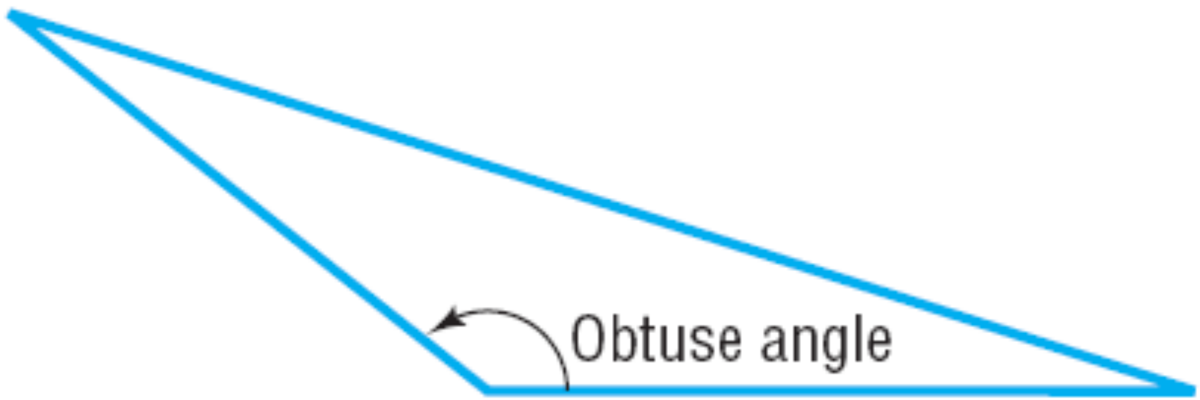
Section 9.2

The Law of Sines

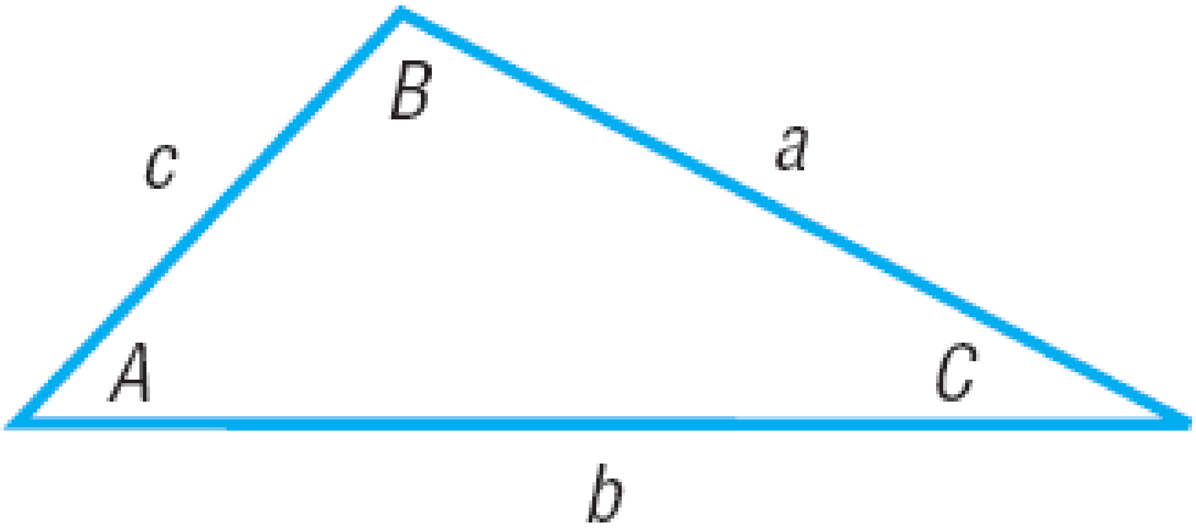
(a) All angles are acute



(b) Two acute angles and one obtuse angle

Oblique Triangle

(None of the angles is a right angle)

