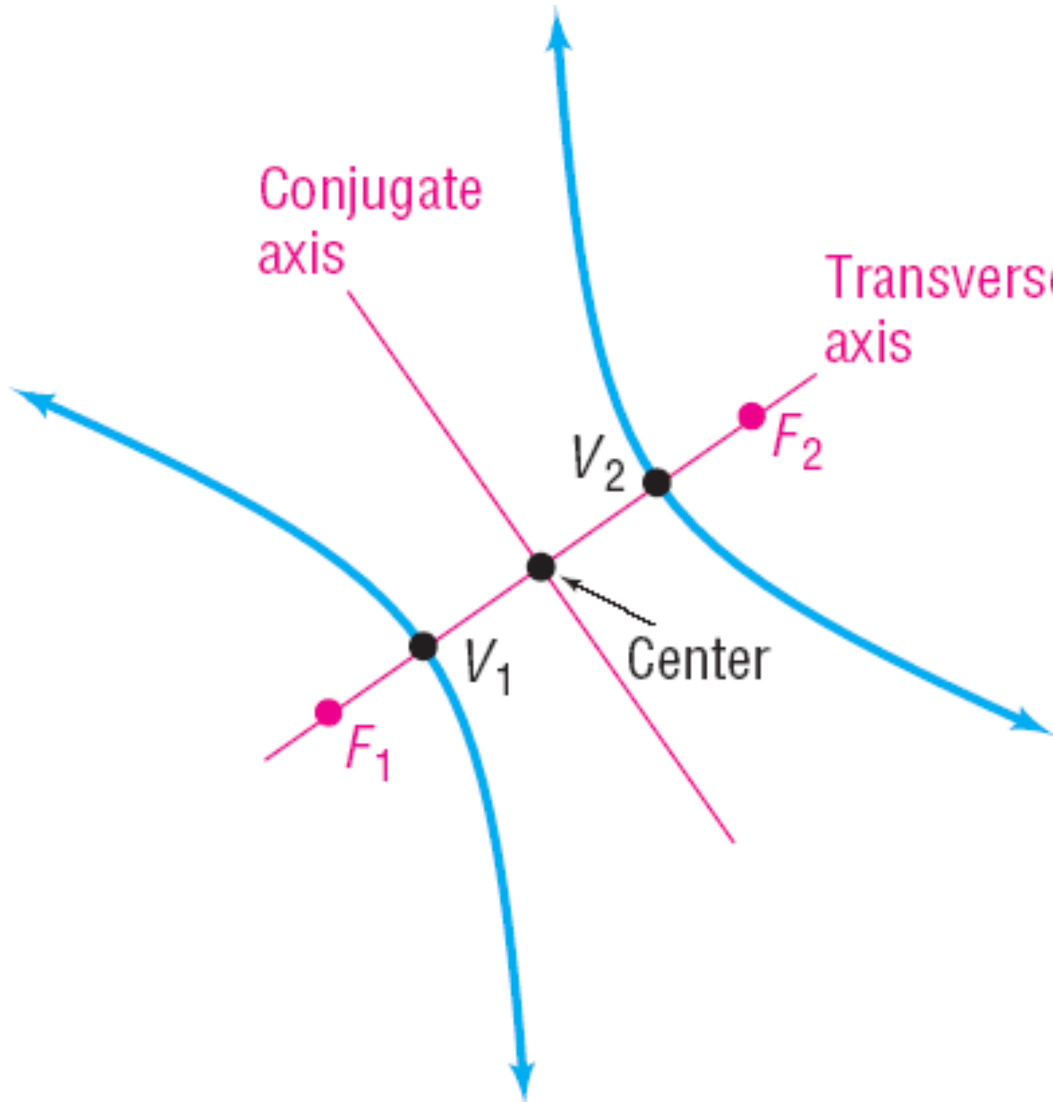
Section 11.4

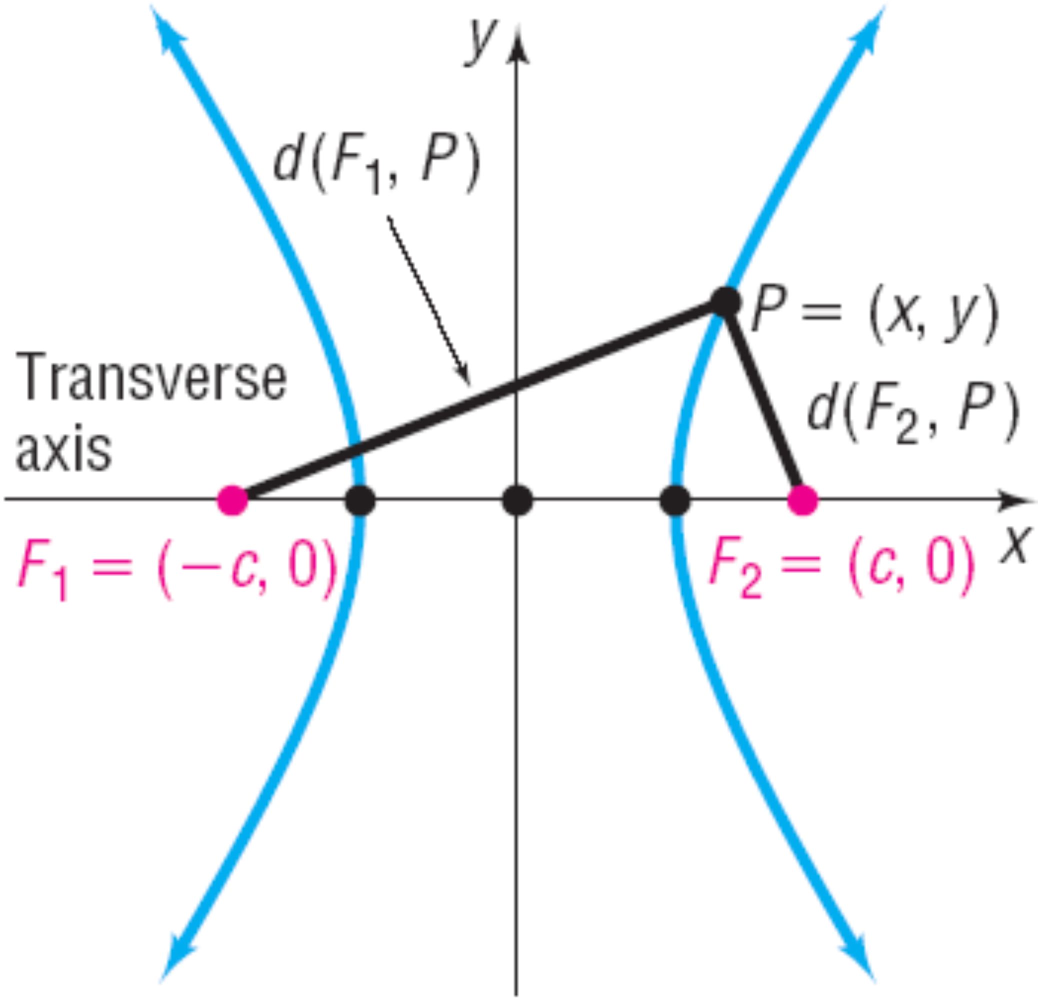
The Hyperbola

A **hyperbola** is the collection of all points in the plane the difference of whose distances from two fixed points, called the **foci**, is a constant.



1 Analyze Hyperbolas with Center at the Origin

$$d(F_1, P) - d(F_2, P) = \pm 2a$$

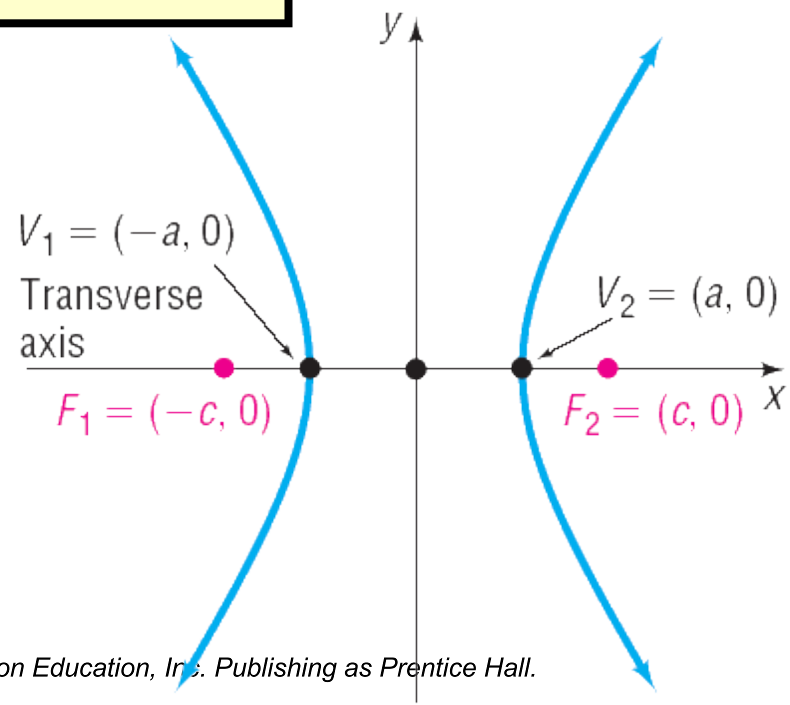


Equation of a Hyperbola Center at (0, 0) Transverse Axis along the x-Axis

