Section 12.1

Systems of Linear Equations; Substitution and Elimination

EXAMPLE **Movie Theater Ticket Sales**

A movie theater sells tickets for \$8.00 each, with seniors receiving a discount of \$2.00. One evening the theater took in \$3580 in revenue. If x represents the number of tickets sold at \$8.00 and y the number of tickets sold at the discounted price of \$6.00, write an equation that relates these variables.

$$\boxed{8x + 6y = 3580}$$

Suppose we also know that 525 tickets were sold. Write another equation relating the variables x and y.

$$\boxed{x + y = 525}$$

EXAMPLE Examples of Systems of Equations

(a) $\begin{cases} 2x + y = 5 & (1) \\ -4x + 6y = -2 & (2) \end{cases}$ Two equations containing two variables, x and y

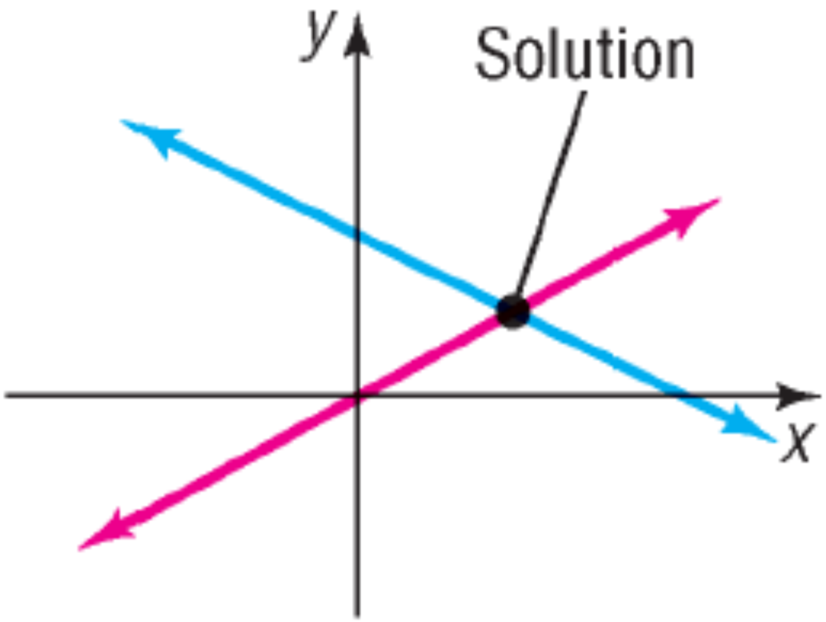
(b) $\begin{cases} x + y^2 = 5 & (1) \\ 2x + y = 4 & (2) \end{cases}$ Two equations containing two variables, x and y

(c) $\begin{cases} x + y + z = 6 & (1) \\ 3x - 2y + 4z = 9 & (2) \\ x - y - z = 0 & (3) \end{cases}$ Three equations containing three variables, x, y, and z

(d) $\begin{cases} x + y + z = 5 & (1) \\ x - y = 2 & (2) \end{cases}$ Two equations containing three variables, x, y, and z

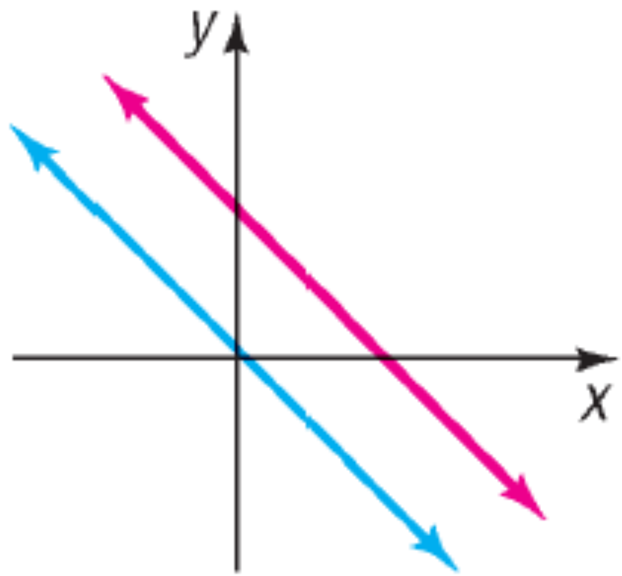
(e) $\begin{cases} x + y + z = 6 & (1) \\ 2x + 2z = 4 & (2) \\ y + z = 2 & (3) \\ x = 4 & (4) \end{cases}$ Four equations containing three variables, x, y, and z

