# Bayesian Methods for Multimedia Signal Processing

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## Goals of this Tutorial

- Provide a basic understanding of underlying principles of probabilistic modeling and Bayesian inference
- Orientation in the broad literature of Bayesian machine learning and statistical signal processing
- Focus on fundamental concepts rather than technical details,

... we avoid heavy use of algebra by a graphical notation
... but there will be some maths



## Goals of this Tutorial

- Model based approach

... rather than description of algorithms for solving specific problems

- Illustrate with examples how certain problems in multimedia signal analysis can be approached using generic tools
- Motivate participants to investigate further

... provide alternative perspective to existing solutions
... and hopefully provide new inspiration



## First Part, Basic Concepts

- Introduction
  - Bayes' Theorem,
  - Trivial toy example to clarify notation
- Graphical Models
  - Bayesian Networks
  - Undirected Graphical models, Markov Random Fields
  - Factor graphs
- Maximum Likelihood, Penalised Likelihood, Bayesian Learning
- Basic Building Blocks in model construction
  - Probability distributions, Exponential family

## Second Part, Models and Applications

- Hidden Markov Models,
  - Tempo tracking, Score-performance matching
  - Inference in Hidden Markov Models
    - Forward Backward Algorithm
    - Viterbi
    - Exact inference by message passing: Belief Propagation
- Linear Dynamical systems, Kalman Filter Models
  - Tracking
  - Computer Accompaniment
  - Kalman Filtering and Smoothing
  - Audio Restoration and Interpolation



- Switching State Space models, Changepoint Models
  - Pitch tracking
  - Particle Filtering
- Nonlinear Dynamical Systems
  - Object tracking in video
  - Particle Filtering, Sequential Monte Carlo
- Markov Random Fields
  - Denoising, Source Separation
  - Markov Chain Monte Carlo, Gibbs sampler
  - Variational Bayes



- Topic-Term Models
  - Latent Semantic indexing
  - Generative aspect model, Latent Dirichlet allocation
- Factorial Models, Sparsity, Model selection
  - Audio Source Separation
  - Polyphonic Pitch Tracking
  - Approximate Inference in Factorial Models
- Final Remarks and Bibliography



## Bayes' Theorem [13, 15]

Thomas Bayes (1702-1761)

What you know about a parameter $\boldsymbol{\lambda}$ after the data $\mathcal{D}$ arrive is what you knew before about $\boldsymbol{\lambda}$ and what the data $\mathcal{D}$ told you.

$$
\begin{aligned}
p(\lambda|\mathcal{D}) &= \frac{p(\mathcal{D}|\lambda)p(\lambda)}{p(\mathcal{D})} \\
\text{Posterior} &= \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}
\end{aligned}
$$

## An application of Bayes' Theorem: "Source Separation"

Given two fair dice with outcomes $\lambda$ and $y$,

$$\mathcal{D} = \lambda + y$$

What is $\lambda$ when $\mathcal{D} = 9$ ?



## An application of Bayes' Theorem: "Source Separation"

$$\mathcal{D} = \lambda + y = 9$$

| $\mathcal{D} = \lambda + y$ | $y = 1$ | $y = 2$ | $y = 3$ | $y = 4$ | $y = 5$ | $y = 6$ |
|---|---|---|---|---|---|---|
| $\lambda = 1$ | 2 | 3 | 4 | 5 | 6 | 7 |
| $\lambda = 2$ | 3 | 4 | 5 | 6 | 7 | 8 |
| $\lambda = \mathbf{3}$ | 4 | 5 | 6 | 7 | 8 | **9** |
| $\lambda = \mathbf{4}$ | 5 | 6 | 7 | 8 | **9** | 10 |
| $\lambda = \mathbf{5}$ | 6 | 7 | 8 | **9** | 10 | 11 |
| $\lambda = \mathbf{6}$ | 7 | 8 | **9** | 10 | 11 | 12 |

Bayes theorem "upgrades" $p(\lambda)$ into $p(\lambda|\mathcal{D})$.

But you have to provide an observation model: $p(\mathcal{D}|\lambda)$



## "Bureaucratical" derivation

Formally we write

$$\begin{aligned} p(\lambda) &= \mathcal{C}(\lambda; [\ 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6\ ]) \\ p(y) &= \mathcal{C}(y; [\ 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6\ ]) \\ p(\mathcal{D}|\lambda, y) &= \delta(\mathcal{D} - (\lambda + y)) \end{aligned}$$

$$\begin{aligned} p(\lambda, y|\mathcal{D}) &= \frac{1}{p(\mathcal{D})} \times p(\mathcal{D}|\lambda, y) \times p(y)p(\lambda) \\ \text{Posterior} &= \frac{1}{\text{Evidence}} \times \text{Likelihood} \times \text{Prior} \end{aligned}$$

Kronecker delta function denoting a degenerate (deterministic) distribution $\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$



## Prior

$$p(y)p(\lambda)$$

| $p(y) \times p(\lambda)$ | $y = 1$ | $y = 2$ | $y = 3$ | $y = 4$ | $y = 5$ | $y = 6$ |
|---|---|---|---|---|---|---|
| $\lambda = 1$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ |
| $\lambda = 2$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ |
| $\lambda = 3$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ |
| $\lambda = 4$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ |
| $\lambda = 5$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ |
| $\lambda = 6$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ | $1/36$ |

- A table with indicies $\lambda$ and $y$
- Each cell denotes the probability $p(\lambda, y)$

## Likelihood

$$p(\mathcal{D}=9|\lambda,y)$$

| $p(\mathcal{D}=9\|\lambda,y)$ | $y=1$ | $y=2$ | $y=3$ | $y=4$ | $y=5$ | $y=6$ |
|---|---|---|---|---|---|---|
| $\lambda=1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\lambda=2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\lambda=3$ | 0 | 0 | 0 | 0 | 0 | **1** |
| $\lambda=4$ | 0 | 0 | 0 | 0 | **1** | 0 |
| $\lambda=5$ | 0 | 0 | 0 | **1** | 0 | 0 |
| $\lambda=6$ | 0 | 0 | **1** | 0 | 0 | 0 |

- A table with indicies $\lambda$ and $y$
- The likelihood is **not** a probability distribution, but a positive function.