An equation of the hyperbola with center at (0, 0), foci at $(-c, 0)$ and $(c, 0)$, and vertices at $(-a, 0)$ and $(a, 0)$ is

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \text{where } b^2 = c^2 - a^2}$$

The transverse axis is the x-axis.



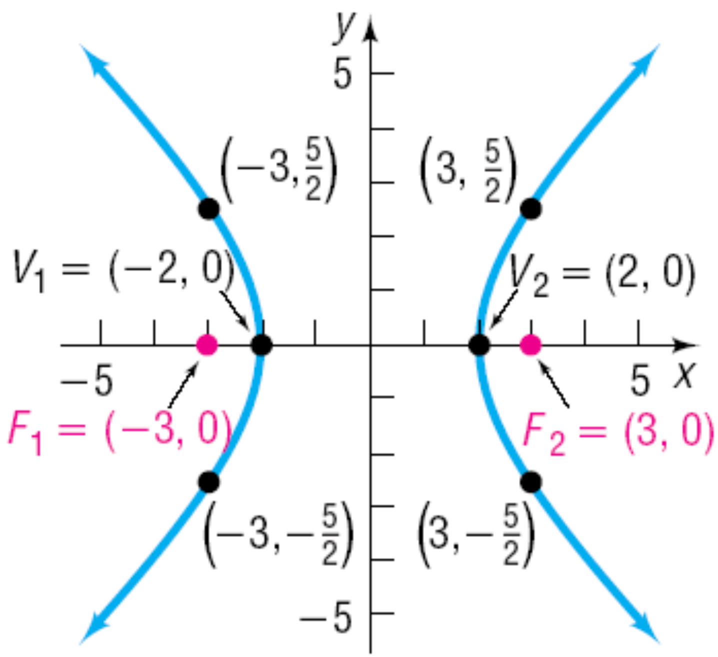
EXAMPLE Finding and Graphing an Equation of a Hyperbola

Finding an equation of the hyperbola with center at the origin, one focus at (3, 0), and one vertex at (–2, 0). Graph the equation.

