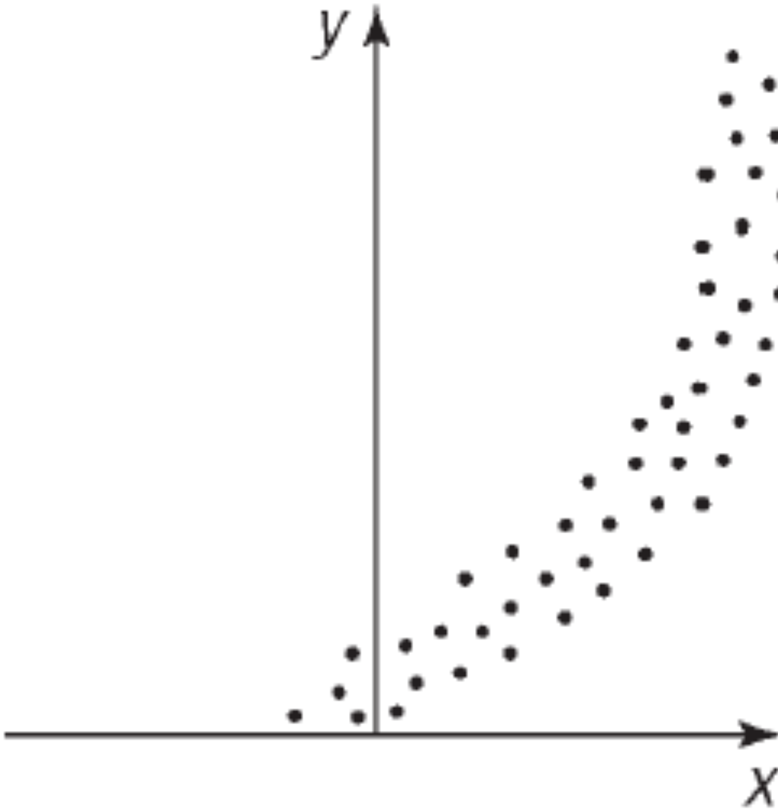
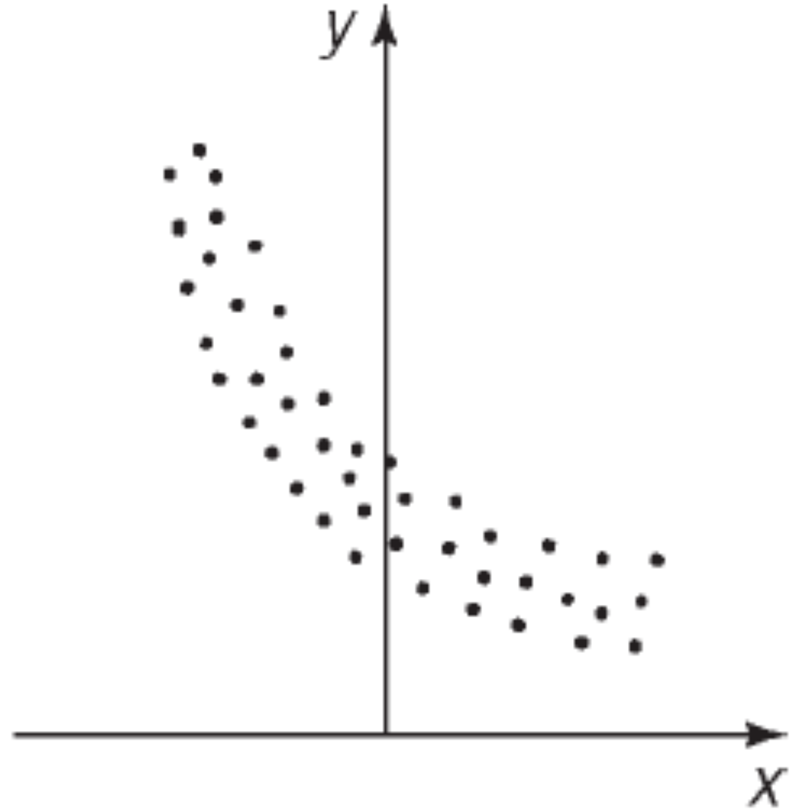
Section 6.9

