L

SELECTED PROOFS

PROOFS OF BASIC LIMIT THEOREMS

An extensive excursion into proofs of limit theorems would be too time consuming to undertake, so we have selected a few proofs of results from Section 1.2 that illustrate some of the basic ideas.

L.1 THEOREM *Let a be any real number, let k be a constant, and suppose that $\lim_{x\to a} f(x) = L_1$ and that $\lim_{x\to a} g(x) = L_2$. Then:*

(*a*) $\lim_{x\to a} k = k$

(*b*) $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$

(*c*) $\lim_{x\to a} [f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1L_2$

PROOF (*a*) We will apply Definition 1.4.1 with $f(x) = k$ and $L = k$. Thus, given $\epsilon > 0$, we must find a number $\delta > 0$ such that

$$|k - k| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta$$

or, equivalently,

$$0 < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta$$

But the condition on the left side of this statement is *always* true, no matter how δ is chosen. Thus, any positive value for δ will suffice.

PROOF (*b*) We must show that given $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|(f(x) + g(x)) - (L_1 + L_2)| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta \tag{1}$$

However, from the limits of f and g in the hypothesis of the theorem we can find numbers δ_1 and δ_2 such that

$$|f(x) - L_1| < \epsilon/2 \quad \text{if} \quad 0 < |x - a| < \delta_1$$
$$|g(x) - L_2| < \epsilon/2 \quad \text{if} \quad 0 < |x - a| < \delta_2$$

Moreover, the inequalities on the left sides of these statements *both* hold if we replace δ_1 and δ_2 by any positive number δ that is less than both δ_1 and δ_2. Thus, for any such δ it follows that

$$|f(x) - L_1| + |g(x) - L_2| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta \tag{2}$$

However, it follows from the triangle inequality [Theorem G.5 of Web Appendix G] that

$$\begin{aligned}|(f(x)+g(x))-(L_1+L_2)| &= |(f(x)-L_1)+(g(x)-L_2)| \\ &\le |f(x)-L_1|+|g(x)-L_2|\end{aligned}$$

so that (1) follows from (2).

PROOF (*c*) We must show that given $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x)g(x)-L_1L_2| < \epsilon \quad \text{if} \quad 0<|x-a|<\delta \tag{3}$$

To find δ it will be helpful to express (3) in a different form. If we rewrite $f(x)$ and $g(x)$ as

$$f(x) = L_1 + (f(x)-L_1) \quad \text{and} \quad g(x) = L_2 + (g(x)-L_2)$$

then the inequality on the left side of (3) can be expressed as (verify)

$$|L_1(g(x)-L_2)+L_2(f(x)-L_1)+(f(x)-L_1)(g(x)-L_2)| < \epsilon \tag{4}$$

Since

$$\lim_{x\to a} f(x) = L_1 \quad \text{and} \quad \lim_{x\to a} g(x) = L_2$$

we can find positive numbers δ_1, δ_2, δ_3, and δ_4 such that

$$\begin{aligned}
|f(x)-L_1| &< \sqrt{\epsilon/3} && \text{if} \quad 0<|x-a|<\delta_1 \\
|f(x)-L_1| &< \frac{\epsilon}{3(1+|L_2|)} && \text{if} \quad 0<|x-a|<\delta_2 \\
|g(x)-L_2| &< \sqrt{\epsilon/3} && \text{if} \quad 0<|x-a|<\delta_3 \\
|g(x)-L_2| &< \frac{\epsilon}{3(1+|L_1|)} && \text{if} \quad 0<|x-a|<\delta_4
\end{aligned} \tag{5}$$

