Section 14.2

Permutations and Combinations

1 Solve Counting Problems Using Permutations Involving *n* Distinct Objects

A **permutation** is an ordered arrangement of r objects chosen from n objects.

Three Types of Permutations

1. The n objects are distinct (different), and repetition is allowed in the selection of r of them. [Distinct, with repetition]
2. The n objects are distinct (different), and repetition is not allowed in the selection of r of them, where $r \leq n$. [Distinct, without repetition]
3. The n objects are not distinct, and we use all of them in the arrangement. [Not distinct]

EXAMPLE

Counting Airport Codes [Permutation: Distinct, with Repetition]

The International Airline Transportation Association (IATA) assigns three-letter codes to represent airport locations. For example, the airport code for Ft. Lauderdale, Florida, is FLL. Notice that repetition is allowed in forming this code. How many airport codes are possible?

We are choosing 3 letters from 26 letters and arranging them in order. In the ordered arrangement a letter may be repeated. This is an example of a permutation with repetition in which 3 objects are chosen from 26 distinct objects.

$$26 \cdot 26 \cdot 26 = 26^3 = 17{,}576$$

THEOREM

Permutations: Distinct Objects with Repetition

The number of ordered arrangements of r objects chosen from n objects, in which the n objects are distinct and repetition is allowed, is n^r.

EXAMPLE

Forming Codes [Permutation: Distinct, without Repetition]

Suppose that we wish to establish a three-letter code using any of the 26 uppercase letters of the alphabet, but we require that no letter be used more than once. How many different three-letter codes are there?

Some of the possibilities are ABC, ABD, ABZ, ACB, CBA, and so on. The task consists of making three selections. The first selection requires choosing from 26 letters. Because no letter can be used more than once, the second selection requires choosing from 25 letters. The third selection requires choosing from 24 letters. (Do you see why?) By the Multiplication Principle, there are

$$26 \cdot 25 \cdot 24 = 15{,}600$$

The notation $\boldsymbol{P(n, r)}$ represents the number of ordered arrangements of r objects chosen from n distinct objects, where $r \leq n$ and repetition is not allowed.

EXAMPLE Lining People Up

In how many ways can 5 people be lined up?

The 5 people are distinct. Once a person is in line, that person will not be repeated elsewhere in the line; and, in lining people up, order is important. We have a permutation of 5 objects taken 5 at a time. We can line up 5 people in

$$P(5, 5) = \underbrace{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{\text{5 factors}} = 120 \text{ ways}$$

$$\begin{aligned} P(n, r) &= \underset{\text{1st}}{n} \cdot \underset{\text{2nd}}{(n-1)} \cdot \underset{\text{3rd}}{(n-2)} \cdot \cdots \cdot \underset{r\text{th}}{[n-(r-1)]} \\ &= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1) \end{aligned}$$

THEOREM

Permutations of *r* Objects Chosen from *n* Distinct Objects without Repetition

The number of arrangements of n objects using $r \leq n$ of them, in which

1. the n objects are distinct,
2. once an object is used it cannot be repeated, and
3. order is important,

is given by the formula

$$P(n, r) = \frac{n!}{(n-r)!} \tag{1}$$

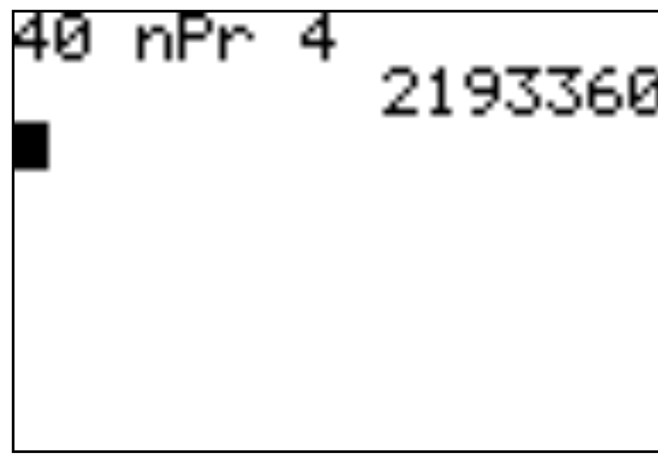
EXAMPLE Computing Permutations

Evaluate:

(a) $P(6,4) = \dfrac{6!}{(6-4)!} = \dfrac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$

(b) $P(7,2) = \dfrac{7!}{(7-2)!} = \dfrac{7!}{5!} = 7 \cdot 6 = 42$

(c) (c) $P(40,4)$ Use a calculator.



$$P(n,r) = \frac{n!}{(n-r)!}$$

EXAMPLE The Birthday Problem

If you have a group of four people that each have a different birthday, how many possible ways could this occur?

$$P(365,4)=\frac{365!}{(365-4)!}=\frac{365!}{361!}$$

$$=\frac{365\cdot 364\cdot 363\cdot 362\cdot 361!}{361!}$$

$$=365\cdot 364\cdot 363\cdot 362=17,458,601,160$$

2 Solve Counting Problems Using Combinations

A **combination** is an arrangement, without regard to order, of r objects selected from n distinct objects without repetition, where $r \leq n$. The notation $\boldsymbol{C(n, r)}$ represents the number of combinations of n distinct objects using r of them.

EXAMPLE Listing Combinations

List all the combinations of the 4 colors, red, green, yellow and blue taken 3 at a time. What is $C(4, 3)$?

red, green, yellow

red, green, blue

red, yellow, blue

green, yellow, blue

$$C(4,3) = 4$$

THEOREM

Number of Combinations of *n* Distinct Objects Taken *r* at a Time

The number of arrangements of n objects using $r \leq n$ of them, in which

1. the n objects are distinct,
2. once an object is used, it cannot be repeated, and
3. order is not important,

is given by the formula

$$C(n, r) = \frac{n!}{(n - r)!r!} \qquad \textbf{(2)}$$

EXAMPLE Using Formula (2)

Find the value of each expression.

