Section 8.2

The Inverse Trigonometric Functions (Continued)

1 Find the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

EXAMPLE Finding the Exact Value of an Expression Involving Inverse Trigonometric Functions

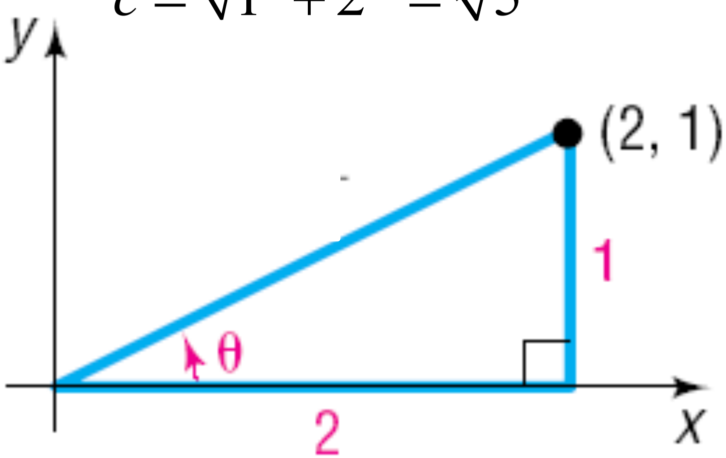
Find the exact value of: $\sin\left(\tan^{-1}\frac{1}{2}\right) = \sin\theta$

Let $\theta = \tan^{-1}\frac{1}{2}$. Then $\tan\theta = \frac{1}{2}$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We seek $\sin\theta$.

Since tangent is positive we draw a triangle in quadrant I and label sides so $\tan\theta = \frac{1}{2}$.

Use the Pythagorean Theorem to determine the hypotenuse.

$$c = \sqrt{1^2 + 2^2} = \sqrt{5}$$



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

EXAMPLE **Finding the Exact Value of an Expression Involving Inverse Trigonometric Functions**

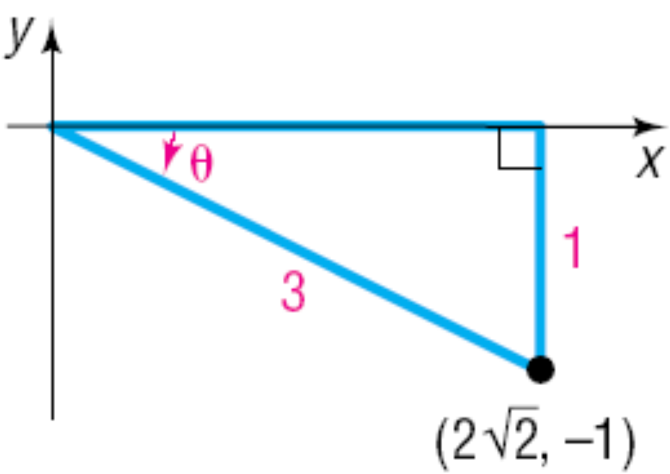
Find the exact value of: $\cos\left[\sin^{-1}\left(-\frac{1}{3}\right)\right] = \cos\theta$

Let $\theta = \sin^{-1}\left(-\frac{1}{3}\right)$. Then $\sin\theta = -\frac{1}{3}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. We seek $\cos\theta$.

Since sine is negative we draw a triangle in quadrant IV and label sides so $\sin\theta = -\frac{1}{3}$.

Use the Pythagorean Theorem to determine x.

$$x = \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$$



$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2\sqrt{2}}{3}$$

EXAMPLE **Finding the Exact Value of an Expression Involving Inverse Trigonometric Functions**

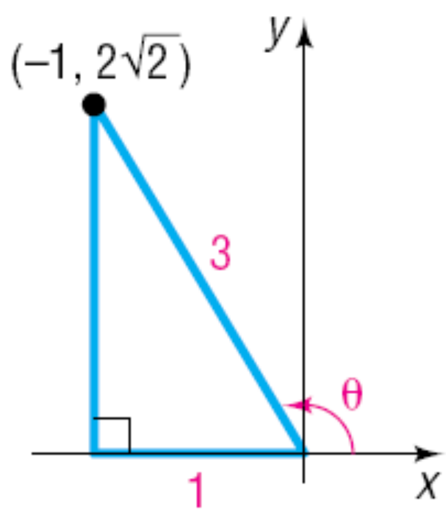
Find the exact value of: $\tan\left[\cos^{-1}\left(-\frac{1}{3}\right)\right] = \tan\theta$

Let $\theta = \cos^{-1}\left(-\frac{1}{3}\right)$. Then $\cos\theta = -\frac{1}{3}$ and $0 \le \theta \le \pi$. We seek $\tan\theta$.

