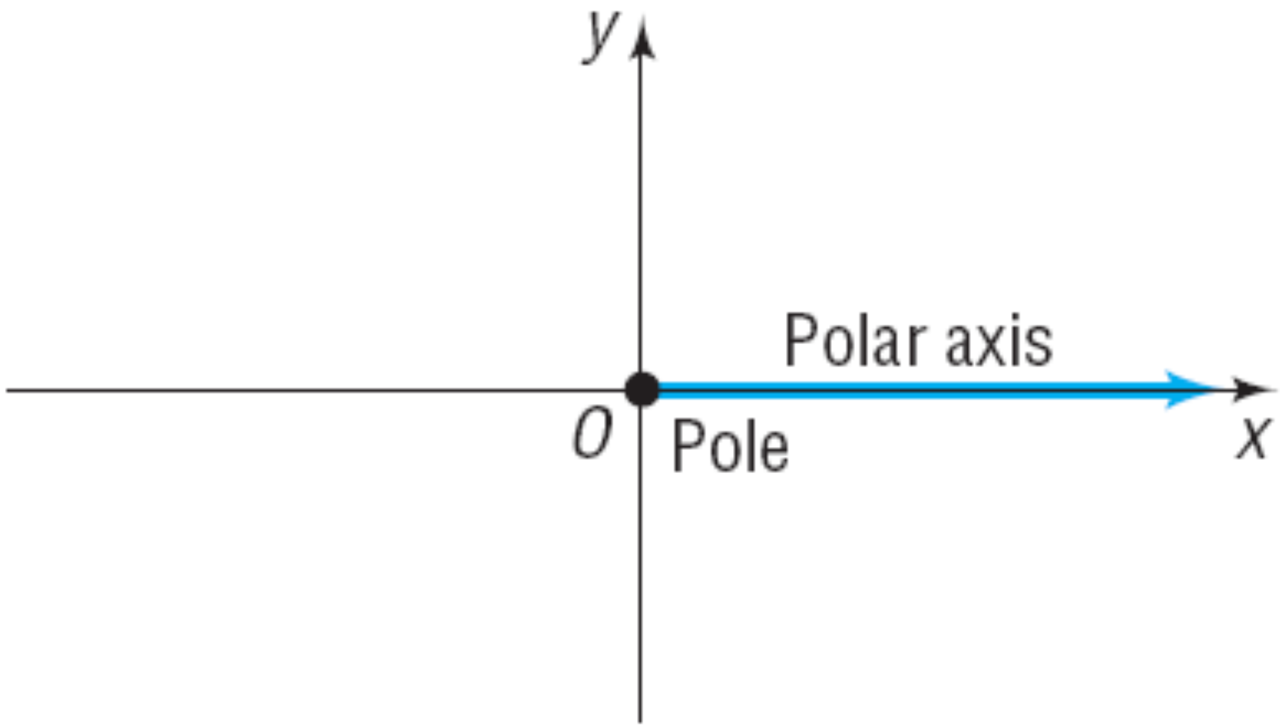
Section 10.1

Polar Coordinates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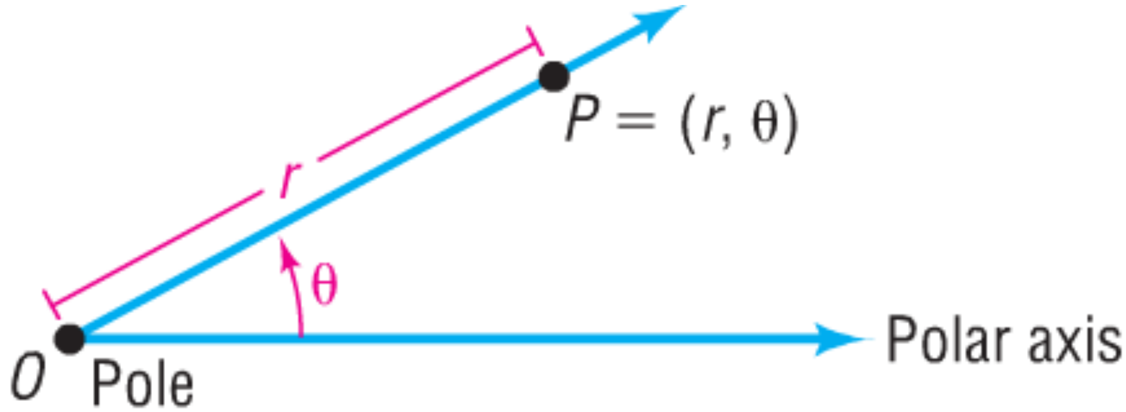
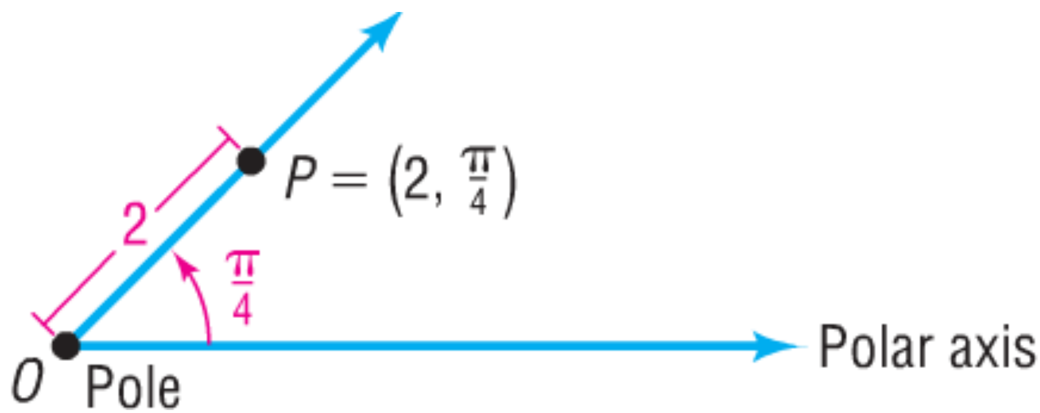
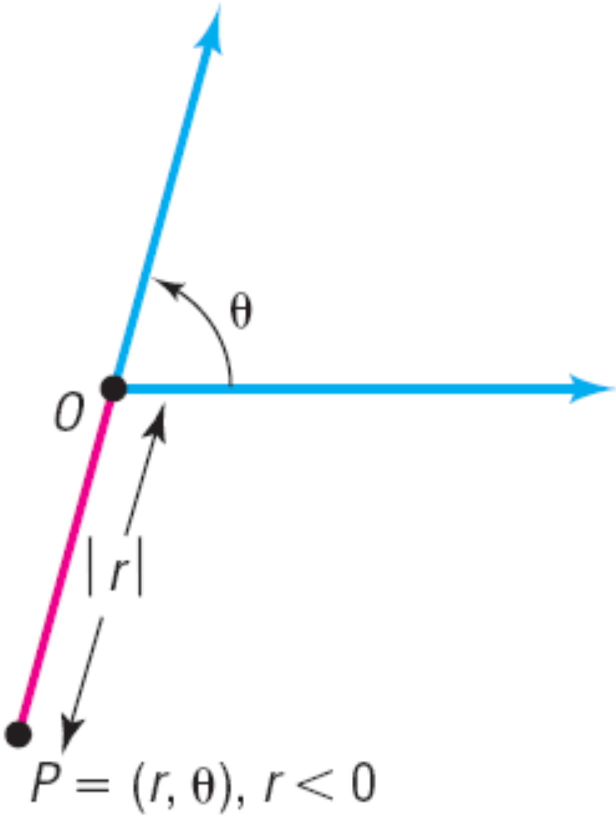
1 Plot Points Using Polar Coordinates

