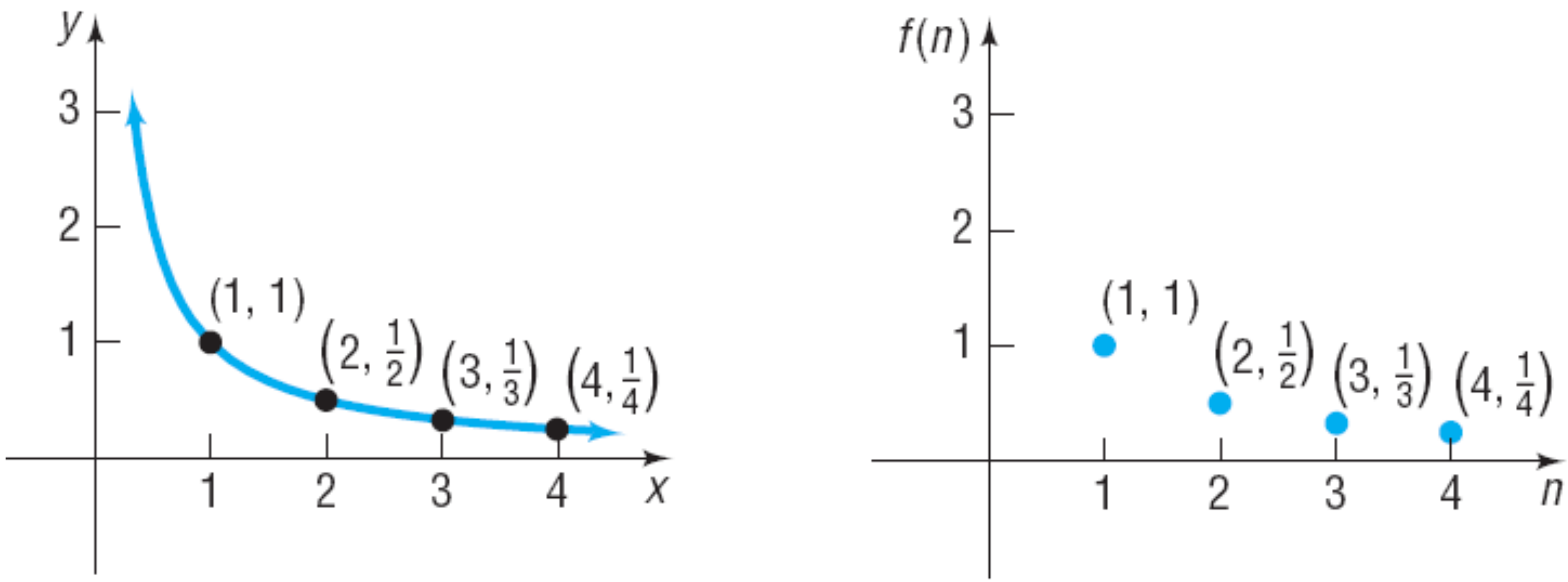
Section 13.1

Sequences

A **sequence** is a function whose domain is the set of positive integers.



(a) $f(x) = \frac{1}{x}, x > 0$ **(b)** $f(n) = \frac{1}{n}$, n a positive integer

1 Write the First Several Terms of a Sequence

EXAMPLE **Writing the First Several Terms of a Sequence**

Write down the first six terms of the following sequence and graph it.