Since cosine is negative we draw a triangle in quadrant II and label sides so $\cos\theta = -\frac{1}{3}$.

Use the Pythagorean Theorem to determine y.

$$y = \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$$

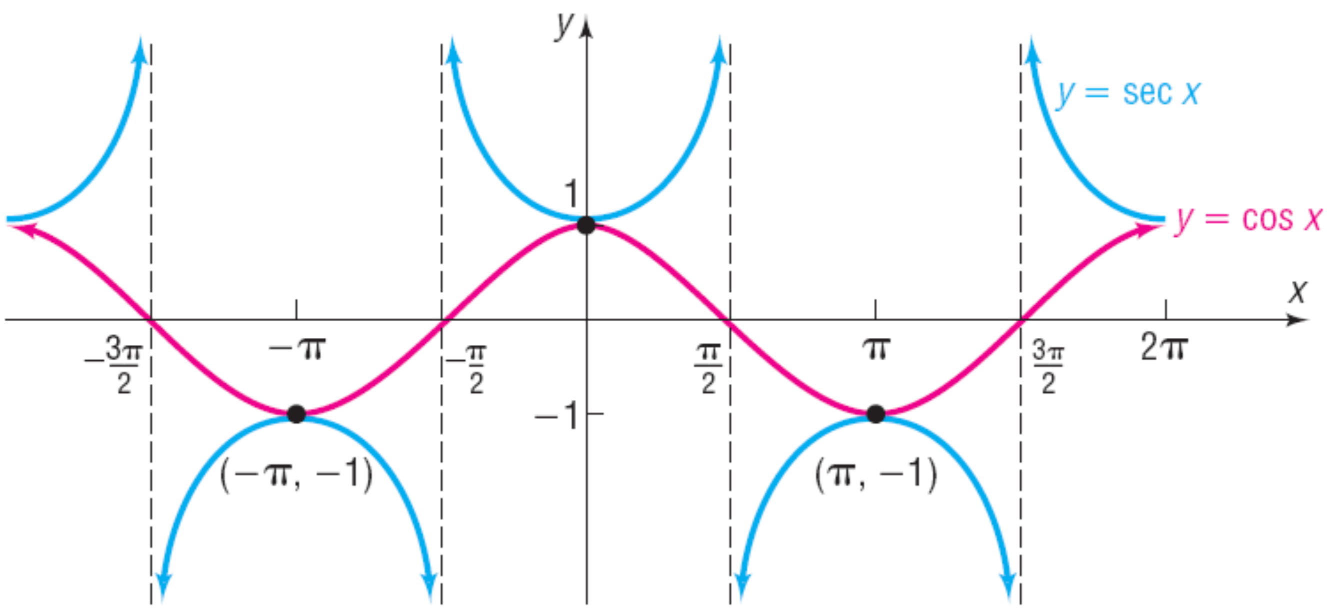


$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{2\sqrt{2}}{-1} = -2\sqrt{2}$$

2 Define the Inverse Secant, Cosecant, and Cotangent Functions

$$y = \sec^{-1} x \quad \text{means} \quad x = \sec y$$

$$\text{where} \quad |x| \geq 1 \quad \text{and} \quad 0 \leq y \leq \pi, \quad y \neq \frac{\pi}{2}^{*}$$

