Section 3.5

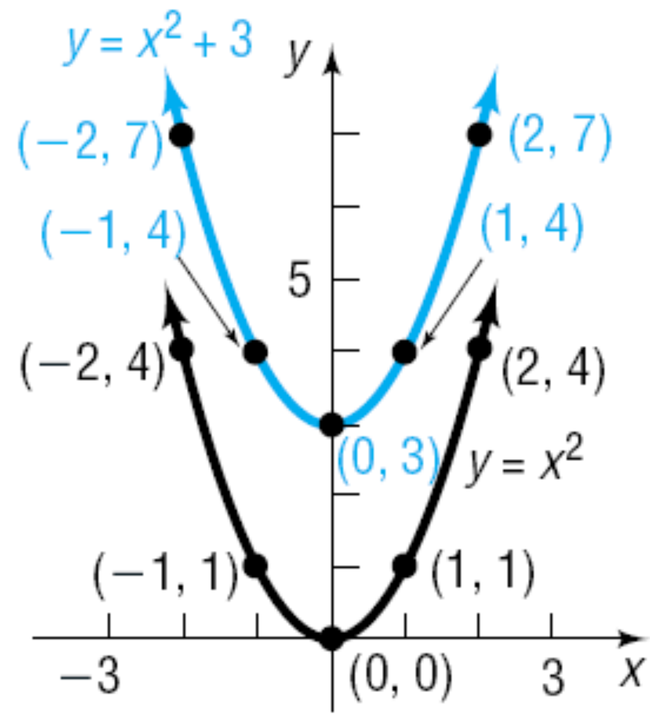
Graphing Techniques; Transformations

1 Graph Functions Using Vertical and Horizontal Shifts

EXAMPLE Vertical Shift Up

Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = x^2 + 3$.

x	$y = f(x) = x^2$	$y = g(x) = x^2 + 3$
-2	4	7
-1	1	4
0	0	3
1	1	4
2	4	7

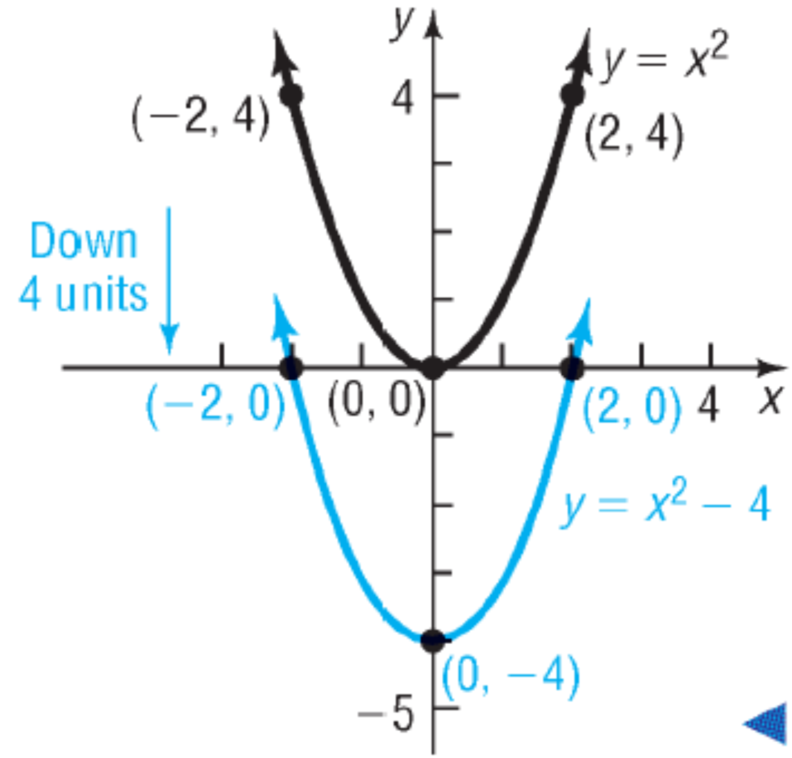


EXAMPLE

Vertical Shift Down

Use the graph of $f(x) = x^2$ to obtain the graph of $h(x) = x^2 - 4$.

x	$y = f(x) = x^2$	$y = g(x) = x^2 - 4$
-2	4	0
-1	1	-3
0	0	-4
1	1	-3
2	4	0



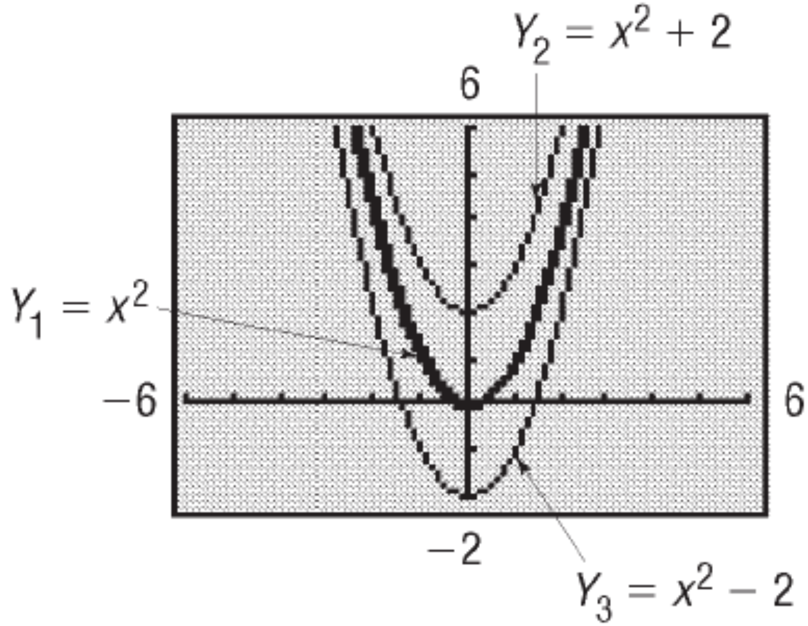
Exploration

On the same screen, graph each of the following functions:

$$Y_1 = x^2$$

$$Y_2 = x^2 + 2$$

$$Y_3 = x^2 - 2$$



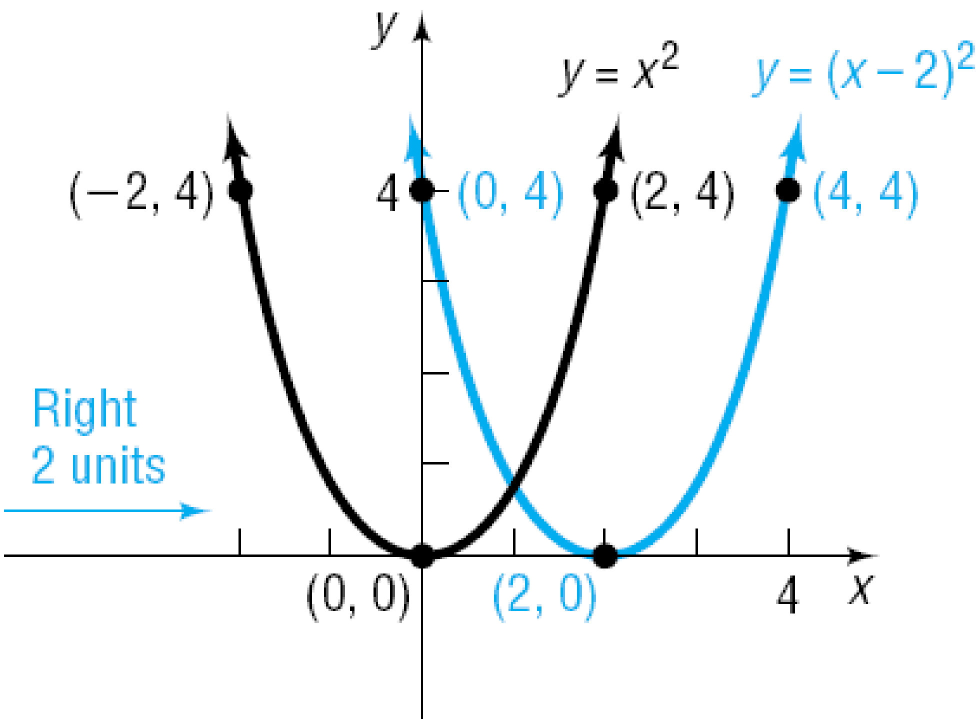
If a positive real number k is added to the output of a function $y = f(x)$, the graph of the new function $y = f(x) + k$ is the graph of f **shifted vertically up** k units.

If a positive real number k is subtracted from the output of a function $y = f(x)$, the graph of the new function $y = f(x) - k$ is the graph of f **shifted vertically down** k units.

EXAMPLE Horizontal Shift to the Right

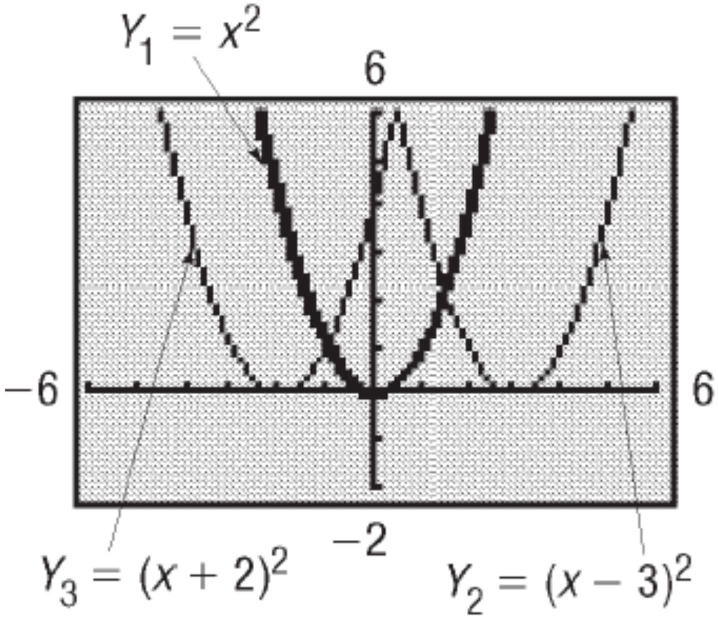
Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = (x - 2)^2$.

x	$y = f(x) = x^2$	$y = g(x) = (x-2)^2$
−2	4	16
0	0	4
2	4	0
4	16	4



Exploration

On the same screen, graph each of the following functions:

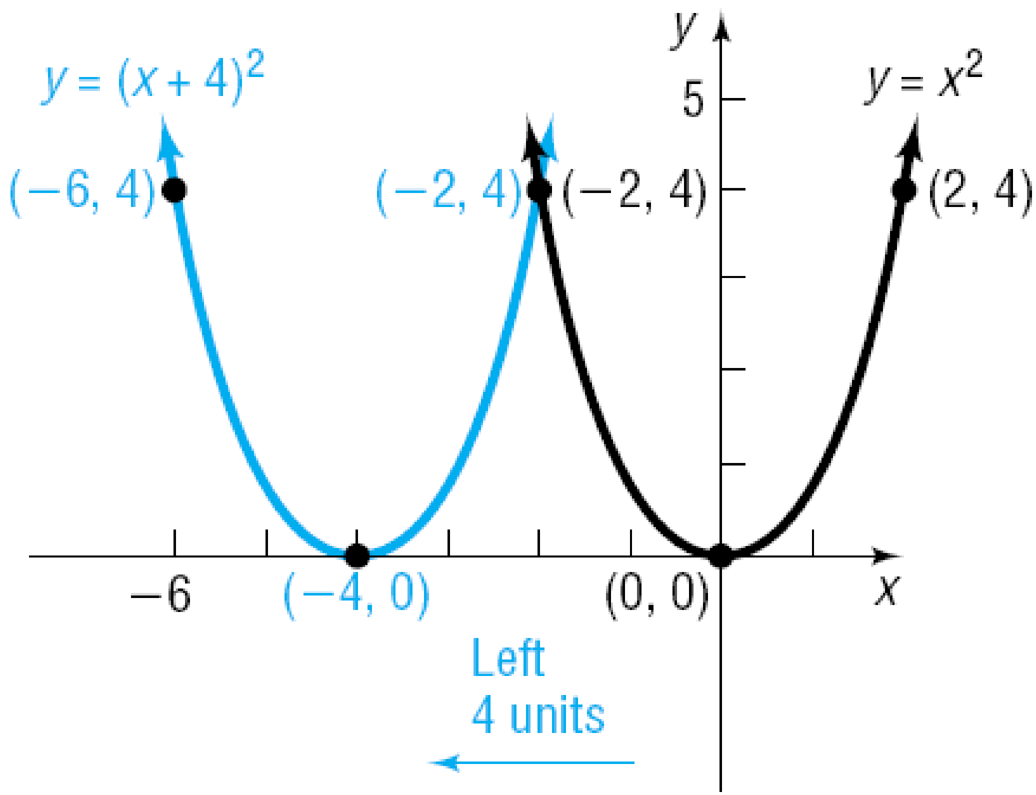


$$Y_1 = x^2$$
$$Y_2 = (x - 3)^2$$
$$Y_3 = (x + 2)^2$$

If the argument x of a function f is replaced by $x - h$, $h > 0$, the graph of the new function $y = f(x - h)$ is the graph of f **shifted horizontally right** h units. If the argument x of a function f is repaced by $x + h$, $h > 0$, the graph of the new function $y = f(x + h)$ is the graph of f **shifted horizontally left** h units..

EXAMPLE Horizontal Shift to the Left

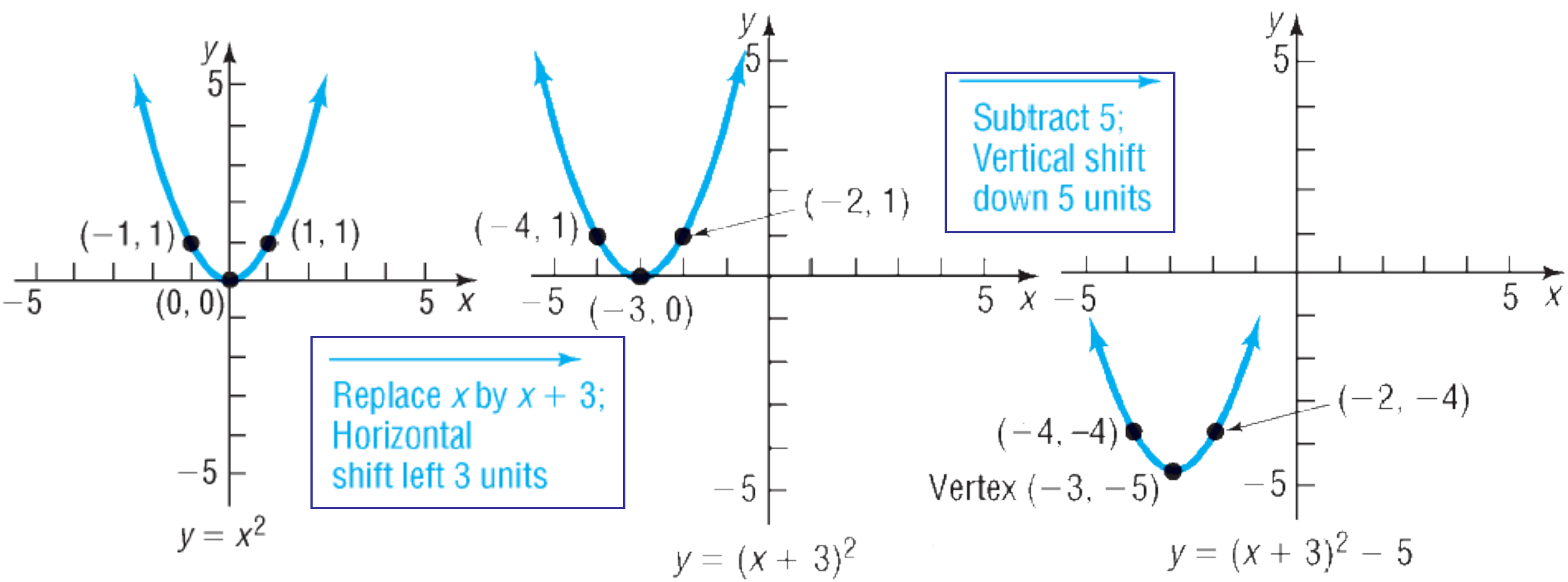
Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = (x + 4)^2$.



EXAMPLE

Combining Vertical and Horizontal Shifts

Graph the function $f(x) = (x + 3)^2 - 5$.

