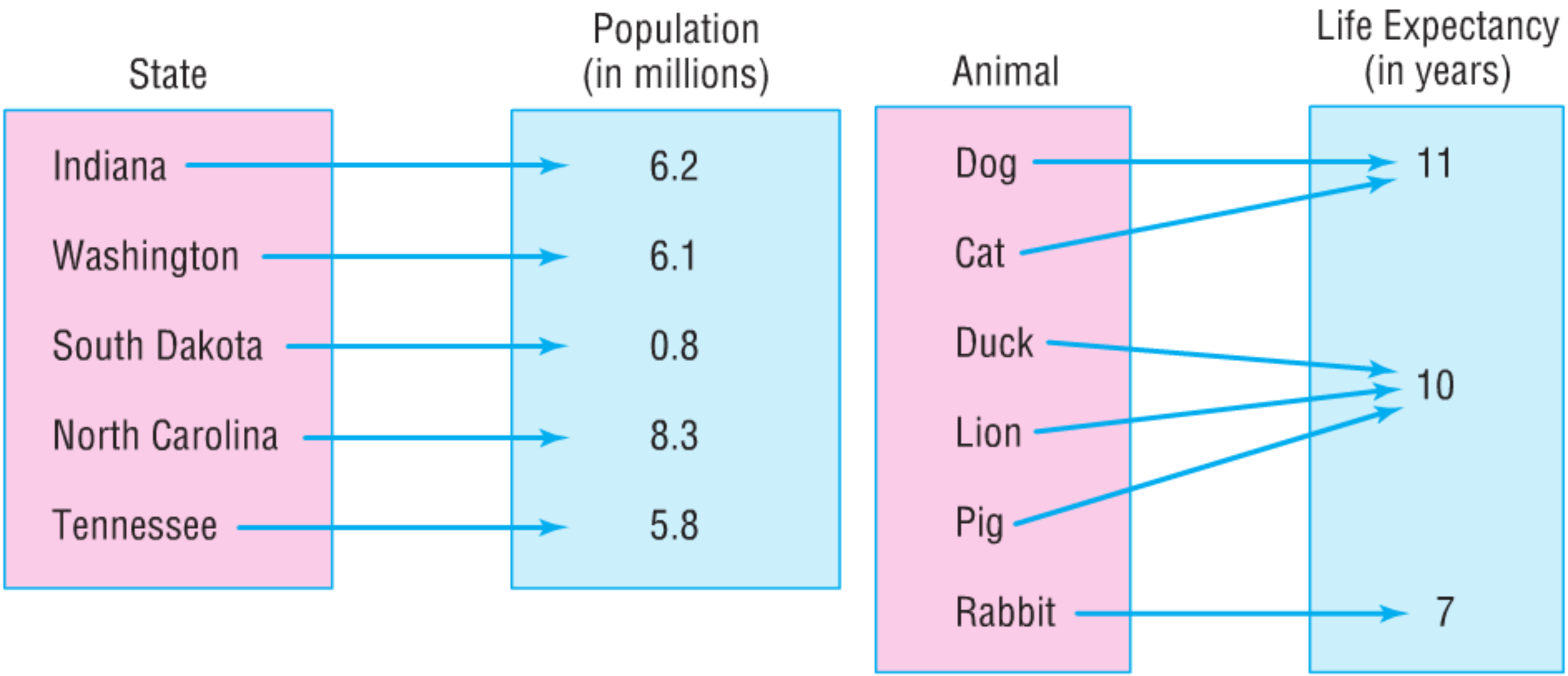
Section 6.2

One-to-One Functions;

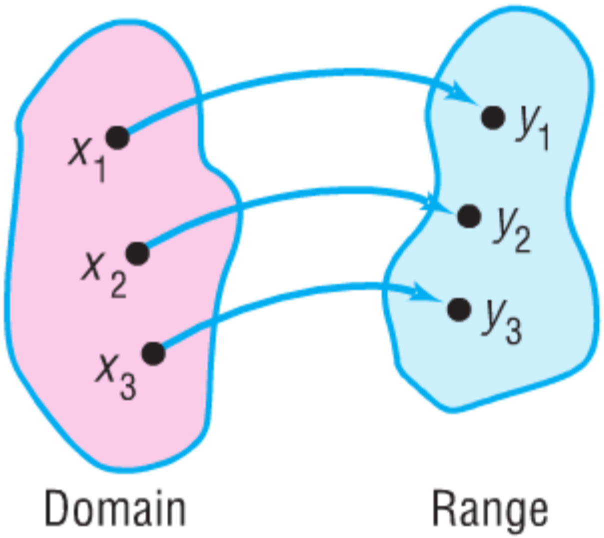
Inverse Functions

1 Determine Whether a Function Is One-to-One

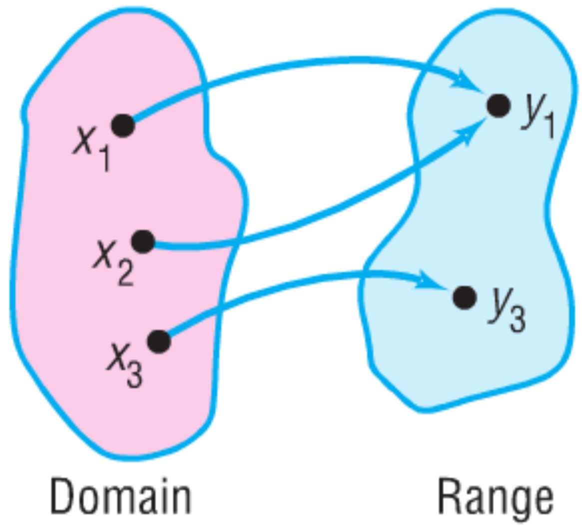

State
Population
(in millions)
Indiana
Washington
South Dakota
North Carolina
Tennessee
6.2
6.1
0.8
8.3
5.8
Animal
Life Expectancy
(in years)
Dog
Cat
Duck
Lion
Pig
Rabbit
11
10
7

DEFINITION

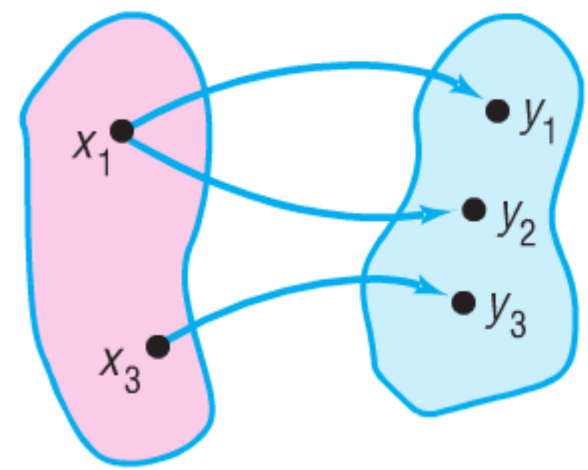
A function is **one-to-one** if any two different inputs in the domain correspond to two different outputs in the range. That is, if x_1 and x_2 are two different inputs of a function f, then f is one-to-one if $f(x_1) \neq f(x_2)$.



