E

INVERSE FUNCTIONS

INVERSE FUNCTIONS

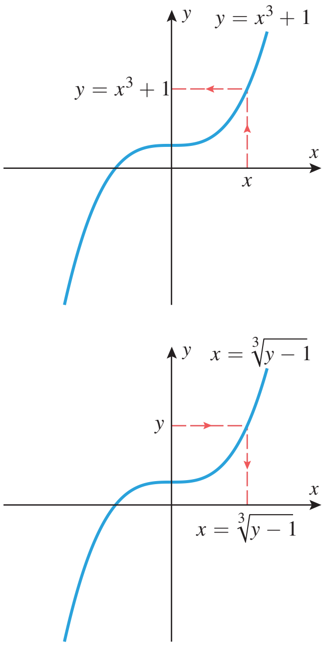
The idea of solving an equation $y = f(x)$ for x as a function of y, say $x = g(y)$, is one of the most important ideas in mathematics. Sometimes, solving an equation is a simple process; for example, using basic algebra the equation

$$y = x^3 + 1 \qquad \boxed{y = f(x)}$$

can be solved for x as a function of y:

$$x = \sqrt[3]{y-1} \qquad \boxed{x = g(y)}$$

The first equation is better for computing y if x is known, and the second is better for computing x if y is known (Figure E.1).



▲ **Figure E.1**

Our primary interest in this section is to identify relationships that may exist between the functions f and g when an equation $y = f(x)$ is expressed as $x = g(y)$, or conversely. For example, consider the functions $f(x) = x^3 + 1$ and $g(y) = \sqrt[3]{y-1}$ discussed above. When these functions are composed in either order, they cancel out the effect of one another in the sense that

$$\begin{aligned} g(f(x)) &= \sqrt[3]{f(x)-1} = \sqrt[3]{(x^3+1)-1} = x \\ f(g(y)) &= [g(y)]^3 + 1 = (\sqrt[3]{y-1}\,)^3 + 1 = y \end{aligned} \tag{1}$$

Pairs of functions with these two properties are so important that there is special terminology for them.

> **E.1 DEFINITION** If the functions f and g satisfy the two conditions
>
> $$g(f(x)) = x \text{ for every } x \text{ in the domain of } f$$
> $$f(g(y)) = y \text{ for every } y \text{ in the domain of } g$$
>
> then we say that ***f is an inverse of g*** and ***g is an inverse of f*** or that ***f and g are inverse functions***.

WARNING

If f is a function, then the -1 in the symbol f^{-1} always denotes an inverse and *never* an exponent. That is,

$$f^{-1}(x) \quad \textit{never means} \quad \frac{1}{f(x)}$$

It can be shown (Exercise 35) that if a function f has an inverse, then that inverse is unique. Thus, if a function f has an inverse, then we are entitled to talk about "the" inverse of f, in which case we denote it by the symbol f^{-1}.

▶ **Example 1** The computations in (1) show that $g(y) = \sqrt[3]{y-1}$ is the inverse of $f(x) = x^3 + 1$. Thus, we can express g in inverse notation as

$$f^{-1}(y) = \sqrt[3]{y-1}$$

and we can express the equations in Definition E.1 as

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(y)) &= y \quad \text{for every } y \text{ in the domain of } f^{-1} \end{aligned} \tag{2}$$

We will call these the ***cancellation equations*** for f and f^{-1}. ◄

CHANGING THE INDEPENDENT VARIABLE

The formulas in (2) use x as the independent variable for f and y as the independent variable for f^{-1}. Although it is often convenient to use different independent variables for f and f^{-1}, there will be occasions on which it is desirable to use the same independent variable for both. For example, if we want to graph the functions f and f^{-1} together in the same xy-coordinate system, then we would want to use x as the independent variable and y as the dependent variable for both functions. Thus, to graph the functions $f(x) = x^3 + 1$ and $f^{-1}(y) = \sqrt[3]{y-1}$ of Example 1 in the same xy-coordinate system, we would change the independent variable y to x, use y as the dependent variable for both functions, and graph the equations

$$y = x^3 + 1 \quad \text{and} \quad y = \sqrt[3]{x-1}$$

We will talk more about graphs of inverse functions later in this section, but for reference we give the following reformulation of the cancellation equations in (2) using x as the independent variable for both f and f^{-1}:

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in the domain of } f^{-1} \end{aligned} \tag{3}$$

► Example 2 Confirm each of the following.