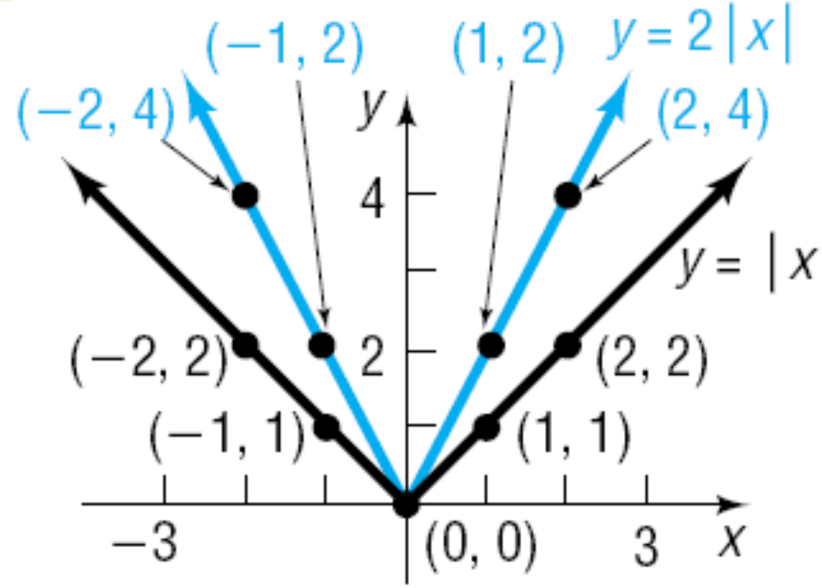


2 Graph Functions Using Compressions and Stretches

EXAMPLE Vertical Stretch

Use the graph of $f(x) = |x|$ to obtain the graph of $g(x) = 2|x|$.

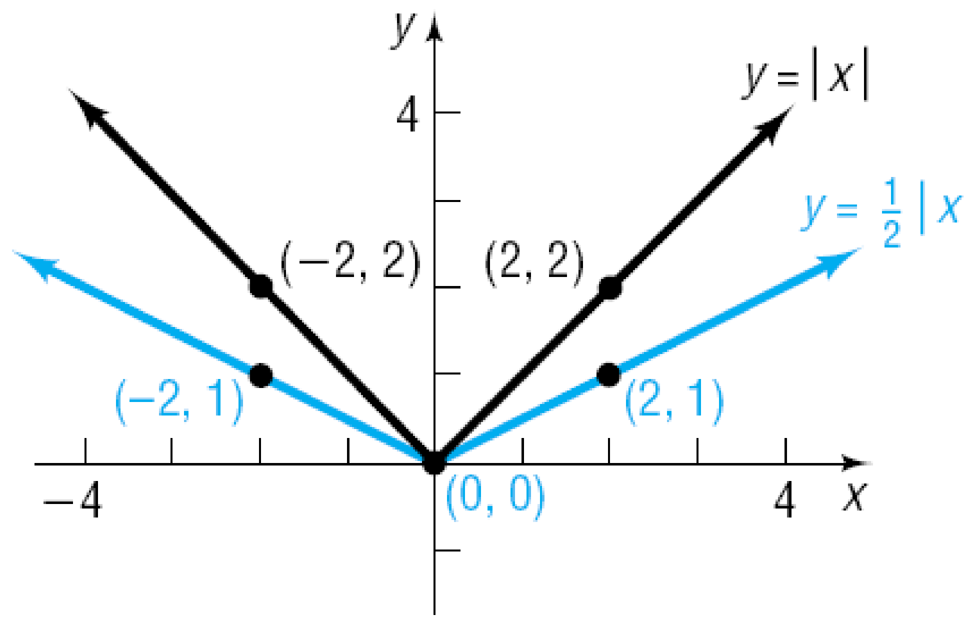
x	$y = f(x) = \|x\|$	$y = g(x) = 2\|x\|$
-2	2	4
-1	1	2
0	0	0
1	1	2
2	2	4



