Section 10.5

The Dot Product

1 Find the Dot Product of Two Vectors

If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$ are two vectors, the **dot product** $\mathbf{v} \cdot \mathbf{w}$ is defined as

$$\boxed{\mathbf{v} \cdot \mathbf{w} = a_1a_2 + b_1b_2 \qquad \textbf{(1)}}$$

EXAMPLE Finding Dot Products

If $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} - 5\mathbf{j}$, find:

(a) $\mathbf{v} \cdot \mathbf{w}$ (b) $\mathbf{w} \cdot \mathbf{v}$ (c) $\mathbf{v} \cdot \mathbf{v}$

(d) $\mathbf{w} \cdot \mathbf{w}$ (e) $\|\mathbf{v}\|$ (f) $\|\mathbf{w}\|$

(a) $\mathbf{v} \bullet \mathbf{w} = -3(2) + 4(-5) = -6 - 20 = -26$

(b) $\mathbf{w} \bullet \mathbf{v} = 2(-3) - 5(4) = -6 - 20 = -26$

(c) $\mathbf{v} \bullet \mathbf{v} = -3(-3) + 4(4) = 9 + 16 = 25$

(d) $\mathbf{w} \bullet \mathbf{w} = 2(2) - 5(-5) = 4 + 25 = 29$

(e) $\|\mathbf{v}\| = \sqrt{(-3)^2 + (4)^2} = \sqrt{9 + 16} = 5$

(f) $\|\mathbf{w}\| = \sqrt{2^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}$

Properties of the Dot Product

If **u**, **v**, and **w** are vectors, then

Commutative Property

$$\boxed{\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}}$$

Distributive Property

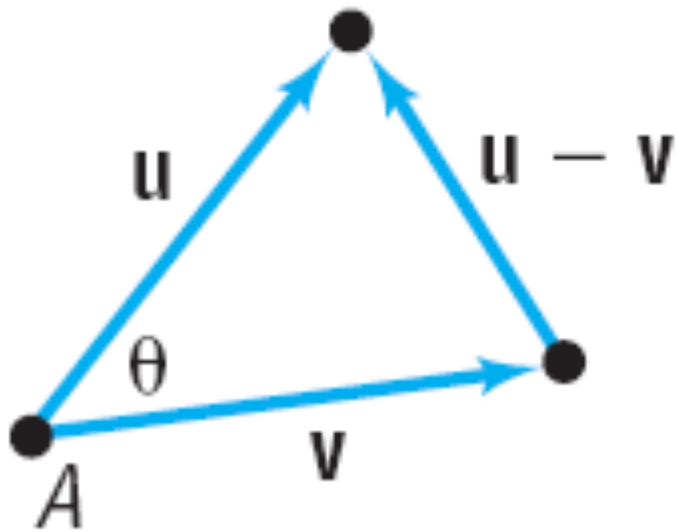
$$\boxed{\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}}$$

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$

$$\mathbf{0} \cdot \mathbf{v} = 0$$

2 Find the Angle between Two Vectors

The Law of Cosines



$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$$

Theorem

Angle between Vectors

If **u** and **v** are two nonzero vectors, the angle θ, $0 \leq \theta \leq \pi$, between **u** and **v** is determined by the formula