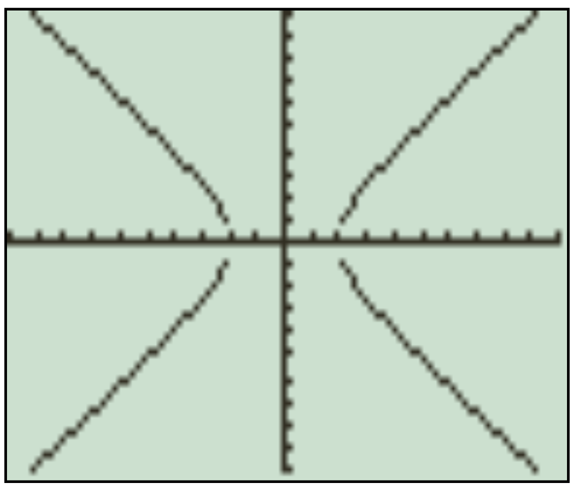
Distance from center to focus is $c = 3$
Distance from center to vertex is $a = 2$.

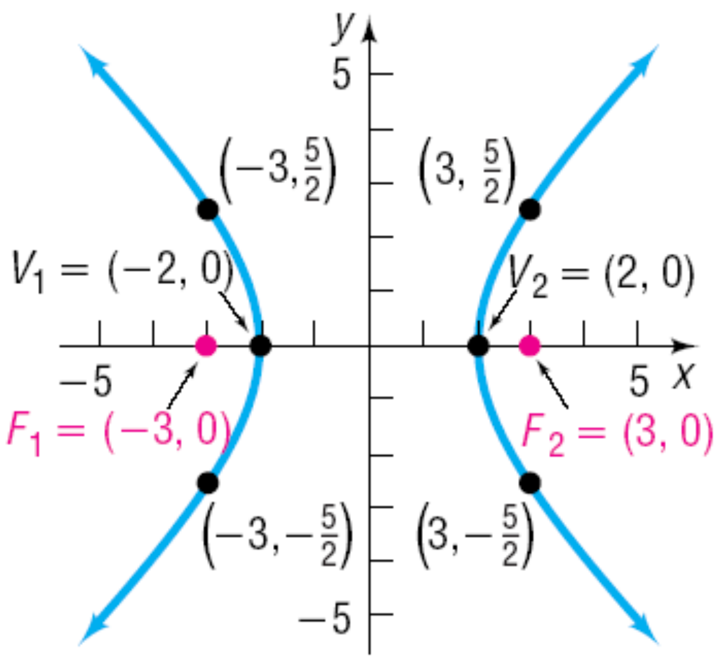
$$b^2 = c^2 - a^2 = 9 - 4 = 5$$

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$



COMMENT To graph the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ discussed in Example 1, we need to graph the two functions $Y_1 = \sqrt{5}\sqrt{\frac{x^2}{4} - 1}$ and $Y_2 = -\sqrt{5}\sqrt{\frac{x^2}{4} - 1}$. Do this and compare what you see with Figure 34. ■





EXAMPLE Analyzing the Equation of a Hyperbola

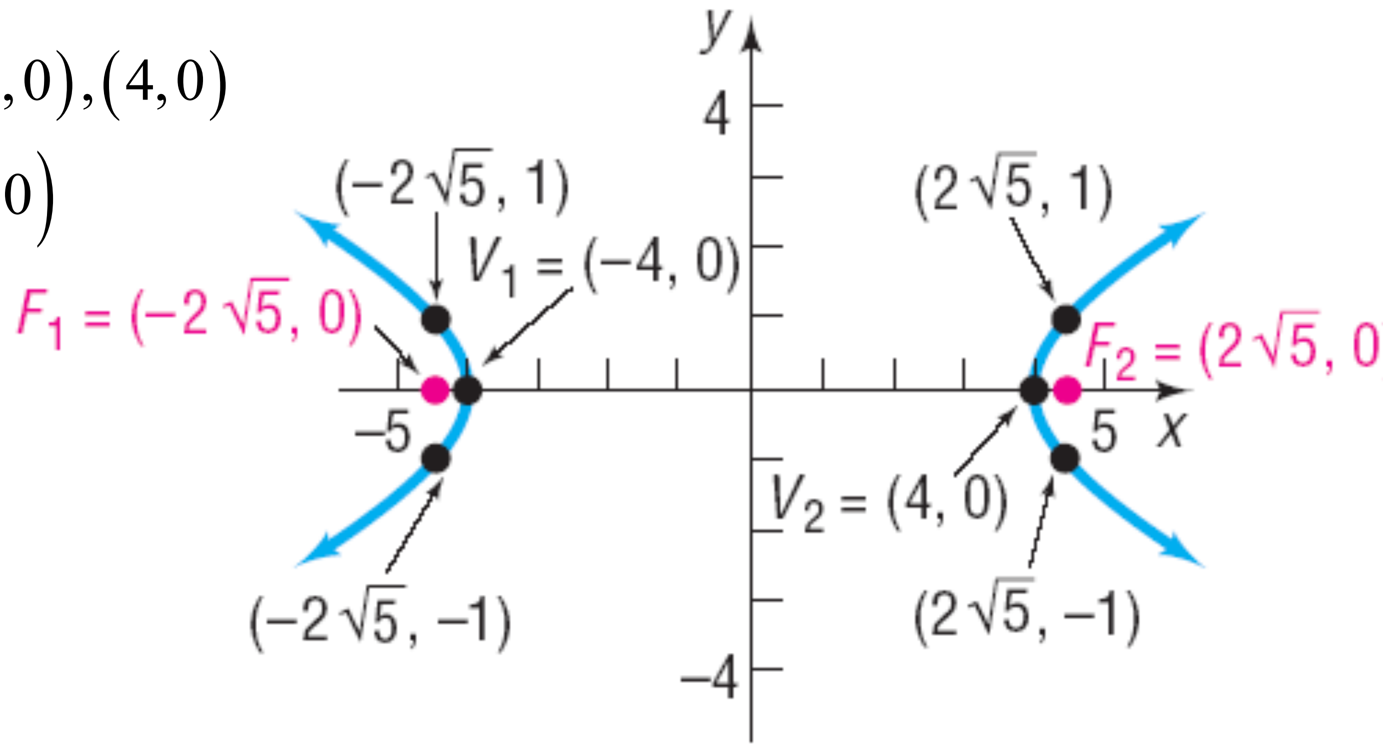
Analyze the equation: $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

