Section 5.3

The Graph of a Rational Function

1 Analyze the Graph of a Rational Function

EXAMPLE

How to Analyze the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x-1}{x^2-4}$

Step 1: Factor the numerator and denominator of R. Find the domain of the rational function.

$$R(x) = \frac{x-1}{(x+2)(x-2)}$$

The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

Step 2: Write R in lowest terms.

Because there are no common factors between the numerator and denominator, R is in lowest terms.

Step 3: Locate the intercepts of the graph. Determine the behavior of the graph of R near each x-intercept using the same procedure as for polynomial functions. Plot each x-intercept and indicate the behavior of the graph near it.

Since 0 is in the domain of R, the y-intercept is $R(0) = \frac{1}{4}$.'

The x-intercepts are the real zeros of the numerator that are in the domain of R. $x - 1 = 0$ so $x = 1$ is the only x-intercept.

Near 1: $$R(x) = \frac{x-1}{(x+2)(x-2)} \approx \frac{x-1}{(1+2)(1-2)} = -\frac{1}{3}(x-1)$$

EXAMPLE

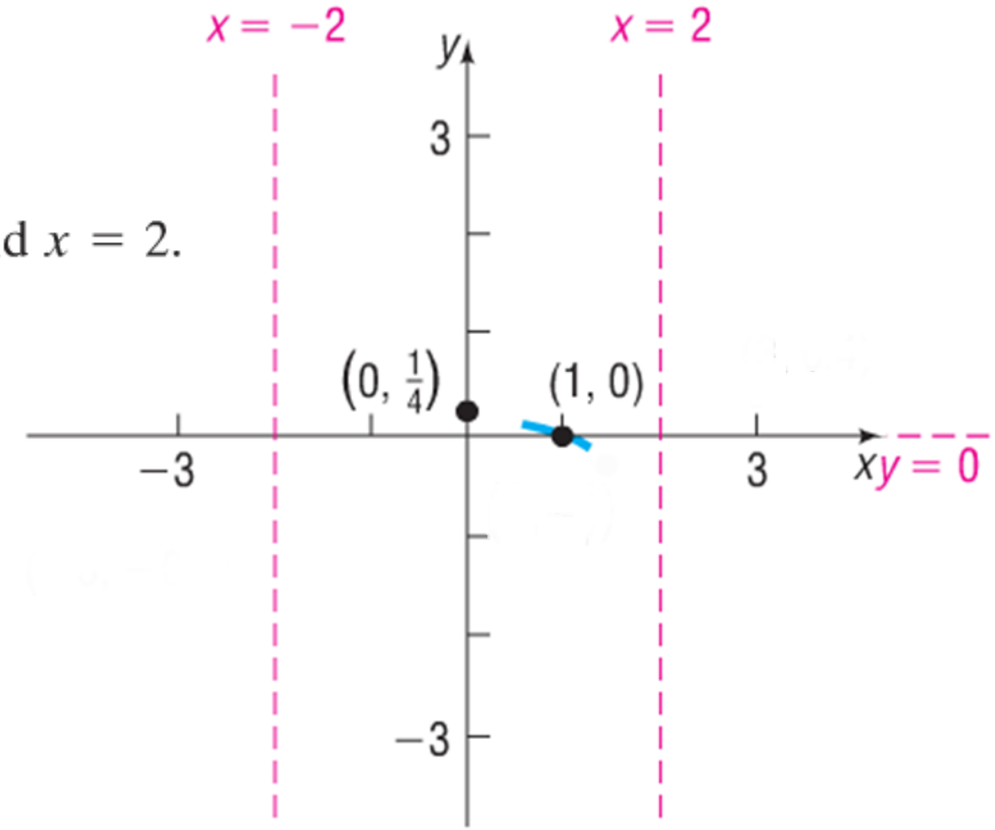
How to Analyze the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x-1}{x^2-4}$

Step 4: Locate the vertical asymptotes. Graph each vertical asymptote using a dashed line.

vertical asymptotes: the lines $x = -2$ and $x = 2$.

Step 5: Locate the horizontal or oblique asymptote, if one exists. Determine points, if any, at which the graph of R intersects this asymptote. Graph the asymptotes using a dashed line. Plot any points at which the graph of R intersects the asymptote.



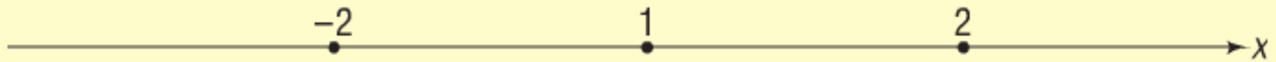
Because the degree of the numerator is less than the degree of the denominator, R is proper and the line $y = 0$ (the x-axis) is a horizontal asymptote of the graph. To determine if the graph of R intersects the horizontal asymptote, solve the equation $R(x) = 0$:

EXAMPLE

How to Analyze the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x - 1}{x^2 - 4}$

Step 6: Use the zeros of the numerator and denominator of R to divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluating R there. Plot the points found.

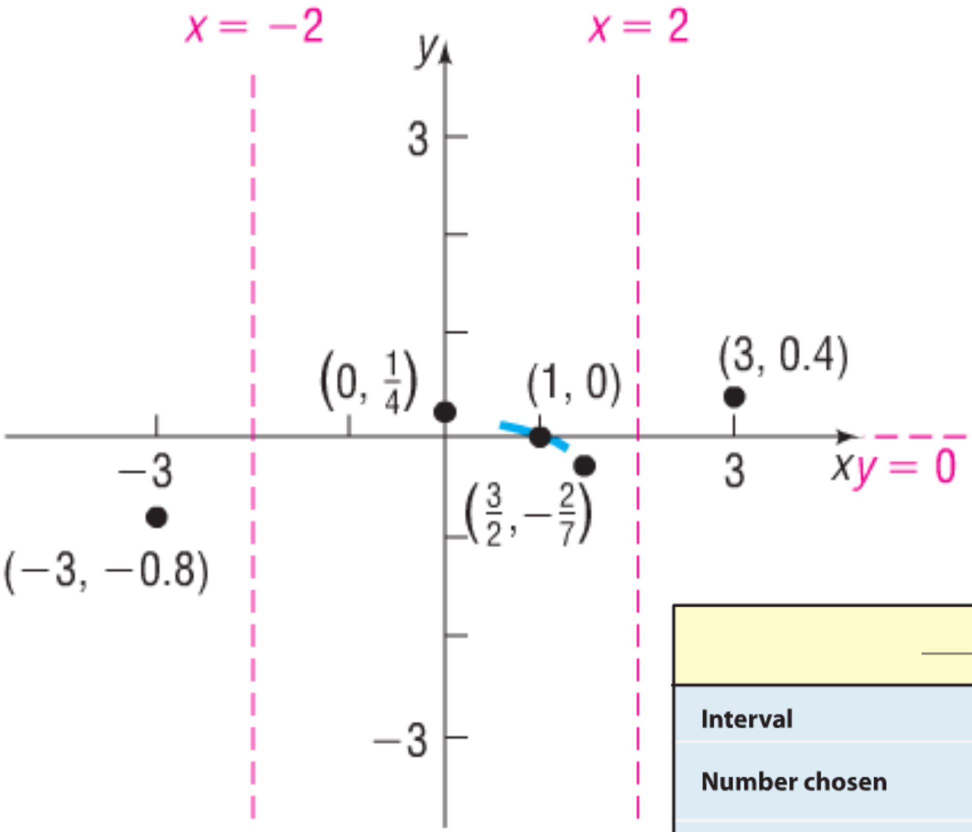


Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 2)$	$(2, \infty)$
Number chosen	-3	0	$\frac{3}{2}$	3
Value of *R*	$R(-3) = -0.8$	$R(0) = \frac{1}{4}$	$R\left(\frac{3}{2}\right) = -\frac{2}{7}$	$R(3) = 0.4$
Location of graph	Below *x*-axis	Above *x*-axis	Below *x*-axis	Above *x*-axis
Point on graph	$(-3, -0.8)$	$\left(0, \frac{1}{4}\right)$	$\left(\frac{3}{2}, -\frac{2}{7}\right)$	$(3, 0.4)$

EXAMPLE

How to Analyze the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x-1}{x^2-4}$



	-2	1	2	x
Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 2)$	$(2, \infty)$
Number chosen	-3	0	$\frac{3}{2}$	3
Value of *R*	$R(-3) = -0.8$	$R(0) = \frac{1}{4}$	$R\left(\frac{3}{2}\right) = -\frac{2}{7}$	$R(3) = 0.4$
Location of graph	Below *x*-axis	Above *x*-axis	Below *x*-axis	Above *x*-axis
Point on graph	$(-3, -0.8)$	$\left(0, \frac{1}{4}\right)$	$\left(\frac{3}{2}, -\frac{2}{7}\right)$	$(3, 0.4)$

Analyze the graph of the rational function: $R(x) = \dfrac{x-1}{x^2-4}$

Step 7: Analyze the behavior of the graph of R near each asymptote and indicate this behavior on the graph.

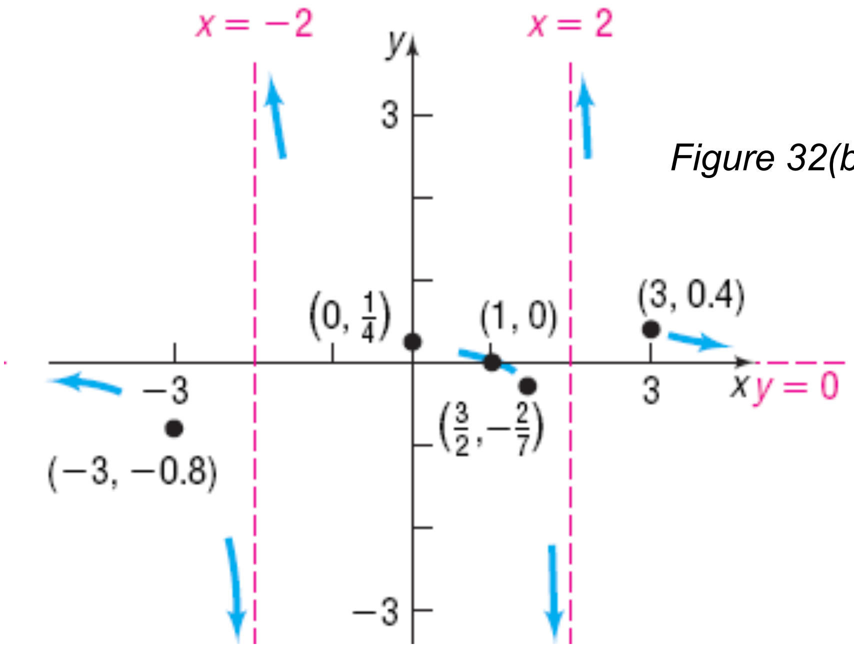


Figure 32(b)