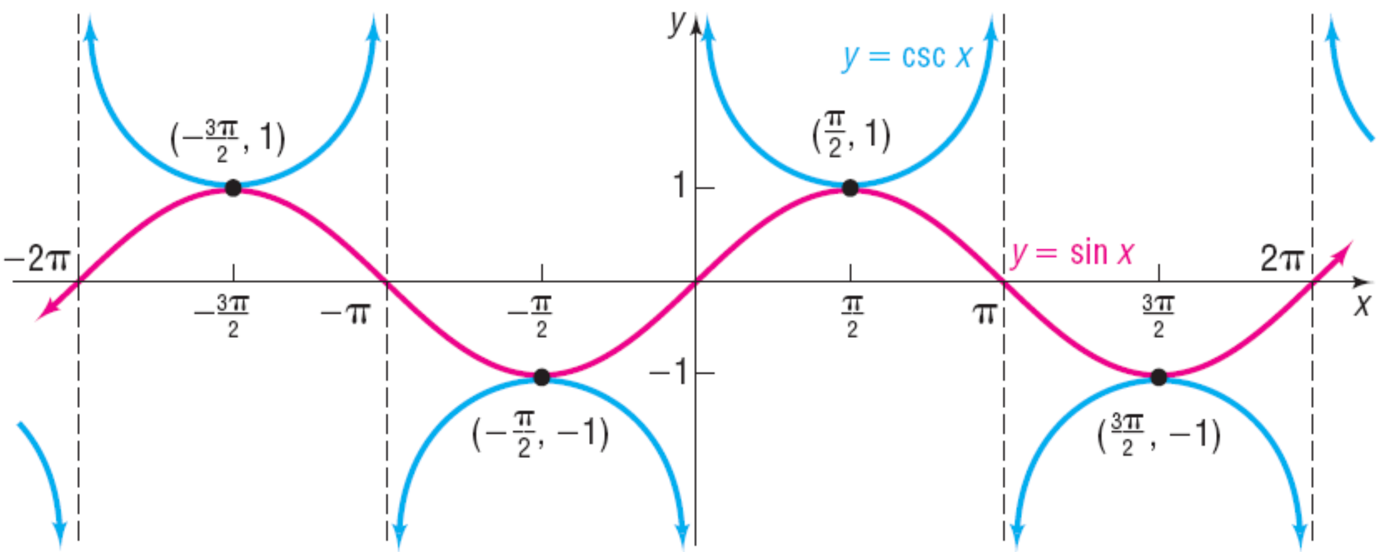


$y = \sec x$, $-\infty < x < \infty$, x not equal

to odd multiples of $\frac{\pi}{2}$, $|y| \geq 1$

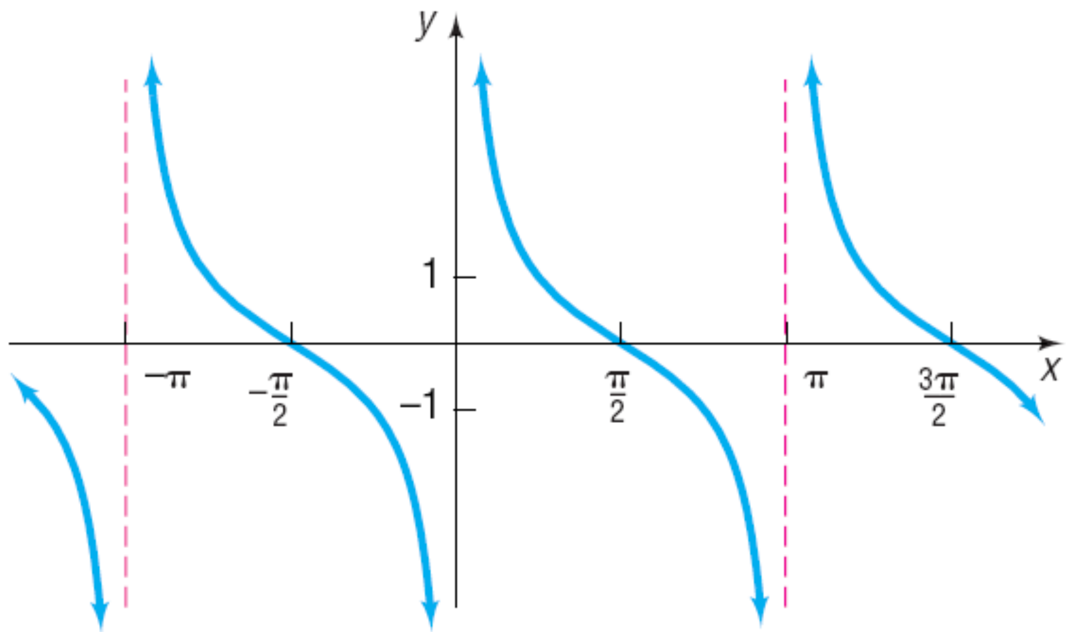
$$y = \csc^{-1} x \quad \text{means} \quad x = \csc y$$

$$\text{where} \quad |x| \geq 1 \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad y \neq 0^{\dagger}$$