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

The results in Example 2 should make sense to you intuitively, since the operations of multiplying by 2 and multiplying by $\frac{1}{2}$ in either order cancel the effect of one another, as do the operations of cubing and taking a cube root.

Solution (a).

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$

$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

Solution (b).

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$

$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x \quad ◄$$

► Example 3 Given that the function f has an inverse and that $f(3) = 5$, find $f^{-1}(5)$.

In general, if a function f has an inverse and $f(a) = b$, then the procedure in Example 3 shows that $a = f^{-1}(b)$; that is, f^{-1} maps each output of f back into the corresponding input (Figure E.2).

Solution. Apply f^{-1} to both sides of the equation $f(3) = 5$ to obtain

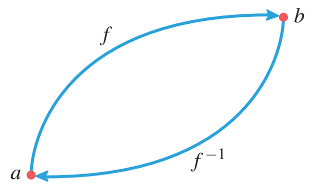
$$f^{-1}(f(3)) = f^{-1}(5)$$

and now apply the first equation in (3) to conclude that $f^{-1}(5) = 3$. ◄

DOMAIN AND RANGE OF INVERSE FUNCTIONS

The equations in (3) imply the following relationships between the domains and ranges of f and f^{-1}:

$$\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned} \tag{4}$$



▲ **Figure E.2** If f maps a to b, then f^{-1} maps b back to a.

One way to show that two sets are the same is to show that each is a subset of the other. Thus we can establish the first equality in (4) by showing that the domain of f^{-1} is a subset of the range of f and that the range of f is a subset of the domain of f^{-1}. We do this as follows: The first equation in (3) implies that f^{-1} is defined at $f(x)$ for all values of x in the domain of f, and this implies that the range of f is a subset of the domain of f^{-1}. Conversely, if x is in the domain of f^{-1}, then the second equation in (3) implies that x is in the range of f because it is the image of $f^{-1}(x)$. Thus, the domain of f^{-1} is a subset of the range of f. We leave the proof of the second equation in (4) as an exercise.

A METHOD FOR FINDING INVERSE FUNCTIONS

At the beginning of this section we observed that solving $y = f(x) = x^3 + 1$ for x as a function of y produces $x = f^{-1}(y) = \sqrt[3]{y-1}$. The following theorem shows that this is not accidental.

E.2 THEOREM *If an equation $y = f(x)$ can be solved for x as a function of y, say $x = g(y)$, then f has an inverse and that inverse is $g(y) = f^{-1}(y)$.*

PROOF Substituting $y = f(x)$ into $x = g(y)$ yields $x = g(f(x))$, which confirms the first equation in Definition E.1, and substituting $x = g(y)$ into $y = f(x)$ yields $y = f(g(y))$, which confirms the second equation in Definition E.1. ■

Theorem E.2 provides us with the following procedure for finding the inverse of a function.

A Procedure for Finding the Inverse of a Function f

Step 1. Write down the equation $y = f(x)$.

Step 2. If possible, solve this equation for x as a function of y.

Step 3. The resulting equation will be $x = f^{-1}(y)$, which provides a formula for f^{-1} with y as the independent variable.

Step 4. If y is acceptable as the independent variable for the inverse function, then you are done, but if you want to have x as the independent variable, then you need to interchange x and y in the equation $x = f^{-1}(y)$ to obtain $y = f^{-1}(x)$.

