Section 12.2

Systems of Linear Equations: Matrices

$$\begin{cases} x + 4y = 14 \\ 3x - 2y = 0 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 4 & 14 \\ 3 & -2 & 0 \end{array}\right]$$

A **matrix** is defined as a rectangular array of numbers,

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} \text{Column } 1 & & \text{Column } 2 & & \text{Column } j & & \text{Column } n \end{array} \\
\begin{array}{c} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } i \\ \vdots \\ \text{Row } m \end{array} &
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}
\end{array}
$$

1 Write the Augmented Matrix of a System of Linear Equations

EXAMPLE

Writing the Augmented Matrix of a System of Linear Equations

(a) $\begin{cases} 3x - 2y = 3 \\ -2x + y = -2 \end{cases}$ $\quad \left[\begin{array}{cc|c} 3 & -2 & 3 \\ -2 & 1 & -2 \end{array}\right]$

(b) $\begin{cases} 3x - 2y + 5z = 0 \\ -2x + 4z + 2 = 0 \\ x + 4y - 7 = 0 \end{cases} = \begin{cases} 3x - 2y + 5z = 0 \\ -2x + 0y + 4z = -2 \\ x + 4y + 0z = 7 \end{cases}$

$$\left[\begin{array}{ccc|c} 3 & -2 & 5 & 0 \\ -2 & 0 & 4 & -2 \\ 1 & 4 & 0 & 7 \end{array}\right]$$

2 Write the System of Equations from the Augmented Matrix

EXAMPLE

Writing the System of Linear Equations from the Augmented Matrix

Write the system of linear equations 1 corresponding to each augmented matrix.

(a) $\left[\begin{array}{cc|c} -2 & 1 & 3 \\ 1 & 1 & -2 \end{array}\right]$

$$\begin{cases} -2x + y = 3 \\ x + y = -2 \end{cases}$$

(b) $\left[\begin{array}{ccc|c} 3 & -2 & 5 & 3 \\ 3 & -2 & 1 & -2 \\ 4 & -2 & 1 & 1 \end{array}\right]$

$$\begin{cases} 3x - 2y + 5z = 3 \\ 3x - 2y + z = -2 \\ 4x - 2y + z = 1 \end{cases}$$

3 Perform Row Operations on a Matrix

Row Operations

1. Interchange any two rows.
2. Replace a row by a nonzero multiple of that row.
3. Replace a row by the sum of that row and a constant nonzero multiple of some other row.

EXAMPLE

Applying a Row Operation to an Augmented Matrix

Apply the row operation $R_2 = -3r_1 + r_2$ to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right] \xrightarrow[R_2 = -3r_1 + r_2]{} \left[\begin{array}{cc|c} 1 & -2 & 2 \\ -3(1) + 3 & (-3)(-2) + (-5) & -3(2) + 9 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

EXAMPLE

Finding a Particular Row Operation

Find a row operation that will result in the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

having a 0 in row 1, column 2.

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 = 2r_2 + r_1} \left[\begin{array}{cc|c} 2(0)+1 & 2(1)+(-2) & 2(3)+2 \\ 0 & 1 & 3 \end{array}\right]$$

$R_1 = 2r_2 + r_1$

$$= \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 3 \end{array}\right]$$

4 Solve a System of Linear Equations Using Matrices

DEFINITION

A matrix is in **row echelon form** when

1. The entry in row 1, column **1** is a 1, and 0's appear below it.
2. The first nonzero entry in each row after the first row is a 1, 0's appear below it, and it appears to the right of the first nonzero entry in any row above.
3. Any rows that contain all 0's to the left of the vertical bar appear at the bottom.

EXAMPLE

How to Solve a System of Linear Equations Using Matrices

Solve: $\begin{cases} 2x + 2y = 6 & (1) \\ x + y + z = 1 & (2) \\ 3x + 4y - z = 13 & (3) \end{cases}$

Step 1: Write the augmented matrix that represents the system.

$$\left[\begin{array}{rrr|r} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

Step 2: Perform row operations that result in the entry in row 1, column 1 becoming 1.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

To get a 1 in row 1, column 1, interchange rows 1 and 2.

