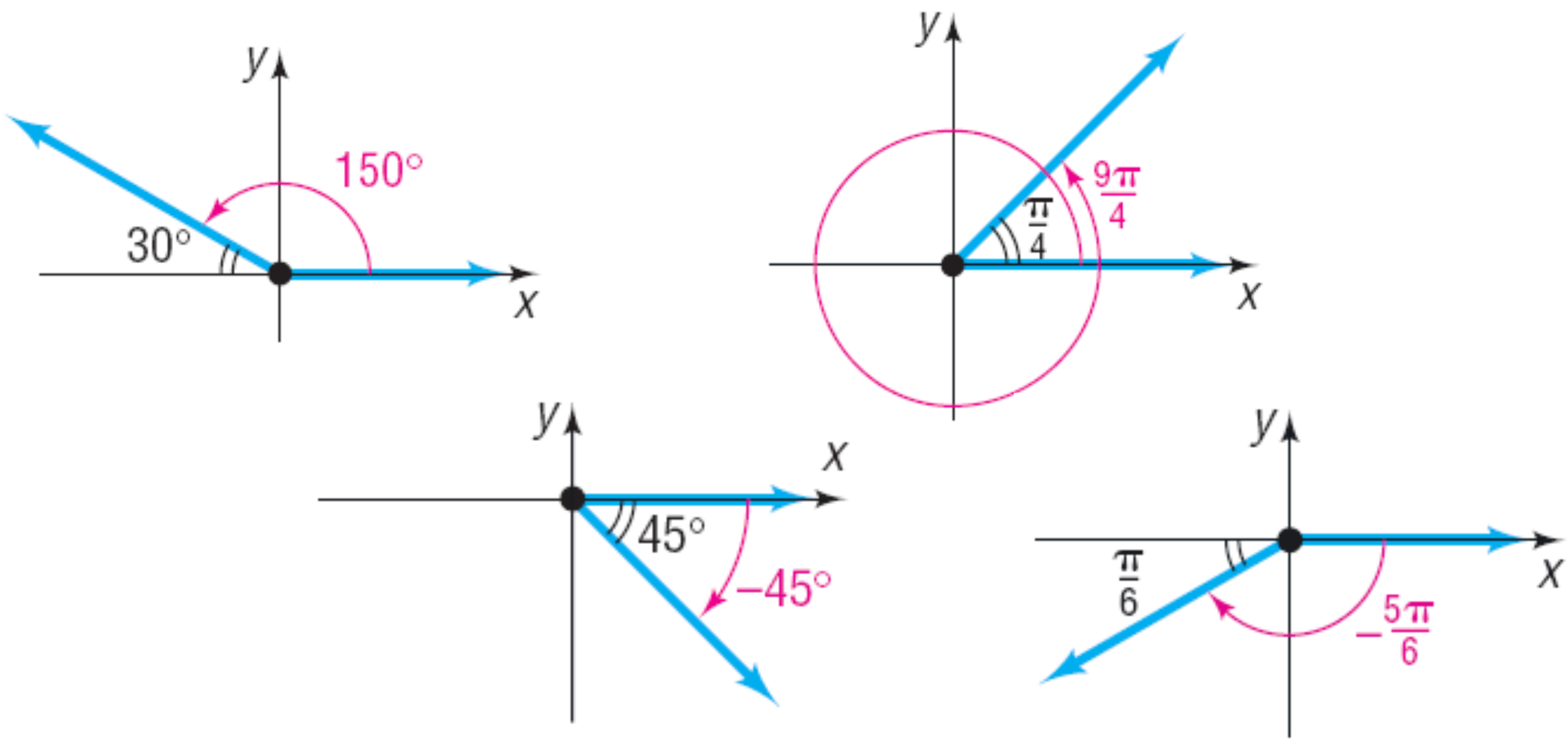




EXAMPLE Finding Reference Angles

Find the reference angle for each of the following angles:

