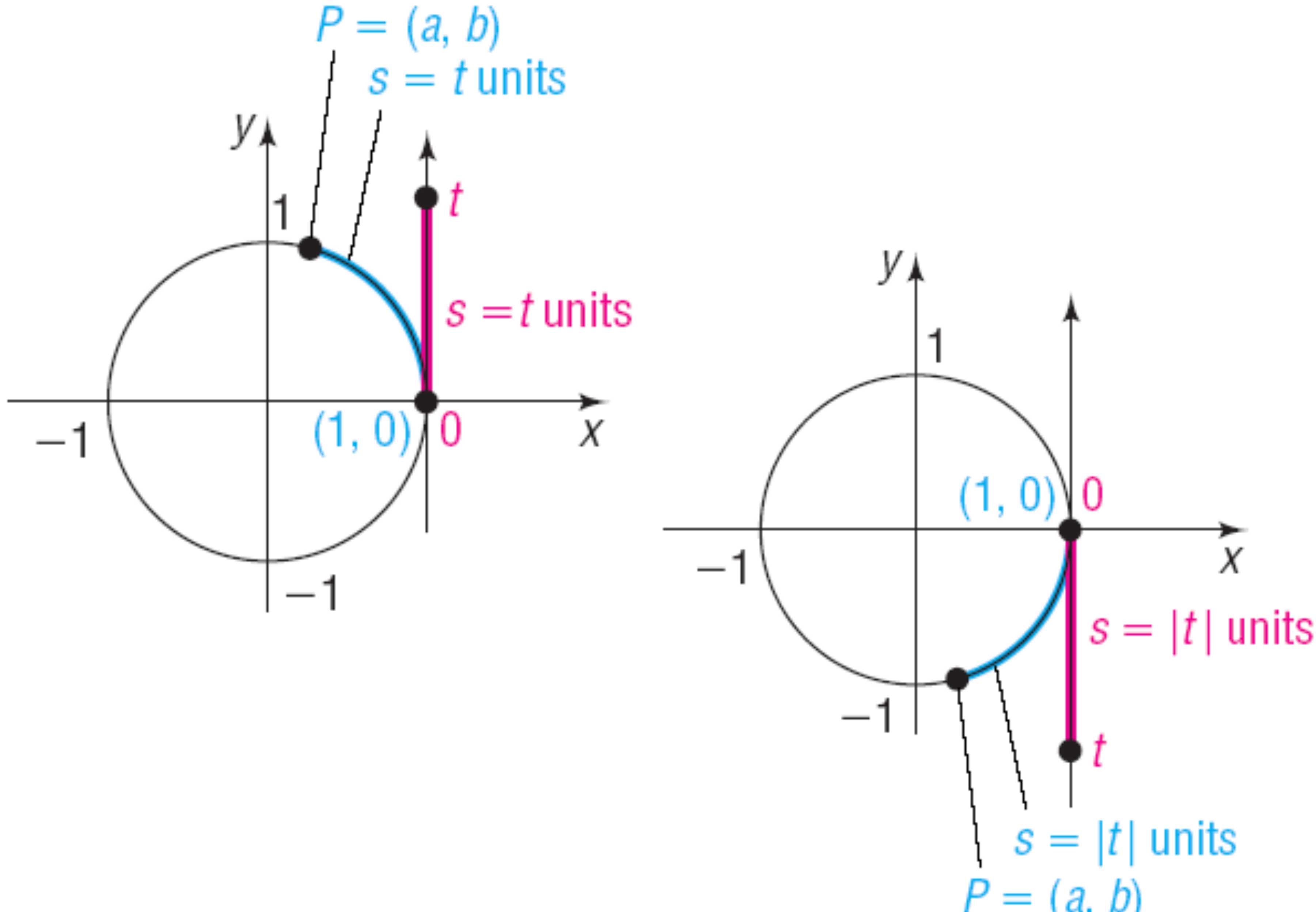
Section 7.5

Unit Circle Approach; Properties of the Trigonometric Functions

1 Find the Exact Values of the Trigonometric Functions Using the Unit Circle



P = (a, b)
s = t units
y
1
t
s = t units
−1
(1, 0)
0
x
−1
y
1
(1, 0)
0
−1
x
s = |t| units
−1
t
s = |t| units
P = (a, b)

Let t be a real number and let $P = (a, b)$ be the point on the unit circle that corresponds to t.