One-to-one function: Each x in the domain has one and only one image in the range. No y in the range is the image of more than one x.



Not a one-to-one function: y_1 is the image of both x_1 and x_2



Not a function: x_1 has two images, y_1 and y_2

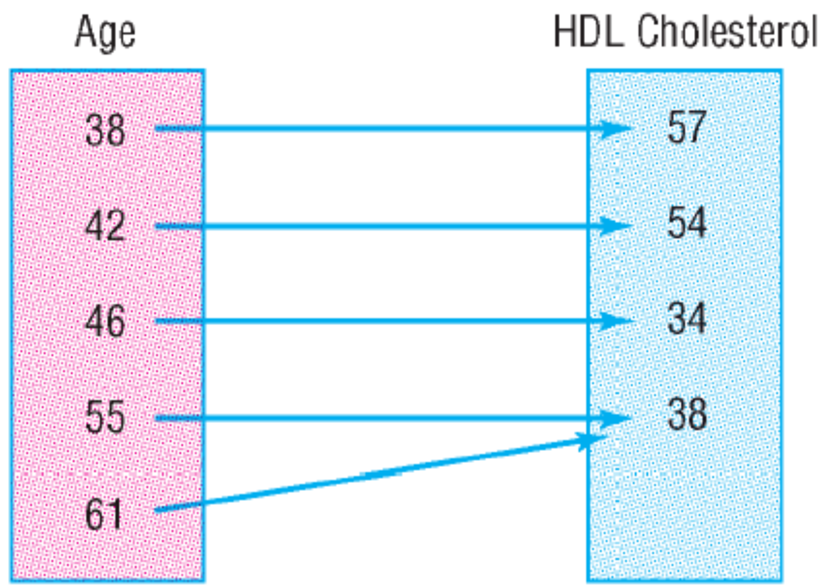
EXAMPLE

Determining Whether a Function Is One-to-One

Determine whether the following functions are one-to-one.

(a) For the following function, the domain represents the age of five males and the range represents their HDL (good) cholesterol (mg/dL).

This function is not one-to-one because two different inputs, 55 and 62, have the same output of 38.



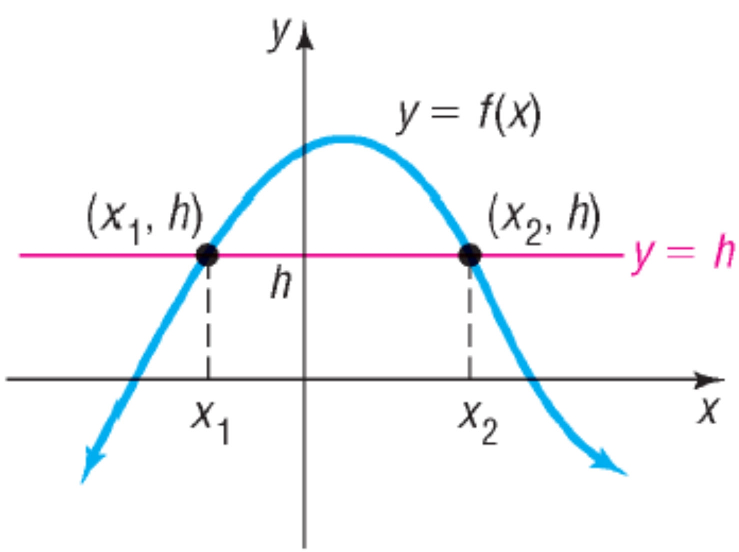
(b) $\{(-2, 6), (-1, 3), (0, 2), (1, 5), (2, 8)\}$

This function is one-to-one because there are no two distinct inputs that correspond to the same output.

THEOREM

Horizontal-line Test

If every horizontal line intersects the graph of a function f in at most one point, then f is one-to-one.

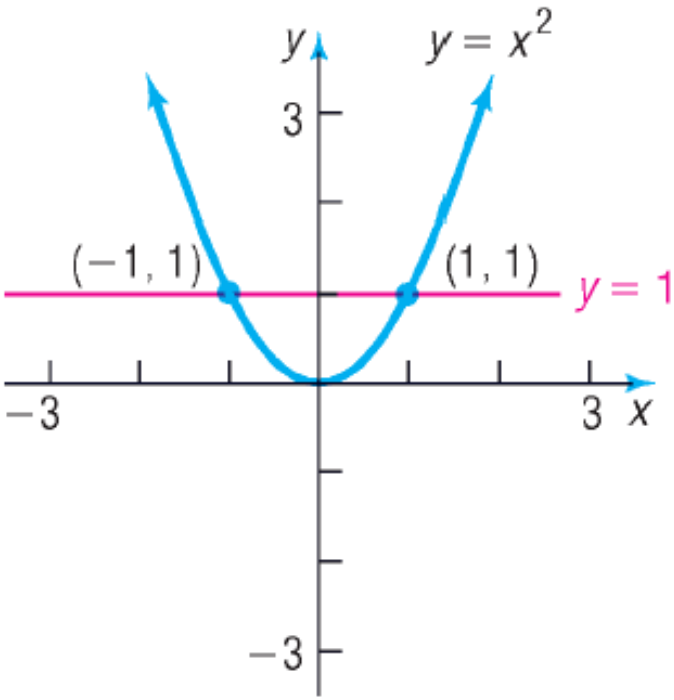


$f(x_1) = f(x_2) = h$ and $x_1 \neq x_2$; f is not a one-to-one function.