## Likelihood $\times$ Prior

$$\phi_{\mathcal{D}}(\lambda,y)=p(\mathcal{D}=9|\lambda,y)p(\lambda)p(y)$$

| $p(\mathcal{D}=9\|\lambda,y)$ | $y=1$ | $y=2$ | $y=3$ | $y=4$ | $y=5$ | $y=6$ |
|---|---|---|---|---|---|---|
| $\lambda=1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\lambda=2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\lambda=3$ | 0 | 0 | 0 | 0 | 0 | **1/36** |
| $\lambda=4$ | 0 | 0 | 0 | 0 | **1/36** | 0 |
| $\lambda=5$ | 0 | 0 | 0 | **1/36** | 0 | 0 |
| $\lambda=6$ | 0 | 0 | **1/36** | 0 | 0 | 0 |



## Evidence

$$\begin{aligned} p(\mathcal{D}=9) &= \sum_{\lambda,y} p(\mathcal{D}=9|\lambda,y)p(\lambda)p(y) \\ &= 0+0+\cdots+1/36+1/36+1/36+1/36+0+\cdots+0 \\ &= 1/9 \end{aligned}$$

| $p(\mathcal{D}=9\|\lambda,y)$ | $y=1$ | $y=2$ | $y=3$ | $y=4$ | $y=5$ | $y=6$ |
|---|---|---|---|---|---|---|
| $\lambda=1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\lambda=2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\lambda=3$ | 0 | 0 | 0 | 0 | 0 | **1/36** |
| $\lambda=4$ | 0 | 0 | 0 | 0 | **1/36** | 0 |
| $\lambda=5$ | 0 | 0 | 0 | **1/36** | 0 | 0 |
| $\lambda=6$ | 0 | 0 | **1/36** | 0 | 0 | 0 |



## Posterior

$$p(\lambda,y|\mathcal{D}=9) = \frac{1}{p(\mathcal{D})}p(\mathcal{D}=9|\lambda,y)p(\lambda)p(y)$$

| $p(\mathcal{D}=9\|\lambda,y)$ | $y=1$ | $y=2$ | $y=3$ | $y=4$ | $y=5$ | $y=6$ |
|---|---|---|---|---|---|---|
| $\lambda=1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\lambda=2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\lambda=3$ | 0 | 0 | 0 | 0 | 0 | **1/4** |
| $\lambda=4$ | 0 | 0 | 0 | 0 | **1/4** | 0 |
| $\lambda=5$ | 0 | 0 | 0 | **1/4** | 0 | 0 |
| $\lambda=6$ | 0 | 0 | **1/4** | 0 | 0 | 0 |

$$1/4 = (1/36)/(1/9)$$

# Marginal Posterior

$$p(\lambda|\mathcal{D}) \quad = \quad \sum_y \frac{1}{p(\mathcal{D})} p(\mathcal{D}|\lambda, y)p(\lambda)p(y)$$

| | $p(\lambda\|\mathcal{D}=9)$ | $y=1$ | $y=2$ | $y=3$ | $y=4$ | $y=5$ | $y=6$ |
|---|---|---|---|---|---|---|---|
| $\lambda=1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\lambda=2$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\lambda=3$ | $\mathbf{1/4}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1/4$ |
| $\lambda=4$ | $\mathbf{1/4}$ | $0$ | $0$ | $0$ | $0$ | $1/4$ | $0$ |
| $\lambda=5$ | $\mathbf{1/4}$ | $0$ | $0$ | $0$ | $1/4$ | $0$ | $0$ |
| $\lambda=6$ | $\mathbf{1/4}$ | $0$ | $0$ | $1/4$ | $0$ | $0$ | $0$ |



# The "proportional to" notation

$$p(\lambda|\mathcal{D}=9) \quad \propto \quad p(\lambda, \mathcal{D}=9) = \sum_y p(\mathcal{D}=9|\lambda, y)p(\lambda)p(y)$$

| | $p(\lambda,\mathcal{D}=9)$ | $y=1$ | $y=2$ | $y=3$ | $y=4$ | $y=5$ | $y=6$ |
|---|---|---|---|---|---|---|---|
| $\lambda=1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\lambda=2$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\lambda=3$ | $1/36$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\mathbf{1/36}$ |
| $\lambda=4$ | $1/36$ | $0$ | $0$ | $0$ | $0$ | $\mathbf{1/36}$ | $0$ |
| $\lambda=5$ | $1/36$ | $0$ | $0$ | $0$ | $\mathbf{1/36}$ | $0$ | $0$ |
| $\lambda=6$ | $1/36$ | $0$ | $0$ | $\mathbf{1/36}$ | $0$ | $0$ | $0$ |



# Exercise

| $p(x_1, x_2)$ | $x_2=1$ | $x_2=2$ |
|---|---|---|
| $x_1=1$ | $0.3$ | $0.3$ |
| $x_1=2$ | $0.1$ | $0.3$ |

1. Find the following quantities
   - Marginals: $p(x_1)$, $p(x_2)$
   - Conditionals: $p(x_1|x_2)$, $p(x_2|x_1)$
   - Posterior: $p(x_1, x_2=2)$, $p(x_1|x_2=2)$
   - Evidence: $p(x_2=2)$
   - $p(\{\})$
   - Max: $p(x_1^*) = \max_{x_1} p(x_1|x_2=1)$
   - Mode: $x_1^* = \arg\max_{x_1} p(x_1|x_2=1)$
   - Max-marginal: $\max_{x_1} p(x_1, x_2)$
2. Are $x_1$ and $x_2$ independent ? (i.e., Is $p(x_1, x_2) = p(x_1)p(x_2)$ ?)



# Answers

| $p(x_1, x_2)$ | $x_2=1$ | $x_2=2$ |
|---|---|---|
| $x_1=1$ | $0.3$ | $0.3$ |
| $x_1=2$ | $0.1$ | $0.3$ |

- Marginals:

| $p(x_1)$ | |
|---|---|
| $x_1=1$ | $0.6$ |
| $x_1=2$ | $0.4$ |

| $p(x_2)$ | $x_2=1$ | $x_2=2$ |
|---|---|---|
| | $0.4$ | $0.6$ |

- Conditionals:

| $p(x_1\|x_2)$ | $x_2=1$ | $x_2=2$ |
|---|---|---|
| $x_1=1$ | $0.75$ | $0.5$ |
| $x_1=2$ | $0.25$ | $0.5$ |

| $p(x_2\|x_1)$ | $x_2=1$ | $x_2=2$ |
|---|---|---|
| $x_1=1$ | $0.5$ | $0.5$ |
| $x_1=2$ | $0.25$ | $0.75$ |

## Answers

| $p(x_1, x_2)$ | $x_2 = 1$ | $x_2 = 2$ |
|---|---|---|
| $x_1 = 1$ | 0.3 | 0.3 |
| $x_1 = 2$ | 0.1 | 0.3 |

- Posterior:

| $p(x_1, x_2 = 2)$ | $x_2 = 2$ |
|---|---|
| $x_1 = 1$ | 0.3 |
| $x_1 = 2$ | 0.3 |

| $p(x_1 \vert x_2 = 2)$ | $x_2 = 2$ |
|---|---|
| $x_1 = 1$ | 0.5 |
| $x_1 = 2$ | 0.5 |