The **sine function** associates with t the y-coordinate of P

$$\boxed{\sin t = b}$$

The **cosine function** associates with t the x-coordinate of P

$$\boxed{\cos t = a}$$

If $a \neq 0$, the **tangent function** is defined as

$$\boxed{\tan t = \frac{b}{a}}$$

If $b \neq 0$, the **cosecant function** is defined as

$$\boxed{\csc t = \frac{1}{b}}$$

If $a \neq 0$, the **secant function** is defined as

$$\boxed{\sec t = \frac{1}{a}}$$

If $b \neq 0$, the **cotangent function** is defined as

$$\boxed{\cot t = \frac{a}{b}}$$

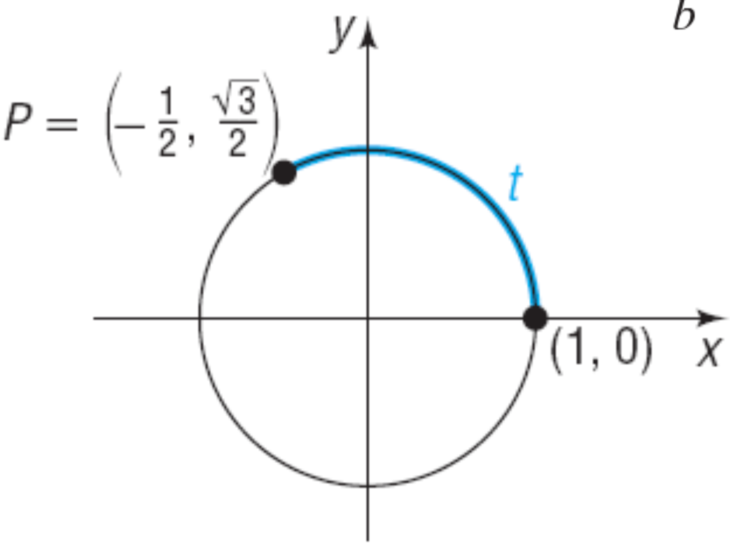
EXAMPLE

Finding the Values of the Trigonometric Functions Using a Point on the Unit Circle

Find the values of sin t, cos t, tan t, csc t, sec t, and cot t if $P = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is the point on the unit circle that corresponds to the real number t.

$$\sin t = b = \frac{\sqrt{3}}{2} \qquad \cos t = a = -\frac{1}{2} \qquad \tan t = \frac{b}{a} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

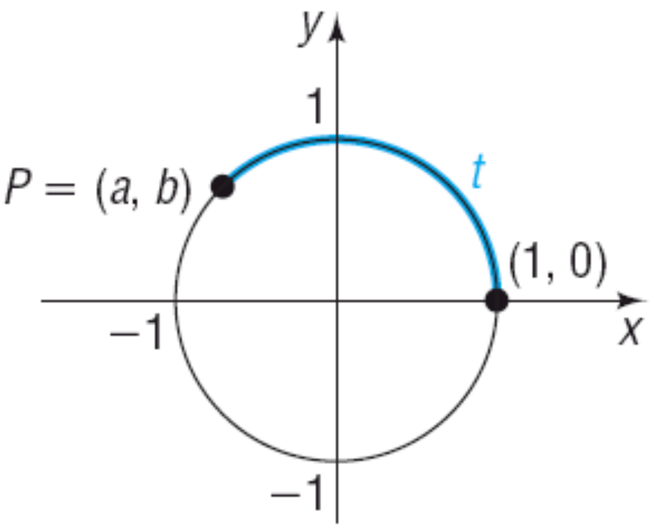
$$\csc t = \frac{1}{b} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