1. If the lines intersect, then the system of equations has one solution, given by the point of intersection. The system is **consistent** and the equations are **independent**.



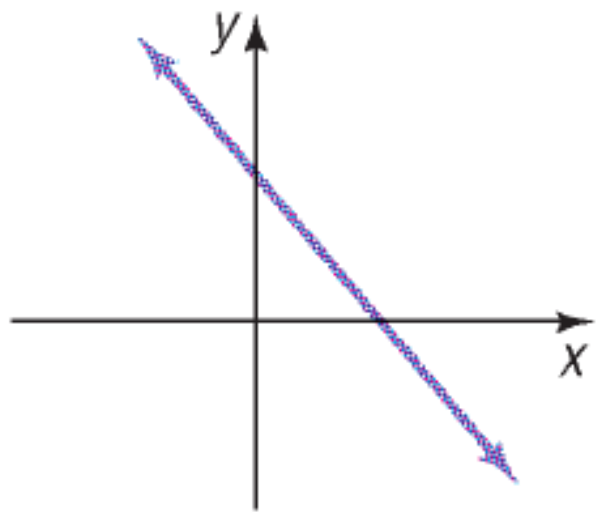
(a) Intersecting lines; system has one solution

2. If the lines are parallel, then the system of equations has no solution, because the lines never intersect. The system is **inconsistent**.



(b) Parallel lines; system has no solution

3. If the lines are coincident, then the system of equations has infinitely many solutions, represented by the totality of points on the line. The system is **consistent** and the equations are **dependent**.



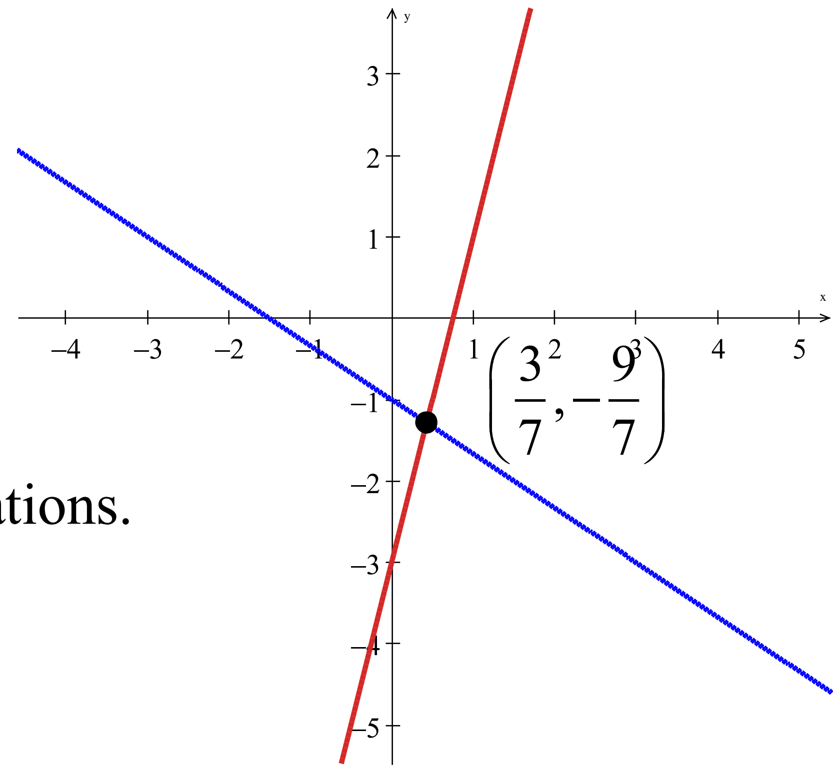
(c) Coincident lines; system has infinitely many solutions

EXAMPLE Graphing a System of Linear Equations

Graph the system: $\begin{cases} 2x+3y=-3 \\ 4x-y=3 \end{cases}$

$$\boxed{y=-\frac{2}{3}x-1}$$

$$\boxed{y=4x-3}$$



The 2 lines intersect at the point $\left(\frac{3}{7},-\frac{9}{7}\right)$

which is the solution to the system of equations.