$$\{a_n\} = \left\{\frac{n-1}{n}\right\}$$

$$a_1 = \frac{1-1}{1} = 0$$

$$a_2 = \frac{2-1}{2} = \frac{1}{2}$$

$$a_3 = \frac{3-1}{3} = \frac{2}{3}$$

$$a_4 = \frac{4-1}{4} = \frac{3}{4}$$

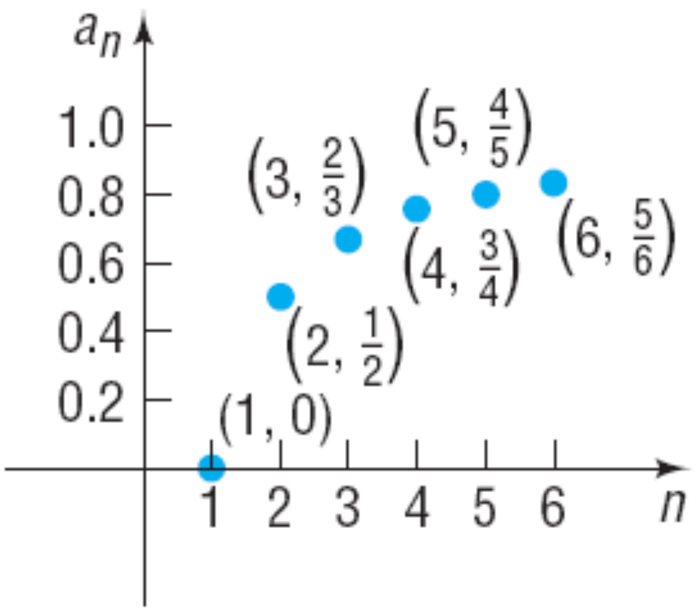
$$a_5 = \frac{5-1}{5} = \frac{4}{5}$$

$$a_6 = \frac{6-1}{6} = \frac{5}{6}$$

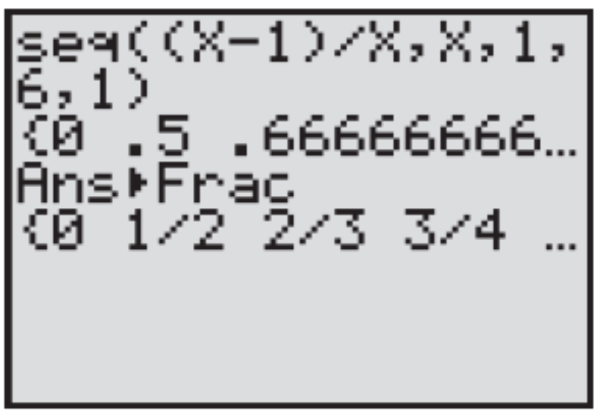
The first six terms of the sequence are

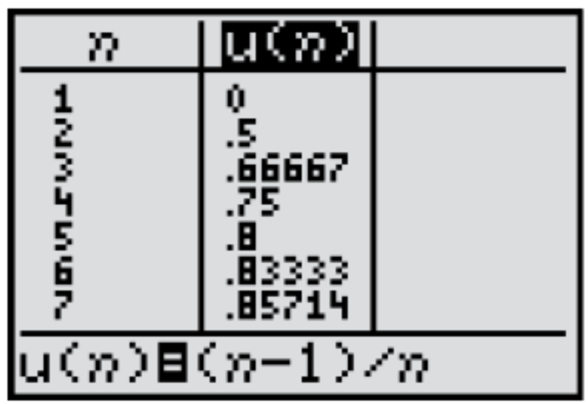
$$a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{2}{3},$$

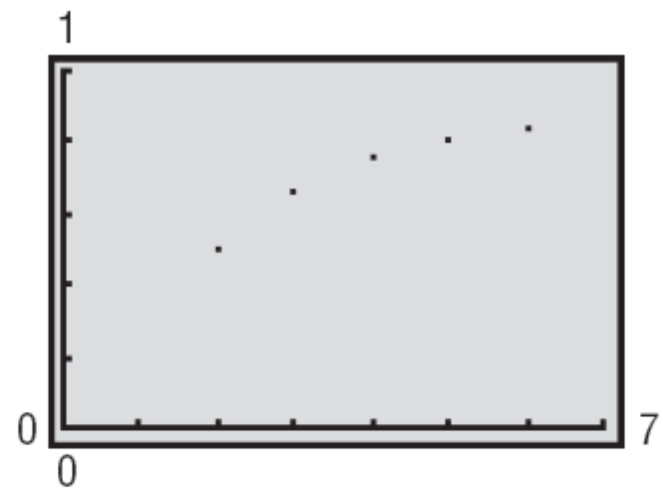
$$a_4 = \frac{3}{4}, \quad a_5 = \frac{4}{5}, \quad a_6 = \frac{5}{6}$$



COMMENT Graphing utilities can be used to write the terms of a sequence and graph them. Figure 3 shows the sequence given in Example 1 generated on a TI-84 Plus graphing calculator. We can see the first few terms of the sequence on the viewing window. You need to press the right arrow key to scroll right to see the remaining terms of the sequence. Figure 4 shows a graph of the sequence. Notice that the first term of the sequence is not visible since it lies on the x-axis. TRACEing the graph will allow you to see the terms of the sequence. The TABLE feature can also be used to generate the terms of the sequence. See Table 1.