$y = \csc x, -\infty < x < \infty$, x not equal to integer multiples of π, $|y| \geq 1$

$$y = \cot^{-1} x \quad \text{means} \quad x = \cot y$$
$$\text{where} \quad -\infty < x < \infty \quad \text{and} \quad 0 < y < \pi$$



$y = \cot x$, $-\infty < x < \infty$, x not equal to integer multiples of π, $-\infty < y < \infty$

EXAMPLE

Finding the Exact Value of an Inverse Cosecant Function

Find the exact value of: $\csc^{-1} 2$

Let $\theta = \csc^{-1} 2$. We seek the angle $\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \neq 0$, whose cosecant equals 2 $\left(\text{or, equivalently, whose sine equals } \frac{1}{2}\right)$.

The only angle θ in the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \neq 0$, whose cosecant is 2 $\left[\sin\theta = \frac{1}{2}\right]$ is $\frac{\pi}{6}$, so $\csc^{-1} 2 = \frac{\pi}{6}$.

3 Use a Calculator to Evaluate $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$

EXAMPLE

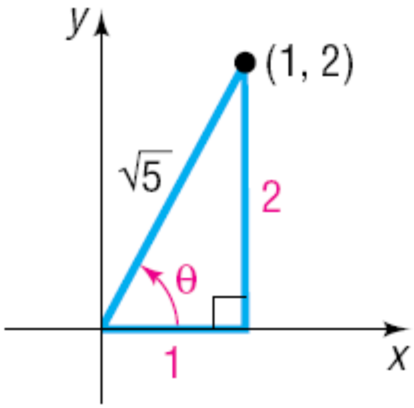
Approximating the Value of Inverse Trigonometric Functions

Use a calculator to approximate each expression in radians rounded to two decimal places.

(a) $\sec^{-1} 3$ (b) $\csc^{-1}(-4)$ (c) $\cot^{-1}\dfrac{1}{2}$ (d) $\cot^{-1}(-2)$

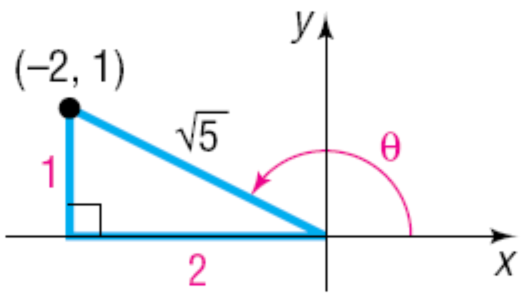
$$\sec^{-1} 3 = \theta = \cos^{-1}\frac{1}{3} \approx 1.23$$

$$\csc^{-1}(-4) = \theta = \sin^{-1}\left(-\frac{1}{4}\right) \approx -0.25$$



$$\cot^{-1}\frac{1}{2} = \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 1.11$$

$$\cot^{-1}(-2) = \theta = \cos^{-1}\left(-\frac{2}{\sqrt{5}}\right) \approx 2.68$$



4 Write a Trigonometric Expression as an Algebraic Expression

EXAMPLE **Writing a Trigonometric Expression as an Algebraic Expression**

Write $\sin(\tan^{-1} u)$ as an algebraic expression containing u.

Let $\theta = \tan^{-1} u$ so that $\tan\theta = u, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, -\infty < u < \infty$.

$$\sin(\tan^{-1} u) = \sin\theta \underset{\uparrow}{=} \sin\theta \cdot \frac{\cos\theta}{\cos\theta} \underset{\uparrow}{=} \tan\theta\cos\theta$$

Multiply by 1: $\frac{\cos\theta}{\cos\theta}$.

$\frac{\sin\theta}{\cos\theta} = \tan\theta$

$$= \frac{\tan\theta}{\sec\theta} \underset{\uparrow}{=} \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{u}{\sqrt{1+u^2}}$$

$\sec^2\theta = 1 + \tan^2\theta$

$\sec\theta > 0$