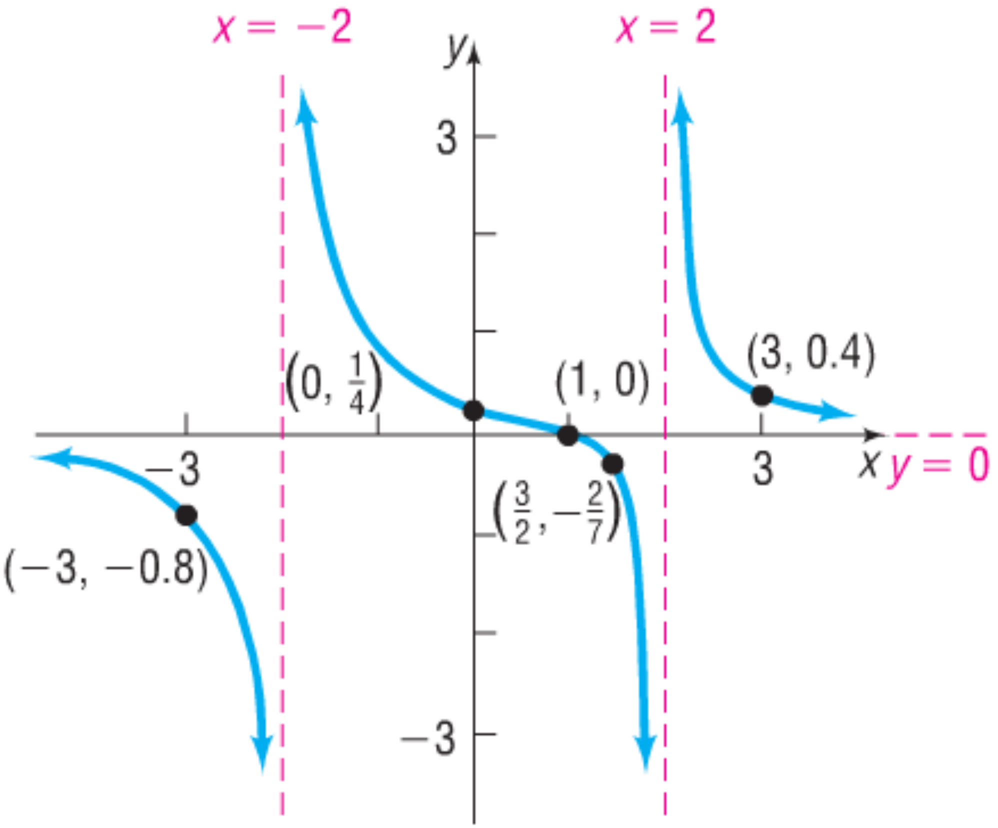
- Since $y = 0$ (the x-axis) is a horizontal asymptote and the graph lies below the x-axis for $x < -2$, we can sketch a portion of the graph by placing a small arrow to the far left and under the x-axis.
- Since the line $x = -2$ is a vertical asymptote and the graph lies below the x-axis for $x < -2$, we place an arrow well below the x-axis and approaching the line $x = -2$ from the left $\left(\lim_{x \to -2^-} R(x) = -\infty\right)$.
- Since the graph is above the x-axis for $-2 < x < 1$ and $x = -2$ is a vertical asymptote, the graph will continue on the right of $x = -2$ at the top $\left(\lim_{x \to -2^+} R(x) = +\infty\right)$. Similar explanations account for the other arrows shown in Figure 32(b).

EXAMPLE

How to Analyze the Graph of a Rational Function

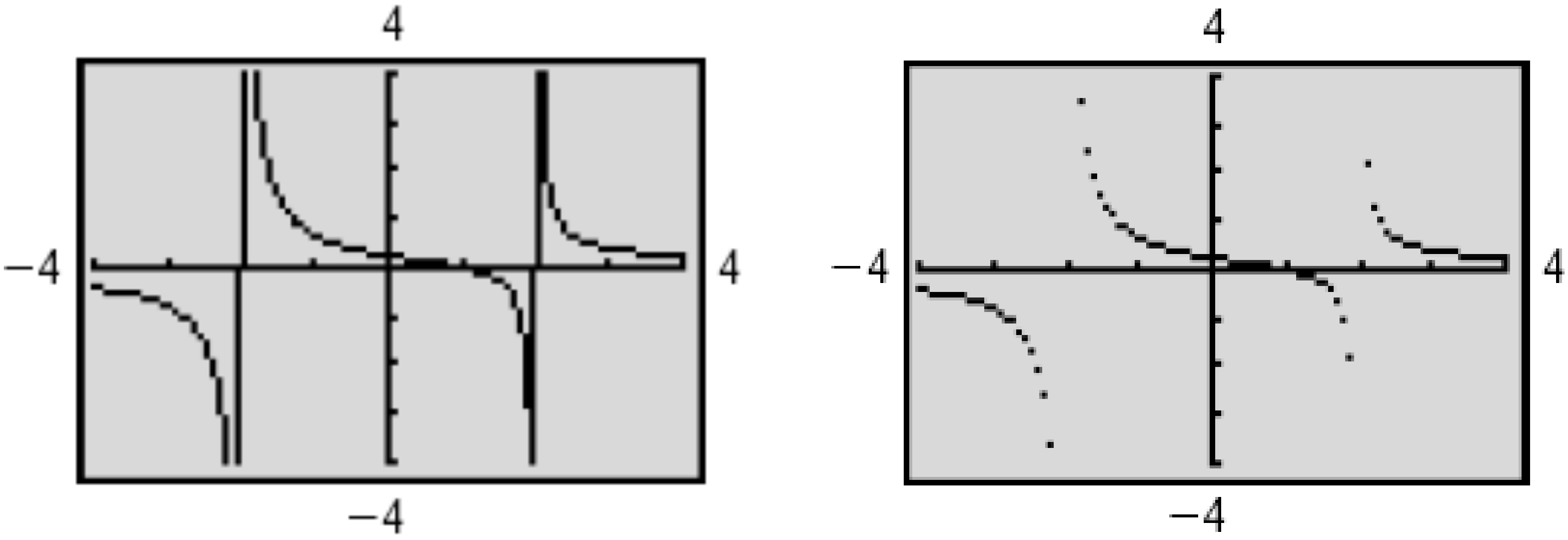
Analyze the graph of the rational function: $R(x) = \dfrac{x-1}{x^2-4}$

Step 8: Use the results obtained in Steps 1 through 7 to graph R.



Exploration

$$\text{Graph } R(x) = \frac{x-1}{x^2-4}$$



Connected mode

Dot mode

SUMMARY Analyzing the Graph of a Rational Function R

STEP 1: Factor the numerator and denominator of R. Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph. The x-intercepts are the zeros of the numerator of R that are in the domain of R. Determine the behavior of the graph of R near each x-intercept.

STEP 4: Determine the vertical asymptotes. Graph each vertical asymptote using a dashed line.

STEP 5: Determine the horizontal or oblique asymptote, if one exists. Determine points, if any, at which the graph of R intersects this asymptote. Graph the asymptote using a dashed line. Plot any points at which the graph of R intersects the asymptote.

STEP 6: Use the zeros of the numerator and denominator of R to divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluating R there. Plot the points found.

STEP 7: Analyze the behavior of the graph of R near each asymptote and indicate this behavior on the graph.

STEP 8: Use the results obtained in Steps 1 through 7 to graph R.

EXAMPLE Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x^2 - 1}{x}$

Step 1: $R(x) = \dfrac{(x+1)(x-1)}{x}$. The domain of R is $\{x \mid x \neq 0\}$.

Step 2: R is in lowest terms.

Step 3: Because x cannot equal 0, there is no y-intercept. The graph has two x-intercepts: -1 and 1.

Near -1: $R(x) = \dfrac{(x+1)(x-1)}{x} \approx \dfrac{(x+1)(-1-1)}{-1} = 2(x+1)$

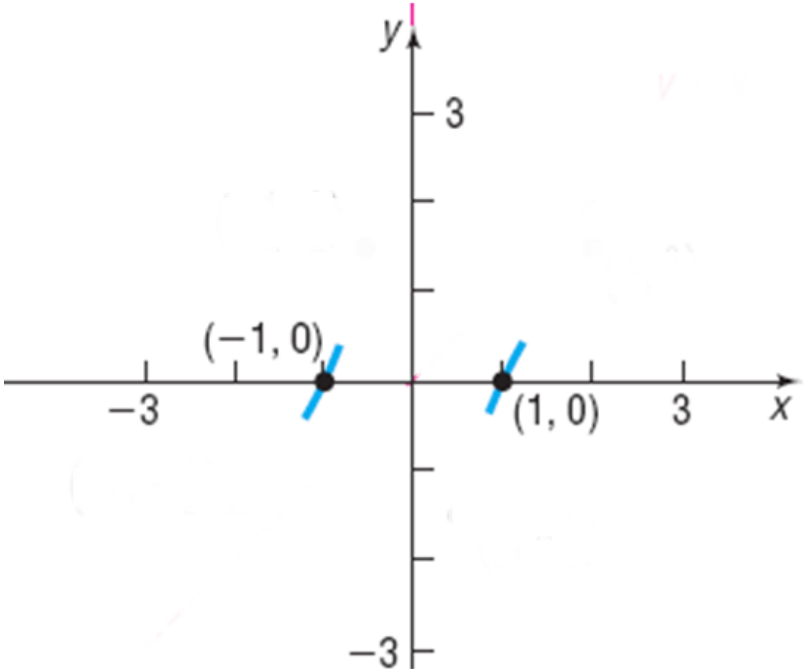
Near 1: $R(x) = \dfrac{(x+1)(x-1)}{x} \approx \dfrac{(1+1)(x-1)}{1} = 2(x-1)$

Plot the point $(-1, 0)$ and indicate a line with positive slope there. Plot the point $(1, 0)$ and indicate a line with positive slope there.

EXAMPLE Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \frac{x^2 - 1}{x}$

STEP 4: The real zero of the denominator with R in lowest terms is 0, so the graph of R has the line $x = 0$ (the y-axis) as a vertical asymptote. Graph $x = 0$ using a dashed line.

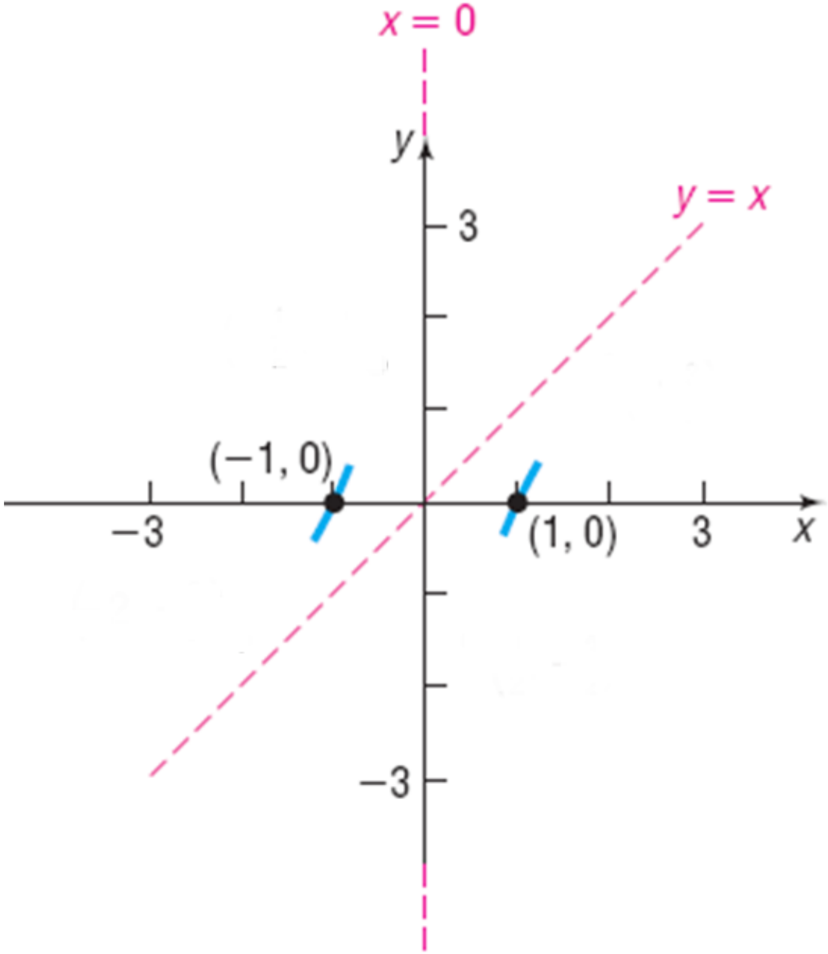


Analyze the graph of the rational function: $R(x) = \dfrac{x^2 - 1}{x}$

STEP 5: Since the degree of the numerator, 2, is one greater than the degree of the denominator, 1, the rational function will have an oblique asymptote. To find the oblique asymptote, we use long division.

$$\begin{array}{r} x \\ x\overline{)x^2 - 1} \\ \underline{x^2 } \\ -1 \end{array}$$

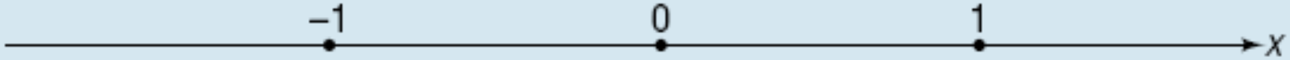
The quotient is x, so the line $y = x$ is an oblique asymptote of the graph. Graph $y = x$ using a dashed line.



