A

TRIGONOMETRY REVIEW

ANGLES

Angles in the plane can be generated by rotating a ray about its endpoint. The starting position of the ray is called the ***initial side*** of the angle, the final position is called the ***terminal side*** of the angle, and the point at which the initial and terminal sides meet is called the ***vertex*** of the angle. We allow for the possibility that the ray may make more than one complete revolution. Angles are considered to be ***positive*** if generated counterclockwise and ***negative*** if generated clockwise (Figure A.1).

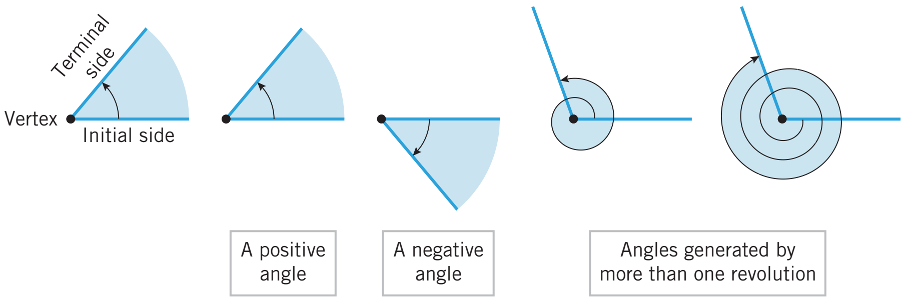


Figure A.1

There are two standard measurement systems for describing the size of an angle: ***degree measure*** and ***radian measure***. In degree measure, one degree (written 1°) is the measure of an angle generated by $1/360$ of one revolution. Thus, there are 360° in an angle of one revolution, 180° in an angle of one-half revolution, 90° in an angle of one-quarter revolution (a *right angle*), and so forth. Degrees are divided into sixty equal parts, called ***minutes***, and minutes are divided into sixty equal parts, called ***seconds***. Thus, one minute (written $1'$) is $1/60$ of a degree, and one second (written $1''$) is $1/60$ of a minute. Smaller subdivisions of a degree are expressed as fractions of a second.

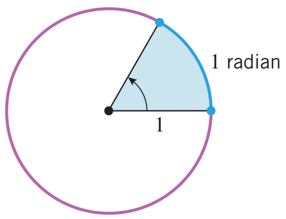


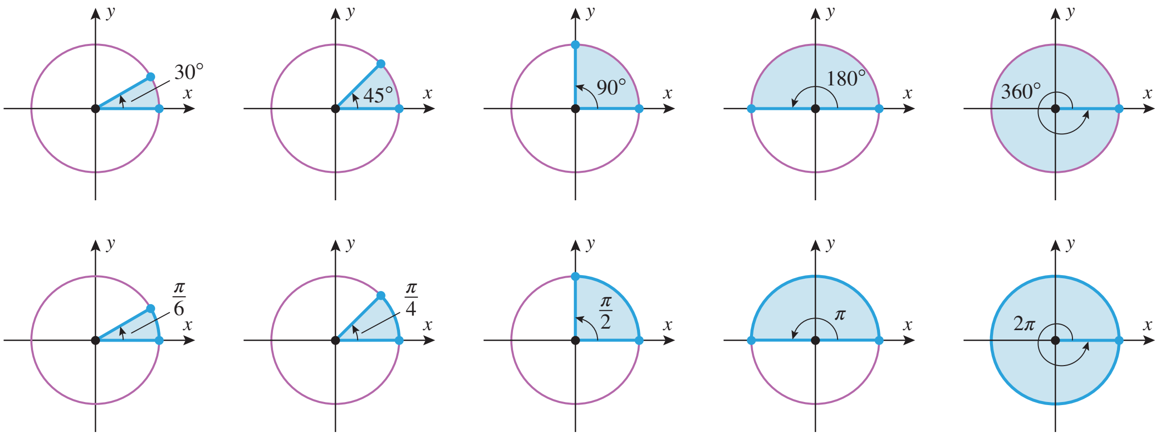
Figure A.2

In radian measure, angles are measured by the length of the arc that the angle subtends on a circle of radius 1 when the vertex is at the center. One unit of arc on a circle of radius 1 is called one ***radian*** (written 1 radian or 1 rad) (Figure A.2), and hence the entire circumference of a circle of radius 1 is 2π radians. It follows that an angle of 360° subtends an arc of 2π radians, an angle of 180° subtends an arc of π radians, an angle of 90° subtends an arc of $\pi/2$ radians, and so forth. Figure A.3 and Table A.1 show the relationship between degree measure and radian measure for some important positive angles.

From the fact that π radians corresponds to 180°, we obtain the following formulas, which are useful for converting from degrees to radians and conversely.

$$1^\circ = \frac{\pi}{180}\text{ rad} \approx 0.01745 \text{ rad} \tag{1}$$

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57^\circ 17' 44.8'' \tag{2}$$



▲ Figure A.3

Observe that in Table A.1, angles in degrees are designated by the degree symbol, but angles in radians have no units specified. This is standard practice—when no units are specified for an angle, it is understood that the units are radians.

Table A.1

DEGREES	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
RADIANS	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

▶ Example 1

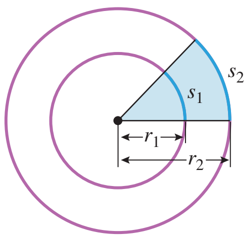
(a) Express 146° in radians. (b) Express 3 radians in degrees.

Solution (a). From (1), degrees can be converted to radians by multiplying by a conversion factor of $\pi/180$. Thus,

$$146^\circ = \left(\frac{\pi}{180}\cdot 146\right)\text{ rad} = \frac{73\pi}{90}\text{ rad} \approx 2.5482\text{ rad}$$

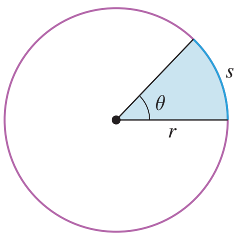
Solution (b). From (2), radians can be converted to degrees by multiplying by a conversion factor of $180/\pi$. Thus,

$$3\text{ rad} = \left(3\cdot\frac{180}{\pi}\right)^\circ = \left(\frac{540}{\pi}\right)^\circ \approx 171.9^\circ \blacktriangleleft$$



$\boxed{\dfrac{s_1}{s_2} = \dfrac{r_1}{r_2}}$

▲ Figure A.4



If θ is in radians, then $\theta = s/r$.

