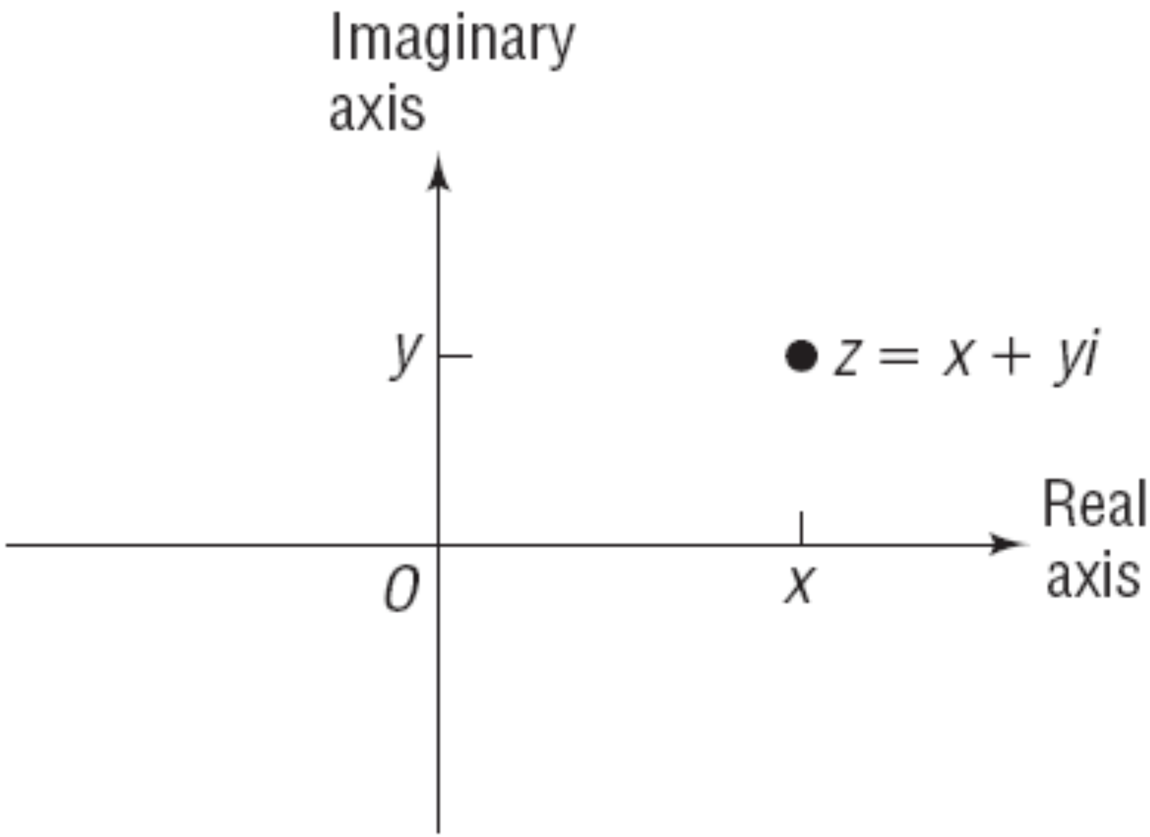
# Section 10.3

# The Complex Plane;

# De Moivre's Theorem

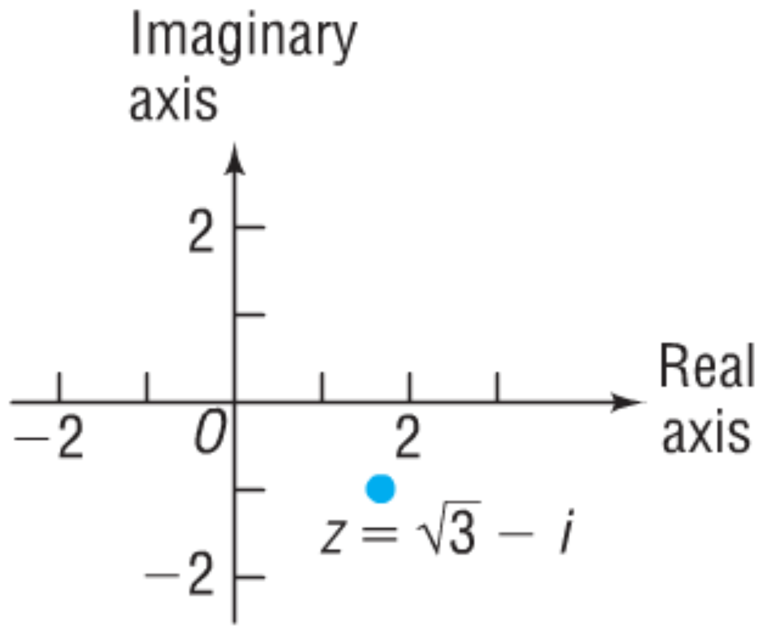
# 1 Plot Points in the Complex Plane

