Section 7.3

Computing the Values of Trigonometric Functions of Acute Angles

1 Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45°$

EXAMPLE

Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45°$

Using the right triangle in Figure 27(a), in which one of the angles is $\frac{\pi}{4} = 45°$, it follows that the other acute angle is also $\frac{\pi}{4} = 45°$, so the triangle is isosceles.

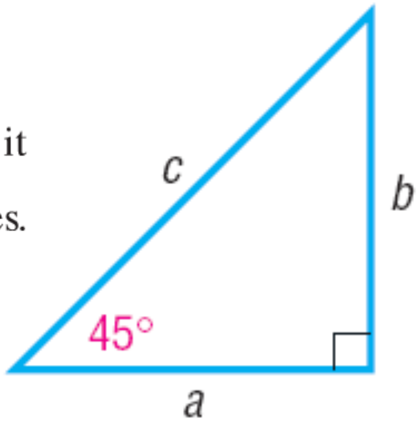
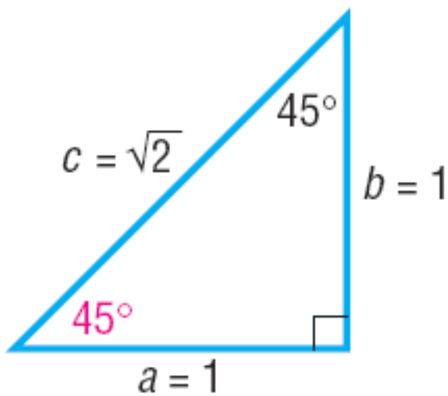


Figure 27 (a)

Let $a = b = 1$

Then, by the Pythagorean Theorem,

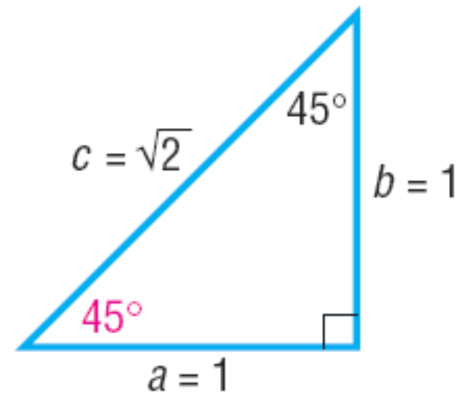
$$c^2 = a^2 + b^2 = 1 + 1 = 2$$
$$c = \sqrt{2}$$



$$\sin\frac{\pi}{4} = \sin 45° = \frac{b}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

EXAMPLE

Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{4} = 45°$



$$\sin\frac{\pi}{4} = \sin 45° = \frac{b}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos\frac{\pi}{4} = \cos 45° = \frac{a}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan\frac{\pi}{4} = \tan 45° = \frac{\sin 45°}{\cos 45°} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$\sec\frac{\pi}{4} = \sec 45° = \frac{1}{\cos 45°} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\cot\frac{\pi}{4} = \cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1$$

$$\csc\frac{\pi}{4} = \csc 45° = \frac{1}{\sin 45°} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

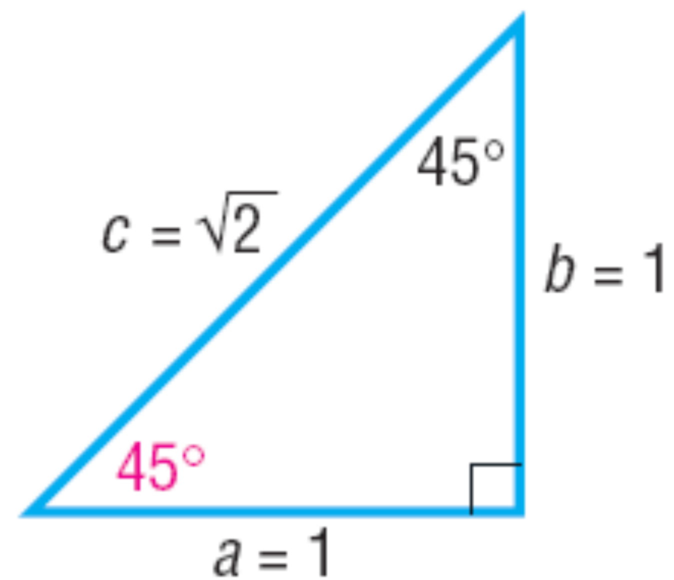
EXAMPLE

Finding the Exact Value of a Trigonometric Expression

(a) $(\sin 45^\circ)(\tan 45^\circ)$ (b) $\left(\sec \frac{\pi}{4}\right)\left(\cot \frac{\pi}{4}\right)$