▲ Figure A.5

RELATIONSHIPS BETWEEN ARC LENGTH, ANGLE, RADIUS, AND AREA

There is a theorem from plane geometry which states that for two concentric circles, the ratio of the arc lengths subtended by a central angle is equal to the ratio of the corresponding radii (Figure A.4). In particular, if s is the arc length subtended on a circle of radius r by a central angle of θ radians, then by comparison with the arc length subtended by that angle on a circle of radius 1 we obtain

$$\frac{s}{\theta} = \frac{r}{1}$$

from which we obtain the following relationships between the central angle θ, the radius r, and the subtended arc length s when θ is in radians (Figure A.5):

$$\theta = s/r \qquad \text{and} \qquad s = r\theta \tag{3–4}$$

The shaded region in Figure A.5 is called a ***sector***. It is a theorem from plane geometry that the ratio of the area A of this sector to the area of the entire circle is the same as the ratio of the central angle of the sector to the central angle of the entire circle; thus, if the angles are in radians, we have

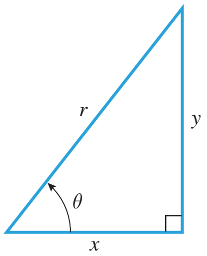
$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

Solving for A yields the following formula for the area of a sector in terms of the radius r and the angle θ in radians:

$$A = \tfrac{1}{2}r^2\theta \tag{5}$$

TRIGONOMETRIC FUNCTIONS FOR RIGHT TRIANGLES

The ***sine***, ***cosine***, ***tangent***, ***cosecant***, ***secant***, and ***cotangent*** of a positive acute angle θ can be defined as ratios of the sides of a right triangle. Using the notation from Figure A.6, these definitions take the following form:

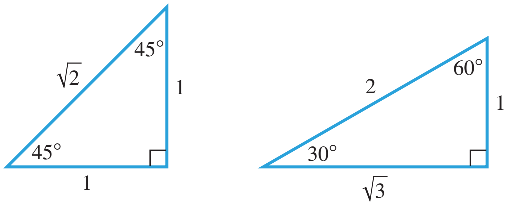


▲ **Figure A.6**

$$\begin{aligned}
\sin\theta &= \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{y}{r}, & \csc\theta &= \frac{\text{hypotenuse}}{\text{side opposite } \theta} = \frac{r}{y} \\
\cos\theta &= \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{x}{r}, & \sec\theta &= \frac{\text{hypotenuse}}{\text{side adjacent to } \theta} = \frac{r}{x} \\
\tan\theta &= \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} = \frac{y}{x}, & \cot\theta &= \frac{\text{side adjacent to } \theta}{\text{side opposite } \theta} = \frac{x}{y}
\end{aligned} \tag{6}$$

We will call sin, cos, tan, csc, sec, and cot the ***trigonometric functions***. Because similar triangles have proportional sides, the values of the trigonometric functions depend only on the size of θ and not on the particular right triangle used to compute the ratios. Moreover, in these definitions it does not matter whether θ is measured in degrees or radians.

▶ **Example 2** Recall from geometry that the two legs of a $45°–45°–90°$ triangle are of equal size and that the hypotenuse of a $30°–60°–90°$ triangle is twice the shorter leg, where the shorter leg is opposite the $30°$ angle. These facts and the Theorem of Pythagoras yield Figure A.7. From that figure we obtain the results in Table A.2. ◀



▶ **Figure A.7**

Table A.2

$\sin 45° = 1/\sqrt{2}$,	$\cos 45° = 1/\sqrt{2}$,	$\tan 45° = 1$
$\csc 45° = \sqrt{2}$,	$\sec 45° = \sqrt{2}$,	$\cot 45° = 1$
$\sin 30° = 1/2$,	$\cos 30° = \sqrt{3}/2$,	$\tan 30° = 1/\sqrt{3}$
$\csc 30° = 2$,	$\sec 30° = 2/\sqrt{3}$,	$\cot 30° = \sqrt{3}$
$\sin 60° = \sqrt{3}/2$,	$\cos 60° = 1/2$,	$\tan 60° = \sqrt{3}$
$\csc 60° = 2/\sqrt{3}$,	$\sec 60° = 2$,	$\cot 60° = 1/\sqrt{3}$

ANGLES IN RECTANGULAR COORDINATE SYSTEMS

Because the angles of a right triangle are between $0°$ and $90°$, the formulas in (6) are not directly applicable to negative angles or to angles greater than $90°$. To extend the trigonometric functions to include these cases, it will be convenient to consider angles in rectangular coordinate systems. An angle is said to be in ***standard position*** in an xy-coordinate system if its vertex is at the origin and its initial side is on the positive x-axis (Figure A.8).

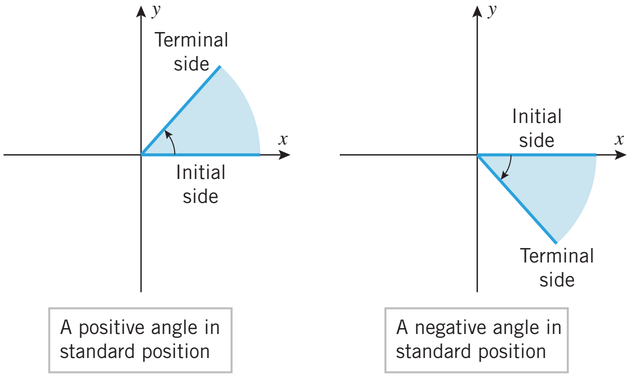


Figure A.8

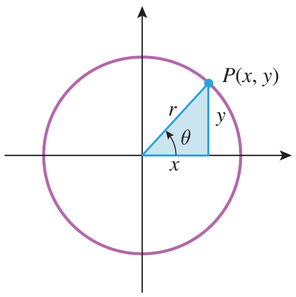


Figure A.9

To define the trigonometric functions of an angle θ in standard position, construct a circle of radius r, centered at the origin, and let $P(x, y)$ be the intersection of the terminal side of θ with this circle (Figure A.9). We make the following definition.