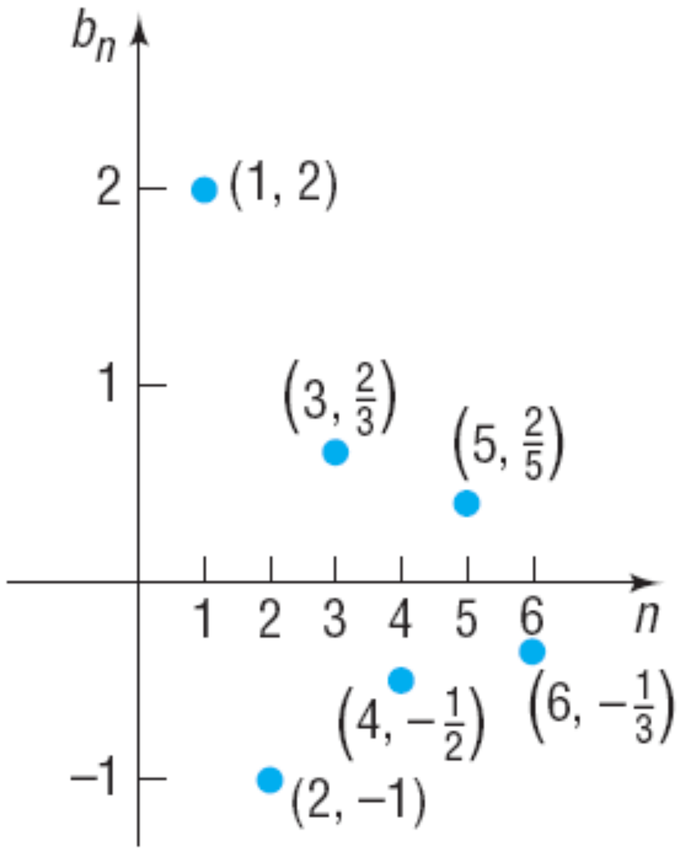


EXAMPLE Writing the First Several Terms of a Sequence

Write down the first six terms of the following sequence and graph it.

$$\{b_n\} = \left\{(-1)^{n+1}\left(\frac{2}{n}\right)\right\}$$

$$b_1 = 2, \quad b_2 = -1, \quad b_3 = \frac{2}{3}, \quad b_4 = -\frac{1}{2}, \quad b_5 = \frac{2}{5}, \quad b_6 = -\frac{1}{3}$$