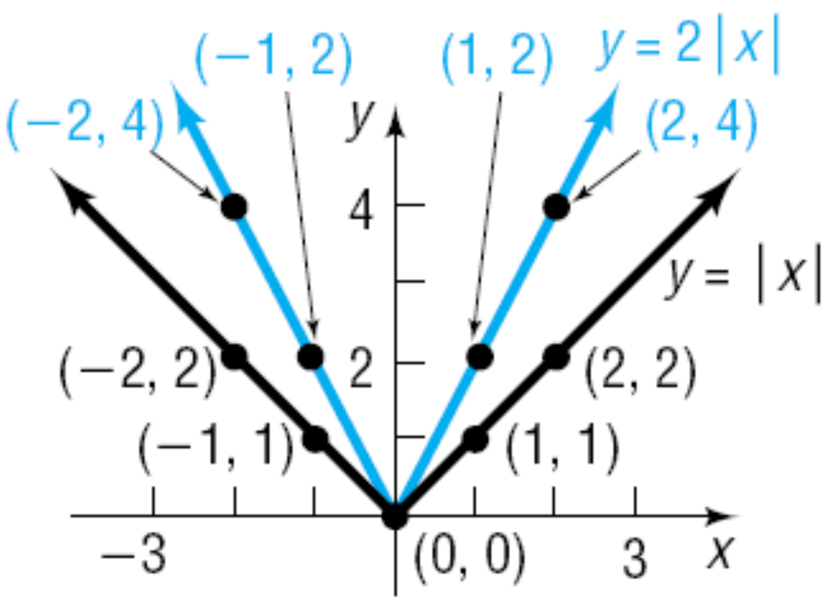
EXAMPLE Vertical Compression

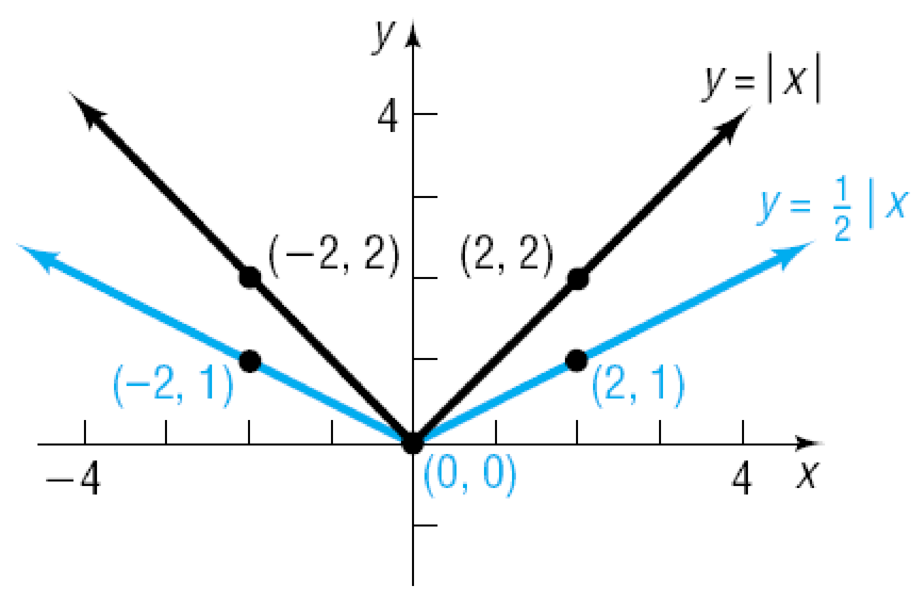
Use the graph of $f(x) = |x|$ to obtain the graph of $g(x) = \frac{1}{2}|x|$.

x	$y = f(x) = \lvert x \rvert$	$y = g(x) = \frac{1}{2}\lvert x \rvert$
-2	2	1
-1	1	$\frac{1}{2}$
0	0	0
1	1	$\frac{1}{2}$
2	2	1



When the right side of a function $y = f(x)$ is multiplied by a positive number a, the graph of the new function $y = af(x)$ is obtained by multiplying each y-coordinate on the graph of $y = f(x)$ by a. The new graph is a **vertically compressed** (if $0 < a < 1$) or a **vertically stretched** (if $a > 1$) version of the graph of $y = f(x)$.



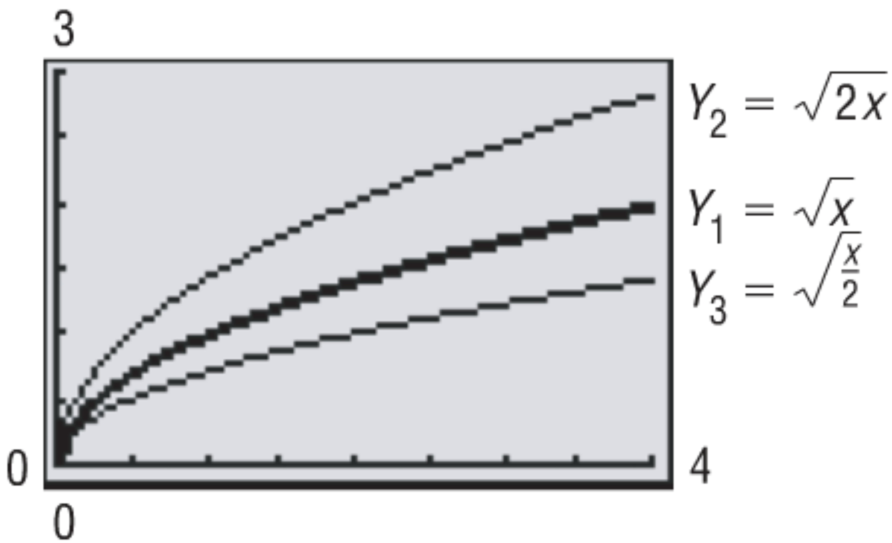


Exploration

On the same screen, graph each of the following functions:

$$Y_1 = f(x) = \sqrt{x} \qquad Y_2 = f(2x) = \sqrt{2x} \qquad Y_3 = f\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x} = \sqrt{\frac{x}{2}}$$

Create a table of values to explore the relation between the *x*- and *y*-coordinates of each function.



X	Y1	Y2
0	0	0
.5	.70711	1
1	1	1.4142
2	1.4142	2
4	2	2.8284
4.5	2.1213	3
9	3	4.2426

Y2 = √(2X)

X	Y1	Y3
0	0	0
1	1	.70711
2	1.4142	1
4	2	1.4142
8	2.8284	2
9	3	2.1213
18	4.2426	3

Y3 = √(X/2)

If the argument x of a function $y = f(x)$ is multiplied by a positive number a, the graph of the new function $y = f(ax)$ is obtained by multiplying each x-coordinate of $y = f(x)$ by $\frac{1}{a}$. A **horizontal compression** results if $a > 1$, and a **horizontal stretch** occurs if $0 < a < 1$.