An alternative way to obtain a formula for $f^{-1}(x)$ with x as the independent variable is to reverse the roles of x and y at the outset and solve the equation $x = f(y)$ for y as a function of x.

▶ **Example 4** Find a formula for the inverse of $f(x) = \sqrt{3x-2}$ with x as the independent variable, and state the domain of f^{-1}.

Solution. Following the procedure stated above, we first write

$$y = \sqrt{3x-2}$$

Then we solve this equation for x as a function of y:

$$y^2 = 3x - 2$$
$$x = \tfrac{1}{3}(y^2 + 2)$$

which tells us that

$$f^{-1}(y) = \tfrac{1}{3}(y^2 + 2) \tag{5}$$

Since we want x to be the independent variable, we reverse x and y in (5) to produce the formula

$$f^{-1}(x) = \tfrac{1}{3}(x^2 + 2) \tag{6}$$

We know from (4) that the domain of f^{-1} is the range of f. In general, this need not be the same as the natural domain of the formula for f^{-1}. Indeed, in this example the natural domain of (6) is $(-\infty, +\infty)$, whereas the range of $f(x) = \sqrt{3x-2}$ is $[0, +\infty)$. Thus, if we want to make the domain of f^{-1} clear, we must express it explicitly by rewriting (6) as

$$f^{-1}(x) = \tfrac{1}{3}(x^2+2), \quad x \geq 0 \blacktriangleleft$$

EXISTENCE OF INVERSE FUNCTIONS

The procedure we gave above for finding the inverse of a function f was based on solving the equation $y = f(x)$ for x as a function of y. This procedure can fail for two reasons—the function f may not have an inverse, or it may have an inverse but the equation $y = f(x)$ cannot be solved explicitly for x as a function of y. Thus, it is important to establish conditions that ensure the existence of an inverse, even if it cannot be found explicitly.

If a function f has an inverse, then it must assign distinct outputs to distinct inputs. For example, the function $f(x) = x^2$ cannot have an inverse because it assigns the same value to $x = 2$ and $x = -2$, namely,

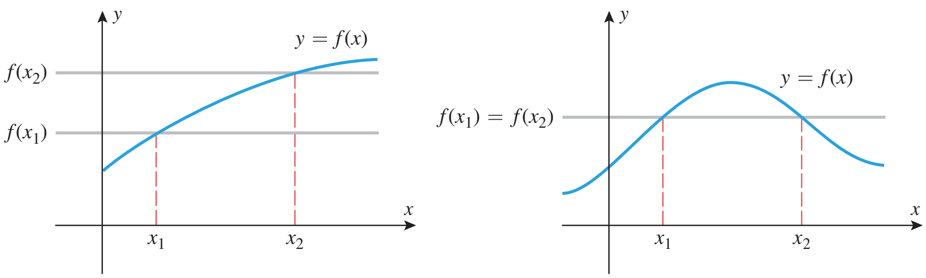
$$f(2) = f(-2) = 4$$

Thus, if $f(x) = x^2$ were to have an inverse, then the equation $f(2) = 4$ would imply that $f^{-1}(4) = 2$, and the equation $f(-2) = 4$ would imply that $f^{-1}(4) = -2$. But this is impossible because $f^{-1}(4)$ cannot have two different values. Another way to see that $f(x) = x^2$ has no inverse is to attempt to find the inverse by solving the equation $y = x^2$ for x as a function of y. We run into trouble immediately because the resulting equation $x = \pm\sqrt{y}$ does not express x as a *single* function of y.

A function that assigns distinct outputs to distinct inputs is said to be ***one-to-one*** or ***invertible***, so we know from the preceding discussion that if a function f has an inverse, then it must be one-to-one. The converse is also true, thereby establishing the following theorem.

E.3 THEOREM *A function has an inverse if and only if it is one-to-one.*

Stated algebraically, a function f is one-to-one if and only if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$; stated geometrically, a function f is one-to-one if and only if the graph of $y = f(x)$ is cut at most once by any horizontal line (Figure E.3). The latter statement together with Theorem E.3 provides the following geometric test for determining whether a function has an inverse.



