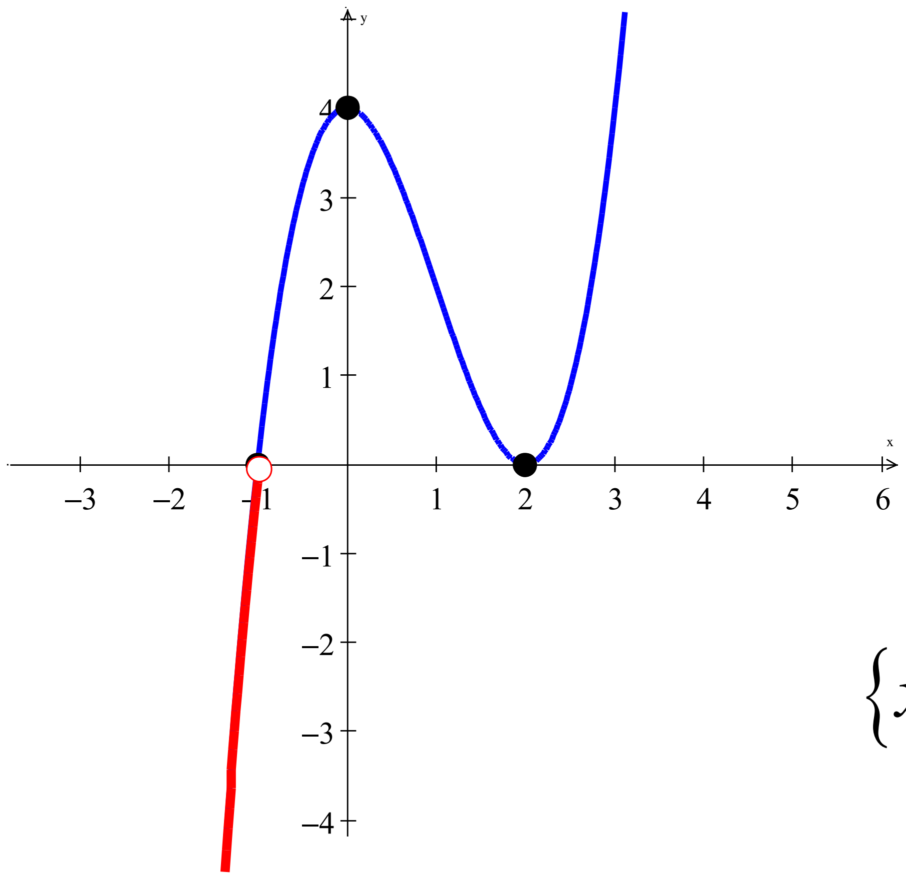
Section 5.4

Polynomial and Rational Inequalities

1 Solve Polynomial Inequalities

EXAMPLE Solving a Polynomial Inequality Using Its Graph

Solve $(x-2)^2(x+1)<0$ by graphing $f(x)=(x-2)^2(x+1)$



x-intercepts are 2 with multiplicity 2 (touches) and -1 with multiplicity 1 (crosses).

y-intercept is 4.

End behavior is like $f(x) = x^3$.

Where is this function less than 0?

$$\{x | x < -1\} \text{ or } (-\infty, -1)$$

EXAMPLE

How to Solve a Polynomial Inequality Algebraically

Solve the inequality $x^4 > x$, and graph the solution set.

Step 1: Write the inequality so that a polynomial expression *f* is on the left side and zero is on the right side.

$$x^4 - x > 0$$

$$x(x^3 - 1) = 0 \quad \text{Factor out x.}$$

Step 2: Determine the real zeros (x-intercepts of the graph) of *f*.

$$x(x - 1)(x^2 + x + 1) = 0 \quad \text{Factor the difference of two cubes.}$$

$$x = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x^2 + x + 1 = 0 \qquad x = 0 \quad \text{or} \quad x = 1$$

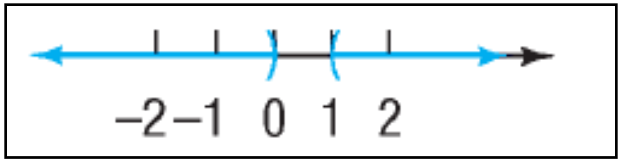
Step 3: Use the zeros found in Step 2 to divide the real number line into intervals.