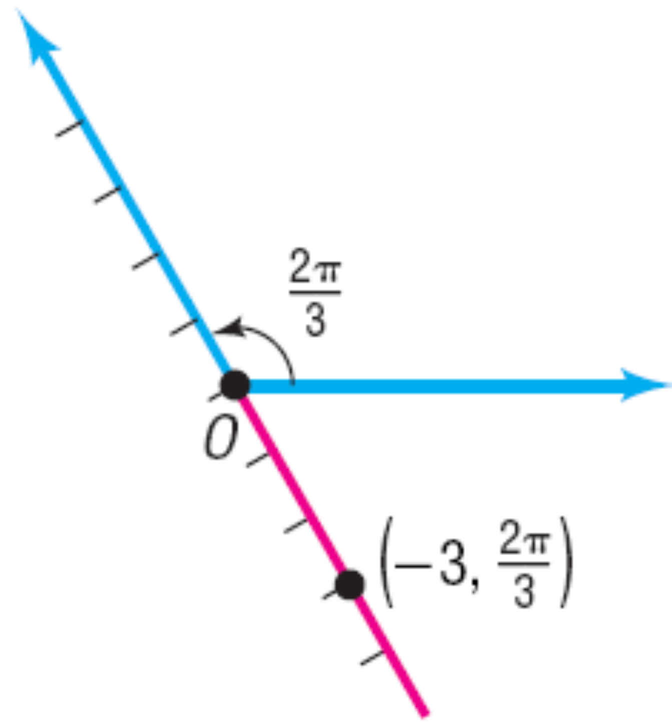


Suppose the polar coordinates of the point are $\left(2, \dfrac{\pi}{4}\right)$





plot the point $\left(-3, \frac{2\pi}{3}\right)$

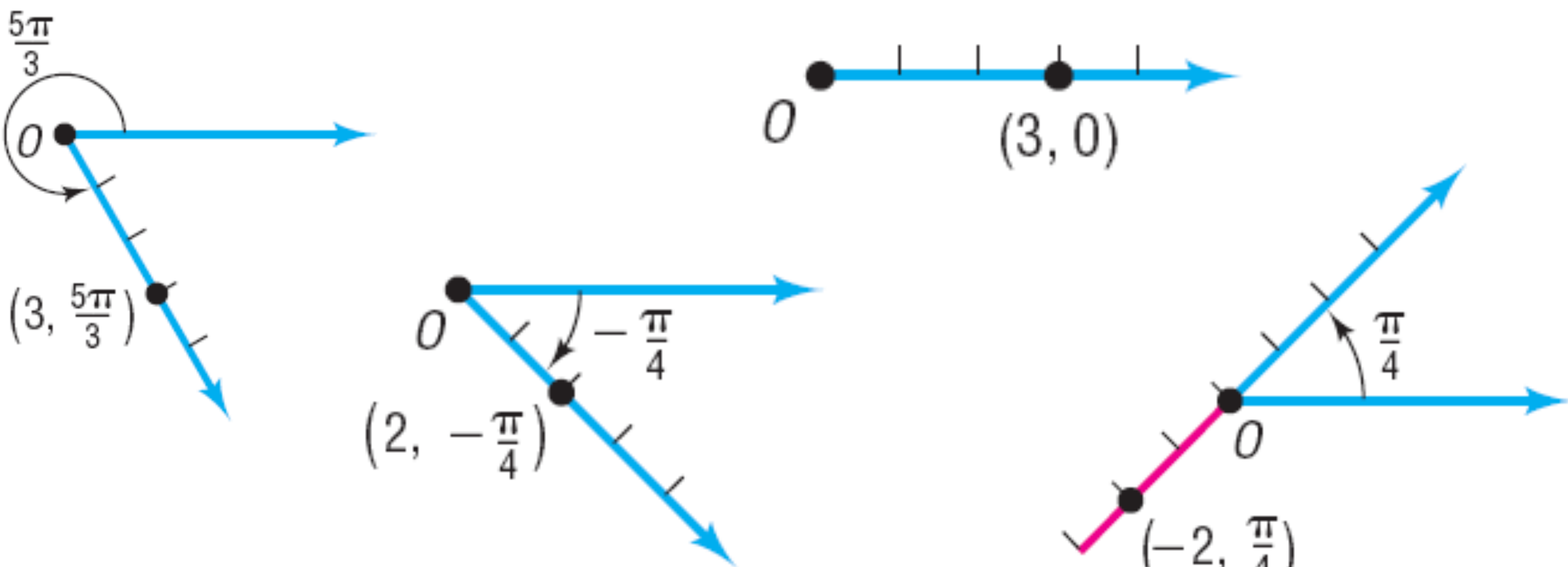


EXAMPLE

Plotting Points Using Polar Coordinates

Plot the points with the following polar coordinates:

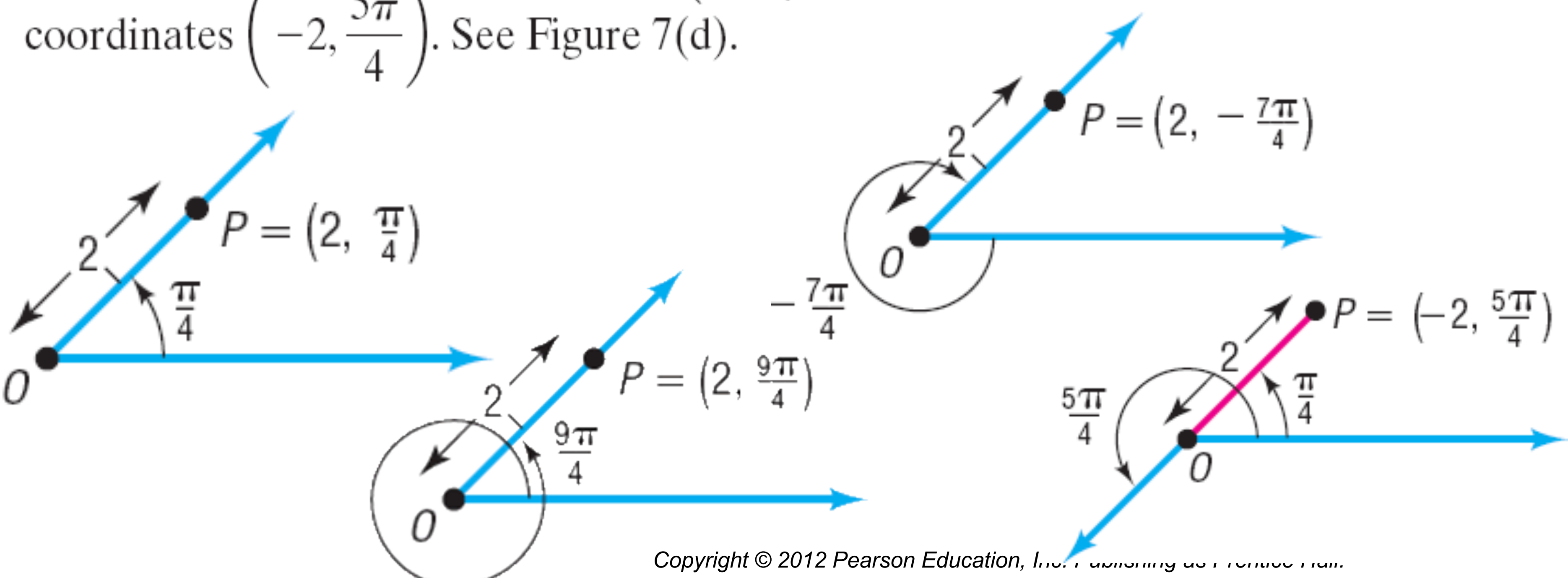
(a) $\left(3, \dfrac{5\pi}{3}\right)$ (b) $\left(2, -\dfrac{\pi}{4}\right)$ (c) $(3, 0)$ (d) $\left(-2, \dfrac{\pi}{4}\right)$



EXAMPLE

Finding Several Polar Coordinates of a Single Point

Consider again the point P with polar coordinates $\left(2, \frac{\pi}{4}\right)$, as shown in Figure 7(a). Because $\frac{\pi}{4}$, $\frac{9\pi}{4}$, and $-\frac{7\pi}{4}$ all have the same terminal side, we also could have located this point P by using the polar coordinates $\left(2, \frac{9\pi}{4}\right)$ or $\left(2, -\frac{7\pi}{4}\right)$, as shown in Figures 7(b) and (c). The point $\left(2, \frac{\pi}{4}\right)$ can also be represented by the polar coordinates $\left(-2, \frac{5\pi}{4}\right)$. See Figure 7(d).

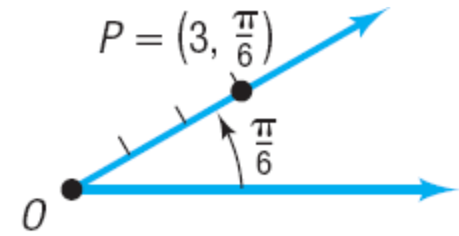


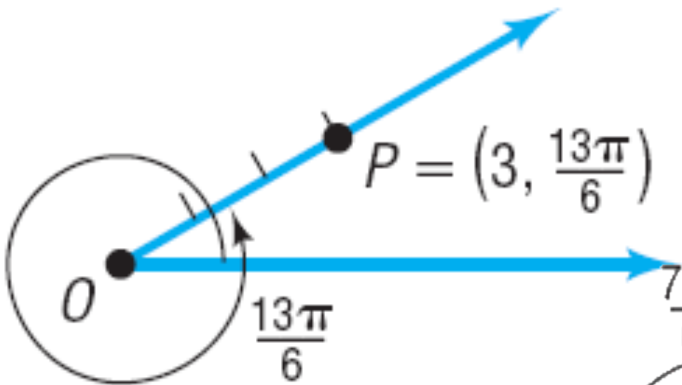
EXAMPLE **Finding Other Polar Coordinates of a Given Point**

