Section 8.5

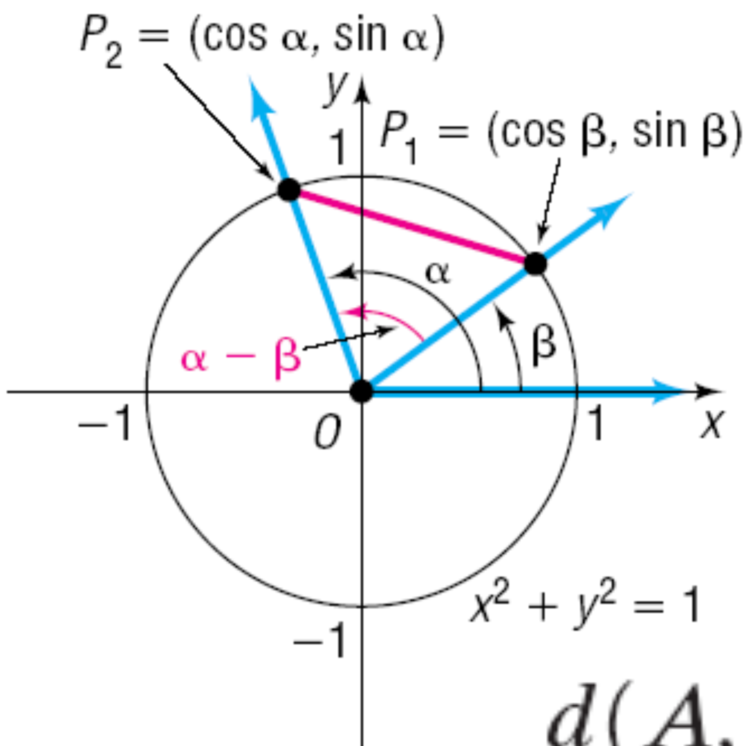
Sum and Difference Formulas

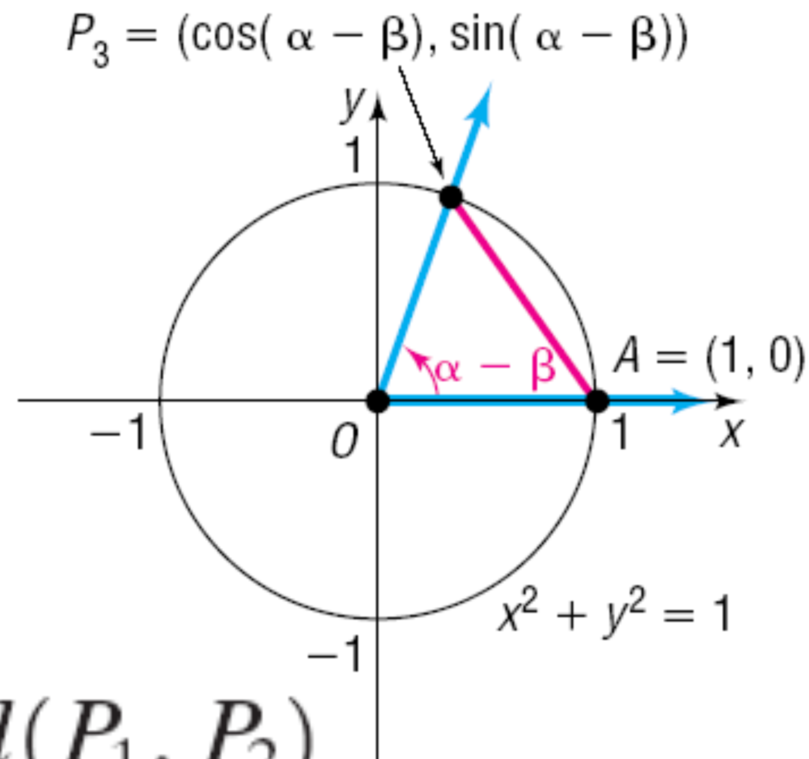
THEOREM

Sum and Difference Formulas for Cosines

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$





$$d(A, P_3) = d(P_1, P_2)$$

1 Use Sum and Difference Formulas to Find Exact Values

EXAMPLE

Using the Sum Formula to Find an Exact Value

Find the exact value of $\cos\frac{7\pi}{12}$.