Moreover, the inequalities on the left sides of these four statements *all* hold if we replace δ_1, δ_2, δ_3, and δ_4 by any positive number δ that is smaller than δ_1, δ_2, δ_3, and δ_4. Thus, for any such δ it follows with the help of the triangle inequality that

$$\begin{aligned}
&|L_1(g(x)-L_2)+L_2(f(x)-L_1)+(f(x)-L_1)(g(x)-L_2)| \\
&\quad \le |L_1(g(x)-L_2)| + |L_2(f(x)-L_1)| + |(f(x)-L_1)(g(x)-L_2)| \\
&\quad = |L_1||g(x)-L_2| + |L_2||f(x)-L_1| + |f(x)-L_1||g(x)-L_2| \\
&\quad < |L_1|\frac{\epsilon}{3(1+|L_1|)} + |L_2|\frac{\epsilon}{3(1+|L_2|)} + \sqrt{\epsilon/3}\sqrt{\epsilon/3} && \boxed{\text{From (5)}} \\
&\quad = \frac{\epsilon}{3}\frac{|L_1|}{1+|L_1|} + \frac{\epsilon}{3}\frac{|L_2|}{1+|L_2|} + \frac{\epsilon}{3} \\
&\quad < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon && \boxed{\text{Since } \frac{|L_1|}{1+|L_1|} < 1 \text{ and } \frac{|L_2|}{1+|L_2|} < 1}
\end{aligned}$$

which shows that (4) holds for the δ selected. ■

Do not be alarmed if the proof of part (c) seems difficult; it takes some experience with proofs of this type to develop a feel for choosing a valid δ. Your initial goal should be to understand the ideas and the computations.

PROOF OF A BASIC CONTINUITY PROPERTY

Next we will prove Theorem 1.5.5 for two-sided limits.

L.2 THEOREM (*Theorem 1.5.5*) *If* $\lim_{x\to c} g(x) = L$ *and if the function* f *is continuous at* L, *then* $\lim_{x\to c} f(g(x)) = f(L)$. *That is,*

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$$

PROOF We must show that given $\epsilon > 0$, we can find a number $\delta > 0$ such that

$$|f(g(x)) - f(L)| < \epsilon \quad \text{if} \quad 0 < |x - c| < \delta \tag{6}$$

Since f is continuous at L, we have

$$\lim_{u \to L} f(u) = f(L)$$

and hence we can find a number $\delta_1 > 0$ such that

$$|f(u) - f(L)| < \epsilon \quad \text{if} \quad |u - L| < \delta_1$$

In particular, if $u = g(x)$, then

$$|f(g(x)) - f(L)| < \epsilon \quad \text{if} \quad |g(x) - L| < \delta_1 \tag{7}$$

But $\lim_{x \to c} g(x) = L$, and hence there is a number $\delta > 0$ such that

$$|g(x) - L| < \delta_1 \quad \text{if} \quad 0 < |x - c| < \delta \tag{8}$$

Thus, if x satisfies the condition on the right side of statement (8), then it follows that $g(x)$ satisfies the condition on the right side of statement (7), and this implies that the condition on the left side of statement (6) is satisfied, completing the proof. ■

PROOF OF THE CHAIN RULE

Next we will prove the chain rule (Theorem 2.6.1), but first we need a preliminary result.

L.3 THEOREM *If f is differentiable at x and if $y = f(x)$, then*

$$\Delta y = f'(x)\Delta x + \epsilon \Delta x$$

where $\epsilon \to 0$ as $\Delta x \to 0$ and $\epsilon = 0$ if $\Delta x = 0$.

PROOF Define

$$\epsilon = \begin{cases} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} - f'(x) & \text{if } \Delta x \neq 0 \\ 0 & \text{if } \Delta x = 0 \end{cases} \tag{9}$$

If $\Delta x \neq 0$, it follows from (9) that

$$\epsilon \Delta x = [f(x + \Delta x) - f(x)] - f'(x)\Delta x \tag{10}$$

But

$$\Delta y = f(x + \Delta x) - f(x) \tag{11}$$

so (10) can be written as

$$\epsilon \Delta x = \Delta y - f'(x)\Delta x$$

or

$$\Delta y = f'(x)\Delta x + \epsilon \Delta x \tag{12}$$

If $\Delta x = 0$, then (12) still holds (why?), so (12) is valid for all values of Δx. It remains to show that $\epsilon \to 0$ as $\Delta x \to 0$. But this follows from the assumption that f is differentiable at x, since

$$\lim_{\Delta x \to 0} \epsilon = \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} - f'(x) \right] = f'(x) - f'(x) = 0 \quad ■$$

We are now ready to prove the chain rule.