1 Solve Systems of Equations by Substitution

EXAMPLE **How to Solve a System of Linear Equations by Substitution**

Solve: $\begin{cases} 2x+3y=-3 \\ 4x-y=3 \end{cases}$

Step 1: Pick one of the equations and solve for one of the variables in terms of the remaining variable(s).

$$y=4x-3$$

Step 2: Substitute the result into the remaining equation(s).

$$2x+3(4x-3)=-3$$

Step 3: If one equation in one variable results, solve this equation. Otherwise, repeat Steps 1 and 2 until a single equation with one variable remains.

$$2x+12x-9=-3$$

$$14x=6 \quad x=\frac{6}{14}=\frac{3}{7}$$

Step 4: Find the values of the remaining variables by back-substitution.

$$y=4\left(\frac{3}{7}\right)-3=-\frac{9}{7}$$

Step 5: Check the solution found.

$$2\left(\frac{3}{7}\right)+3\left(-\frac{9}{7}\right)=-3 \qquad 4\left(\frac{3}{7}\right)-\left(-\frac{9}{7}\right)=3$$

Solution:

$$\left(\frac{3}{7},-\frac{9}{7}\right)$$

2 Solve Systems of Equations by Elimination

Rules for Obtaining an Equivalent System of Equations

1. Interchange any two equations of the system.
2. Multiply (or divide) each side of an equation by the same nonzero constant.
3. Replace any equation in the system by the sum (or difference) of that equation and a nonzero multiple of any other equation in the system.

EXAMPLE How to Solve a System of Linear Equations by Elimination

Solve: $\begin{cases} 2x+3y=-1 \\ 2x-y=3 \end{cases}$

Step 1: Multiply both sides of one or both equations by a nonzero constant so that the coefficients of one of the variables are additive inverses.

$$\begin{cases} 2x+3y=-1 \\ 3(2x-y)=3(3) \end{cases}$$

Step 2: Add the equations to eliminate the variable. Solve the resulting equation for the remaining unknown.

$$\begin{cases} 2x+3y=-1 \\ 6x-3y=9 \end{cases}$$

$$8x=8$$

$$x=1$$

Solution: $(1,-1)$

Step 3: Back-substitute the value of the variable found in Step 2 into one of the *original* equations to find the value of the remaining variable.

$$2(1)-y=3$$

$$y=-1$$

Step 4: Check the solution found.

$$2(1)+3(-1)=-1 \qquad 2(1)-(-1)=3$$

EXAMPLE Movie Theater Ticket Sales

A movie theater sells tickets for \$8.00 each, with seniors receiving a discount of \$2.00. One evening the theater sold 525 tickets and took in \$3580 in revenue. How many of each type of ticket were sold?

If x represents the number of tickets sold at \$8.00 and y the number of tickets sold at the discounted price of \$6.00, then the given information results in the system of equations

$$\begin{cases} 8x + 6y = 3580 \\ x + y = 525 \end{cases}$$

We use the method of elimination. First, multiply the second equation by −6, and then add the equations.

$$\begin{cases} 8x + 6y = 3580 \\ -6x - 6y = -3150 \end{cases}$$

$$2x = 430 \quad \text{Add the equations.}$$

$$x = 215$$

Since $x + y = 525$, then $y = 525 - x = 525 - 215 = 310$. So 215 nondiscounted tickets and 310 senior discount tickets were sold.