(a) $(\sin 45^\circ)(\tan 45^\circ) = \frac{\sqrt{2}}{2} \cdot 1 = \frac{\sqrt{2}}{2}$

(b) $\left(\sec \frac{\pi}{4}\right)\left(\cot \frac{\pi}{4}\right) = \sqrt{2} \cdot 1 = \sqrt{2}$



2 Find the Exact Values of the Trigonometric Functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$

EXAMPLE

Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$

Find the exact values of the six trigonometric functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$.

Form a right triangle in which one of the angles is $\frac{\pi}{6} = 30°$. It then follows that the third angle is $\frac{\pi}{3} = 60°$. Figure 28(a) illustrates such a triangle with hypotenuse of length 2. Our problem is to determine a and b.

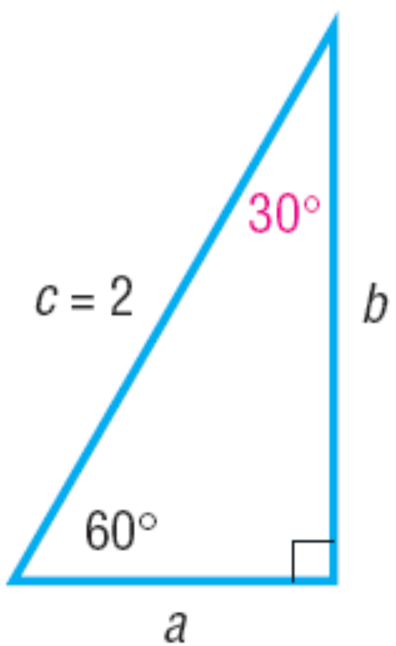


Figure 28 (a)

Make a second congruent triangle.

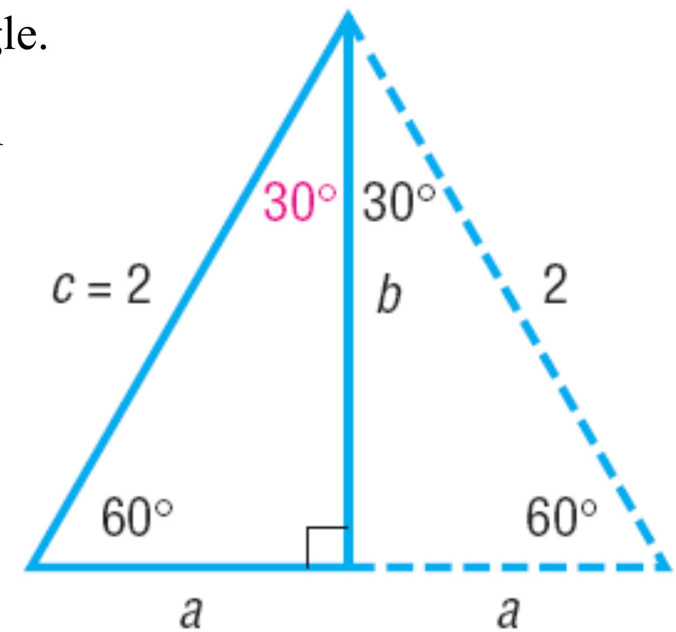
This triangle is equilateral so all sides are 2. 2a = 2 so a = 1.

Use Pythagorean Theorem to find b.

$$1^2 + b^2 = 2^2$$

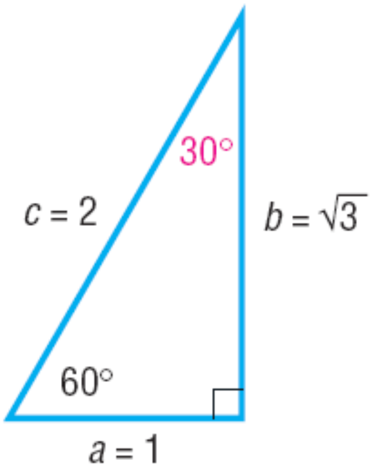
$$b^2 = 4 - 1 = 3$$

$$b = \sqrt{3}$$



EXAMPLE

Finding the Exact Values of the Trigonometric Functions of $\frac{\pi}{6} = 30°$ and $\frac{\pi}{3} = 60°$