$$\boxed{\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}}$$

EXAMPLE Finding the Angle θ between Two Vectors

Find the angle θ between $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

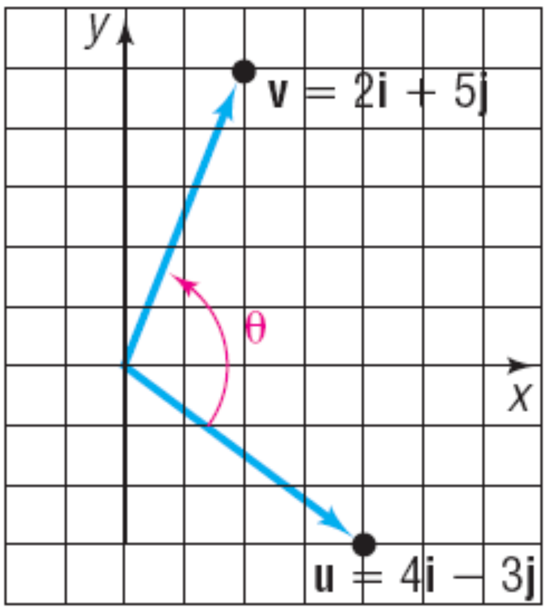
$$\mathbf{u} \cdot \mathbf{v} = 4(2) + (-3)(5) = -7$$

$$\|\mathbf{u}\| = \sqrt{4^2 + (-3)^2} = 5$$

$$\|\mathbf{v}\| = \sqrt{2^2 + 5^2} = \sqrt{29}$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-7}{5\sqrt{29}} \approx -0.26$$

We find that $\theta \approx 105°$.



$$\boxed{\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}}$$

3 Determine Whether Two Vectors Are Parallel

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be **parallel** if there is a nonzero scalar α so that $\mathbf{v} = \alpha\mathbf{w}$. In this case, the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is 0 or π.

EXAMPLE

Determining Whether Vectors Are Parallel

The vectors $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 2\mathbf{j}$ are parallel, since $\mathbf{v} = \frac{1}{2}\mathbf{w}$. Furthermore, since

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{18 + 2}{\sqrt{10}\,\sqrt{40}} = \frac{20}{\sqrt{400}} = 1$$

the angle θ between $\mathbf{v}$ and $\mathbf{w}$ is 0.

4 Determine Whether Two Vectors Are Orthogonal

If the angle θ between two nonzero vectors **v** and **w** is $\frac{\pi}{2}$, the vectors **v** and **w** are called **orthogonal**.

v is orthogonal to w.

Theorem

Two vectors **v** and **w** are orthogonal if and only if

$$\boxed{\mathbf{v} \cdot \mathbf{w} = 0}$$

EXAMPLE

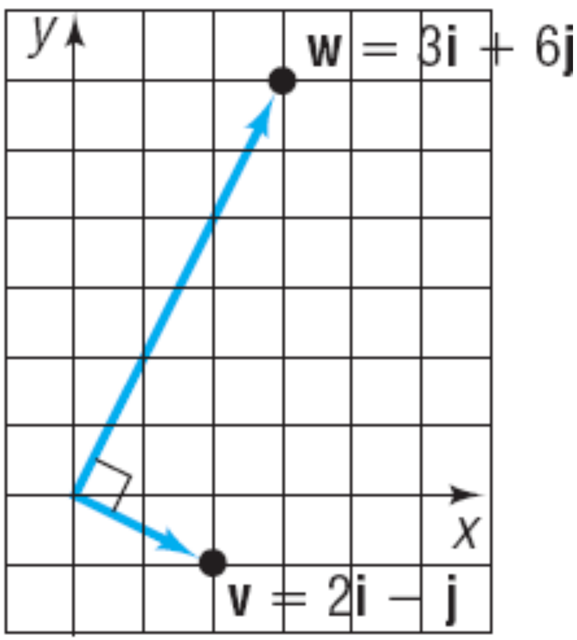
Determining Whether Two Vectors Are Orthogonal

