Section 14.3

Probability

EXAMPLE Tossing a Fair Coin

In tossing a fair coin, we know that the outcome is either a head or a tail. On any particular throw, we cannot predict what will happen, but, if we toss the coin many times, we observe that the number of times that a head comes up is approximately equal to the number of times that we get a tail. It seems reasonable, therefore, to assign a probability of $\frac{1}{2}$ that a head comes up and a probability of $\frac{1}{2}$ that a tail comes up.

1 Construct Probability Models

A **probability model** with the sample space

$$S = \{e_1, e_2, \ldots, e_n\}$$

where $e_1, e_2, \ldots, e_n$ are the possible outcomes and $P(e_1), P(e_2), \ldots, P(e_n)$ are the respective probabilities of these outcomes, requires that

$$P(e_1) \geq 0, P(e_2) \geq 0, \ldots, P(e_n) \geq 0 \qquad \textbf{(1)}$$

$$\sum_{i=1}^{n} P(e_i) = P(e_1) + P(e_2) + \cdots + P(e_n) = 1 \qquad \textbf{(2)}$$

EXAMPLE Determining Probability Models

In a bag of M&Ms,™ the candies are colored red, green, blue, brown, yellow, and orange. A candy is drawn from the bag and the color is recorded. The sample space of this experiment is {red, green, blue, brown, yellow, orange}. Determine which of the following are probability models.

(a)

Outcome	Probability
red	0.3
green	0.15
blue	0
brown	0.15
yellow	0.2
orange	0.2

(b)

Outcome	Probability
red	0.1
green	0.1
blue	0.1
brown	0.4
yellow	0.2
orange	0.3

(a) This model is a probability model since all the outcomes have probabilities that are nonnegative and the sum of the probabilities is 1.

(b) This model is not a probability model because the sum of the probabilities is not 1.

EXAMPLE Determining Probability Models

In a bag of M&Ms,™ the candies are colored red, green, blue, brown, yellow, and orange. A candy is drawn from the bag and the color is recorded. The sample space of this experiment is {red, green, blue, brown, yellow, orange}. Determine which of the following are probability models.

(c)

Outcome	Probability
red	0.3
green	−0.3
blue	0.2
brown	0.4
yellow	0.2
orange	0.2

(d)

Outcome	Probability
red	0
green	0
blue	0
brown	0
yellow	1
orange	0

(c) This model is not a probability model because P(green) is less than 0. Recall, all probabilities must be nonnegative.

(d) This model is a probability model because all the outcomes have probabilities that are nonnegative, and the sum of the probabilities is 1. Notice that $P(\text{yellow}) = 1$, meaning that this outcome will occur with 100% certainty each time that the experiment is repeated. This means that the bag of M&Ms™ has only yellow candies.

EXAMPLE Constructing a Probability Model

An experiment consists of rolling a fair die once. A die is a cube with each face having either 1, 2, 3, 4, 5, or 6 dots on it. See Figure 8. Construct a probability model for this experiment.

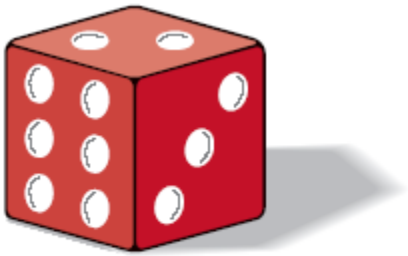
A sample space S consists of all the possibilities that can occur. Because rolling the die will result in one of six faces showing, the sample space S consists of

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$P(1) = \frac{1}{6} \qquad P(2) = \frac{1}{6}$$

$$P(3) = \frac{1}{6} \qquad P(4) = \frac{1}{6}$$

$$P(5) = \frac{1}{6} \qquad P(6) = \frac{1}{6}$$



EXAMPLE Constructing a Probability Model

An experiment consists of tossing a coin. The coin is weighted so that heads (H) is three times as likely to occur as tails (T). Construct a probability model for this experiment.

The sample space S is $S = \{H, T\}$. If x denotes the probability that a tail occurs,

$$P(T) = x \quad \text{and} \quad P(H) = 3x$$

$$\begin{aligned} P(T) + P(H) = x + 3x &= 1 \\ 4x &= 1 \\ x &= \frac{1}{4} \end{aligned}$$

$$P(T) = \frac{1}{4} \qquad P(H) = \frac{3}{4}$$

2 Compute Probabilities of Equally Likely Outcomes

THEOREM

Probability for Equally Likely Outcomes

If an experiment has n equally likely outcomes and if the number of ways that an event E can occur is m, then the probability of E is

$$P(E) = \frac{\text{Number of ways that } E \text{ can occur}}{\text{Number of possible outcomes}} = \frac{m}{n} \qquad \textbf{(3)}$$