- Evidence:

$$p(x_2 = 2) = \sum_{x_1} p(x_1, x_2 = 2) = 0.6$$

- Normalisation constant:

$$p(\{\}) = \sum_{x_1} \sum_{x_2} p(x_1, x_2) = 1$$



## Answers

| $p(x_1, x_2)$ | $x_2 = 1$ | $x_2 = 2$ |
|---|---|---|
| $x_1 = 1$ | 0.3 | 0.3 |
| $x_1 = 2$ | 0.1 | 0.3 |

- Max: (get the value)

$$\max_{x_1} p(x_1 | x_2 = 1) = 0.75$$

- Mode: (get the index)

$$\operatorname*{argmax}_{x_1} p(x_1 | x_2 = 1) = 1$$

- Max-marginal: (get the "skyline") $\max_{x_1} p(x_1, x_2)$

| $\max_{x_1} p(x_1, x_2)$ | $x_2 = 1$ | $x_2 = 2$ |
|---|---|---|
| | 0.3 | 0.3 |



## Another application of Bayes' Theorem: "Model Selection"

Given an unknown number of fair dice with outcomes $\lambda_1, \lambda_2, \dots, \lambda_n$,

$$\mathcal{D} = \sum_{i=1}^{n} \lambda_i$$

How many dice are there when $\mathcal{D} = 9$ ?

Assume that any number $n$ is equally likely



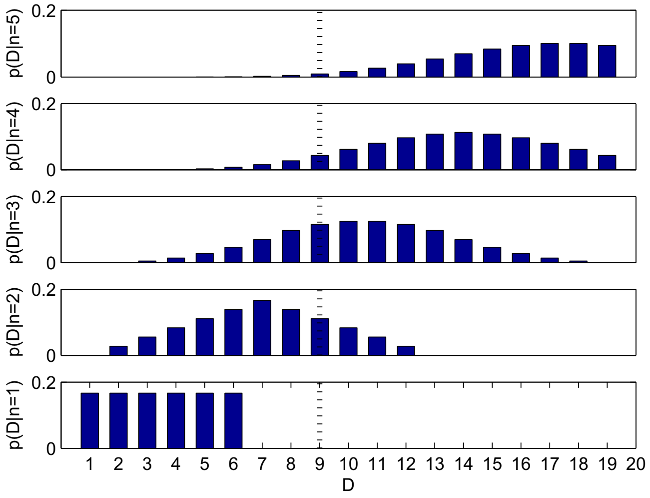
## Another application of Bayes' Theorem: "Model Selection"

Given all $n$ are equally likely (i.e., $p(n)$ is flat), we calculate (formally)

$$p(n | \mathcal{D} = 9) = \frac{p(\mathcal{D} = 9 | n) p(n)}{p(\mathcal{D})} \propto p(\mathcal{D} = 9 | n)$$

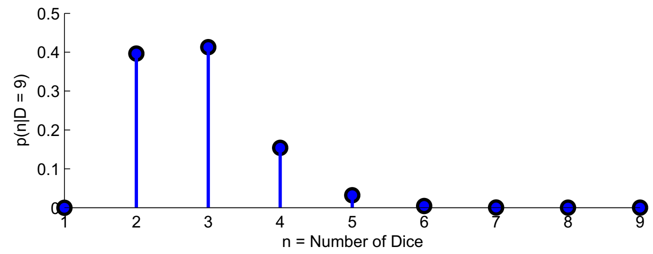
$$\begin{aligned}
p(\mathcal{D} | n = 1) &= \sum_{\lambda_1} p(\mathcal{D} | \lambda_1) p(\lambda_1) \\
p(\mathcal{D} | n = 2) &= \sum_{\lambda_1} \sum_{\lambda_2} p(\mathcal{D} | \lambda_1, \lambda_2) p(\lambda_1) p(\lambda_2) \\
\dots \\
p(\mathcal{D} | n = n') &= \sum_{\lambda_1, \dots, \lambda_{n'}} p(\mathcal{D} | \lambda_1, \dots, \lambda_{n'}) \prod_{i=1}^{n'} p(\lambda_i)
\end{aligned}$$

$$p(\mathcal{D}|n) = \sum_{\boldsymbol{\lambda}} p(\mathcal{D}|\boldsymbol{\lambda}, n)p(\boldsymbol{\lambda}|n)$$






## Another application of Bayes' Theorem: "Model Selection"




- Complex models are more flexible but they spread their probability mass
- Bayesian inference inherently prefers "simpler models" – Occam's razor
- Computational burden: We need to sum over all parameters $\lambda$



## Probabilistic Inference

A huge spectrum of applications – all boil down to computation of

- **expectations** of functions under probability distributions: **Integration**

$$\langle f(x) \rangle = \int_{\mathcal{X}} dx p(x) f(x) \qquad \langle f(x) \rangle = \sum_{x \in \mathcal{X}} p(x) f(x)$$

- **modes** of functions under probability distributions: **Optimization**

$$x^* = \operatorname*{argmax}_{x \in \mathcal{X}} p(x) f(x)$$

- any "mix" of the above: e.g.,

$$x^* = \operatorname*{argmax}_{x \in \mathcal{X}} p(x) = \operatorname*{argmax}_{x \in \mathcal{X}} \int_{\mathcal{Z}} dz p(z) p(x|z)$$



## Divide and Conquer

Probabilistic modelling provides a methodology that puts a clear division between

- What to solve : Model Construction
  - Both an Art and Science
  - Highly domain specific
- How to solve : Inference Algorithm
  - Mechanical (In theory! not in practice)
  - Generic

## Applications of Probability Models

- Classification
- Optimal Decision, given a loss function
- Finding interesting (hidden) structure
  - Clustering, Segmentation
  - Dimensionality Reduction
  - Outlier Detection
- Finding a compact representation $=$ Data Compression
- Prediction



## Probability Models

$+$

## Inference Algorithms

$=$

## Bayesian Numerical Methods



## Graphical Models

- formal languages for specification of probability models and associated inference algorithms
- historically, introduced in probabilistic expert systems (Pearl 1988) as a visual guide for representing expert knowledge
- today, a standard tool in machine learning, statistics and signal processing



## Graphical Models

- provide graph based algorithms for derivations and computation
- pedagogical insight/motivation for model/algorithm construction
  - Statistics:
    "Kalman filter models and hidden Markov models (HMM) are equivalent upto parametrisation"
  - Signal processing:
    "Fast Fourier transform is an instance of sum-product algorithm on a factor graph"
  - Computer Science:
    "Backtracking in Prolog is equivalent to inference in Bayesian networks with deterministic tables"
- Automated tools for code generation start to emerge, making the design/implement/test cycle shorter

## Important types of Graphical Models

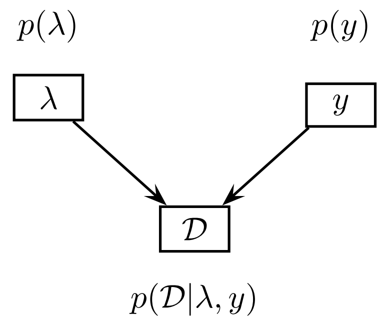
- Useful for Model Construction
  - **Directed Acyclic Graphs (DAG), Bayesian Networks**
  - **Undirected Graphs, Markov Networks, Random Fields**
  - Influence diagrams
  - ...
- Useful for Inference
  - **Factor Graphs**
  - Junction/Clique graphs
  - Region graphs
  - ...