The vectors

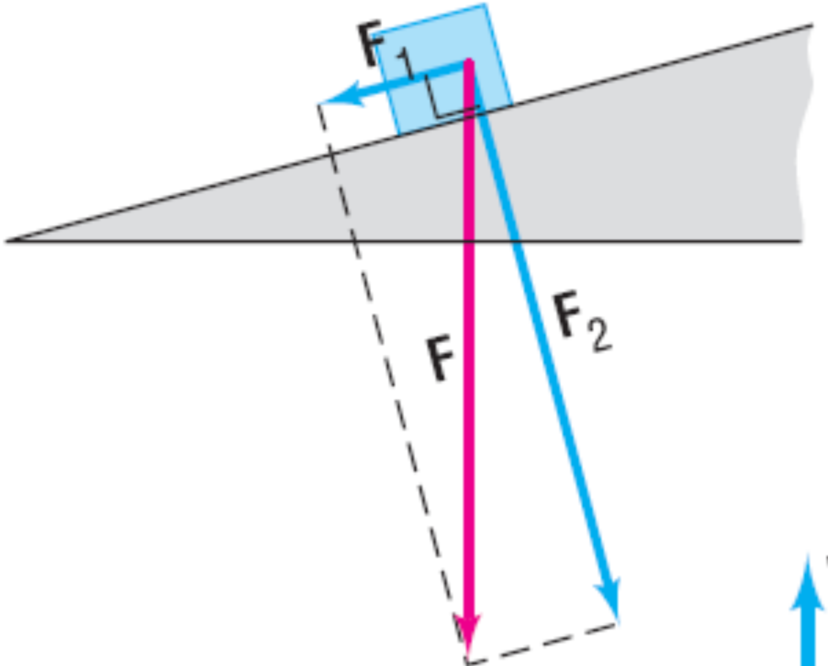
$$\mathbf{v} = 2\mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{w} = 3\mathbf{i} + 6\mathbf{j}$$

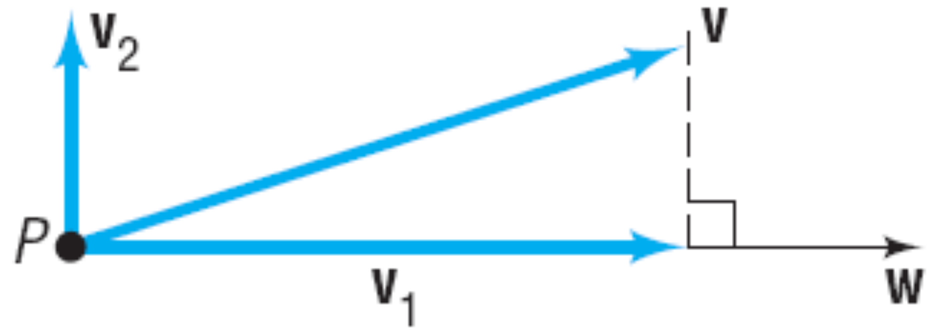
are orthogonal, since

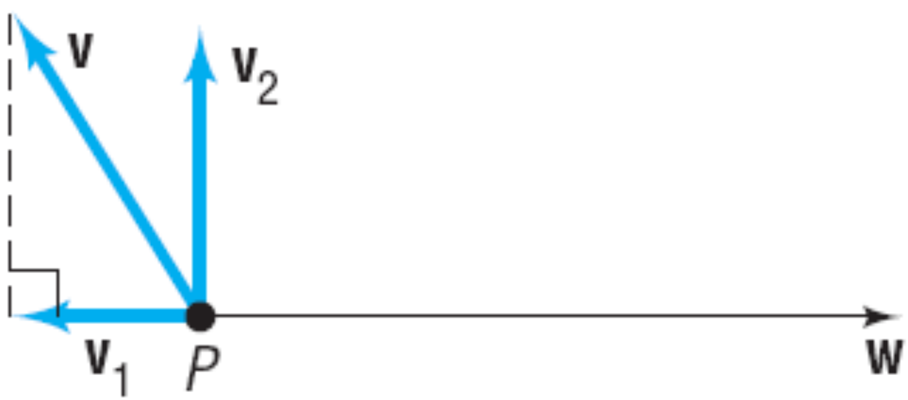
$$\mathbf{v} \cdot \mathbf{w} = 6 - 6 = 0$$