$a^2 = 16$ and $b^2 = 4$ $\qquad$ $c^2 = a^2 + b^2 = 16 + 4 = 20$

Center: $(0,0)$

Vertices: $(-4,0),(4,0)$

Foci: $\left(\pm 2\sqrt{5},0\right)$

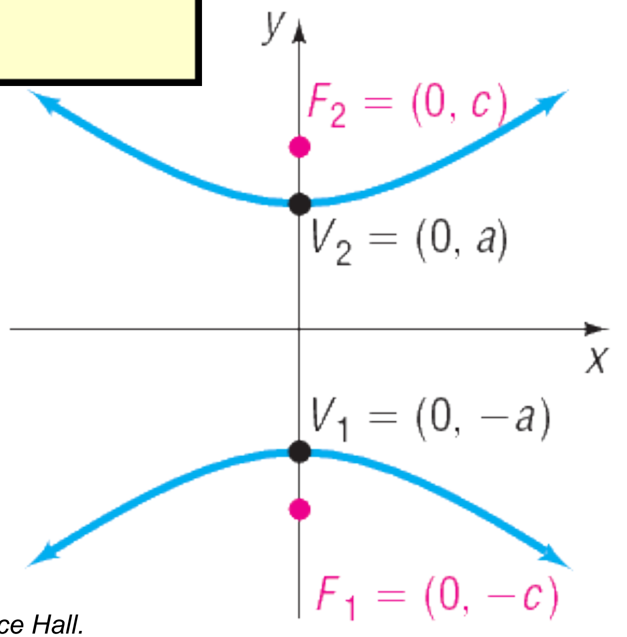


Equation of a Hyperbola; Center at (0, 0); Transverse Axis along the *y*-Axis

An equation of the hyperbola with center at (0, 0), foci at $(0, -c)$ and $(0, c)$, and vertices at $(0, -a)$ and $(0, a)$ is

$$\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \qquad \text{where } b^2 = c^2 - a^2}$$

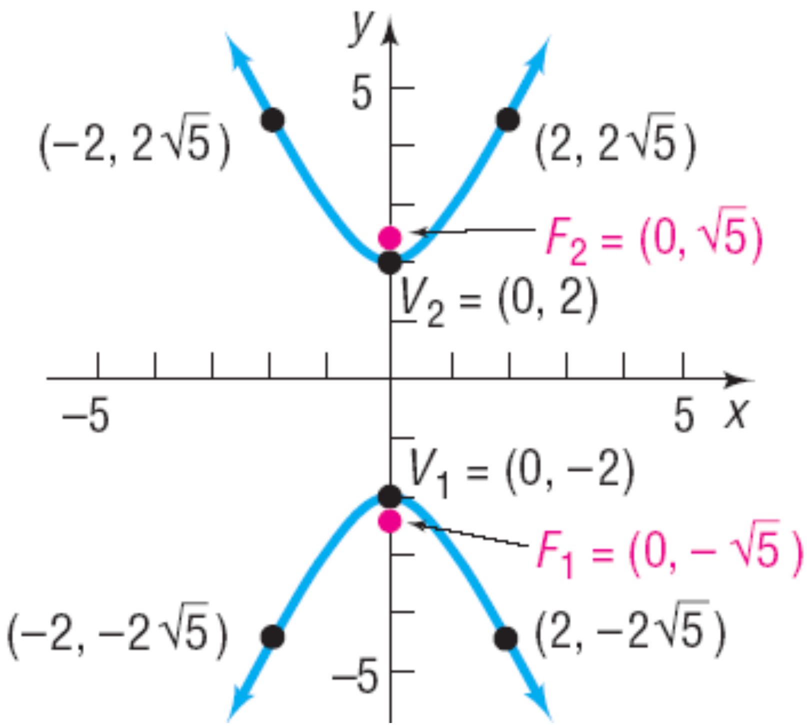
The transverse axis is the y-axis.



EXAMPLE Analyzing the Equation of a Hyperbola

Analyze the equation: $y^2 - 4x^2 = 4$ $\qquad \dfrac{y^2}{4} - x^2 = 1$

$a^2 = 4, b^2 = 1 \qquad c^2 = a^2 + b^2 = 5$



Center: $(0,0)$

Vertices: $(0,-2),(0,2)$

Foci: $\left(0,\pm\sqrt{5}\right)$

EXAMPLE Finding an Equation of a Hyperbola

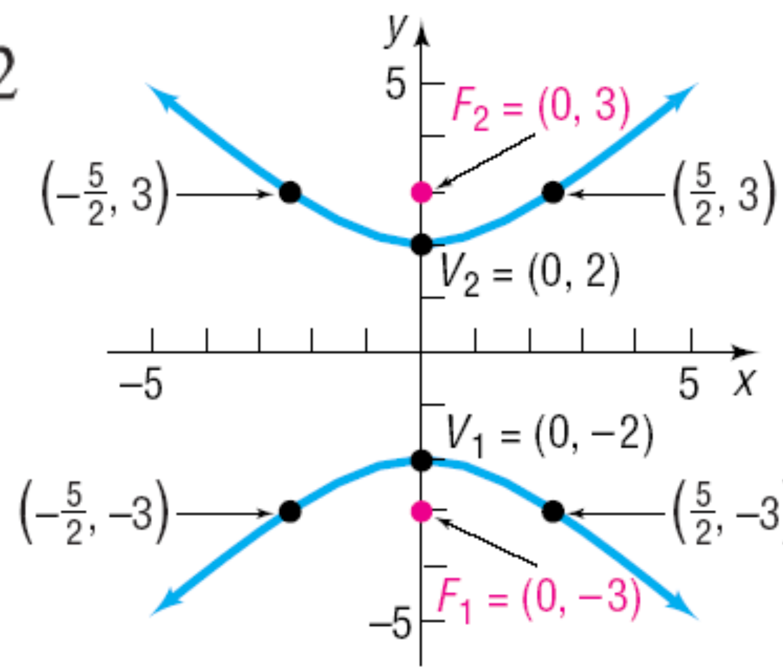
Find an equation of the hyperbola having one vertex at (0, 2) and foci at (0, –3) and (0, 3). Graph the equation.