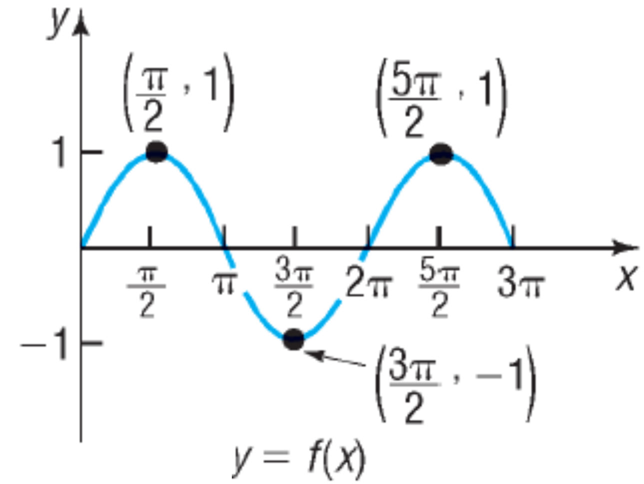
EXAMPLE

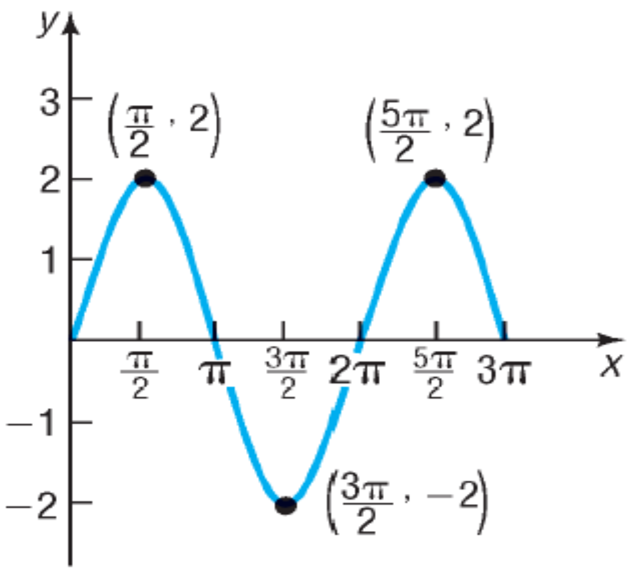
Graphing Using Stretches and Compressions

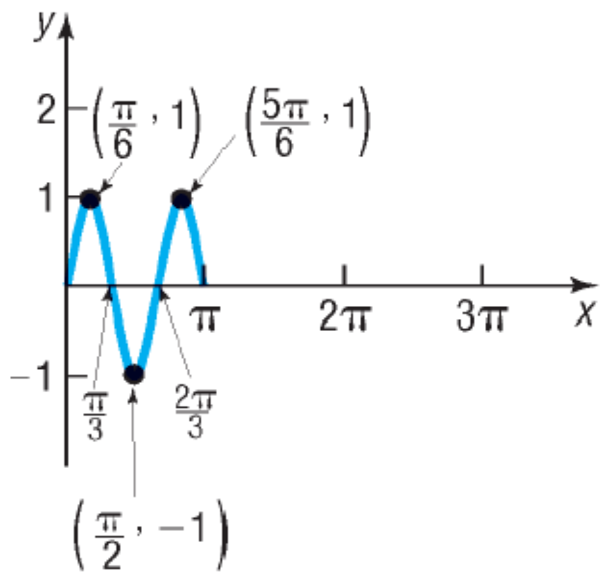
The graph of $y = f(x)$ is given.

Use this graph to find the graphs of

(a) $y = 2f(x)$ (b) $y = f(3x)$





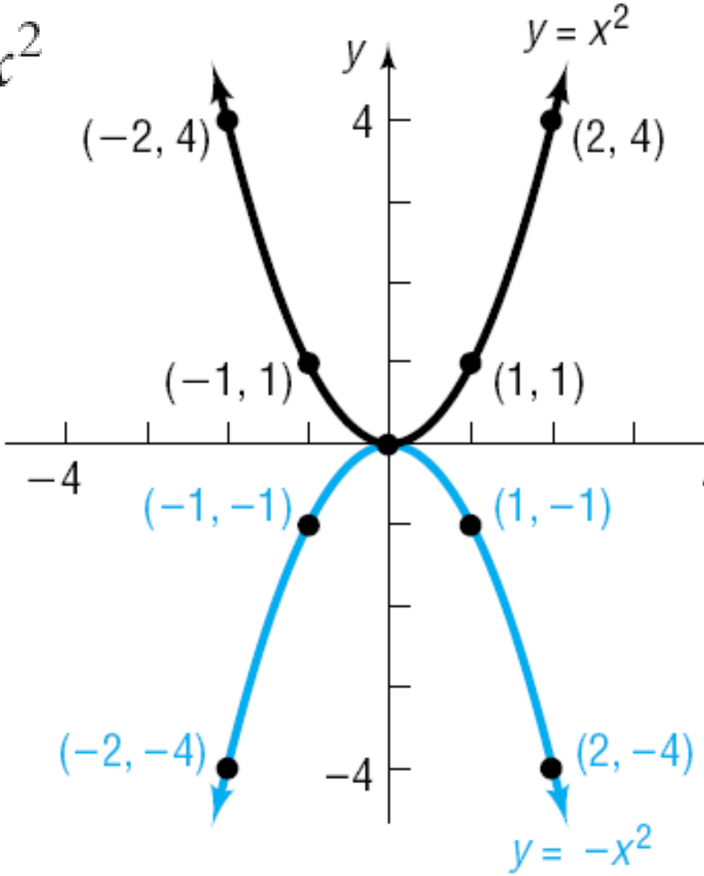


3 Graph Functions Using Reflections about the *x*-Axis and the *y*-Axis

EXAMPLE Reflection about the x-Axis

Graph the function: $f(x) = -x^2$

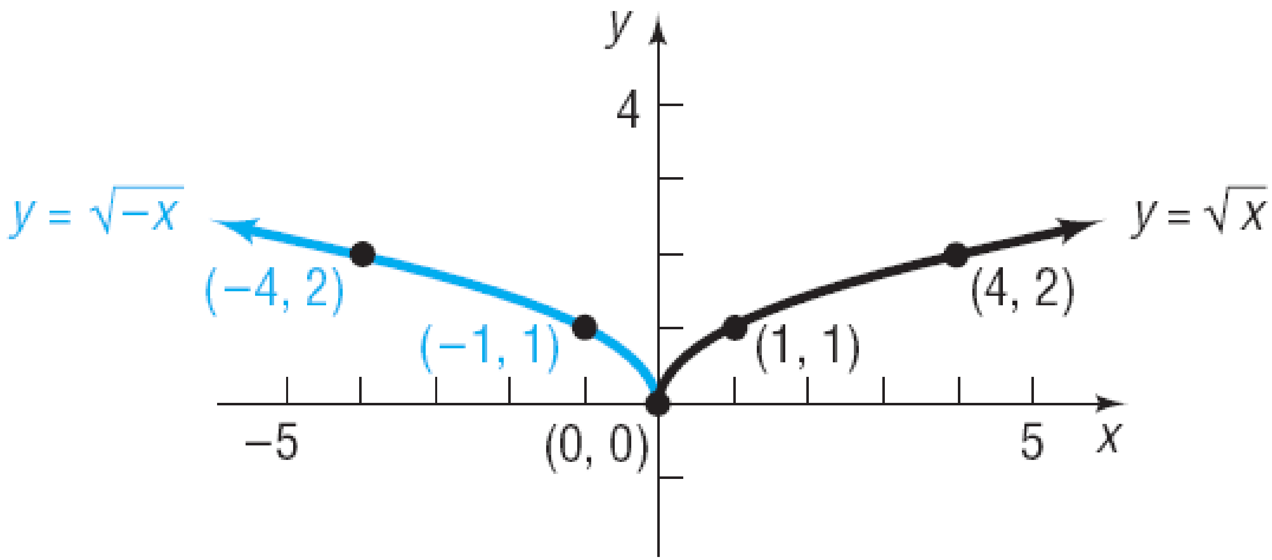
x	$y = x^2$	$y = -x^2$
-2	4	-4
-1	1	-1
0	0	0
1	1	-1
2	4	-4