A.1 DEFINITION

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y}$$

Note that the formulas in this definition agree with those in (6), so there is no conflict with the earlier definition of the trigonometric functions for triangles. However, this definition applies to all angles (except for cases where a zero denominator occurs).

In the special case where $r = 1$, we have $\sin\theta = y$ and $\cos\theta = x$, so the terminal side of the angle θ intersects the unit circle at the point $(\cos\theta, \sin\theta)$ (Figure A.10). It follows from Definition A.1 that the remaining trigonometric functions of θ are expressible as (verify)

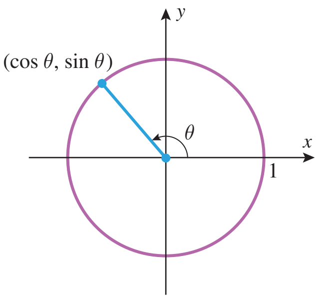


Figure A.10

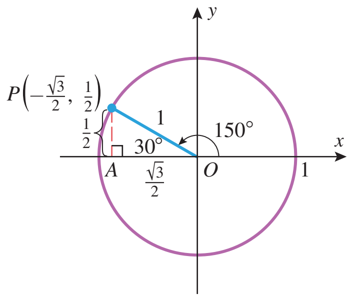
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta} \qquad (7\text{–}10)$$

These observations suggest the following procedure for evaluating the trigonometric functions of common angles:

- Construct the angle θ in standard position in an xy-coordinate system.
- Find the coordinates of the intersection of the terminal side of the angle and the unit circle; the x- and y-coordinates of this intersection are the values of $\cos\theta$ and $\sin\theta$, respectively.
- Use Formulas (7) through (10) to find the values of the remaining trigonometric functions from the values of $\cos\theta$ and $\sin\theta$.

Example 3 Evaluate the trigonometric functions of $\theta = 150^\circ$.

Solution. Construct a unit circle and place the angle $\theta = 150^\circ$ in standard position (Figure A.11). Since $\angle AOP$ is 30° and $\triangle OAP$ is a 30°–60°–90° triangle, the leg AP has length $\frac{1}{2}$ (half the hypotenuse) and the leg OA has length $\sqrt{3}/2$ by the Theorem of



▲ **Figure A.11**

Pythagoras. Thus, the coordinates of P are $(-\sqrt{3}/2, 1/2)$, from which we obtain

$$\sin 150^\circ = \frac{1}{2}, \quad \cos 150^\circ = -\frac{\sqrt{3}}{2}, \quad \tan 150^\circ = \frac{\sin 150^\circ}{\cos 150^\circ} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$$

$$\csc 150^\circ = \frac{1}{\sin 150^\circ} = 2, \quad \sec 150^\circ = \frac{1}{\cos 150^\circ} = -\frac{2}{\sqrt{3}}$$

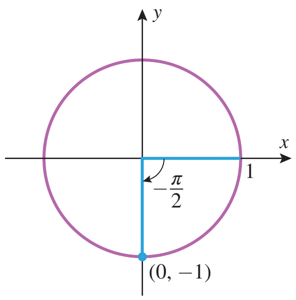
$$\cot 150^\circ = \frac{1}{\tan 150^\circ} = -\sqrt{3} \blacktriangleleft$$

▶ **Example 4** Evaluate the trigonometric functions of $\theta = 5\pi/6$.

Solution. Since $5\pi/6 = 150^\circ$, this problem is equivalent to that of Example 3. From that example we obtain

$$\sin\frac{5\pi}{6} = \frac{1}{2}, \quad \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}, \quad \tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$$

$$\csc\frac{5\pi}{6} = 2, \quad \sec\frac{5\pi}{6} = -\frac{2}{\sqrt{3}}, \quad \cot\frac{5\pi}{6} = -\sqrt{3} \blacktriangleleft$$



▲ **Figure A.12**

▶ **Example 5** Evaluate the trigonometric functions of $\theta = -\pi/2$.

Solution. As shown in Figure A.12, the terminal side of $\theta = -\pi/2$ intersects the unit circle at the point $(0, -1)$, so

$$\sin(-\pi/2) = -1, \quad \cos(-\pi/2) = 0$$

and from Formulas (7) through (10),

$$\tan(-\pi/2) = \frac{\sin(-\pi/2)}{\cos(-\pi/2)} = \frac{-1}{0} \quad \text{(undefined)}$$

$$\cot(-\pi/2) = \frac{\cos(-\pi/2)}{\sin(-\pi/2)} = \frac{0}{-1} = 0$$

$$\sec(-\pi/2) = \frac{1}{\cos(-\pi/2)} = \frac{1}{0} \quad \text{(undefined)}$$

$$\csc(-\pi/2) = \frac{1}{\sin(-\pi/2)} = \frac{1}{-1} = -1 \blacktriangleleft$$

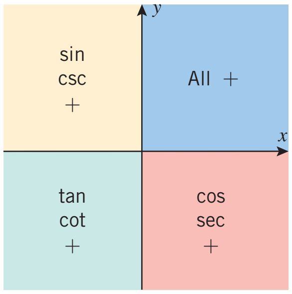
You should be able to obtain all of the results in Table A.3 by the methods illustrated in the last three examples. The dashes indicate quantities that are undefined.

REMARK It is only in special cases that exact values for trigonometric functions can be obtained; usually, a calculating utility or a computer program will be required.

