Section R.4

Polynomials

1 Recognize Monomials

A **monomial** in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form

$$\boxed{ax^k}$$

where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is called the **coefficient** of the monomial. If $a \neq 0$, then k is called the **degree** of the monomial.

EXAMPLE Examples of Monomials

Monomial	Coefficient	Degree	
(a) $6x^2$	6	2	
(b) $-\sqrt{2}x^3$	$-\sqrt{2}$	3	
(c) 3	3	0	Since $3 = 3 \cdot 1 = 3x^0, x \neq 0$
(d) $-5x$	-5	1	Since $-5x = -5x^1$
(e) x^4	1	4	Since $x^4 = 1 \cdot x^4$

EXAMPLE Examples of Nonmonomial Expressions

(a) $3x^{1/2}$ is not a monomial, since the exponent of the variable x is $\frac{1}{2}$ and $\frac{1}{2}$ is not a nonnegative integer.

(b) $4x^{-3}$ is not a monomial, since the exponent of the variable x is -3 and -3 is not a nonnegative integer.

2 Recognize Polynomials

A **polynomial** in one variable is an algebraic expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants,* called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is called the **leading coefficient,** $a_n x^n$ is called the **leading term,** and n is the **degree** of the polynomial.

EXAMPLE

Examples of Polynomials

Polynomial	Coefficients	Degree
$-8x^3 + 4x^2 - 6x + 2$	$-8, 4, -6, 2$	3
$3x^2 - 5 = 3x^2 + 0 \cdot x + (-5)$	$3, 0, -5$	2
$8 - 2x + x^2 = 1 \cdot x^2 + (-2)x + 8$	$1, -2, 8$	2
$5x + \sqrt{2} = 5x^1 + \sqrt{2}$	$5, \sqrt{2}$	1
$3 = 3 \cdot 1 = 3 \cdot x^0$	3	0
0	0	No degree

3 Add and Subtract Polynomials

EXAMPLE Adding Polynomials

Find the sum of the polynomials:

$$8x^3 - 2x^2 + 6x - 2 \quad \text{and} \quad 3x^4 - 2x^3 + x^2 + x$$

Horizontal Addition: The idea here is to group the like terms and then combine them.

$$(8x^3 - 2x^2 + 6x - 2) + (3x^4 - 2x^3 + x^2 + x)$$

$$= 3x^4 + (8x^3 - 2x^3) + (-2x^2 + x^2) + (6x + x) - 2$$

$$= 3x^4 + 6x^3 - x^2 + 7x - 2$$

Vertical Addition: The idea here is to vertically line up the like terms in each polynomial and then add the coefficients.

$$\begin{array}{rrrrrr}
 & x^4 & x^3 & x^2 & x^1 & x^0 \\
 & & 8x^3 & -\ 2x^2 & +\ 6x & -\ 2 \\
+ & 3x^4 & -\ 2x^3 & +\ x^2 & +\ x & \\
\hline
 & 3x^4 & +\ 6x^3 & -\ x^2 & +\ 7x & -\ 2
\end{array}$$

EXAMPLE Subtracting Polynomials

Find the difference: $(3x^4 - 4x^3 + 6x^2 - 1) - (2x^4 - 8x^2 - 6x + 5)$

Horizontal Subtraction:

$$
\begin{aligned}
&(3x^4 - 4x^3 + 6x^2 - 1) - (2x^4 - 8x^2 - 6x + 5)\\
&= 3x^4 - 4x^3 + 6x^2 - 1 + \underbrace{(-2x^4 + 8x^2 + 6x - 5)}_{\text{Be sure to change the sign of each term in the second polynomial.}}\\
&= (3x^4 - 2x^4) + (-4x^3) + (6x^2 + 8x^2) + 6x + (-1 - 5)\\
&= x^4 - 4x^3 + 14x^2 + 6x - 6
\end{aligned}
$$

Group like terms.

Vertical Subtraction: We line up like terms, change the sign of each coefficient of the second polynomial, and add.

$$
\begin{array}{rrrrrcrrrrr}
x^4 & x^3 & x^2 & x^1 & x^0 & & x^4 & x^3 & x^2 & x^1 & x^0\\
3x^4 & -\,4x^3 & +\,6x^2 & & -\,1 & = & 3x^4 & -\,4x^3 & +\,6x^2 & & -\,1\\
-\,[2x^4 & & -\,8x^2 & -\,6x & +\,5] & = + & -2x^4 & & +\,8x^2 & +\,6x & -\,5\\
\hline
 & & & & & & x^4 & -\,4x^3 & +\,14x^2 & +\,6x & -\,6
\end{array}
$$

4 Multiply Polynomials

EXAMPLE **Multiplying Polynomials**

Find the product: $(2x + 5)(x^2 - x + 2)$

Horizontal Multiplication:

$$(2x + 5)(x^2 - x + 2) = 2x(x^2 - x + 2) + 5(x^2 - x + 2)$$

Distributive Property

$$= (2x \cdot x^2 - 2x \cdot x + 2x \cdot 2) + (5 \cdot x^2 - 5 \cdot x + 5 \cdot 2)$$

Distributive Property

$$= (2x^3 - 2x^2 + 4x) + (5x^2 - 5x + 10)$$

Law of Exponents

$$= 2x^3 + 3x^2 - x + 10$$

Combine like terms.