$$\sin\frac{\pi}{6} = \sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \cos\frac{\pi}{3} = \cos 60° = \frac{1}{2}$$

$$\cos\frac{\pi}{6} = \cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{3} = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{6} = \tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\frac{\pi}{3} = \cot 60° = \frac{\sqrt{3}}{3}$$

$$\csc\frac{\pi}{6} = \csc 30° = \frac{1}{\sin 30°} = \frac{1}{\frac{1}{2}} = 2 \qquad \sec\frac{\pi}{3} = \sec 60° = 2$$

$$\sec\frac{\pi}{6} = \sec 30° = \frac{1}{\cos 30°} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \csc\frac{\pi}{3} = \csc 60° = \frac{2\sqrt{3}}{3}$$

$$\cot\frac{\pi}{6} = \cot 30° = \frac{1}{\tan 30°} = \frac{1}{\frac{\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \sqrt{3} \qquad \tan\frac{\pi}{3} = \tan 60° = \sqrt{3}$$

θ (Radians)	θ (Degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

EXAMPLE

Finding the Exact Value of a Trigonometric Expression

Find the exact value of each expression.

(a) $\sin^2\dfrac{\pi}{3}+\tan^2\dfrac{\pi}{4}$ (b) $\sin 60^\circ \cos 45^\circ$ (c) $\cos\dfrac{\pi}{3}-\cot\dfrac{\pi}{4}$

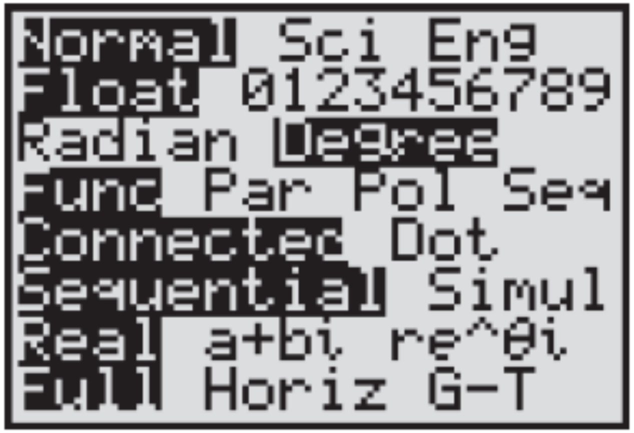
(a) $\sin^2\dfrac{\pi}{3}+\tan^2\dfrac{\pi}{4}=\left(\dfrac{\sqrt{3}}{2}\right)^2+(1)^2=\dfrac{3}{4}+1=\dfrac{7}{4}$

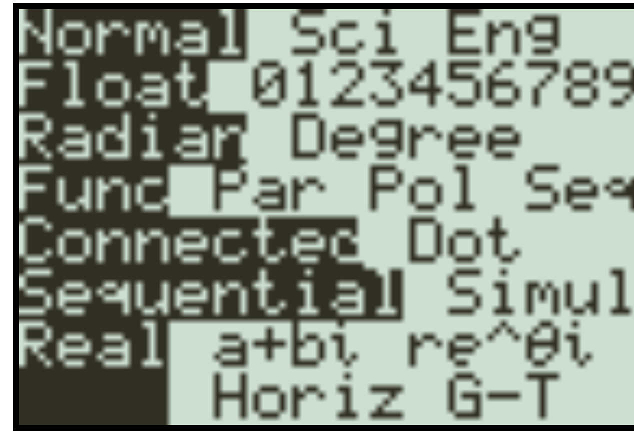
(b) $\sin 60^\circ \cos 45^\circ=\left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right)=\dfrac{\sqrt{6}}{4}$

