Section 12.3

Systems of Linear Equations: Determinants

1 Evaluate 2 by 2 Determinants

DEFINITION

If a, b, c, and d are four real numbers, the symbol

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called a **2 by 2 determinant.** Its value is the number $ad - bc$; that is,

$$\boxed{D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \textbf{(1)}}$$

bc

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

ad

Minus

EXAMPLE Evaluating a 2 × 2 Determinant

Evaluate: $$\begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} = (-2)(-1) - (4)(3) = 2 - 12 = -10$$

bc

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

ad

Minus

2 Use Cramer's Rule to Solve a System of Two Equations Containing Two Variables

Cramer's Rule for Two Equations Containing Two Variables

The solution to the system of equations

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases}$$

is given by

$$x = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

provided that

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

Cramer's Rule

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases} \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$D_x = \begin{vmatrix} s & b \\ t & d \end{vmatrix} \qquad D_y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$$

if $D \neq 0$,

$$\boxed{x = \frac{D_x}{D}, \qquad y = \frac{D_y}{D}}$$

EXAMPLE

Solving a System of Linear Equations Using Determinants

Use Cramer's Rule, if applicable, to solve the system

$$\begin{cases} 3x - 6y = 24 \\ 5x + 4y = 12 \end{cases} \qquad x = \frac{168}{42} = 4 \qquad y = \frac{-84}{42} = -2$$

$$D = \begin{vmatrix} 3 & -6 \\ 5 & 4 \end{vmatrix} = (3)(4) - (5)(-6) = 12 + 30 = 42$$

$$D_x = \begin{vmatrix} 24 & -6 \\ 12 & 4 \end{vmatrix} = (24)(4) - (12)(-6) = 96 + 72 = 168$$

$$D_y = \begin{vmatrix} 3 & 24 \\ 5 & 12 \end{vmatrix} = (3)(12) - (5)(24) = 36 - 120 = -84$$

Solution: $(4, -2)$

$$\boxed{x = \frac{D_x}{D}, \qquad y = \frac{D_y}{D}}$$

3 Evaluate 3 by 3 Determinants

A **3 by 3 determinant** is symbolized by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

in which $a_{11}, a_{12}, \ldots,$ are real numbers.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\substack{\text{2 by 2}\\ \text{determinant}\\ \text{left after}\\ \text{removing row}\\ \text{and column}\\ \text{containing } a_{11}}} \overset{\text{Minus}}{-} a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\substack{\text{2 by 2}\\ \text{determinant}\\ \text{left after}\\ \text{removing row}\\ \text{and column}\\ \text{containing } a_{12}}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\substack{\text{2 by 2}\\ \text{determinant}\\ \text{left after}\\ \text{removing row}\\ \text{and column}\\ \text{containing } a_{13}}}$$

EXAMPLE

Finding Minors of a 3 by 3 Determinant

For the determinant $A = \begin{vmatrix} -1 & 2 & 0 \\ 3 & 2 & 1 \\ -2 & 1 & -1 \end{vmatrix}$ find: (a) M_{12} (b) M_{23}

(a) $A = \begin{vmatrix} -1 & 2 & 0 \\ 3 & 2 & 1 \\ -2 & 1 & -1 \end{vmatrix}$ $\quad M_{12} = \begin{vmatrix} 3 & 1 \\ -2 & -1 \end{vmatrix} = (3)(-1)-(-2)(1) = -1$

(b) $A = \begin{vmatrix} -1 & 2 & 0 \\ 3 & 2 & 1 \\ -2 & 1 & -1 \end{vmatrix}$ $\quad M_{23} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = (-1)(1)-(-2)(2) = 3$

For an n by n determinant A, the **cofactor** of entry a_{ij}, denoted by A_{ij}, is given by

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}}$$

where M_{ij} is the minor of entry a_{ij}.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Expand down column 2.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{3+1} a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + (-1)^{3+3} a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Expand across row 3.

EXAMPLE

Evaluating a 3 × 3 Determinant

Find the value of the 3 by 3 determinant: $\begin{vmatrix} \boxed{1 \quad 2 \quad 1} \\ 3 \quad 5 \quad 1 \\ 2 \quad 6 \quad 7 \end{vmatrix}$

$$= (-1)^{1+1}(1)\begin{vmatrix} 5 & 1 \\ 6 & 7 \end{vmatrix} + (-1)^{1+2}(2)\begin{vmatrix} 3 & 1 \\ 2 & 7 \end{vmatrix} + (-1)^{1+3}(1)\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix}$$

$$= 29 - 38 + 8 = -1$$

4 Use Cramer's Rule to Solve a System of Three Equations Containing Three Variables

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases} \qquad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