Analyze the graph of the rational function: $R(x) = \frac{x^2 - 1}{x}$

Step 6: The zeros of the numerator are -1 and 1; the zero of the denominator is 0. Use these values to divide the x-axis into four intervals:

$$(-\infty, -1) \qquad (-1, 0) \qquad (0, 1) \qquad (1, \infty)$$

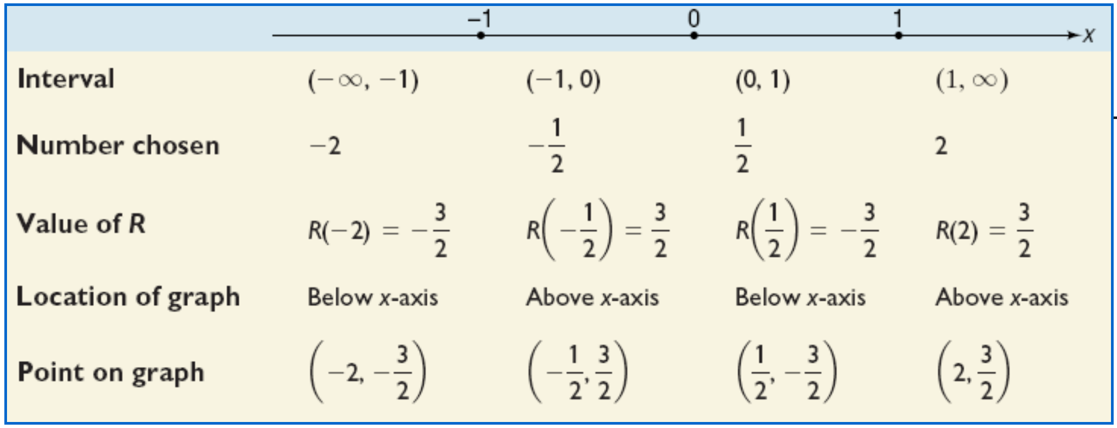


Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Number chosen	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
Value of *R*	$R(-2) = -\frac{3}{2}$	$R\left(-\frac{1}{2}\right) = \frac{3}{2}$	$R\left(\frac{1}{2}\right) = -\frac{3}{2}$	$R(2) = \frac{3}{2}$
Location of graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$\left(-2, -\frac{3}{2}\right)$	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$\left(\frac{1}{2}, -\frac{3}{2}\right)$	$\left(2, \frac{3}{2}\right)$

Analyze the graph of the rational function: $R(x) = \dfrac{x^2 - 1}{x}$

Step 7: Since the graph of R is below the x-axis for $x < -1$ and is above the x-axis for $x > 1$, and since the graph of R does not intersect the oblique asymptote $y = x$, the graph of R will approach the line $y = x$ as shown in Figure 34(b).

Since the graph of R is above the x-axis for $-1 < x < 0$, the graph of R will approach the vertical asymptote $x = 0$ at the top to the left of $x = 0$ $\left[\lim_{x \to 0^-} R(x) = \infty\right]$; since the graph of R is below the x-axis for $0 < x < 1$, the graph of R will approach the vertical asymptote $x = 0$ at the bottom to the right of $x = 0$ $\left[\lim_{x \to 0^+} R(x) = -\infty\right]$. See Figure 34(b).



Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Number chosen	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
Value of R	$R(-2) = -\frac{3}{2}$	$R\left(-\frac{1}{2}\right) = \frac{3}{2}$	$R\left(\frac{1}{2}\right) = -\frac{3}{2}$	$R(2) = \frac{3}{2}$
Location of graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$\left(-2, -\frac{3}{2}\right)$	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$\left(\frac{1}{2}, -\frac{3}{2}\right)$	$\left(2, \frac{3}{2}\right)$

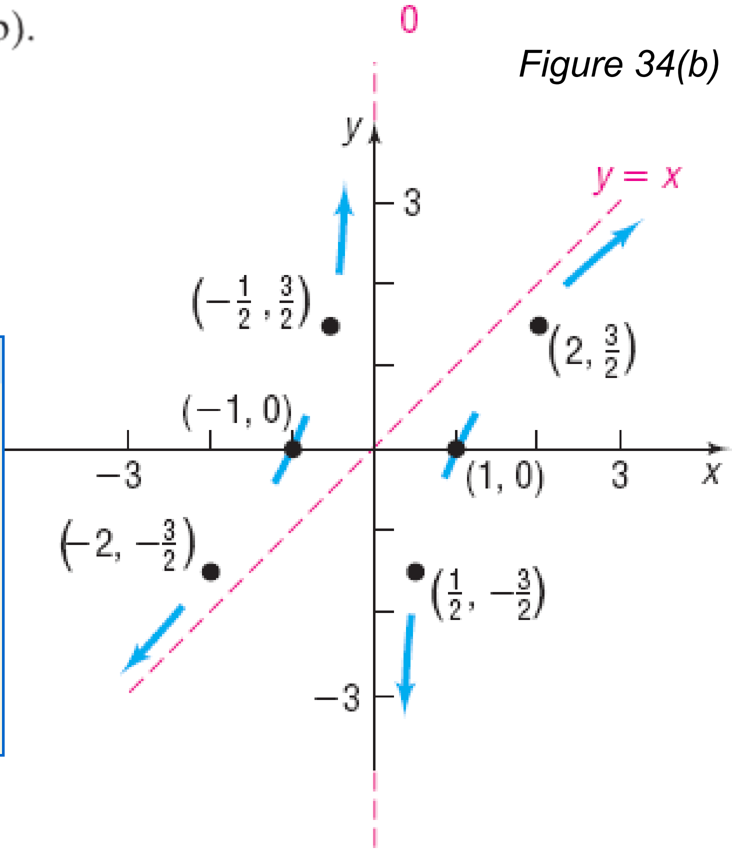
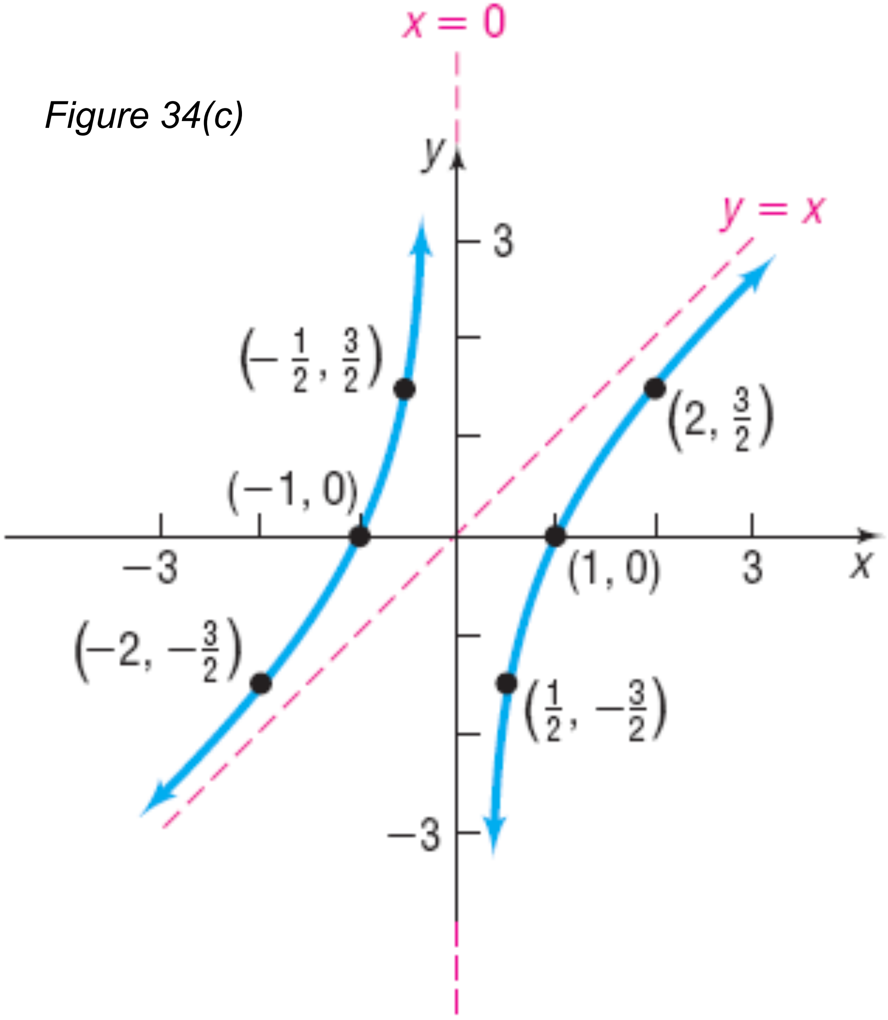


Figure 34(b)

Analyze the graph of the rational function: $R(x) = \dfrac{x^2 - 1}{x}$

STEP 8: The complete graph is given in Figure 34(c).

Figure 34(c)



EXAMPLE Analyzing the Graph of a Rational Function

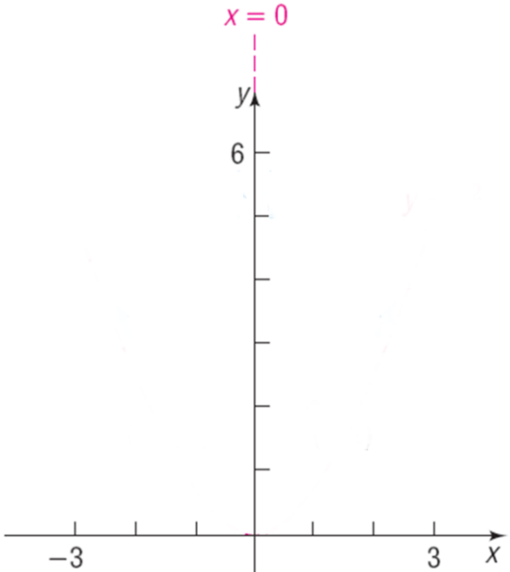
Analyze the graph of the rational function: $R(x) = \dfrac{x^4 + 1}{x^2}$

Step 1: R is completely factored. The domain of R is $\{x \mid x \neq 0\}$.

