Section 6.1

Composite Functions

1 Form a Composite Function

Suppose that an oil tanker is leaking oil and we want to be able to determine the area o the circular oil patch around the ship. It is determined that the oil is leaking from the tanker in such a way that the radius of the circular oil patch around the ship is increasing at a rate of 3 feet per minute.

$$r(t) = 3t$$

$$A(r) = \pi r^2$$

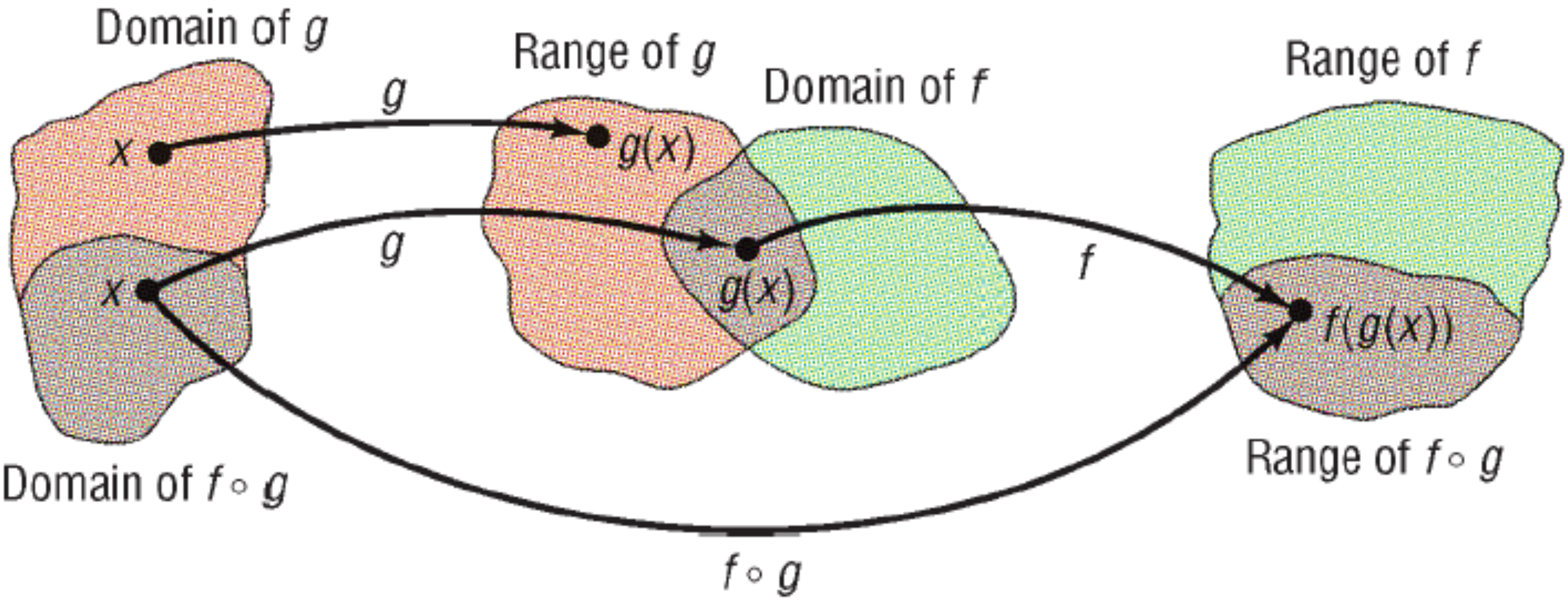
In general, we can find the area of the oil patch as a function of time t by evaluating $A(r(t))$ and obtaining $A(r(t)) = A(3t) = \pi(3t)^2 = 9\pi t^2$. The function $A(r(t))$ is a special type of function called a *composite function.*