(c) $\cos\dfrac{\pi}{3}-\cot\dfrac{\pi}{4}=\dfrac{1}{2}-1=-\dfrac{1}{2}$

3 Use a Calculator to Approximate the Values of Trigonometric Functions of Acute Angles

EXAMPLE

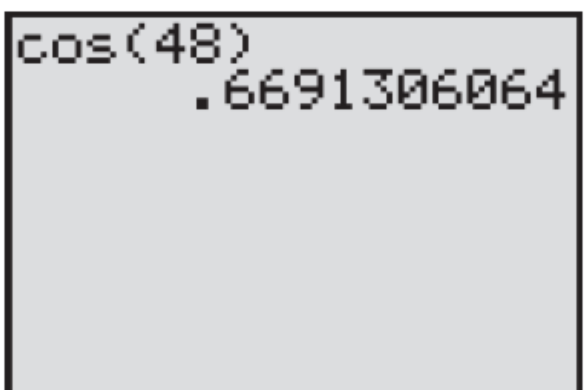


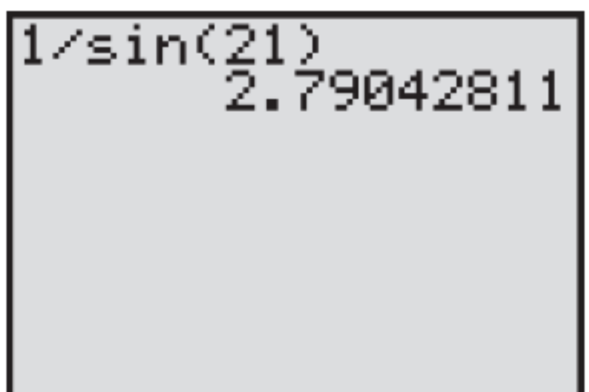


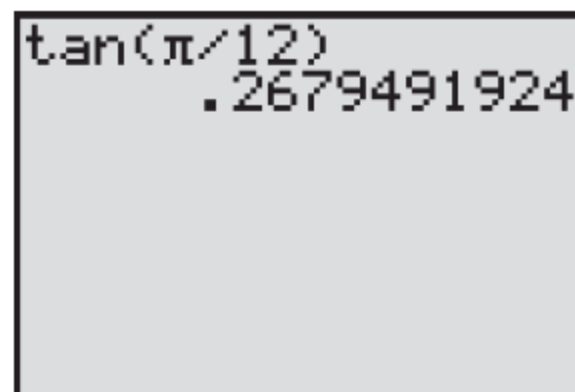
Using a Calculator to Approximate the Values of Trigonometric Functions

Use a calculator to find the approximate value of:

(a) $\cos 48^\circ$ (b) $\csc 21^\circ$ (c) $\tan \frac{\pi}{12}$







4 Model and Solve Applied Problems Involving Right Triangles

EXAMPLE Constructing a Rain Gutter

A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, the sides are bent up at an angle θ. See Figure 30.

(a) Express the area A of the opening as a function of θ.

$$A = \text{area of the two triangles} + \text{area of the rectangle}$$

$$A(\theta) = 16 \sin\theta \cos\theta + 16 \sin\theta = 16 \sin\theta(\cos\theta + 1)$$

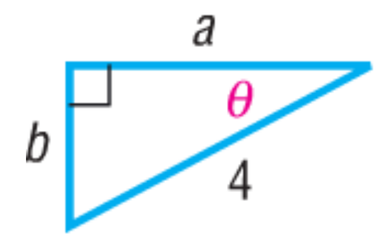
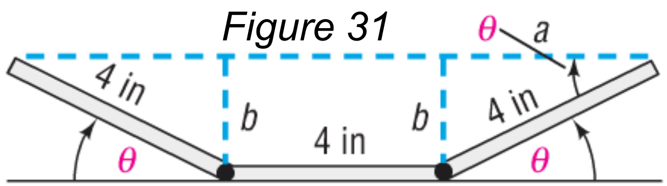


Figure 30

Figure 31



$$\text{area of rectangle} = 4b = 4(4 \sin\theta) = 16 \sin\theta$$

(a) Look again at Figure 30. The area A of the opening is the sum of the areas of two congruent right triangles and one rectangle. Look at Figure 31, which shows the triangle on the right in Figure 30 redrawn. We see that

$$\cos\theta = \frac{a}{4}, \quad \text{so} \quad a = 4\cos\theta \qquad \sin\theta = \frac{b}{4}, \quad \text{so} \quad b = 4\sin\theta$$

$$\text{area of triangle} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}ab = \frac{1}{2}(4\cos\theta)(4\sin\theta) = 8\sin\theta\cos\theta$$

EXAMPLE Constructing a Rain Gutter

(b) Find the area A of the opening for $\theta = 30°$, $\theta = 45°$, $\theta = 60°$, and $\theta = 75°$.