$$\cos\frac{7\pi}{12} = \cos\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$

$$= \cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\sin\frac{\pi}{3} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

EXAMPLE

Using the Difference Formula to Find an Exact Value

Find the exact value of $\cos 15°$.

$$= \cos\left(45° - 30°\right) = \cos 45° \cos 30° + \sin 45° \sin 30°$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\boxed{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

2 Use Sum and Difference Formulas to Establish Identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

Seeing the Concept

Graph $Y_1 = \cos\left(\frac{\pi}{2} - x\right)$ and $Y_2 = \sin x$ on the same screen. Does this demonstrate the result 3(a)? How would you demonstrate the result 3(b)?

THEOREM

Sum and Difference Formulas for Sines

$$\boxed{\begin{aligned} \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}}$$

EXAMPLE

Using the Sum Formula to Find an Exact Value

Find the exact value of $\sin\frac{19\pi}{12}$.

$$\sin\frac{19\pi}{12} = \sin\left(\frac{16\pi}{12} + \frac{3\pi}{12}\right) = \sin\left(\frac{4\pi}{3} + \frac{\pi}{4}\right)$$

$$= \sin\frac{4\pi}{3}\cos\frac{\pi}{4} + \cos\frac{4\pi}{3}\sin\frac{\pi}{4} = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{-\sqrt{6} - \sqrt{2}}{4}$$

$$\boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

EXAMPLE

Using the Difference Formula to Find an Exact Value

Find the exact value of $\cos 40° \cos 80° - \sin 40° \sin 80°.$

$$\cos 40° \cos 80° - \sin 40° \sin 80° = \cos\left(40° + 80°\right) = \cos 120°$$

$$= -\frac{1}{2}$$

$$\boxed{\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta}$$

EXAMPLE Finding Exact Values

If it is known that $\sin\alpha = \frac{3}{5}$, $\frac{\pi}{2} < \alpha < \pi$, and that $\sin\beta = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$, $\frac{3\pi}{2} < \beta < 2\pi$, find the exact value of

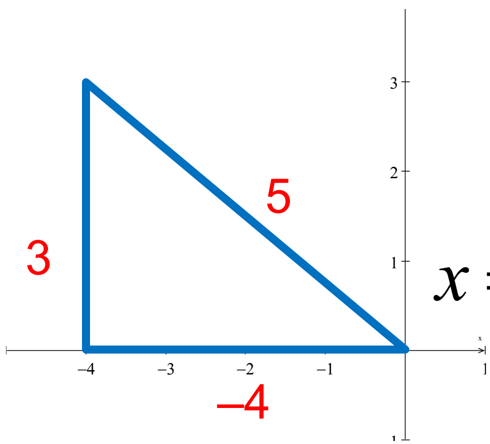
(a) $\cos\alpha$ (b) $\cos\beta$ (c) $\cos(\alpha+\beta)$

(a) $\cos\alpha = \frac{x}{r} = -\frac{4}{5}$

(b) $\cos\beta = \frac{x}{r} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

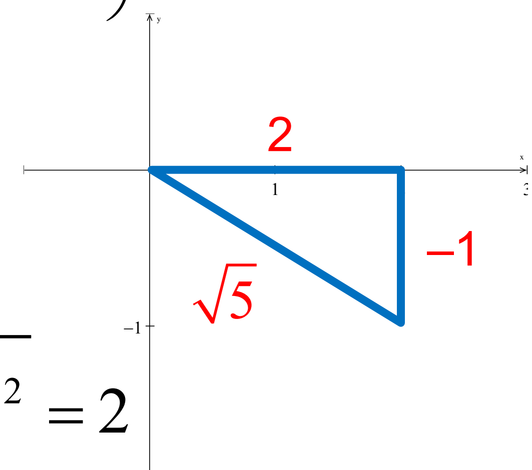
(c) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{4}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) - \left(\frac{3}{5}\right)\left(-\frac{\sqrt{5}}{5}\right) = -\frac{5\sqrt{5}}{25} = -\frac{\sqrt{5}}{5}$$



