Section 8.6

Double-angle and Half-angle Formulas

THEOREM

Double-angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

1 Use Double-angle Formulas to Find Exact Values

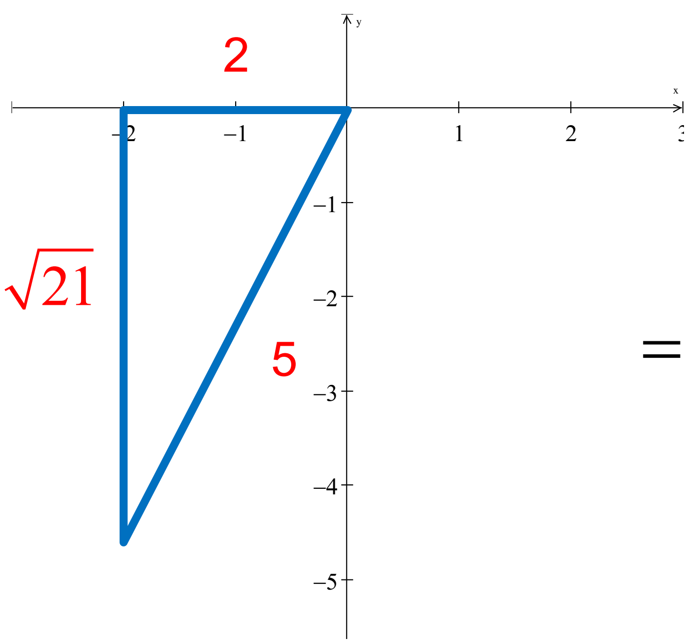
EXAMPLE Finding Exact Values Using the Double-angle Formulas

If $\cos\theta = -\frac{2}{5}, \pi < \theta < \frac{3\pi}{2},$ find the exact value of:

(a) $\sin(2\theta)$ (b) $\cos(2\theta)$

$$y = \sqrt{5^2 - 2^2} = \sqrt{21}$$

(a) $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right) = \frac{4\sqrt{21}}{25}$



(b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \left(-\frac{2}{5}\right)^2 - \left(-\frac{\sqrt{21}}{5}\right)^2 = \frac{4}{25} - \frac{21}{25} = -\frac{17}{25}$$

2 Use Double-angle Formulas to Establish Identities

EXAMPLE **Establishing Identities**

(a) Develop a formula for $\tan(2\theta)$ in terms of $\tan\theta$.

(b) Develop a formula for $\sin(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

(a) In the sum formula for $\tan(\alpha + \beta)$, let $\alpha = \beta = \theta$.

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\boxed{\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}}$$

(b) To get a formula for $\sin(3\theta)$, we write 3θ as $2\theta + \theta$ and use the sum formula.

$$\begin{aligned}
\sin(3\theta) &= \sin(2\theta + \theta) = \sin(2\theta)\cos\theta + \cos(2\theta)\sin\theta \\
&= (2\sin\theta\cos\theta)(\cos\theta) + (\cos^2\theta - \sin^2\theta)(\sin\theta) \\
&= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta \\
&= 3\sin\theta\cos^2\theta - \sin^3\theta
\end{aligned}$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\boxed{\sin^2\theta = \frac{1 - \cos(2\theta)}{2}}$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\boxed{\cos^2\theta = \frac{1 + \cos(2\theta)}{2}}$$

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\dfrac{1 - \cos(2\theta)}{2}}{\dfrac{1 + \cos(2\theta)}{2}}$$

$$\boxed{\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}}$$

EXAMPLE **Establishing an Identity**

Write an equivalent expression for $\cos^4\theta$ that does not involve any powers of sine or cosine greater than 1.

$$\cos^4\theta = (\cos^2\theta)^2 = \left(\frac{1+\cos(2\theta)}{2}\right)^2 = \frac{1}{4}[1 + 2\cos(2\theta) + \cos^2(2\theta)]$$

$$= \frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta = \frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\left\{\frac{1+\cos[2(2\theta)]}{2}\right\}$$

$$= \frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}[1+\cos(4\theta)] \quad = \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$$

$$\boxed{\cos^2\theta = \frac{1+\cos(2\theta)}{2}}$$

EXAMPLE **Solving a Trigonometric Equation Using Identities**

Solve the equation: $\cos 2\theta \cos\theta - \sin 2\theta \sin\theta = \frac{1}{2}$

$$\cos 2\theta \cos\theta - \sin 2\theta \sin\theta = \cos\left(2\theta + \theta\right) = \cos 3\theta = \frac{1}{2}$$

$$\cos 2\theta \cos\theta - \sin 2\theta \sin\theta = \cos\left(2\theta + \theta\right) = \cos 3\theta = \frac{1}{2}$$

$$3\theta = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad 3\theta = \frac{2\pi}{3} + 2k\pi$$

$$\theta = \frac{\pi}{9} + \frac{2k\pi}{3} \quad \text{or} \quad \theta = \frac{2\pi}{9} + \frac{2k\pi}{3}$$