One-to-one, since $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

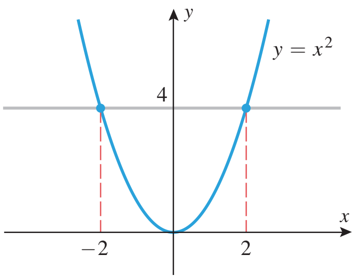
Not one-to-one, since $f(x_1) = f(x_2)$ and $x_1 \neq x_2$

▶ **Figure E.3**

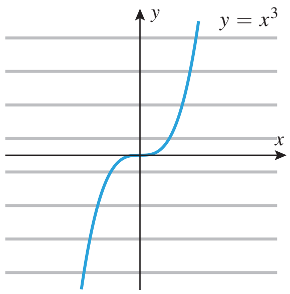
E.4 THEOREM (*The Horizontal Line Test*) *A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*

▶ **Example 5** Use the horizontal line test to show that $f(x) = x^2$ has no inverse but that $f(x) = x^3$ does.

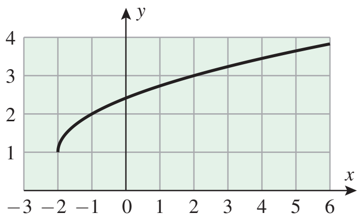
Solution. Figure E.4 shows a horizontal line that cuts the graph of $y = x^2$ more than once, so $f(x) = x^2$ is not invertible. Figure E.5 shows that the graph of $y = x^3$ is cut at most once by any horizontal line, so $f(x) = x^3$ is invertible. [Recall from Example 2 that the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.] ◀



▲ **Figure E.4**



▲ **Figure E.5**



▲ **Figure E.6**

The function $f(x) = x^3$ in Figure E.5 is an example of an increasing function. Give an example of a decreasing function and compute its inverse.

▶ **Example 6** Explain why the function f that is graphed in Figure E.6 has an inverse, and find $f^{-1}(3)$.

Solution. The function f has an inverse since its graph passes the horizontal line test. To evaluate $f^{-1}(3)$, we view $f^{-1}(3)$ as that number x for which $f(x) = 3$. From the graph we see that $f(2) = 3$, so $f^{-1}(3) = 2$. ◀

INCREASING OR DECREASING FUNCTIONS ARE INVERTIBLE

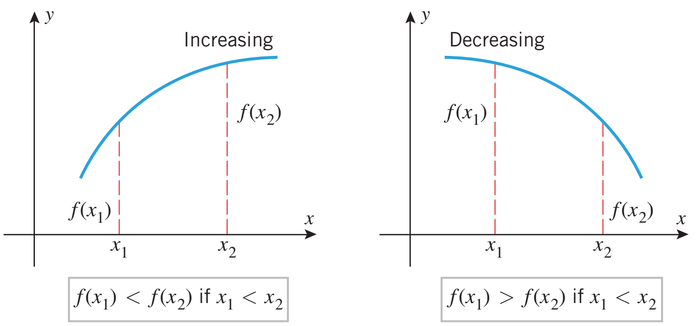
A function whose graph is always rising as it is traversed from left to right is said to be an ***increasing function***, and a function whose graph is always falling as it is traversed from left to right is said to be a ***decreasing function***. If x_1 and x_2 are points in the domain of a function f, then f is increasing if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2$$

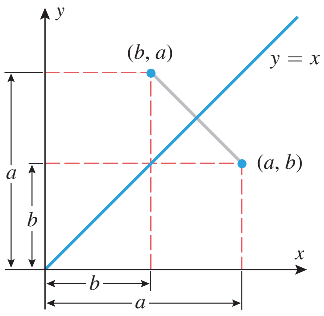
and f is decreasing if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2$$

(Figure E.7). It is evident geometrically that increasing and decreasing functions pass the horizontal line test and hence are invertible.