EXAMPLE

Multiplying Polynomials

Find the product: $(2x + 5)(x^2 - x + 2)$

Vertical Multiplication: The idea here is very much like multiplying a two-digit number by a three-digit number.

$$
\begin{array}{rrrrl}
 & x^2 & - x & + 2 & \\
 & & 2x & + 5 & \\
\hline
2x^3 & - 2x^2 & + 4x & & \text{This line is } 2x(x^2 - x + 2). \\
(+) & 5x^2 & - 5x & + 10 & \text{This line is } 5(x^2 - x + 2). \\
\hline
2x^3 & + 3x^2 & - x & + 10 & \text{Sum of the above two lines.}
\end{array}
$$

5 Know Formulas for Special Products

FOIL

Outer

First

$$(ax + b)(cx + d) = ax(cx + d) + b(cx + d)$$

Inner

Last

$$= \overbrace{ax \cdot cx}^{\text{First}} + \overbrace{ax \cdot d}^{\text{Outer}} + \overbrace{b \cdot cx}^{\text{Inner}} + \overbrace{b \cdot d}^{\text{Last}}$$

$$= acx^2 + adx + bcx + bd$$

$$= acx^2 + (ad + bc)x + bd$$

EXAMPLE

Using FOIL

(a) $(x-3)(x+3) = x^2 + 3x - 3x - 9 = x^2 - 9$

F O I L

(b) $(x+2)^2 = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$

(c) $(x-3)^2$

$= (x-3)(x-3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$

(d) $(x+3)(x+1) = x^2 + x + 3x + 3 = x^2 + 4x + 3$

(e) $(2x+1)(3x+4)$

$= 6x^2 + 8x + 3x + 4 = 6x^2 + 11x + 4$

Difference of Two Squares

$$(x - a)(x + a) = x^2 - a^2$$

Squares of Binomials, or Perfect Squares

$$(x + a)^2 = x^2 + 2ax + a^2$$
$$(x - a)^2 = x^2 - 2ax + a^2$$

EXAMPLE **Using Special Product Formulas**

(a) $(x - 5)(x + 5) = x^2 - 5^2 = x^2 - 25$

Difference of two squares

(b) $(x + 7)^2 = x^2 + 2 \cdot 7 \cdot x + 7^2 = x^2 + 14x + 49$

Square of a binomial

(c) $(2x + 1)^2 = (2x)^2 + 2 \cdot 1 \cdot 2x + 1^2 = 4x^2 + 4x + 1$

Notice that we used 2x in place of x in formula

(d) $(3x - 4)^2 = (3x)^2 - 2 \cdot 4 \cdot 3x + 4^2 = 9x^2 - 24x + 16$

Replace x by 3x in formula

EXAMPLE Cubing a Binomial

(a) $(x+2)^3 = (x+2)(x+2)^2 = (x+2)(x^2+4x+4)$

$$= (x^3+4x^2+4x) + (2x^2+8x+8)$$

$$= x^3+6x^2+12x+8$$

(b) $(x-1)^3 = (x-1)(x-1)^2 = (x-1)(x^2-2x+1)$

$$= (x^3-2x^2+x) - (x^2-2x+1)$$

$$= x^3-3x^2+3x-1$$

Cubes of Binomials, or Perfect Cubes

$$\boxed{\begin{aligned}(x+a)^3 &= x^3+3ax^2+3a^2x+a^3\\ (x-a)^3 &= x^3-3ax^2+3a^2x-a^3\end{aligned}}$$

EXAMPLE Forming the Difference of Two Cubes

$$\begin{aligned}(x-1)(x^2+x+1) &= x(x^2+x+1)-1(x^2+x+1)\\ &= x^3+x^2+x-x^2-x-1\\ &= x^3-1\end{aligned}$$

EXAMPLE Forming the Sum of Two Cubes

$$\begin{aligned}(x+2)(x^2-2x+4) &= x(x^2-2x+4)+2(x^2-2x+4)\\ &= x^3-2x^2+4x+2x^2-4x+8\\ &= x^3+8\end{aligned}$$

Difference of Two Cubes

$$\boxed{(x-a)(x^2+ax+a^2)=x^3-a^3}$$

Sum of Two Cubes

$$\boxed{(x+a)(x^2-ax+a^2)=x^3+a^3}$$

6 Divide Polynomials Using Long Division

EXAMPLE

Divide 842 by 15.

$$\begin{array}{rrl}
 & 56 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow & 15\overline{)842} & \leftarrow \text{Dividend} \\
 & \underline{75} & \leftarrow 5 \cdot 15 \text{ (subtract)} \\
 & 92 & \\
 & \underline{90} & \leftarrow 6 \cdot 15 \text{ (subtract)} \\
 & 2 & \leftarrow \text{Remainder}
\end{array}$$