Step 2: R is in lowest terms.

Step 3: There is no y-intercept. Since $x^4 + 1 = 0$ has no real solutions, there are no x-intercepts.

Step 4: R is in lowest terms, so $x = 0$ (the y-axis) is a vertical asymptote of R. Graph the line $x = 0$ using dashes.

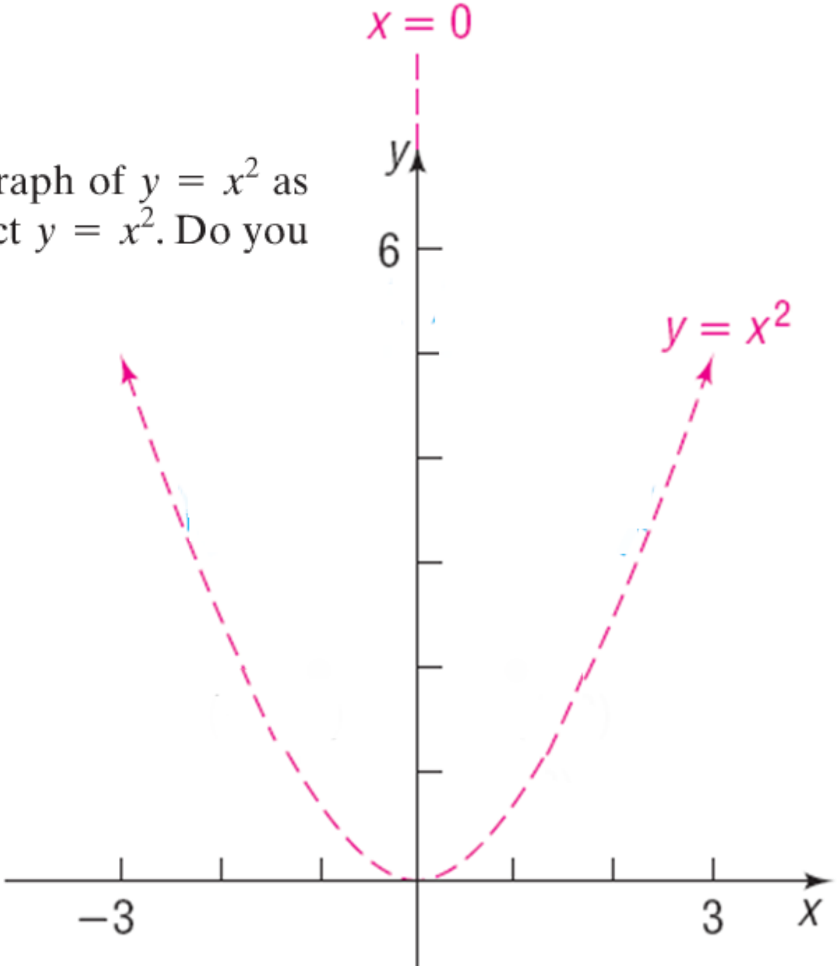


Analyze the graph of the rational function: $R(x) = \dfrac{x^4 + 1}{x^2}$

STEP 5: Since the degree of the numerator, 4, is two more than the degree of the denominator, 2, the rational function will not have a horizontal or oblique asymptote. We use long division to find the end behavior of R.

$$\begin{array}{r} x^2 \\ x^2\overline{)x^4 + 1} \\ \underline{x^4} \\ 1 \end{array}$$

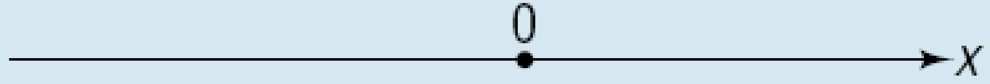
The quotient is x^2, so the graph of R will approach the graph of $y = x^2$ as $x \to -\infty$ and as $x \to \infty$. The graph of R does not intersect $y = x^2$. Do you know why? Graph $y = x^2$ using dashes.



Analyze the graph of the rational function: $R(x) = \dfrac{x^4 + 1}{x^2}$

STEP 6: The numerator has no real zeros, and the denominator has one real zero at 0. We divide the x-axis into the two intervals

$$(-\infty, 0) \qquad (0, \infty)$$

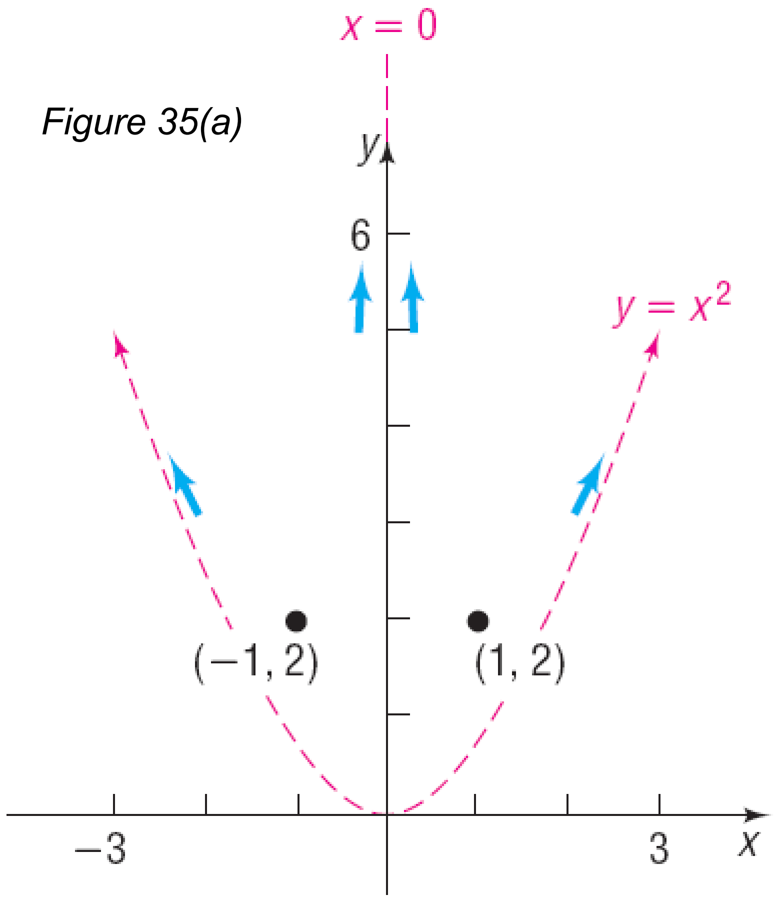


Interval	$(-\infty, 0)$	$(0, \infty)$
Number chosen	-1	1
Value of *R*	$R(-1) = 2$	$R(1) = 2$
Location of graph	Above x-axis	Above x-axis
Point on graph	$(-1, 2)$	$(1, 2)$

Analyze the graph of the rational function: $R(x) = \dfrac{x^4 + 1}{x^2}$

STEP 7: Since the graph of R is above the x-axis and does not intersect $y = x^2$, we place arrows above $y = x^2$ as shown in Figure 35(a). Also, since the graph of R is above the x-axis, it will approach the vertical asymptote $x = 0$ at the top to the left of $x = 0$ and at the top to the right of $x = 0$. See Figure 35(a).

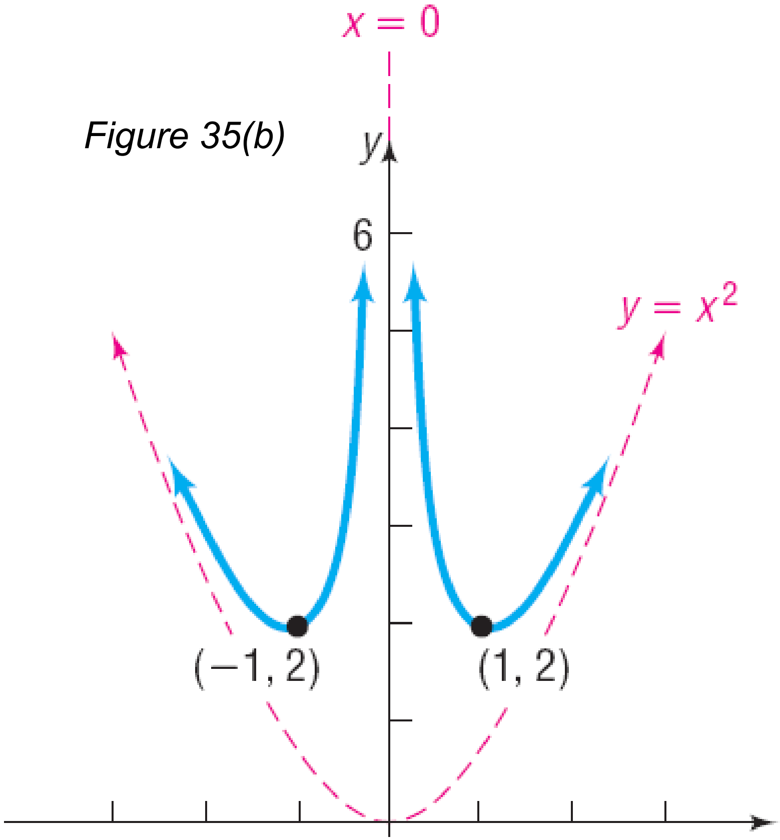
Figure 35(a)



Analyze the graph of the rational function: $R(x) = \dfrac{x^4 + 1}{x^2}$

Step 8: Figure 35(b) shows the complete graph.

Figure 35(b)



Seeing the Concept

Graph $R(x) = \dfrac{x^4 + 1}{x^2}$ and compare what you see with Figure 35(b). Use MINIMUM to find the two turning points. Enter $Y_2 = x^2$ and ZOOM-OUT. What do you see?

EXAMPLE Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

STEP 1: Factor R to get $R(x) = \dfrac{3x(x-1)}{(x+4)(x-3)}$

STEP 2: R is in lowest terms.

STEP 3: The y-intercept is $R(0) = 0$. Plot the point $(0, 0)$. Since the real solutions of the equation $3x(x-1) = 0$ are $x = 0$ and $x = 1$, the graph has two x-intercepts, 0 and 1. We determine the behavior of the graph of R near each x-intercept.