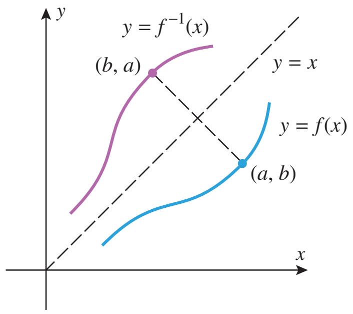


▶ **Figure E.7**



The points (a, b) and (b, a) are reflections about $y = x$.

▲ **Figure E.8**



The graphs of f and f^{-1} are reflections about $y = x$.

▲ **Figure E.9**

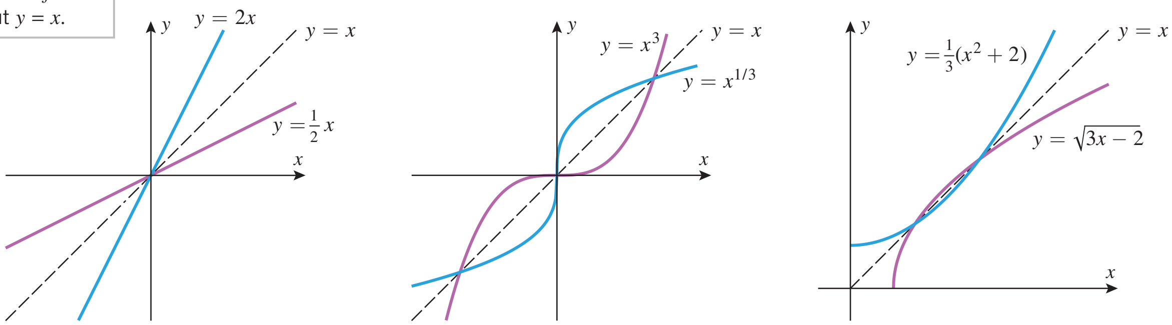
GRAPHS OF INVERSE FUNCTIONS

Our next objective is to explore the relationship between the graphs of f and f^{-1}. For this purpose, it will be desirable to use x as the independent variable for both functions so we can compare the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

If (a, b) is a point on the graph $y = f(x)$, then $b = f(a)$. This is equivalent to the statement that $a = f^{-1}(b)$, which means that (b, a) is a point on the graph of $y = f^{-1}(x)$. In short, reversing the coordinates of a point on the graph of f produces a point on the graph of f^{-1}. Similarly, reversing the coordinates of a point on the graph of f^{-1} produces a point on the graph of f (verify). However, the geometric effect of reversing the coordinates of a point is to reflect that point about the line $y = x$ (Figure E.8), and hence the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about this line (Figure E.9). In summary, we have the following result.

E.5 THEOREM *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*

► **Example 7** Figure E.10 shows the graphs of the inverse functions discussed in Examples 2 and 4. ◄



▲ **Figure E.10**

RESTRICTING DOMAINS FOR INVERTIBILITY

If a function g is obtained from a function f by placing restrictions on the domain of f, then g is called a ***restriction*** of f. Thus, for example, the function

$$g(x) = x^3, \quad x \geq 0$$

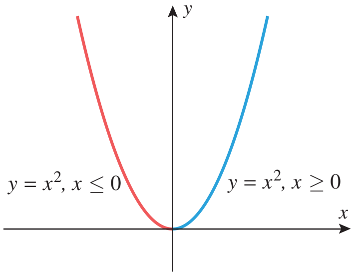
is a restriction of the function $f(x) = x^3$. More precisely, it is called the restriction of x^3 to the interval $[0, +\infty)$.