## Directed Graphical models (DAG)



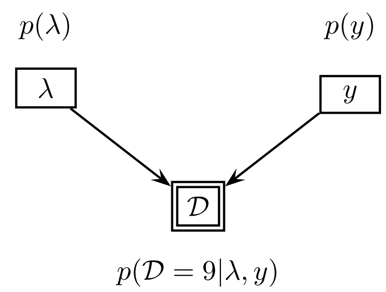
## DAG Example: Two dice




$$p(\mathcal{D}, \lambda, y) = p(\mathcal{D}|\lambda, y)p(\lambda)p(y)$$



## DAG with observations




$$\phi_{\mathcal{D}}(\lambda, y) = p(\mathcal{D} = 9|\lambda, y)p(\lambda)p(y)$$

## Directed Graphical models

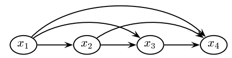
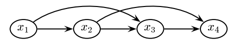
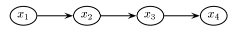
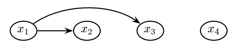
- Each random variable is associated with a node in the graph,
- We draw an arrow from $A \to B$ if $p(B| \dots, A, \dots)$ ($A \in \text{parent}(B)$),
- The edges tell us *qualitatively* about the factorization of the joint probability
- For $N$ random variables $x_1, \dots, x_N$, the distribution admits

$$p(x_1, \dots, x_N) = \prod_{i=1}^{N} p(x_i|\text{parent}(x_i))$$

- Describes in a compact way an algorithm to "generate" the data – "Generative models"



## Examples

| Model | Structure | factorization |
|---|---|---|
| Full |  | $p(x_1)p(x_2\|x_1)p(x_3\|x_1,x_2)p(x_4\|x_1,x_2,x_3)$ |
| Markov(2) |  | $p(x_1)p(x_2\|x_1)p(x_3\|x_1,x_2)p(x_4\|x_2,x_3)$ |
| Markov(1) |  | $p(x_1)p(x_2\|x_1)p(x_3\|x_2)p(x_4\|x_3)$ |
| |  | $p(x_1)p(x_2\|x_1)p(x_3\|x_1)p(x_4)$ |
| Factorized |  | $p(x_1)p(x_2)p(x_3)p(x_4)$ |

Removing edges eliminates a term from the conditional probability factors.



# Undirected Graphical Models



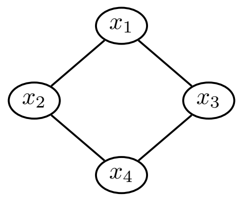
## Undirected Graphical Models

- Define a distribution by non-negative *local compatibility functions* $\phi(x_\alpha)$

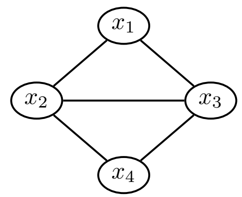
$$p(\mathbf{x}) = \frac{1}{Z} \prod_{\alpha} \phi(x_\alpha)$$

where $\alpha$ runs over **cliques** : fully connected subsets

- Examples

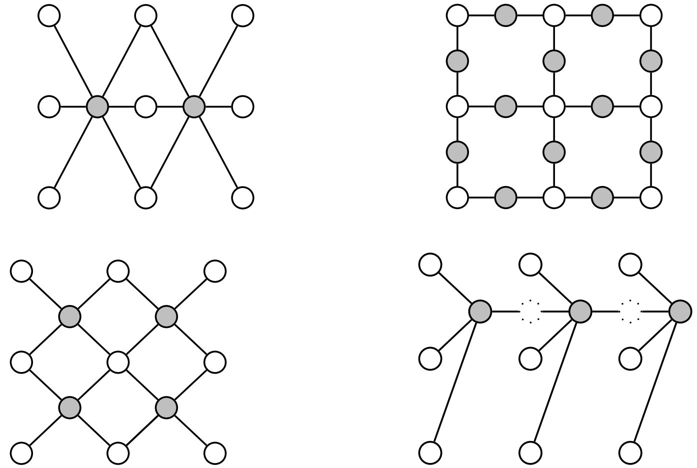


$p(\mathbf{x}) = \frac{1}{Z}\phi(x_1,x_2)\phi(x_1,x_3)\phi(x_2,x_4)\phi(x_3,x_4)$



$p(\mathbf{x}) = \frac{1}{Z}\phi(x_1,x_2,x_3)\phi(x_2,x_3,x_4)$

## Possible Model Topologies



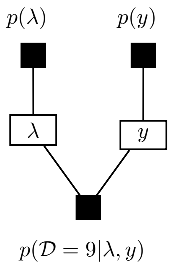


## Factor graphs



## Factor graphs [14]

- A bipartite graph. A powerful graphical representation of the inference problem
  - **Factor nodes**: Black squares. Factor potentials (local functions) defining the posterior.
  - **Variable nodes**: White Nodes. Define collections of random variables
  - **Edges**: denote membership. A variable node is connected to a factor node if a member variable is an argument of the local function.