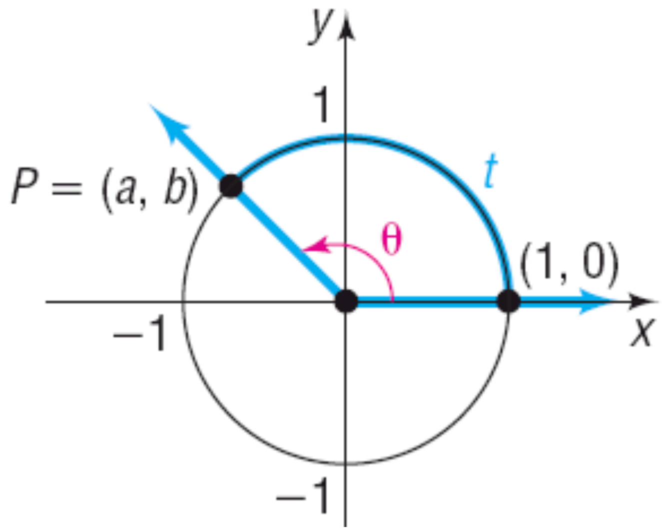


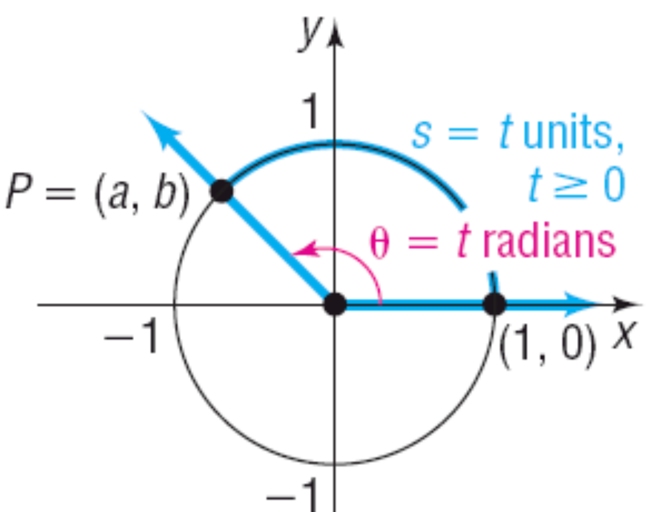
$$\sec t = \frac{1}{a} = \frac{1}{-\frac{1}{2}} = -2$$

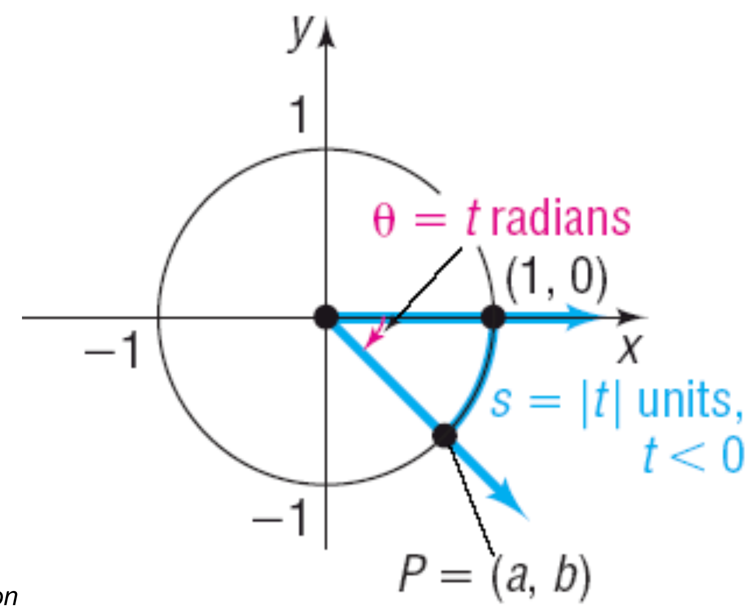
$$\cot t = \frac{a}{b} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Trigonometric Functions of Angles





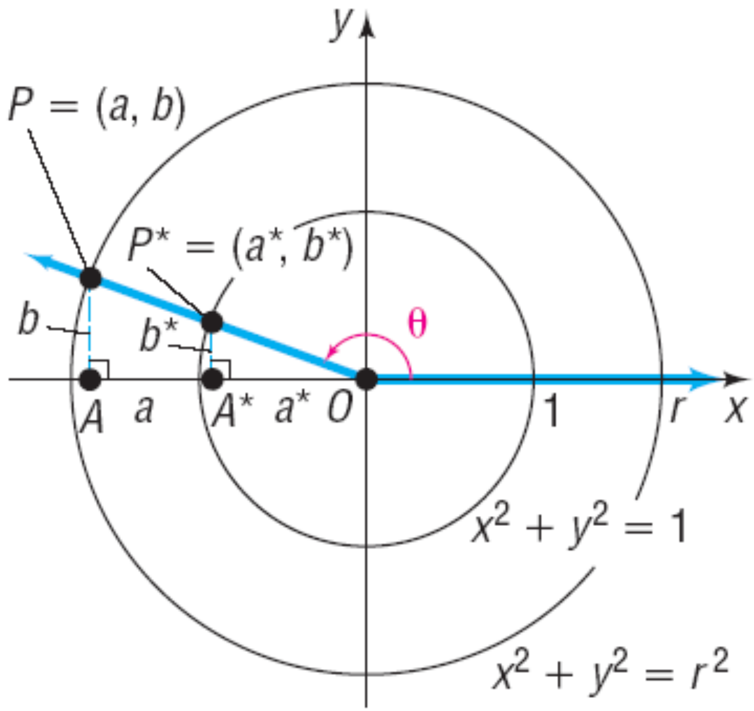




DEFINITION

If $\theta = t$ radians, the six **trigonometric functions of the angle** $\boldsymbol{\theta}$ are defined as

$$\boxed{\begin{array}{lll} \sin\theta = \sin t & \cos\theta = \cos t & \tan\theta = \tan t \\ \csc\theta = \csc t & \sec\theta = \sec t & \cot\theta = \cot t \end{array}}$$