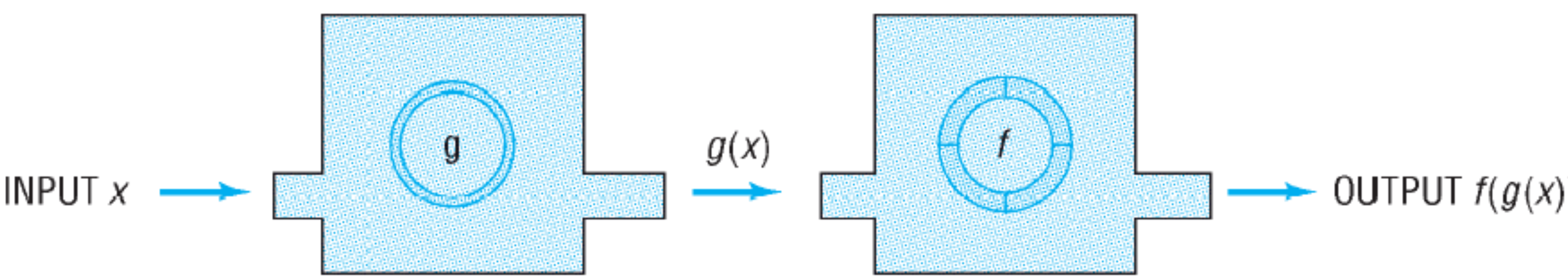
DEFINITION

Given two functions f and g, the **composite function**, denoted by $f \circ g$ (read as "f composed with g"), is defined by

$$\boxed{(f \circ g)(x) = f(g(x))}$$

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f.


Domain of g
Range of g
Domain of f
Range of f
g
x
$g(x)$
g
x
$g(x)$
f
$f(g(x))$
Domain of $f \circ g$
Range of $f \circ g$
$f \circ g$


INPUT x
g
$g(x)$
f
OUTPUT $f(g(x))$

EXAMPLE

Evaluating a Composite Function

Suppose that $f(x) = 2x^2 + 3$ $g(x) = 4x^3 + 1$. Find:

(a) $(f \circ g)(1)$ (b) $(g \circ f)(1)$ (c) $(f \circ f)(-2)$ (d) $(g \circ g)(-1)$

(a) $(f \circ g)(1) = f(g(1)) = f(5) = 2(5)^2 + 3 = 53$

$g(1) = 4(1)^3 + 1 = 5$

(b) $(g \circ f)(1) = g(f(1)) = g(5) = 4(5)^3 + 1 = 501$

$f(1) = 2(1)^2 + 3 = 5$

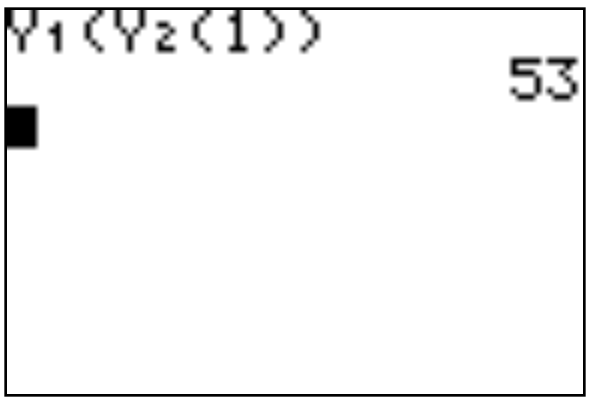
(c) $(f \circ f)(-2) = f(f(-2)) = f(11) = 2(11)^2 + 3 = 245$