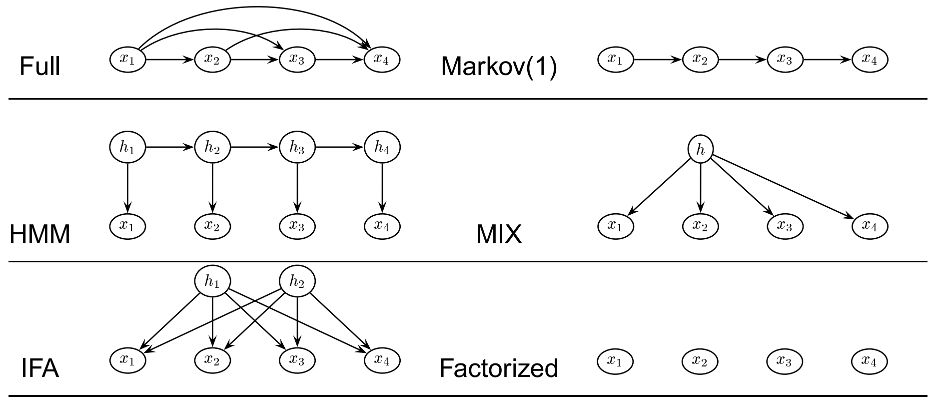



$$\phi_{\mathcal{D}}(\lambda,y) \quad = \quad p(\mathcal{D}=9|\lambda,y)p(\lambda)p(y) = \phi_1(\lambda,y)\phi_2(\lambda)\phi_3(y)$$

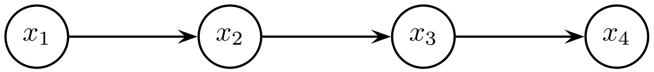


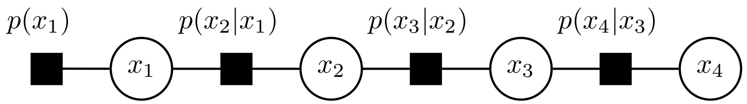
## Exercise

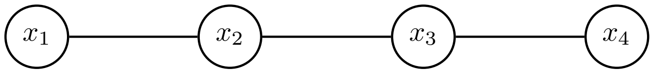
- For the following Graphical models, write down the factors of the joint distribution and plot an equivalent factor graph and an undirected graph.



# Answer (Markov(1))



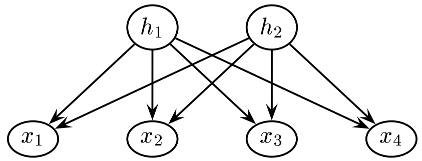



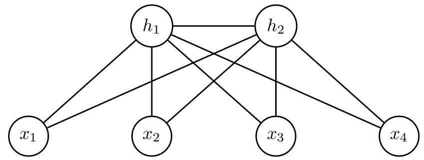

$$\underbrace{p(x_1)p(x_2|x_1)}_{\phi(x_1,x_2)}\underbrace{p(x_3|x_2)}_{\phi(x_2,x_3)}\underbrace{p(x_4|x_3)}_{\phi(x_3,x_4)}$$



# Answer (IFA – Factorial)

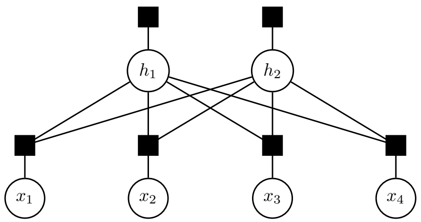



$$p(h_1)p(h_2)\prod_{i=1}^{4} p(x_i|h_1,h_2)$$

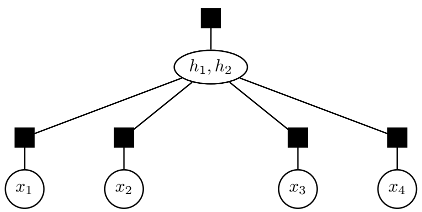





# Answer (IFA – Factorial)




- We can also cluster nodes together






# Inference and Learning

- Data set
$$\mathcal{D} = \{x_1, \dots x_N\}$$
- Model with parameter $\lambda$
$$p(\mathcal{D}|\lambda)$$
- Maximum Likelihood (ML)
$$\lambda^{\text{ML}} = \arg\max_{\lambda} \log p(\mathcal{D}|\lambda)$$
- Predictive distribution
$$p(x_{N+1}|\mathcal{D}) \approx p(x_{N+1}|\lambda^{\text{ML}})$$

## Regularisation

- Prior

$$p(\lambda)$$

- Maximum a-posteriori (MAP) : Regularised Maximum Likelihood

$$\lambda^{\text{MAP}} = \arg\max_{\lambda} \log p(\mathcal{D}|\lambda)p(\lambda)$$

- Predictive distribution

$$p(x_{N+1}|\mathcal{D}) \approx p(x_{N+1}|\lambda^{\text{MAP}})$$



## Bayesian Learning

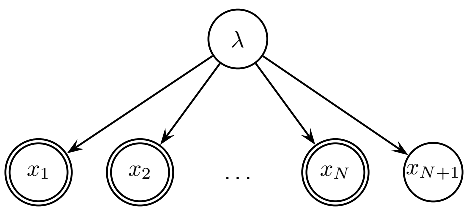
- We treat parameters on the same footing as all other variables
- We integrate over unknown parameters rather than using point estimates (remember the many-dice example)
  - Avoids overfitting
  - Natural setup for online adaptation
  - Model selection
    - (arguably) many problems in music processing are model selection problems



## Bayesian Learning

- Predictive distribution

$$p(x_{N+1}|\mathcal{D}) = \int d\lambda \;\; p(x_{N+1}|\lambda)p(\lambda|\mathcal{D})$$

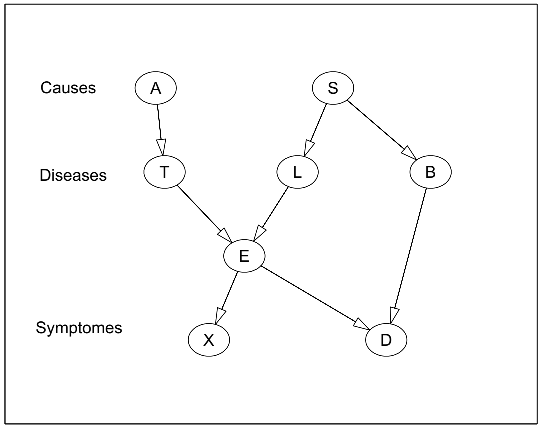



- Bayesian learning is just inference ...



