Section 5.6

Complex Zeros;

Fundamental Theorem of Algebra

DEFINITION

A variable in the complex number system is referred to as a **complex variable.** A **complex polynomial function** f of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are complex numbers, $a_n \neq 0$, n is a nonnegative integer, and x is a complex variable. As before, a_n is called the **leading coefficient** of f. A complex number r is called a **complex zero** of f if $f(r) = 0$.

Fundamental Theorem of Algebra

Every complex polynomial function $f(x)$ of degree $n \geq 1$ has at least one complex zero.

THEOREM

Every complex polynomial function $f(x)$ of degree $n \geq 1$ can be factored into n linear factors (not necessarily distinct) of the form

$$f(x) = a_n(x - r_1)(x - r_2) \cdot \cdots \cdot (x - r_n) \qquad \textbf{(2)}$$

where $a_n, r_1, r_2, \ldots, r_n$ are complex numbers. That is, every complex polynomial function of degree $n \geq 1$ has exactly n complex zeros, some of which may repeat.

1 Use the Conjugate Pairs Theorem

CONJUGATE PAIRS THEOREM

Let $f(x)$ be a polynomial whose coefficients are real numbers. If $r = a + bi$ is a zero of f, the complex conjugate $\bar{r} = a - bi$ is also a zero of f.

COROLLARY

A polynomial f of odd degree with real coefficients has at least one real zero.

EXAMPLE

Using the Conjugate Pairs Theorem

A polynomial of degree 5 whose coefficients are real numbers has the zeros -2, $-3i$, and $2 + 4i$. Find the remaining two zeros.

Since f has coefficients that are real numbers, complex zeros appear as conjugate pairs. It follows that $3i$, the conjugate of $-3i$, and $2 - 4i$, the conjugate of $2 + 4i$, are the two remaining zeros.

2 Find a Polynomial Function with Specified Zeros

EXAMPLE

Finding a Polynomial Function Whose Zeros Are Given

Find a polynomial f of degree 4 whose coefficients are real numbers and that has the zeros 1, 1, $4 + i$.

Since $4 + i$ is a zero, by the Conjugate Pairs Theorem, $4 - i$ is also a zero.

By the factor theorem: $f(x) = a(x-1)(x-1)[x-(4+i)][x-(4-i)]$

$$= a(x^2 - 2x + 1)[x^2 - (4+i)x - (4-i)x + (4+i)(4-i)]$$

$$= a(x^2 - 2x + 1)(x^2 - 4x - ix - 4x + ix + 16 + 4i - 4i - i^2)$$

$$= a(x^2 - 2x + 1)(x^2 - 8x + 17)$$

$$= a(x^4 - 10x^3 + 34x^2 - 42x + 17)$$

Exploration

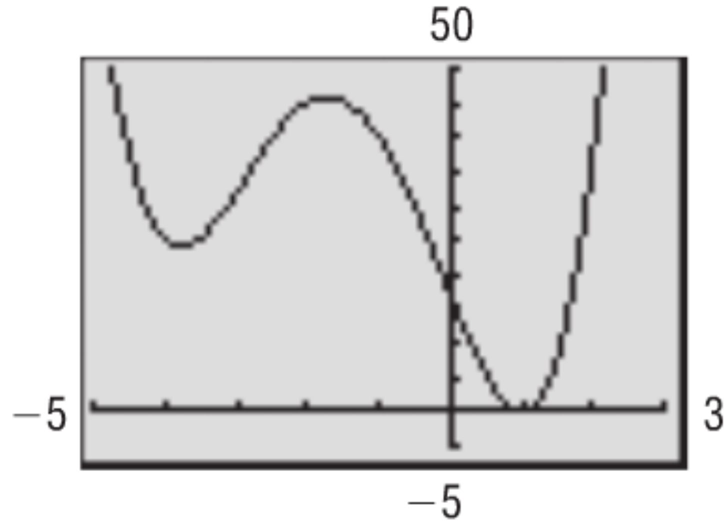

Figure 51

Graph the function f for $a = 1$.
Does the value of a affect the zeros of f? How does the value of a affect the graph of f? What information about f is sufficient to uniquely determine a?

Result A quick analysis of the polynomial function f tells us what to expect:

At most three turning points.

For large $|x|$, the graph will behave like $y = x^4$.

A repeated real zero at 1 of even multiplicity, so the graph will touch the x-axis at 1.

The only x-intercept is at 1; the y-intercept is 17.

Figure 51 shows the complete graph. (Do you see why? The graph has exactly three turning points.) The value of a causes a stretch or compression; a reflection also occurs if $a < 0$. The zeros are not affected.

If any point other than an x-intercept on the graph of f is known, then a can be determined. For example, if (2, 3) is on the graph, then $f(2) = 3 = a(37)$, so $a = 3/37$. Why won't an x-intercept work?

3 Find the Complex Zeros of a Polynomial Function

EXAMPLE Finding the Complex Zeros of a Polynomial Function

Find the complex zeros of the polynomial function and write f in factored form.

$$f(x) = x^4 + 2x^3 + x^2 - 8x - 20$$

STEP 1: The degree of f is 4 so there will be 4 complex zeros.

STEP 2: The potential rational zeros are $\frac{p}{q}: \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20.$

$$\begin{array}{r|rrrrr} 2 & 1 & 2 & 1 & -8 & -20 \\ & & 2 & 8 & 18 & 20 \\ \hline & 1 & 4 & 9 & 10 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 4 & 9 & 10 \\ & & -2 & -4 & -10 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

$$f(x) = (x-2)\left(x^3 + 4x^2 + 9x + 10\right)$$

$$f(x) = (x-2)(x+2)\left(x^2 + 2x + 5\right)$$

$$(x-2)(x+2)\left(x^2 + 2x + 5\right) = 0$$

$$x - 2 = 0 \text{ or } x + 2 = 0 \text{ or } x^2 + 2x + 5 = 0$$

$$x = 2 \text{ or } x = -2 \text{ or } x = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)}$$

EXAMPLE Finding the Complex Zeros of a Polynomial Function

Find the complex zeros of the polynomial function and write f in factored form.

$$f(x) = x^4 + 2x^3 + x^2 - 8x - 20$$

$$f(x) = (x-2)(x+2)(x^2 + 2x + 5)$$

$$x = 2 \text{ or } x = -2 \text{ or } x = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

The four complex zeros of f are $\{-2, 2, -1+2i, -1-2i\}$.

$$f(x) = (x-2)(x+2)(x-(-1+2i))(x-(-1-2i))$$