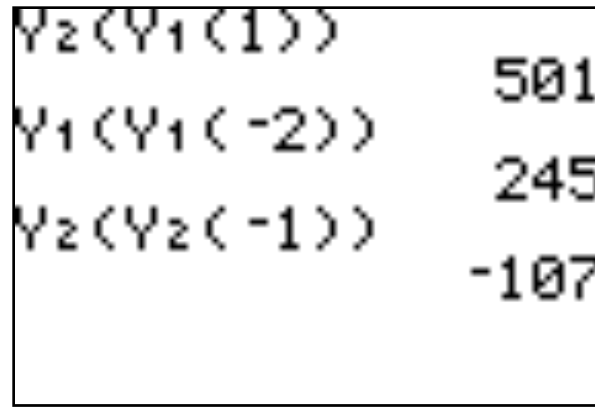
$f(-2) = 2(-2)^2 + 3 = 11$

(d) $(g \circ g)(-1) = g(g(-1)) = g(-3) = 4(-3)^3 + 1 = -107$

$g(-1) = 4(-1)^3 + 1 = -3$

COMMENT Graphing calculators can be used to evaluate composite functions.*





2 Find the Domain of a Composite Function

EXAMPLE Finding a Composite Function and Its Domain

Suppose that $f(x) = 2x^2 - x + 4$ and $g(x) = 4x + 1$.

Find: (a) $f \circ g$ (b) $g \circ f$

Then find the domain of each composite function.

(a) $(f \circ g)(x) = f(g(x)) = f(4x+1) = 2(4x+1)^2 - (4x+1) + 4$

$g(x) = 4x + 1$

$= 2(16x^2 + 8x + 1) - 4x - 1 + 4$

$= 32x^2 + 16x + 2 - 4x + 3$

$= 32x^2 + 12x + 5$

The domain of g is all real numbers as is the domain of the composite function, so the domain of $f \circ g$ is the set of all real numbers.

EXAMPLE Finding a Composite Function and Its Domain

Suppose that $f(x) = 2x^2 - x + 4$ and $g(x) = 4x + 1$.

Find: (a) $f \circ g$ (b) $g \circ f$

Then find the domain of each composite function.

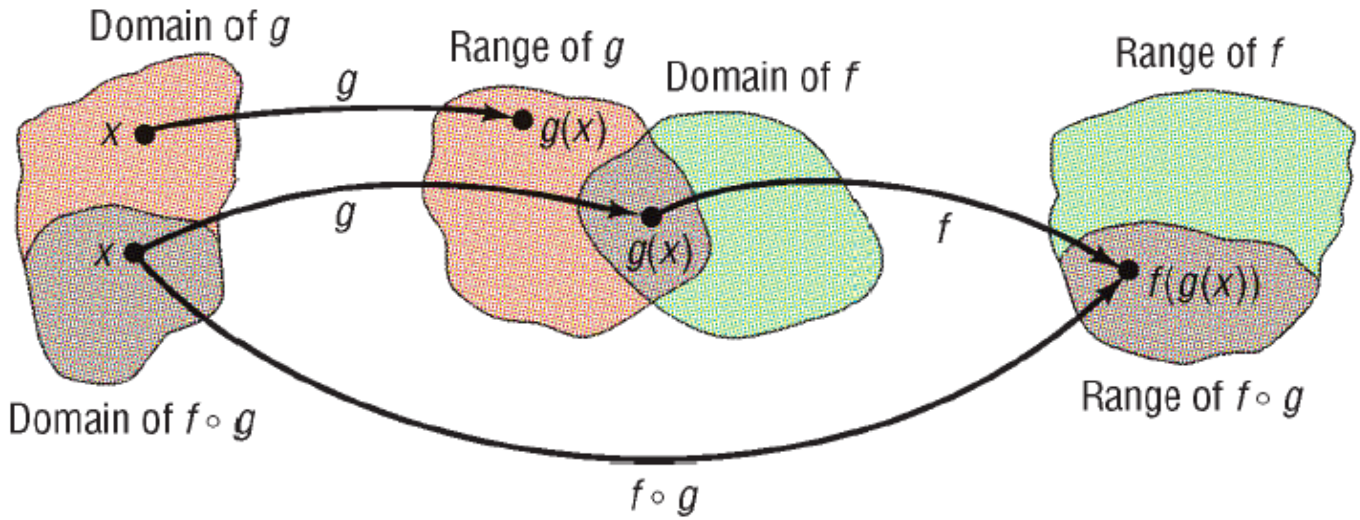
(b) $(g \circ f)(x) = g(f(x)) = g(2x^2 - x + 4) = 4(2x^2 - x + 4) + 1$

$f(x) = 2x^2 - x + 4$

$$= 8x^2 - 4x + 16 + 1$$

$$= 8x^2 - 4x + 17$$

The domain of f is all real numbers as is the domain of the composite function, so the domain of $g \circ f$ is the set of all real numbers.