(a) $150°$ (b) $-45°$ (c) $\dfrac{9\pi}{4}$ (d) $-\dfrac{5\pi}{6}$



5 Use a Reference Angle to Find the Exact Value of a Trigonometric Function

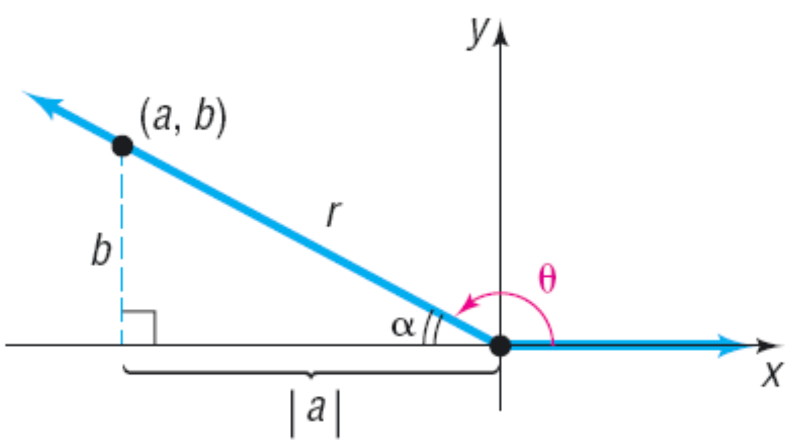
THEOREM

Reference Angles

If θ is an angle that lies in a quadrant and if α is its reference angle, then

$$\boxed{\begin{array}{lll} \sin\theta = \pm\sin\alpha & \cos\theta = \pm\cos\alpha & \tan\theta = \pm\tan\alpha \\ \csc\theta = \pm\csc\alpha & \sec\theta = \pm\sec\alpha & \cot\theta = \pm\cot\alpha \end{array}}$$

where the + or − sign depends on the quadrant in which θ lies.



EXAMPLE

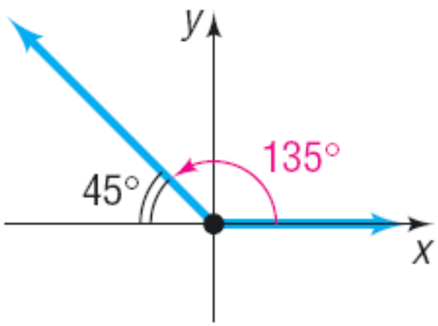
Using the Reference Angle to Find the Exact Value of a Trigonometric Function

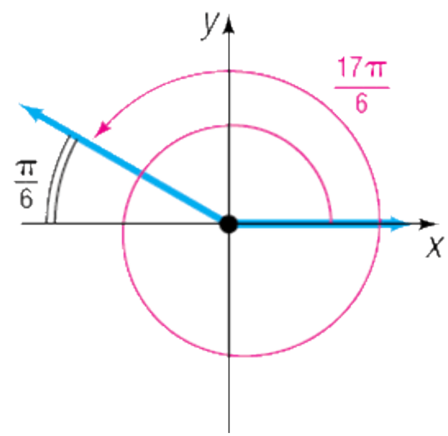
Find the exact value of each of the following trigonometric functions using reference angles.

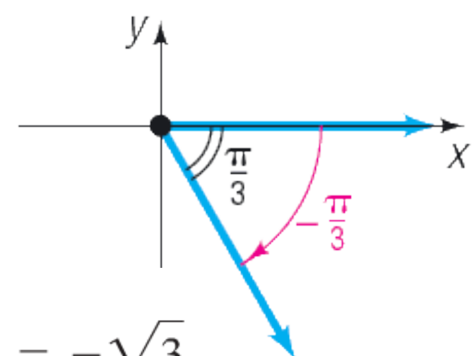
(a) $\sin 135^\circ$ (b) $\cos 600^\circ$ (c) $\cos \frac{17\pi}{6}$ (d) $\tan\left(-\frac{\pi}{3}\right)$

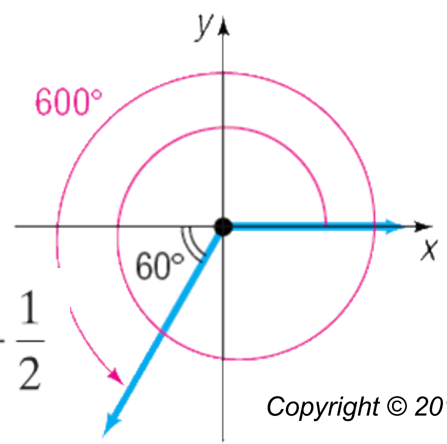
$$\sin 135^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}$$

$$\cos \frac{17\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$









$$\cos 600^\circ = -\cos 60^\circ = -\frac{1}{2}$$

$$\tan\left(-\frac{\pi}{3}\right) = -\tan \frac{\pi}{3} = -\sqrt{3}$$

Finding the Values of the Trigonometric Functions of Any Angle

- If the angle θ is a quadrantal angle, draw the angle, pick a point on its terminal side, and apply the definition of the trigonometric functions.
- If the angle θ lies in a quadrant:

  1. Find the reference angle α of θ.

  2. Find the value of the trigonometric function at α.

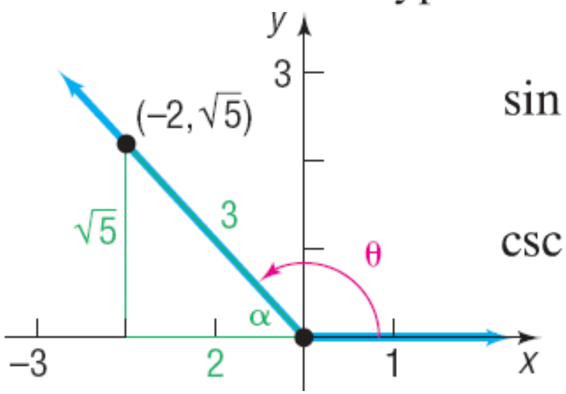
  3. Adjust the sign ($+$ or $-$) of the value of the trigonometric function based on the quadrant in which θ lies.

6 Find the Exact Values of Trigonometric Functions of an Angle, Given Information about the Functions

EXAMPLE **Finding the Exact Values of Trigonometric Functions**

Given that $\cos\theta = -\frac{2}{3}, \frac{\pi}{2} < \theta < \pi$, find the exact value of each of the remaining trigonometric functions.

The angle θ lies in quadrant II, so we know that $\sin\theta$ and $\csc\theta$ are positive and the other four trigonometric functions are negative. If α is the reference angle for θ, then $\cos\alpha = \frac{2}{3} = \frac{\text{adjacent}}{\text{hypotenuse}}$.



$$\sin\alpha = \frac{\sqrt{5}}{3} \qquad \cos\alpha = \frac{2}{3} \qquad \tan\alpha = \frac{\sqrt{5}}{2}$$

$$\csc\alpha = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} \qquad \sec\alpha = \frac{3}{2} \qquad \cot\alpha = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Now assign the appropriate signs to each of these values to find the values of the trigonometric functions of θ.

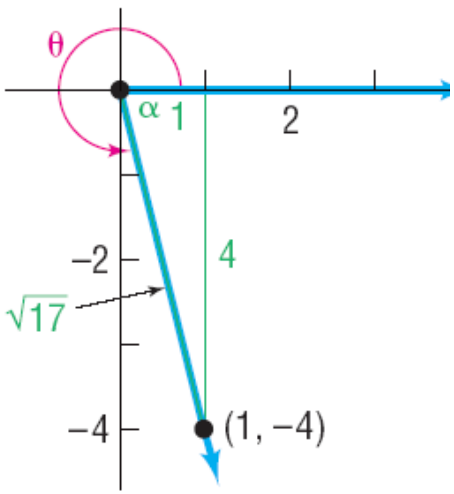
$$\sin\theta = \frac{\sqrt{5}}{3} \qquad \cos\theta = -\frac{2}{3} \qquad \tan\theta = -\frac{\sqrt{5}}{2}$$

$$\csc\theta = \frac{3\sqrt{5}}{5} \qquad \sec\theta = -\frac{3}{2} \qquad \cot\theta = -\frac{2\sqrt{5}}{5}$$

EXAMPLE Finding the Exact Values of Trigonometric Functions

If $\tan\theta = -4$ and $\sin\theta < 0$, find the exact value of each of the remaining trigonometric functions of θ.

Since $\tan\theta = -4 < 0$ and $\sin\theta < 0$, it follows that θ lies in quadrant IV.



$$\sin\alpha = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17} \qquad \cos\alpha = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17} \qquad \tan\alpha = \frac{4}{1} = 4$$

$$\csc\alpha = \frac{\sqrt{17}}{4} \qquad \sec\alpha = \frac{\sqrt{17}}{1} = \sqrt{17} \qquad \cot\alpha = \frac{1}{4}$$

Assign the appropriate sign to each of these to obtain the values of the trigonometric functions of θ.

$$\sin\theta = -\frac{4\sqrt{17}}{17} \qquad \cos\theta = \frac{\sqrt{17}}{17} \qquad \tan\theta = -4$$

$$\csc\theta = -\frac{\sqrt{17}}{4} \qquad \sec\theta = \sqrt{17} \qquad \cot\theta = -\frac{1}{4}$$