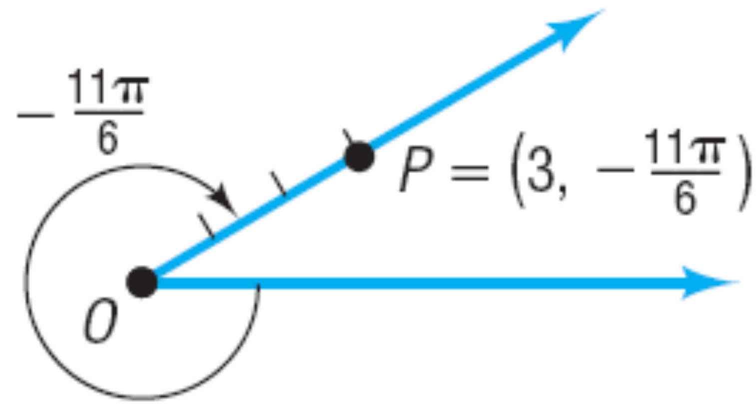
Plot the point P with polar coordinates $\left(3, \dfrac{\pi}{6}\right)$, and find other polar coordinates (r, θ) of this same point for which:

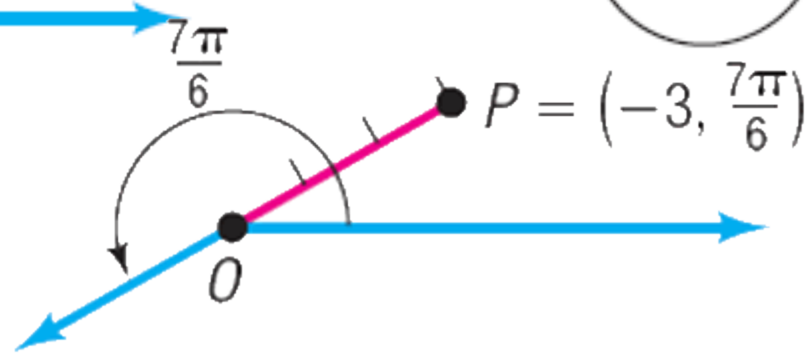
(a) $r > 0, \quad 2\pi \leq \theta < 4\pi$ (b) $r < 0, \quad 0 \leq \theta < 2\pi$

(c) $r > 0, \quad -2\pi \leq \theta < 0$









SUMMARY

A point with polar coordinates (r, θ), θ in radians can also be represented by either of the following:

$$(r, \theta + 2\pi k) \quad \text{or} \quad (-r, \theta + \pi + 2\pi k) \qquad k \text{ any integer}$$

The polar coordinates of the pole are $(0, \theta)$, where θ can be any angle.

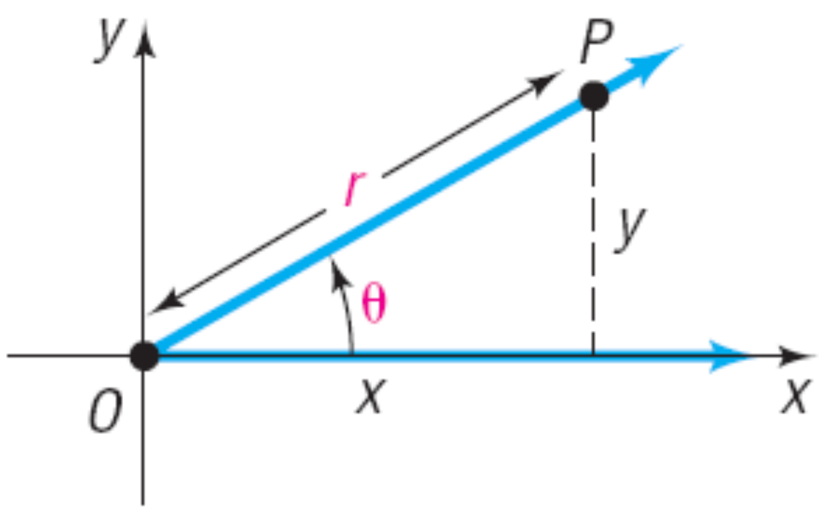
2 Convert from Polar Coordinates to Rectangular Coordinates

THEOREM

Conversion from Polar Coordinates to Rectangular Coordinates

If P is a point with polar coordinates (r, θ), the rectangular coordinates (x, y) of P are given by

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

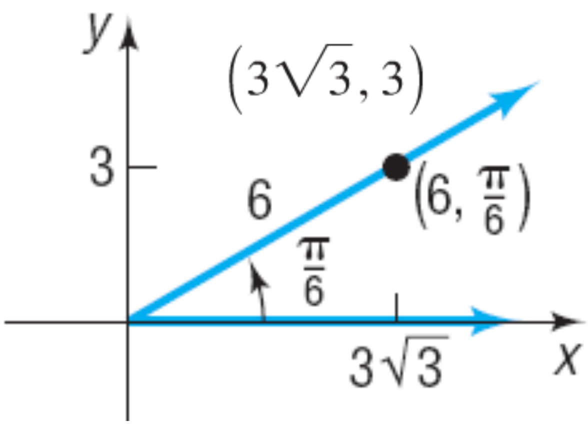


EXAMPLE

Converting from Polar Coordinates to Rectangular Coordinates

Find the rectangular coordinates of the points with the following polar coordinates:

(a) $\left(6, \frac{\pi}{6}\right)$ (b) $\left(-4, -\frac{\pi}{4}\right)$

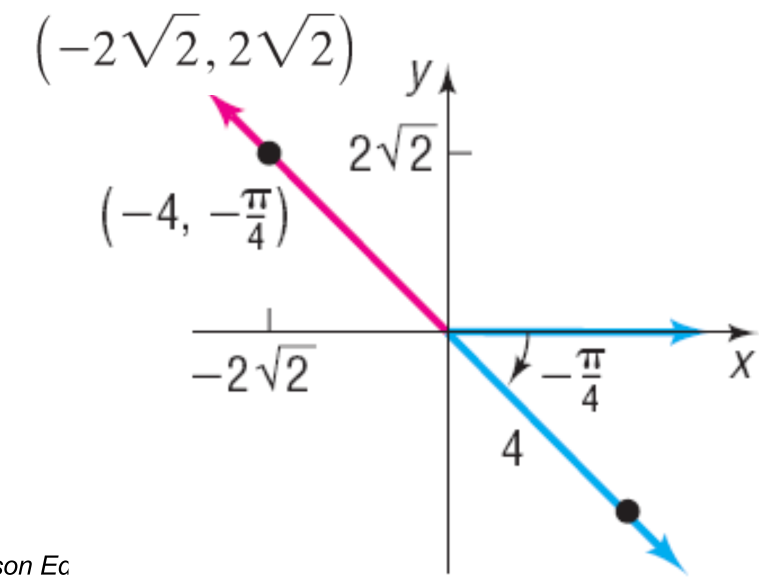


$$x = r\cos\theta = 6\cos\frac{\pi}{6} = 6\cdot\frac{\sqrt{3}}{2} = 3\sqrt{3}$$

$$y = r\sin\theta = 6\sin\frac{\pi}{6} = 6\cdot\frac{1}{2} = 3$$

$$x = r\cos\theta = -4\cos\left(-\frac{\pi}{4}\right) = -4\cdot\frac{\sqrt{2}}{2} = -2\sqrt{2}$$

$$y = r\sin\theta = -4\sin\left(-\frac{\pi}{4}\right) = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$



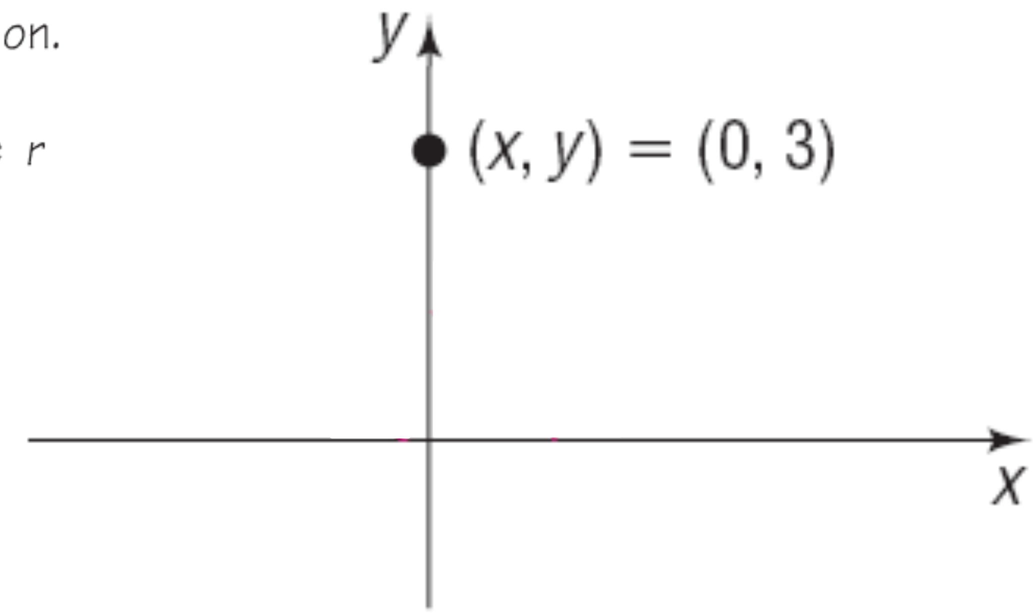
3 Convert from Rectangular Coordinates to Polar Coordinates

EXAMPLE

How to Convert from Rectangular Coordinates to Polar Coordinates with the Point on a Coordinate Axis

Find polar coordinates of a point whose rectangular coordinates are $(0, 3)$.

Step 1: Plot the point (x, y) and note the quadrant the point lies in or the coordinate axis the point lies on.



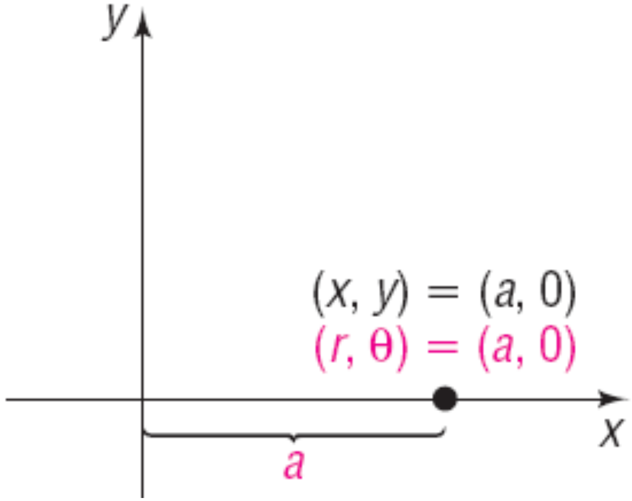
Step 2: Determine the distance r from the origin to the point.

Step 3: Determine θ.