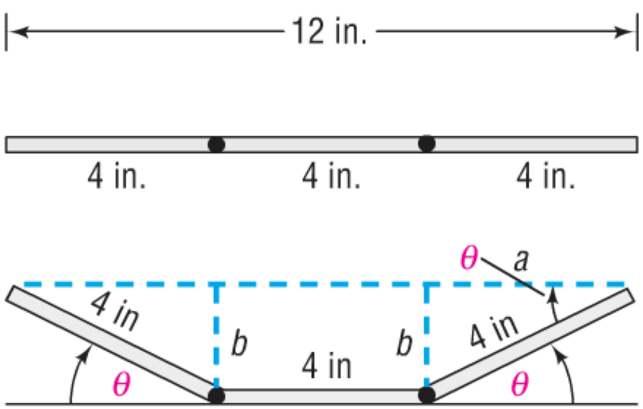
(c) Graph $A = A(\theta)$. Find the angle θ that makes A largest. (This bend will allow the most water to flow through the gutter.)

$$A(\theta) = 16 \sin \theta(\cos \theta + 1)$$

(b) For $\theta = 30°$: $\quad A(30°) = 16 \sin 30°(\cos 30° + 1)$

$$= 16\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} + 1\right) = 4\sqrt{3} + 8 \approx 14.93$$

The area of the opening for $\theta = 30°$ is about 14.93 square inches.



EXAMPLE Constructing a Rain Gutter

(b) Find the area A of the opening for $\theta = 30°$, $\theta = 45°$, $\theta = 60°$, and $\theta = 75°$.

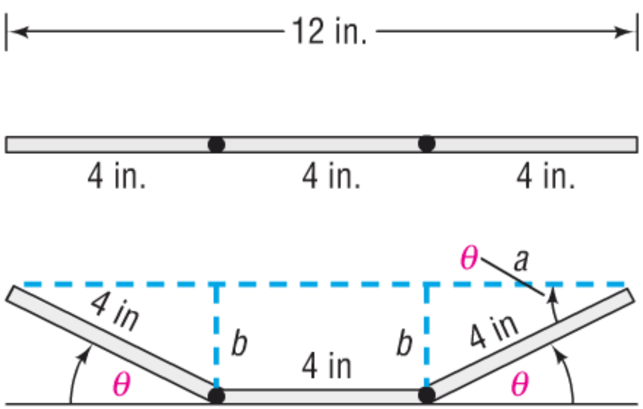
(c) Graph $A = A(\theta)$. Find the angle θ that makes A largest. (This bend will allow the most water to flow through the gutter.)

$$A(\theta) = 16 \sin \theta(\cos \theta + 1)$$

For $\theta = 45°$: $\quad A(45°) = 16 \sin 45°(\cos 45° + 1)$

$$= 16\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} + 1\right) = 8 + 8\sqrt{2} \approx 19.31$$

The area of the opening for $\theta = 45°$ is about 19.31 square inches.



EXAMPLE Constructing a Rain Gutter

(b) Find the area A of the opening for $\theta = 30°$, $\theta = 45°$, $\theta = 60°$, and $\theta = 75°$.

(c) Graph $A = A(\theta)$. Find the angle θ that makes A largest. (This bend will allow the most water to flow through the gutter.)

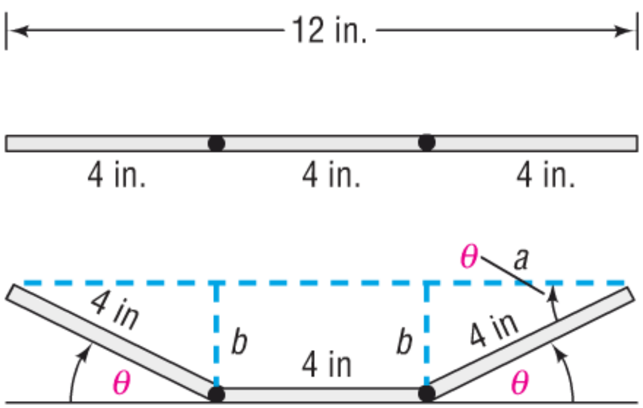
$$A(\theta) = 16 \sin \theta(\cos \theta + 1)$$

For $\theta = 60°$:

$$A(60°) = 16 \sin 60°(\cos 60° + 1)$$

$$= 16\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} + 1\right) = 12\sqrt{3} \approx 20.78$$

The area of the opening for $\theta = 60°$ is about 20.78 square inches.