3 Identify Inconsistent Systems of Equations Containing Two Variables

EXAMPLE

An Inconsistent System of Linear Equations

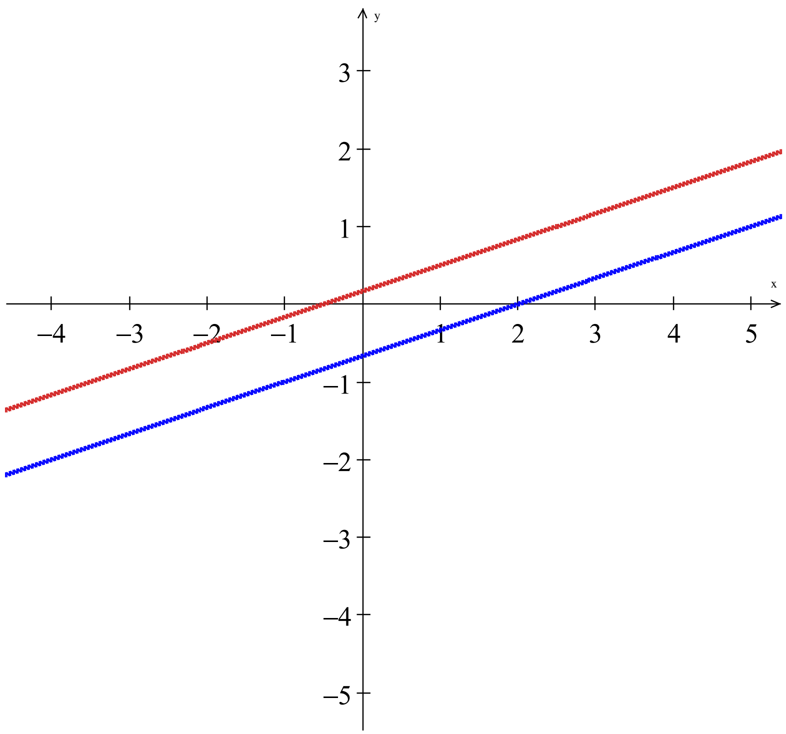
Solve: $\begin{cases} x - 3y = 2 \\ -2x + 6y = 1 \end{cases}$

$$x = 3y + 2$$

$$-2(3y + 2) + 6y = 1$$

$$-6y - 4 + 6y = 1$$

$$-4 = 1$$



Since this statement is false we conclude there is no solution. We say the system is inconsistent.

4 Express the Solution of a System of Dependent Equations Containing Two Variables

EXAMPLE

Solving a System of Dependent Equations

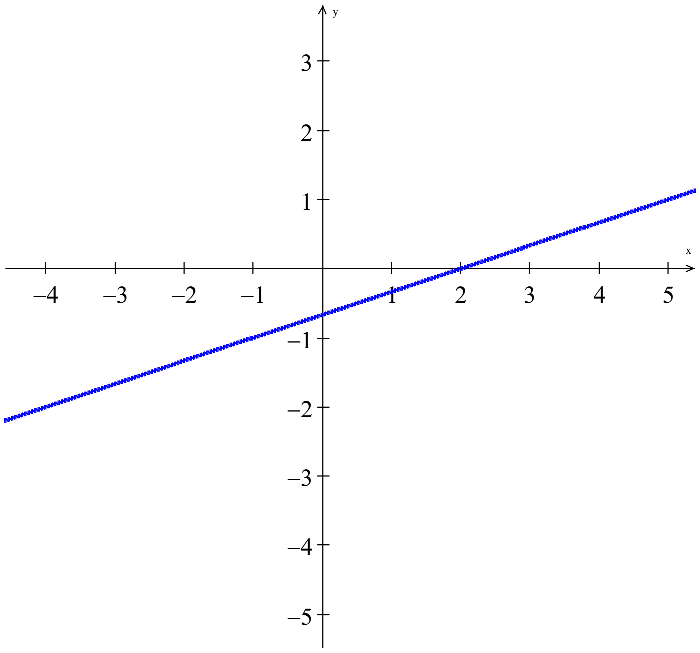
Solve: $\begin{cases} x - 3y = 2 \\ -2x + 6y = -4 \end{cases}$