Looking at the points given we see that the center is at (0,0) and the transverse axis is along the *y*-axis.

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad c = 3, a = 2$$

$$b^2 = c^2 - a^2 = 9 - 4 = 5$$

$$\frac{y^2}{4} - \frac{x^2}{5} = 1$$

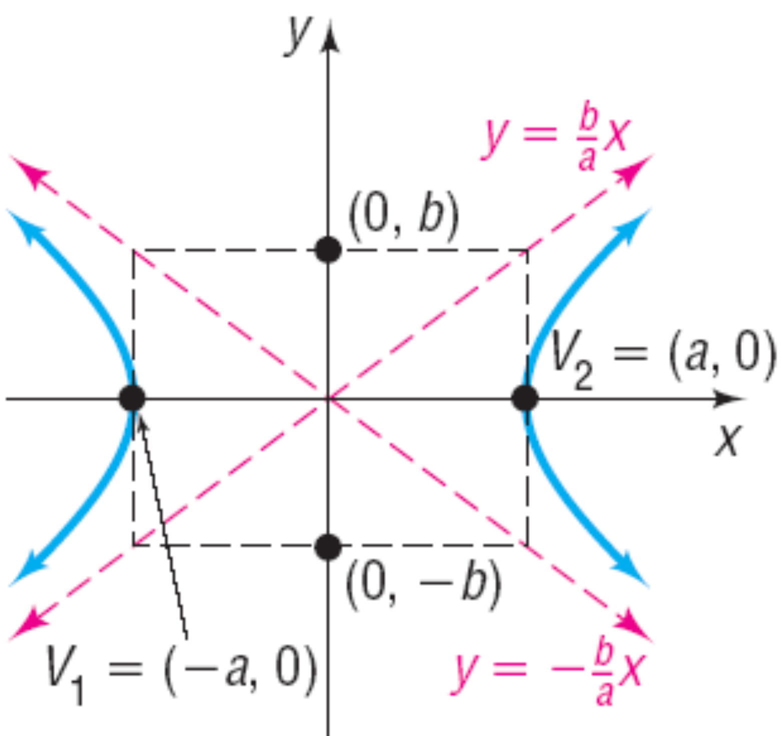


2 Find the Asymptotes of a Hyperbola

Asymptotes of a Hyperbola

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has the two oblique asymptotes

$$\boxed{y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x}$$



Asymptotes of a Hyperbola

The hyperbola $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ has the two oblique asymptotes

$$\boxed{y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x}$$

EXAMPLE Analyzing the Equation of a Hyperbola

Analyze the equation: $\dfrac{y^2}{4} - x^2 = 1$

Since the y^2 term is positive, the transverse axis is along the y-axis.