Sometimes it is possible to create an invertible function from a function that is not invertible by restricting the domain appropriately. For example, we showed earlier that $f(x) = x^2$ is not invertible. However, consider the restricted functions

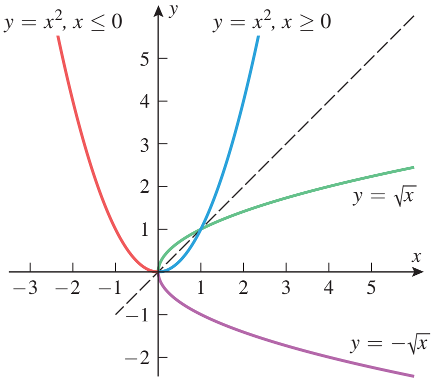
$$f_1(x) = x^2, \quad x \geq 0 \quad \text{and} \quad f_2(x) = x^2, \quad x \leq 0$$

the union of whose graphs is the complete graph of $f(x) = x^2$ (Figure E.11). These restricted functions are each one-to-one (hence invertible), since their graphs pass the horizontal line test. As illustrated in Figure E.12, their inverses are

$$f_1^{-1}(x) = \sqrt{x} \quad \text{and} \quad f_2^{-1}(x) = -\sqrt{x}$$



▲ **Figure E.11**



▲ **Figure E.12**

EXERCISE SET E　Graphing Utility

1. In (a)–(d), determine whether f and g are inverse functions.
(a) $f(x) = 4x,\ g(x) = \frac{1}{4}x$
(b) $f(x) = 3x + 1,\ g(x) = 3x - 1$
(c) $f(x) = \sqrt[3]{x - 2},\ g(x) = x^3 + 2$
(d) $f(x) = x^4,\ g(x) = \sqrt[4]{x}$

2. Check your answers to Exercise 1 with a graphing utility by determining whether the graphs of f and g are reflections of one another about the line $y = x$.

3. In each part, use the horizontal line test to determine whether the function f is one-to-one.
(a) $f(x) = 3x + 2$
(b) $f(x) = \sqrt{x - 1}$
(c) $f(x) = |x|$
(d) $f(x) = x^3$
(e) $f(x) = x^2 - 2x + 2$
(f) $f(x) = \sin x$

4. In each part, generate the graph of the function f with a graphing utility, and determine whether f is one-to-one.
(a) $f(x) = x^3 - 3x + 2$　(b) $f(x) = x^3 - 3x^2 + 3x - 1$

FOCUS ON CONCEPTS

5. In each part, determine whether the function f defined by the table is one-to-one.

(a)

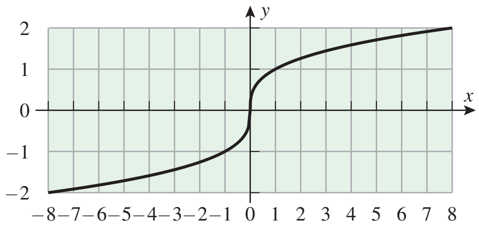
x	1	2	3	4	5	6
$f(x)$	−2	−1	0	1	2	3

(b)

x	1	2	3	4	5	6
$f(x)$	4	−7	6	−3	1	4

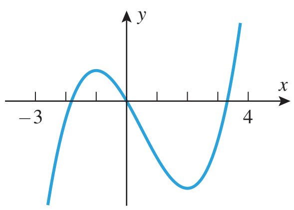
6. A face of a broken clock lies in the xy-plane with the center of the clock at the origin and 3:00 in the direction of the positive x-axis. When the clock broke, the tip of the hour hand stopped on the graph of $y = f(x)$, where f is a function that satisfies $f(0) = 0$.
(a) Are there any times of the day that cannot appear in such a configuration? Explain.
(b) How does your answer to part (a) change if f must be an invertible function?
(c) How do your answers to parts (a) and (b) change if it was the tip of the minute hand that stopped on the graph of f?

7. (a) The accompanying figure shows the graph of a function f over its domain $-8 \le x \le 8$. Explain why f has an inverse, and use the graph to find $f^{-1}(2)$, $f^{-1}(-1)$, and $f^{-1}(0)$.
(b) Find the domain and range of f^{-1}.
(c) Sketch the graph of f^{-1}.