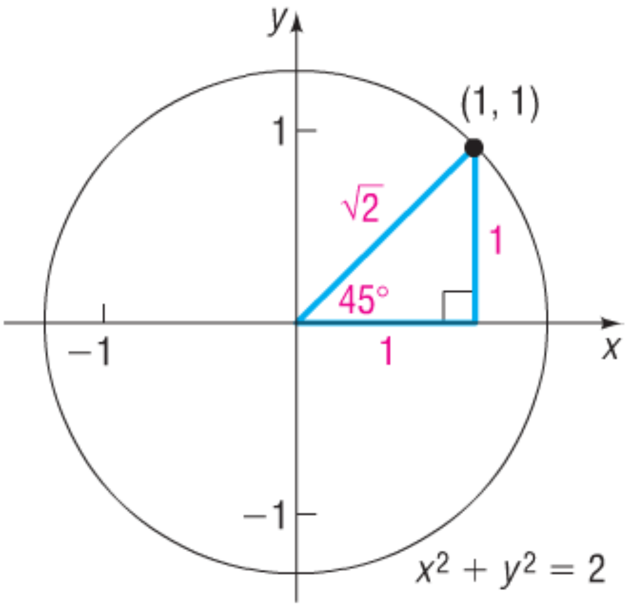


THEOREM

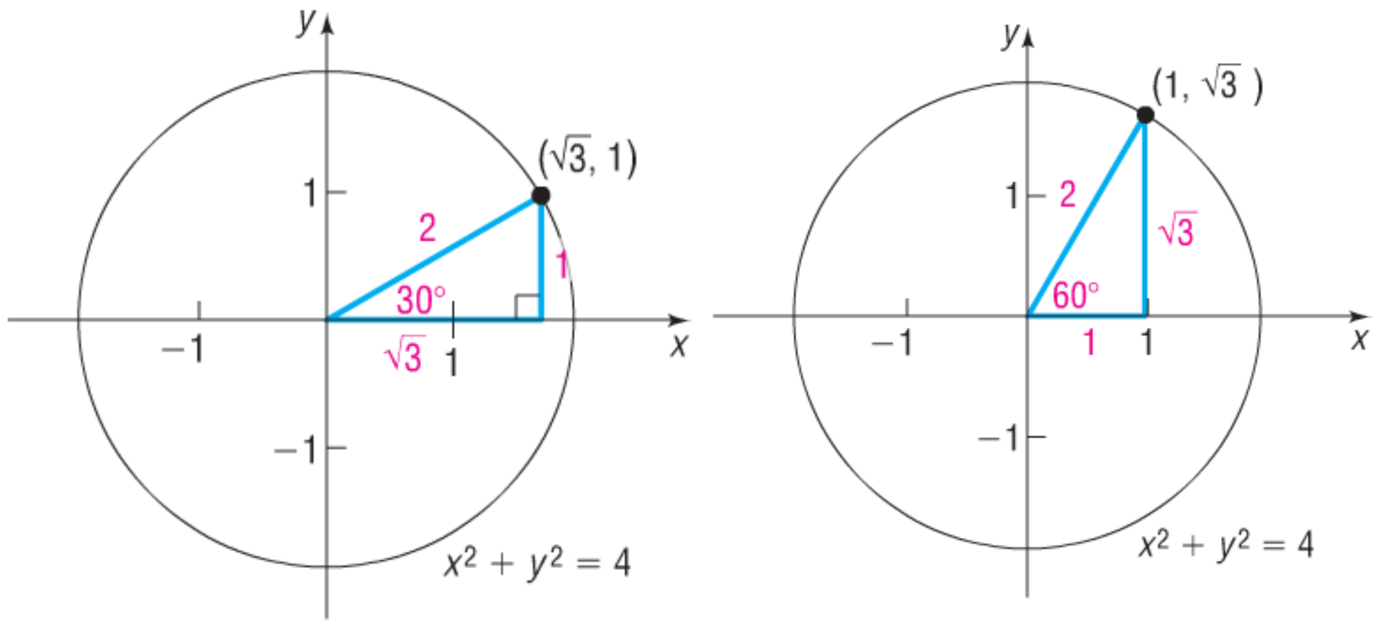
For an angle θ in standard position, let $P = (a, b)$ be any point on the terminal side of θ that is also on the circle $x^2 + y^2 = r^2$. Then