Building Exponential, Logarithmic, and Logistic Models from Data



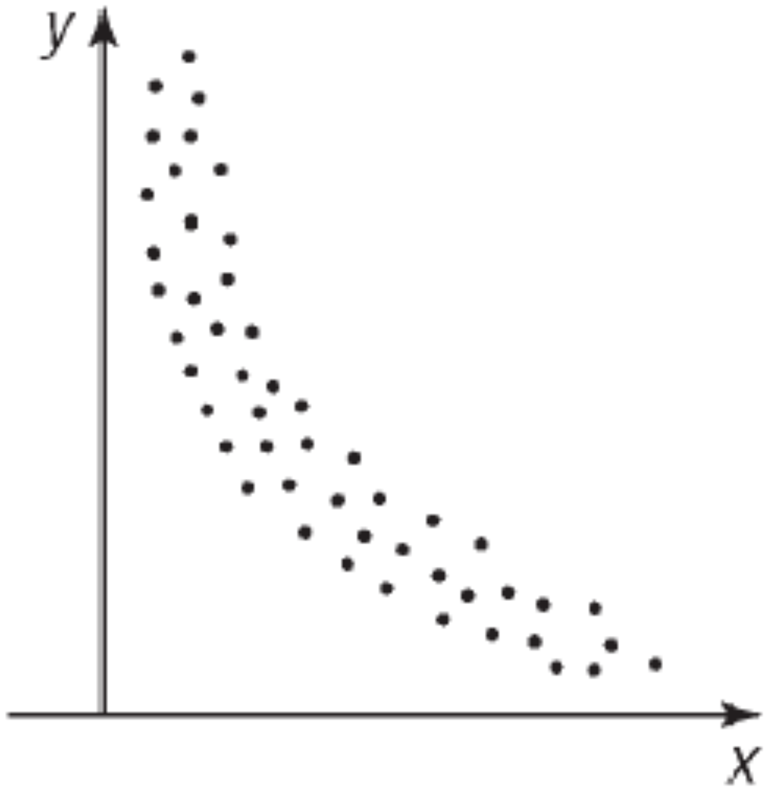
$y = ab^x, a > 0, b > 1$

Exponential



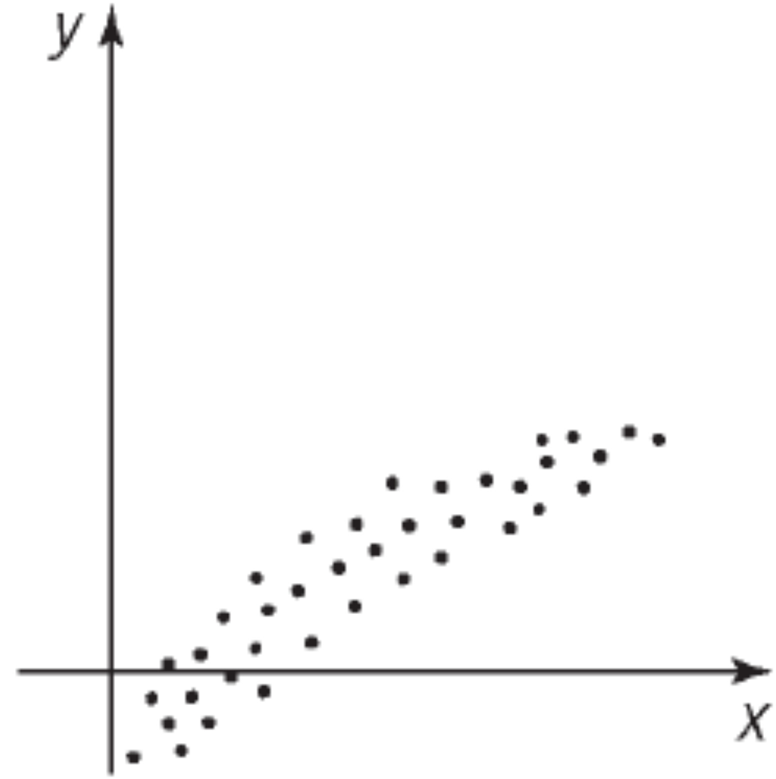
$y = ab^x, 0 < b < 1, a > 0$

Exponential



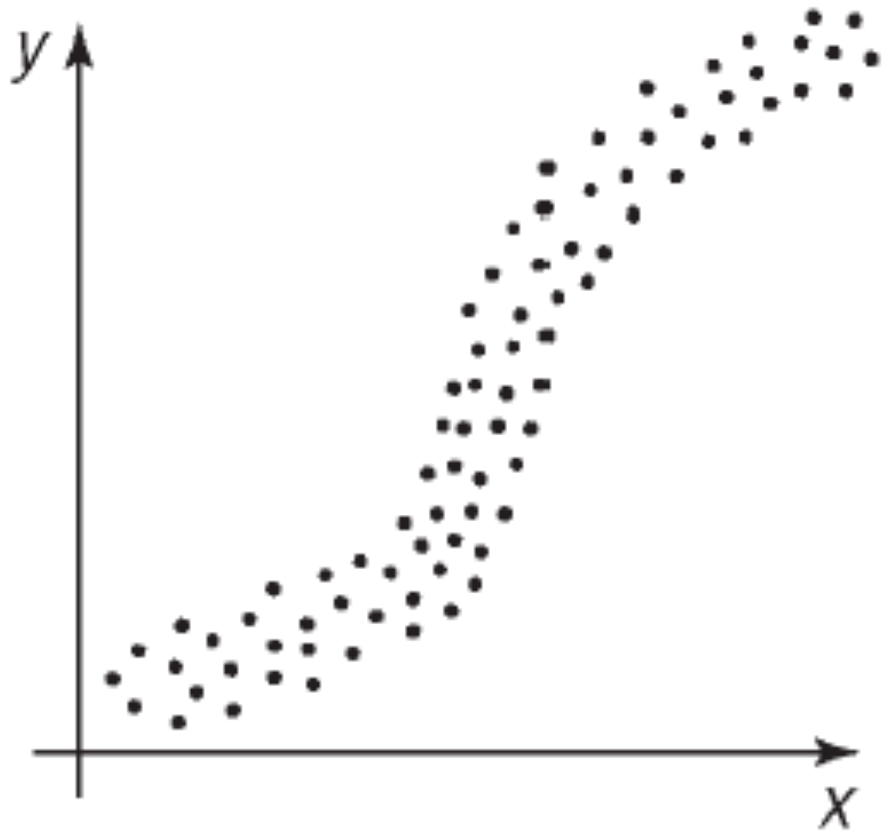
$y = a + b \ln x, a > 0, b < 0$

Logarithmic



$y = a + b \ln x, a > 0, b > 0$

Logarithmic



$$y = \frac{c}{1 + ae^{-bx}},\ a > 0,\ b > 0,\ c > 0$$

Logistic

1 Build an Exponential Model from Data

EXAMPLE Fitting an Exponential Function to Data

Kathleen is interested in finding a function that explains the growth of cell phone usage in the United States. She gathers data on the number (in millions) of U.S. cell phone subscribers from 1985 through 2005. The data are shown in Table 9.

(a) Using a graphing utility, draw a scatter diagram with year as the independent variable.

(b) Using a graphing utility, fit an exponential function to the data.

(c) Express the function found in part (b) in the form $A = A_0 e^{kt}$.