EXAMPLE **Using the Horizontal-line Test**

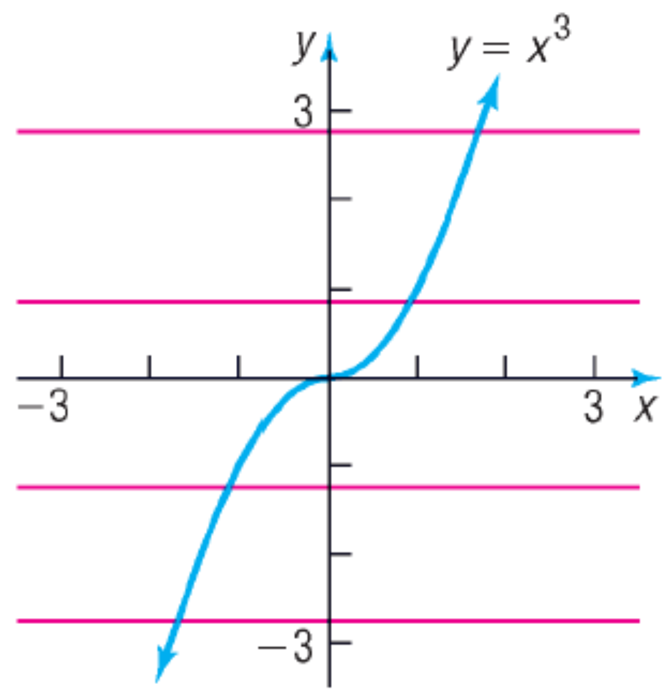
For each function, use the graph to determine whether the function is one-to-one.

(a) $f(x) = x^2$



A horizontal line intersects the graph twice; f is not one-to-one

(b) $g(x) = x^3$



Every horizontal line intersects the graph exactly once; g is one-to-one

THEOREM

A function that is increasing on an interval I is a one-to-one function in I.

A function that is decreasing on an interval I is a one-to-one function on I.

2 Determine the Inverse of a Function Defined by a Map or a Set of Ordered Pairs

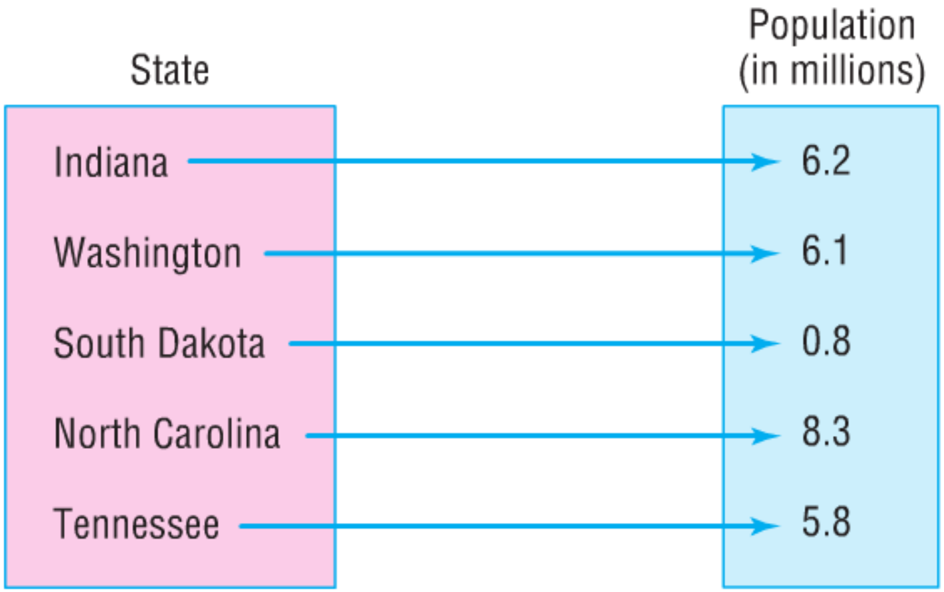
DEFINITION

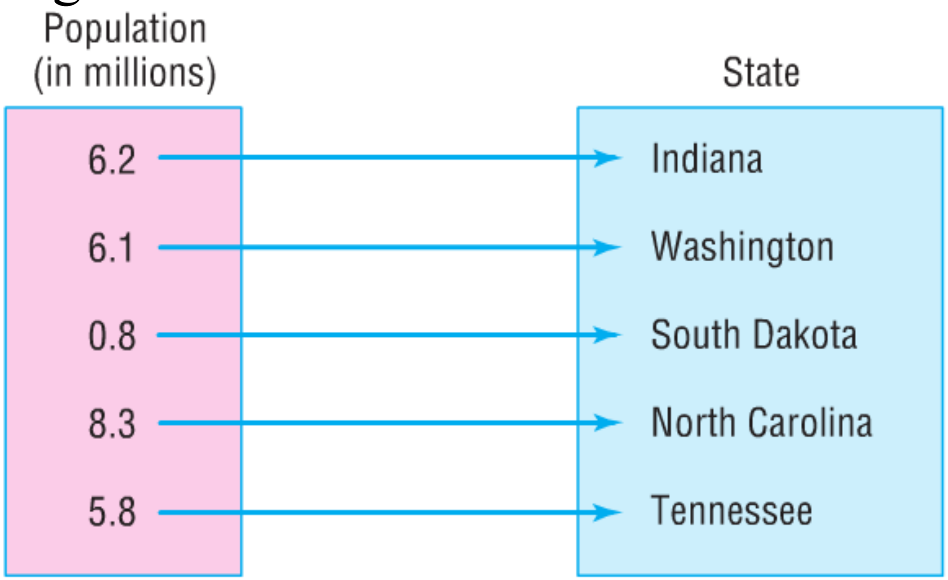
Suppose f is a one-to-one function. Then, to each x in the domain of f, there is exactly one y in the range (because f is a function); and to each y in the range of f, there is exactly one x in the domain (because f is one-to-one). The correspondence from the range of f back to the domain of f is called the **inverse function of** f.

EXAMPLE **Finding the Inverse of a Function Defined by a Map**

Find the inverse of the following function. Let the domain of the function represent certain states, and let the range represent the state's population. State the domain and the range of the inverse function.