L.4 THEOREM (*Theorem 2.6.1*) *If g is differentiable at the point x and f is differentiable at the point $g(x)$, then the composition $f \circ g$ is differentiable at the point x. Moreover, if $y = f(g(x))$ and $u = g(x)$, then*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

PROOF Since g is differentiable at x and $u = g(x)$, it follows from Theorem L.3 that

$$\Delta u = g'(x)\Delta x + \epsilon_1 \Delta x \tag{13}$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$. And since $y = f(u)$ is differentiable at $u = g(x)$, it follows from Theorem L.3 that

$$\Delta y = f'(u)\Delta u + \epsilon_2 \Delta u \tag{14}$$

where $\epsilon_2 \to 0$ as $\Delta u \to 0$.

Factoring out the Δu in (14) and then substituting (13) yields

$$\Delta y = [f'(u) + \epsilon_2][g'(x)\Delta x + \epsilon_1 \Delta x]$$

or

$$\Delta y = [f'(u) + \epsilon_2][g'(x) + \epsilon_1]\Delta x$$

or if $\Delta x \neq 0$,

$$\frac{\Delta y}{\Delta x} = [f'(u) + \epsilon_2][g'(x) + \epsilon_1] \tag{15}$$

But (13) implies that $\Delta u \to 0$ as $\Delta x \to 0$, and hence $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$. Thus, from (15)

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u)g'(x)$$

or

$$\frac{dy}{dx} = f'(u)g'(x) = \frac{dy}{du} \cdot \frac{du}{dx} \quad \blacksquare$$

PROOF THAT RELATIVE EXTREMA OCCUR AT CRITICAL POINTS

In this subsection we will prove Theorem 4.2.2, which states that the relative extrema of a function occur at critical points.

L.5 THEOREM (*Theorem 4.2.2*) *Suppose that f is a function defined on an open interval containing the point x_0. If f has a relative extremum at $x = x_0$, then $x = x_0$ is a critical point of f; that is, either $f'(x_0) = 0$ or f is not differentiable at x_0.*

PROOF Suppose that f has a relative maximum at x_0. There are two possibilities—either f is differentiable at x_0 or it is not. If it is not, then x_0 is a critical point for f and we are done. If f is differentiable at x_0, then we must show that $f'(x_0) = 0$. We will do this by showing that $f'(x_0) \geq 0$ and $f'(x_0) \leq 0$, from which it follows that $f'(x_0) = 0$. From the definition of a derivative we have

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

so that

$$f'(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \tag{16}$$

and

$$f'(x_0) = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \tag{17}$$

Because f has a relative maximum at x_0, there is an open interval (a, b) containing x_0 in which $f(x) \leq f(x_0)$ for all x in (a, b).

Assume that h is sufficiently small so that $x_0 + h$ lies in the interval (a, b). Thus,

$$f(x_0 + h) \le f(x_0) \quad \text{or equivalently} \quad f(x_0 + h) - f(x_0) \le 0$$

Thus, if h is negative,

$$\frac{f(x_0 + h) - f(x_0)}{h} \ge 0 \tag{18}$$

and if h is positive,

$$\frac{f(x_0 + h) - f(x_0)}{h} \le 0 \tag{19}$$

But an expression that never assumes negative values cannot approach a negative limit and an expression that never assumes positive values cannot approach a positive limit, so that

$$f'(x_0) = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \ge 0 \qquad \boxed{\text{From (17) and (18)}}$$

and

$$f'(x_0) = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \le 0 \qquad \boxed{\text{From (16) and (19)}}$$

Since $f'(x_0) \ge 0$ and $f'(x_0) \le 0$, it must be that $f'(x_0) = 0$.