When the right side of the function $y = f(x)$ is multiplied by -1, the graph of the new function $y = -f(x)$ is the **reflection about the x-axis** of the graph of the function $y = f(x)$.

EXAMPLE Reflection about the x-Axis

Graph the function: $f(x) = \sqrt{-x}$



When the graph of the function $y = f(x)$ is known, the graph of the new function $y = f(-x)$ is the **reflection about the *y*-axis** of the graph of the function $y = f(x)$.

SUMMARY OF GRAPHING TECHNIQUES

To Graph:	Draw the Graph of f and:	Functional Change to $f(x)$
Vertical shifts		
$y = f(x) + k, \quad k > 0$	Raise the graph of f by k units.	Add k to $f(x)$.
$y = f(x) - k, \quad k > 0$	Lower the graph of f by k units	Subtract k from $f(x)$.

Horizontal shifts		
$y = f(x + h), \quad h > 0$	Shift the graph of f to the left h units.	Replace x by $x + h$.
$y = f(x - h), \quad h > 0$	Shift the graph of f to the right h units.	Replace x by $x - h$.

Summary of Graphing Techniques

To Graph:	Draw the Graph of f and:	Functional Change to $f(x)$
Compressing or stretching		
$y = af(x), \quad a > 0$	Multiply each y-coordinate of $y = f(x)$ by a. Stretch the graph of f vertically if $a > 1$. Compress the graph of f vertically if $0 < a < 1$.	Multiply $f(x)$ by a.
$y = f(ax), \quad a > 0$	Multiply each x-coordinate of $y = f(x)$ by $\frac{1}{a}$. Stretch the graph of f horizontally if $0 < a < 1$. Compress the graph of f horizontally if $a > 1$.	Replace x by ax.

Summary of Graphing Techniques

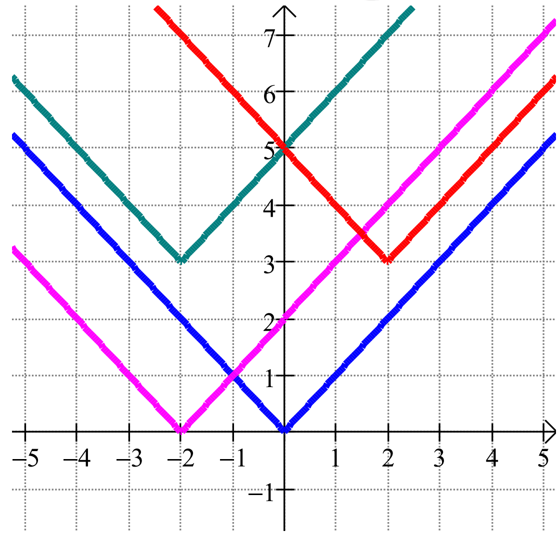
To Graph:	Draw the Graph of f and:	Functional Change to $f(x)$
Reflection about the x-axis $y = -f(x)$	Reflect the graph of f about the x-axis.	Multiply $f(x)$ by -1.
Reflection about the y-axis $y = f(-x)$	Reflect the graph of f about the y-axis.	Replace x by $-x$.

EXAMPLE

Determining the Function Obtained from a Series of Transformations

Find the function that is finally graphed after the following three transformations are applied to the graph of $\boxed{y = |x|}$.

1. Shift left 2 units.
2. Shift up 3 units.
3. Reflect about the y-axis.



1. Shift left 2 units: Replace x by $x + 2$. $\boxed{y = |x + 2|}$

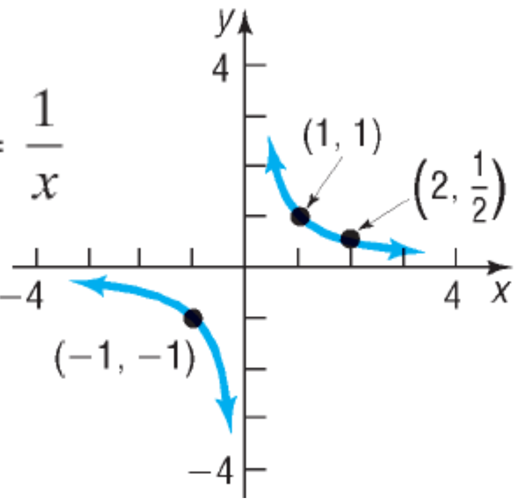
2. Shift up 3 units: Add 3. $\boxed{y = |x + 2| + 3}$