$$x = \sqrt{5^2 - 3^2} = 4$$

$$x = \sqrt{\left(\sqrt{5}\right)^2 - 1^2} = 2$$



EXAMPLE Establishing an Identity

Establish the identity: $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$

$$\cos\left(\frac{\pi}{2}+\theta\right)=\cos\frac{\pi}{2}\cos\theta-\sin\frac{\pi}{2}\sin\theta$$

$$=(0)\cos\theta-(1)\sin\theta=-\sin\theta$$

THEOREM

Sum and Difference Formulas for Tangents

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

EXAMPLE Establishing an Identity

Prove the identity: $\tan\left(2\pi-\theta\right)=-\tan\theta$

$$\tan\left(2\pi-\theta\right)=\frac{\tan 2\pi-\tan\theta}{1+\tan 2\pi\tan\theta}$$

$$=\frac{0-\tan\theta}{1+\left(0\right)\tan\theta}=-\tan\theta$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

EXAMPLE **Establishing an Identity**

Prove the identity: $\tan\left(\frac{\pi}{4}+\theta\right)=\cot\left(\frac{\pi}{4}-\theta\right)$

$$\tan\left(\frac{\pi}{4}+\theta\right)=\frac{\sin\left(\frac{\pi}{4}+\theta\right)}{\cos\left(\frac{\pi}{4}+\theta\right)}=\frac{\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta}{\cos\frac{\pi}{4}\cos\theta-\sin\frac{\pi}{4}\sin\theta}$$

$$=\frac{\frac{\sqrt{2}}{2}\cos\theta+\frac{\sqrt{2}}{2}\sin\theta}{\frac{\sqrt{2}}{2}\cos\theta-\frac{\sqrt{2}}{2}\sin\theta}=\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$$

This equals the left hand side

$$\cot\left(\frac{\pi}{4}-\theta\right)=\frac{\cos\left(\frac{\pi}{4}-\theta\right)}{\sin\left(\frac{\pi}{4}-\theta\right)}=\frac{\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta}{\sin\frac{\pi}{4}\cos\theta-\cos\frac{\pi}{4}\sin\theta}=\frac{\frac{\sqrt{2}}{2}\cos\theta+\frac{\sqrt{2}}{2}\sin\theta}{\frac{\sqrt{2}}{2}\cos\theta-\frac{\sqrt{2}}{2}\sin\theta}$$

3 Use Sum and Difference Formulas Involving Inverse Trigonometric Functions

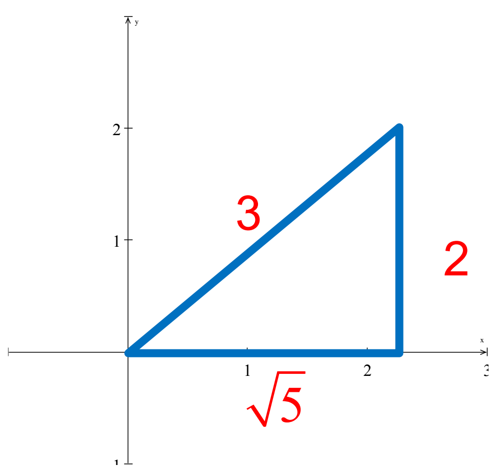
EXAMPLE

Finding the Exact Value of an Expression Involving Inverse Trigonometric Functions

Find the exact value of: $\cos\left(\sin^{-1}\frac{2}{3}+\tan^{-1}\left(-\frac{3}{4}\right)\right)=\cos(\alpha+\beta)$