So, $\dfrac{842}{15} = 56 + \dfrac{2}{15}$.

$$\boxed{(\text{Quotient})(\text{Divisor}) + \text{Remainder} = \text{Dividend}}$$

EXAMPLE Dividing Two Polynomials

Find the quotient and the remainder when

$$3x^3 + 4x^2 + x + 7 \quad \text{is divided by} \quad x^2 + 1$$

Step 1: Divide the leading term of the dividend, $3x^3$, by the leading term of the divisor, x^2. Enter the result, $3x$, over the term $3x^3$, as follows:

$$\begin{array}{r} 3x\phantom{{}^3 + 4x^2 + x + 7} \\ x^2+1\overline{)\,3x^3 + 4x^2 + x + 7} \end{array}$$

Step 2: Multiply $3x$ by $x^2 + 1$ and enter the result below the dividend.

$$\begin{array}{rl} & 3x \\ x^2+1\big) & 3x^3 + 4x^2 + x + 7 \\ & \underline{3x^3 \phantom{{}+4x^2} + 3x \phantom{{}+7}} \end{array}$$

$\leftarrow 3x \cdot (x^2 + 1) = 3x^3 + 3x$

Notice that we align the $3x$ term under the x to make the next step easier.

EXAMPLE Dividing Two Polynomials

Find the quotient and the remainder when

$$3x^3 + 4x^2 + x + 7 \quad \text{is divided by} \quad x^2 + 1$$

Step 3: Subtract and bring down the remaining terms.

$$
\begin{array}{r|l}
 & \phantom{3x^3 + {}}3x \\ \cline{2-2}
x^2 + 1 & 3x^3 + 4x^2 + x + 7 \\
 & \underline{3x^3 \phantom{{}+ 4x^2} + 3x \phantom{{}+ 7}} \\
 & \phantom{3x^3 + {}} 4x^2 - 2x + 7
\end{array}
$$

← Subtract (change the signs and add).

← Bring down the $4x^2$ and the 7.

Step 4: Repeat Steps 1–3 using $4x^2 - 2x + 7$ as the dividend.

$$
\begin{array}{r|l}
 & 3x + 4 \\ \cline{2-2}
x^2 + 1 & 3x^3 + 4x^2 + x + 7 \\
 & \underline{3x^3 \phantom{{}+ 4x^2} + 3x \phantom{{}+ 7}} \\
 & \phantom{3x^3 + {}} 4x^2 - 2x + 7 \\
 & \phantom{3x^3 + {}} \underline{4x^2 \phantom{{}- 2x} + 4} \\
 & -2x + 3
\end{array}
$$

← Divide $4x^2$ by x^2 to get 4.

← Multiply $x^2 + 1$ by 4; subtract.

Since x^2 does not divide $-2x$ evenly (that is, the result is not a monomial), the process ends. The quotient is $3x + 4$, and the remainder is $-2x + 3$.

EXAMPLE Dividing Two Polynomials

Find the quotient and the remainder when

$$x^4 - 3x^3 + 2x - 5 \quad \text{is divided by} \quad x^2 - x + 1$$

$$
\begin{array}{rrrl}
 & & x^2 - 2x - 3 & \leftarrow \text{Quotient} \\ \cline{3-3}
\text{Divisor} \rightarrow & x^2 - x + 1 & \big) \, x^4 - 3x^3 + 2x - 5 & \leftarrow \text{Dividend} \\
\text{Subtract} \rightarrow & & x^4 - x^3 + x^2 & \\ \cline{3-3}
 & & -2x^3 - x^2 + 2x - 5 & \\
\text{Subtract} \rightarrow & & -2x^3 + 2x^2 - 2x & \\ \cline{3-3}
 & & -3x^2 + 4x - 5 & \\
\text{Subtract} \rightarrow & & -3x^2 + 3x - 3 & \\ \cline{3-3}
 & & x - 2 & \leftarrow \text{Remainder}
\end{array}
$$

THEOREM

Let Q be a polynomial of positive degree and let P be a polynomial whose degree is greater than or equal to the degree of Q. The remainder after dividing P by Q is either the zero polynomial or a polynomial whose degree is less than the degree of the divisor Q.

7 Work with Polynomials in Two Variables

EXAMPLE

Examples of Polynomials in Two Variables

$3x^2 + 2x^3y + 5$
Two variables, degree is 4.

$\pi x^3 - y^2$
Two variables, degree is 3.

$x^4 + 4x^3y - xy^3 + y^4$
Two variables, degree is 4.

EXAMPLE

Using a Special Product Formula

To multiply $(2x - y)^2$ use the Squares of Binomials formula with $2x$ instead of x and y instead of a.

$$(2x - y)^2 = (2x)^2 - 2 \cdot y \cdot 2x + y^2 = 4x^2 - 4xy + y^2$$