5 Decompose a Vector into Two Orthogonal Vectors


F1
F
F2


v2
v
P
v1
w


v
v2
v1
P
w

THEOREM

If **v** and **w** are two nonzero vectors, the vector projection of **v** onto **w** is

$$\boxed{\mathbf{v}_1 = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}}$$

The decomposition of **v** into $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to **w** and $\mathbf{v}_2$ is perpendicular to **w**, is

$$\boxed{\mathbf{v}_1 = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} \qquad \mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1}$$

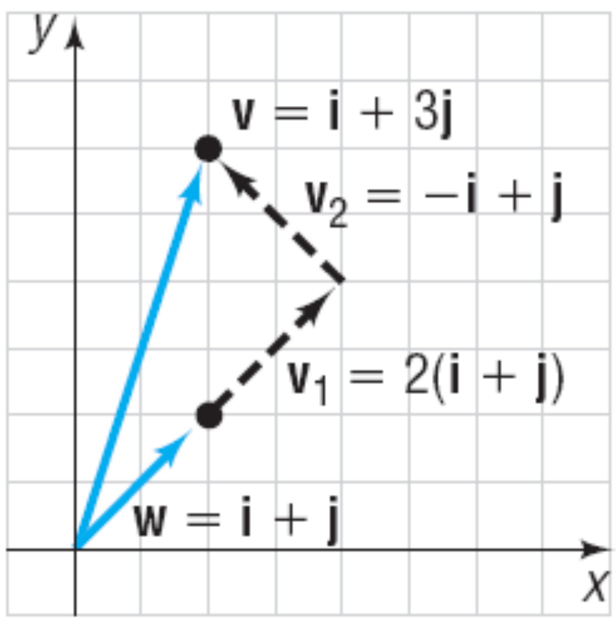
EXAMPLE

Decomposing a Vector into Two Orthogonal Vectors

Find the vector projection of $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ onto $\mathbf{w} = \mathbf{i} + \mathbf{j}$. Decompose $\mathbf{v}$ into two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.

$$\mathbf{v}_1 = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{1 + 3}{\left(\sqrt{2}\right)^2}\mathbf{w} = 2\mathbf{w} = 2(\mathbf{i} + \mathbf{j})$$

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (\mathbf{i} + 3\mathbf{j}) - 2(\mathbf{i} + \mathbf{j}) = -\mathbf{i} + \mathbf{j}$$



EXAMPLE

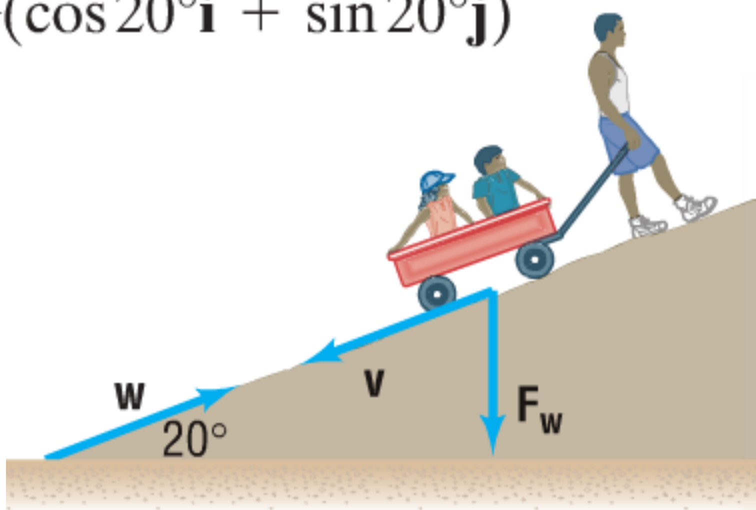
Finding the Force Required to Hold a Wagon on a Hill

A wagon with two small children as occupants that weighs 100 pounds is on a hill with a grade of 20°. What is the magnitude of the force that is required to keep the wagon from rolling down the hill?