(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.

(e) Using the solution to part (b) or (c), predict the number of U.S. cell phone subscribers in 2009.

(f) Interpret the value of k found in part (c).

Table on next slide

Table 9

Year, x	Number of Subscribers (in millions), y
1985 ($x = 1$)	0.34
1986 ($x = 2$)	0.68
1987 ($x = 3$)	1.23
1988 ($x = 4$)	2.07
1989 ($x = 5$)	3.51
1990 ($x = 6$)	5.28
1991 ($x = 7$)	7.56
1992 ($x = 8$)	11.03
1993 ($x = 9$)	16.01
1994 ($x = 10$)	24.13
1995 ($x = 11$)	33.76
1996 ($x = 12$)	44.04
1997 ($x = 13$)	55.31
1998 ($x = 14$)	69.21
1999 ($x = 15$)	86.05
2000 ($x = 16$)	109.48
2001 ($x = 17$)	128.37
2002 ($x = 18$)	140.77
2003 ($x = 19$)	158.72
2004 ($x = 20$)	182.14
2005 ($x = 21$)	207.90
2006 ($x = 22$)	233.00
2007 ($x = 23$)	255.40
2008 ($x = 24$)	270.33

(a) Enter the data into the graphing utility, letting 1 represent 1985, 2 represent 1986, and so on. We obtain the scatter diagram shown in Figure 45. (next slide)

Figure 45

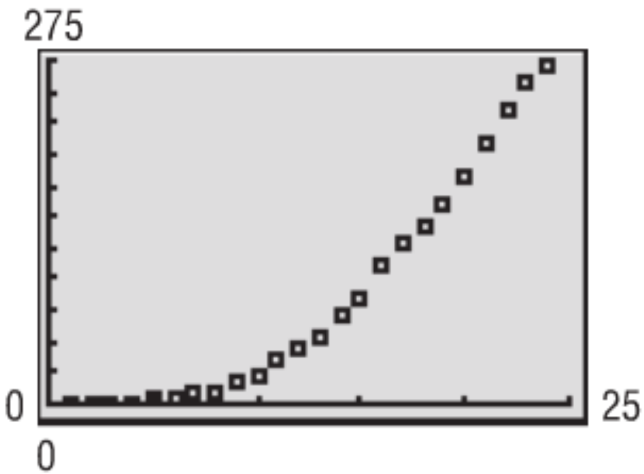
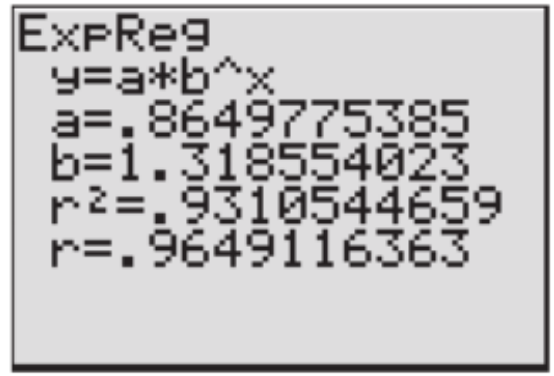


Figure 46



(b) A graphing utility fits the data in Figure 45 to an exponential function of the form $y = ab^x$ using the EXPonential REGression option. From Figure 46 we find that $y = ab^x = 0.86498(1.31855)^x$. Notice that $|r|$ is close to 1, indicating a good fit.