# Complex plane



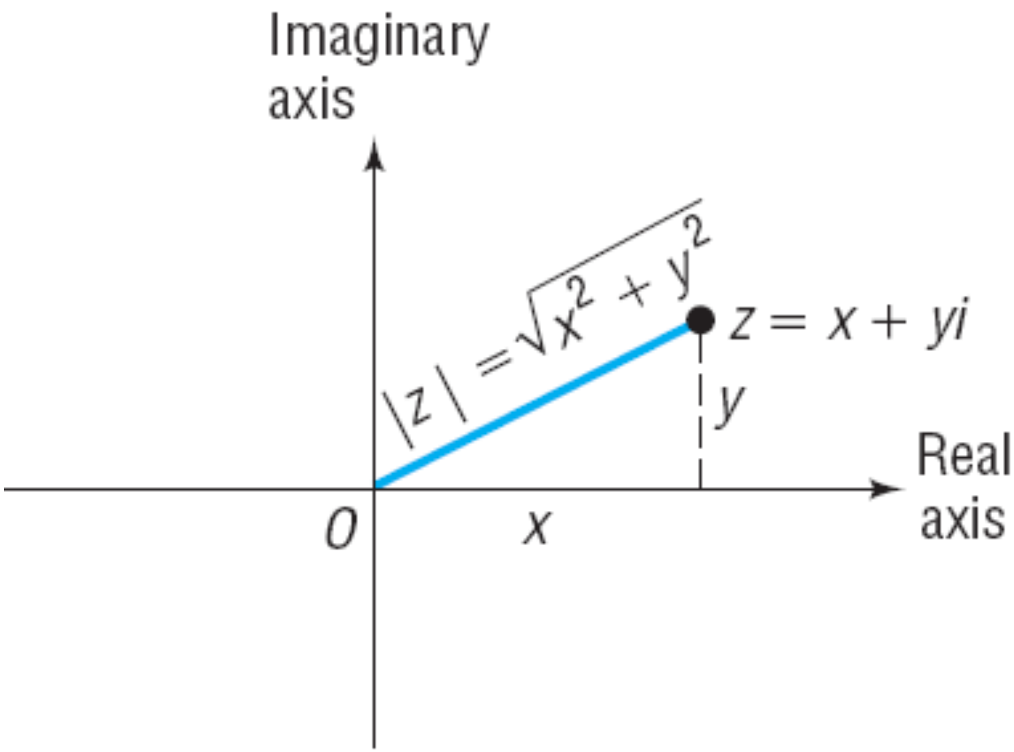
## EXAMPLE Plotting a Point in the Complex Plane

Plot the point corresponding to $z = \sqrt{3} - i$ in the complex plane.