The signs of the trigonometric functions of an angle are determined by the quadrant in which the terminal side of the angle falls. For example, if the terminal side falls in the first quadrant, then x and y are positive in Definition A.1, so all of the trigonometric functions have positive values. If the terminal side falls in the second quadrant, then x is negative

Table A.3

	$\theta = 0$ (0°)	$\pi/6$ (30°)	$\pi/4$ (45°)	$\pi/3$ (60°)	$\pi/2$ (90°)	$2\pi/3$ (120°)	$3\pi/4$ (135°)	$5\pi/6$ (150°)	π (180°)	$3\pi/2$ (270°)	2π (360°)
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1/2	$-1/\sqrt{2}$	$-\sqrt{3}/2$	–1	0	1
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	—	$-\sqrt{3}$	–1	$-1/\sqrt{3}$	0	—	0
$\csc\theta$	—	2	$\sqrt{2}$	$2/\sqrt{3}$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	—	–1	—
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	—	–2	$-\sqrt{2}$	$-2/\sqrt{3}$	–1	—	1
$\cot\theta$	—	$\sqrt{3}$	1	$1/\sqrt{3}$	0	$-1/\sqrt{3}$	–1	$-\sqrt{3}$	—	0	—



▲ **Figure A.13**

and y is positive, so sin and csc are positive, but all other trigonometric functions are negative. The diagram in Figure A.13 shows which trigonometric functions are positive in the various quadrants. You will find it instructive to check that the results in Table A.3 are consistent with Figure A.13.

TRIGONOMETRIC IDENTITIES

A ***trigonometric identity*** is an equation involving trigonometric functions that is true for all angles for which both sides of the equation are defined. One of the most important identities in trigonometry can be derived by applying the Theorem of Pythagoras to the triangle in Figure A.9 to obtain

$$x^2 + y^2 = r^2$$

Dividing both sides by r^2 and using the definitions of $\sin\theta$ and $\cos\theta$ (Definition A.1), we obtain the following fundamental result:

$$\sin^2\theta + \cos^2\theta = 1 \tag{11}$$

The following identities can be obtained from (11) by dividing through by $\cos^2\theta$ and $\sin^2\theta$, respectively, then applying Formulas (7) through (10):

$$\tan^2\theta + 1 = \sec^2\theta \tag{12}$$

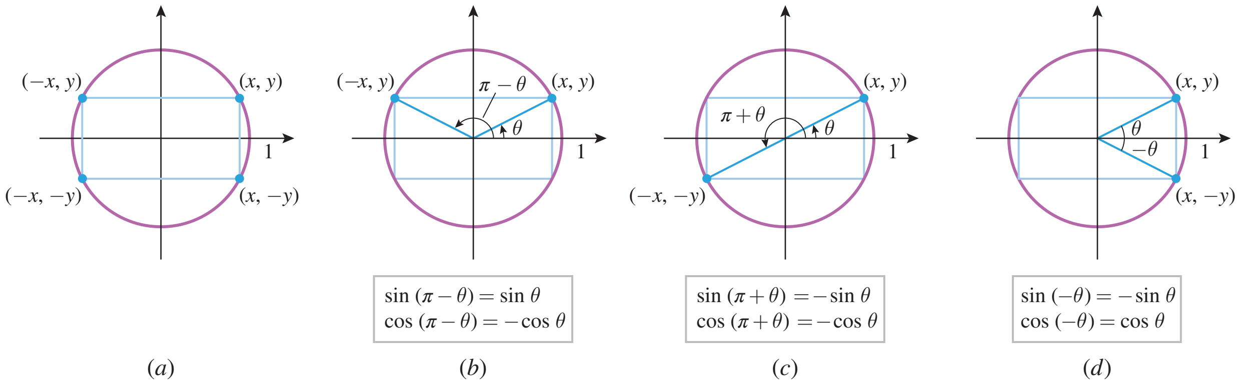
$$1 + \cot^2\theta = \csc^2\theta \tag{13}$$

If (x, y) is a point on the unit circle, then the points $(-x, y)$, $(-x, -y)$, and $(x, -y)$ also lie on the unit circle (why?), and the four points form corners of a rectangle with sides parallel to the coordinate axes (Figure A.14a). The x- and y-coordinates of each corner represent the cosine and sine of an angle in standard position whose terminal side passes through the corner; hence we obtain the identities in parts (b), (c), and (d) of Figure A.14 for sine and cosine. Dividing those identities leads to identities for the tangent. In summary:

$$\sin(\pi-\theta) = \sin\theta, \quad \sin(\pi+\theta) = -\sin\theta, \quad \sin(-\theta) = -\sin\theta \tag{14–16}$$

$$\cos(\pi-\theta) = -\cos\theta, \quad \cos(\pi+\theta) = -\cos\theta, \quad \cos(-\theta) = \cos\theta \tag{17–19}$$

$$\tan(\pi-\theta) = -\tan\theta, \quad \tan(\pi+\theta) = \tan\theta, \quad \tan(-\theta) = -\tan\theta \tag{20–22}$$



▲ **Figure A.14**

Two angles in standard position that have the same terminal side must have the same values for their trigonometric functions since their terminal sides intersect the unit circle at the same point. In particular, two angles whose radian measures differ by a multiple of 2π have the same terminal side and hence have the same values for their trigonometric functions. This yields the identities

$$\sin\theta = \sin(\theta + 2\pi) = \sin(\theta - 2\pi) \tag{23}$$

$$\cos\theta = \cos(\theta + 2\pi) = \cos(\theta - 2\pi) \tag{24}$$

and more generally,

$$\sin\theta = \sin(\theta \pm 2n\pi), \quad n = 0, 1, 2, \ldots \tag{25}$$

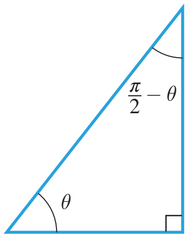
$$\cos\theta = \cos(\theta \pm 2n\pi), \quad n = 0, 1, 2, \ldots \tag{26}$$

Identity (21) implies that

$$\tan\theta = \tan(\theta + \pi) \qquad \text{and} \qquad \tan\theta = \tan(\theta - \pi) \tag{27–28}$$