To find the inverse we interchange the elements of the domain with the elements of the range.





The domain of the inverse function is {0.8, 5.8, 6.1, 6.2, 8.3}.

The range of the inverse function is {Indiana, Washington, South Dakota, North Carolina, Tennessee}.

EXAMPLE

Finding the Inverse of a Function Defined by a Set of Ordered Pairs

Find the inverse of the following one-to-one function:

$$\{(-5,1),(3,3),(0,0),(2,-4),(7,-8)\}$$

State the domain and range of the function and its inverse.

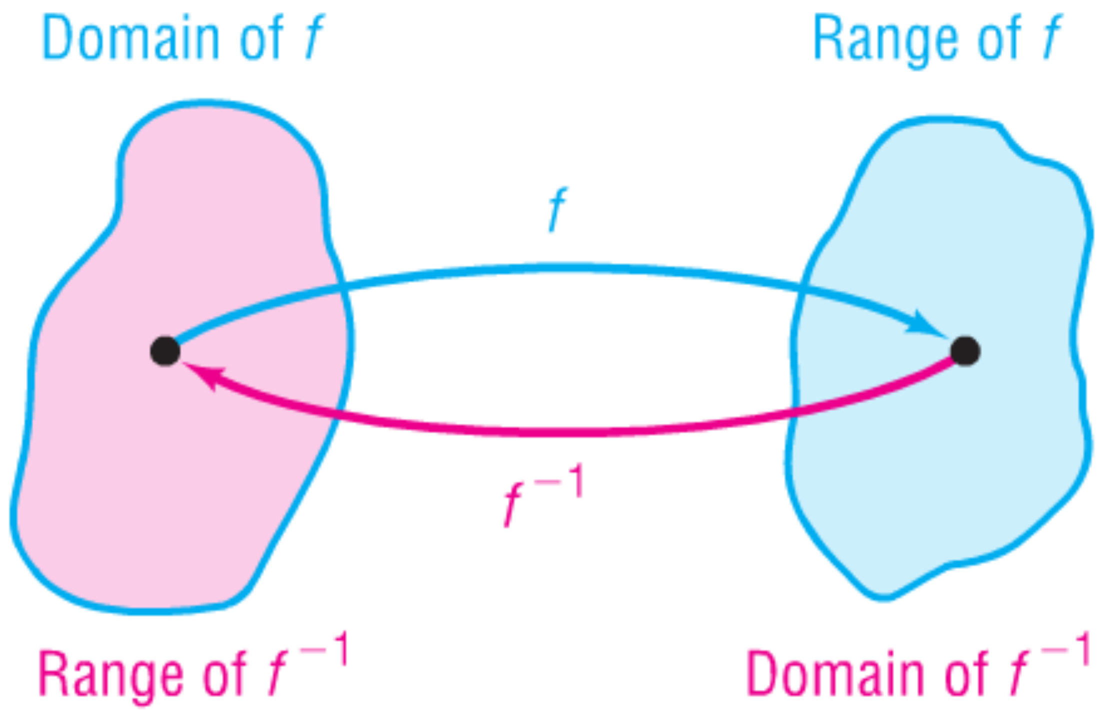
The inverse is found by interchanging the entries in each ordered pair:

$$\{(1,-5),(3,3),(0,0),(-4,2),(-8,7)\}$$

The domain of the function is $\{-5, 0, 2, 3, 7\}$.

The range of the function is $\{-8, -4, 0 ,1, 3)$.
This is also the domain of the inverse function.

The range of the inverse function is $\{-5, 0, 2, 3, 7\}$.



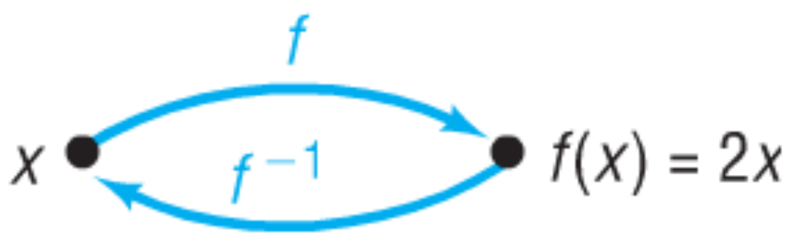
Domain of f = Range of f^{-1} Range of f = Domain of f^{-1}

$\boxed{\text{Input } x \text{ from domain of } f} \xrightarrow{\text{Apply } f} \boxed{f(x)} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(f(x)) = x}$

$\boxed{\text{Input } x \text{ from domain of } f^{-1}} \xrightarrow{\text{Apply } f^{-1}} \boxed{f^{-1}(x)} \xrightarrow{\text{Apply } f} \boxed{f(f^{-1}(x)) = x}$

$$f^{-1}(f(x)) = x \quad \text{where } x \text{ is in the domain of } f$$
$$f(f^{-1}(x)) = x \quad \text{where } x \text{ is in the domain of } f^{-1}$$

$$f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}(2x) = x \quad \text{and} \quad f(f^{-1}(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x$$



$$f^{-1}(2x) = \frac{1}{2}(2x) = x$$

EXAMPLE Verifying Inverse Functions

(a) We verify that the inverse of $g(x) = x^3$ is $g^{-1}(x) = \sqrt[3]{x}$ by showing that

$$g^{-1}(g(x)) = g^{-1}(x^3) = \sqrt[3]{x^3} = x \quad \text{for all } x \text{ in the domain of } g$$

$$g(g^{-1}(x)) = g(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x \quad \text{for all } x \text{ in the domain of } g^{-1}.$$

(b) We verify that the inverse of $f(x) = 2x + 3$ is $f^{-1}(x) = \frac{1}{2}(x - 3)$ by showing that

$$f^{-1}(f(x)) = f^{-1}(2x + 3) = \frac{1}{2}[(2x + 3) - 3] = \frac{1}{2}(2x) = x \quad \text{for all } x \text{ in the domain of } f$$

$$f(f^{-1}(x)) = f\left(\frac{1}{2}(x - 3)\right) = 2\left[\frac{1}{2}(x - 3)\right] + 3 = (x - 3) + 3 = x \quad \text{for all } x \text{ in the domain of } f^{-1}.$$

EXAMPLE **Verifying Inverse Functions**

Verify that the inverse of $f(x)=\dfrac{3}{x+5}$ is $f^{-1}(x)=\dfrac{3}{x}-5.$

For what values of x is $f^{-1}\left(f(x)\right)=x$?