$$\sin\theta = \frac{b}{r} \qquad \cos\theta = \frac{a}{r} \qquad \tan\theta = \frac{b}{a}, \quad a \neq 0$$

$$\csc\theta = \frac{r}{b}, \quad b \neq 0 \qquad \sec\theta = \frac{r}{a}, \quad a \neq 0 \qquad \cot\theta = \frac{a}{b}, \quad b \neq 0$$



y
(1, 1)
1
$\sqrt{2}$
1
45°
−1
1
x
−1
$x^2 + y^2 = 2$



y
($\sqrt{3}$, 1)
1
2
1
30°
−1
$\sqrt{3}$
1
x
−1
$x^2 + y^2 = 4$
y
(1, $\sqrt{3}$)
1
2
$\sqrt{3}$
60°
−1
1
1
x
−1
$x^2 + y^2 = 4$

(a) **(b)**

EXAMPLE

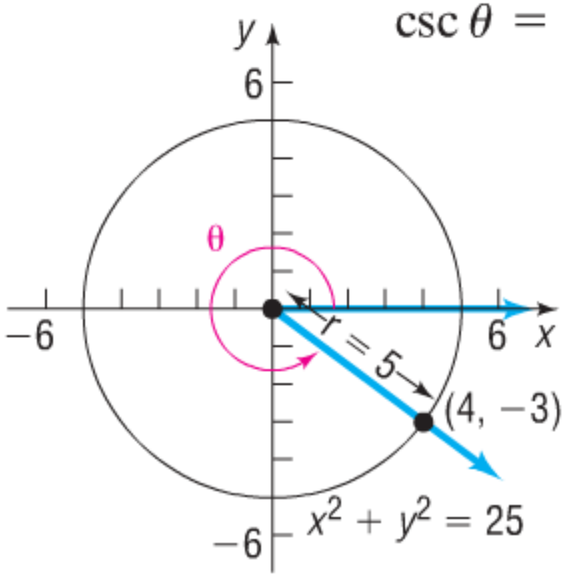
Finding the Exact Values of the Six Trigonometric Functions

Find the exact values of each of the six trigonometric functions of an angle θ if $(4, -3)$ is a point on its terminal side.

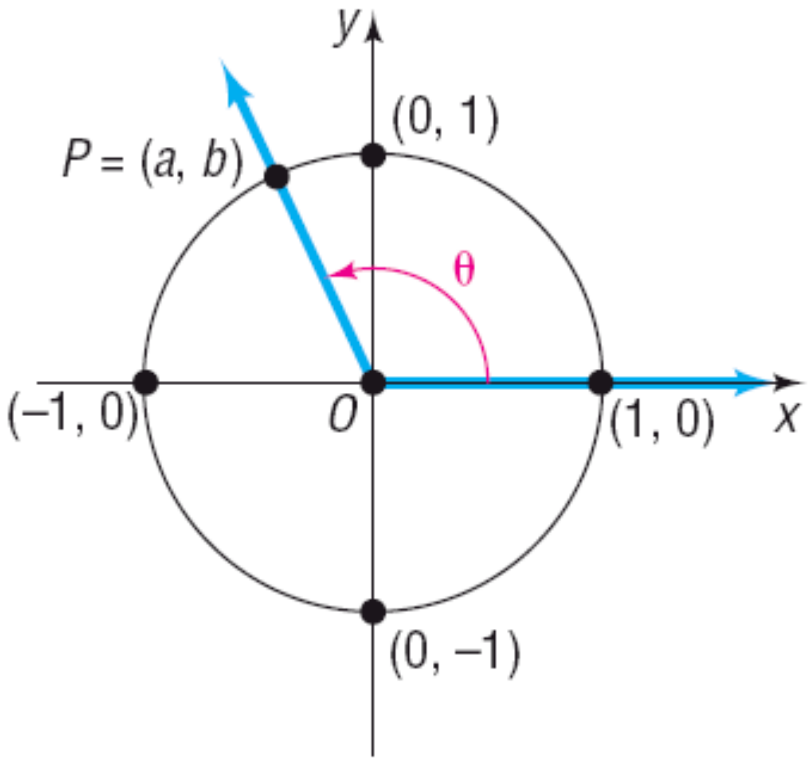
$$r = \sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$\sin\theta = \frac{b}{r} = -\frac{3}{5} \qquad \cos\theta = \frac{a}{r} = \frac{4}{5} \qquad \tan\theta = \frac{b}{a} = -\frac{3}{4}$$

$$\csc\theta = \frac{r}{b} = -\frac{5}{3} \qquad \sec\theta = \frac{r}{a} = \frac{5}{4} \qquad \cot\theta = \frac{a}{b} = -\frac{4}{3}$$