A similar argument applies if f has a relative minimum at x_0. ■

PROOFS OF TWO SUMMATION FORMULAS

We will prove parts (a) and (b) of Theorem 5.4.2. The proof of part (c) is similar to that of part (b) and is omitted.

L.6 THEOREM (*Theorem 5.4.2*)

(*a*) $$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(*b*) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(*c*) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

PROOF (*a*) Writing

$$\sum_{k=1}^{n} k$$

two ways, with summands in increasing order and in decreasing order, and then adding, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} k &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\
\sum_{k=1}^{n} k &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\
\hline
2\sum_{k=1}^{n} k &= (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \\
&= n(n+1)
\end{aligned}$$

Thus,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

PROOF (*b*) Note that

$$(k+1)^3 - k^3 = k^3 + 3k^2 + 3k + 1 - k^3 = 3k^2 + 3k + 1$$

So,

$$\sum_{k=1}^{n} [(k+1)^3 - k^3] = \sum_{k=1}^{n} (3k^2 + 3k + 1) \tag{20}$$

The sum in (21) is an example of a ***telescoping sum***, since the cancellation of each of the two parts of an interior summand with parts of its neighboring summands allows the entire sum to collapse like a telescope.

Writing out the left side of (20) with the index running *down* from $k = n$ to $k = 1$, we have

$$\begin{aligned}\sum_{k=1}^{n} [(k+1)^3 - k^3] &= [(n+1)^3 - n^3] + \cdots + [4^3 - 3^3] + [3^3 - 2^3] + [2^3 - 1^3] \\ &= (n+1)^3 - 1 \end{aligned} \tag{21}$$

Combining (21) and (20), and expanding the right side of (20) by using Theorem 5.4.1 and part (*a*) of this theorem yields

$$\begin{aligned}(n+1)^3 - 1 &= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\ &= 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n\end{aligned}$$

So,

$$\begin{aligned}3\sum_{k=1}^{n} k^2 &= [(n+1)^3 - 1] - 3\frac{n(n+1)}{2} - n \\ &= (n+1)^3 - 3(n+1)\left(\frac{n}{2}\right) - (n+1) \\ &= \frac{n+1}{2}[2(n+1)^2 - 3n - 2] \\ &= \frac{n+1}{2}[2n^2 + n] = \frac{n(n+1)(2n+1)}{2}\end{aligned}$$

Thus,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \blacksquare$$

PROOF OF THE LIMIT COMPARISON TEST

L.7 THEOREM (*Theorem 9.5.4*) *Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that*

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k}$$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

PROOF We need only show that $\sum b_k$ converges when $\sum a_k$ converges and that $\sum b_k$ diverges when $\sum a_k$ diverges, since the remaining cases are logical implications of these (why?). The idea of the proof is to apply the comparison test to $\sum a_k$ and suitable multiples of $\sum b_k$. For this purpose let ϵ be any positive number. Since

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k}$$

it follows that eventually the terms in the sequence $\{a_k/b_k\}$ must be within ϵ units of ρ; that is, there is a positive integer K such that for $k \geq K$ we have

$$\rho - \epsilon < \frac{a_k}{b_k} < \rho + \epsilon$$

In particular, if we take $\epsilon = \rho/2$, then for $k \geq K$ we have

$$\frac{1}{2}\rho < \frac{a_k}{b_k} < \frac{3}{2}\rho \quad \text{or} \quad \frac{1}{2}\rho b_k < a_k < \frac{3}{2}\rho b_k$$

Thus, by the comparison test we can conclude that

$$\sum_{k=K}^{\infty} \frac{1}{2}\rho b_k \quad \text{converges if} \quad \sum_{k=K}^{\infty} a_k \quad \text{converges} \tag{22}$$

$$\sum_{k=K}^{\infty} \frac{3}{2}\rho b_k \quad \text{diverges if} \quad \sum_{k=K}^{\infty} a_k \quad \text{diverges} \tag{23}$$