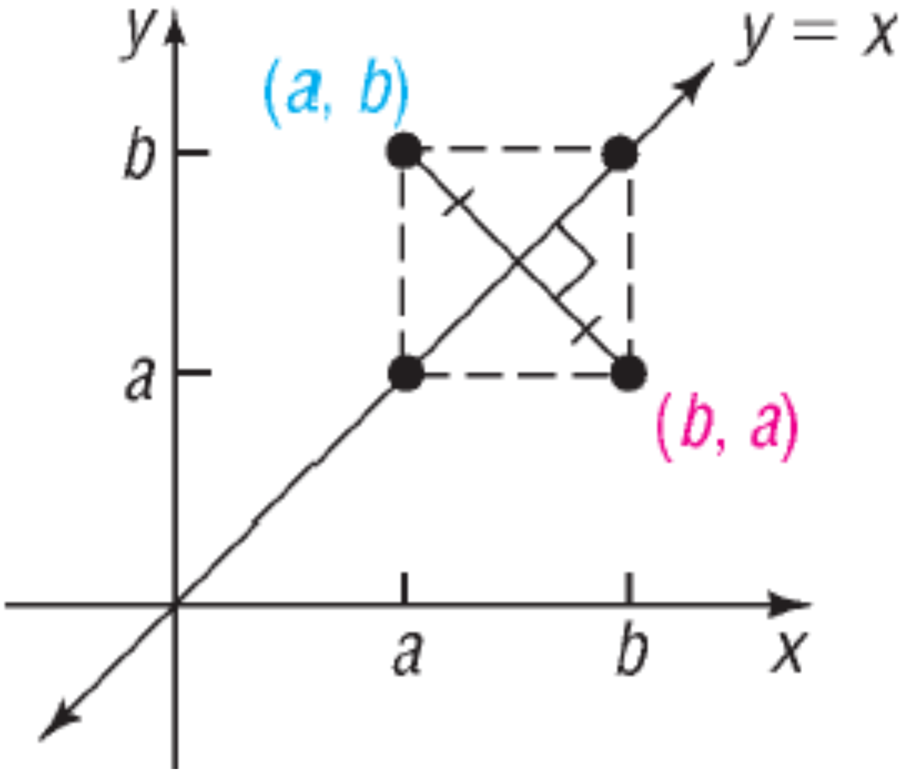
For what values of x is $f\left(f^{-1}(x)\right)=x$?

Note the domain of f is $\{x|x\neq -5\}$ and the domain of g is $\{x|x\neq 0\}$.

$$f^{-1}\left(f(x)\right)= \frac{3}{\dfrac{3}{x+5}}-5 = \frac{3(x+5)}{3}-5=x \text{ provided } x\neq -5$$

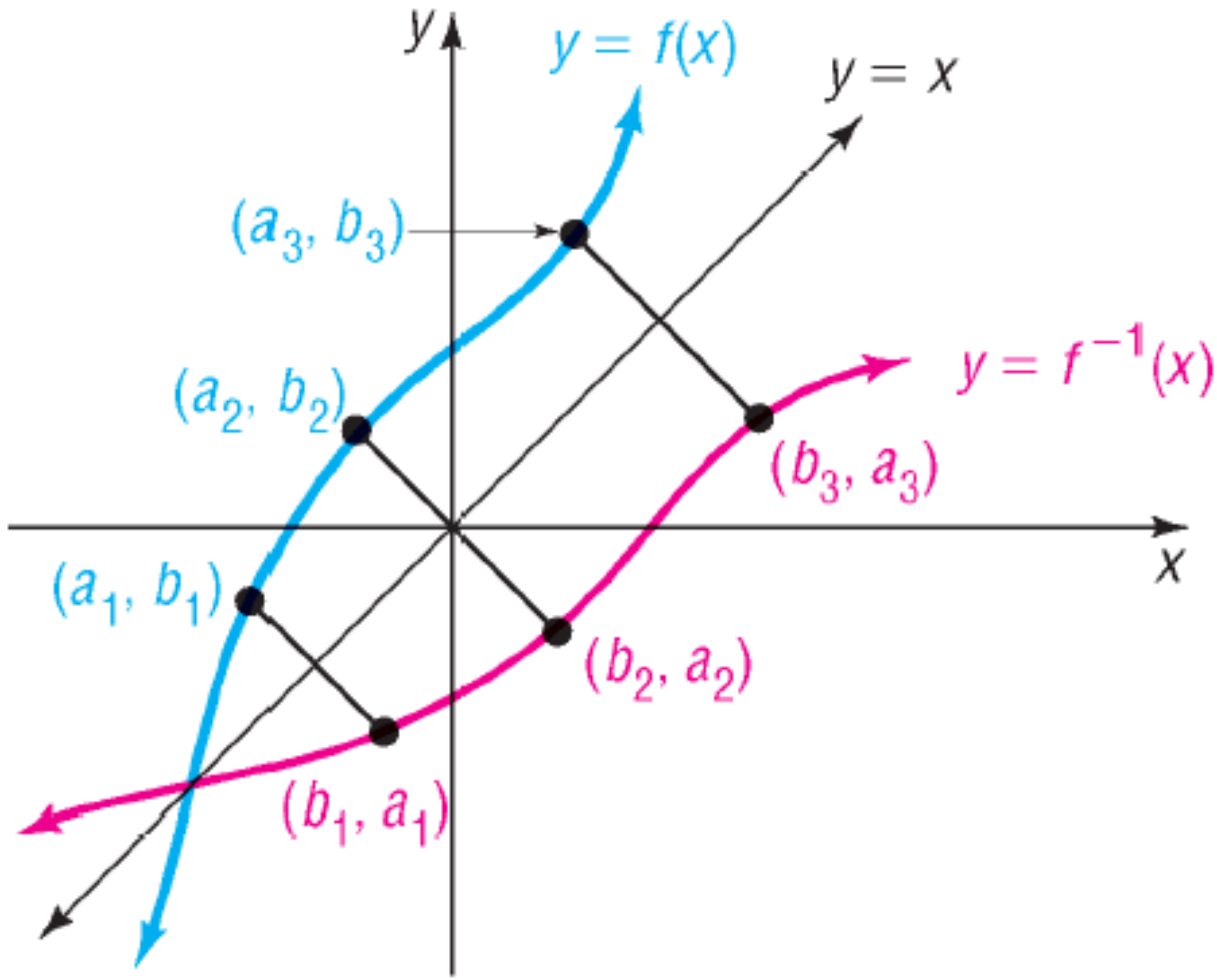
$$f\left(f^{-1}(x)\right)= \frac{3}{\dfrac{3}{x}-5+5} = \frac{3}{\dfrac{3}{x}}=x \text{ provided } x\neq 0$$