EXAMPLE Constructing a Rain Gutter

(b) Find the area A of the opening for $\theta = 30°$, $\theta = 45°$, $\theta = 60°$, and $\theta = 75°$.

(c) Graph $A = A(\theta)$. Find the angle θ that makes A largest. (This bend will allow the most water to flow through the gutter.)

$$A(\theta) = 16 \sin \theta(\cos \theta + 1)$$

For $\theta = 75°$: $\quad A(75°) = 16 \sin 75°(\cos 75° + 1) \approx 19.45$

The area of the opening for $\theta = 75°$ is about 19.45 square inches.

(c) Figure 32 shows the graph of $A = A(\theta)$. Using MAXIMUM, the angle θ that makes A largest is $60°$.

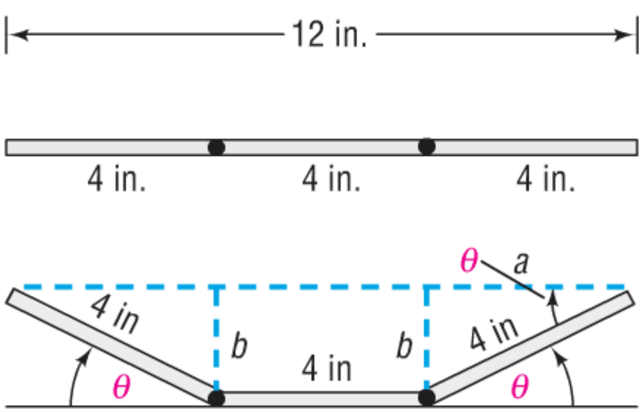
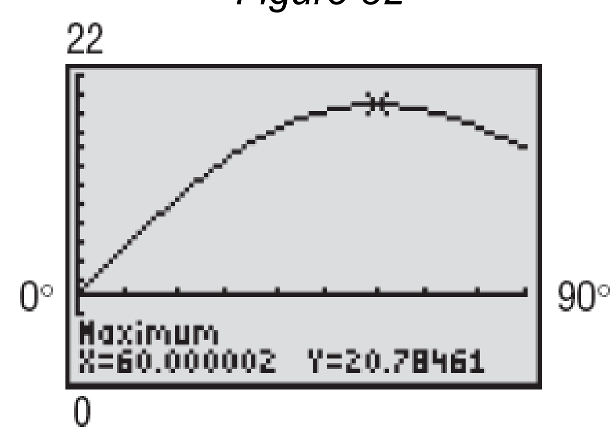


Figure 32



EXAMPLE Finding the Width of a River

A surveyor can measure the width of a river by setting up a transit* at a point C on one side of the river and taking a sighting of a point A on the other side. Refer to Figure 37. After turning through an angle of 90° at C, the surveyor walks a distance of 200 meters to point B. Using the transit at B, the angle β is measured and found to be 20°. What is the width of the river rounded to the nearest meter?

