Section R.8

*n*th Roots;

Rational Exponents

1 Work with *n*th Roots

The **principal** $\boldsymbol{n}$**th root of a real number** $\boldsymbol{a}$**,** $n \geq 2$ an integer, symbolized by $\sqrt[n]{a}$, is defined as follows:

$$\boxed{\sqrt[n]{a} = b \quad \text{means} \quad a = b^n}$$

where $a \geq 0$ and $b \geq 0$ if n is even and a, b are any real numbers if n is odd.

EXAMPLE **Simplifying Principal *n*th Roots**

(a) $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$

(b) $\sqrt[3]{-64} = \sqrt[3]{(-4)^3} = -4$

(c) $\sqrt[4]{\frac{1}{16}} = \sqrt[4]{\left(\frac{1}{2}\right)^4} = \frac{1}{2}$

(d) $\sqrt[6]{(-2)^6} = |-2| = 2$

In general, if $n \geq 2$ is an integer and a is a real number, we have

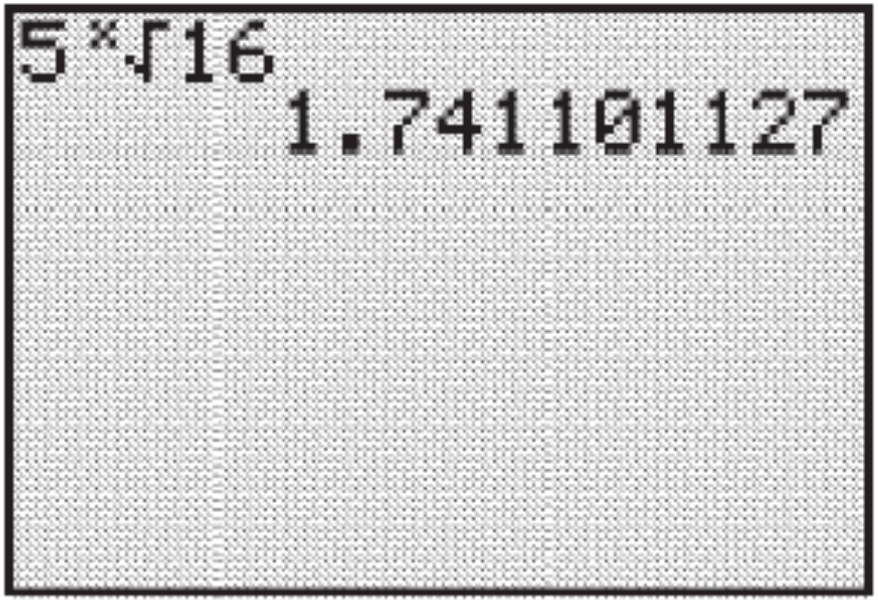
$$\sqrt[n]{a^n} = a \qquad \text{if } n \geq 3 \text{ is odd}$$

$$\sqrt[n]{a^n} = |a| \qquad \text{if } n \geq 2 \text{ is even}$$

EXAMPLE

Using a Calculator to Approximate Roots

Use a calculator to approximate $\sqrt[5]{16}$.



2 Simplify Radicals

Properties of Radicals

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

EXAMPLE Simplifying Radicals

(a) $\sqrt{32} = \sqrt{16 \cdot 2} \quad \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$

Factor out 16, a perfect square.

(2a)

(b) $\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = \sqrt[3]{2^3} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$

Factor out 8, a perfect cube.

(2a)

EXAMPLE Simplifying Radicals

(c) $\sqrt[3]{-16x^4} = \sqrt[3]{-8 \cdot 2 \cdot x^3 \cdot x} = \sqrt[3]{(-8x^3)(2x)}$

Factor perfect cubes inside radical.

Group perfect cubes.

$$= \sqrt[3]{(-2x)^3 \cdot 2x} = \sqrt[3]{(-2x)^3} \cdot \sqrt[3]{2x} = -2x\sqrt[3]{2x}$$

(d) $\sqrt[4]{\dfrac{16x^5}{81}} = \sqrt[4]{\dfrac{2^4x^4x}{3^4}} = \sqrt[4]{\left(\dfrac{2x}{3}\right)^4 \cdot x} = \sqrt[4]{\left(\dfrac{2x}{3}\right)^4} \cdot \sqrt[4]{x} = \left|\dfrac{2x}{3}\right| \sqrt[4]{x}$

EXAMPLE Combining Like Radicals

(a) $-8\sqrt{12}+\sqrt{3} = -8\sqrt{4\cdot 3}+\sqrt{3}$

$= -8\cdot\sqrt{4}\sqrt{3}+\sqrt{3}$

$= -16\sqrt{3}+\sqrt{3} = -15\sqrt{3}$

(b) $\sqrt[3]{8x^4}+\sqrt[3]{-x}+4\sqrt[3]{27x}$

$= \sqrt[3]{2^3x^3x}+\sqrt[3]{-1\cdot x}+4\sqrt[3]{3^3x}$

$= \sqrt[3]{(2x)^3}\cdot\sqrt[3]{x}+\sqrt[3]{-1}\cdot\sqrt[3]{x}+4\sqrt[3]{3^3}\cdot\sqrt[3]{x}$