3 Obtain the Graph of the Inverse Function from the Graph of the Function



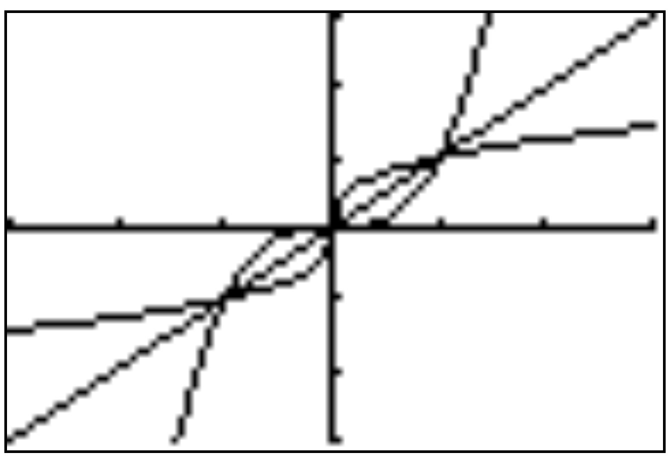
Theorem

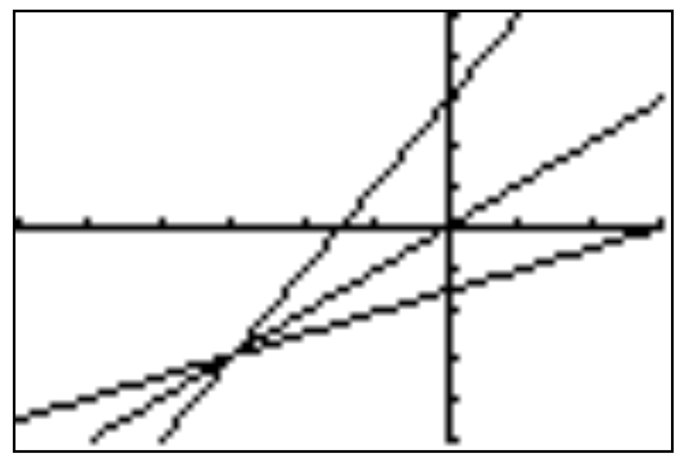
The graph of a function f and the graph of its inverse f^{-1} are symmetric with respect to the line $y = x$.


y
$y = f(x)$
$y = x$
(a_3, b_3)
(a_2, b_2)
$y = f^{-1}(x)$
(b_3, a_3)
x
(a_1, b_1)
(b_2, a_2)
(b_1, a_1)

Seeing the Concept

Simultaneously graph $Y_1 = x$, $Y_2 = x^3$, and $Y_3 = \sqrt[3]{x}$ on a square screen with $-3 \le x \le 3$. What do you observe about the graphs of $Y_2 = x^3$, its inverse $Y_3 = \sqrt[3]{x}$, and the line $Y_1 = x$? Repeat this experiment by simultaneously graphing $Y_1 = x$, $Y_2 = 2x + 3$, and $Y_3 = \frac{1}{2}(x - 3)$ on a square screen with $-6 \le x \le 3$. Do you see the symmetry of the graph of Y_2 and its inverse Y_3 with respect to the line $Y_1 = x$?





Graphing the Inverse Function

The graph in Figure 16(a) is that of a one-to-one function $y = f(x)$. Draw the graph of its inverse.

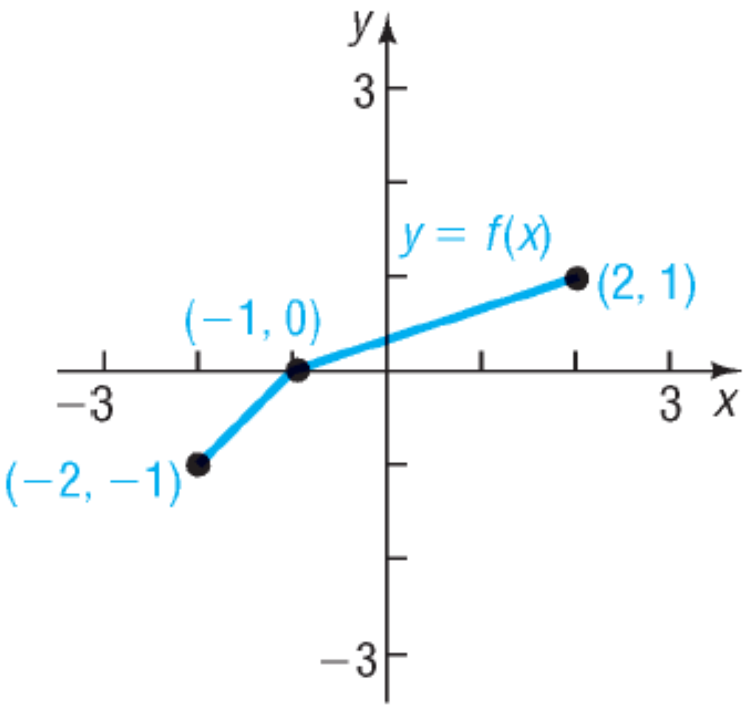
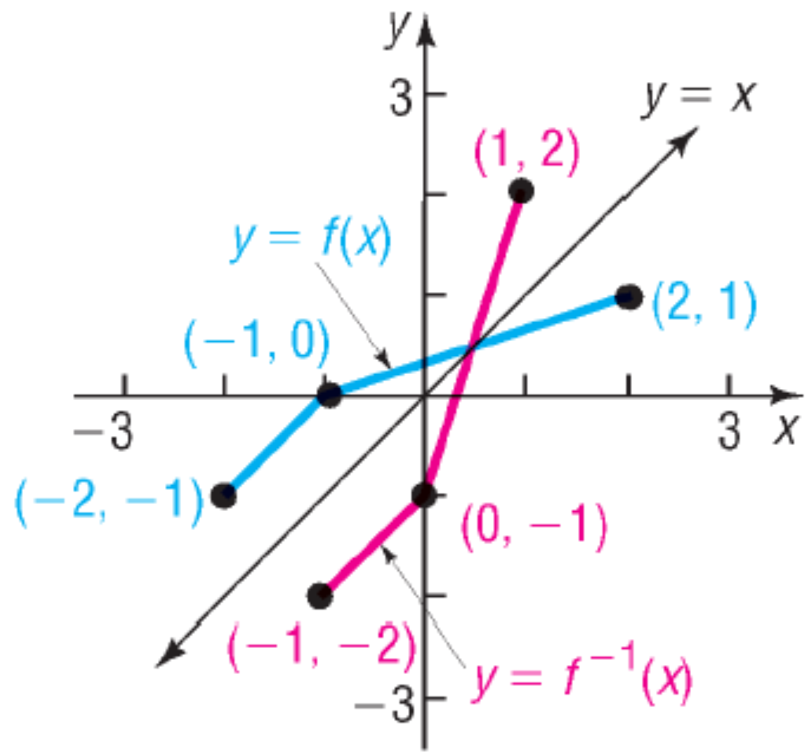