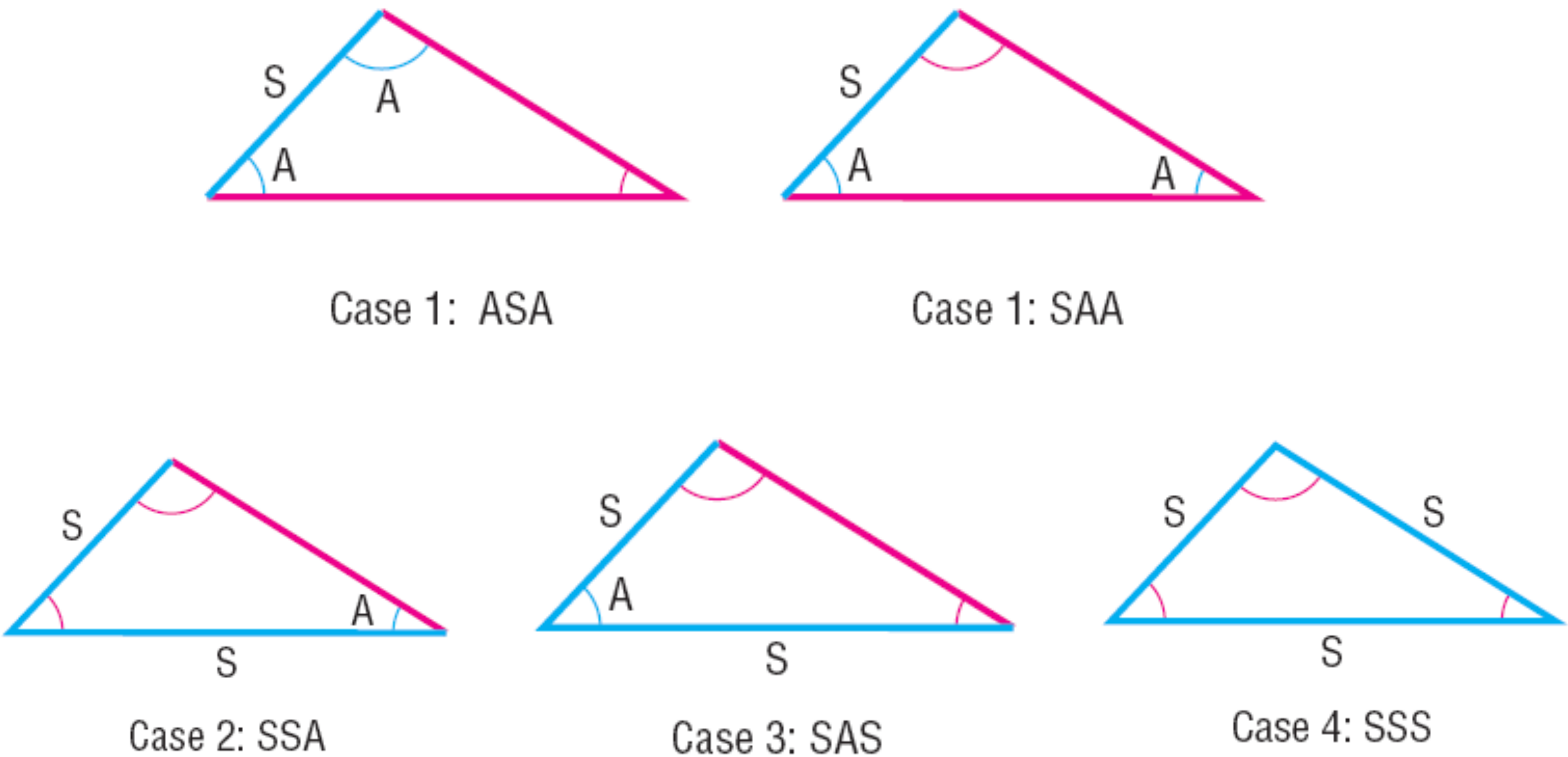


Case 1: One side and two angles are known (ASA or SAA).

Case 2: Two sides and the angle opposite one of them are known (SSA).

Case 3: Two sides and the included angle are known (SAS).

Case 4: Three sides are known (SSS).



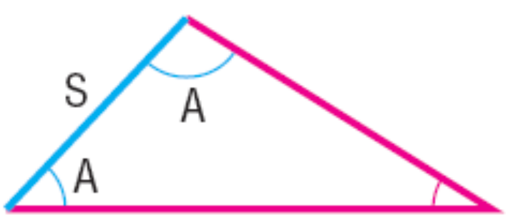
The **Law of Sines** is used to solve triangles for which Case 1 or 2 holds.

THEOREM

Law of Sines

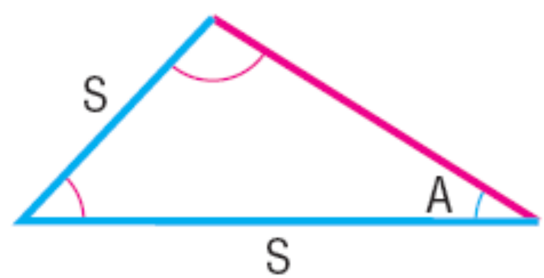
For a triangle with sides a, b, c and opposite angles A, B, C, respectively,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \textbf{(1)}$$



Case 1: ASA

Case 1: SAA



Case 2: SSA

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \frac{\sin A}{a} = \frac{\sin C}{c} \qquad \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$A + B + C = 180^\circ$$

1 Solve SAA or ASA Triangles

EXAMPLE Using the Law of Sines to Solve an SAA Triangle

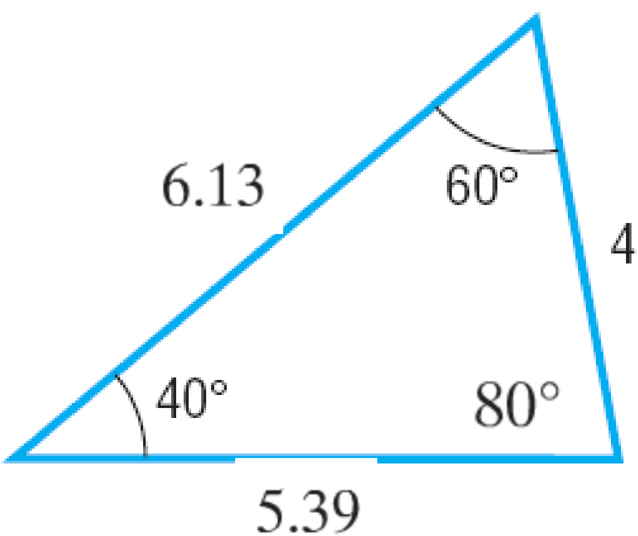
Solve the triangle: $A = 40°, B = 60°, a = 4$

$$40° + 60° + C = 180° \qquad C = 80°$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \frac{\sin A}{a} = \frac{\sin C}{c}$$