2 Know the Domain and Range of the Trigonometric Functions



$$\sin\theta = b \qquad \cos\theta = a \qquad \tan\theta = \frac{b}{a}, \quad a \neq 0$$

$$\csc\theta = \frac{1}{b}, \quad b \neq 0 \qquad \sec\theta = \frac{1}{a}, \quad a \neq 0 \qquad \cot\theta = \frac{a}{b}, \quad b \neq 0$$

The domain of the sine function is the set of all real numbers.

The domain of the cosine function is the set of all real numbers.

The domain of the tangent function is the set of all real numbers, except odd multiples of $\frac{\pi}{2}\,(90^\circ)$.

The domain of the secant function is the set of all real numbers, except odd multiples of $\frac{\pi}{2}\,(90^\circ)$.

The domain of the cotangent function is the set of all real numbers, except integer multiples of π (180°).

The domain of the cosecant function is the set of all real numbers, except integer multiples of π (180°).

$$-1 \leq \sin\theta \leq 1 \quad \text{and} \quad -1 \leq \cos\theta \leq 1$$

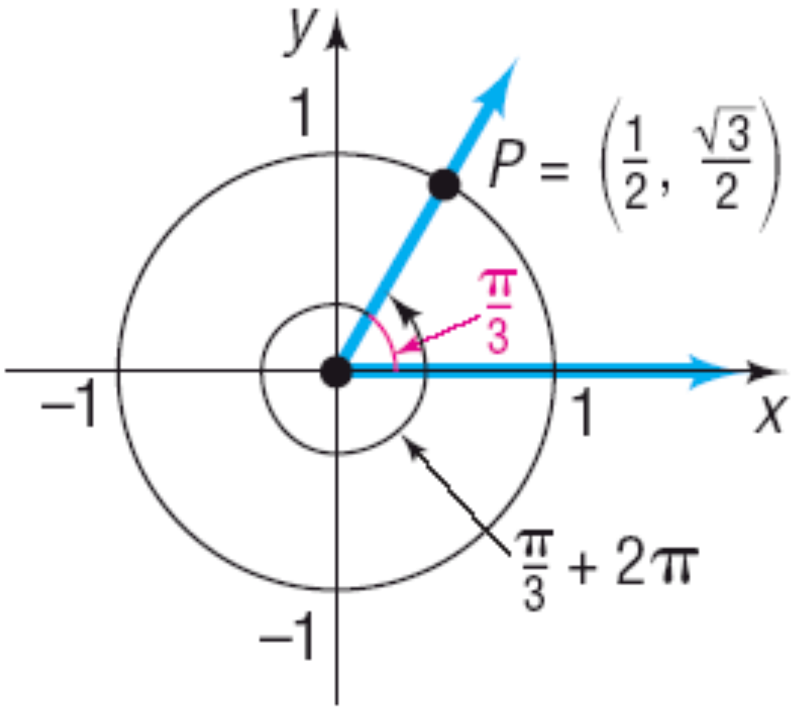
$$\csc\theta \leq -1 \quad \text{or} \quad \csc\theta \geq 1$$

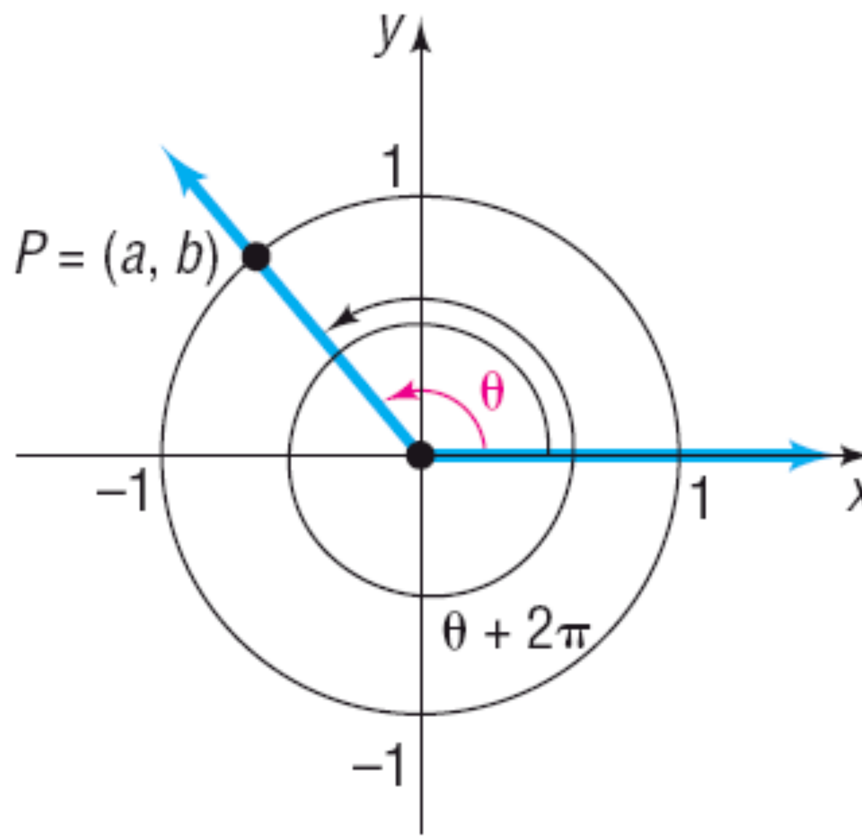
$$\sec\theta \leq -1 \quad \text{or} \quad \sec\theta \geq 1$$