Let $z = x + yi$ be a complex number. The **magnitude** or **modulus** of $z$, denoted by $|z|$, is defined as the distance from the origin to the point $(x, y)$. That is,

$$|z| = \sqrt{x^2 + y^2} \qquad \textbf{(1)}$$

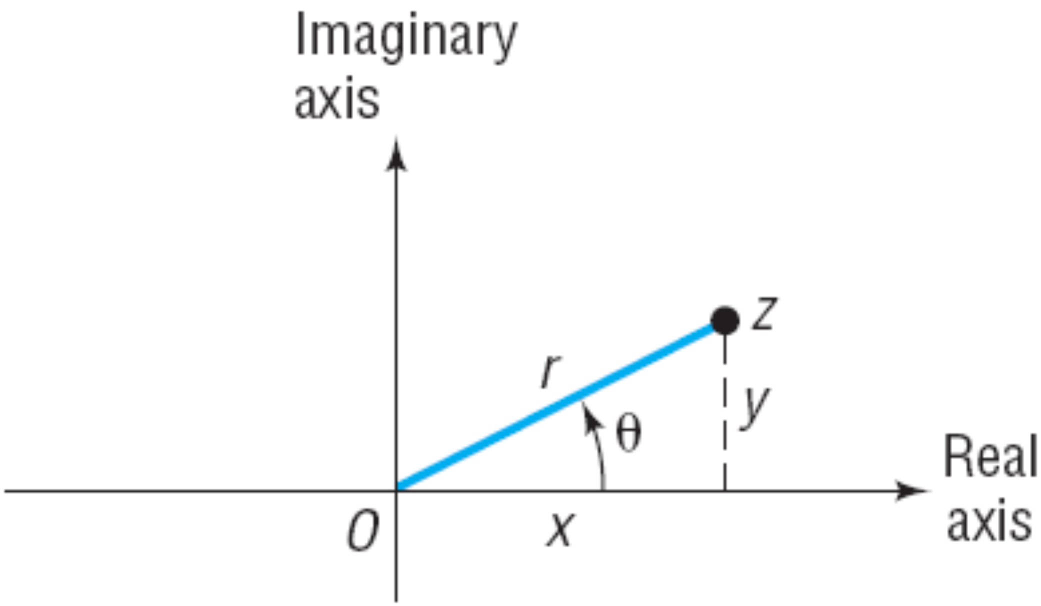



$$\boxed{|z| = \sqrt{z\bar{z}}}$$

## 2 Convert a Complex Number between Rectangular Form and Polar Form

If $r \geq 0$ and $0 \leq \theta < 2\pi$, the complex number $z = x + yi$ may be written in **polar form** as

$$z = x + yi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta) \qquad \textbf{(4)}$$




$z = x + yi = r(\cos\theta + i\sin\theta)$,
$r \geq 0$, $0 \leq \theta < 2\pi$

$$|z| = r$$

EXAMPLE

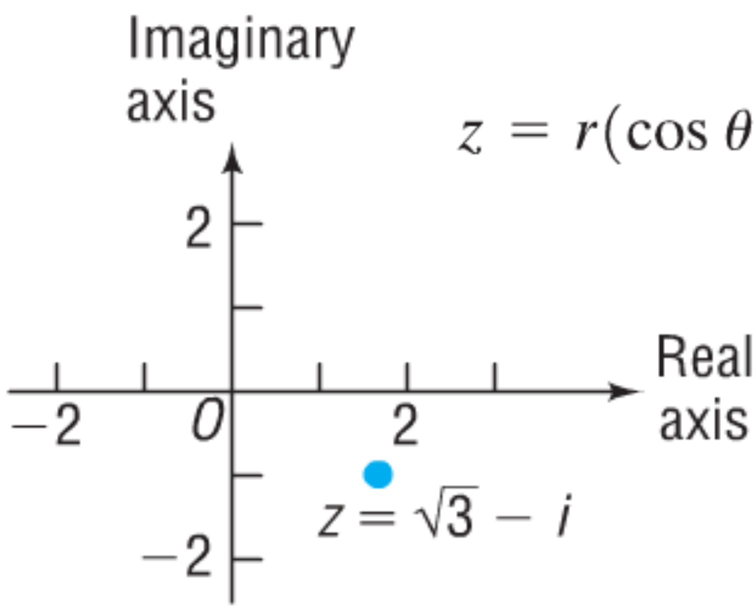
# Writing a Complex Number in Polar Form