Because $a = 4$, $A = 40°$, $B = 60°$, and $C = 80°$, we have

$$\frac{\sin 40°}{4} = \frac{\sin 60°}{b} \qquad \frac{\sin 40°}{4} = \frac{\sin 80°}{c}$$



$$b = \frac{4\sin 60°}{\sin 40°} \approx 5.39$$

$$c = \frac{4\sin 80°}{\sin 40°} \approx 6.13$$

EXAMPLE Using the Law of Sines to Solve an ASA Triangle

Solve the triangle: $A = 35°, B = 15°, c = 5$

$35° + 15° + C = 180°$ $\quad C = 130°$

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

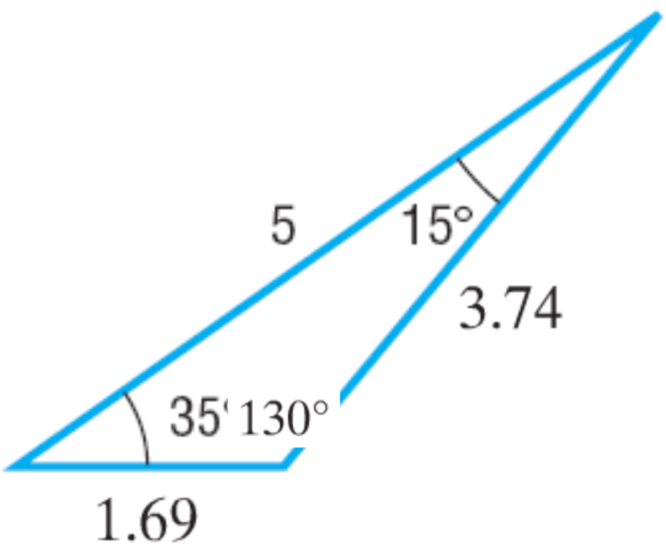
$$\frac{\sin 35°}{a} = \frac{\sin 130°}{5}$$

$$a = \frac{5 \sin 35°}{\sin 130°} \approx 3.74$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

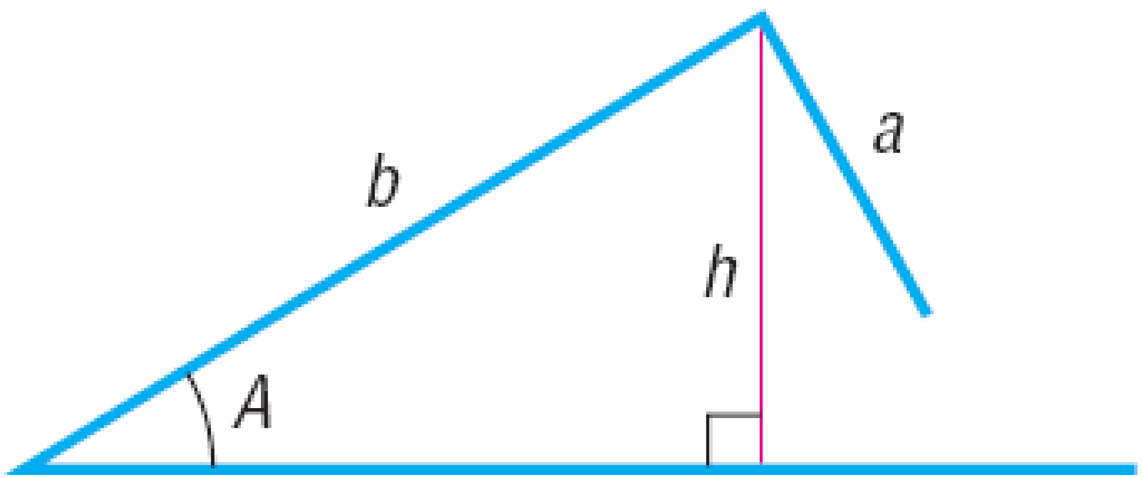
$$\frac{\sin 15°}{b} = \frac{\sin 130°}{5}$$

$$b = \frac{5 \sin 15°}{\sin 130°} \approx 1.69$$



2 Solve SSA Triangles

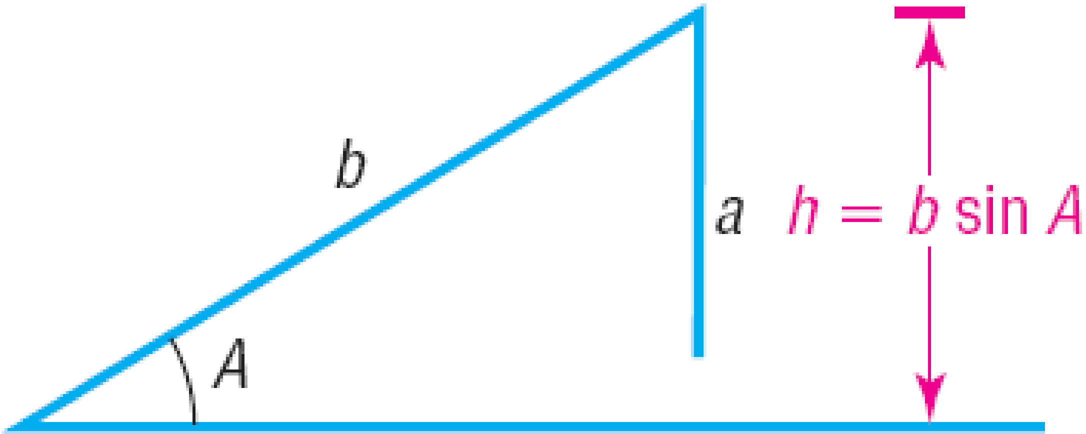
SSA --- The Ambiguous Case



$$\sin A = \frac{h}{b}$$

No Triangle If $a < h = b \sin A$, then side a is not sufficiently long to form a triangle. See Figure 14.