Figure 16(a)



4 Find the Inverse of a Function Defined by an Equation

EXAMPLE How to Find the Inverse Function

Find the inverse of $f(x) = 2x + 3$. Graph f and f^{-1} on the same coordinate axes.

Step 1: Replace $f(x)$ with y. In $y = f(x)$, interchange the variables x and y to obtain $x = f(y)$. This equation defines the inverse function f^{-1} implicitly.

Replace $f(x)$ with y in $f(x) = 2x + 3$ and obtain $y = 2x + 3$. Now interchange the variables x and y to obtain

Step 2: If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}, $y = f^{-1}(x)$.

$$x = 2y + 3$$

$$2y = x - 3$$

$$y = \frac{1}{2}(x - 3)$$

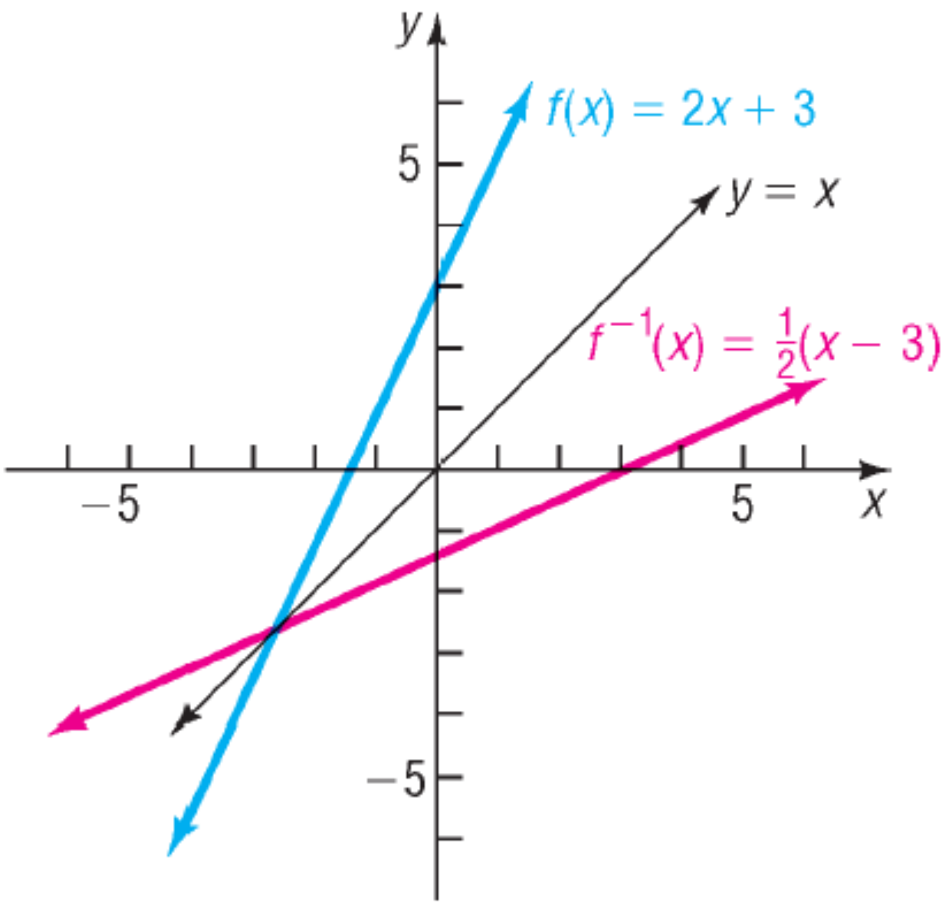
$$f^{-1}(x) = \frac{1}{2}(x - 3)$$

Step 3: Check the result by showing that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}\left(f\left(x\right)\right) = \frac{1}{2}\left(\left(2x+3\right)-3\right) = x = 2\left(\frac{1}{2}\left(x-3\right)\right)+3 = f\left(f^{-1}\left(x\right)\right)$$

EXAMPLE **How to Find the Inverse Function**

Find the inverse of $f(x) = 2x + 3$. Also find the domain and range of f and f^{-1}. Graph f and f^{-1} on the same coordinate axes.



Procedure for Finding the Inverse of a One-to-One Function

STEP 1: In $y = f(x)$, interchange the variables x and y to obtain

$$x = f(y)$$

This equation defines the inverse function f^{-1} implicitly.

STEP 2: If possible, solve the implicit equation for y in terms of x to obtain the explicit form of f^{-1}

$$y = f^{-1}(x)$$

STEP 3: Check the result by showing that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

EXAMPLE Finding the Inverse Function

The function $$f(x) = \frac{2x + 1}{x - 1}, \qquad x \neq 1$$

is one-to-one. Find its inverse and check the result.