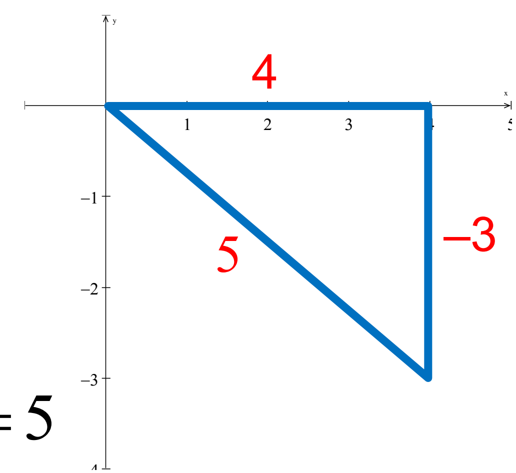
$$\sin\alpha=\frac{2}{3}, 0\le\alpha\le\frac{\pi}{2} \qquad \tan\beta=-\frac{3}{4}, -\frac{\pi}{2}\le\beta\le 0$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=\left(\frac{\sqrt{5}}{3}\right)\left(\frac{4}{5}\right)-\left(\frac{2}{3}\right)\left(-\frac{3}{5}\right)=\frac{4\sqrt{5}+6}{15}$$



$$x=\sqrt{3^2-2^2}=\sqrt{5}$$

$$r=\sqrt{3^2+4^2}=5$$



EXAMPLE **Writing a Trigonometric Expression as an Algebraic Expression**

Write $\sin(\sin^{-1} u + \cos^{-1} v)$ as an algebraic expression containing u and v (that is, without any trigonometric functions). Give the restrictions on u and v.

First, for $\sin^{-1} u$, we have $-1 \leq u \leq 1$, and for $\cos^{-1} v$, we have $-1 \leq v \leq 1$. Now let $\alpha = \sin^{-1} u$ and $\beta = \cos^{-1} v$. Then

$$\sin\alpha = u \qquad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \quad -1 \leq u \leq 1$$

$$\cos\beta = v \qquad 0 \leq \beta \leq \pi \quad -1 \leq v \leq 1$$

Since $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, we know that $\cos\alpha \geq 0$. As a result,

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - u^2}$$

Similarly, since $0 \leq \beta \leq \pi$, we know that $\sin\beta \geq 0$. Then

$$\sin\beta = \sqrt{1 - \cos^2\beta} = \sqrt{1 - v^2}$$

As a result,

$$\begin{aligned}\sin(\sin^{-1} u + \cos^{-1} v) &= \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= uv + \sqrt{1 - u^2}\sqrt{1 - v^2}\end{aligned}$$

4 Solve Trigonometric Equations Linear in Sine and Cosine

EXAMPLE

Solving a Trigonometric Equation Linear in $\sin\theta$ and $\cos\theta$

Solve the equation: $\sin\theta + \cos\theta = 1, \quad 0 \le \theta < 2\pi$

$(\sin\theta + \cos\theta)^2 = 1$ Square each side.

$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1$ Remove parentheses.

$2\sin\theta\cos\theta = 0$ $\sin^2\theta + \cos^2\theta = 1$ $\sin\theta\cos\theta = 0$

Setting each factor equal to zero, we obtain

$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = 0$$

The apparent solutions are

$$\theta = 0, \qquad \theta = \pi, \qquad \theta = \frac{\pi}{2}, \qquad \theta = \frac{3\pi}{2}$$

EXAMPLE

Solving a Trigonometric Equation Linear in sin θ and cos θ

Solve the equation: $\sin\theta + \cos\theta = 1, \quad 0 \le \theta < 2\pi$

Because we squared both sides of the original equation, we must check these apparent solutions to see if any are extraneous.

$$\theta = 0: \quad \sin 0 + \cos 0 = 0 + 1 = 1 \qquad \text{A solution}$$

$$\theta = \pi: \quad \sin\pi + \cos\pi = 0 + (-1) = -1 \qquad \text{Not a solution}$$

$$\theta = \frac{\pi}{2}: \quad \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1 \qquad \text{A solution}$$

$$\theta = \frac{3\pi}{2}: \quad \sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = -1 + 0 = -1 \qquad \text{Not a solution}$$

The values $\theta = \pi$ and $\theta = \frac{3\pi}{2}$ are extraneous. The solution set is $\left\{0, \frac{\pi}{2}\right\}$.

SUMMARY Sum and Difference Formulas

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$