▲ **Figure Ex-7**

8. (a) Explain why the function f graphed in the accompanying figure has no inverse function on its domain $-3 \le x \le 4$.
(b) Subdivide the domain into three adjacent intervals on each of which the function f has an inverse.



◀ **Figure Ex-8**

9–16 Find a formula for $f^{-1}(x)$. ■

9. $f(x) = 7x - 6$

10. $f(x) = \dfrac{x + 1}{x - 1}$

11. $f(x) = 3x^3 - 5$

12. $f(x) = \sqrt[5]{4x + 2}$

13. $f(x) = 3/x^2,\quad x < 0$

14. $f(x) = 5/(x^2 + 1),\quad x \ge 0$

15. $f(x) = \begin{cases} 5/2 - x, & x < 2 \\ 1/x, & x \geq 2 \end{cases}$

16. $f(x) = \begin{cases} 2x, & x \leq 0 \\ x^2, & x > 0 \end{cases}$

17–20 Find a formula for $f^{-1}(x)$, and state the domain of the function f^{-1}. ■

17. $f(x) = (x+2)^4, \quad x \geq 0$

18. $f(x) = \sqrt{x+3}$

19. $f(x) = -\sqrt{3-2x}$

20. $f(x) = x - 5x^2, \quad x \geq 1$

21–24 True–False Determine whether the statement is true or false. Explain your answer. ■

21. If f is an invertible function such that $f(2) = 2$, then $f^{-1}(2) = \frac{1}{2}$.

22. If f and g are inverse functions, then f and g have the same domain.

23. A one-to-one function is invertible.

24. If the graph of a function f is symmetric about the line $y = x$, then f is invertible.

25. Let $f(x) = ax^2 + bx + c$, $a > 0$. Find f^{-1} if the domain of f is restricted to
(a) $x \geq -b/(2a)$ (b) $x \leq -b/(2a)$.

FOCUS ON CONCEPTS

26. The formula $F = \frac{9}{5}C + 32$, where $C \geq -273.15$ expresses the Fahrenheit temperature F as a function of the Celsius temperature C.
(a) Find a formula for the inverse function.
(b) In words, what does the inverse function tell you?
(c) Find the domain and range of the inverse function.

27. (a) One meter is about 6.214×10^{-4} miles. Find a formula $y = f(x)$ that expresses a length y in meters as a function of the same length x in miles.
(b) Find a formula for the inverse of f.
(c) Describe what the formula $x = f^{-1}(y)$ tells you in practical terms.

28. Let $f(x) = x^2$, $x > 1$, and $g(x) = \sqrt{x}$.
(a) Show that $f(g(x)) = x$, $x > 1$, and $g(f(x)) = x$, $x > 1$.
(b) Show that f and g are *not* inverses by showing that the graphs of $y = f(x)$ and $y = g(x)$ are not reflections of one another about $y = x$.
(c) Do parts (a) and (b) contradict one another? Explain.

29. (a) Show that $f(x) = (3-x)/(1-x)$ is its own inverse.
(b) What does the result in part (a) tell you about the graph of f?

30. Sketch the graph of a function that is one-to-one on $(-\infty, +\infty)$, yet not increasing on $(-\infty, +\infty)$ and not decreasing on $(-\infty, +\infty)$.

31. Let $f(x) = 2x^3 + 5x + 3$. Find x if $f^{-1}(x) = 1$.

32. Let $f(x) = \dfrac{x^3}{x^2+1}$. Find x if $f^{-1}(x) = 2$.

33. Prove that if $a^2 + bc \neq 0$, then the graph of

$$f(x) = \frac{ax+b}{cx-a}$$

is symmetric about the line $y = x$.

34. (a) Prove: If f and g are one-to-one, then so is the composition $f \circ g$.
(b) Prove: If f and g are one-to-one, then

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

35. Prove: A one-to-one function f cannot have two different inverses.