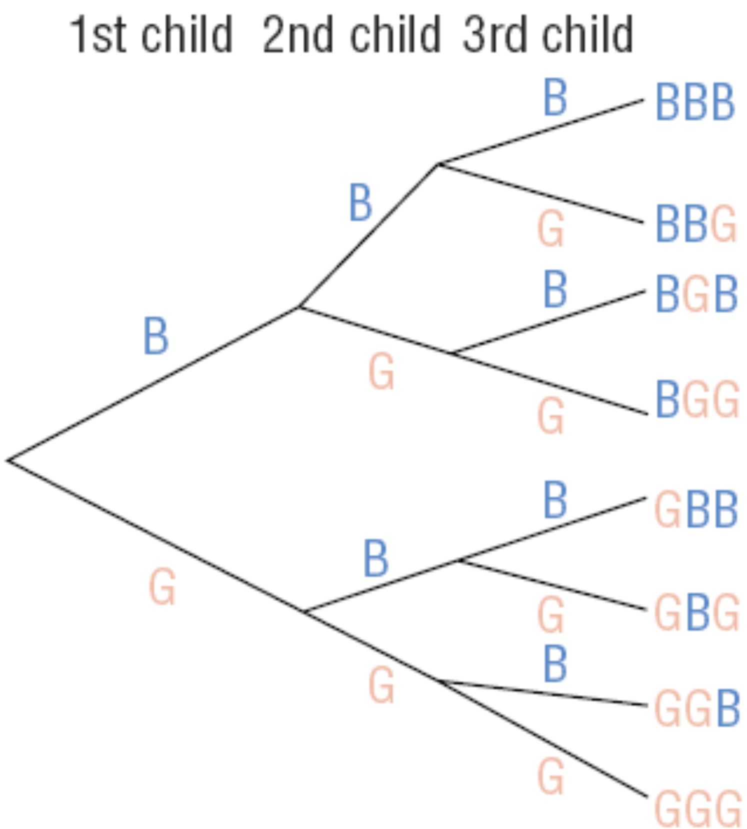
If S is the sample space of this experiment,

$$P(E) = \frac{n(E)}{n(S)} \qquad \textbf{(4)}$$

EXAMPLE **Calculating Probabilities of Events Involving Equally Likely Outcomes**

Calculate the probability that in a 3-child family there are 2 boys and 1 girl. Assume equally likely outcomes.

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$



$$E = \{BBG, BGB, GBB\}$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

$$\boxed{P(E) = \frac{n(E)}{n(S)}}$$

EXAMPLE Computing Compound Probabilities

Consider the experiment of drawing a card from a standard 52 card deck of playing cards (ace through king, four suits). Let E represent the event "draw a heart" and let F represent the event "draw an ace".

(a) Write the event E and F. What is $n(E \cap F)$?

(b) Write the event E or F. What is $n(E \cup F)$?

(c) Compute $P(E)$. Compute $P(F)$.

(d) Compute $P(E \cap F)$.

(e) Compute $P(E \cup F)$.

$n(E) = 13$ and $n(F) = 4$

(a) The word "and" means intersection so $n(E \text{ and } F)$ is $n(E \cap F) = 1$ (The ace of hearts)

(b) The word "of" means union so $n(E \text{ or } F)$ is $n(E \cup F) = 16$ (All hearts plus the aces of clubs, diamonds and spades)

EXAMPLE Computing Compound Probabilities

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(a) Write the event E and F. What is $n(E \cap F)$?

(b) Write the event E or F. What is $n(E \cup F)$?

(c) Compute $P(E)$. Compute $P(F)$.

(d) Compute $P(E \cap F)$.

(e) Compute $P(E \cup F)$.

$$n(E \cap F) = 1$$

$$n(E \cup F) = 16$$

(c) $P(E) = \dfrac{13}{52} = \dfrac{1}{4}$ $\qquad$ $P(F) = \dfrac{4}{52} = \dfrac{1}{13}$

(d) $P(E \cap F) = \dfrac{n(E \cap F)}{n(S)} = \dfrac{1}{52}$

(e) $P(E \cup F) = \dfrac{n(E \cup F)}{n(S)} = \dfrac{16}{52} = \dfrac{4}{13}$

3 Find Probabilities of the Union of Two Events

THEOREM

For any two events E and F,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) \qquad \textbf{(5)}$$