$$x = 3y + 2$$

$$-2(3y+2) + 6y = -4$$

$$-6y - 4 + 6y = -4$$

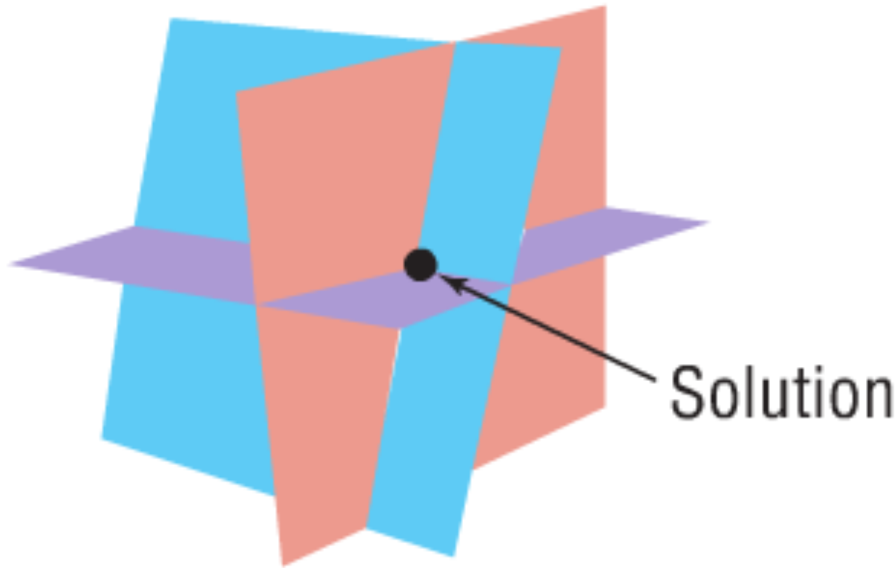
$$-4 = -4$$



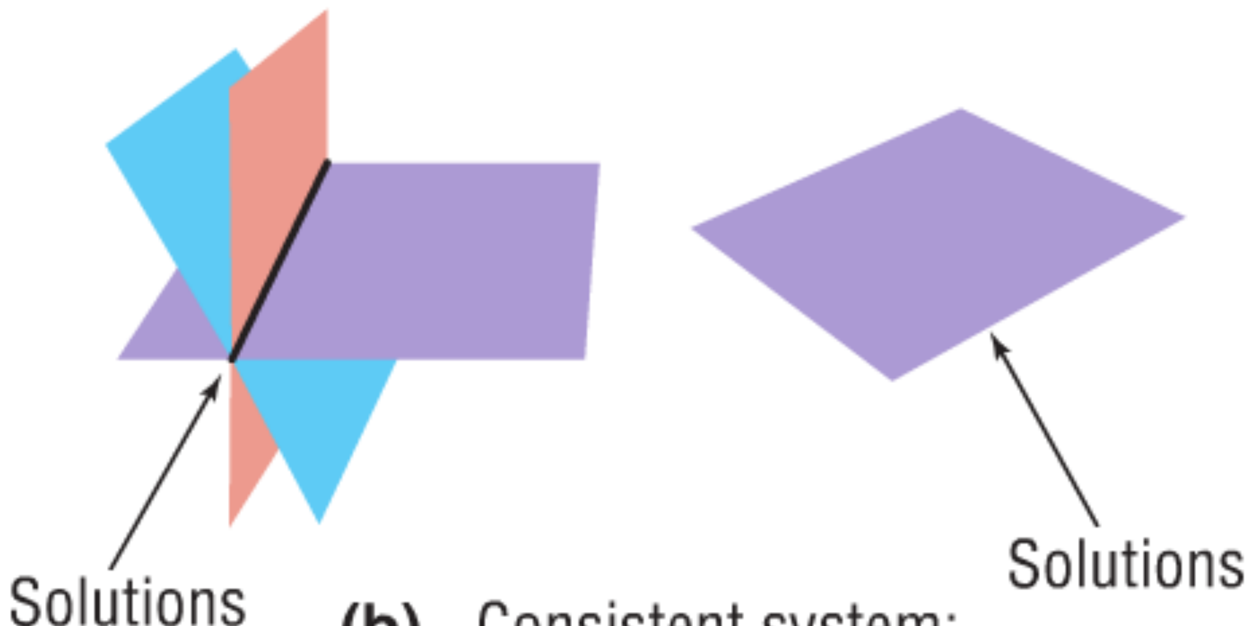
Solution: $\{(x, y) \mid x = 3y + 2\}$

Since this statement is true but we have no variables, the two equations are equivalent so the equations are dependent.

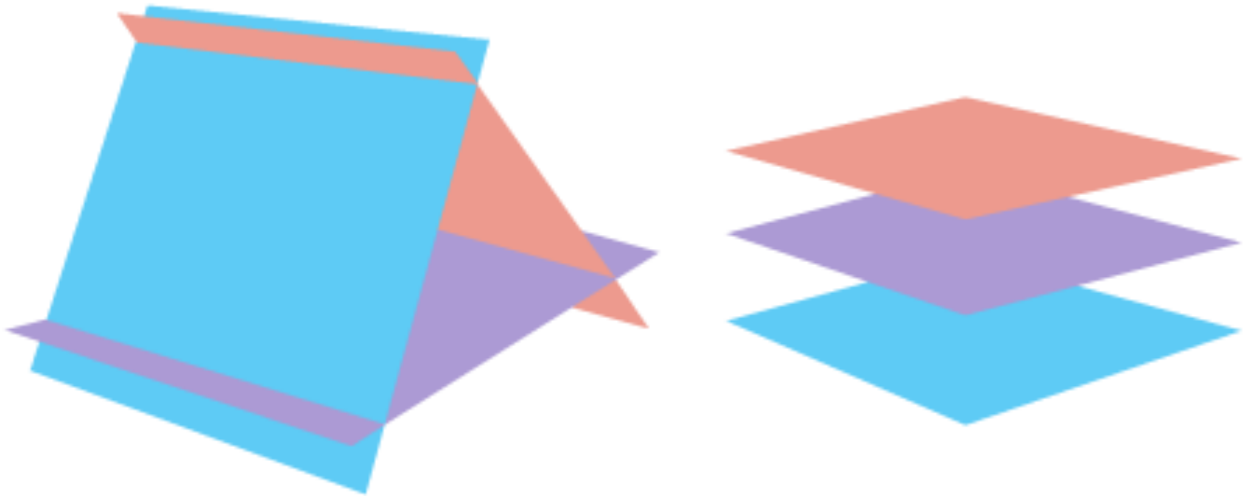
5 Solve Systems of Three Equations Containing Three Variables



(a) Consistent system; one solution



(b) Consistent system; infinite number of solutions



(c) Inconsistent system;
no solution

EXAMPLE Solving a System of Three Linear Equations with Three Variables

Use the method of elimination to solve the system of equations.

$$\begin{cases} 2x + y + z = 4 \\ -3x + 2y - 2z = -10 \\ x - 2y + 3z = 7 \end{cases}$$

$$\begin{array}{r} 2x + y + z = 4 \\ -2x + 4y - 6z = -14 \\ \hline 5y - 5z = -10 \end{array}$$

$$\begin{array}{r} -3x + 2y - 2z = -10 \\ 3x - 6y + 9z = 21 \\ \hline -4y + 7z = 11 \end{array}$$

$$\begin{array}{r} 20y - 20z = -40 \\ -20y + 35z = 55 \\ \hline 15z = 15 \end{array} \qquad z = 1$$