But the convergence or divergence of a series is not affected by deleting finitely many terms or by multiplying the general term by a nonzero constant, so (22) and (23) imply that

$$\sum_{k=1}^{\infty} b_k \quad \text{converges if} \quad \sum_{k=1}^{\infty} a_k \quad \text{converges}$$

$$\sum_{k=1}^{\infty} b_k \quad \text{diverges if} \quad \sum_{k=1}^{\infty} a_k \quad \text{diverges} \quad \blacksquare$$

PROOF OF THE RATIO TEST

L.8 THEOREM (*Theorem 9.5.5*) *Let $\sum u_k$ be a series with positive terms and suppose that*

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}$$

(*a*) *If $\rho < 1$, the series converges.*

(*b*) *If $\rho > 1$ or $\rho = +\infty$, the series diverges.*

(*c*) *If $\rho = 1$, the series may converge or diverge, so that another test must be tried.*

PROOF (*a*) The number ρ must be nonnegative since it is the limit of u_{k+1}/u_k, which is positive for all k. In this part of the proof we assume that $\rho < 1$, so that $0 \leq \rho < 1$.

We will prove convergence by showing that the terms of the given series are eventually less than the terms of a convergent geometric series. For this purpose, choose any real number r such that $0 < \rho < r < 1$. Since the limit of u_{k+1}/u_k is ρ, and $\rho < r$, the terms of the sequence $\{u_{k+1}/u_k\}$ must eventually be less than r. Thus, there is a positive integer K such that for $k \geq K$ we have

$$\frac{u_{k+1}}{u_k} < r \quad \text{or} \quad u_{k+1} < ru_k$$

This yields the inequalities

$$\begin{aligned} u_{K+1} &< ru_K \\ u_{K+2} &< ru_{K+1} < r^2 u_K \\ u_{K+3} &< ru_{K+2} < r^3 u_K \\ u_{K+4} &< ru_{K+3} < r^4 u_K \\ &\vdots \end{aligned} \tag{24}$$

But $0 < r < 1$, so

$$ru_K + r^2 u_K + r^3 u_K + \cdots$$

is a convergent geometric series. From the inequalities in (24) and the comparison test it follows that

$$u_{K+1} + u_{K+2} + u_{K+3} + \cdots$$

must also be a convergent series. Thus, $u_1 + u_2 + u_3 + \cdots + u_k + \cdots$ converges by Theorem 9.4.3(c).

PROOF (b) In this part we will prove divergence by showing that the limit of the general term is not zero. Since the limit of u_{k+1}/u_k is ρ and $\rho > 1$, the terms in the sequence $\{u_{k+1}/u_k\}$ must eventually be greater than 1. Thus, there is a positive integer K such that for $k \geq K$ we have

$$\frac{u_{k+1}}{u_k} > 1 \quad \text{or} \quad u_{k+1} > u_k$$

This yields the inequalities

$$\begin{aligned} &u_{K+1} > u_K \\ &u_{K+2} > u_{K+1} > u_K \\ &u_{K+3} > u_{K+2} > u_K \\ &u_{K+4} > u_{K+3} > u_K \\ &\qquad\vdots \end{aligned} \tag{25}$$

Since $u_K > 0$, it follows from the inequalities in (25) that $\lim_{k\to+\infty} u_k \neq 0$, and thus the series $u_1 + u_2 + \cdots + u_k + \cdots$ diverges by part (a) of Theorem 9.4.1. The proof in the case where $\rho = +\infty$ is omitted.

PROOF (c) The divergent harmonic series and the convergent p-series with $p = 2$ both have $\rho = 1$ (verify), so the ratio test does not distinguish between convergence and divergence when $\rho = 1$. ■

PROOF OF THE REMAINDER ESTIMATION THEOREM

L.9 THEOREM (*Theorem 9.7.4*) *If the function f can be differentiated $n+1$ times on an interval containing the number x_0, and if M is an upper bound for $|f^{(n+1)}(x)|$ on the interval, that is, $|f^{(n+1)}(x)| \leq M$ for all x in the interval, then*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

for all x in the interval.