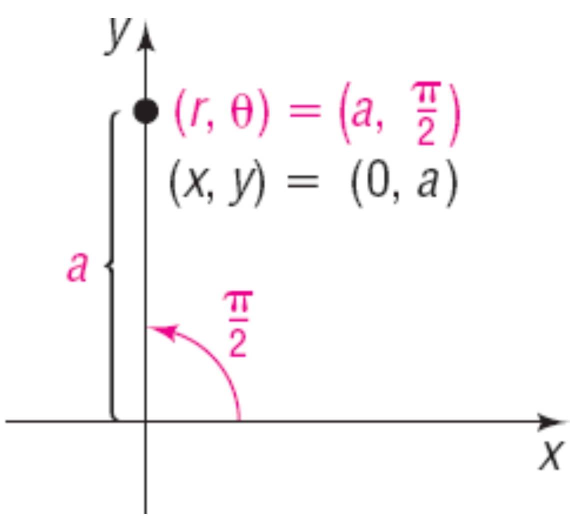
$$r = 3$$

$$\theta = \frac{\pi}{2}$$

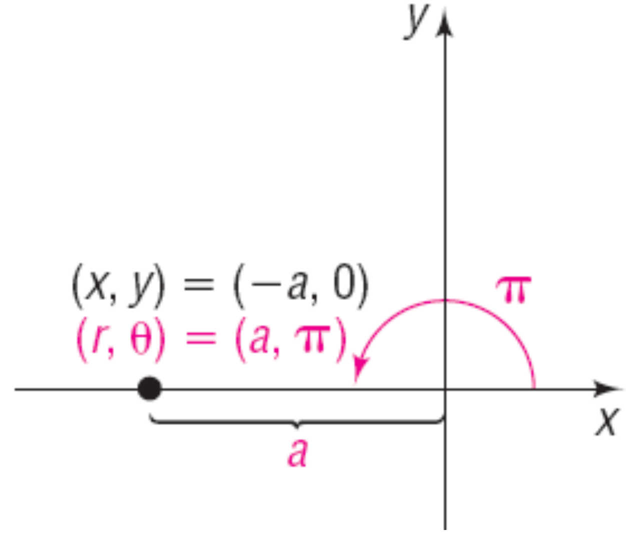
Polar coordinates for this point can be given by $\left(3, \frac{\pi}{2}\right)$. Other possible representations include $\left(-3, -\frac{\pi}{2}\right)$ and $\left(3, \frac{5\pi}{2}\right)$.



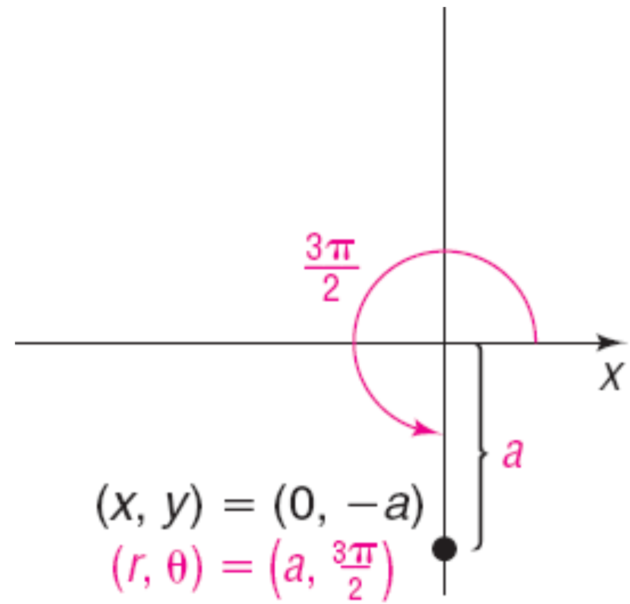
(a) $(x, y) = (a, 0), a > 0$



(b) $(x, y) = (0, a), a > 0$



(c) $(x, y) = (-a, 0), a > 0$

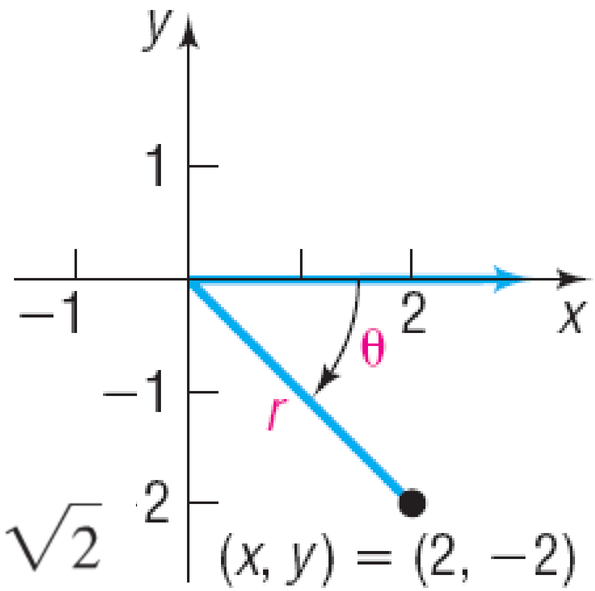


(d) $(x, y) = (0, -a), a > 0$

EXAMPLE How to Convert from Rectangular Coordinates to Polar Coordinates with the Point in a Quadrant

Find the polar coordinates of a point whose rectangular coordinates are $(2, -2)$.

Step 1: Plot the point (x, y) and note the quadrant the point lies in or the coordinate axis the point lies on.



Step 2: Determine the distance r from the origin to the point using $r = \sqrt{x^2 + y^2}$.

$$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

Step 3: Determine θ.

$$\tan\theta = \frac{y}{x}, \text{ so } \theta = \tan^{-1}\frac{y}{x}, -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

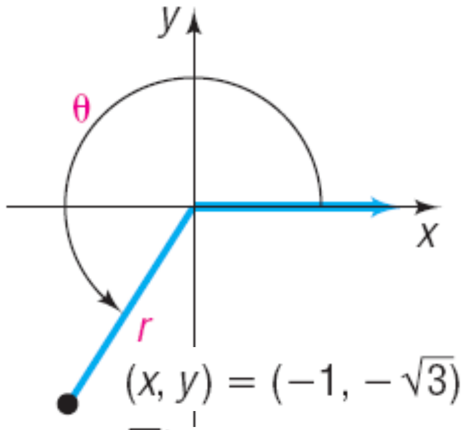
A set of polar coordinates for the point $(2, -2)$ is $\left(2\sqrt{2}, -\frac{\pi}{4}\right)$. Other possible representations include $\left(2\sqrt{2}, \frac{7\pi}{4}\right)$ and $\left(-2\sqrt{2}, \frac{3\pi}{4}\right)$.

EXAMPLE **Converting from Rectangular Coordinates to Polar Coordinates**

Find polar coordinates of a point whose rectangular coordinates are $\left(-1, -\sqrt{3}\right)$.

Step 1: See Figure 17. The point lies in quadrant III.

A set of polar coordinates for this point is $\left(2, \dfrac{4\pi}{3}\right)$.