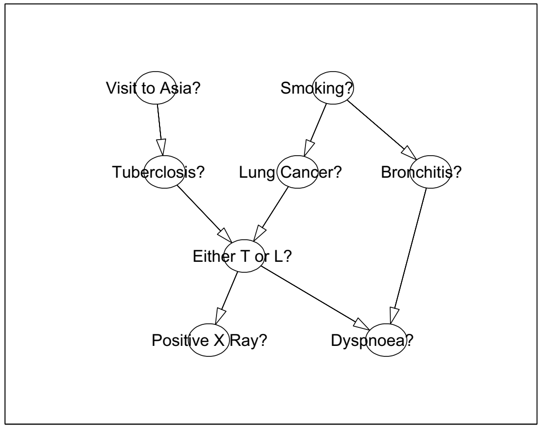
# Example Applications and Models

# Medical Expert Systems



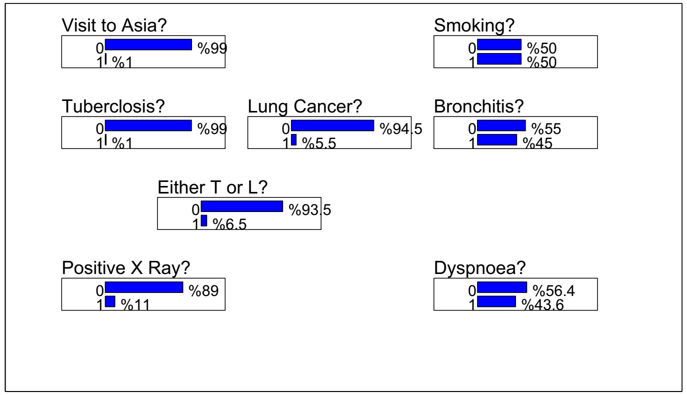



# Medical Expert Systems



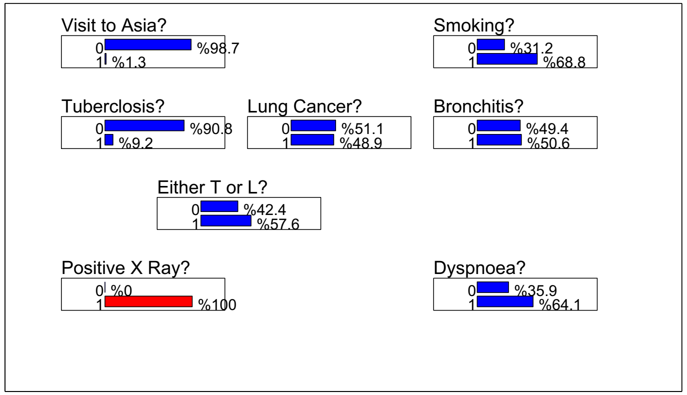



# Medical Expert Systems

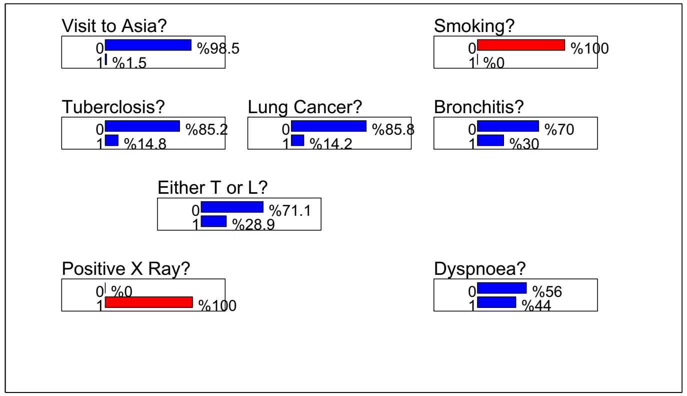





# Medical Expert Systems



# Medical Expert Systems

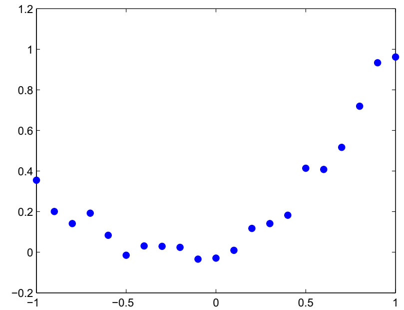





# Model Selection: Variable selection in Polynomial Regression

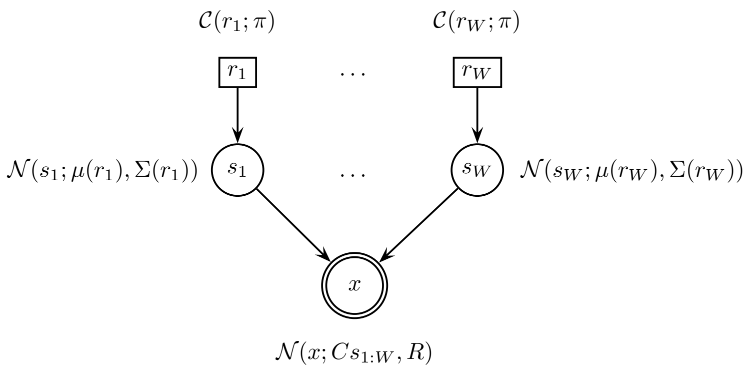
- Given $\mathcal{D} = \{t_j, x(t_j)\}_{j=1\ldots J}$, what is the order $N$ of the polynomial?

$$x(t) = \sum_{i=0}^{N} s_{i+1} t^i + \epsilon(t)$$





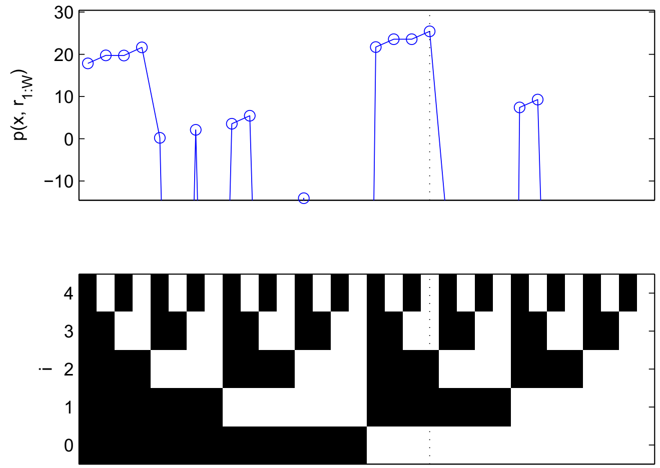
# Bayesian Variable Selection




- Generalized Linear Model – Column's of $C$ are the basis vectors
- The exact posterior is a mixture of $2^W$ Gaussians
- When $W$ is large, computation of posterior features becomes intractable.