$$-\infty < \tan\theta < \infty \quad \text{and} \quad -\infty < \cot\theta < \infty$$

Function	Symbol	Domain	Range
sine	$f(\theta) = \sin\theta$	All real numbers	All real numbers from -1 to 1, inclusive
cosine	$f(\theta) = \cos\theta$	All real numbers	All real numbers from -1 to 1, inclusive
tangent	$f(\theta) = \tan\theta$	All real numbers, except odd integer multiples of $\frac{\pi}{2}(90^\circ)$	All real numbers
cosecant	$f(\theta) = \csc\theta$	All real numbers, except integer multiples of $\pi(180^\circ)$	All real numbers greater than or equal to 1 or less than or equal to -1
secant	$f(\theta) = \sec\theta$	All real numbers, except odd integer multiples of $\frac{\pi}{2}(90^\circ)$	All real numbers greater than or equal to 1 or less than or equal to -1
cotangent	$f(\theta) = \cot\theta$	All real numbers, except integer multiples of $\pi(180^\circ)$	All real numbers

3 Use the Periodic Properties to Find the Exact Values of the Trigonometric Functions


y
1
$P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{3}$
−1
1
x
$\frac{\pi}{3} + 2\pi$
−1


y
1
$P = (a, b)$
θ
−1
1
x
θ + 2π
−1

$$\sin(\theta + 2\pi k) = \sin\theta \qquad \cos(\theta + 2\pi k) = \cos\theta$$

where k is any integer

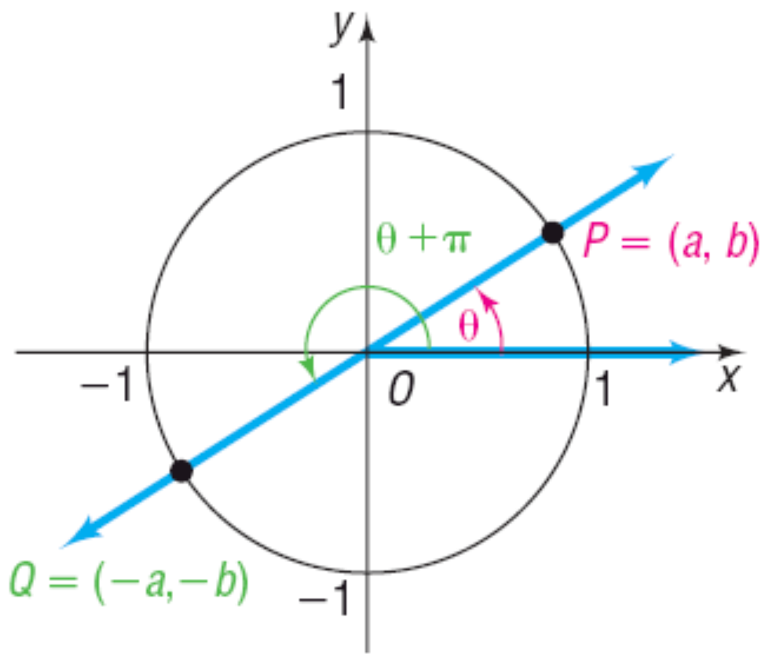
A function f is called **periodic** if there is a positive number p such that, whenever θ is in the domain of f, so is $\theta + p$, and

$$\boxed{f(\theta + p) = f(\theta)}$$

If there is a smallest such number p, this smallest value is called the **(fundamental) period** of f.