Write an expression for $z = \sqrt{3} - i$ in polar form.

$$r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$

$$\sin\theta = \frac{y}{r} = \frac{-1}{2}, \qquad \cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2}, \qquad 0 \le \theta < 2\pi$$

The angle $\theta, 0 \le \theta < 2\pi$, that satisfies both equations is $\theta = \frac{11\pi}{6}$. With $\theta = \frac{11\pi}{6}$ and $r = 2$, the polar form of $z = \sqrt{3} - i$ is

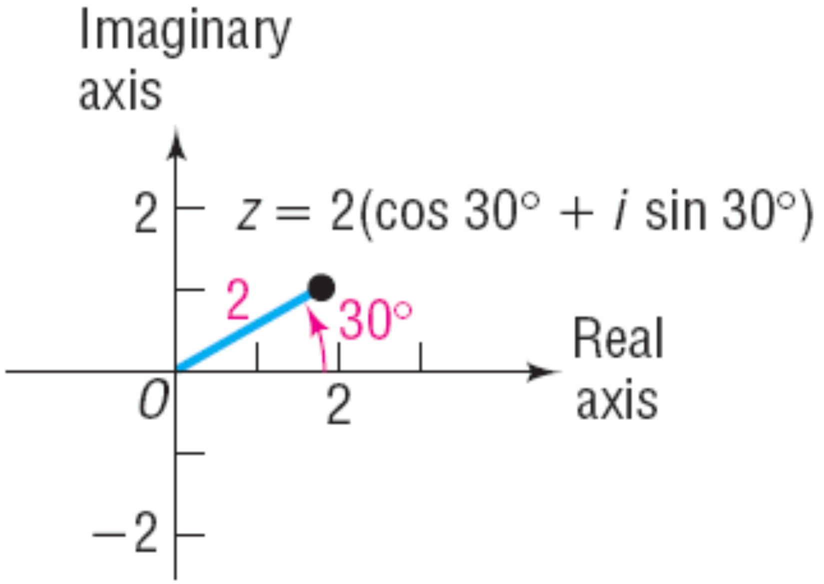



$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

EXAMPLE

## Plotting a Point in the Complex Plane and Converting from Polar to Rectangular Form

Plot the point corresponding to $z = 2(\cos 30° + i \sin 30°)$ in the complex plane, and write an expression for $z$ in rectangular form.




$$z = 2(\cos 30° + i \sin 30°) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

## 3 Find Products and Quotients of Complex Numbers in Polar Form

## THEOREM

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. Then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \quad \textbf{(5)}$$

If $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad \textbf{(6)}$$

EXAMPLE

## Finding Products and Quotients of Complex Numbers in Polar Form

If $z = 4(\cos 35° + i\sin 35°)$ and $w = 2(\cos 80° + i\sin 80°)$ find

(a) $zw$ (b) $\dfrac{z}{w}$ Leave answer in polar form.

$$\text{(a)}\ \ zw = \left[4(\cos 35° + i\sin 35°)\right]\left[2(\cos 80° + i\sin 80°)\right]$$

$$= (4\cdot 2)\left[\cos(35° + 80°) + i\sin(35° + 80°)\right] = 8\cos 115° + i\sin 115°$$

$$\text{(b)}\ \ \frac{z}{w} = \frac{4(\cos 35° + i\sin 35°)}{2(\cos 80° + i\sin 80°)} = \frac{4}{2}\left[\cos(35° - 80°) + i\sin(35° - 80°)\right]$$

$$= 2\left[\cos(-45°) + i\sin(-45°)\right] = 2(\cos 45° - i\sin 45°)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

# 4 Use De Moivre's Theorem

## THEOREM

### De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$\boxed{z^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$$

where $n \geq 1$ is a positive integer.