Identity (27) is just (21) with the terms in the sum reversed, and identity (28) follows from (21) by substituting $\theta - \pi$ for θ. These two identities state that adding or subtracting π from an angle does not affect the value of the tangent of the angle. It follows that the same is true for any multiple of π; thus,

$$\tan\theta = \tan(\theta \pm n\pi), \quad n = 0, 1, 2, \ldots \tag{29}$$



▲ **Figure A.15**

Figure A.15 shows complementary angles θ and $(\pi/2) - \theta$ of a right triangle. It follows from (6) that

$$\sin\theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{\text{side adjacent to } (\pi/2) - \theta}{\text{hypotenuse}} = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{\text{side opposite } (\pi/2) - \theta}{\text{hypotenuse}} = \sin\left(\frac{\pi}{2} - \theta\right)$$

which yields the identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \quad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \tag{30–32}$$

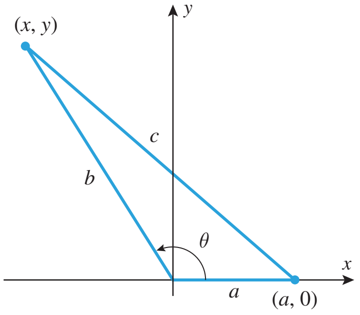
where the third identity results from dividing the first two. These identities are also valid for angles that are not acute and for negative angles as well.

THE LAW OF COSINES

The next theorem, called the ***law of cosines***, generalizes the Theorem of Pythagoras. This result is important in its own right and is also the starting point for some important trigonometric identities.

> **A.2 THEOREM (*Law of Cosines*)** *If the sides of a triangle have lengths a, b, and c, and if θ is the angle between the sides with lengths a and b, then*
> $$c^2 = a^2 + b^2 - 2ab\cos\theta$$

PROOF Introduce a coordinate system so that θ is in standard position and the side of length a falls along the positive x-axis. As shown in Figure A.16, the side of length a extends from the origin to $(a, 0)$ and the side of length b extends from the origin to some point (x, y). From the definition of $\sin\theta$ and $\cos\theta$ we have $\sin\theta = y/b$ and $\cos\theta = x/b$, so



▲ **Figure A.16**

$$y = b\sin\theta, \quad x = b\cos\theta \tag{33}$$

From the distance formula in Theorem I.1 of Web Appendix I, we obtain

$$c^2 = (x - a)^2 + (y - 0)^2$$

so that, from (33),

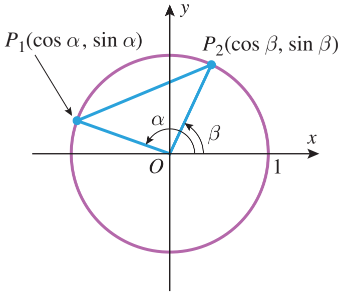
$$\begin{aligned} c^2 &= (b\cos\theta - a)^2 + b^2\sin^2\theta \\ &= a^2 + b^2(\cos^2\theta + \sin^2\theta) - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

which completes the proof. ■

We will now show how the law of cosines can be used to obtain the following identities, called the ***addition formulas*** for sine and cosine:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{34}$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{35}$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{36}$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{37}$$



▲ **Figure A.17**

We will derive (37) first. In our derivation we will assume that $0 \le \beta < \alpha < 2\pi$ (Figure A.17). As shown in the figure, the terminal sides of α and β intersect the unit circle at the points $P_1(\cos\alpha, \sin\alpha)$ and $P_2(\cos\beta, \sin\beta)$. If we denote the lengths of the sides of triangle OP_1P_2 by OP_1, P_1P_2, and OP_2, then $OP_1 = OP_2 = 1$ and, from the distance formula in Theorem I.1 of Web Appendix I,

$$\begin{aligned} (P_1P_2)^2 &= (\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 \\ &= (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \end{aligned}$$

But angle $P_2OP_1 = \alpha - \beta$, so that the law of cosines yields

$$\begin{aligned} (P_1P_2)^2 &= (OP_1)^2 + (OP_2)^2 - 2(OP_1)(OP_2)\cos(\alpha - \beta) \\ &= 2 - 2\cos(\alpha - \beta) \end{aligned}$$

Equating the two expressions for $(P_1P_2)^2$ and simplifying, we obtain

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

which completes the derivation of (37).

We can use (31) and (37) to derive (36) as follows:

$$\begin{aligned}
\sin(\alpha - \beta) &= \cos\left[\frac{\pi}{2} - (\alpha - \beta)\right] = \cos\left[\left(\frac{\pi}{2} - \alpha\right) - (-\beta)\right] \\
&= \cos\left(\frac{\pi}{2} - \alpha\right)\cos(-\beta) + \sin\left(\frac{\pi}{2} - \alpha\right)\sin(-\beta) \\
&= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta - \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta \\
&= \sin\alpha\cos\beta - \cos\alpha\sin\beta
\end{aligned}$$

Identities (34) and (35) can be obtained from (36) and (37) by substituting $-\beta$ for β and using the identities

$$\sin(-\beta) = -\sin\beta, \quad \cos(-\beta) = \cos\beta$$

We leave it for you to derive the identities

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \tag{38–39}$$

Identity (38) can be obtained by dividing (34) by (35) and then simplifying. Identity (39) can be obtained from (38) by substituting $-\beta$ for β and simplifying.