$= 2x\sqrt[3]{x}-1\cdot\sqrt[3]{x}+12\sqrt[3]{x}$

$= (2x+11)\sqrt[3]{x}$

3 Rationalize Denominators

If Denominator Contains the Factor	Multiply by	To Obtain Denominator Free of Radicals
$\sqrt{3}$	$\sqrt{3}$	$\left(\sqrt{3}\right)^2 = 3$
$\sqrt{3} + 1$	$\sqrt{3} - 1$	$\left(\sqrt{3}\right)^2 - 1^2 = 3 - 1 = 2$
$\sqrt{2} - 3$	$\sqrt{2} + 3$	$\left(\sqrt{2}\right)^2 - 3^2 = 2 - 9 = -7$
$\sqrt{5} - \sqrt{3}$	$\sqrt{5} + \sqrt{3}$	$\left(\sqrt{5}\right)^2 - \left(\sqrt{3}\right)^2 = 5 - 3 = 2$
$\sqrt[3]{4}$	$\sqrt[3]{2}$	$\sqrt[3]{4} \cdot \sqrt[3]{2} = \sqrt[3]{8} = 2$

EXAMPLE Rationalizing Denominators

Rationalize the denominator of each expression:

(a) $$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\left(\sqrt{3}\right)^2} = \frac{\sqrt{3}}{3}$$

(b) $$\frac{5}{4\sqrt{2}} = \frac{5}{4\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{4\left(\sqrt{2}\right)^2} = \frac{5\sqrt{2}}{4 \cdot 2} = \frac{5\sqrt{2}}{8}$$

(c) $$\frac{\sqrt{2}}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{2}}{\sqrt{3} - 3\sqrt{2}} \cdot \frac{\sqrt{3} + 3\sqrt{2}}{\sqrt{3} + 3\sqrt{2}} = \frac{\sqrt{2}\left(\sqrt{3} + 3\sqrt{2}\right)}{\left(\sqrt{3}\right)^2 - \left(3\sqrt{2}\right)^2}$$

$$= \frac{\sqrt{2}\,\sqrt{3} + 3\left(\sqrt{2}\right)^2}{3 - 18} = \frac{\sqrt{6} + 6}{-15} = -\frac{6 + \sqrt{6}}{15}$$

4 Simplify Expressions with Rational Exponents

If a is a real number and $n \geq 2$ is an integer, then

$$a^{1/n} = \sqrt[n]{a}$$

provided that $\sqrt[n]{a}$ exists.

EXAMPLE

Writing Expressions Containing Fractional Exponents as Radicals

(a) $4^{1/2} = \sqrt{4} = 2$

(b) $8^{1/2} = \sqrt{8} = 2\sqrt{2}$

(c) $(-27)^{1/3} = \sqrt[3]{-27} = -3$

(d) $16^{1/3} = \sqrt[3]{16} = 2\sqrt[3]{2}$

If a is a real number and m and n are integers containing no common factors, with $n \geq 2$, then

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

provided that $\sqrt[n]{a}$ exists.

EXAMPLE

(a) $4^{3/2} = \left(\sqrt{4}\right)^3 = 2^3 = 8$

(b) $(-8)^{4/3}$

$= \left(\sqrt[3]{-8}\right)^4 = (-2)^4 = 16$

(c) $(32)^{-2/5}$

$= \left(\sqrt[5]{32}\right)^{-2} = 2^{-2} = \dfrac{1}{4}$

(d) $25^{6/4}$

$= 25^{3/2} = \left(\sqrt{25}\right)^3 = 5^3 = 125$

EXAMPLE

Simplifying Expressions Containing Rational Exponents

Simplify each expression. Express your answer so that only positive exponents occur. Assume that the variables are positive.

(a) $$(x^{2/3}y)(x^{-2}y)^{1/2} = (x^{2/3}y)\left[(x^{-2})^{1/2}y^{1/2}\right] = x^{2/3}yx^{-1}y^{1/2}$$
$$= (x^{2/3}\cdot x^{-1})(y\cdot y^{1/2}) = x^{-1/3}y^{3/2} = \frac{y^{3/2}}{x^{1/3}}$$

(b) $$\left(\frac{2x^{1/3}}{y^{2/3}}\right)^{-3} = \left(\frac{y^{2/3}}{2x^{1/3}}\right)^{3} = \frac{(y^{2/3})^3}{(2x^{1/3})^3} = \frac{y^2}{2^3(x^{1/3})^3} = \frac{y^2}{8x}$$

(c) $$\left(\frac{9x^2y^{1/3}}{x^{1/3}y}\right)^{1/2} = \left(\frac{9x^{2-(1/3)}}{y^{1-(1/3)}}\right)^{1/2} = \left(\frac{9x^{5/3}}{y^{2/3}}\right)^{1/2} = \frac{9^{1/2}(x^{5/3})^{1/2}}{(y^{2/3})^{1/2}} = \frac{3x^{5/6}}{y^{1/3}}$$

EXAMPLE

Writing an Expression as a Single Quotient

Write the following expression as a single quotient in which only positive exponents appear.

$$(x^2+1)^{1/2} + x\cdot\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x$$

$$(x^2+1)^{1/2} + x\cdot\frac{1}{\cancel{2}}(x^2+1)^{-1/2}\cdot \cancel{2}x = (x^2+1)^{1/2} + \frac{x^2}{(x^2+1)^{1/2}}$$

$$= \frac{(x^2+1)^{1/2}(x^2+1)^{1/2} + x^2}{(x^2+1)^{1/2}} = \frac{(x^2+1)+x^2}{(x^2+1)^{1/2}} = \frac{2x^2+1}{(x^2+1)^{1/2}}$$

EXAMPLE

Factoring an Expression Containing Rational Exponents

Factor: $\frac{4}{3}x^{1/3}(2x+1)+2x^{4/3} = \frac{4x^{1/3}(2x+1)}{3}+\frac{6x^{4/3}}{3}$

$$= \frac{4x^{1/3}(2x+1)+6x^{4/3}}{3} = \frac{2x^{1/3}[2(2x+1)+3x]}{3} = \frac{2x^{1/3}(7x+2)}{3}$$