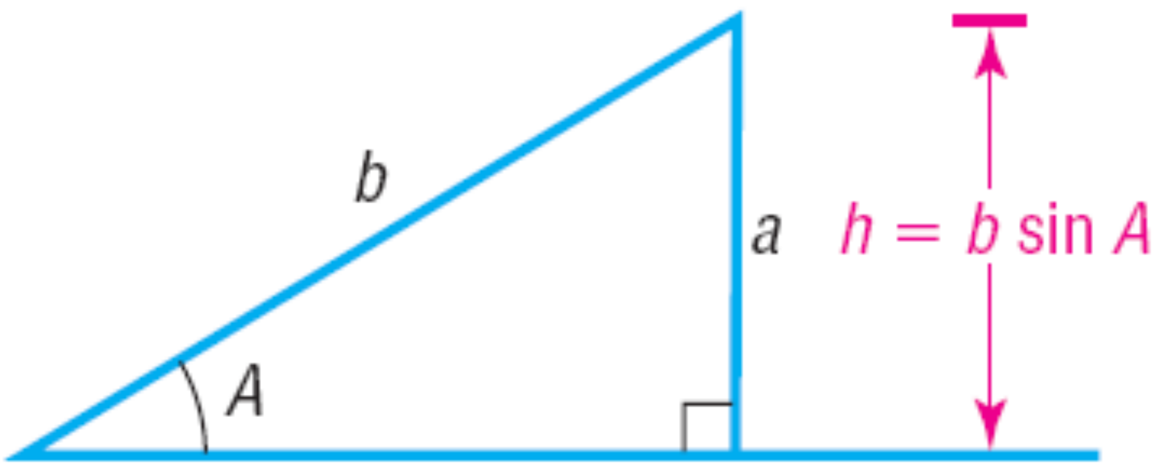
Figure 14
$a < h = b \sin A$



One Right Triangle If $a = h = b \sin A$, then side a is just long enough to form a right triangle. See Figure 15.

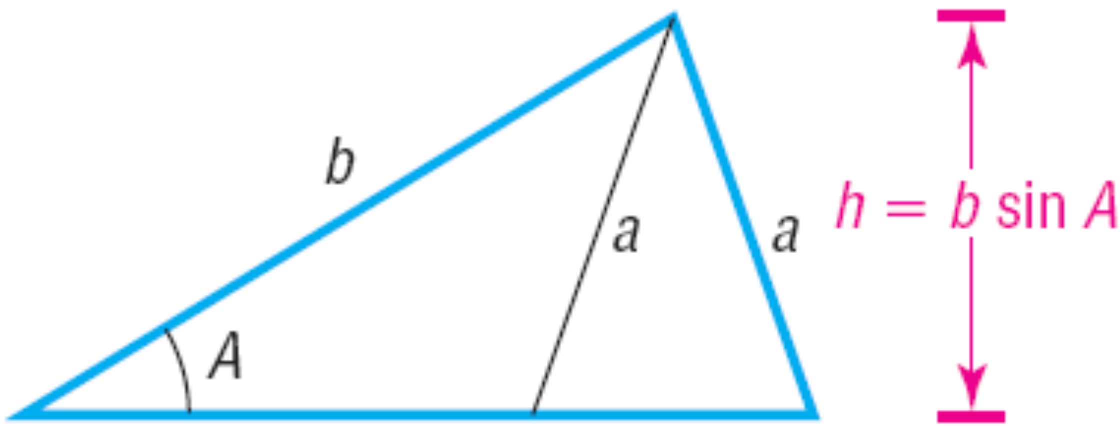
Figure 15

$a = h = b \sin A$



Two Triangles If $h = b \sin A < a$, and $a < b$ two distinct triangles can be formed from the given information. See Figure 16.

Figure 16
$b \sin A < a$ and $a < b$



One Triangle If $a \geq b$, only one triangle can be formed. See Figure 17.

Figure 17
$a \geq b$

EXAMPLE

Using the Law of Sines to Solve an SSA Triangle (One Solution)

Solve the triangle: $a = 3, b = 2, A = 40°$

$$\frac{\sin 40°}{3} = \frac{\sin B}{2} \qquad \sin B = \frac{2 \sin 40°}{3} \approx 0.43$$

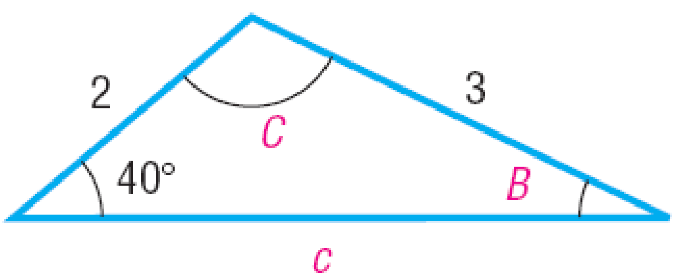
There are two angles B, $0° < B < 180°$, for which $\sin B \approx 0.43$.

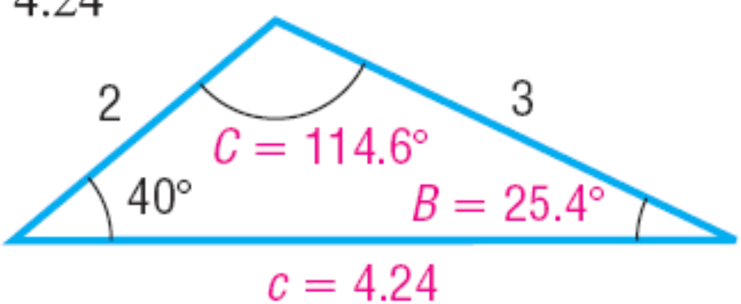
$$B_1 \approx 25.4° \quad \text{and} \quad B_2 \approx 180° - 25.4° = 154.6°$$