PROOF We are assuming that f can be differentiated $n+1$ times on an interval containing the number x_0 and that

$$|f^{(n+1)}(x)| \leq M \tag{26}$$

for all x in the interval. We want to show that

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1} \tag{27}$$

for all x in the interval, where

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{28}$$

In our proof we will need the following two properties of $R_n(x)$:

$$R_n(x_0) = R_n'(x_0) = \cdots = R_n^{(n)}(x_0) = 0 \tag{29}$$

$$R_n^{(n+1)}(x) = f^{(n+1)}(x) \quad \text{for all } x \text{ in the interval} \tag{30}$$

These properties can be obtained by analyzing what happens if the expression for $R_n(x)$ in Formula (28) is differentiated j times and x_0 is then substituted in that derivative. If $j < n$, then the jth derivative of the summation in Formula (28) consists of a constant term $f^{(j)}(x_0)$ plus terms involving powers of $x - x_0$ (verify). Thus, $R_n^{(j)}(x_0) = 0$ for $j < n$, which proves all but the last equation in (29). For the last equation, observe that the nth derivative of the summation in (28) is the constant $f^{(n)}(x_0)$, so $R_n^{(n)}(x_0) = 0$. Formula (30) follows from the observation that the $(n+1)$-st derivative of the summation in (28) is zero (why?).

Now to the main part of the proof. For simplicity we will give the proof for the case where $x \geq x_0$ and leave the case where $x < x_0$ for the reader. It follows from (26) and (30) that $|R_n^{(n+1)}(x)| \leq M$, and hence

$$-M \leq R_n^{(n+1)}(x) \leq M$$

Thus,

$$\int_{x_0}^{x} -M\,dt \leq \int_{x_0}^{x} R_n^{(n+1)}(t)\,dt \leq \int_{x_0}^{x} M\,dt \tag{31}$$

However, it follows from (29) that $R_n^{(n)}(x_0) = 0$, so

$$\int_{x_0}^{x} R_n^{(n+1)}(t)\,dt = R_n^{(n)}(t)\Big]_{x_0}^{x} = R_n^{(n)}(x)$$

Thus, performing the integrations in (31) we obtain the inequalities

$$-M(x - x_0) \leq R_n^{(n)}(x) \leq M(x - x_0)$$

Now we will integrate again. Replacing x by t in these inequalities, integrating from x_0 to x, and using $R_n^{(n-1)}(x_0) = 0$ yields

$$-\frac{M}{2}(x - x_0)^2 \leq R_n^{(n-1)}(x) \leq \frac{M}{2}(x - x_0)^2$$

If we keep repeating this process, then after $n+1$ integrations we will obtain

$$-\frac{M}{(n+1)!}(x - x_0)^{n+1} \leq R_n(x) \leq \frac{M}{(n+1)!}(x - x_0)^{n+1}$$

which we can rewrite as

$$|R_n(x)| \leq \frac{M}{(n+1)!}(x - x_0)^{n+1}$$

This completes the proof of (27), since the absolute value signs can be omitted in that formula when $x \geq x_0$ (which is the case we are considering). ■

PROOF OF THE EQUALITY OF MIXED PARTIALS

L.10 THEOREM (*Theorem 13.3.2*) *Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.*

PROOF Suppose that f is a function of two variables with f_{xy} and f_{yx} both continuous on some open disk. Let (x, y) be a point in that disk and define the function

$$w(\Delta x, \Delta y) = f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) - f(x, y + \Delta y) + f(x, y)$$