Near 0: $$R(x) = \frac{3x(x-1)}{(x+4)(x-3)} \approx \frac{3x(0-1)}{(0+4)(0-3)} = \frac{1}{4}x$$

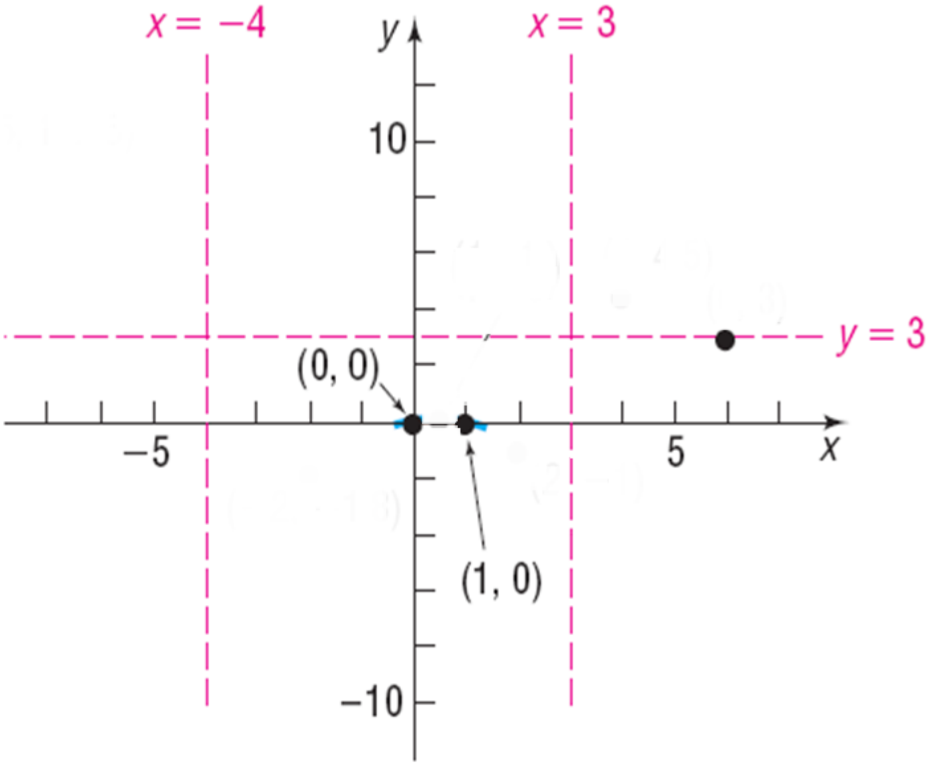
Near 1: $$R(x) = \frac{3x(x-1)}{(x+4)(x-3)} \approx \frac{3(1)(x-1)}{(1+4)(1-3)} = -\frac{3}{10}(x-1)$$

Plot the point $(0, 0)$ and show a line with positive slope there. Plot the point $(1, 0)$ and show a line with negative slope there.

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

STEP 4: R is in lowest terms. The real solutions of the equation $(x + 4)(x - 3) = 0$ are $x = -4$ and $x = 3$, so the graph of R has two vertical asymptotes, the lines $x = -4$ and $x = 3$. Graph these lines using dashes.

STEP 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 3, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 3$.



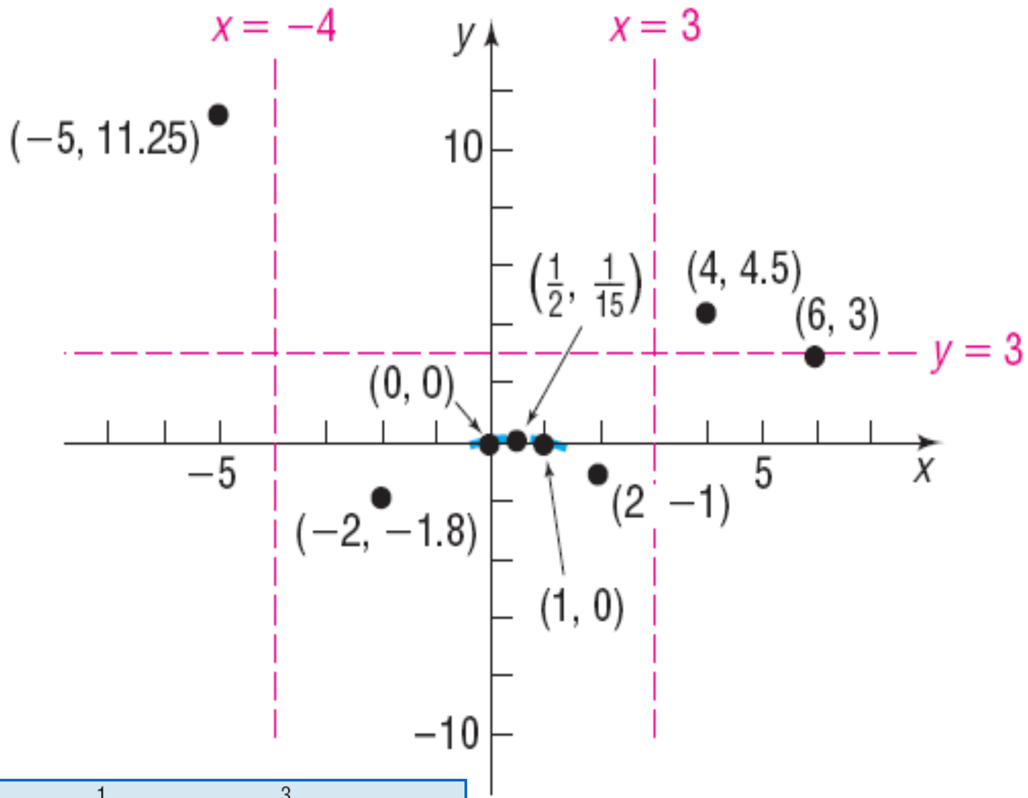
Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

Step 6: The real zeros of the numerator, 0 and 1, and the real zeros of the denominator, −4 and 3, divide the x-axis into five intervals:

$$(-\infty, -4) \quad (-4, 0) \quad (0, 1) \quad (1, 3) \quad (3, \infty)$$

Interval	$(-\infty, -4)$	$(-4, 0)$	$(0, 1)$	$(1, 3)$	$(3, \infty)$
Number chosen	-5	-2	$\frac{1}{2}$	2	4
Value of *R*	$R(-5) = 11.25$	$R(-2) = -1.8$	$R\left(\frac{1}{2}\right) = \frac{1}{15}$	$R(2) = -1$	$R(4) = 4.5$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$(-5, 11.25)$	$(-2, -1.8)$	$\left(\frac{1}{2}, \frac{1}{15}\right)$	$(2, -1)$	$(4, 4.5)$

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

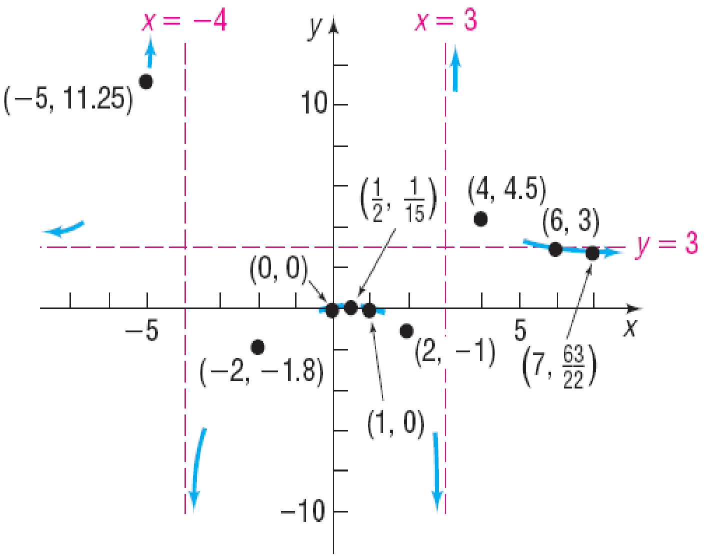


	−4	0	1	3	
Interval	$(-\infty, -4)$	$(-4, 0)$	$(0, 1)$	$(1, 3)$	$(3, \infty)$
Number chosen	-5	-2	$\frac{1}{2}$	2	4
Value of R	$R(-5) = 11.25$	$R(-2) = -1.8$	$R\left(\frac{1}{2}\right) = \frac{1}{15}$	$R(2) = -1$	$R(4) = 4.5$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$(-5, 11.25)$	$(-2, -1.8)$	$\left(\frac{1}{2}, \frac{1}{15}\right)$	$(2, -1)$	$(4, 4.5)$

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

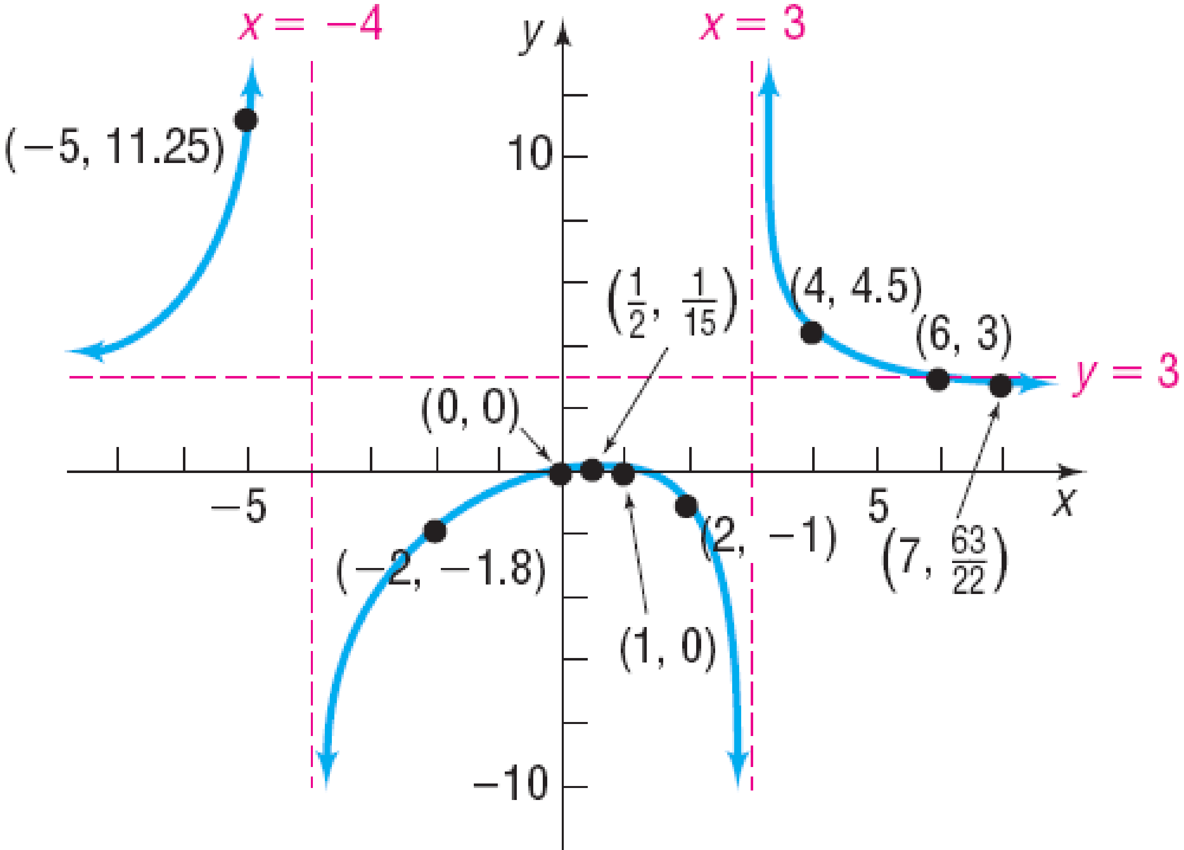
Step 7:

- Since the graph of R is above the x-axis for $x < -4$ and only crosses the line $y = 3$ at $(6, 3)$, as x approaches $-\infty$ the graph of R will approach the horizontal asymptote $y = 3$ from above $\left(\lim\limits_{x \to -\infty} R(x) = 3\right)$.
- The graph of R will approach the vertical asymptote $x = -4$ at the top to the left of $x = -4$ $\left(\lim\limits_{x \to -4^-} R(x) = +\infty\right)$ and at the bottom to the right of $x = -4$ $\left(\lim\limits_{x \to -4^+} R(x) = -\infty\right)$.
- The graph of R will approach the vertical asymptote $x = 3$ at the bottom to the left of $x = 3$ $\left(\lim\limits_{x \to 3^-} R(x) = -\infty\right)$ and at the top to the right of $x = 3$ $\left(\lim\limits_{x \to 3^+} R(x) = +\infty\right)$.
- We do not know whether the graph of R crosses or touches the line $y = 3$ at $(6, 3)$. To see whether the graph, in fact, crosses or touches the line $y = 3$, we plot an additional point to the right of $(6, 3)$. We use $x = 7$ to find $R(7) = \dfrac{63}{22} < 3$. The graph crosses $y = 3$ at $x = 6$. Because $(6, 3)$ is the only point where the graph of R intersects the asymptote $y = 3$, the graph must approach the line $y = 3$ from below as $x \to \infty$ $\left(\lim\limits_{x \to \infty} R(x) = 3\right)$. See Figure 36(b).