The second possibility, $B_2 \approx 154.6°$, is ruled out, because $A = 40°$ makes $A + B_2 \approx 194.6° > 180°$. Now, using $B_1 \approx 25.4°$, we find that

$$C = 180° - A - B_1 \approx 180° - 40° - 25.4° = 114.6°$$

$$\frac{\sin 40°}{3} = \frac{\sin 114.6°}{c} \qquad c = \frac{3 \sin 114.6°}{\sin 40°} \approx 4.24$$





EXAMPLE

Using the Law of Sines to Solve an SSA Triangle (Two Solutions)

Solve the triangle: $a = 6, b = 8, A = 35°$

$$\frac{\sin 35°}{6} = \frac{\sin B}{8}$$

$$\sin B = \frac{8 \sin 35°}{6} \approx 0.76$$

$$B_1 \approx 49.9°$$

$$B_2 \approx 180° - 49.9° = 130.1°$$

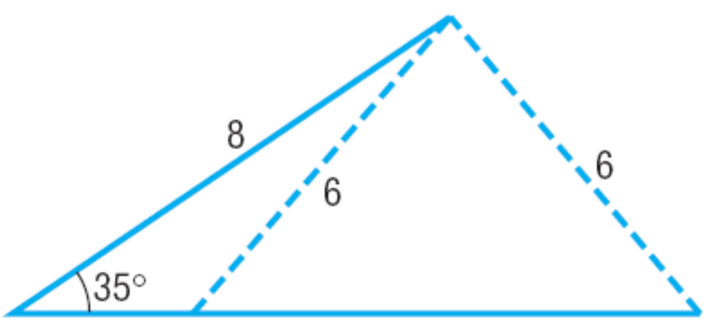
$$C_1 = 180° - A - B_1 \approx 95.1°$$

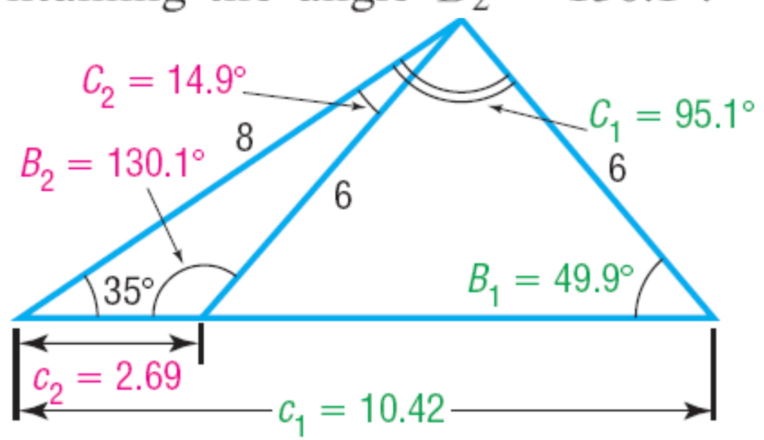
$$\frac{\sin 35°}{6} = \frac{\sin 95.1°}{c_1} \qquad c_1 = \frac{6 \sin 95.1°}{\sin 35°} \approx 10.42$$

$$C_2 = 180° - A - B_2 \approx 14.9°$$

$$\frac{\sin 35°}{6} = \frac{\sin 14.9°}{c_2} \qquad c_2 = \frac{6 \sin 14.9°}{\sin 35°} \approx 2.69$$

For both choices of B, we have $A + B < 180°$. There are two triangles, one containing the angle $B_1 \approx 49.9°$ and the other containing the angle $B_2 \approx 130.1°$.





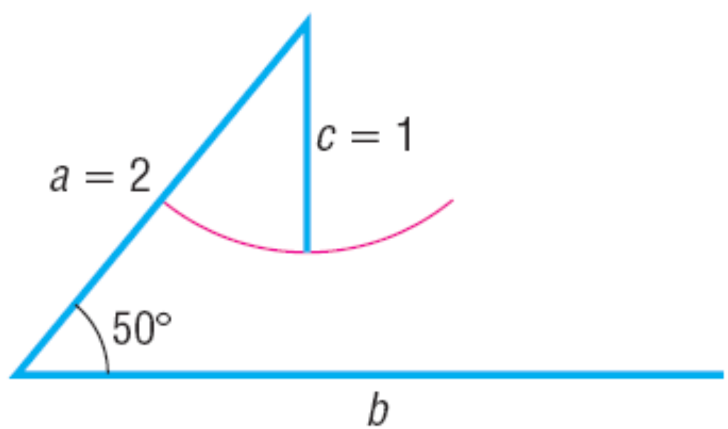
EXAMPLE

Using the Law of Sines to Solve an SSA Triangle (No Solution)

Solve the triangle: $a = 2, c = 1, C = 50^\circ$

$$\frac{\sin A}{2} = \frac{\sin 50^\circ}{1} \qquad \sin A = 2\sin 50^\circ \approx 1.53$$

Since there is no angle A for which $\sin A > 1$, there can be no triangle with the given measurements. Figure 20 illustrates the measurements given. Notice that, no matter how we attempt to position side c, it will never touch side b to form a triangle.