Use the real zeros to separate the real number line into three intervals:

$$(-\infty, 0) \qquad (0, 1) \qquad (1, \infty)$$

Solve the inequality $x^4 > x$, and graph the solution set.

Step 4: Select a number in each interval, evaluate f at the number, and determine whether f is positive or negative. If f is positive, all values of f in the interval are positive. If f is negative, all values of f in the interval are negative.



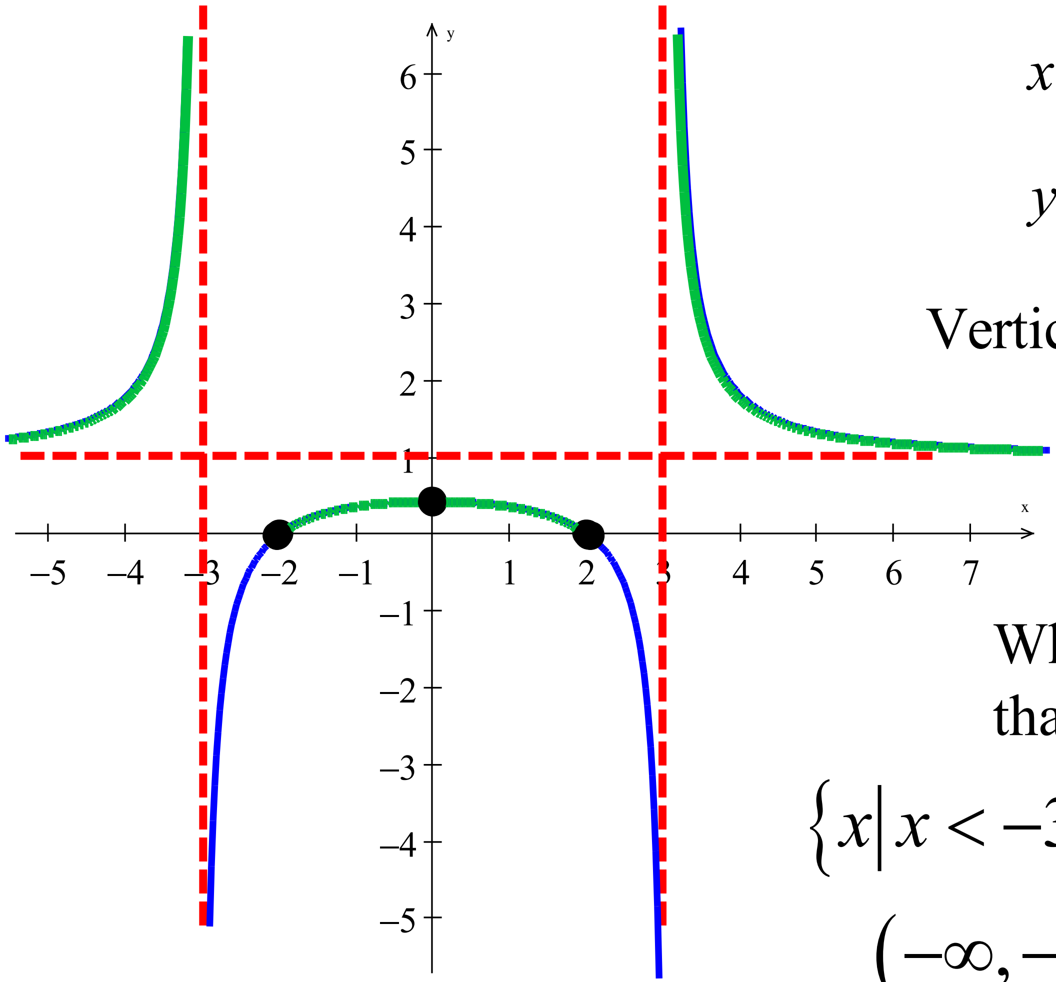
Since we want to know where $f(x)$ is positive, we conclude that $f(x) > 0$ for all numbers x for which $x < 0$ or $x > 1$. Because the original inequality is strict, numbers x that satisfy the equation $x^4 = x$ are not solutions. The solution set of the inequality $x^4 > x$ is $\{x \mid x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, 0) \cup (1, \infty)$.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Number Chosen	-1	$\frac{1}{2}$	2
Value of f	$f(-1) = 2$	$f\left(\frac{1}{2}\right) = -\frac{7}{16}$	$f(2) = 14$
Conclusion	Positive	Negative	Positive

2 Solve Rational Inequalities

EXAMPLE Solving a Rational Inequality Using Its Graph

Solve $\dfrac{x^2-4}{x^2-9} \geq 0$ by graphing $f(x) = \dfrac{x^2-4}{x^2-9}$.



x-intercepts are -2 and 2.

y-intercept is $\dfrac{4}{9}$.