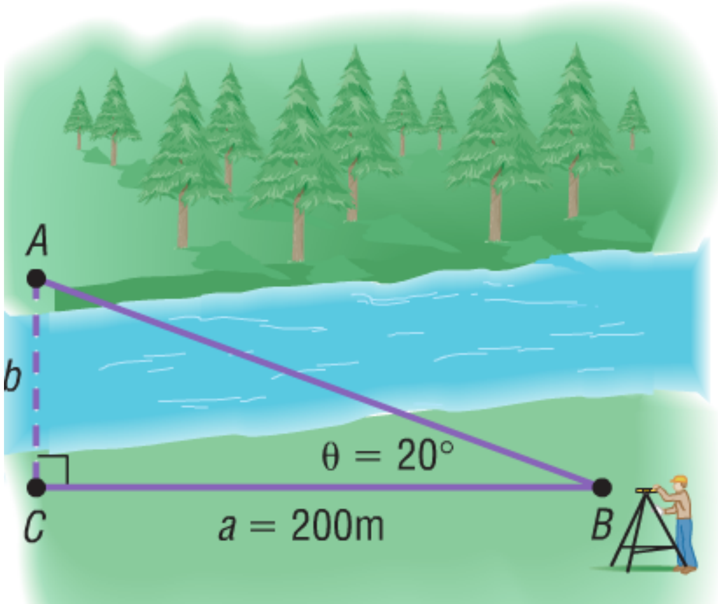


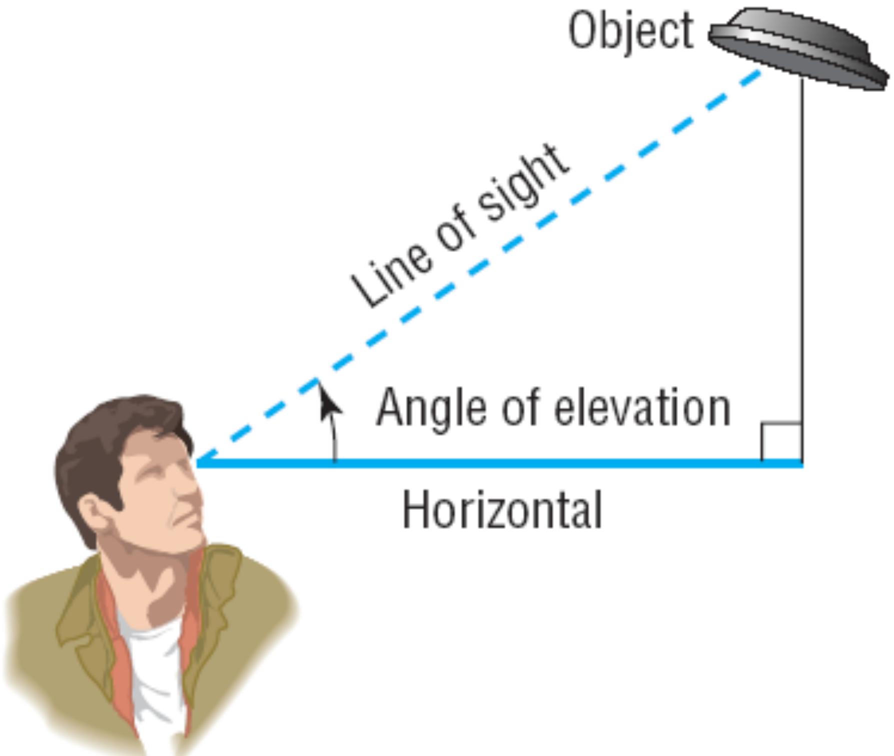
Figure 37

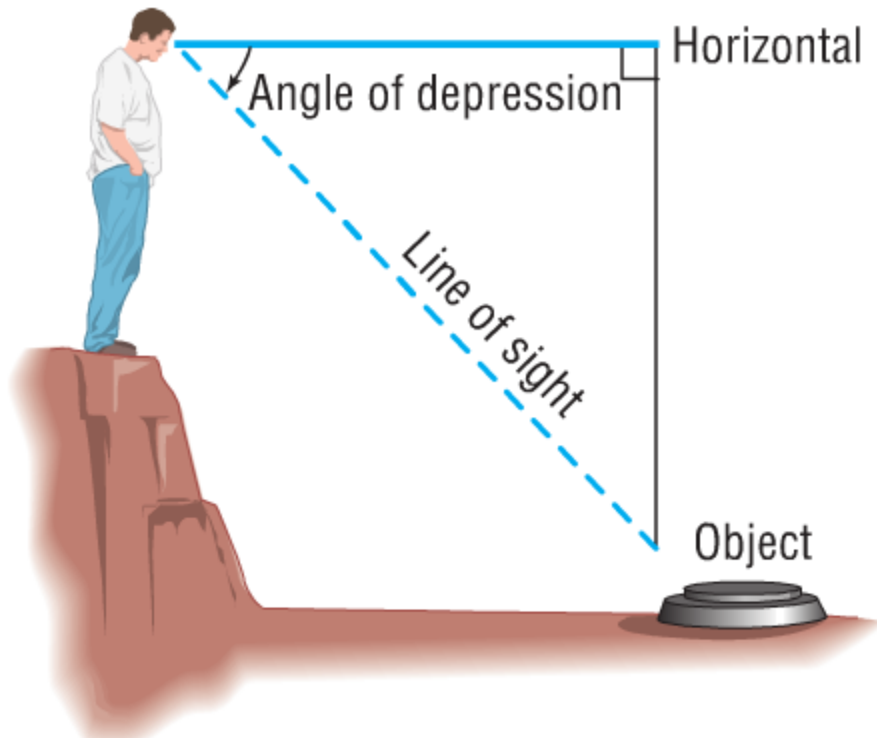
$$\tan \theta = \frac{b}{a}$$

$$\tan 20^\circ = \frac{b}{200}$$

$$b = 200 \tan 20^\circ \approx 72.79 \text{ meters}$$

The width of the river is 73 meters, rounded to the nearest meter.