In the special case where $\alpha = \beta$, identities (34), (35), and (38) yield the ***double-angle formulas***

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \tag{40}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \tag{41}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \tag{42}$$

By using the identity $\sin^2\alpha + \cos^2\alpha = 1$, (41) can be rewritten in the alternative forms

$$\cos 2\alpha = 2\cos^2\alpha - 1 \quad \text{and} \quad \cos 2\alpha = 1 - 2\sin^2\alpha \tag{43–44}$$

If we replace α by $\alpha/2$ in (43) and (44) and use some algebra, we obtain the ***half-angle formulas***

$$\cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2} \quad \text{and} \quad \sin^2\frac{\alpha}{2} = \frac{1 - \cos\alpha}{2} \tag{45–46}$$

We leave it for the exercises to derive the following ***product-to-sum formulas*** from (34) through (37):

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)] \tag{47}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \tag{48}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \tag{49}$$

We also leave it for the exercises to derive the following ***sum-to-product formulas***:

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \tag{50}$$

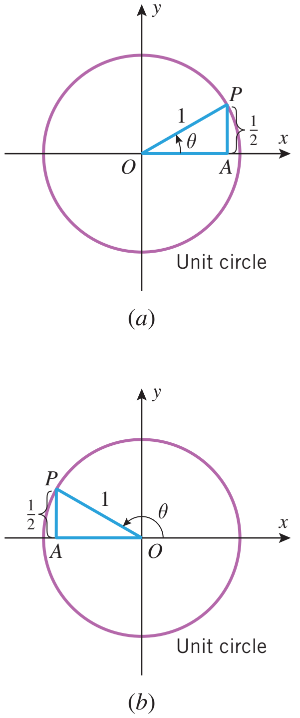
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \tag{51}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \tag{52}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \tag{53}$$

FINDING AN ANGLE FROM THE VALUE OF ITS TRIGONOMETRIC FUNCTIONS

There are numerous situations in which it is necessary to find an unknown angle from a known value of one of its trigonometric functions. The following example illustrates a method for doing this.



▲ Figure A.18

► Example 6 Find θ if $\sin\theta = \frac{1}{2}$.

Solution. We begin by looking for positive angles that satisfy the equation. Because $\sin\theta$ is positive, the angle θ must terminate in the first or second quadrant. If it terminates in the first quadrant, then the hypotenuse of $\triangle OAP$ in Figure A.18*a* is double the leg AP, so

$$\theta = 30^\circ = \frac{\pi}{6} \text{ radians}$$

If θ terminates in the second quadrant (Figure A.18*b*), then the hypotenuse of $\triangle OAP$ is double the leg AP, so $\angle AOP = 30^\circ$, which implies that

$$\theta = 180^\circ - 30^\circ = 150^\circ = \frac{5\pi}{6} \text{ radians}$$

Now that we have found these two solutions, all other solutions are obtained by adding or subtracting multiples of 360° (2π radians) to or from them. Thus, the entire set of solutions is given by the formulas

$$\theta = 30^\circ \pm n\cdot 360^\circ, \quad n = 0, 1, 2, \ldots$$

and

$$\theta = 150^\circ \pm n\cdot 360^\circ, \quad n = 0, 1, 2, \ldots$$

or in radian measure,

$$\theta = \frac{\pi}{6} \pm n\cdot 2\pi, \quad n = 0, 1, 2, \ldots$$

and

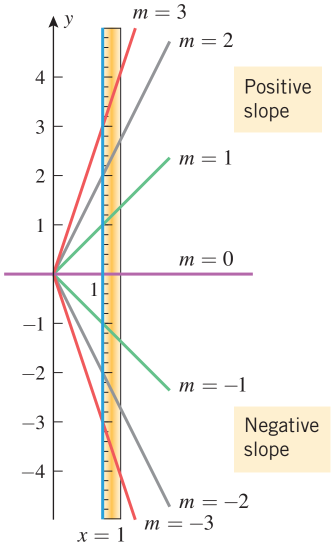
$$\theta = \frac{5\pi}{6} \pm n\cdot 2\pi, \quad n = 0, 1, 2, \ldots \blacktriangleleft$$

ANGLE OF INCLINATION

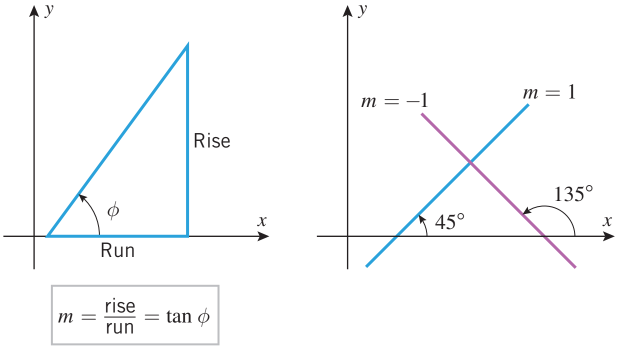
The slope of a nonvertical line L is related to the angle that L makes with the positive x-axis. If ϕ is the smallest positive angle measured counterclockwise from the x-axis to L, then the slope of the line can be expressed as

$$m = \tan\phi \tag{54}$$

(Figure A.19*a*). The angle ϕ, which is called the ***angle of inclination*** of the line, satisfies $0^\circ \le \phi < 180^\circ$ in degree measure (or, equivalently, $0 \le \phi < \pi$ in radian measure). If ϕ is an acute angle, then $m = \tan\phi$ is positive and the line slopes up to the right, and if ϕ is an obtuse angle, then $m = \tan\phi$ is negative and the line slopes down to the right. For example, a line whose angle of inclination is 45° has slope $m = \tan 45^\circ = 1$, and a line whose angle of inclination is 135° has a slope of $m = \tan 135^\circ = -1$ (Figure A.19*b*). Figure A.20 shows a convenient way of using the line $x = 1$ as a "ruler" for visualizing the relationship between lines of various slopes.



▲ **Figure A.20**



(*a*) (*b*)

▲ **Figure A.19**

EXERCISE SET A

1–2 Express the angles in radians.

1. (a) 75° (b) 390° (c) 20° (d) 138°

2. (a) 420° (b) 15° (c) 225° (d) 165°

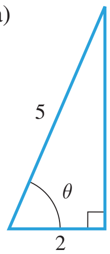
3–4 Express the angles in degrees.

3. (a) $\pi/15$ (b) 1.5 (c) $8\pi/5$ (d) 3π

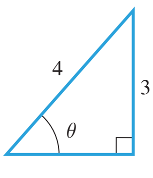
4. (a) $\pi/10$ (b) 2 (c) $2\pi/5$ (d) $7\pi/6$

5–6 Find the exact values of all six trigonometric functions of θ.