(c) To express $y = ab^x$ in the form $A = A_0e^{kt}$, where $x = t$ and $y = A$, proceed as follows:

$$ab^x = A_0e^{kt} \quad x = t$$

When $x = t = 0$, we find that $a = A_0$. This leads to

$$a = A_0 \qquad b^x = e^{kt}$$
$$b^x = (e^k)^t$$
$$b = e^k \qquad x = t$$

Since $y = ab^x = 0.86498(1.31855)^x$, we find that $a = 0.86498$ and $b = 1.31855$.

$$a = A_0 = 0.86498 \quad \text{and} \quad b = e^k = 1.31855$$

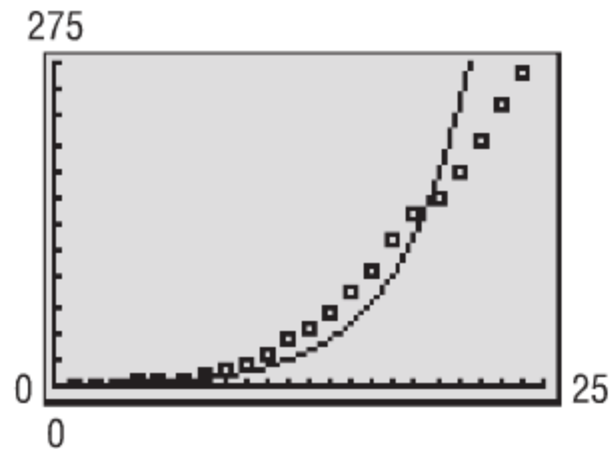
We want to find k, so we rewrite $e^k = 1.31855$ as a logarithm and obtain

$$k = \ln(1.31855) \approx 0.2765$$

As a result, $A = A_0e^{kt} = 0.86498e^{0.2765t}$.

(d) See Figure 47 for the graph of the exponential function of best fit.

Figure 47



(e) Let $t = 25$ (end of 2009) in the function found in part (c). The predicted number (in millions) of cell phone subscribers in the United States in 2009 is

$$A_0e^{kt} = 0.86498e^{0.2765(25)} \approx 869$$

This prediction (869 million) far exceeds what the U.S. population was in 2009 (currently the U.S. population is about 304 million). See the answer in part (f).

(f) The value of $k = 0.2765$ represents the growth rate of the number of cell phone subscribers in the United States. Over the period 1985 through 2008, the number of cell phone subscribers grew at an annual rate of 27.65% compounded continuously. This growth rate is not sustainable as we learned in part (e). In Problem 10 you are asked to build a better model from these data.

2 Build a Logarithmic Model from Data

EXAMPLE Fitting a Logarithmic Function to Data

Jodi, a meteorologist, is interested in finding a function that explains the relation between the height of a weather balloon (in kilometers) and the atmospheric pressure (measured in millimeters of mercury) on the balloon. She collects the data shown in Table 10.

(a) Using a graphing utility, draw a scatter diagram of the data with atmospheric pressure as the independent variable.

(b) It is known that the relation between atmospheric pressure and height follows a logarithmic model. Using a graphing utility, fit a logarithmic function to the data.

(c) Draw the logarithmic function found in part (b) on the scatter diagram.

(d) Use the function found in part (b) to predict the height of the weather balloon if the atmospheric pressure is 560 millimeters of mercury.