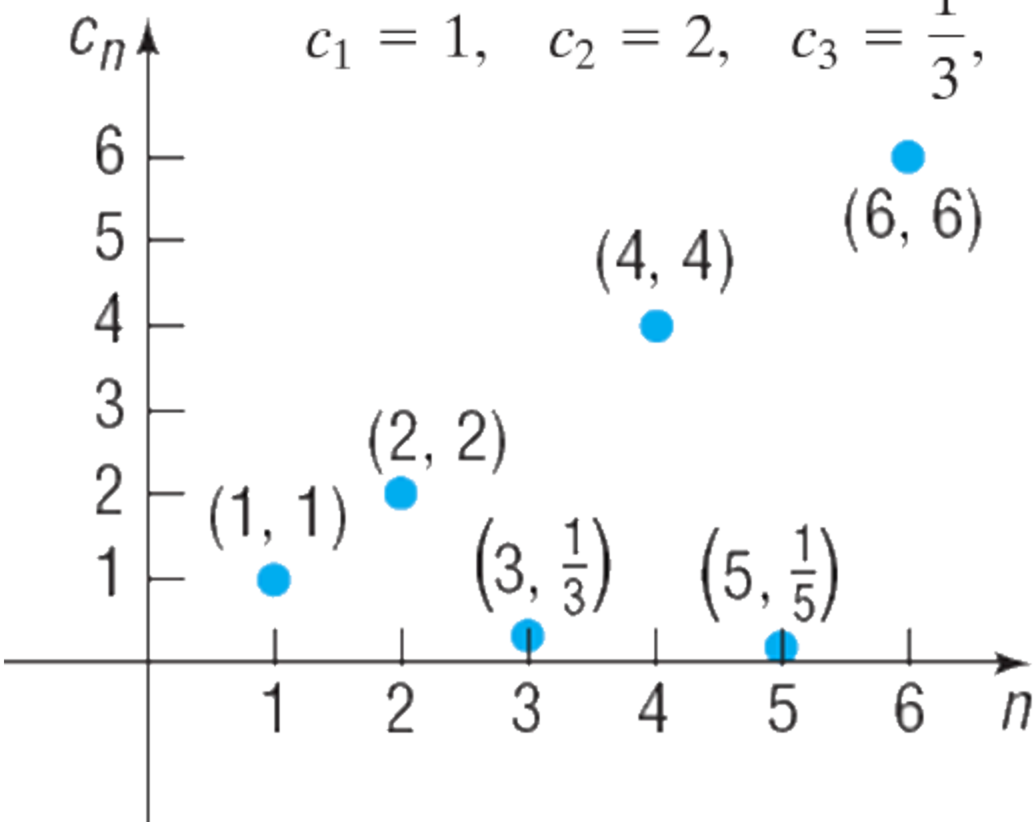


EXAMPLE **Writing the First Several Terms of a Sequence**

Write down the first six terms of the following sequence and graph it.

$$\{c_n\} = \left\{ \begin{array}{ll} n & \text{if } n \text{ is even} \\ \dfrac{1}{n} & \text{if } n \text{ is odd} \end{array} \right\}$$

$$c_1 = 1, \quad c_2 = 2, \quad c_3 = \frac{1}{3}, \quad c_4 = 4, \quad c_5 = \frac{1}{5}, \quad c_6 = 6$$



EXAMPLE

Determining a Sequence from a Pattern

(a) $e, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \ldots$ $\qquad a_n = \frac{e^n}{n}$

(b) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ $\qquad b_n = \frac{1}{3^{n-1}}$

(c) $1, 3, 5, 7, \ldots$ $\qquad c_n = 2n - 1$

(d) $1, 4, 9, 16, 25, \ldots$ $\qquad d_n = n^2$

(e) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$ $\qquad e_n = (-1)^{n+1}\left(\frac{1}{n}\right)$

The Factorial Symbol

If $n \geq 0$ is an integer, the **factorial symbol** $\boldsymbol{n!}$ is defined as follows:

$$\boxed{\begin{array}{l} 0! = 1 \qquad 1! = 1 \\ n! = n(n-1)\cdot \cdots \cdot 3 \cdot 2 \cdot 1 \qquad \text{if } n \geq 2 \end{array}}$$

n	*n!*
0	1
1	1
2	2
3	6
4	24
5	120
6	720

$$\boxed{n! = n(n-1)!}$$

2 Write the Terms of a Sequence Defined by a Recursive Formula

EXAMPLE

Writing the Terms of a Recursively Defined Sequence

Write down the first five terms of the following recursively defined sequence.

$$s_1 = 1, \qquad s_n = ns_{n-1}$$

$$s_1 = 1$$

$$s_2 = 2 \cdot 1 = 2$$

$$s_3 = 3 \cdot 2 = 6$$

$$s_4 = 4 \cdot 6 = 24$$

$$s_5 = 5 \cdot 24 = 120$$

EXAMPLE

Writing the Terms of a Recursively Defined Sequence

Write down the first five terms of the following recursively defined sequence.

$$u_1 = 1, \qquad u_2 = 1, \qquad u_{n+2} = u_n + u_{n+1}$$

Fibonacci sequence

$$u_1 = 1$$
$$u_2 = 1$$
$$u_3 = u_1 + u_2 = 1 + 1 = 2$$
$$u_4 = u_2 + u_3 = 1 + 2 = 3$$
$$u_5 = u_3 + u_4 = 2 + 3 = 5$$

3 Use Summation Notation

summation notation

$$\boxed{a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k}$$

EXAMPLE **Expanding Summation Notation**

Write out each sum.

