Section 8.7

Product-to-Sum and Sum-to-Product Formulas

1 Express Products as Sums

THEOREM

Product-to-Sum Formulas

$$\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

EXAMPLE **Expressing Products as Sums**

Express each of the following products as a sum containing only sines or cosines.

(a) $$\sin(3\theta)\sin(7\theta) = \frac{1}{2}\left[\cos(3\theta-7\theta)-\cos(3\theta+7\theta)\right]$$

$$=\frac{1}{2}\left[\cos(-4\theta)-\cos(10\theta)\right]=\frac{1}{2}\left[\cos(4\theta)-\cos(10\theta)\right]$$

(b) $$\cos\theta\cos(5\theta) = \frac{1}{2}\left[\cos(\theta-5\theta)+\cos(\theta+5\theta)\right]$$

$$=\frac{1}{2}\left[\cos(-4\theta)+\cos(6\theta)\right]=\frac{1}{2}\left[\cos(4\theta)+\cos(6\theta)\right]$$

(c) $$\sin(2\theta)\cos(7\theta) = \frac{1}{2}\left[\sin(2\theta+7\theta)+\sin(2\theta-7\theta)\right]$$

$$=\frac{1}{2}\left[\sin(9\theta)+\sin(-5\theta)\right]$$

$$=\frac{1}{2}\left[\sin(9\theta)-\sin(5\theta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

2 Express Sums as Products

THEOREM

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

EXAMPLE **Expressing Sums (or Differences) as a Product**

Express each sum or difference as a product of sines and/or cosines.

(a) $$\sin(4\theta)-\sin(6\theta)=2\sin\frac{4\theta-6\theta}{2}\cos\frac{4\theta+6\theta}{2}$$

$$=2\sin\frac{-2\theta}{2}\cos\frac{10\theta}{2}=-2\sin\theta\cos 5\theta$$

(b) $$\cos(2\theta)+\cos(8\theta)=2\cos\frac{2\theta+8\theta}{2}\cos\frac{2\theta-8\theta}{2}$$

$$=2\cos\frac{10\theta}{2}\cos\frac{-6\theta}{2}$$

$$=2\cos 5\theta\cos 3\theta$$

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$