STEP 1: Replace $f(x)$ with y and interchange the variables x and y in

$$y = \frac{2x + 1}{x - 1}$$

to obtain

$$x = \frac{2y + 1}{y - 1}$$

STEP 2: Solve for y. $\quad x(y - 1) = 2y + 1 \qquad xy - x = 2y + 1$

$$xy - 2y = x + 1 \qquad (x - 2)y = x + 1 \qquad y = \frac{x + 1}{x - 2}$$

$$f^{-1}(x) = \frac{x + 1}{x - 2} \quad x \neq 2$$

EXAMPLE Finding the Inverse Function

The function $$f(x) = \frac{2x+1}{x-1}, \qquad x \neq 1$$

is one-to-one. Find its inverse and check the result.

Step 3: ✓**Check:** $$f^{-1}(x) = \frac{x+1}{x-2} \quad x \neq 2$$

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x+1}{x-1}\right) = \frac{\dfrac{2x+1}{x-1}+1}{\dfrac{2x+1}{x-1}-2} = \frac{2x+1+x-1}{2x+1-2(x-1)} = \frac{3x}{3} = x \quad x \neq 1$$

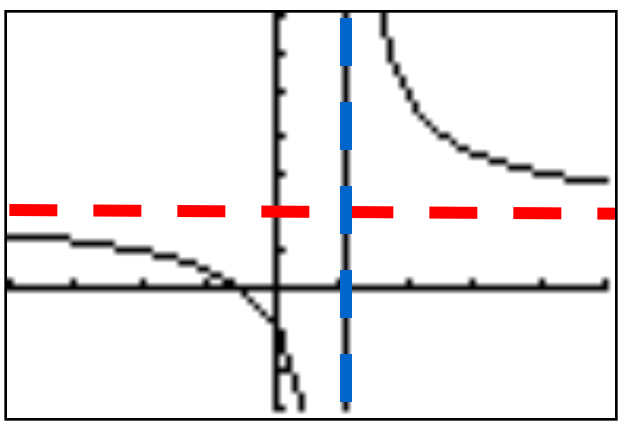
$$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right) = \frac{2\left(\dfrac{x+1}{x-2}\right)+1}{\dfrac{x+1}{x-2}-1} = \frac{2(x+1)+x-2}{x+1-(x-2)} = \frac{3x}{3} = x \quad x \neq 2$$

Exploration

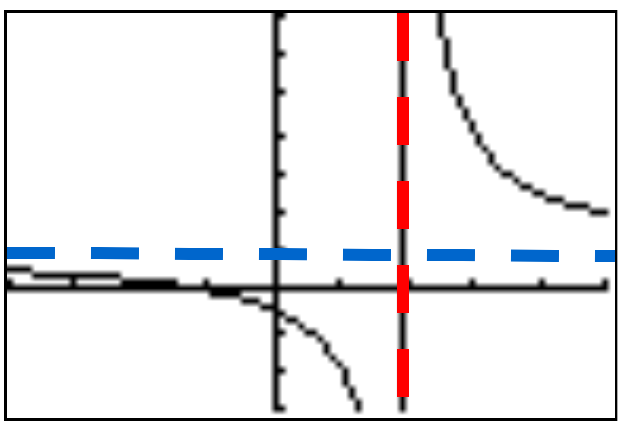
If $f(x) = \dfrac{2x + 1}{x - 1}$, then $f^{-1}(x) = \dfrac{x + 1}{x - 2}$.

Compare the vertical and horizontal asymptotes of f and f^{-1}.

$f(x) = \dfrac{2x + 1}{x - 1}$



$f^{-1}(x) = \dfrac{x + 1}{x - 2}$



EXAMPLE **Finding the Inverse of a Domain-restricted Function**

Find the inverse of $y = f(x) = x^2$ if $x \geq 0$. Graph f and f^{-1}.

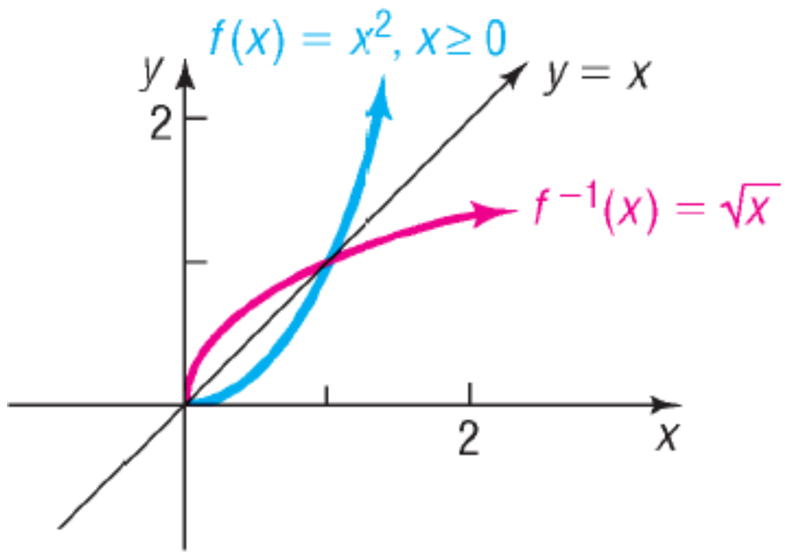
Step 1: In the equation $y = x^2$, $x \geq 0$, interchange the variables x and y. The result is

$$x = y^2 \qquad y \geq 0$$

Step 2: Solve for y to get the explicit form of the inverse. Since $y \geq 0$, only one solution for y is obtained: $y = \sqrt{x}$. So $f^{-1}(x) = \sqrt{x}$.

Step 3: **✓Check:** $f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = |x| = x \quad \text{since } x \geq 0$

$$f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$



SUMMARY

1. If a function f is one-to-one, then it has an inverse function f^{-1}.

2. Domain of f = Range of f^{-1}; Range of f = Domain of f^{-1}.

3. To verify that f^{-1} is the inverse of f, show that $f^{-1}(f(x)) = x$ for every x in the domain of f and $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}.

4. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.