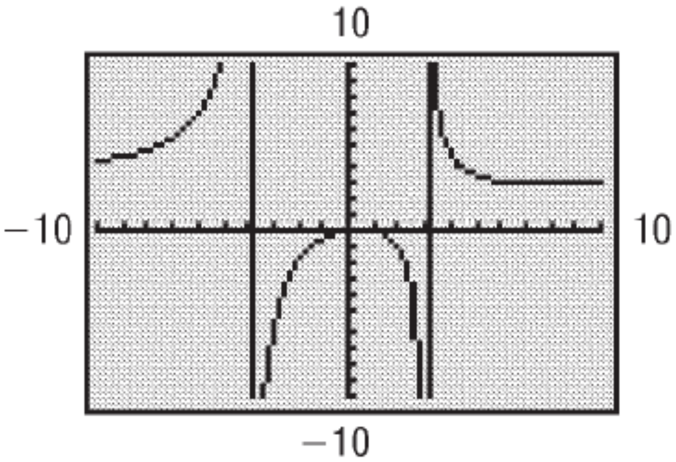
Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

STEP 8: The complete graph is shown in Figure 36(c).

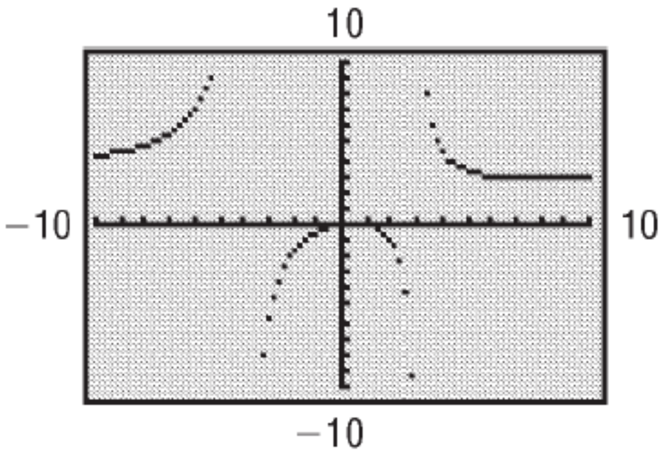


Exploration

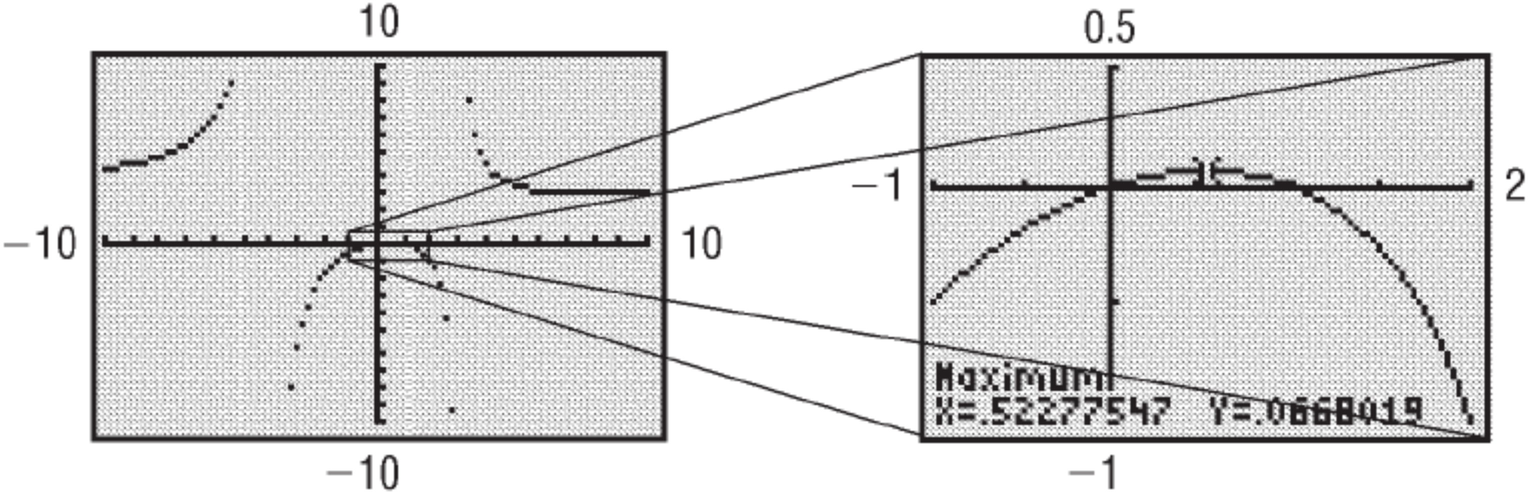
Graph $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$



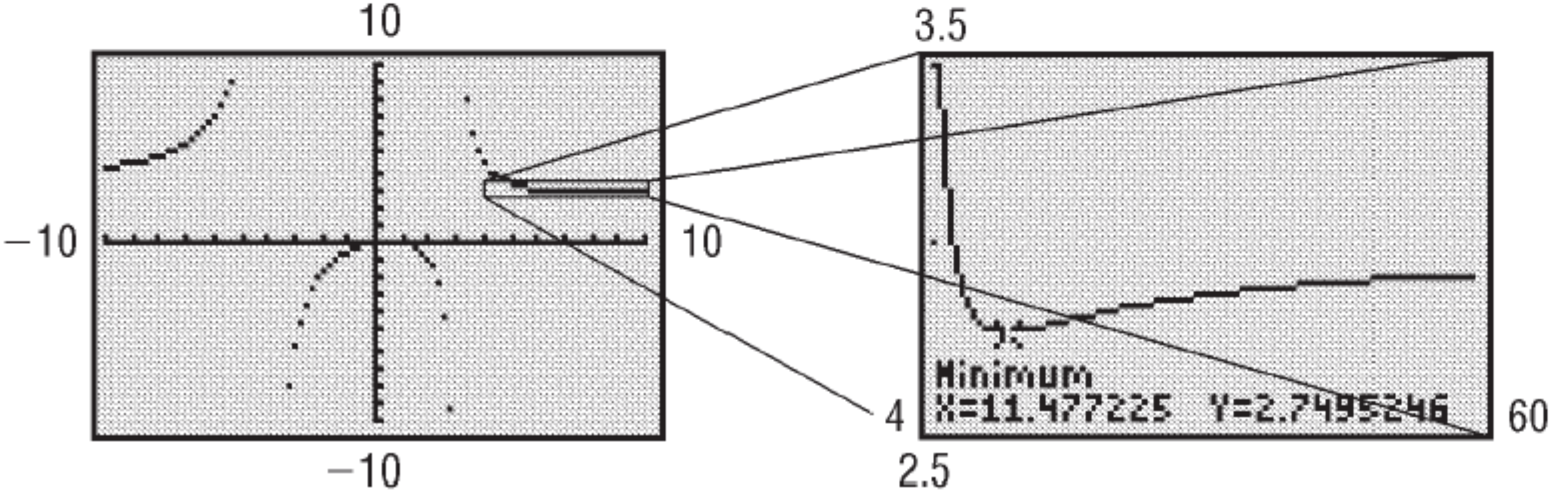
Connected mode



Dot mode



```
Maximum
X=.52277547  Y=.0668019
```


10
−10
10
−10
3.5
4
2.5
60
Minimum
X=11.477225 Y=2.7495246

EXAMPLE **Analyzing the Graph of a Rational Function with a Hole**

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

STEP 1: Factor R and obtain $R(x) = \dfrac{(2x - 1)(x - 2)}{(x + 2)(x - 2)}$

The domain of R is $\{x | x \neq -2, x \neq 2\}$.

STEP 2: In lowest terms, $R(x) = \dfrac{(2x - 1)\cancel{(x - 2)}}{(x + 2)\cancel{(x - 2)}}$

STEP 3: The y-intercept is $R(0) = -\frac{1}{2}$. Plot the point $\left(0, -\frac{1}{2}\right)$.
The graph has one x-intercept: $\frac{1}{2}$.

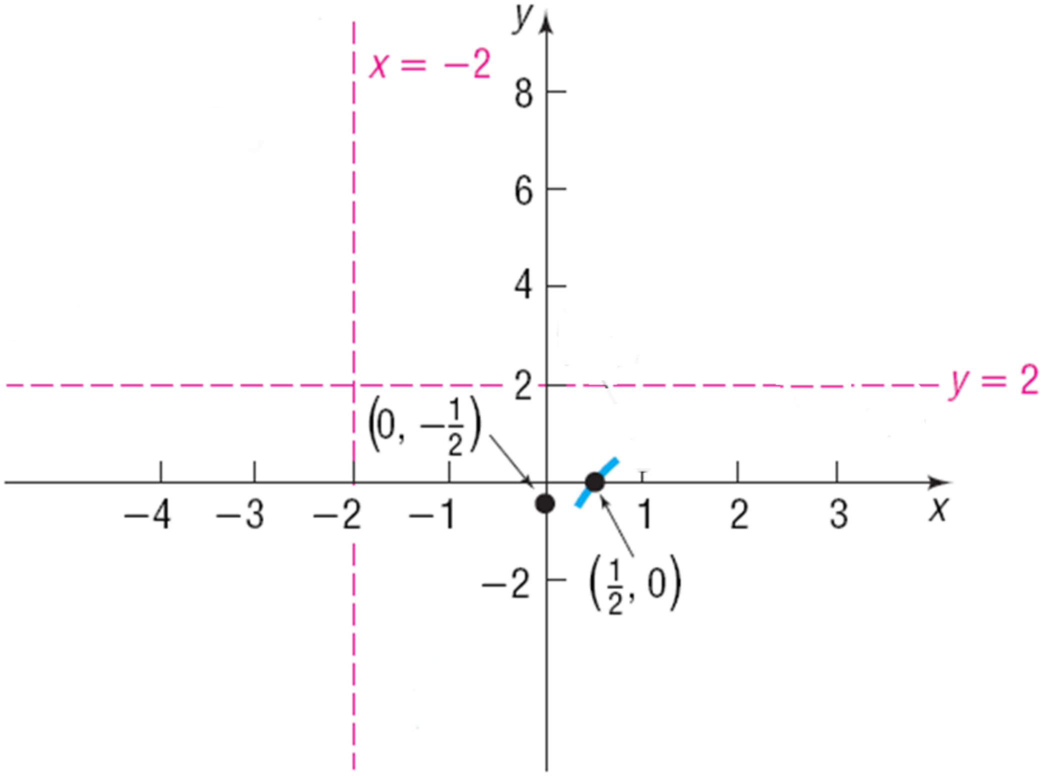
Near $\frac{1}{2}$: $R(x) = \dfrac{2x - 1}{x + 2} \approx \dfrac{2x - 1}{\frac{1}{2} + 2} = \dfrac{2}{5}(2x - 1)$

Plot the point $\left(\frac{1}{2}, 0\right)$ showing a line with positive slope.