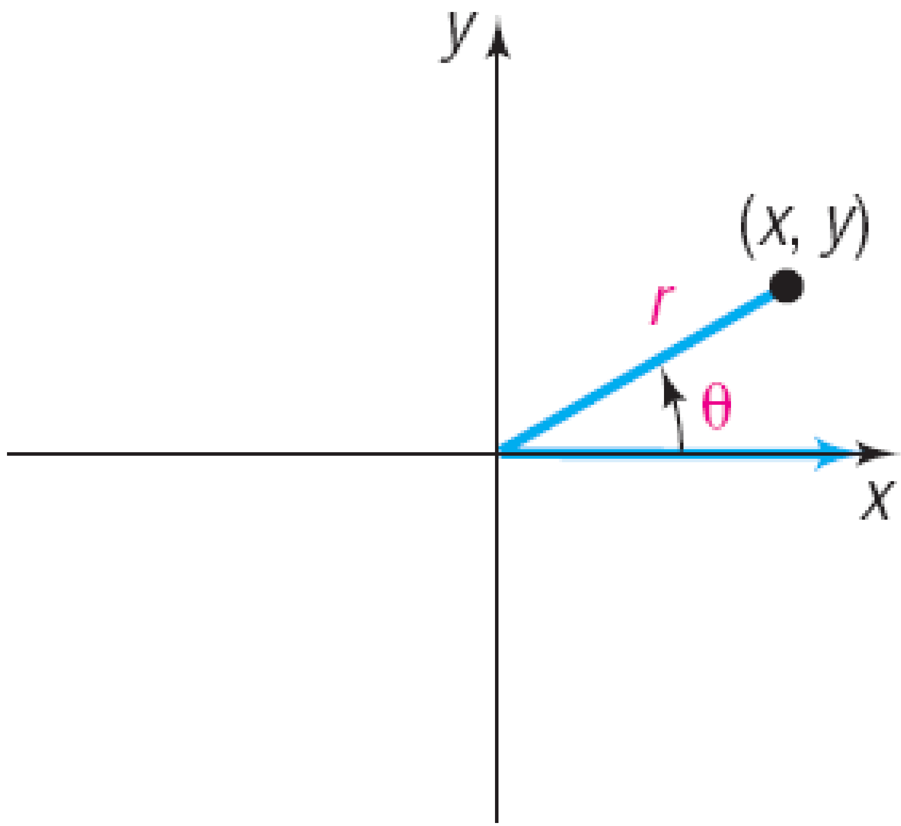
Step 2: The distance r from the origin to the point $\left(-1, -\sqrt{3}\right)$ is

$$r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

Step 3: To find θ, we use $\alpha = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{-\sqrt{3}}{-1} = \tan^{-1}\sqrt{3}, -\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}$.

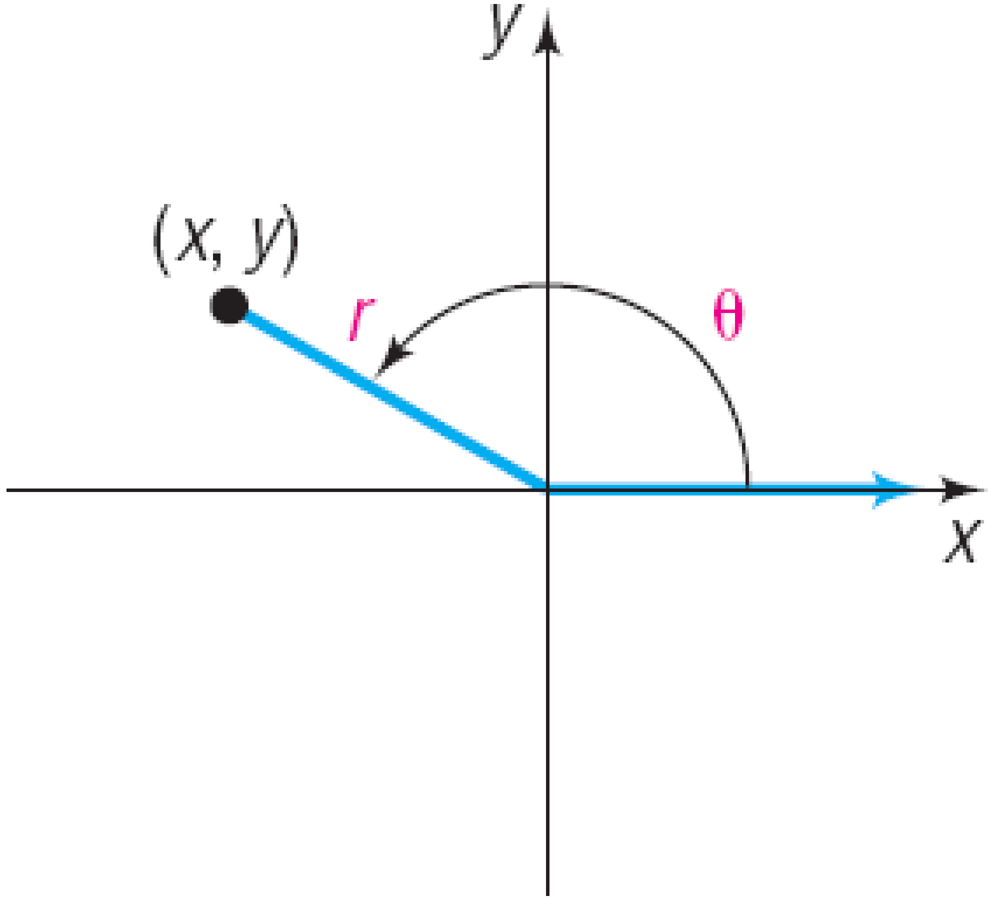
Since the point $\left(-1, -\sqrt{3}\right)$ lies in quadrant III and the inverse tangent function gives an angle in quadrant I, we add π to the result to obtain an angle in quadrant III.

$$\theta = \pi + \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \pi + \tan^{-1}\sqrt{3} = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