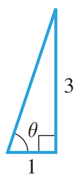
5. (a)



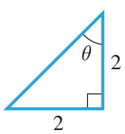
(b)



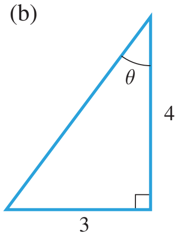
(c)



6. (a)



(b)



(c)

7–12 The angle θ is an acute angle of a right triangle. Solve the problems by drawing an appropriate right triangle. Do *not* use a calculator.

7. Find $\sin\theta$ and $\cos\theta$ given that $\tan\theta = 3$.

8. Find $\sin\theta$ and $\tan\theta$ given that $\cos\theta = \frac{2}{3}$.

9. Find $\tan\theta$ and $\csc\theta$ given that $\sec\theta = \frac{5}{2}$.

10. Find $\cot\theta$ and $\sec\theta$ given that $\csc\theta = 4$.

11. Find the length of the side adjacent to θ given that the hypotenuse has length 6 and $\cos\theta = 0.3$.

12. Find the length of the hypotenuse given that the side opposite θ has length 2.4 and $\sin\theta = 0.8$.

13–14 The value of an angle θ is given. Find the values of all six trigonometric functions of θ without using a calculator.

13. (a) 225° (b) -210° (c) $5\pi/3$ (d) $-3\pi/2$

14. (a) 330° (b) -120° (c) $9\pi/4$ (d) -3π

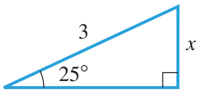
15–16 Use the information to find the exact values of the remaining five trigonometric functions of θ.

15. (a) $\cos\theta = \frac{3}{5},\ 0 < \theta < \pi/2$
(b) $\cos\theta = \frac{3}{5},\ -\pi/2 < \theta < 0$
(c) $\tan\theta = -1/\sqrt{3},\ \pi/2 < \theta < \pi$
(d) $\tan\theta = -1/\sqrt{3},\ -\pi/2 < \theta < 0$
(e) $\csc\theta = \sqrt{2},\ 0 < \theta < \pi/2$
(f) $\csc\theta = \sqrt{2},\ \pi/2 < \theta < \pi$

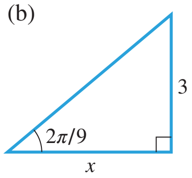
16. (a) $\sin\theta = \frac{1}{4},\ 0 < \theta < \pi/2$
(b) $\sin\theta = \frac{1}{4},\ \pi/2 < \theta < \pi$
(c) $\cot\theta = \frac{1}{3},\ 0 < \theta < \pi/2$
(d) $\cot\theta = \frac{1}{3},\ \pi < \theta < 3\pi/2$
(e) $\sec\theta = -\frac{5}{2},\ \pi/2 < \theta < \pi$
(f) $\sec\theta = -\frac{5}{2},\ \pi < \theta < 3\pi/2$

17–18 Use a calculating utility to find x to four decimal places.

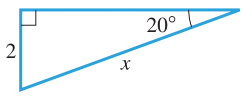
17. (a)



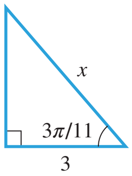
(b)



18. (a)



(b)



19. In each part, let θ be an acute angle of a right triangle. Express the remaining five trigonometric functions in terms of a.
(a) $\sin\theta = a/3$ (b) $\tan\theta = a/5$ (c) $\sec\theta = a$

20–27 Find all values of θ (in radians) that satisfy the given equation. Do not use a calculator.

20. (a) $\cos\theta = -1/\sqrt{2}$ (b) $\sin\theta = -1/\sqrt{2}$

21. (a) $\tan\theta = -1$ (b) $\cos\theta = \frac{1}{2}$

22. (a) $\sin\theta = -\frac{1}{2}$ (b) $\tan\theta = \sqrt{3}$

23. (a) $\tan\theta = 1/\sqrt{3}$ (b) $\sin\theta = -\sqrt{3}/2$

24. (a) $\sin\theta = -1$ (b) $\cos\theta = -1$

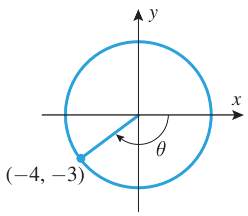
25. (a) $\cot\theta = -1$ (b) $\cot\theta = \sqrt{3}$

26. (a) $\sec\theta = -2$ (b) $\csc\theta = -2$

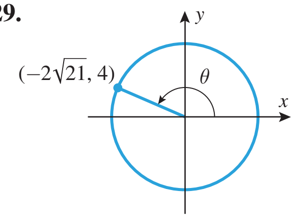
27. (a) $\csc\theta = 2/\sqrt{3}$ (b) $\sec\theta = 2/\sqrt{3}$

28–29 Find the values of all six trigonometric functions of θ.

28.



29.



30. Find all values of θ (in radians) such that
(a) $\sin\theta = 1$ (b) $\cos\theta = 1$ (c) $\tan\theta = 1$
(d) $\csc\theta = 1$ (e) $\sec\theta = 1$ (f) $\cot\theta = 1$.

31. Find all values of θ (in radians) such that
(a) $\sin\theta = 0$ (b) $\cos\theta = 0$ (c) $\tan\theta = 0$
(d) $\csc\theta$ is undefined (e) $\sec\theta$ is undefined
(f) $\cot\theta$ is undefined.

32. How could you use a ruler and protractor to approximate $\sin 17°$ and $\cos 17°$?

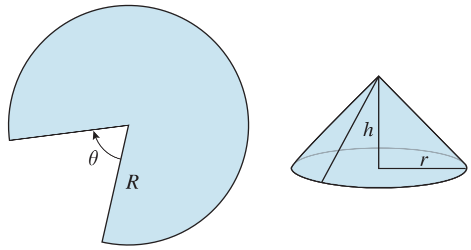
33. Find the length of the circular arc on a circle of radius 4 cm subtended by an angle of
(a) $\pi/6$ (b) $150°$.

34. Find the radius of a circular sector that has an angle of $\pi/3$ and a circular arc length of 7 units.

35. A point P moving counterclockwise on a circle of radius 5 cm traverses an arc length of 2 cm. What is the angle swept out by a radius from the center to P?