Analyze the graph of the rational function: $R(x) = \frac{2x^2 - 5x + 2}{x^2 - 4}$

STEP 4: Since $x + 2$ is the only factor of the denominator of $R(x)$ *in lowest terms*, the graph has one vertical asymptote, $x = -2$. However, the rational function is undefined at both $x = 2$ and $x = -2$. Graph the line $x = -2$ using dashes.

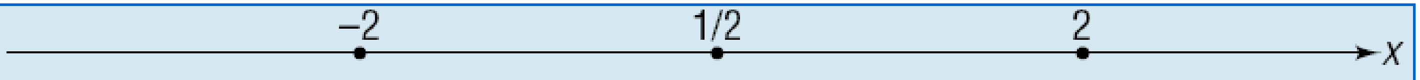
STEP 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 2, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 2$. Graph the line $y = 2$ using dashes.



Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

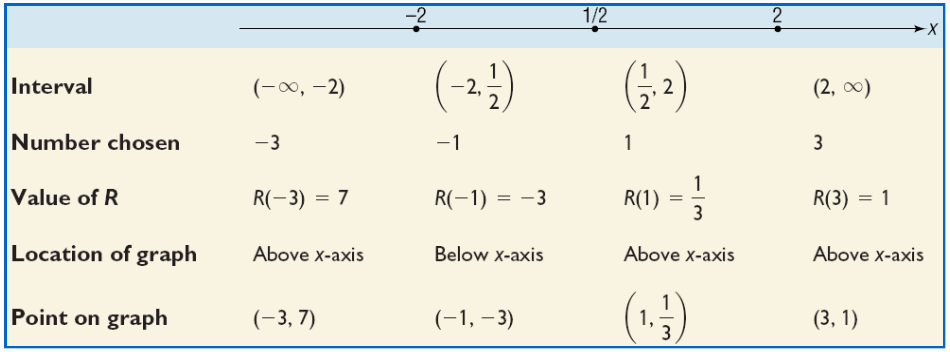
Step 6: Look at the factored expression for R in Step 1. The real zeros of the numerator and denominator, $-2, \frac{1}{2}$, and 2, divide the x-axis into four intervals:

$$(-\infty, -2) \qquad \left(-2, \frac{1}{2}\right) \qquad \left(\frac{1}{2}, 2\right) \qquad (2, \infty)$$



Interval	$(-\infty, -2)$	$\left(-2, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 2\right)$	$(2, \infty)$
Number chosen	-3	-1	1	3
Value of *R*	$R(-3) = 7$	$R(-1) = -3$	$R(1) = \frac{1}{3}$	$R(3) = 1$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Above x-axis
Point on graph	$(-3, 7)$	$(-1, -3)$	$\left(1, \frac{1}{3}\right)$	$(3, 1)$

Step 7:
- From Table 15 we know that the graph of R is above the x-axis for $x < -2$.
- From Step 5 we know that the graph of R does not intersect the asymptote $y = 2$. Therefore, the graph of R will approach $y = 2$ from above as $x \to -\infty$ and will approach the vertical asymptote $x = -2$ at the top from the left.
- Since the graph of R is below the x-axis for $-2 < x < \frac{1}{2}$, the graph of R will approach $x = -2$ at the bottom from the right.
- Finally, since the graph of R is above the x-axis for $x > \frac{1}{2}$ and does not intersect the horizontal asymptote $y = 2$, the graph of R will approach $y = 2$ from below as $x \to \infty$. See Figure 40(a).



-2 $\quad$ $1/2$ $\quad$ 2 $\quad$ x

Interval	$(-\infty, -2)$	$\left(-2, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 2\right)$	$(2, \infty)$
Number chosen	-3	-1	1	3
Value of R	$R(-3) = 7$	$R(-1) = -3$	$R(1) = \frac{1}{3}$	$R(3) = 1$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Above x-axis
Point on graph	$(-3, 7)$	$(-1, -3)$	$\left(1, \frac{1}{3}\right)$	$(3, 1)$

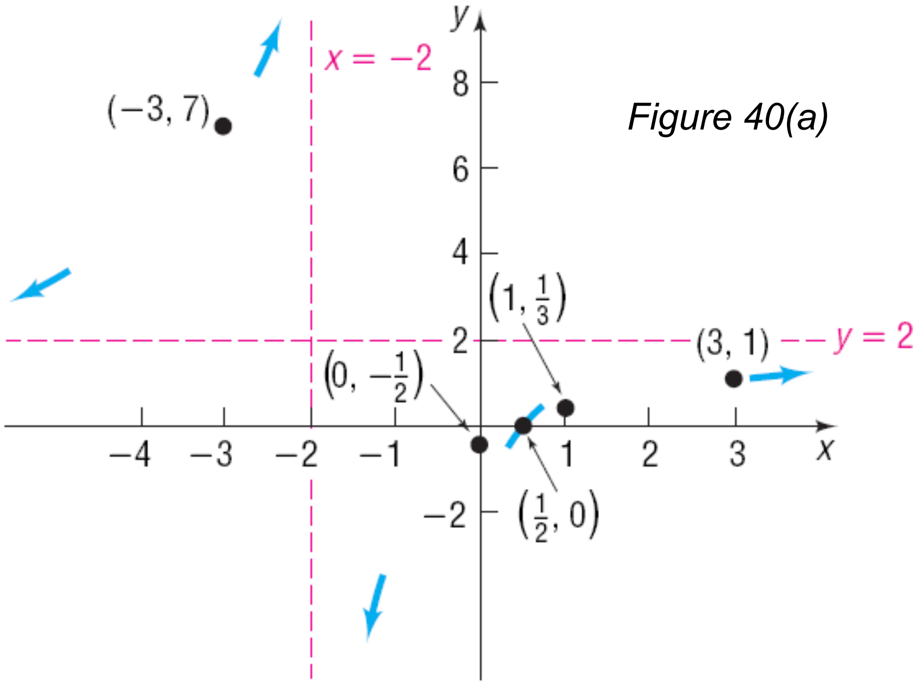


Figure 40(a)

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

STEP 8: See Figure 40(b) for the complete graph. Since R is not defined at 2, there is a hole at the point $\left(2, \frac{3}{4}\right)$.

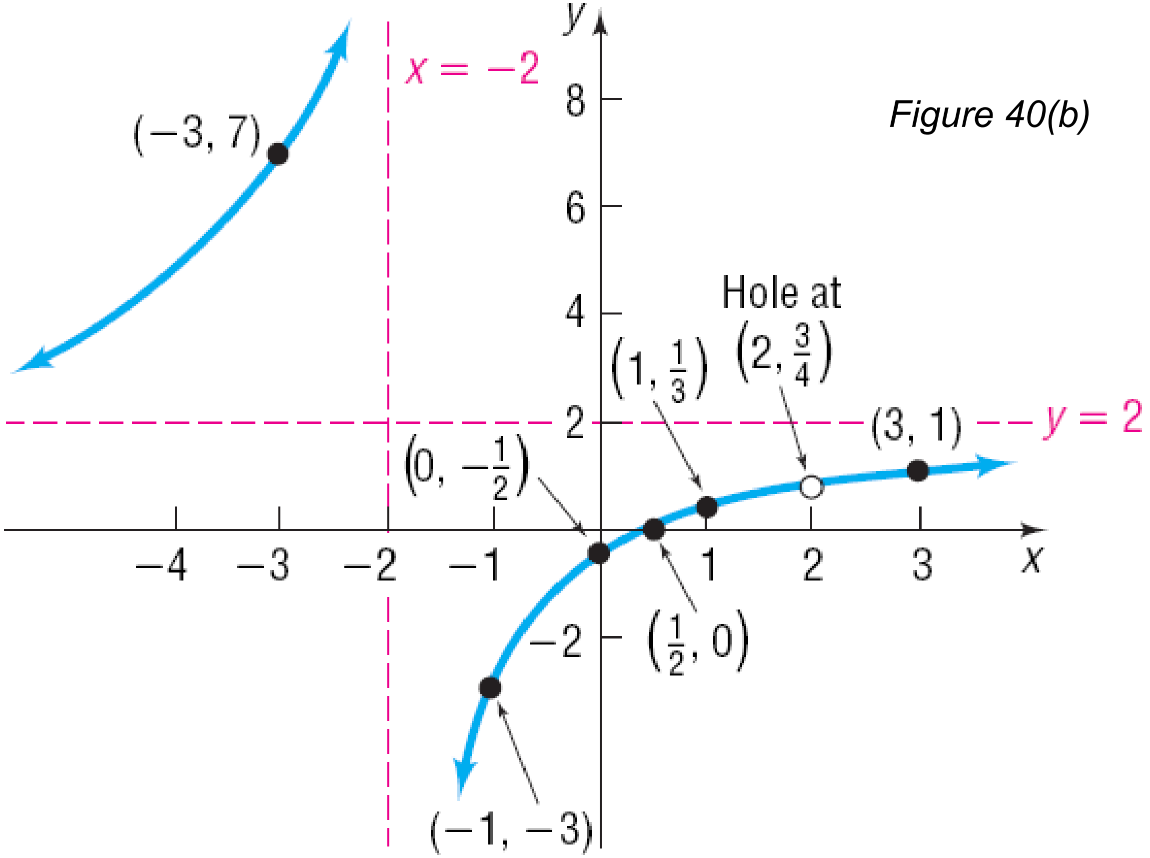


Figure 40(b)