(a) $$\sum_{k=1}^{n}\frac{k}{k+1} = \frac{1}{1+1}+\frac{2}{2+1}+\frac{3}{3+1}+\ldots+\frac{n}{n+1}$$

(b) $$\sum_{k=0}^{n}\left(k^2-1\right) = \left(1^2-1\right)+\left(2^2-1\right)+\left(3^2-1\right)+\ldots+\left(n^2-1\right)$$

EXAMPLE

Writing a Sum in Summation Notation

Express each sum using summation notation.

(a) $1^2 + 2^2 + 3^2 + \cdots + 9^2$

(b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$

(a) $\sum_{k=1}^{9} k^2$

(b) $\sum_{k=1}^{n} \frac{1}{2^{n-1}}$

4 Find the Sum of a Sequence

THEOREM

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences and c is a real number, then:

$$\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c\sum_{k=1}^{n} a_k \quad \textbf{(1)}$$

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \quad \textbf{(2)}$$

$$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \quad \textbf{(3)}$$

$$\sum_{k=j+1}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j} a_k, \quad \text{where } 0 < j < n \quad \textbf{(4)}$$

THEOREM

Formulas for Sums of Sequences

$$\sum_{k=1}^{n} c = \underbrace{c + c + \cdots + c}_{n \text{ terms}} = cn \qquad c \text{ is a real number} \tag{5}$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{8}$$

EXAMPLE Finding the Sum of a Sequence

$$\text{(a)} \quad \sum_{k=1}^{5}(3k) = 3\sum_{k=1}^{5}k = 3\left(\frac{5(5+1)}{2}\right) = 3(15) = 45$$

$$\text{(b)} \quad \sum_{k=1}^{10}(k^3+1) = \sum_{k=1}^{10}k^3 + \sum_{k=1}^{10}1 = \left(\frac{10(10+1)}{2}\right)^2 + 1(10)$$

$$= 3025 + 10 = 3035$$

$$\boxed{\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n}a_k + \sum_{k=1}^{n}b_k}$$

$$\boxed{\sum_{k=1}^{n}c = cn}$$

$$\boxed{\sum_{k=1}^{n}k = \frac{n(n+1)}{2}}$$

$$\boxed{\sum_{k=1}^{n}(ca_k) = c\sum_{k=1}^{n}a_k}$$

$$\boxed{\sum_{k=1}^{n}k^3 = \left[\frac{n(n+1)}{2}\right]^2}$$

EXAMPLE

Finding the Sum of a Sequence

(c) $$\sum_{k=1}^{24}(k^2 - 7k + 2) = \sum_{k=1}^{24} k^2 - \sum_{k=1}^{24}(7k) + \sum_{k=1}^{24} 2$$

$$= \sum_{k=1}^{24} k^2 - 7\sum_{k=1}^{24} k + \sum_{k=1}^{24} 2$$

$$= \frac{24(24+1)(2\cdot 24+1)}{6} - 7\left(\frac{24(24+1)}{2}\right) + 2(24)$$

$$= 4900 - 2100 + 48 = 2848$$

$$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} c = cn$$

$$\sum_{k=1}^{n}(ca_k) = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

EXAMPLE

Finding the Sum of a Sequence

$$\text{(d)}\quad \sum_{k=6}^{20}(4k^2) = 4\sum_{k=6}^{20}k^2 = 4\left[\sum_{k=1}^{20}k^2 - \sum_{k=1}^{5}k^2\right]$$

$$= 4\left[\frac{20(21)(41)}{6} - \frac{5(6)(11)}{6}\right] = 4[2870 - 55] = 11{,}260$$

$$\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n}(ca_k) = c\sum_{k=1}^{n}a_k$$

$$\sum_{k=j+1}^{n}a_k = \sum_{k=1}^{n}a_k - \sum_{k=1}^{j}a_k$$