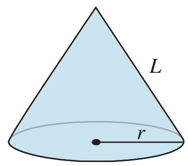
36. Find a formula for the area A of a circular sector in terms of its radius r and arc length s.

37. As shown in the accompanying figure, a right circular cone is made from a circular piece of paper of radius R by cutting out a sector of angle θ radians and gluing the cut edges of the remaining piece together. Find
(a) the radius r of the base of the cone in terms of R and θ
(b) the height h of the cone in terms of R and θ.



▲ **Figure Ex-37**

38. As shown in the accompanying figure, let r and L be the radius of the base and the slant height of a right circular cone. Show that the lateral surface area, S, of the cone is $S = \pi r L$. [*Hint:* As shown in the figure in Exercise 37, the lateral surface of the cone becomes a circular sector when cut along a line from the vertex to the base and flattened.]



◀ **Figure Ex-38**

39. Two sides of a triangle have lengths of 3 cm and 7 cm and meet at an angle of $60°$. Find the area of the triangle.

40. Let ABC be a triangle whose angles at A and B are $30°$ and $45°$. If the side opposite the angle B has length 9, find the lengths of the remaining sides and the size of the angle C.

41. A 10-foot ladder leans against a house and makes an angle of $67°$ with level ground. How far is the top of the ladder above the ground? Express your answer to the nearest tenth of a foot.

42. From a point 120 feet on level ground from a building, the angle of elevation to the top of the building is $76°$. Find the height of the building. Express your answer to the nearest foot.

43. An observer on level ground is at a distance d from a building. The angles of elevation to the bottoms of the windows on the second and third floors are α and β, respectively. Find the distance h between the bottoms of the windows in terms of α, β, and d.

44. From a point on level ground, the angle of elevation to the top of a tower is α. From a point that is d units closer to the tower, the angle of elevation is β. Find the height h of the tower in terms of α, β, and d.

45–46 Do *not* use a calculator in these exercises.

45. If $\cos\theta = \frac{2}{3}$ and $0 < \theta < \pi/2$, find
(a) $\sin 2\theta$ (b) $\cos 2\theta$.

46. If $\tan\alpha = \frac{3}{4}$ and $\tan\beta = 2$, where $0 < \alpha < \pi/2$ and $0 < \beta < \pi/2$, find
(a) $\sin(\alpha - \beta)$ (b) $\cos(\alpha + \beta)$.

47. Express $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

48–58 Derive the given identities.

48. $\dfrac{\cos\theta\sec\theta}{1+\tan^2\theta} = \cos^2\theta$

49. $\dfrac{\cos\theta\tan\theta + \sin\theta}{\tan\theta} = 2\cos\theta$

50. $2\csc 2\theta = \sec\theta\csc\theta$ **51.** $\tan\theta + \cot\theta = 2\csc 2\theta$

52. $\dfrac{\sin 2\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\cos\theta} = \sec\theta$

53. $\dfrac{\sin\theta + \cos 2\theta - 1}{\cos\theta - \sin 2\theta} = \tan\theta$

54. $\sin 3\theta + \sin\theta = 2\sin 2\theta\cos\theta$

55. $\sin 3\theta - \sin\theta = 2\cos 2\theta\sin\theta$

56. $\tan\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{\sin\theta}$ **57.** $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta}$

58. $\cos\left(\dfrac{\pi}{3}+\theta\right) + \cos\left(\dfrac{\pi}{3}-\theta\right) = \cos\theta$

59–60 In these exercises, refer to an arbitrary triangle ABC in which the side of length a is opposite angle A, the side of length b is opposite angle B, and the side of length c is opposite angle C.

59. Prove: The area of a triangle ABC can be written as

$$\text{area} = \tfrac{1}{2}bc\sin A$$

Find two other similar formulas for the area.

60. Prove the ***law of sines***: In any triangle, the ratios of the sides to the sines of the opposite angles are equal; that is,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

61. Use identities (34) through (37) to express each of the following in terms of $\sin\theta$ or $\cos\theta$.
(a) $\sin\left(\dfrac{\pi}{2}+\theta\right)$ (b) $\cos\left(\dfrac{\pi}{2}+\theta\right)$
(c) $\sin\left(\dfrac{3\pi}{2}-\theta\right)$ (d) $\cos\left(\dfrac{3\pi}{2}+\theta\right)$

62. Derive identities (38) and (39).

63. Derive identity
(a) (47) (b) (48) (c) (49).

64. If $A = \alpha + \beta$ and $B = \alpha - \beta$, then $\alpha = \frac{1}{2}(A+B)$ and $\beta = \frac{1}{2}(A-B)$ (verify). Use this result and identities (47) through (49) to derive identity
(a) (50) (b) (52) (c) (53).

65. Substitute $-\beta$ for β in identity (50) to derive identity (51).

66. (a) Express $3\sin\alpha + 5\cos\alpha$ in the form

$$C\sin(\alpha + \phi)$$

(b) Show that a sum of the form

$$A\sin\alpha + B\cos\alpha$$

can be rewritten in the form $C\sin(\alpha+\phi)$.

67. Show that the length of the diagonal of the parallelogram in the accompanying figure is

$$d = \sqrt{a^2 + b^2 + 2ab\cos\theta}$$

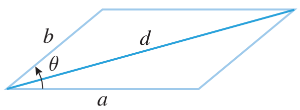


Figure Ex-67

68–69 Find the angle of inclination of the line with slope m to the nearest degree. Use a calculating utility, where needed.

68. (a) $m = \frac{1}{2}$ (b) $m = -1$
(c) $m = 2$ (d) $m = -57$

69. (a) $m = -\frac{1}{2}$ (b) $m = 1$
(c) $m = -2$ (d) $m = 57$

70–71 Find the angle of inclination of the line to the nearest degree. Use a calculating utility, where needed.

70. (a) $3y = 2 - \sqrt{3}x$ (b) $y - 4x + 7 = 0$

71. (a) $y = \sqrt{3}x + 2$ (b) $y + 2x + 5 = 0$