Cramer's Rule for Three Equations Containing Three Variables

$$\boxed{x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}}$$

where

$$D_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix} \qquad D_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \qquad D_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

EXAMPLE Using Cramer's Rule

Use Cramer's Rule, if applicable, to solve the following system:

$$\begin{cases} x+2y+z=1 \\ 3x+5y+z=3 \\ 2x+6y+7z=1 \end{cases} \qquad D=\begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 2 & 6 & 7 \end{vmatrix}$$

$$=(-1)^{1+1}(1)\begin{vmatrix} 5 & 1 \\ 6 & 7 \end{vmatrix}+(-1)^{1+2}(2)\begin{vmatrix} 3 & 1 \\ 2 & 7 \end{vmatrix}+(-1)^{1+3}(1)\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix}=-1$$

$$D_x=\begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 1 & 6 & 7 \end{vmatrix}$$

$$=(-1)^{1+1}(1)\begin{vmatrix} 5 & 1 \\ 6 & 7 \end{vmatrix}+(-1)^{1+2}(2)\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}+(-1)^{1+3}(1)\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix}=2$$

$$x=\frac{D_x}{D} \qquad y=\frac{D_y}{D} \qquad z=\frac{D_z}{D}$$

EXAMPLE **Using Cramer's Rule** $D = -1 \quad D_x = 2$

Use Cramer's Rule, if applicable, to solve the following system:

$$\begin{cases} x + 2y + z = 1 \\ 3x + 5y + z = 3 \\ 2x + 6y + 7z = 1 \end{cases} \quad x = \frac{D_x}{D} = \frac{2}{-1} = -2, \quad y = \frac{D_y}{D} = \frac{-2}{-1} = 2, \quad z = \frac{D_z}{D} = \frac{1}{-1} = -1$$

$$D_y = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 3 & 1 \\ 2 & 1 & 7 \end{vmatrix} \qquad D_z = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 3 \\ 2 & 6 & 1 \end{vmatrix}$$

Solution: $(-2, 2, -1)$

$$= (-1)^{1+1}(1)\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix} + (-1)^{1+2}(1)\begin{vmatrix} 3 & 1 \\ 2 & 7 \end{vmatrix} + (-1)^{1+3}(1)\begin{vmatrix} 3 & 3 \\ 2 & 1 \end{vmatrix} = -2$$

$$= (-1)^{1+1}(1)\begin{vmatrix} 5 & 3 \\ 6 & 1 \end{vmatrix} + (-1)^{1+2}(2)\begin{vmatrix} 3 & 3 \\ 2 & 1 \end{vmatrix} + (-1)^{1+3}(1)\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix} = 1$$

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$

Cramer's Rule with Inconsistent or Dependent Systems

- If $D = 0$ and at least one of the determinants D_x, D_y, or D_z is different from 0, then the system is inconsistent and the solution set is $\varnothing$ or $\{ \ \}$.
- If $D = 0$ and all the determinants D_x, D_y, and D_z equal 0, then the system is consistent and dependent so that there are infinitely many solutions. The system must be solved using row reduction techniques.

5 Know Properties of Determinants

Properties of Determinants

The value of a determinant changes sign if any two rows (or any two columns) are interchanged. **(11)**

EXAMPLE

Demonstrating Theorem (11)

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2 \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

Properties of Determinants

If all the entries in any row (or any column) equal 0, the value of the determinant is 0. **(12)**

If any two rows (or any two columns) of a determinant have corresponding entries that are equal, the value of the determinant is 0. **(13)**

EXAMPLE

Demonstrating Theorem (13)

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + (-1)^{1+3} \cdot 3 \cdot \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 1(-3) - 2(-6) + 3(-3) = -3 + 12 - 9 = 0$$

Properties of Determinants

If any row (or any column) of a determinant is multiplied by a nonzero number k, the value of the determinant is also changed by a factor of k. **(14)**

EXAMPLE **Demonstrating Theorem (14)**

$$\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = 6 - 8 = -2$$

$$\begin{vmatrix} k & 2k \\ 4 & 6 \end{vmatrix} = 6k - 8k = -2k = k(-2) = k\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix}$$

Properties of Determinants

If the entries of any row (or any column) of a determinant are multiplied by a nonzero number k and the result is added to the corresponding entries of another row (or column), the value of the determinant remains unchanged. **(15)**

EXAMPLE

Demonstrating Theorem (15)

$$\begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} = -14 \qquad \begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} \to \begin{vmatrix} -7 & 0 \\ 5 & 2 \end{vmatrix} = -14$$

Multiply row 2 by -2 and add to row 1.