Periodic Properties

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta \qquad \tan(\theta + \pi) = \tan\theta$$
$$\csc(\theta + 2\pi) = \csc\theta \qquad \sec(\theta + 2\pi) = \sec\theta \qquad \cot(\theta + \pi) = \cot\theta$$



$$\tan\theta = \frac{b}{a} = \frac{-b}{-a} = \tan(\theta + \pi)$$

EXAMPLE

Using Periodic Properties to Find Exact Values

Find the exact value of:

(a) $\cos 480^\circ$ (b) $\tan\frac{11\pi}{4}$ (c) $\sin\frac{7\pi}{3}$

(a) $\cos 480^\circ = \cos(120^\circ + 360^\circ) = \cos 120^\circ = -\frac{1}{2}$

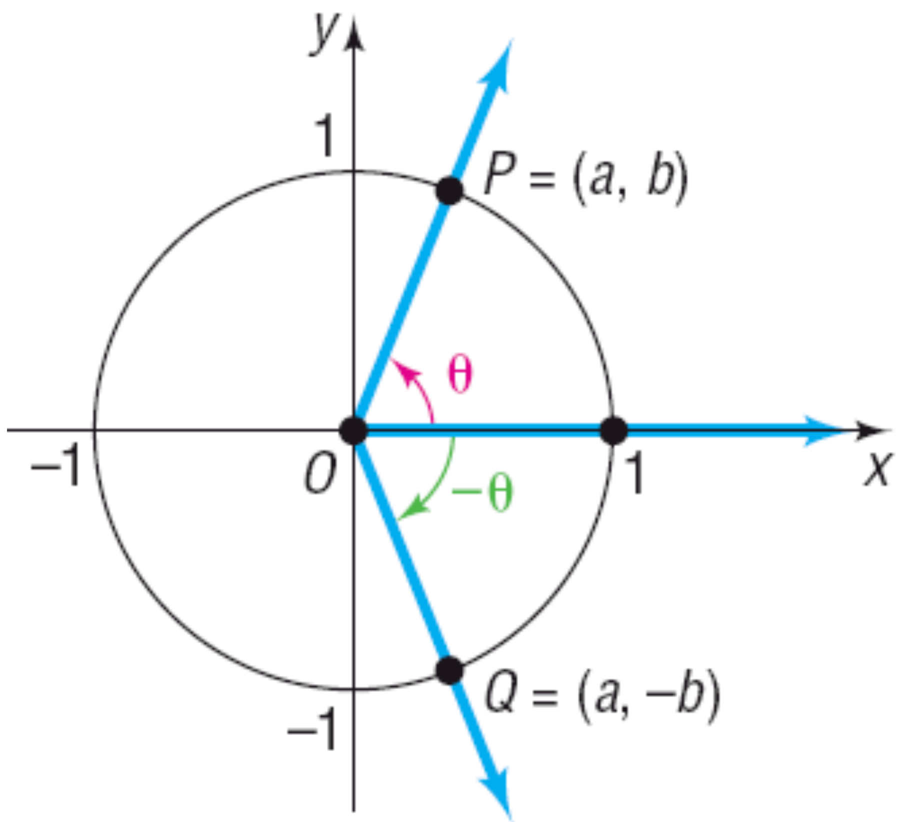
(b) $\tan\frac{11\pi}{4} = \tan\left(\frac{3\pi}{4} + 2\pi\right) = \tan\frac{3\pi}{4} = -1$

(c) $\sin\frac{7\pi}{3} = \sin\left(\frac{\pi}{3} + 2\pi\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

4 Use Even–Odd Properties to Find the Exact Values of the Trigonometric Functions

Even–Odd Properties

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$



EXAMPLE

Finding Exact Values Using Even–Odd Properties

Find the exact value of:

(a) $\cos(-60^\circ)$ (b) $\sin(-390^\circ)$ (c) $\tan\left(-\frac{37\pi}{4}\right)$

$$\text{(a)}\quad \cos(-60^\circ) = \cos 60^\circ = \frac{1}{2}$$

$$\text{(b)}\quad \sin(-390^\circ) = -\sin 390^\circ = -\sin(30^\circ + 360^\circ) = -\sin 30^\circ = -\frac{1}{2}$$

$$\text{(c)}\quad \tan\left(-\frac{37\pi}{4}\right) = -\tan\left(\frac{37\pi}{4}\right) = -\tan\left(\frac{\pi}{4} + \frac{36\pi}{4}\right)$$

$$= -\tan\left(\frac{\pi}{4} + 9\pi\right) = -\tan\frac{\pi}{4} = -1$$