3 Solve Applied Problems

EXAMPLE Finding the Height of a Mountain

To measure the height of a mountain, a surveyor takes two sightings of the peak at a distance 900 meters apart on a direct line to the mountain.* See Figure 31(a). The first observation results in an angle of elevation of 47° and the second results in an angle of elevation of 35°. If the transit is 2 meters high, what is the height h of the mountain?

$$C + 47^\circ = 180^\circ$$

$$C = 133^\circ$$

$$A + C + 35^\circ = 180^\circ$$

$$A = 12^\circ$$

$$\frac{\sin 12^0}{900} = \frac{\sin 133^0}{c}$$

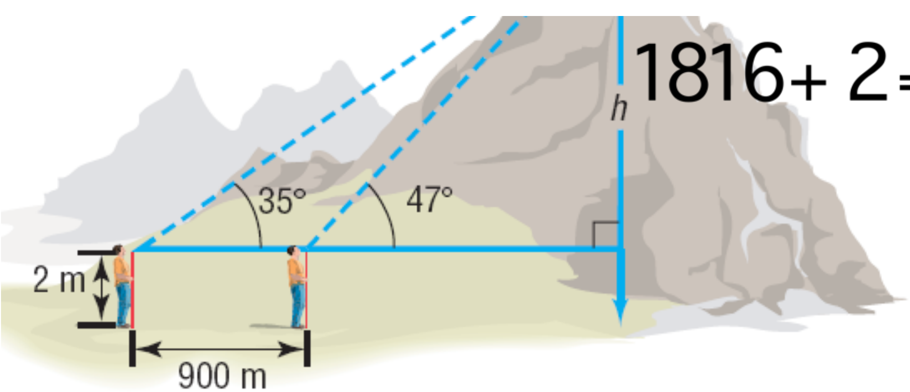
$$c = \frac{900 \sin 133^\circ}{\sin 12^\circ} \approx 3165.86$$

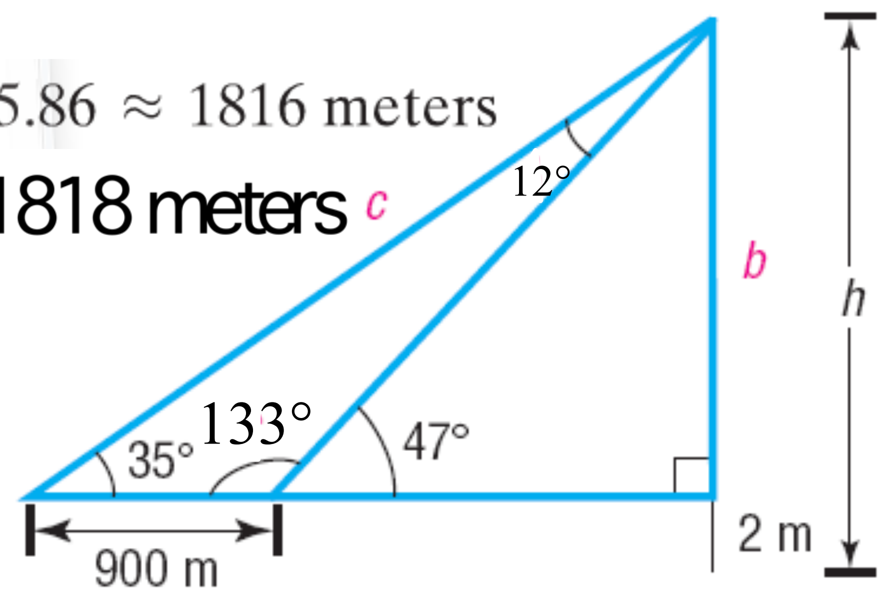
$$\sin 35^\circ = \frac{b}{c}$$

$c = 3165.86$

$$b = 3165.86 \sin 35^\circ \approx 1815.86 \approx 1816 \text{ meters}$$

$$1816 + 2 = 1818 \text{ meters}$$



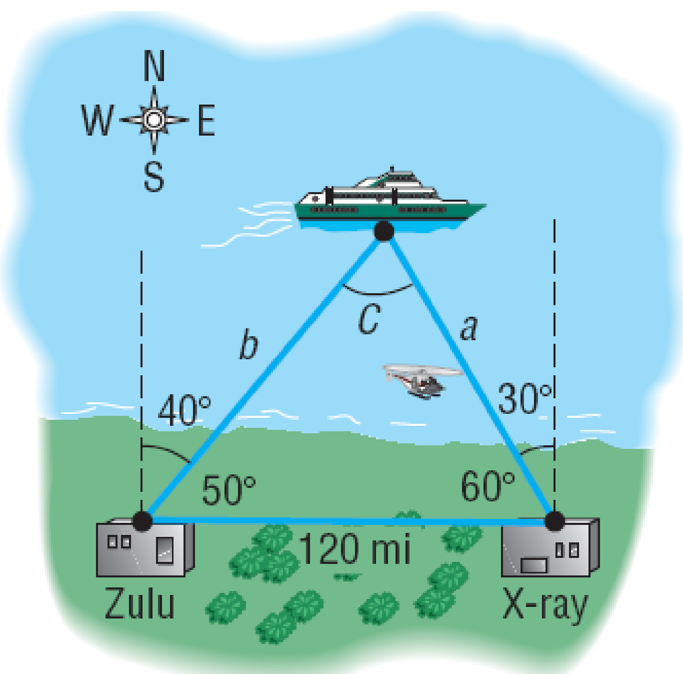


EXAMPLE Rescue at Sea

Coast Guard Station Zulu is located 120 miles due west of Station X-ray. A ship at sea sends an SOS call that is received by each station. The call to Station Zulu indicates that the bearing of the ship from Zulu is N40°E (40° east of north). The call to Station X-ray indicates that the bearing of the ship from X-ray is N30°W (30° west of north).

(a) How far is each station from the ship?