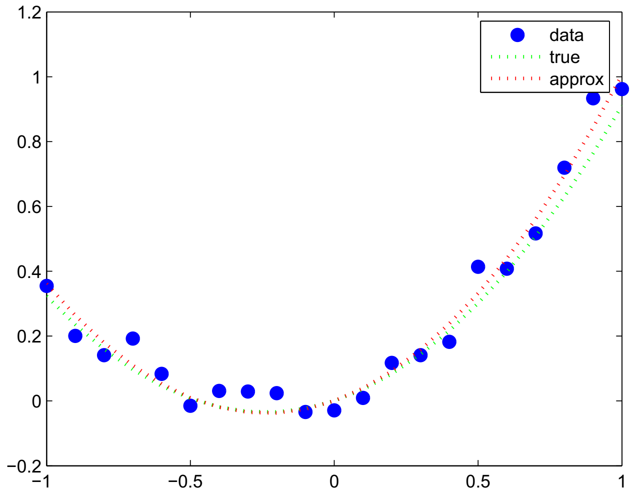


# Regression



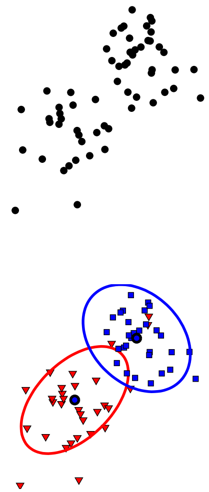

All on Configurations All off

# Regression



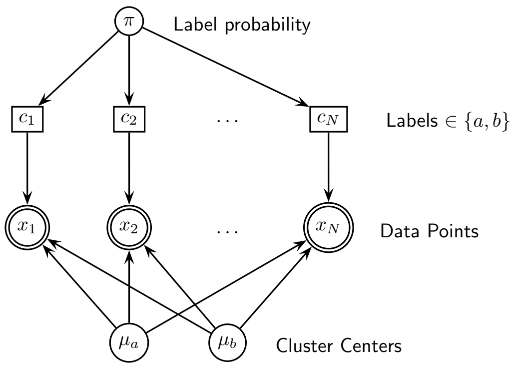


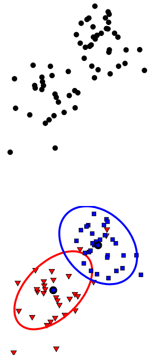

# Clustering





# Clustering



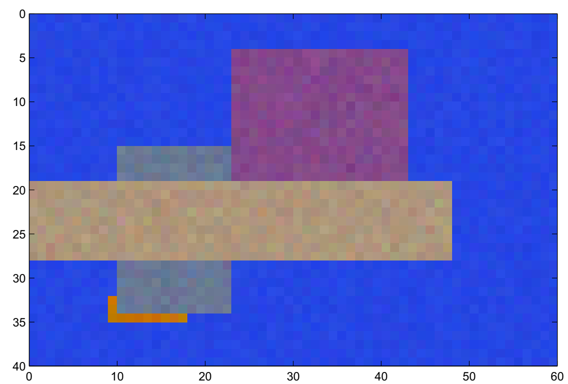



$$(\mu_a^*, \mu_b^*, \pi^*) = \underset{\mu_a,\mu_b,\pi}{\operatorname{argmax}} \sum_{c_{1:N}} \prod_{i=1}^{N} p(x_i|\mu_a, \mu_b, c_i)p(c_i|\pi)$$

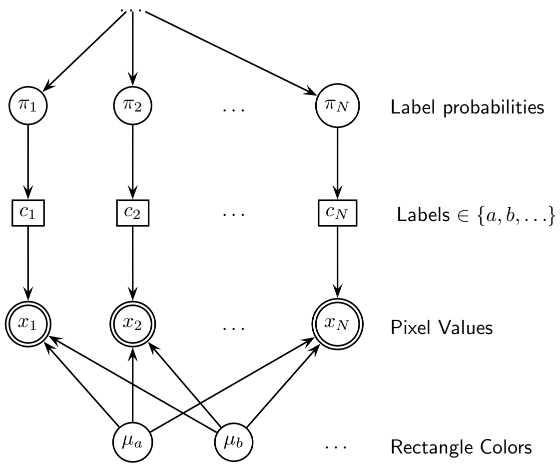


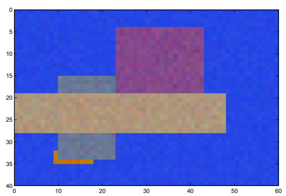
# Computer vision / Cognitive Science

How many rectangles are there in this image?



# Computer vision / Cognitive Science

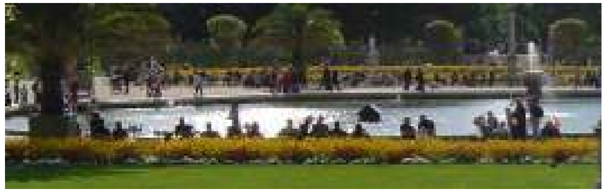


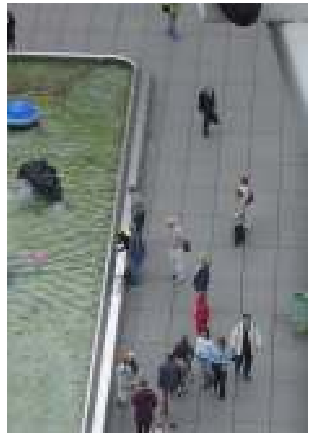





# Computer Vision

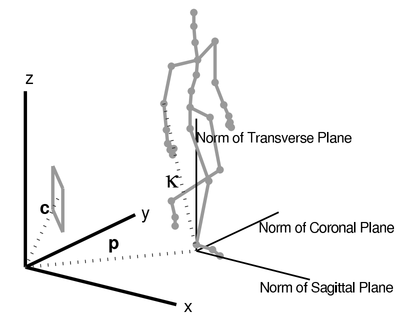
How many people are there in these images?

