



THE DISCRIMINANT

We saw in Section 10.5 that the graph of a quadratic equation in x and y is often a conic section. We were able to determine the type of conic section by using rotation of axes to put the equation either in standard form, or in the form of a translated conic. The next result shows that it is possible to determine the nature of the graph directly from the equation itself.

O.1 THEOREM Consider a second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (1)$$

- (a) If $B^2 - 4AC < 0$, the equation represents an ellipse, a circle, a point, or else has no graph.
- (b) If $B^2 - 4AC > 0$, the equation represents a hyperbola or a pair of intersecting lines.
- (c) If $B^2 - 4AC = 0$, the equation represents a parabola, a line, a pair of parallel lines, or else has no graph.

The quantity $B^2 - 4AC$ in this theorem is called the **discriminant** of the quadratic equation. To see why this theorem is true, we need a fact about the discriminant. It can be shown (Exercise 33 of Section 10.5) that if the coordinate axes are rotated through any angle θ , and if

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \quad (2)$$

is the equation resulting from (1) after rotation, then

$$B^2 - 4AC = B'^2 - 4A'C' \quad (3)$$

In other words, the discriminant of a quadratic equation is not altered by rotating the coordinate axes. For this reason the discriminant is said to be **invariant** under a rotation of coordinate axes. In particular, if we choose the angle of rotation to eliminate the cross-product term, then (2) becomes

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \quad (4)$$

and since $B' = 0$, (3) tells us that

$$B^2 - 4AC = -4A'C' \quad (5)$$

PROOF OF (a) If $B^2 - 4AC < 0$, then from (5), $A'C' > 0$, so (4) can be divided through by $A'C'$ and written in the form

$$\frac{1}{C'} \left(x'^2 + \frac{D'}{A'} x' \right) + \frac{1}{A'} \left(y'^2 + \frac{E'}{C'} y' \right) = -\frac{F'}{A'C'}$$

O2 Appendix O: The Discriminant

Since $A'C' > 0$, the numbers A' and C' have the same sign. We assume that this sign is positive, since Equation (4) can be multiplied through by -1 to achieve this, if necessary. By completing the squares, we can rewrite the last equation in the form

$$\frac{(x' - h)^2}{(\sqrt{C'})^2} + \frac{(y' - k)^2}{(\sqrt{A'})^2} = K$$

There are three possibilities: $K > 0$, in which case the graph is either a circle or an ellipse, depending on whether or not the denominators are equal; $K < 0$, in which case there is no graph, since the left side is nonnegative for all x' and y' ; or $K = 0$, in which case the graph is the single point (h, k) , since the equation is satisfied only by $x' = h$ and $y' = k$. The proofs of parts (b) and (c) require a similar kind of analysis. ■

► **Example 1** Use the discriminant to identify the graph of

$$8x^2 - 3xy + 5y^2 - 7x + 6 = 0$$

Solution. We have

$$B^2 - 4AC = (-3)^2 - 4(8)(5) = -151$$

Since the discriminant is negative, the equation represents an ellipse, a point, or else has no graph. (Why can't the graph be a circle?) ◀

In cases where a quadratic equation represents a point, a line, a pair of parallel lines, a pair of intersecting lines, or has no graph, we say that equation represents a **degenerate conic section**. Thus, if we allow for possible degeneracy, it follows from Theorem O.1 that *every quadratic equation has a conic section as its graph*.

EXERCISE SET O CAS

1–5 Use the discriminant to identify the graph of the given equation. ■

1. $x^2 - xy + y^2 - 2 = 0$

2. $x^2 + 4xy - 2y^2 - 6 = 0$

3. $x^2 + 2\sqrt{3}xy + 3y^2 + 2\sqrt{3}x - 2y = 0$

4. $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$

5. $34x^2 - 24xy + 41y^2 - 25 = 0$

6. Each of the following represents a degenerate conic section. Where possible, sketch the graph.

(a) $x^2 - y^2 = 0$

(b) $x^2 + 3y^2 + 7 = 0$

(c) $8x^2 + 7y^2 = 0$

(d) $x^2 - 2xy + y^2 = 0$

(e) $9x^2 + 12xy + 4y^2 - 36 = 0$

(f) $x^2 + y^2 - 2x - 4y = -5$

7. Prove parts (b) and (c) of Theorem O.1.

 **8.** Consider the conic whose equation is

$$x^2 + xy + 2y^2 - x + 3y + 1 = 0$$

- Use the discriminant to identify the conic.
- Graph the equation by solving for y in terms of x and graphing both solutions.
- Your CAS may be able to graph the equation in the form given. If so, graph the equation in this way.

 **9.** Consider the conic whose equation is

$$2x^2 + 9xy + y^2 - 6x + y - 4 = 0$$

- Use the discriminant to identify the conic.
- Graph the equation by solving for y in terms of x and graphing both solutions.
- Your CAS may be able to graph the equation in the form given. If so, graph the equation in this way.