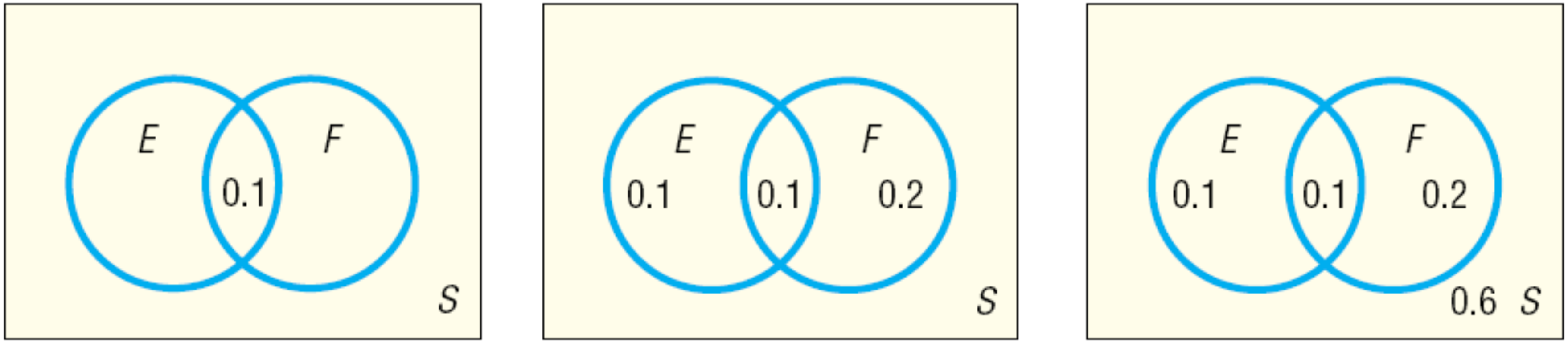
EXAMPLE

Computing Probabilities of the Union of Two Events

If $P(E) = 0.2$, $P(F) = 0.3$, and $P(E \cap F) = 0.1$, find the probability of E or F; that is, find $P(E \cup F)$.

$$\text{Probability of } E \text{ or } F = P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
$$= 0.2 + 0.3 - 0.1 = 0.4$$

A Venn diagram can sometimes be used to obtain probabilities.



$$\boxed{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

THEOREM

Mutually Exclusive Events

If E and F are mutually exclusive events,

$$\boxed{P(E \cup F) = P(E) + P(F) \qquad \textbf{(6)}}$$

EXAMPLE Computing Probabilities of the Union of Two Mutually Exclusive Events

If $P(E) = 0.4$ and $P(F) = 0.25$, and E and F are mutually exclusive, find $P(E \cup F)$.

Since E and F are mutually exclusive, use formula (6).

$$P(E \cup F) = P(E) + P(F) = 0.4 + 0.25 = 0.65$$

$$\boxed{P(E \cup F) = P(E) + P(F)}$$

4 Use the Complement Rule to Find Probabilities

Complement of an Event

Let S denote the sample space of an experiment, and let E denote an event. The **complement of** E, denoted $\overline{E}$, is the set of all outcomes in the sample space S that are not outcomes in the event E.

THEOREM

Computing Probabilities of Complementary Events

If E represents any event and $\overline{E}$ represents the complement of E, then

$$P(\overline{E}) = 1 - P(E) \tag{7}$$

EXAMPLE

Computing Probabilities Using Complements

On the local news the weather reporter stated that the probability of rain tomorrow is 30%. What is the probability that it will not rain?

$$P(\text{no rain}) = 1 - P(\text{rain}) = 1 - 0.3 = 0.7 = 70\%$$

EXAMPLE Birthday Problem

What is the probability that in a group of 10 people at least 2 people have the same birthday? Assume that there are 365 days in a year.

$$n(S) = 365^{10}$$

We wish to find the probability of the event E: "at least two people have the same birthday." It is difficult to count the elements in this set; it is much easier to count the elements of the complementary event $\overline{E}$: "no two people have the same birthday."

$$n(\overline{E}) = 365 \cdot 364 \cdot 363 \cdot \cdots \cdot 356$$

The probability of the event $\overline{E}$ is

$$P(\overline{E}) = \frac{n(\overline{E})}{n(S)} = \frac{365 \cdot 364 \cdot 363 \cdot \cdots \cdot 356}{365^{10}} \approx 0.883$$

The probability of two or more people in a group of 10 people having the same birthday is then

$$P(E) = 1 - P(\overline{E}) \approx 1 - 0.883 = 0.117$$

The birthday problem can be solved for any group size. The following table gives the probabilities for two or more people having the same birthday for various group sizes. Notice that the probability is greater than $\frac{1}{2}$ for any group of 23 or more people.

	Number of People											
	5	10	15	20	21	22	23	24	25	30	40	50
Probability That Two or More Have the Same Birthday	0.027	0.117	0.253	0.411	0.444	0.476	0.507	0.538	0.569	0.706	0.891	0.970

	Number of People			
	60	70	80	90
Probability That Two or More Have the Same Birthday	0.994	0.99916	0.99991	0.99999