(b) If a helicopter capable of flying 200 miles per hour is dispatched from the nearest station to the ship, how long will it take to reach the ship?



$$C = 180° - 50° - 60° = 70°$$

$$\frac{\sin 50°}{a} = \frac{\sin 70°}{120}$$

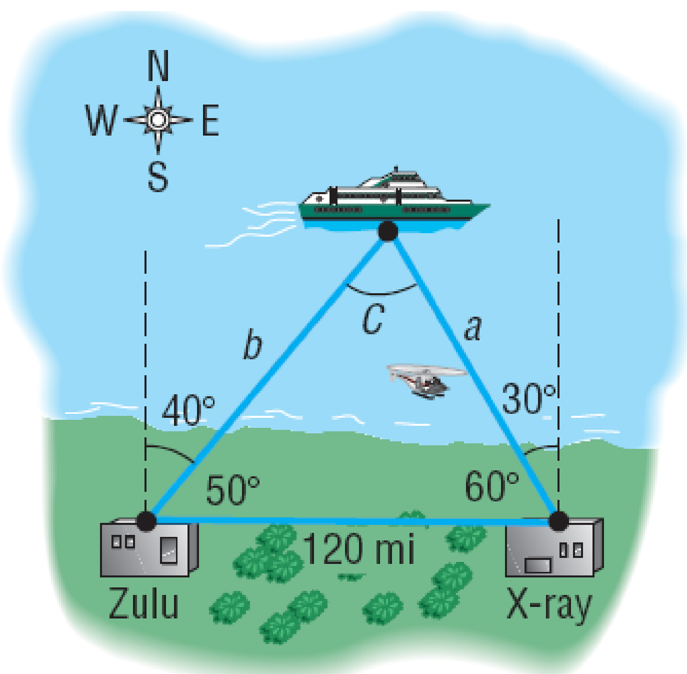
$$a = \frac{120 \sin 50°}{\sin 70°} \approx 97.82 \text{ miles}$$

EXAMPLE Rescue at Sea

Coast Guard Station Zulu is located 120 miles due west of Station X-ray. A ship at sea sends an SOS call that is received by each station. The call to Station Zulu indicates that the bearing of the ship from Zulu is N40°E (40° east of north). The call to Station X-ray indicates that the bearing of the ship from X-ray is N30°W (30° west of north).

(a) How far is each station from the ship?

(b) If a helicopter capable of flying 200 miles per hour is dispatched from the nearest station to the ship, how long will it take to reach the ship?



$$C = 180° - 50° - 60° = 70°$$

$$\frac{\sin 60°}{b} = \frac{\sin 70°}{120}$$

$$b = \frac{120 \sin 60°}{\sin 70°} \approx 110.59 \text{ miles}$$

EXAMPLE Rescue at Sea

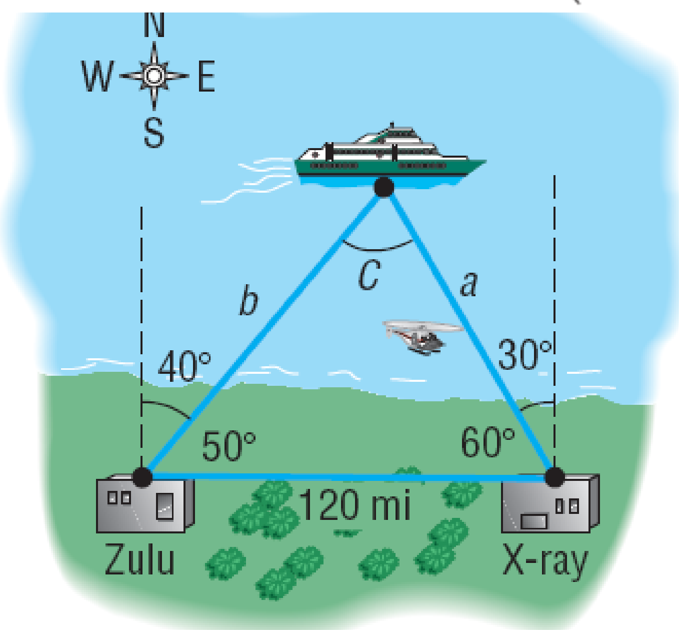
(a) How far is each station from the ship?

(b) If a helicopter capable of flying 200 miles per hour is dispatched from the nearest station to the ship, how long will it take to reach the ship?

Station Zulu is about 111 miles from the ship, and Station X-ray is about 98 miles from the ship.

(b) The time t needed for the helicopter to reach the ship from Station X-ray is found by using the formula

$$(\text{Rate}, r)(\text{Time}, t) = \text{Distance}, a$$



$$t = \frac{a}{r} = \frac{97.82}{200} \approx 0.49 \text{ hour} \approx 29 \text{ minutes}$$

It will take about 29 minutes for the helicopter to reach the ship.