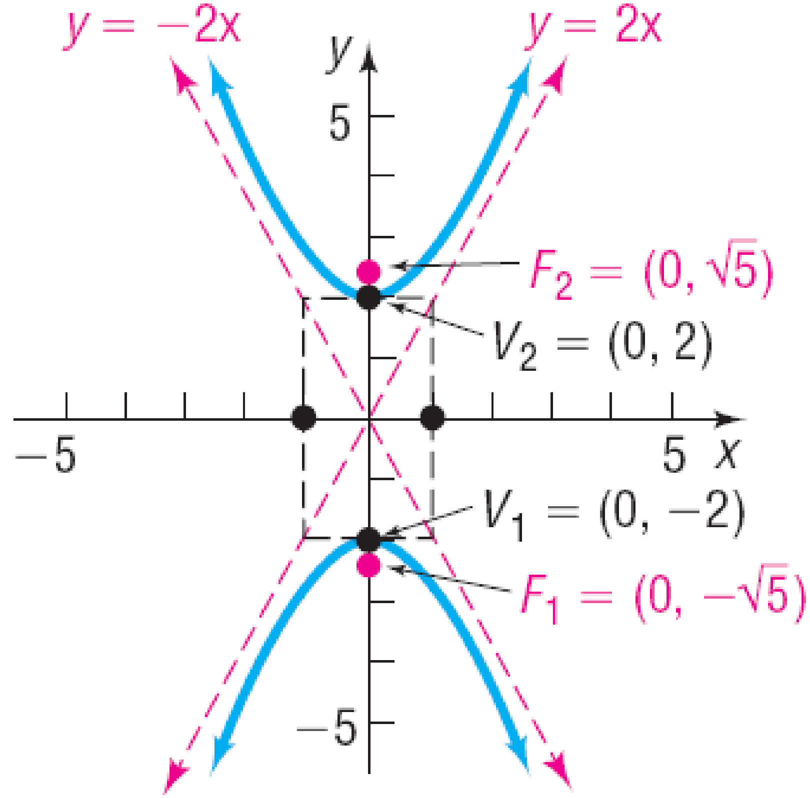
$a^2 = 4, b^2 = 1$

$c^2 = a^2 + b^2 = 5$

Center: $(0,0)$

Vertices: $(0,-2),(0,2)$

Foci: $\left(0,\pm\sqrt{5}\right)$



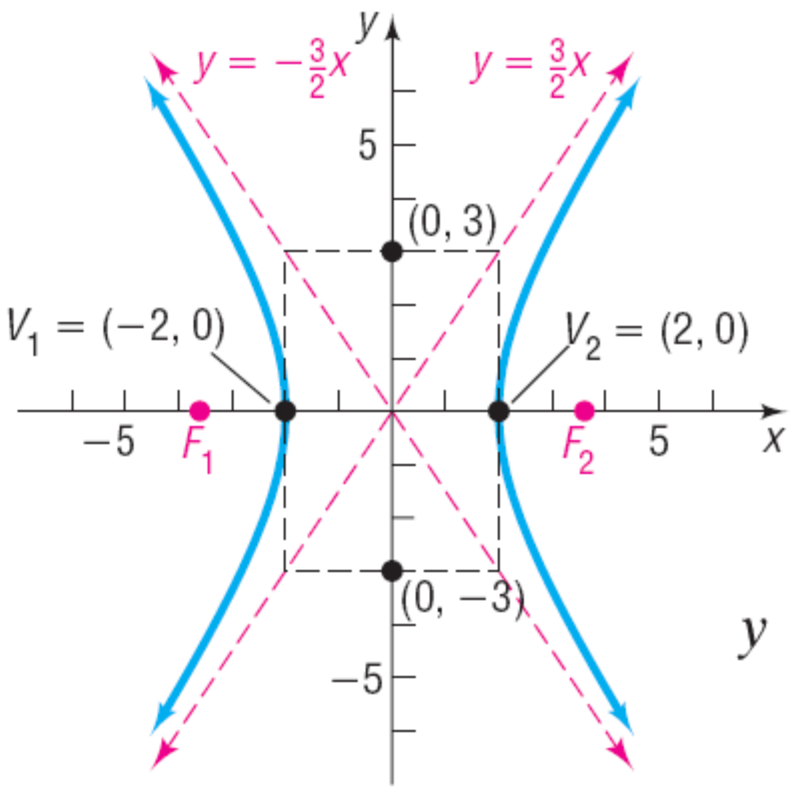
asymptotes are the lines $y = \dfrac{a}{b}x = 2x$ and $y = -\dfrac{a}{b}x = -2x$.

EXAMPLE **Analyzing the Equation of a Hyperbola**

Analyze the equation: $9x^2 - 4y^2 = 36$ $\quad \dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

$$a^2 = 4, b^2 = 9$$

$$c^2 = a^2 + b^2 = 13$$



Center: $(0,0)$

Vertices: $(-2,0),(2,0)$

Foci: $\left(\pm\sqrt{13},0\right)$

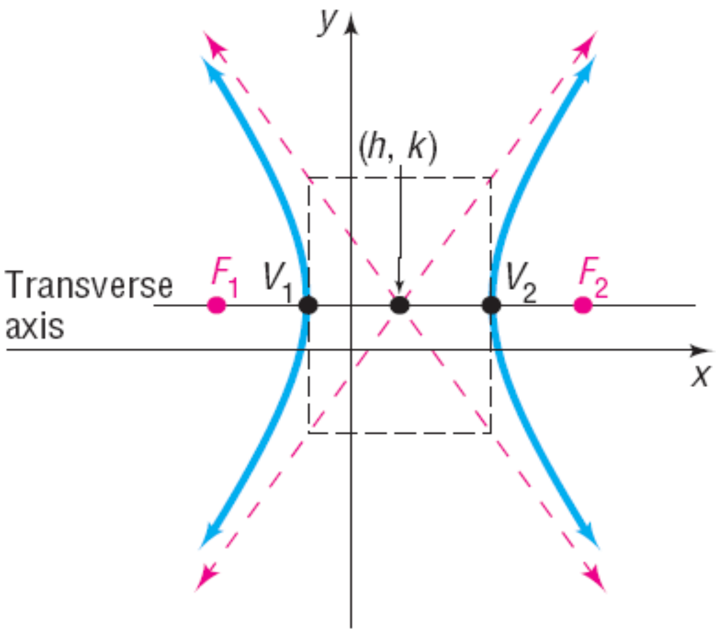
Asymptotes:

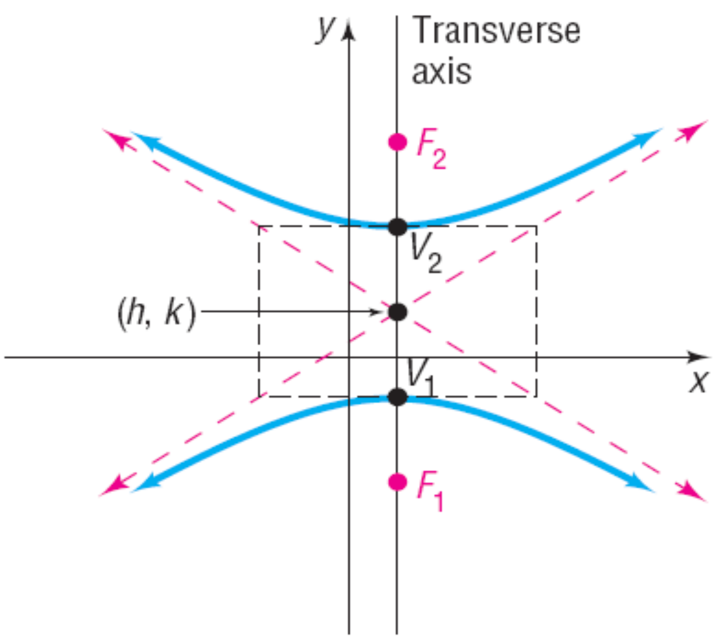
$$y = \frac{b}{a}x = \frac{3}{2}x \quad \text{and} \quad y = -\frac{b}{a}x = -\frac{3}{2}x$$

3 Analyze Hyperbolas with Center at (h, k)

HYPERBOLAS WITH CENTER AT (h, k) AND TRANSVERSE AXIS PARALLEL TO A COORDINATE AXIS

Center	Transverse Axis	Foci	Vertices	Equation	Asymptotes
(h, k)	Parallel to the x-axis	$(h \pm c, k)$	$(h \pm a, k)$	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1, \quad b^2 = c^2 - a^2$	$y - k = \pm\dfrac{b}{a}(x - h)$
(h, k)	Parallel to the y-axis	$(h, k \pm c)$	$(h, k \pm a)$	$\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1, \quad b^2 = c^2 - a^2$	$y - k = \pm\dfrac{a}{b}(x - h)$





EXAMPLE **Finding an Equation of a Hyperbola, Center Not at the Origin**

Find an equation for the hyperbola with center at (1, –2), one focus at (4, –2), and one vertex at (3, –2). Graph the equation by hand.

Center, focus and vertex are on $y = -2$ so tranvserse axis is parallel to x-axis.