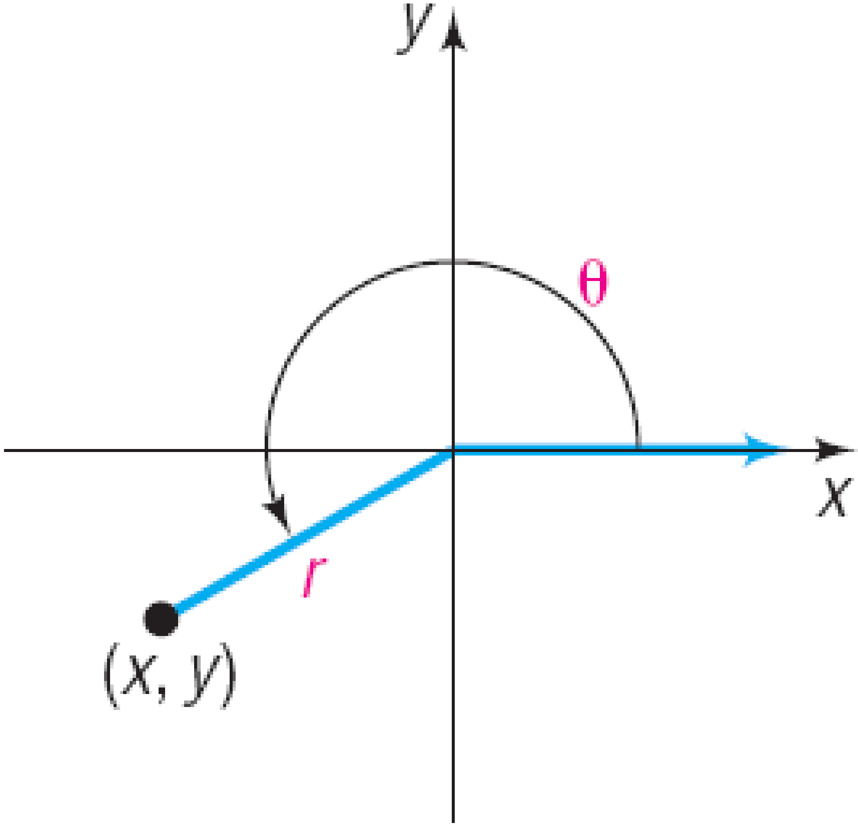


$$r = \sqrt{x^2 + y^2}$$

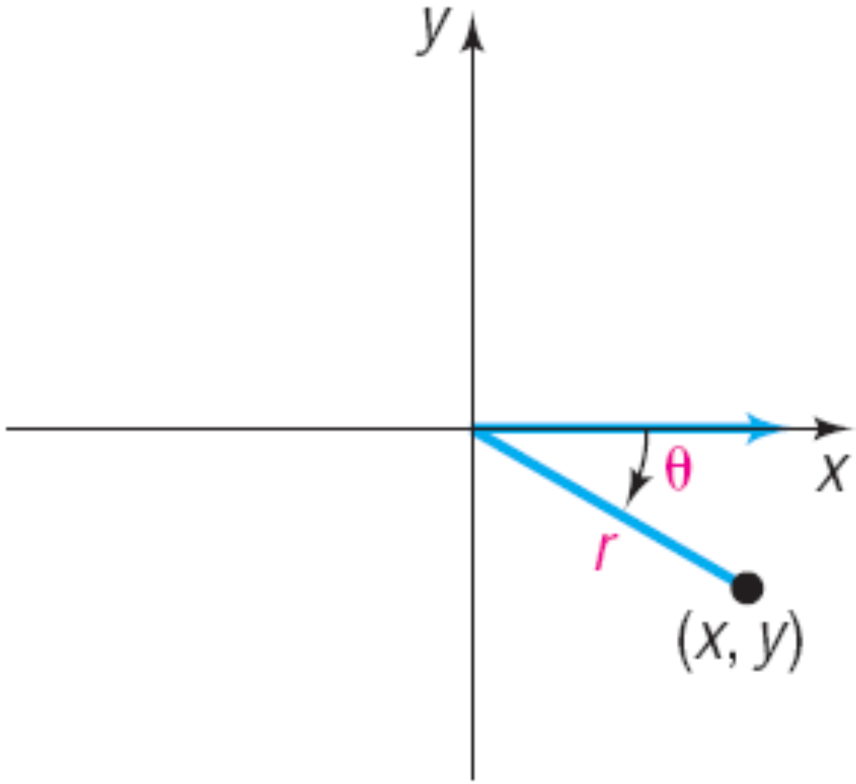
$$\theta = \tan^{-1} \frac{y}{x}$$



$$r = \sqrt{x^2 + y^2}$$
$$\theta = \pi + \tan^{-1}\frac{y}{x}$$



$$r = \sqrt{x^2 + y^2}$$
$$\theta = \pi + \tan^{-1}\frac{y}{x}$$



$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1} \frac{y}{x}$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad \text{if } x \neq 0$$

Steps for Converting from Rectangular to Polar Coordinates

STEP 1: Always plot the point (x, y) first, as we did in Examples 5, 6, and 7. Note the quadrant the point lies in or the coordinate axis the point lies on.

STEP 2: If $x = 0$ or $y = 0$, use your illustration to find r. If $x \neq 0$ and $y \neq 0$, then $r = \sqrt{x^2 + y^2}$.

STEP 3: Find θ. If $x = 0$ or $y = 0$, use your illustration to find θ. If $x \neq 0$ and $y \neq 0$, note the quadrant in which the point lies.

$$\text{Quadrant I or IV:} \quad \theta = \tan^{-1}\frac{y}{x}$$

$$\text{Quadrant II or III:} \quad \theta = \pi + \tan^{-1}\frac{y}{x}$$

4 Transform Equations between Polar and Rectangular Forms

EXAMPLE

Transforming an Equation from Polar to Rectangular Form

Transform the equation $r = 4\sin\theta$ from polar coordinates to rectangular coordinates, and identify the graph.

$$rr = 4r\sin\theta$$

$$x^2 + y^2 = 4y \quad \text{This is a circle.}$$

$$\boxed{r^2 = x^2 + y^2}$$

$$\boxed{y = r\sin\theta}$$

EXAMPLE

Transforming an Equation from Rectangular to Polar Form

Transform the equation $3x^2 + 3y^2 = 2x$ from rectangular to polar coordinates.

$$3\left(x^2 + y^2\right) = 2x$$

$$3r^2 = 2r\cos\theta$$

$$3r = 2\cos\theta$$

$$\boxed{r^2 = x^2 + y^2}$$

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$