**EXAMPLE**

## Using De Moivre's Theorem

Write $\left[2\left(\cos 15^\circ + i\sin 15^\circ\right)\right]^4$ in standard form $a+bi$.

$$\left[2\left(\cos 15^\circ + i\sin 15^\circ\right)\right]^4 = 2^4\left(\cos\left(4\cdot 15^\circ\right) + i\sin\left(4\cdot 15^\circ\right)\right)$$

$$16\left(\cos 60^\circ + i\sin 60^\circ\right)$$

$$16\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 8 + 8\sqrt{3}i$$

$$\boxed{z^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$$

**EXAMPLE**

# Using De Moivre's Theorem

Write $(2+2i)^6$ in standard form $a+bi$.

First we need $2+2i$ in polar form.

$$r=\sqrt{2^2+2^2}=\sqrt{8}=2\sqrt{2} \quad \text{This is in quadrant I.} \quad \theta=\tan^{-1}\frac{2}{2}=\frac{\pi}{4}$$

$$(2+2i)^6=\left[2\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^6=\left(2\sqrt{2}\right)^6\left(\cos\left(6\cdot\frac{\pi}{4}\right)+i\sin\left(6\cdot\frac{\pi}{4}\right)\right)$$

$$=512\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=512(0-1i)=-512i$$

$$\boxed{z^n=r^n[\cos(n\theta)+i\sin(n\theta)]}$$

## 5 Find Complex Roots

## THEOREM

### Finding Complex Roots

Let $w = r(\cos\theta_0 + i\sin\theta_0)$ be a complex number and let $n \geq 2$ be an integer. If $w \neq 0$, there are $n$ distinct complex roots of $w$, given by the formula

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \qquad \textbf{(8)}$$

where $k = 0, 1, 2, \ldots, n - 1$.

## EXAMPLE Finding Complex Cube Roots

Find the complex cube roots of $-1 + \sqrt{3}i$. Leave your answers in polar form, with the argument in degrees.

$$-1 + \sqrt{3}i = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2(\cos 120^\circ + i \sin 120^\circ)$$

$$z_k = \sqrt[3]{2}\left[\cos\left(\frac{120^\circ}{3} + \frac{360^\circ k}{3}\right) + i \sin\left(\frac{120^\circ}{3} + \frac{360^\circ k}{3}\right)\right]$$

$$= \sqrt[3]{2}\left[\cos(40^\circ + 120^\circ k) + i \sin(40^\circ + 120^\circ k)\right] \qquad k = 0, 1, 2$$

$$z_0 = \sqrt[3]{2}\left[\cos(40^\circ + 120^\circ \cdot 0) + i \sin(40^\circ + 120^\circ \cdot 0)\right] = \sqrt[3]{2}(\cos 40^\circ + i \sin 40^\circ)$$

$$z_1 = \sqrt[3]{2}\left[\cos(40^\circ + 120^\circ \cdot 1) + i \sin(40^\circ + 120^\circ \cdot 1)\right] = \sqrt[3]{2}(\cos 160^\circ + i \sin 160^\circ)$$

$$z_2 = \sqrt[3]{2}\left[\cos(40^\circ + 120^\circ \cdot 2) + i \sin(40^\circ + 120^\circ \cdot 2)\right] = \sqrt[3]{2}(\cos 280^\circ + i \sin 280^\circ)$$

$$\boxed{z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right) + i \sin\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]}$$

EXAMPLE **Finding Complex Cube Roots**

Find the complex cube roots of $-1 + \sqrt{3}i$. Leave your answers in polar form, with the argument in degrees.