Table on next slide

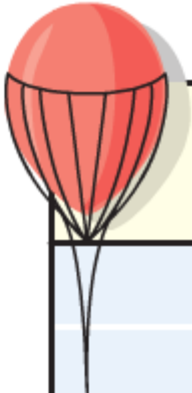
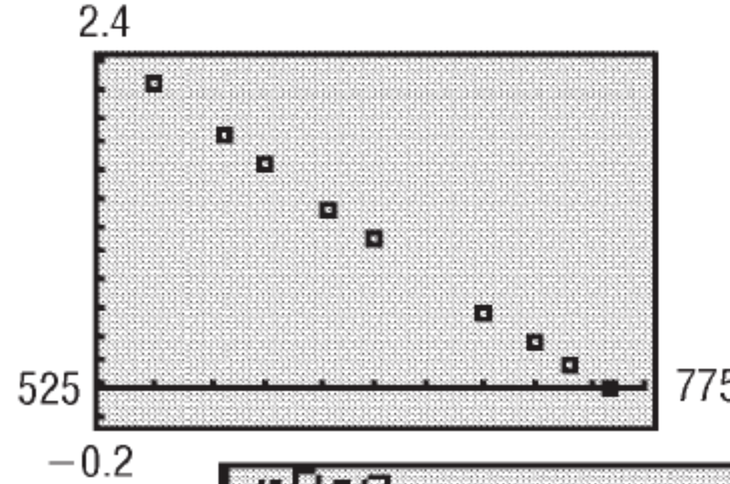
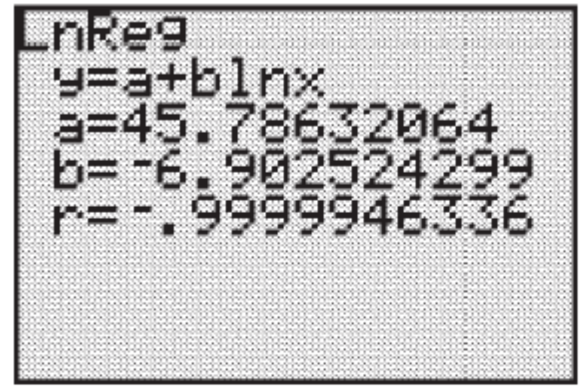
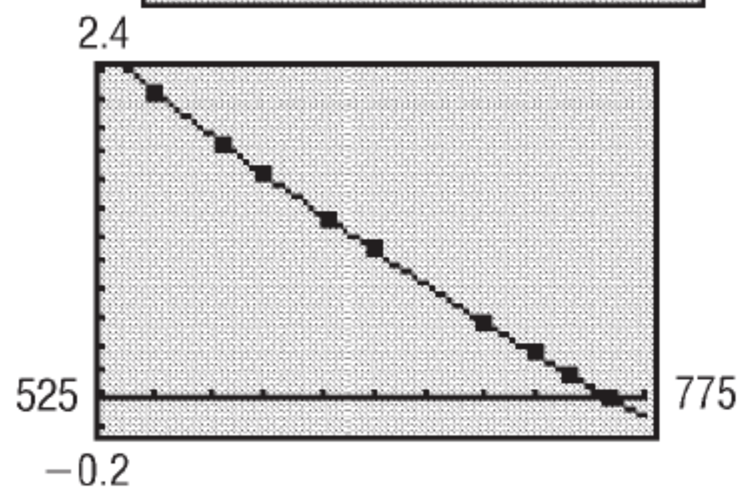


Table 10

Atmospheric Pressure, p	Height, h
760	0
740	0.184
725	0.328
700	0.565
650	1.079
630	1.291
600	1.634
580	1.862
550	2.235







(d) $h(560) = 45.7863 - 6.9025\ln(560)$

≈ 2.108 kilometers

3 Build a Logistic Model from Data

EXAMPLE Fitting a Logistic Function to Data

The data in Table 11 represent the amount of yeast biomass in a culture after t hours.

(a) Using a graphing utility, draw a scatter diagram of the data with time as the independent variable.

(b) Using a graphing utility, fit a logistic function to the data.

(c) Using a graphing utility, graph the function found in part (b) on the scatter diagram.

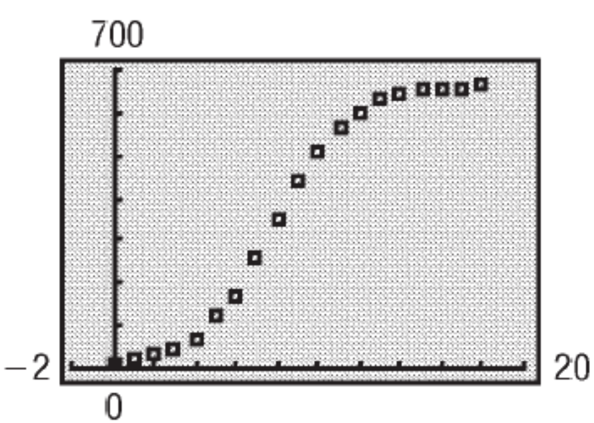
(d) What is the predicted carrying capacity of the culture?