$$5y - 5(1) = -10 \qquad y = -1$$

$$x - 2(-1) + 3(1) = 7 \qquad x = 2$$

Solution: $(2, -1, 1)$

6 Identify Inconsistent Systems of Equations Containing Three Variables

EXAMPLE Identify an Inconsistent System of Linear Equations

Solve: $\begin{cases} 2x-3y-z=0 \\ -x+2y+z=5 \\ 3x-4y-z=1 \end{cases}$

$$\begin{array}{r} 2x-3y-z=0 \\ -2x+4y+2z=10 \\ \hline y+z=10 \end{array}$$

$$\begin{array}{r} -3x+6y+3z=15 \\ 3x-4y-z=1 \\ \hline 2y+2z=16 \end{array}$$

$$\begin{array}{r} -2y-2z=-20 \\ \hline 0=-4 \end{array}$$

Since this statement is false we conclude there is no solution. We say the system is inconsistent.

7 Express the Solution of a System of Dependent Equations Containing Three Variables

EXAMPLE Solving a System of Dependent Equations

Solve: $\begin{cases} x+y+2z=1 \\ 2x-y+z=2 \\ 4x+y+5z=4 \end{cases}$

$$\begin{array}{r} -2x-2y-4z=-2 \\ 2x-y+z=2 \\ \hline -3y-3z=0 \end{array}$$

$$\begin{array}{r} -4x-4y-8z=-4 \\ 4x+y+5z=4 \\ \hline -3y-3z=0 \end{array}$$

$$\begin{array}{r} \\ \hline 0=0 \end{array}$$

Since this statement is true but we have no variables, the equations are dependent.

$-3y-3z=0$ so $y=-z$

$x=-y-2z+1$ so $x=-z+1$

$$\{(x,y,z)\,|\,x=-z+1, y=-z, z \text{ is any real number}\}$$

EXAMPLE **Curve Fitting**

Find real numbers a, b, and c so that the graph of the quadratic function $y = ax^2 + bx + c$ contains the points $(-1, -4)$, $(1, 6)$, and $(3, 0)$.

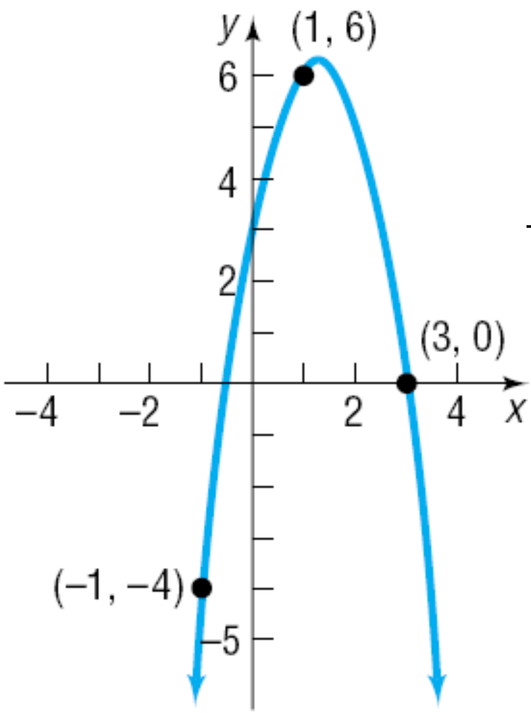
We require that the three points satisfy the equation $y = ax^2 + bx + c$.

For the point $(-1, -4)$ we have: $-4 = a(-1)^2 + b(-1) + c$

For the point $(1, 6)$ we have: $6 = a(1)^2 + b(1) + c$

For the point $(3, 0)$ we have: $0 = a(3)^2 + b(3) + c$

$$\begin{cases} a - b + c = -4 \\ a + b + c = 6 \\ 9a + 3b + c = 0 \end{cases}$$



$$\begin{array}{r} a - b + c = -4 \\ a + b + c = 6 \\ \hline 2a + 2c = 2 \end{array} \qquad \begin{array}{r} 3a - 3b + 3c = -12 \\ 9a + 3b + c = 0 \\ \hline 12a + 4c = -12 \\ -4a - 4c = -4 \\ \hline 8a = -16 \end{array}$$

$$c = 1 - a = 3$$

$$b = -a - c + 6 = 5 \qquad a = -2$$