3. Reflect about the y-axis: Replace x by $-x$. $\boxed{y = |-x + 2| + 3}$

EXAMPLE Combining Graphing Procedures

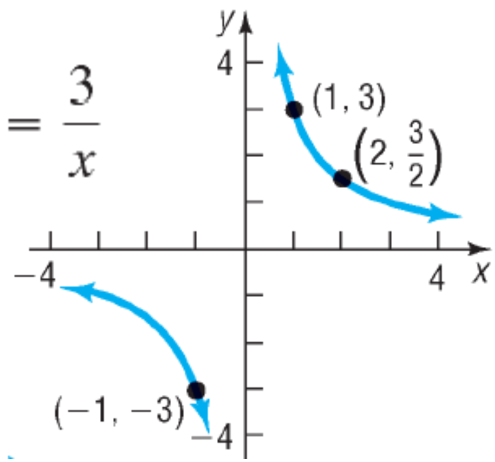
Graph the function: $f(x) = \dfrac{3}{x-2} + 1$

Step 1: $y = \dfrac{1}{x}$



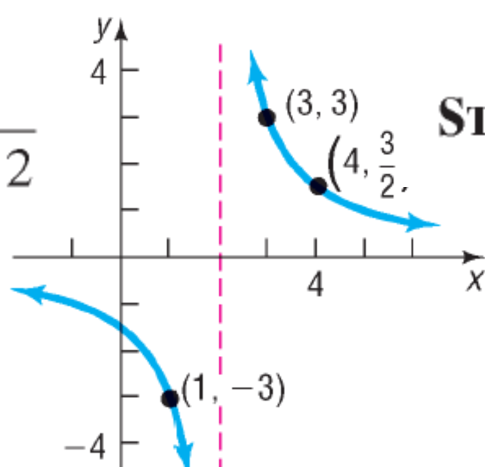
Step 2: $y = \dfrac{3}{x}$

Multiply by 3;
Vertical stretch



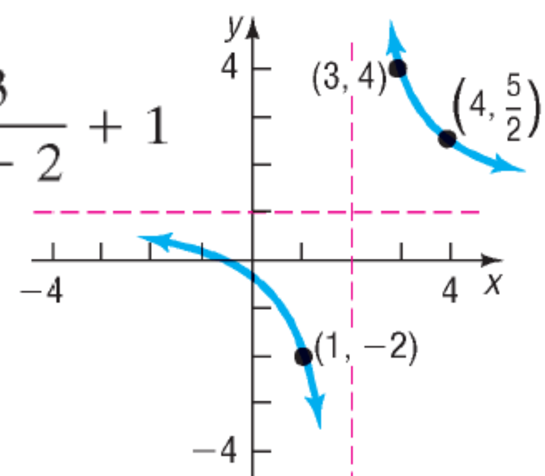
Step 3: $y = \dfrac{3}{x-2}$

Replace x by $x - 2$;
Horizontal shift
right 2 units



Step 4: $y = \dfrac{3}{x-2} + 1$

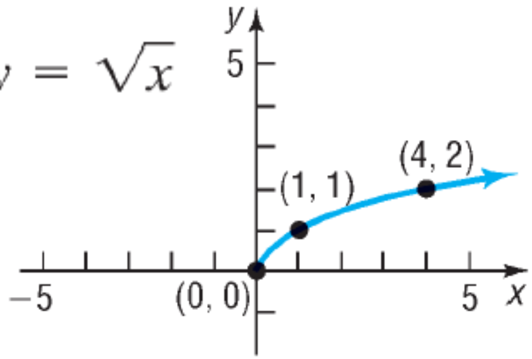
Add 1;
Vertical shift
up 1 unit



EXAMPLE Combining Graphing Procedures

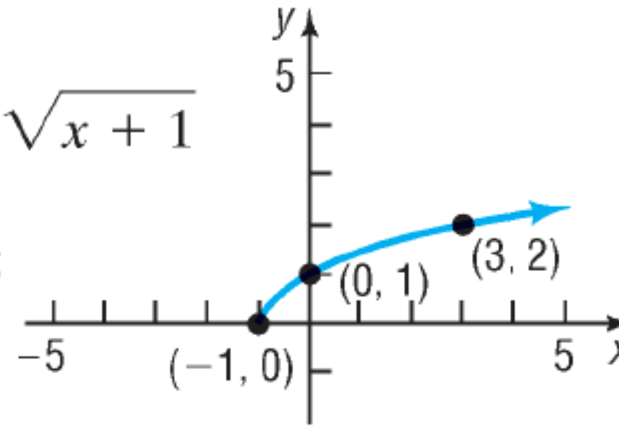
Graph the function: $f(x) = \sqrt{1 - x} + 2$

STEP 1: $y = \sqrt{x}$



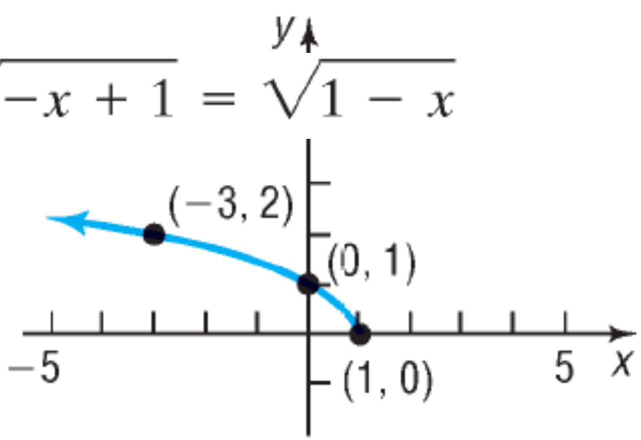
STEP 2: $y = \sqrt{x + 1}$

Replace x by $x + 1$;
Horizontal shift
left 1 unit



STEP 3: $y = \sqrt{-x + 1} = \sqrt{1 - x}$

Replace x by $-x$;
Reflect
about y-axis



STEP 4: $y = \sqrt{1 - x} + 2$

Add 2;
Vertical shift
up 2 units