1. $g(x)$ must be defined so any x not in the domain of g must be excluded.
2. $f(g(x))$ must be defined so any x for which $g(x)$ is not in the domain of f must be excluded.

EXAMPLE Finding the Domain of $f \circ g$

Find the domain of $(f \circ g)(x)$ if $f(x)=\dfrac{3}{x-5}$ and $g(x)=\dfrac{2}{x+1}$.

We first need the domain of g which is $\{x \mid x \neq -1\}$.

So we must exclude -1 from the domain of $f \circ g$.

The domain of f is $x \neq 5$ so if we put $g(x)$ in this function for x, $g(x) \neq 5$.

Find value where $g(x)=\dfrac{2}{x+1}=5$ $\quad 2=5(x+1)$ $\quad 2=5x+5$

$$x=-\frac{3}{5}$$

The domain of $f \circ g$ is $\left\{x \,\middle|\, x \neq -1, x \neq -\dfrac{3}{5}\right\}$.

EXAMPLE Finding a Composite Function and Its Domain

Suppose that $f(x)=\frac{3}{x-5}$ and $g(x)=\frac{2}{x+1}$.

Find: (a) $f \circ g$ (b) $f \circ f$ and the domain of the composite function.

Note the domain of f is $\{x|x\neq 5\}$ and the domain of g is $\{x|x\neq -1\}$.

(a) $$(f\circ g)(x)=f(g(x))=f\left(\frac{2}{x+1}\right)=\frac{3}{\frac{2}{x+1}-5}$$

$$g(x)=\frac{2}{x+1}$$

$$\frac{3(x+1)}{2-5(x+1)}=\frac{3x+3}{2-5x-5}=\frac{3x+3}{-5x-3}=-\frac{3x+3}{5x+3}$$

The domain of $f\circ g$ is the excluded value for the domain of g which is -1 and also the value that would cause the composite function to have division by 0 so the domain of $f\circ g$ is $\left\{x\middle|x\neq -1, x\neq -\frac{3}{5}\right\}$.

EXAMPLE **Finding a Composite Function and Its Domain**

Suppose that $f(x) = \dfrac{3}{x-5}$ and $g(x) = \dfrac{2}{x+1}$.

Find: (a) $f \circ g$ (b) $f \circ f$ and the domain of the composite function.

Note the domain of f is $\{x \mid x \neq 5\}$ and the domain of g is $\{x \mid x \neq -1\}$.

(b) $$(f \circ f)(x) = f(f(x)) = f\left(\frac{3}{x-5}\right) = \frac{3}{\dfrac{3}{x-5} - 5}$$

$$f(x) = \frac{3}{x-5}$$

$$\frac{3(x-5)}{3-5(x-5)} = \frac{3x-15}{3-5x+25} = \frac{3x-15}{-5x+28} = \frac{3x-15}{-5x+28}$$

The domain of $f \circ f$ is the excluded value for the domain of f which is 5 and also the value that would cause the composite function to have division by 0 so the domain of $f \circ f$ is $\left\{x \,\middle|\, x \neq 5, x \neq \dfrac{28}{5}\right\}$.