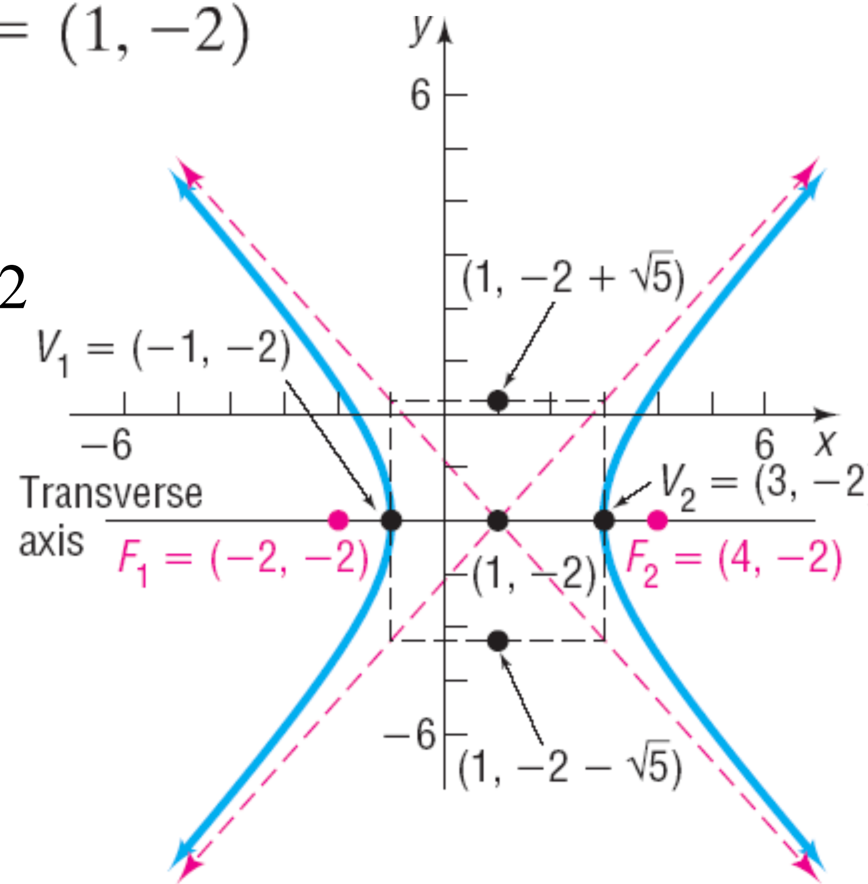
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad (h, k) = (1, -2)$$