Step 3: Perform row operations that leave the entry in row 1, column 1 a 1, while causing the entries in column 1 below row 1 to become 0's.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right]$$

$R_2 = -2r_1 + r_2$

$R_3 = -3r_1 + r_3$

EXAMPLE

How to Solve a System of Linear Equations Using Matrices

Solve: $\begin{cases} 2x + 2y = 6 & (1) \\ x + y + z = 1 & (2) \\ 3x + 4y - z = 13 & (3) \end{cases}$

Step 4: Perform row operations that result in the entry in row 2, column 2 becoming 1 with 0's below it.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

Step 5: Repeat Step 4, placing a 1 in row 3, column 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Step 6: The matrix on the right in Step 5 is the row echelon form of the augmented matrix. Use back-substitution to solve the original system.

$z = -2 \qquad y - 4(-2) = 10 \qquad y = 2$

$$x + y + z = 1$$

$$x + 2 + (-2) = 1 \qquad x = 1$$

The solution of the system is $x = 1$, $y = 2$, $z = -2$ or, using an ordered triplet, $(1, 2, -2)$.

Matrix Method for Solving a System of Linear Equations (Row Echelon Form)

STEP 1: Write the augmented matrix that represents the system.

STEP 2: Perform row operations that place the entry 1 in row 1, column 1.

STEP 3: Perform row operations that leave the entry 1 in row 1, column 1 unchanged, while causing 0's to appear below it in column 1.

STEP 4: Perform row operations that place the entry 1 in row 2, column 2, but leave the entries in columns to the left unchanged. If it is impossible to place a 1 in row 2, column 2, proceed to place a 1 in row 2, column 3. Once a 1 is in place, perform row operations to place 0's below it. (Place any rows that contain only 0's on the left side of the vertical bar, at the bottom of the matrix.)

STEP 5: Now repeat Step 4, placing a 1 in the next row, but one column to the right. Continue until the bottom row or the vertical bar is reached.

STEP 6: The matrix that results is the row echelon form of the augmented matrix. Analyze the system of equations corresponding to it to solve the original system.

EXAMPLE Solving a System of Linear Equations Using Matrices (Row Echelon Form)

Solve: $\begin{cases} x - y + z = 8 \\ 2x + 3y - z = -2 \\ 3x - 2y - 9z = 9 \end{cases}$ $\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right]$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{array}\right]$$

$R_2 = -2r_1 + r_2$
$R_3 = -3r_1 + r_3$

$$\to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{array}\right]$$

$R_3 = -5r_2 + r_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{array}\right] \xrightarrow[R_3 = \frac{1}{57} r_3]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{cases} x - y + z = 8 & (1) \\ y - 12z = -15 & (2) \\ z = 1 & (3) \end{cases}$$

Using $z = 1$, we back-substitute to get

$$\begin{cases} x - y + 1 = 8 & (1) \\ y - 12(1) = -15 & (2) \end{cases} \xrightarrow[\text{Simplify.}]{} \begin{cases} x - y = 7 & (1) \\ y = -3 & (2) \end{cases}$$

We get $y = -3$ and, back-substituting into $x - y = 7$, we find that $x = 4$. The solution of the system is $x = 4$, $y = -3$, $z = 1$ or, using an ordered triplet, $(4, -3, 1)$.

Reduced Row Echelon Form

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

EXAMPLE **Solving a Dependent System of Linear Equations Using Matrices**

Solve: $\begin{cases} 6x - y - z = 4 \\ -12x + 2y + 2z = -8 \\ 5x + y - z = 3 \end{cases}$

$$\left[\begin{array}{rrr|r} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 = -1r_3 + r_1} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \xrightarrow[R_3 = -5r_1 + r_3]{R_2 = 12r_1 + r_2} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 = -\frac{1}{22}r_2} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array}\right] \xrightarrow{R_3 = -11r_2 + r_3} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

EXAMPLE Solving a Dependent System of Linear Equations Using Matrices

Solve: $\begin{cases} 6x - y - z = 4 \\ -12x + 2y + 2z = -8 \\ 5x + y - z = 3 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x - 2y = 1 & (1) \\ y - \frac{1}{11}z = -\frac{2}{11} & (2) \end{cases}$$

$$y = \frac{1}{11}z - \frac{2}{11}$$

$$x = 2y + 1 = 2\left(\frac{1}{11}z - \frac{2}{11}\right) + 1 = \frac{2}{11}z + \frac{7}{11}$$

$$\left\{(x, y, z) \middle| x = \frac{2}{11}z + \frac{7}{11},\ y = \frac{1}{11}z - \frac{2}{11},\ z \text{ any real number}\right\}$$

EXAMPLE Solving an Inconsistent System of Linear Equations Using Matrices

Solve: $\begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 0 & 1 & 1 & -6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & -3 & -3 & -9 \end{array}\right]$

$R_2 = -2r_1 + r_2$
$R_3 = -1r_1 + r_3$

Interchange rows 2 and 3.