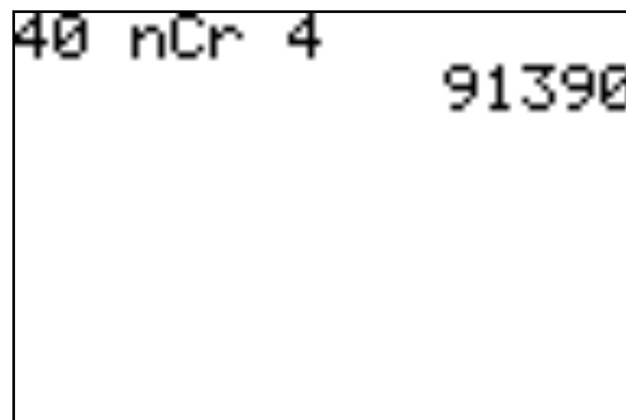
(a) $C(4, 2) = \dfrac{4!}{(4-2)!2!} = \dfrac{4!}{2!2!} = \dfrac{4 \cdot 3 \cdot 2!}{2 \cdot 2!} = 6$

(b) $C(5, 2) = \dfrac{5!}{(5-2)!2!} = \dfrac{5!}{3!2!} = \dfrac{5 \cdot 4 \cdot 3!}{2 \cdot 3!} = 10$

(c) $C(n, n) = \dfrac{n!}{(n-n)!n!} = 1$

(d) $C(n, 0) = \dfrac{n!}{(n-0)!0!} = 1$

(e) $C(40, 4)$ Use a calculator.



EXAMPLE Forming Committees

How many different committees of 4 people can be formed from a pool of 8 people?

$$C(8,4)=\frac{8!}{(8-4)!4!}=\frac{8\cdot 7\cdot 6\cdot 5\cdot 4!}{4!4!}=\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}=70$$

$$\boxed{C(n,r)=\frac{n!}{(n-r)!r!}}$$

EXAMPLE Forming Committees

How many ways can a committee consisting of 3 boys and 2 girls be formed if there are 7 boys and 10 girls eligible to serve on the committee?

$$C(7,3)\cdot C(10,2)=\frac{7!}{4!3!}\cdot\frac{10!}{8!2!}$$

$$=\frac{7\cdot 6\cdot 5\cdot 4!}{4!3\cdot 2}\cdot\frac{10\cdot 9\cdot 8!}{8!2}=35\cdot 45=1575$$

$$\boxed{C(n,r)=\frac{n!}{(n-r)!r!}}$$

3 Solve Counting Problems Using Permutations Involving *n* Nondistinct Objects

EXAMPLE **Forming Different Words**

How many different words (real or imaginary) can be formed using all the letters in the word REARRANGE?

Each word formed will have 9 letters: 3 R's, 2 A's, 2 E's, 1 N, and 1 G. To construct each word, we need to fill in 9 positions with the 9 letters:

$$\overline{1} \quad \overline{2} \quad \overline{3} \quad \overline{4} \quad \overline{5} \quad \overline{6} \quad \overline{7} \quad \overline{8} \quad \overline{9}$$

The process of forming a word consists of five tasks:

Task 1: Choose the positions for the 3 R's.

Task 2: Choose the positions for the 2 A's.

Task 3: Choose the positions for the 2 E's.

Task 4: Choose the position for the 1 N.

Task 5: Choose the position for the 1 G.

The process of forming a word consists of five tasks:

Task 1: Choose the positions for the 3 R's.

Task 2: Choose the positions for the 2 A's.

Task 3: Choose the positions for the 2 E's.

Task 4: Choose the position for the 1 N.

Task 5: Choose the position for the 1 G.

Task 1 can be done in $C(9, 3)$ ways. There then remain 6 positions to be filled, so Task 2 can be done in $C(6, 2)$ ways. There remain 4 positions to be filled, so Task 3 can be done in $C(4, 2)$ ways. There remain 2 positions to be filled, so Task 4 can be done in $C(2, 1)$ ways. The last position can be filled in $C(1, 1)$ way. Using the Multiplication Principle, the number of possible words that can be formed is

$$C(9,3) \cdot C(6,2) \cdot C(4,2) \cdot C(2,1) \cdot C(1,1) = \frac{9!}{3! \cdot \cancel{6!}} \cdot \frac{\cancel{6!}}{2! \cdot \cancel{4!}} \cdot \frac{\cancel{4!}}{2! \cdot \cancel{2!}} \cdot \frac{\cancel{2!}}{1! \cdot \cancel{1!}} \cdot \frac{\cancel{1!}}{0! \cdot 1!}$$

$$= \frac{9!}{3! \cdot 2! \cdot 2! \cdot 1! \cdot 1!} = 15{,}120$$

THEOREM

Permutations Involving *n* Objects That Are Not Distinct

The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, . . . , and n_k are of a kth kind is given by

$$\boxed{\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!} \qquad \textbf{(3)}}$$

where $n = n_1 + n_2 + \cdots + n_k$.

EXAMPLE **Arranging Flags**

How many different vertical arrangements are there of 10 flags if 5 are white, 4 are blue and 2 are red?

$$\frac{10!}{5!4!2!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!4\cdot 3\cdot 2\cdot 2} = 10\cdot 9\cdot 7 = 630$$

$$\boxed{\frac{n!}{n_1!\cdot n_2!\cdot\cdots\cdot n_k!}}$$