Distance from center to a focus is $c = 3$

Distance from center to a vertex is $a = 2$

$$b^2 = c^2 - a^2 = 9 - 4 = 5.$$

$$\frac{(x-1)^2}{4} - \frac{(y+2)^2}{5} = 1$$



EXAMPLE Analyzing the Equation of a Hyperbola

Analyze the equation: $-x^2 + 4y^2 - 2x - 16y + 11 = 0$

$$-(x^2 + 2x) + 4(y^2 - 4y) = -11$$

$$-(x^2 + 2x + 1) + 4(y^2 - 4y + 4) = -11 - 1 + 16$$

$$-(x + 1)^2 + 4(y - 2)^2 = 4$$

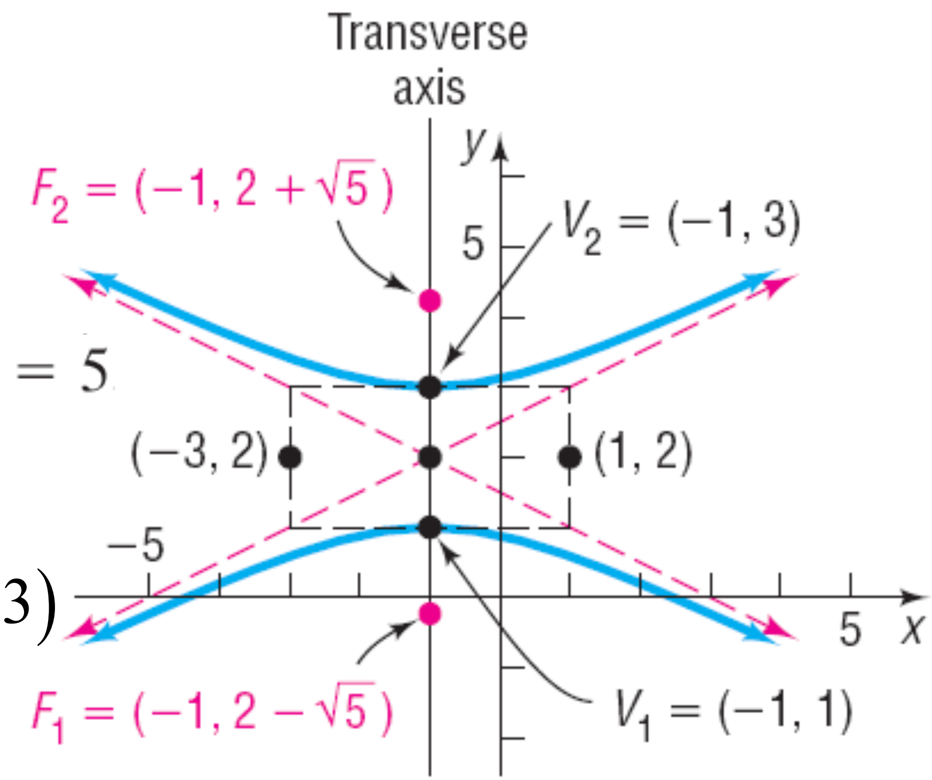
$$(y - 2)^2 - \frac{(x + 1)^2}{4} = 1$$

$a^2 = 1$ and $b^2 = 4$, so $c^2 = a^2 + b^2 = 5$

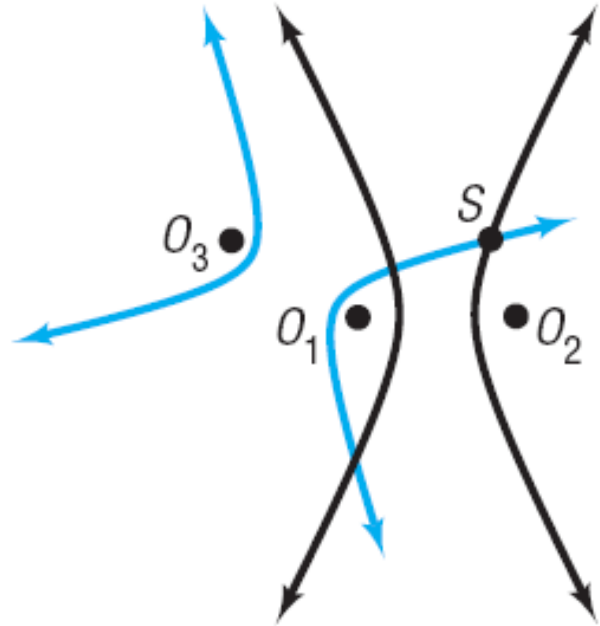
Center: $(-1, 2)$

Vertices: $(-1, 2 \pm 1) = (-1, 1), (-1, 3)$

Foci: $\left(-1, 2 \pm \sqrt{5}\right)$



4 Solve Applied Problems Involving Hyperbolas



Look at Figure 48. Suppose that three microphones are located at points O_1, O_2, and O_3 (the foci of the two hyperbolas). In addition, suppose that a gun is fired at S and the microphone at O_1 records the gun shot 1 second after the microphone at O_2. Because sound travels at about 1100 feet per second, we conclude that the microphone at O_1 is 1100 feet farther from the gunshot than O_2. We can model this situation by saying that S lies on the same branch of a hyperbola with foci at O_1 and O_2. (Do you see why? The difference of the distances from S to O_1 and from S to O_2 is the constant 1100.) If the third microphone at O_3 records the gunshot 2 seconds after O_1, then S will lie on a branch of a second hyperbola with foci at O_1 and O_3. In this case, the constant difference will be 2200. The intersection of the two hyperbolas will identify the location of S.

EXAMPLE

Lightning Strikes

Because the distance between the two people is 1 mile (5280 feet) and each person is at a focus of the hyperbola, we have

$$2c = 5280$$
$$c = \frac{5280}{2} = 2640$$

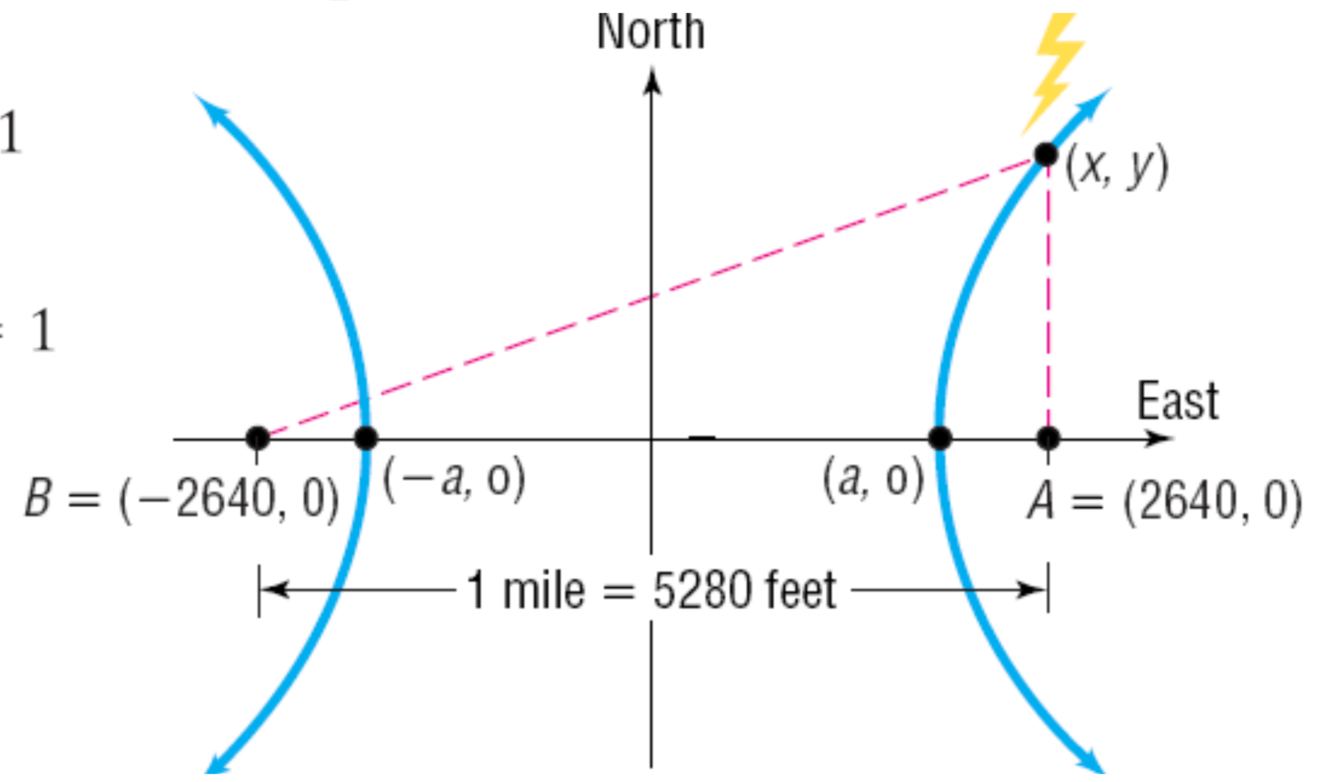
$$\frac{x^2}{550^2} - \frac{y^2}{6,667,100} = 1$$

$$\frac{2640^2}{550^2} - \frac{y^2}{6,667,100} = 1$$

$$-\frac{y^2}{6,667,100} = -22.04$$

$$y^2 = 146,942,884$$

$$y = 12,122$$



The lightning strike occurred 12,122 feet north of the person standing at point A.