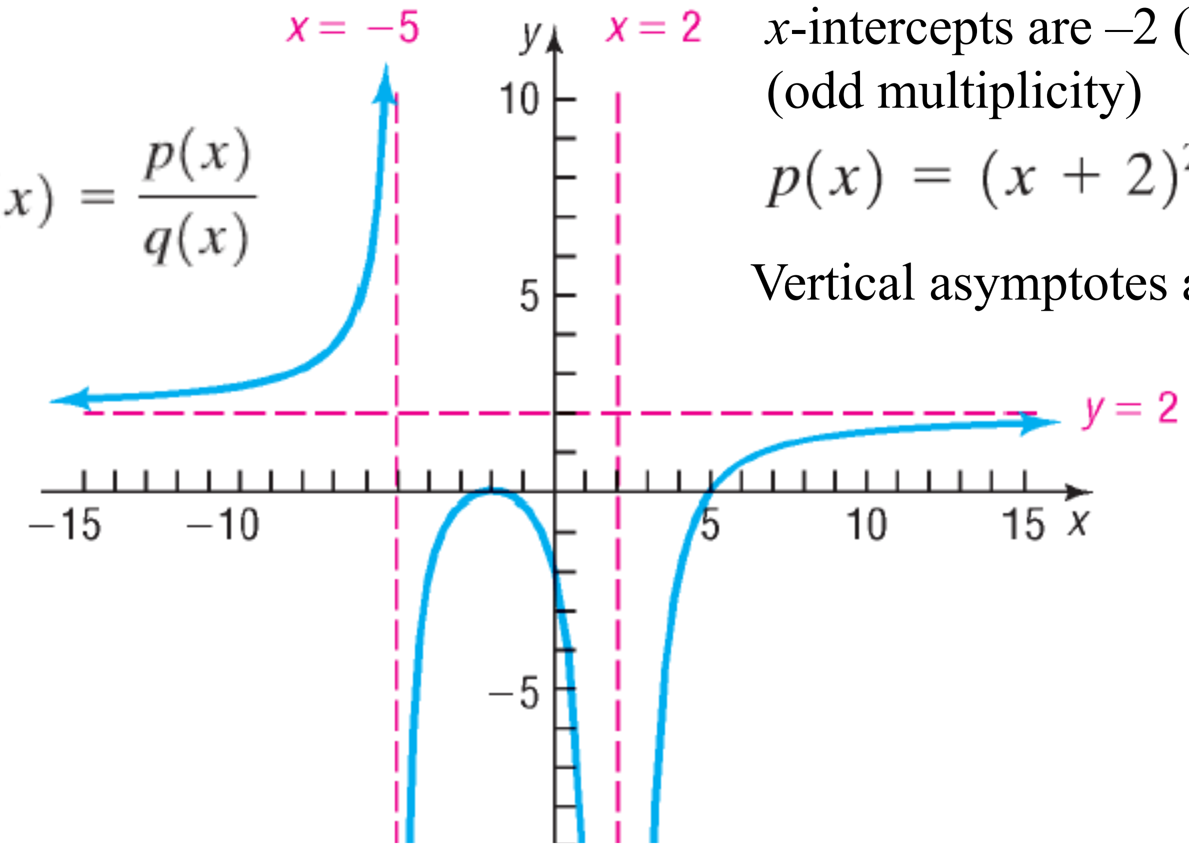
Exploration

Graph $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$. Do you see the hole at $\left(2, \dfrac{3}{4}\right)$? TRACE along the graph. Did you obtain an ERROR at $x = 2$? Are you convinced that an algebraic analysis of a rational function is required in order to accurately interpret the graph obtained with a graphing utility?

EXAMPLE Constructing a Rational Function from Its Graph

Make up a rational function that might have the graph shown

$$R(x) = \frac{p(x)}{q(x)}$$



x-intercepts are –2 (even multiplicity) and 5 (odd multiplicity)

$$p(x) = (x + 2)^2(x - 5)$$

Vertical asymptotes at $x = –5$ and $x = 2$.

Since $R(x)$ approaches ∞ to the left of $x = -5$ and $R(x)$ approaches $-\infty$ to the right of $x = -5$, we know that $(x + 5)$ is a factor of odd multiplicity in $q(x)$. Also, $R(x)$ approaches $-\infty$ on both sides of $x = 2$, so $(x - 2)$ is a factor of even multiplicity in $q(x)$. A possibility for the denominator is $q(x) = (x + 5)(x - 2)^2$.

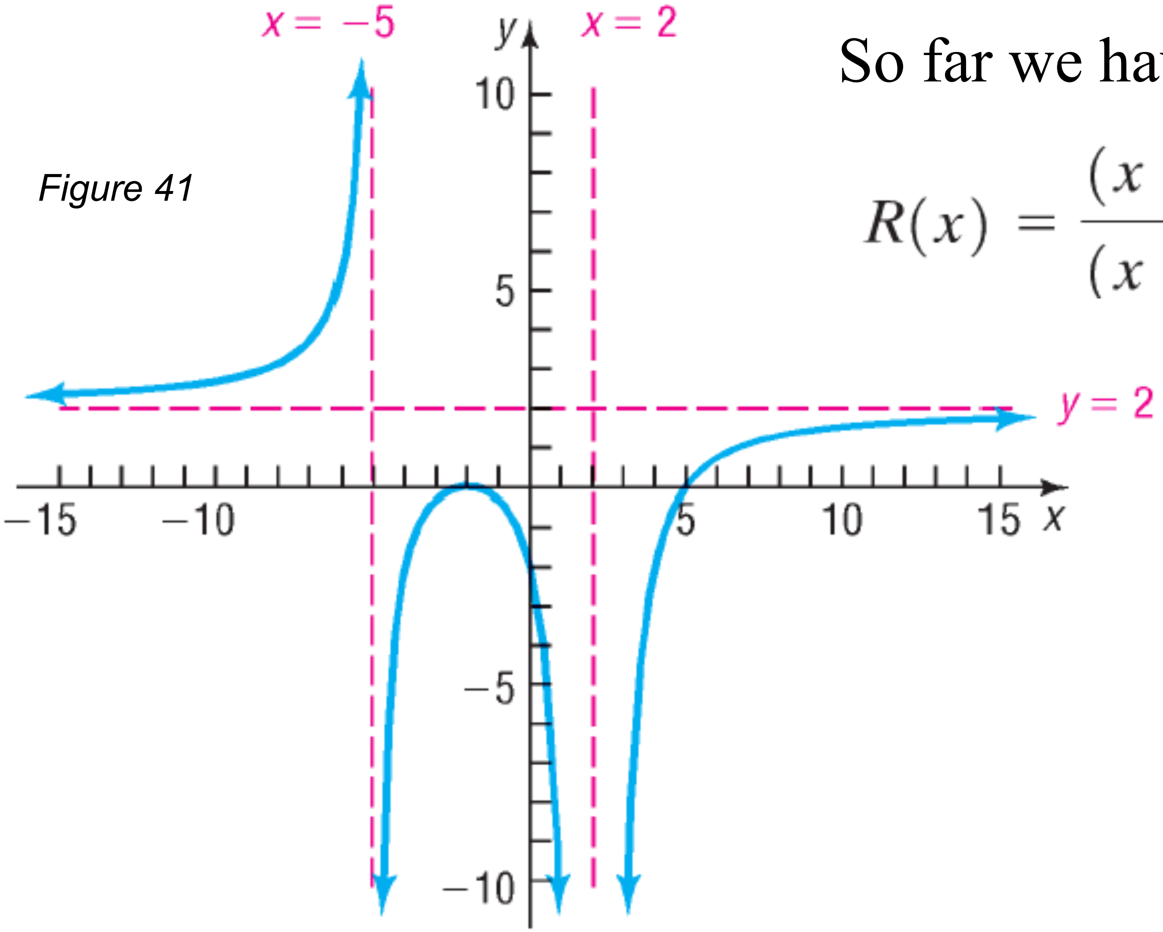


Figure 41

So far we have:

$$R(x) = \frac{(x+2)^2(x-5)}{(x+5)(x-2)^2}$$

The horizontal asymptote of the graph given in Figure 41 is $y = 2$, so we know that the degree of the numerator must equal the degree of the denominator and the quotient of leading coefficients must be $\frac{2}{1}$. This leads to

$$R(x) = \frac{2(x+2)^2(x-5)}{(x+5)(x-2)^2}$$

2 Solve Applied Problems Involving Rational Functions

EXAMPLE

Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2}\text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

(a) Express the cost of material for the can as a function of the radius *r* of the can.

(b) Use a graphing utility to graph the function $C = C(r)$.

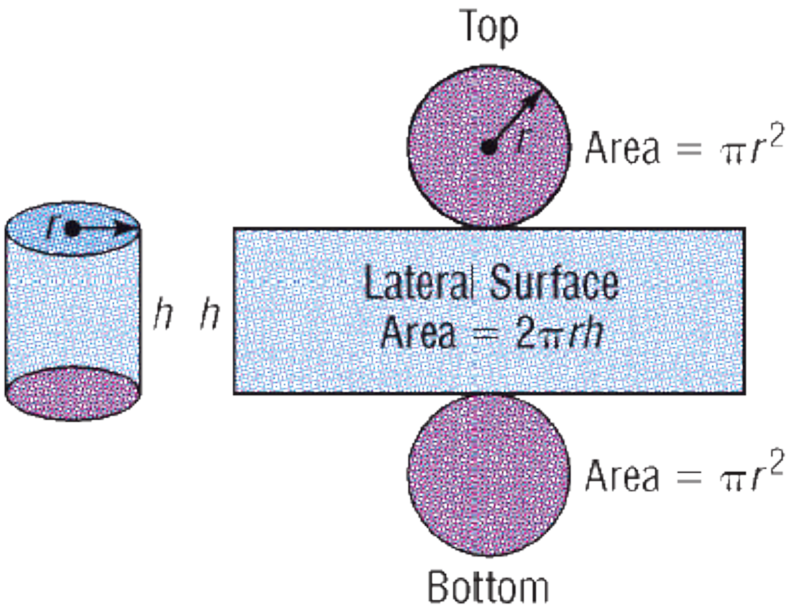
(c) What value of *r* will result in the least cost?

(d) What is this least cost?

$$C = \text{Cost of the top and bottom} + \text{Cost of the side}$$

$$= \underbrace{2(\pi r^2)}_{\text{Total area of top and bottom}} \underbrace{(0.05)}_{\text{Cost/unit area}} + \underbrace{(2\pi r h)}_{\text{Total area of side}} \underbrace{(0.02)}_{\text{Cost/unit area}}$$

$$= 0.10\pi r^2 + 0.04\pi r h$$



EXAMPLE

Finding the Least Cost of a Can

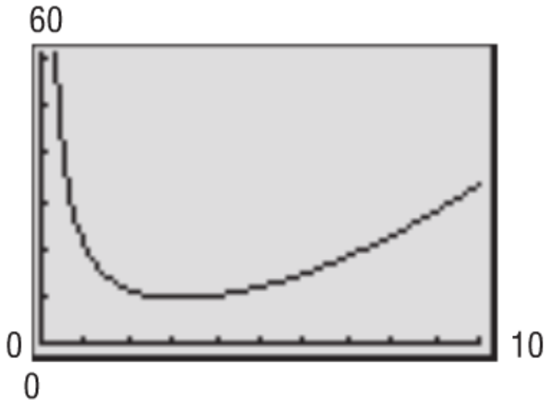
Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2}\text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

(a) Express the cost of material for the can as a function of the radius r of the can.

(b) Use a graphing utility to graph the function $C = C(r)$.

$$V = \pi r^2 h.$$

$$500 = \pi r^2 h \quad \text{so} \quad h = \frac{500}{\pi r^2}$$



Substituting this expression for h, the cost C, in cents, as a function of the radius r is

$$C(r) = 0.10\pi r^2 + 0.04\pi r \cdot \frac{500}{\pi r^2} = 0.10\pi r^2 + \frac{20}{r} = \frac{0.10\pi r^3 + 20}{r}$$

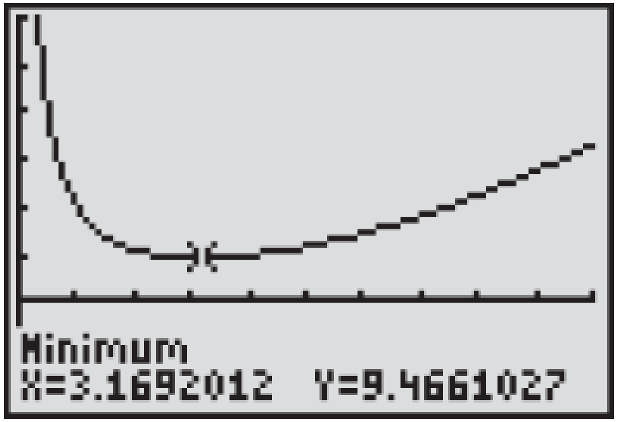
EXAMPLE

Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2}\text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

(c) What value of r will result in the least cost?

(d) What is this least cost?



Minimum
X=3.1692012 Y=9.4661027

(c) Using the MINIMUM command, the cost is least for a radius of about 3.17 centimeters.

(d) The least cost is $C(3.17) \approx 9.47¢$.