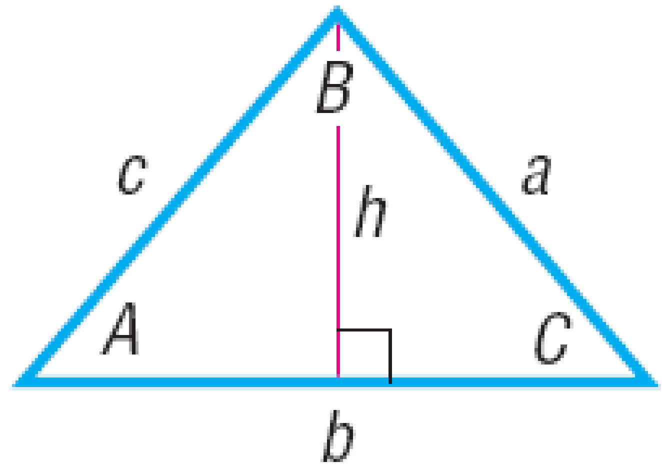
Proof of the Law of Sines

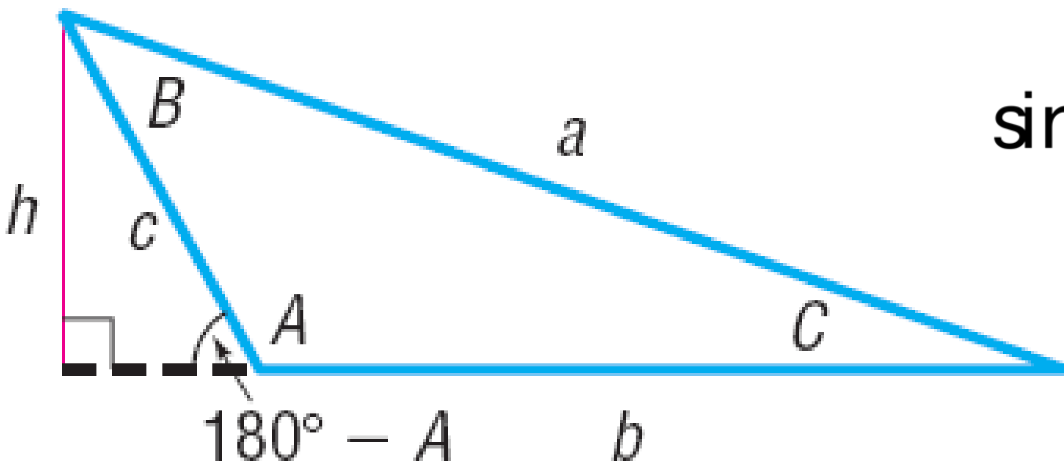
$$\sin C = \frac{h}{a}$$

$$h = a\sin C$$

$$\sin A = \frac{h}{c}$$

$$h = c\sin A$$



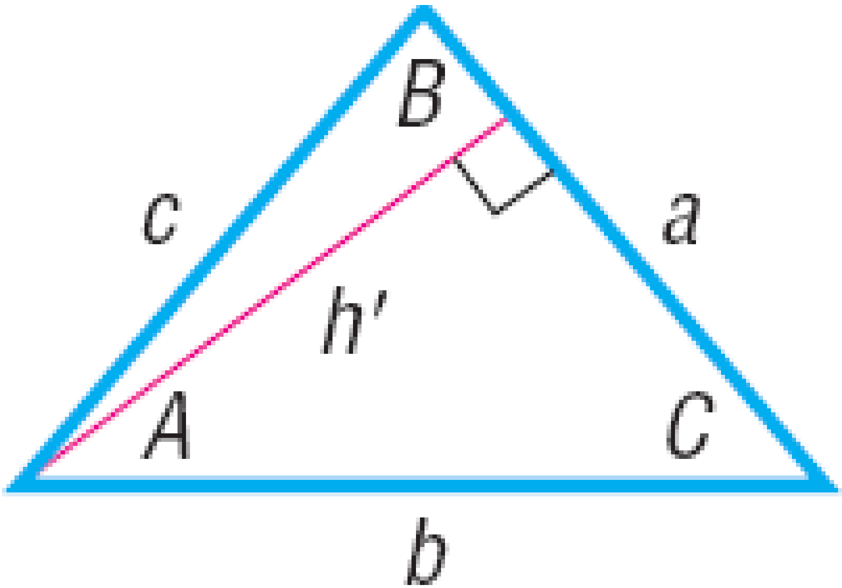


$$\sin(180° - A) = \sin A = \frac{h}{c}$$

$$h = c\sin A$$

$$a\sin C = c\sin A$$

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

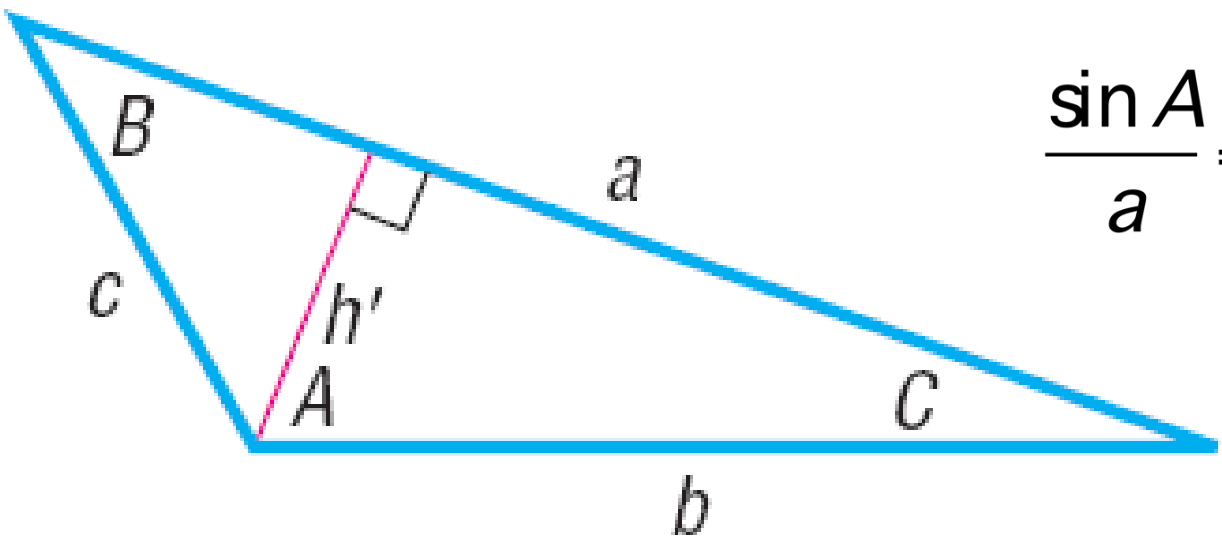


$$\sin B = \frac{h'}{c} \qquad \sin C = \frac{h'}{b}$$

$$h' = c \sin B \qquad h' = b \sin C$$

$$c \sin B = b \sin C$$

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$