(e) Use the function found in part (b) to predict the population of the culture at $t = 19$ hours.

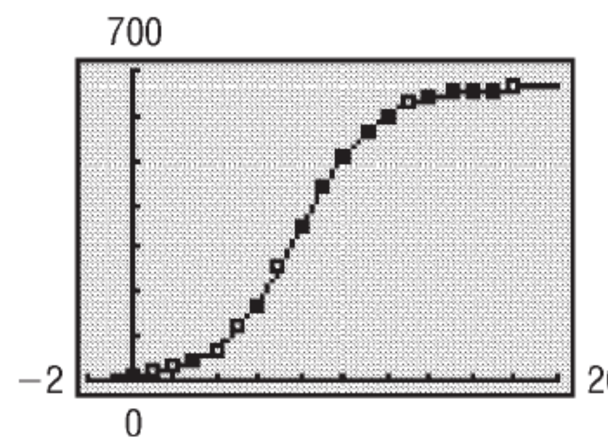
Table on next slide

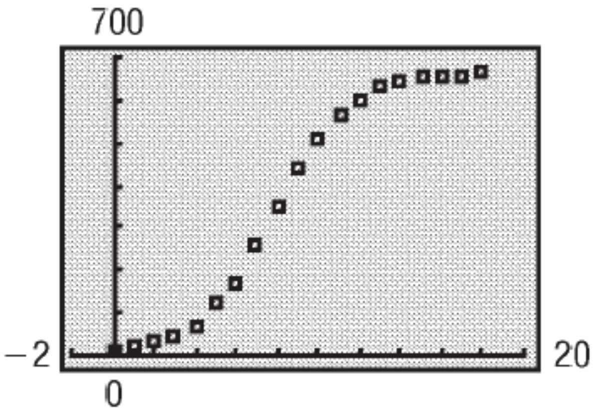
Table 11

Time (in hours)	Yeast Biomass	Time (in hours)	Yeast Biomass
0	9.6	10	513.3
1	18.3	11	559.7
2	29.0	12	594.8
3	47.2	13	629.4
4	71.1	14	640.8
5	119.1	15	651.1
6	174.6	16	655.9
7	257.3	17	659.6
8	350.7	18	661.8
9	441.0		




```
Logistic
 y=c/(1+ae^(-bx))
 a=71.57629487
 b=.5469947267
 c=663.0219908
```






```
Logistic
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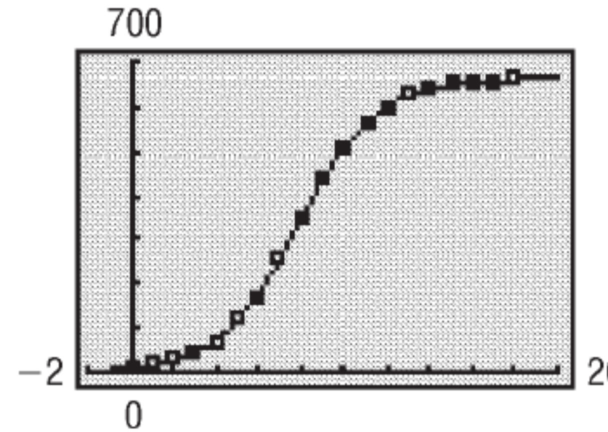

$$y = \frac{663.0}{1 + 71.6e^{-0.5470x}}$$

(d) Based on the logistic growth model found in part (b), the carrying capacity of the culture is 663.

(e) Using the logistic growth model found in part (b), the predicted amount of yeast biomass at $t = 19$ hours is

$$y = \frac{663.0}{1 + 71.6e^{-0.5470(19)}} \approx 661.5$$