EXAMPLE

Showing That Two Composite Functions Are Equal

If $f(x) = -2x+1$ and $g(x) = -\frac{1}{2}(x-1)$. Show that $(f \circ g)(x) = (g \circ f)(x) = x$ for every x in the domain of $f \circ g$ and $g \circ f$.

$$(f \circ g)(x) = f(g(x)) = f\left(-\frac{1}{2}(x-1)\right) = -2\left(-\frac{1}{2}(x-1)\right)+1$$

$$g(x) = -\frac{1}{2}(x-1)$$

$$= -2\left(-\frac{1}{2}x+\frac{1}{2}\right)+1 = x-1+1 = x$$

$$(g \circ f)(x) = g(f(x)) = g(-2x+1) = -\frac{1}{2}(-2x+1-1)$$

$$f(x) = -2x+1$$

$$= -\frac{1}{2}(-2x) = x$$

Seeing the Concept

Using a graphing calculator, let

$$Y_1 = f(x) = 3x - 4$$

$$Y_2 = g(x) = \frac{1}{3}(x + 4)$$

$$Y_3 = f \circ g, Y_4 = g \circ f$$

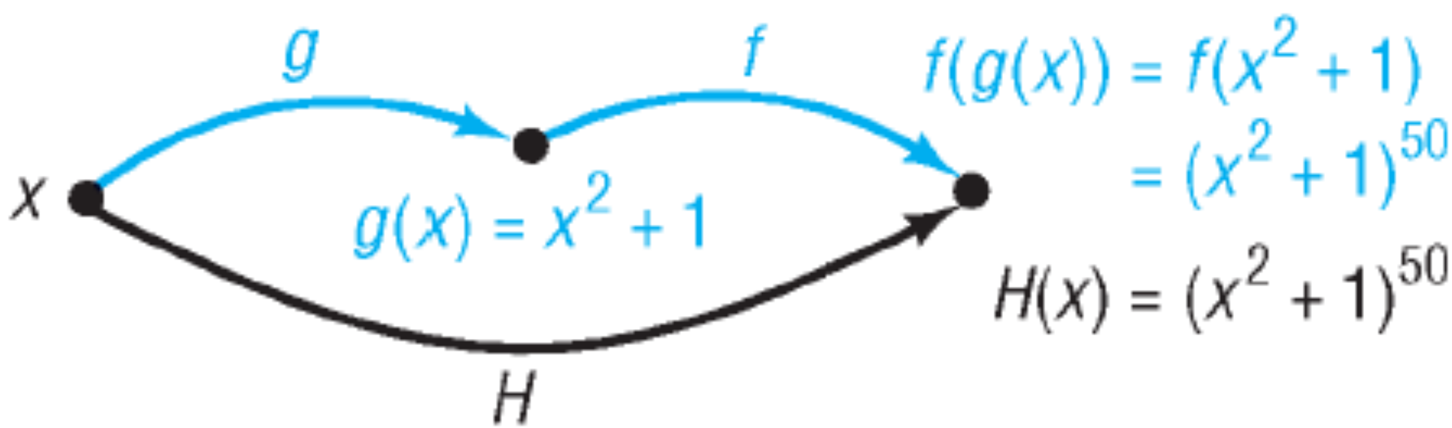
Using the viewing window $-3 \leq x \leq 3$, $-2 \leq y \leq 2$, graph only Y_3 and Y_4. What do you see? TRACE to verify that $Y_3 = Y_4$.

Calculus Application

EXAMPLE Finding the Components of a Composite Function

Find functions f and g such that $f \circ g = H$ if $H(x) = (x^2 + 1)^{50}$.

$$f(x) = x^{50} \qquad g(x) = x^2 + 1$$



EXAMPLE Finding the Components of a Composite Function

Find functions f and g such that $f \circ g = H$ if $H(x) = \dfrac{3}{(x-5)^2}$

$g(x) = x - 5$ $\qquad\qquad$ $f(x) = \dfrac{3}{x^2}$

$$(f \circ g)(x) = f(g(x)) = \frac{3}{(x-5)^2} = H(x)$$