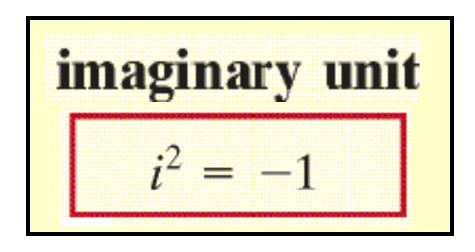
Section 1.3 Complex Numbers Quadratic Equations in the Complex Number System



Complex numbers are numbers of the form a + bi, where a and b are real numbers. The real number a is called the **real part** of the number a + bi; the real number b is called the **imaginary part** of a + bi.



Equality of Complex Numbers

$$a + bi = c + di$$
 if and only if $a = c$ and $b = d$

Sum of Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Difference of Complex Numbers

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Adding and Subtracting Complex Numbers

a)
$$(3-2i)+(5-4i)$$

$$(3+5) + [(-2) + (-4)] i$$

$$8-6i$$

b)
$$(3-2i)-(5-4i)$$

$$(3-5) + [(-2) - (-4)] i$$

$$-2+2 i$$

Multiplying Complex Numbers

$$(3-2i)(5-4i)$$

$$3(5) + 3(-4i) + (-2i)(5) + (-2i)(-4i)$$

$$15 - 12i - 10i + 8i^2 = 15 - 22i + 8(-1) = 7 - 22i$$

Product of Complex Numbers

$$(a+bi)\cdot(c+di)=(ac-bd)+(ad+bc)i$$

Conjugates

If z = a + bi is a complex number, then its **conjugate**, denoted by \overline{z} , is defined as

$$\overline{z} = \overline{a + bi} = a - bi$$

EXAMPLE

Multiplying a Complex Number by Its Conjugate

$$z = 3 - 2i$$
 $\overline{z} = 3 + 2i$ $z\overline{z} = (3 - 2i)(3 + 2i)$

$$z\overline{z} = 9 + 6i - 6i - 4i^2 = 9 - 4(-1) = 13$$

Theorem

The product of a complex number and its conjugate is a nonnegative real number.

That is, if z = a + bi, then

$$z\overline{z} = a^2 + b^2$$

Writing the Reciprocal of a Complex Number in Standard Form

Write $\frac{1}{3-2i}$ in standard form a+bi;

That is, find the reciprocal of 3-2i.

$$\frac{1}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{3+2i}{9+4} = \frac{3}{13} + \frac{2}{13}i$$

Writing the Quotient of Two Complex Numbers in Standard Form

Write in standard form: $\frac{3-2i}{5-4i}$

$$\frac{3-2i}{5-4i} \cdot \frac{5+4i}{5+4i} = \frac{15+12i-10i-8i^2}{25+16} = \frac{23+2i}{41}$$

$$=\frac{23}{41}+\frac{2}{41}i$$

Writing Other Expressions in Standard Form

If z = 3 - i and w = 2 - 5i, write each of the following in standard form:

a)
$$\frac{z}{w} = \frac{3-i}{2-5i} \cdot \frac{2+5i}{2+5i} = \frac{6+15i-2i-5i^2}{4+25} = \frac{11}{29} + \frac{13}{29}i$$

b)
$$\overline{z+w} = \overline{(3-i)+(2-5i)} = \overline{5-6i} = 5+6i$$

c)
$$z + \overline{z} = (3-i)+(3+i)=6$$

Theorem

The conjugate of a real number is the real number itself.

Theorem

The conjugate of the conjugate of a complex number is the complex number itself.

$$(\overline{\overline{z}}) = z$$

The conjugate of the sum of two complex numbers equals the sum of their conjugates.

$$\overline{z+w}=\overline{z}+\overline{w}$$

The conjugate of the product of two complex numbers equals the product of their conjugates.

$$\overline{z \cdot w} = \overline{z} \cdot \overline{w}$$

Powers of i

The powers of *i* follow a pattern that is useful to know.

$$i^{1} = i$$
 $i^{5} = i^{4} \cdot i = 1 \cdot i = i$
 $i^{2} = -1$ $i^{6} = i^{4} \cdot i^{2} = -1$
 $i^{3} = i^{2} \cdot i = -i$ $i^{7} = i^{4} \cdot i^{3} = -i$
 $i^{4} = i^{2} \cdot i^{2} = (-1)(-1) = 1$ $i^{8} = i^{4} \cdot i^{4} = 1$

And so on. The powers of *i* repeat with every fourth power.

EXAMPLE

Evaluating Powers of i

a)
$$i^{33} = i^{32} \cdot i$$
 b) $i^{82} = i^{80} \cdot i^{2}$

$$= (i^{4})^{8} \cdot i = (1)^{8} \cdot i = i$$

$$= (i^{4})^{20} \cdot i^{2} = (1)^{20} \cdot (-1) = -1$$
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Writing the Power of a Complex Number in Standard Form

Write $(3-2i)^3$ in standard form.