$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

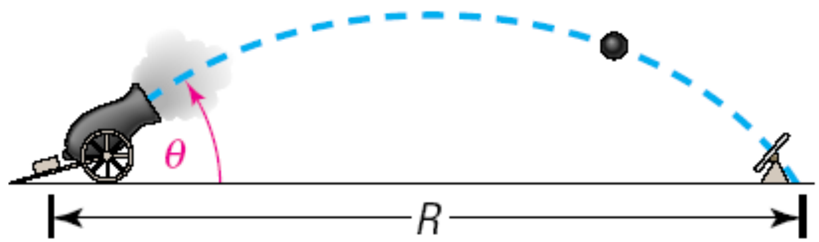
EXAMPLE Projectile Motion

An object is propelled upward at an angle θ to the horizontal with an initial velocity of v_0 feet per second. See Figure 28. If air resistance is ignored, the **range** R, the horizontal distance that the object travels, is given by

$$R = \frac{1}{16} v_0^2 \sin\theta \cos\theta$$

(a) Show that $R = \dfrac{1}{32} v_0^2 \sin(2\theta)$.

(b) Find the angle θ for which R is a maximum.



$$R(\theta) = \frac{1}{16} v_0^2 \sin\theta\cos\theta = \frac{1}{16} v_0^2 \frac{2\sin\theta\cos\theta}{2} = \frac{1}{32} v_0^2 \sin(2\theta)$$

Since the largest value of a sine function is 1, occurring when the argument 2θ is $90°$, it follows that for maximum R we must have

$$2\theta = 90°$$
$$\theta = 45°$$

An inclination to the horizontal of $45°$ results in the maximum range.

3 Use Half-angle Formulas to Find Exact Values

$$\sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2} \qquad \cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2} \qquad \tan^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{1 + \cos\alpha}$$

THEOREM

Half-angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}}$$

EXAMPLE **Finding Exact Values Using Half-angle Formulas**

Use a Half-angle Formula to find the exact value of:

(a) $\sin 22.5°$ (b) $\cos \dfrac{5\pi}{12}$

(a) $$\sin 22.5° = \sin \frac{45°}{2} = \frac{\sqrt{1-\cos 45°}}{2} = \frac{\sqrt{1-\frac{\sqrt{2}}{2}}}{2} = \sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{2}}}{2}$$

(b) $$\cos \frac{5\pi}{12} = \cos \frac{\frac{5\pi}{6}}{2} = \frac{\sqrt{1+\cos \frac{5\pi}{6}}}{2}$$

$$= \frac{\sqrt{1-\frac{\sqrt{3}}{2}}}{2} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

EXAMPLE Finding Exact Values Using Half-angle Formulas

If $\cos\alpha = -\dfrac{1}{5}, \dfrac{\pi}{2} < \alpha < \pi,$ find the exact value of:

(a) $$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{1}{5}\right)}{2}} = \sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{15}}{5}$$

(b) $$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+\left(-\frac{1}{5}\right)}{2}} = \sqrt{\frac{2}{5}} = \frac{\sqrt{2}}{\sqrt{5}} = \frac{\sqrt{10}}{5}$$

(c) $$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\frac{\sqrt{15}}{5}}{\frac{\sqrt{10}}{5}} = \frac{\sqrt{15}}{\sqrt{10}} = \frac{\sqrt{150}}{10} = \frac{5\sqrt{6}}{10} = \frac{\sqrt{6}}{2}$$

$$\frac{\pi}{2} < \alpha < \pi, \text{ so } \frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2} \text{ so } \frac{\alpha}{2} \text{ is in quadrant I}$$

Half-angle Formulas for $\tan \frac{\alpha}{2}$

$$\tan\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 + \cos\alpha}$$