Vertical asymptotes at $x = -3$ and $x = 3$

Horizontal asymptote at $y = 1$

Where is this function greater than or equal to 0?

$$\{x \mid x < -3 \text{ or } -2 \leq x \leq 2 \text{ or } x > 3\}$$

$$(-\infty, -3) \cup [-2, 2] \cup (3, \infty)$$

EXAMPLE How to Solve a Rational Inequality Algebraically

Solve the inequality $\dfrac{4x+5}{x+2} \geq 3$, and graph the solution set.

Step 1: Write the inequality so that a rational expression f is on the left side and zero is on the right side.

$$\frac{4x+5}{x+2} - 3 \geq 0 \qquad \frac{4x+5}{x+2} - 3 \cdot \frac{x+2}{x+2} \geq 0$$

$$\frac{4x+5-3x-6}{x+2} \geq 0 \qquad \frac{x-1}{x+2} \geq 0$$

Step 2: Determine the real zeros (x-intercepts of the graph) of f and the real numbers for which f is undefined.

The zero of $f(x) = \dfrac{x-1}{x+2}$ is 1.

Also, f is undefined for $x = -2$.

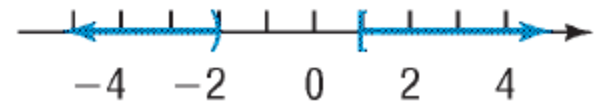
Step 3: Use the zeros and undefined values found in Step 2 to divide the real number line into intervals.

Use the zero and undefined value to separate the real number line into three intervals:

$$(-\infty, -2) \qquad (-2, 1) \qquad (1, \infty)$$

Solve the inequality $\dfrac{4x + 5}{x + 2} \geq 3$, and graph the solution set.

Step 4: *Select a number in each interval, evaluate f at the number, and determine whether f is positive or negative. If f is positive, all values of f in the interval are positive. If f is negative, all values of f in the interval are negative.*



Interval	$(\infty, -2)$	$(-2, 1)$	$(1, \infty)$
Number Chosen	-3	0	2
Value of f	$f(-3) = 4$	$f(0) = -\frac{1}{2}$	$f(2) = \frac{1}{4}$
Conclusion	Positive	Negative	Positive

Since we want to know where $f(x)$ is positive or zero, we conclude that $f(x) \geq 0$ for all numbers x for which $x < -2$ or $x \geq 1$. Notice we do not include -2 in the solution because -2 is not in the domain of f. The solution set of the inequality $\dfrac{4x + 5}{x + 2} \geq 3$ is $\{x \mid x < -2 \text{ or } x \geq 1\}$ or, using interval notation, $(-\infty, -2) \cup [1, \infty)$.

SUMMARY Steps for Solving Polynomial and Rational Inequalities Algebraically

STEP 1: Write the inequality so that a polynomial or rational expression f is on the left side and zero is on the right side in one of the following forms:

$$f(x) > 0 \qquad f(x) \geq 0 \qquad f(x) < 0 \qquad f(x) \leq 0$$

For rational expressions, be sure that the left side is written as a single quotient and find the domain of f.

STEP 2: Determine the real numbers at which the expression f equals zero and, if the expression is rational, the real numbers at which the expression f is undefined.

STEP 3: Use the numbers found in Step 2 to separate the real number line into intervals.

STEP 4: Select a number in each interval and evaluate f at the number.

(a) If the value of f is positive, then $f(x) > 0$ for all numbers x in the interval.

(b) If the value of f is negative, then $f(x) < 0$ for all numbers x in the interval.

If the inequality is not strict ($\geq$ or $\leq$), include the solutions of $f(x) = 0$ that are in the domain of f in the solution set. Be careful to exclude values of x where f is undefined.