Object
Line of sight
Angle of elevation
Horizontal


Horizontal
Angle of depression
Line of sight
Object

EXAMPLE Finding the Height of a Cloud

Meteorologists find the height of a cloud using an instrument called a **ceilometer**. A ceilometer consists of a **light projector** that directs a vertical light beam up to the cloud base and a **light detector** that scans the cloud to detect the light beam. See Figure 39(a). On December 1, 2004, at Midway Airport in Chicago, a ceilometer with a base of 300 feet was employed to find the height of the cloud cover. If the angle of elevation of the light detector is 75°, what is the height of the cloud cover?

$$\tan 75^\circ = \frac{h}{300}$$

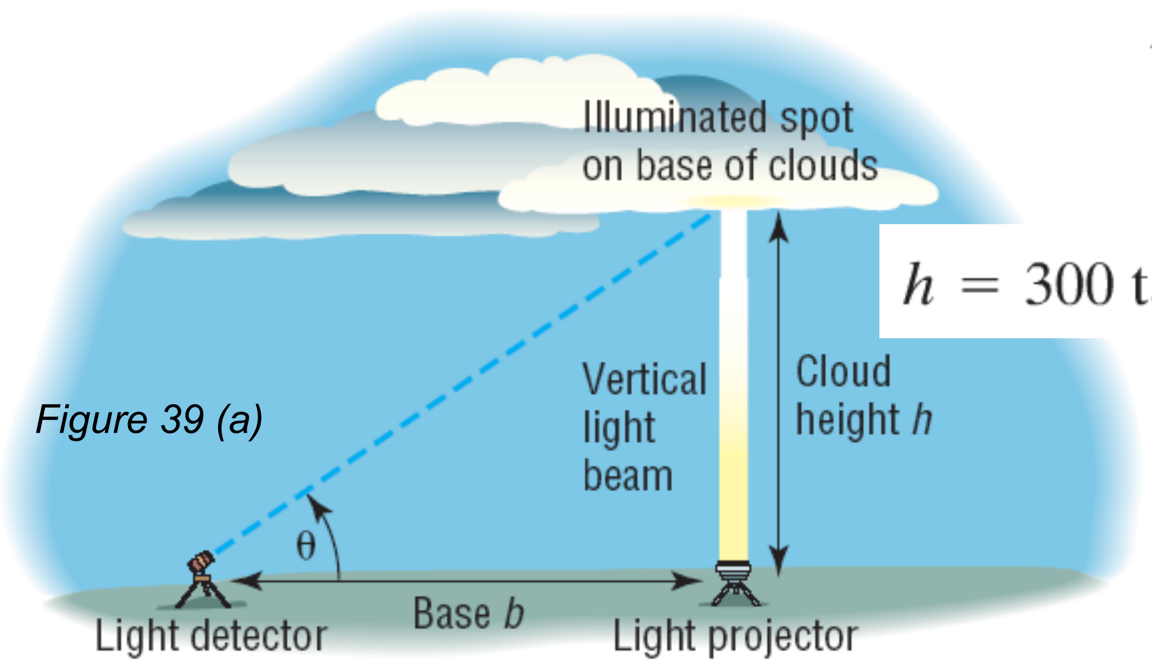


Figure 39 (a)

$$h = 300 \tan 75^\circ \approx 1120 \text{ feet}$$

The ceiling (height to the base of the cloud cover) was approximately 1120 feet.

EXAMPLE Finding the Height of a Statue on a Building

Adorning the top of the Board of Trade building in Chicago is a statue of Ceres, the Roman goddess of wheat. From street level, two observations are taken 400 feet from the center of the building. The angle of elevation to the base of the statue is found to be 55.1° and the angle of elevation to the top of the statue is 56.5°. See Figure 36(a). What is the height of the statue?

$$\tan 55.1^\circ = \frac{b}{400}$$

$$b = 400 \tan 55.1^\circ \approx 573.39$$

$$\tan 56.5^\circ = \frac{b'}{400}$$

$$b' = 400 \tan 56.5^\circ \approx 604.33$$

The height of the statue is approximately $604.33 - 573.39 = 30.94$ feet ≈ 31 feet.

Figure 36(a)