Now fix y and Δy and let

$$g(x) = f(x, y + \Delta y) - f(x, y)$$

so that

$$w(\Delta x, \Delta y) = g(x + \Delta x) - g(x) \tag{32}$$

Since f is differentiable on an open disk containing (x, y), the function g will be differentiable on some interval containing x and $x + \Delta x$ for Δx small enough. The Mean-Value Theorem then applies to g on this interval, and thus there is a c between x and Δx with

$$g(x + \Delta x) - g(x) = g'(c)\Delta x$$

But

$$g'(c) = f_x(c, y + \Delta y) - f_x(c, y)$$

so from Equation (32)

$$w(\Delta x, \Delta y) = g(x + \Delta x) - g(x) = g'(c)\Delta x = (f_x(c, y + \Delta y) - f_x(c, y))\Delta x \tag{33}$$

Now let $h(y) = f_x(c, y)$. Since f_x is differentiable on an open disk containing (x, y), h will be differentiable on some interval containing y and $y + \Delta y$ for Δy small enough. Applying the Mean-Value Theorem to h on this interval gives a d between y and $y + \Delta y$ with

$$h(y + \Delta y) - h(y) = h'(d)\Delta y$$

But $h'(d) = f_{xy}(c, d)$, so by (33) and the definition of h we have

$$\begin{aligned} w(\Delta x, \Delta y) &= (f_x(c, y + \Delta y) - f_x(c, y))\Delta x \\ &= (h(y + \Delta y) - h(y))\Delta x = h'(d)\Delta y \Delta x \\ &= f_{xy}(c, d)\Delta y \Delta x \end{aligned}$$

and

$$f_{xy}(c, d) = \frac{w(\Delta x, \Delta y)}{\Delta y \Delta x} \tag{34}$$

Since c lies between x and Δx and d lies between y and Δy, (c, d) approaches (x, y) as $(\Delta x, \Delta y)$ approaches $(0, 0)$. It then follows from the continuity of f_{xy} and (34) that

$$f_{xy}(x, y) = \lim_{(\Delta x, \Delta y) \to (0,0)} f_{xy}(c, d) = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{w(\Delta x, \Delta y)}{\Delta y \Delta x}$$

In similar fashion to the above argument, it can be shown that

$$f_{yx}(x, y) = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{w(\Delta x, \Delta y)}{\Delta y \Delta x}$$

and the result follows. ■

PROOF OF THE TWO-VARIABLE CHAIN RULE FOR DERIVATIVES

L.11 THEOREM (*Theorem 13.5.1*) *If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

PROOF Let Δx, Δy, and Δz denote the changes in x, y, and z, respectively, that correspond to a change of Δt in t. Then

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}, \quad \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \quad \frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

Since $f(x, y)$ is differentiable at $(x(t), y(t))$, it follows from (5) in Section 13.4 that

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{35}$$

where the partial derivatives are evaluated at $(x(t), y(t))$ and where $\epsilon(\Delta x, \Delta y)$ satisfies $\epsilon(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ and $\epsilon(0, 0) = 0$. Dividing both sides of (35) by Δt yields

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\epsilon(\Delta x, \Delta y)\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} \tag{36}$$

Since

$$\lim_{\Delta t \to 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{|\Delta t|} = \lim_{\Delta t \to 0} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} = \sqrt{\left(\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\right)^2 + \left(\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}\right)^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

we have

$$\begin{aligned}\lim_{\Delta t\to 0}\left|\frac{\epsilon(\Delta x,\Delta y)\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta t}\right| &= \lim_{\Delta t\to 0}\frac{|\epsilon(\Delta x,\Delta y)|\sqrt{(\Delta x)^2+(\Delta y)^2}}{|\Delta t|}\\ &= \lim_{\Delta t\to 0}|\epsilon(\Delta x,\Delta y)|\cdot\lim_{\Delta t\to 0}\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{|\Delta t|}\\ &= 0\cdot\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=0\end{aligned}$$

Therefore,

$$\lim_{\Delta t\to 0}\frac{\epsilon(\Delta x,\Delta y)\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta t}=0$$

Taking the limit as $\Delta t\to 0$ of both sides of (36) then yields the equation

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt} \quad \blacksquare$$