$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & -27 \end{array}\right]$

$R_3 = 3r_2 + r_3$

The system is inconsistent.

EXAMPLE Solving a System of Linear Equations Using Matrices

Solve: $\begin{cases} x - 2y + z = 0 \\ 2x + 2y - 3z = -3 \\ y - z = -1 \\ -x + 4y + 2z = 13 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 2 & -3 & -3 \\ 0 & 1 & -1 & -1 \\ -1 & 4 & 2 & 13 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 6 & -5 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 3 & 13 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 6 & -5 & -3 \\ 0 & 2 & 3 & 13 \end{array}\right]$$

$R_2 = -2r_1 + r_2$
$R_4 = r_1 + r_4$

Interchange rows 2 and 3.

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 5 & 15 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 = -6r_2 + r_3$
$R_4 = -2r_2 + r_4$

$R_4 = -5r_3 + r_4$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 = 2r_2 + r_1]{} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 = r_3 + r_1 \\ R_2 = r_3 + r_2}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x = 1,\ y = 2,\ z = 3$ or, using an ordered triplet, $(1, 2, 3)$

EXAMPLE Financial Planning

Adam and Michelle require an additional $25,000 in annual income (beyond their pension benefits). They are rather risk averse and have narrowed their investment choices down to Treasury notes that yield 3%, Treasury bonds that yield 5%, or corporate bonds that yield 6%. If they have $600,000 to invest and want the amount invested in Treasury notes to equal the total amount invested in Treasury bonds and corporate bonds, how much should be placed in each investment?

Let n, b, and c represent the amounts invested in Treasury notes, Treasury bonds, and corporate bonds, respectively. There is a total of $600,000 to invest, which means that the sum of the amounts invested in Treasury notes, Treasury bonds, and corporate bonds should equal $600,000. The first equation is

$$n + b + c = 600{,}000 \quad (1)$$

If $100,000 was invested in Treasury notes, the income would be 0.03($100,000) = $3000. In general, if n dollars was invested in Treasury notes, the income would be $0.03n$. Since the total income is to be $25,000, the second equation is

$$0.03n + 0.05b + 0.06c = 25{,}000 \quad (2)$$

The amount invested in Treasury notes should equal the amount invested in Treasury bonds and corporate bonds, so the third equation is

$$n = b + c \quad \text{or} \quad n - b - c = 0 \quad (3)$$

$$\begin{cases} n + b + c = 600{,}000 & (1) \\ 0.03n + 0.05b + 0.06c = 25{,}000 & (2) \\ n - b - c = 0 & (3) \end{cases} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 0.03 & 0.05 & 0.06 & 25{,}000 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 = -0.03r_1 + r_2 \\ R_3 = -r_1 + r_2}]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 0 & 0.02 & 0.03 & 7000 \\ 0 & -2 & -2 & -600{,}000 \end{array}\right] \xrightarrow[R_2 = \frac{1}{0.02}r_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 0 & 1 & 1.5 & 350{,}000 \\ 0 & -2 & -2 & -600{,}000 \end{array}\right]$$

$$\xrightarrow{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 0 & 1 & 1.5 & 350{,}000 \\ 0 & 0 & 1 & 100{,}000 \end{array}\right] \qquad \begin{cases} n + b + c = 600{,}000 & (1) \\ b + 1.5c = 350{,}000 & (2) \\ c = 100{,}000 & (3) \end{cases}$$

From equation (3), we determine that Adam and Michelle should invest \$100,000 in corporate bonds. Back-substitute \$100,000 into equation (2) to find that $b = 200{,}000$, so Adam and Michelle should invest \$200,000 in Treasury bonds. Back-substitute these values into equation (1) and find that $n = \$300{,}000$, so \$300,000 should be invested in Treasury notes.