$$\mathbf{F}_{\mathrm{w}} = -100\mathbf{j} \qquad \mathbf{w} = \cos 20°\mathbf{i} + \sin 20°\mathbf{j}$$

$$\mathbf{v} = \frac{\mathbf{F}_{\mathrm{w}} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{-100(\sin 20°)}{\left(\sqrt{\cos^2 20° + \sin^2 20°}\right)^2}(\cos 20°\mathbf{i} + \sin 20°\mathbf{j})$$

$$= -34.2(\cos 20°\mathbf{i} + \sin 20°\mathbf{j})$$



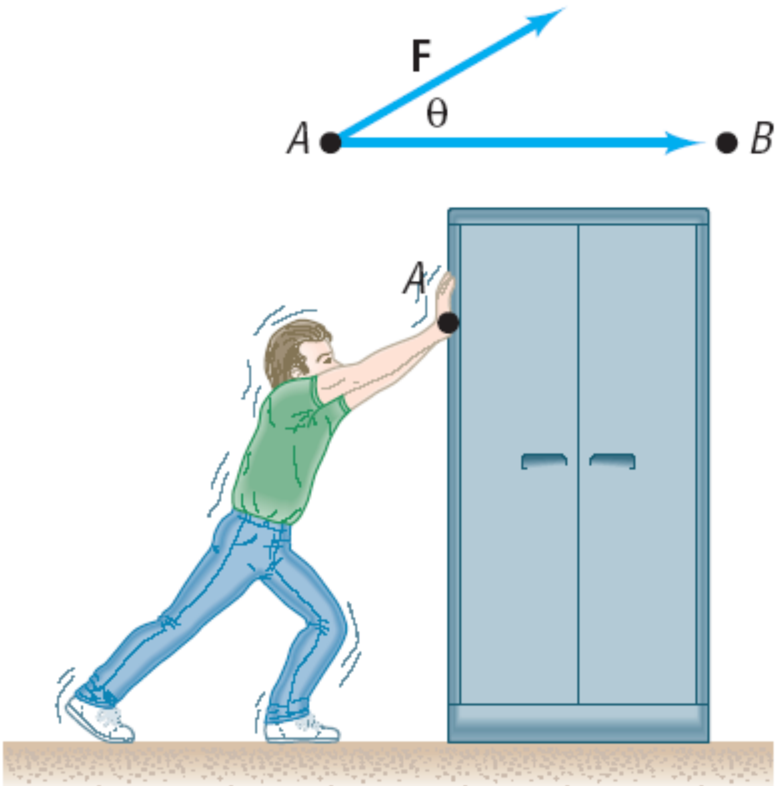
The magnitude of $\mathbf{v}$ is 34.2 pounds, so the magnitude of the force required to keep the wagon from rolling down the hill is 34.2 pounds.

6 Compute Work

In elementary physics, the **work** W done by a constant force $\mathbf{F}$ in moving an object from a point A to a point B is defined as

$$W = (\text{magnitude of force})(\text{distance}) = \|\mathbf{F}\|\|\overrightarrow{AB}\|$$

Figure 71



$$W = \mathbf{F} \cdot \overrightarrow{AB}$$

EXAMPLE

Computing Work

Figure 72(a) shows a girl pulling a wagon with a force of 50 pounds. How much work is done in moving the wagon 100 feet if the handle makes an angle of 30° with the ground?

We position the vectors in a coordinate system in such a way that the wagon is moved from $(0, 0)$ to $(100, 0)$. The motion is from $A = (0, 0)$ to $B = (100, 0)$, so $\overrightarrow{AB} = 100\mathbf{i}$. The force vector $\mathbf{F}$, as shown in Figure 73(b), is

$$\mathbf{F} = 50(\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) = 50\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 25\left(\sqrt{3}\mathbf{i} + \mathbf{j}\right)$$

$$W = \mathbf{F} \cdot \overrightarrow{AB} = 25\left(\sqrt{3}\mathbf{i} + \mathbf{j}\right) \cdot 100\mathbf{i} = 2500\sqrt{3} \text{ foot-pounds}$$