We use the special product formula for $(x + a)^3$.

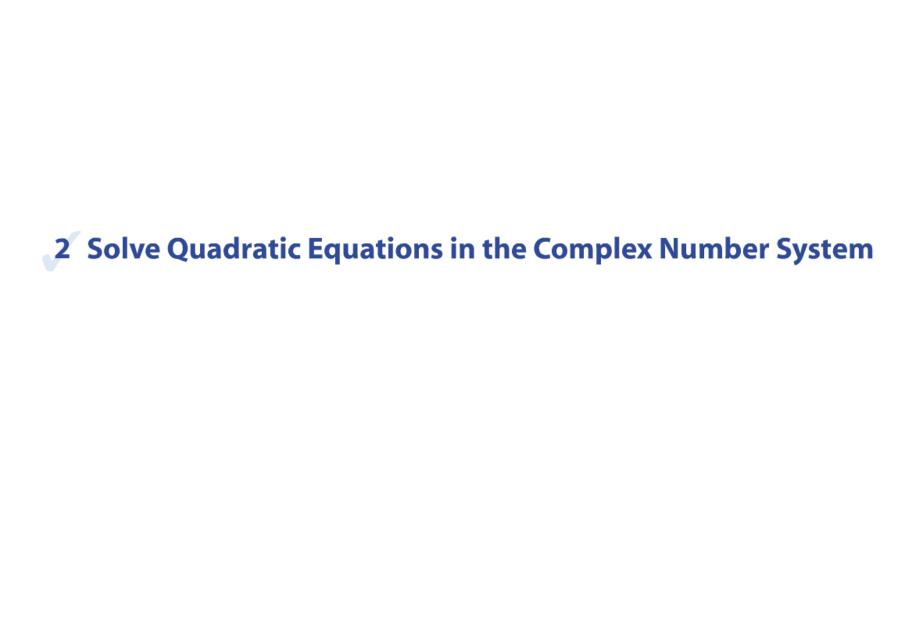
$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$

$$(3-2i)^{3} = (3)^{3} + 3(-2i)(3)^{2} + 3(-2i)^{2}(3) + (-2i)^{3}$$

$$= 27 - 54i + 36i^{2} - 8i^{3}$$

$$= 27 - 54i + 36(-1) - 8(-i)$$

$$= -9 - 46i$$



If N is a positive real number, we define the **principal square root of** -N, denoted by $\sqrt{-N}$, as

$$\sqrt{-N} = \sqrt{N}i$$

Evaluating the Square Root of a Negative Number

a)
$$\sqrt{-1} = \sqrt{1} \ i = i$$

b)
$$\sqrt{-9} = \sqrt{9} \ i = 3i$$

c)
$$\sqrt{-18} = \sqrt{18} \ i = 3\sqrt{2}i$$

Solving Equations

Solve the equation in the complex number system

a)
$$x^2 = 9$$
 $x = \pm \sqrt{9} = \pm 3$

b)
$$x^{2} + 25 = 0$$
 $x^{2} = -25$
$$x = \pm \sqrt{-25}$$

$$x = \pm \sqrt{25} i = \pm 5i$$

Theorem

In the complex number system, the solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \ne 0$, are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{9}$$

Solving a Quadratic Equation in the Complex Number System

Solve the equation $x^2 - 6x + 13 = 0$ in the complex number system.

$$b^2 - 4ac = (-6)^2 - 4(1)(13) = 36 - 52 = -16$$

$$x = \frac{-(-6) \pm \sqrt{-16}}{2(1)} = \frac{6 \pm \sqrt{16}i}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

Character of the Solutions of a Quadratic Equation

In the complex number system, consider a quadratic equation $ax^2 + bx + c = 0$ with real coefficients.

- 1. If $b^2 4ac > 0$, the equation has two unequal real solutions.
- 2. If $b^2 4ac = 0$, the equation has a repeated real solution, a double root.
- 3. If $b^2 4ac < 0$, the equation has two complex solutions that are not real. The solutions are conjugates of each other.

Determining the Character of the Solutions of a Quadratic Equation

Without solving, determine the character of the solutions of each equation.

a)
$$2x^2 + 3x + 1 = 0$$
 $b^2 - 4ac = (3)^2 - 4(2)(1) = 1$

The solutions are two unequal real numbers.

b)
$$5x^2 - 2x + 4 = 0$$
 $b^2 - 4ac = (-2)^2 - 4(5)(4) = -76$

The solutions are two non-real complex numbers that are conjugates of each other.

c)
$$4x^2-4x+1=0$$
 $b^2-4ac=(-4)^2-4(4)(1)=0$

The solution is a repeated real number, that is, a double root.