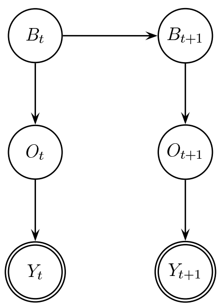


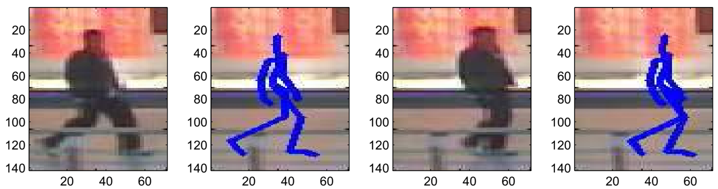




# Visual Tracking

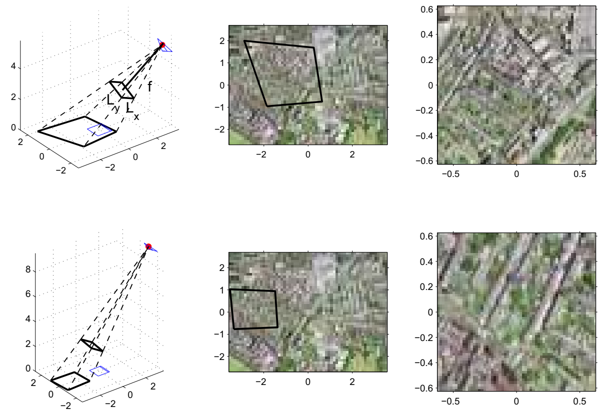


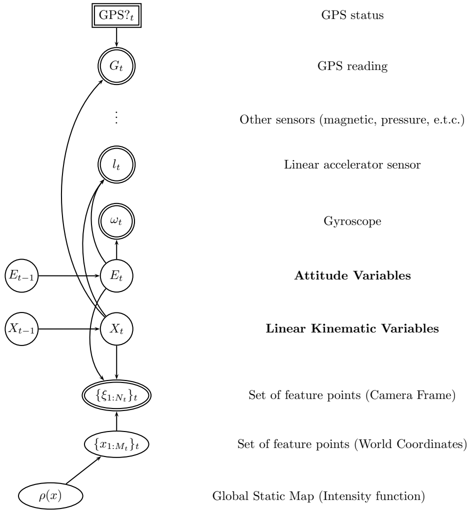




# Navigation, Robotics



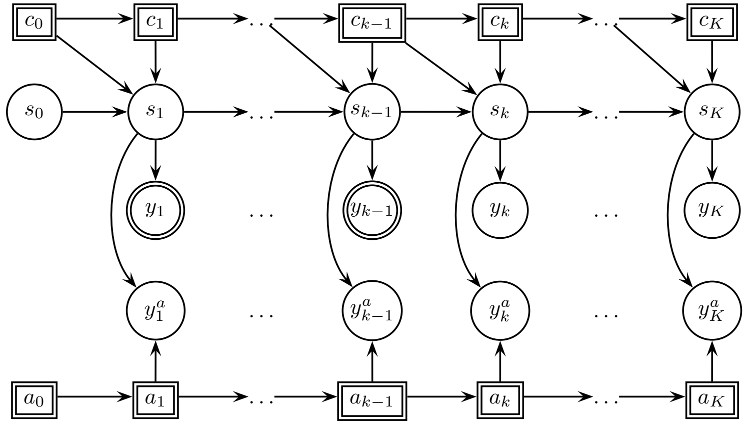
# Navigation, Robotics






# Computer Accompaniment

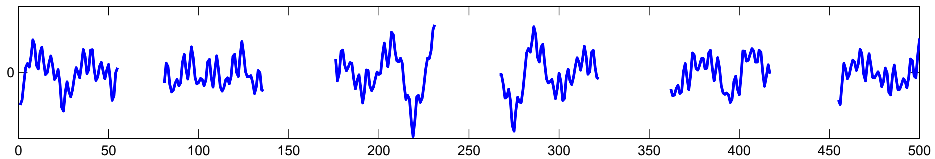
(Music Plus One, Raphael 2000 [18], Dannenberg and Raphael 2006)





# Audio Restoration

- During download or transmission, some samples of audio are lost
- Estimate missing samples given clean ones

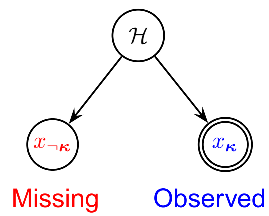


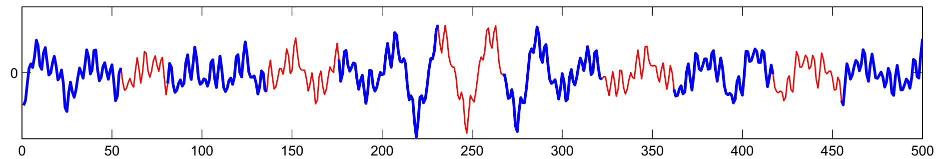


# Examples: Audio Restoration

$$p(x_{\neg\kappa}|x_{\kappa}) \propto \int d\mathcal{H} p(x_{\neg\kappa}|\mathcal{H}) p(x_{\kappa}|\mathcal{H}) p(\mathcal{H})$$

$$\mathcal{H} \equiv \text{(parameters, hidden states)}$$



## Restoration

(Cemgil and Godsill 2005 [5])

- Piano
  - Signal with missing samples (37%)
  - Reconstruction, 7.68 dB improvement
  - Original
- Trumpet
  - Signal with missing samples (37%)
  - Reconstruction, 7.10 dB improvement
  - Original



## Basic Building Blocks



## Probability Distributions : Exponential Family

- Following distributions are used often as elementary building blocks:
  - Gaussian
  - Gamma, Inverse Gamma, (Exponential, Chi-square, Wishart)
  - Dirichlet
  - Discrete (Categorical), Bernoulli, multinomial
- All of those distributions can be written as

$$p(x|\theta) = \exp\{\theta^\top \psi(x) - A(\theta)\}$$

$$A(\theta) = \log \int_{\mathcal{X}^n} dx \; \exp(\theta^\top \psi(x)) \quad \text{log-partition function}$$

$$\theta \quad \text{canonical parameters}$$

$$\psi(x) \quad \text{sufficient statistics}$$



## Example: Bernoulli

Binary (Bernoulli) random variable $c = \{0, 1\}$ with probability of sucsess $w$

$$p(c = 1|w) = w \qquad p(c = 0|w) = 1 - w$$

We write

$$\begin{aligned} p(c|w) &= w^c(1-w)^{1-c} \\ &= \exp\left(c \log w + (1-c)\log(1-w)\right) \\ &= \exp\left(\log(\frac{w}{1-w})c + \log(1-w)\right) \\ &= \mathcal{C}(c; w) \end{aligned}$$

$\mathcal{C}$ stays for categorical

## Example, Univariate Gaussian

The Gaussian distribution with mean $m$ and covariance $S$ has the form

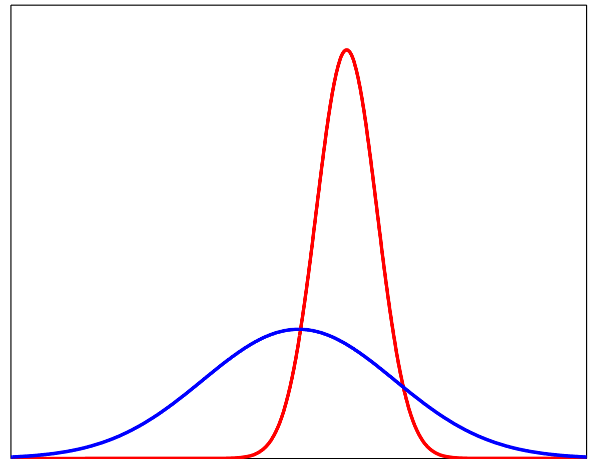
$$\begin{aligned}
\mathcal{N}(x; m, S) &= (2\pi S)^{-1/2} \exp\{-\frac{1}{2}(x-m)^2/S\} \\
&= \exp\{-\frac{1}{2}(x^2 + m^2 - 2xm)/S - \frac{1}{2}\log(2\pi S)\} \\
&= \exp\left\{\frac{m}{S}x - \frac{1}{2S}x^2 - \left(\frac{1}{2}\log(2\pi S) + \frac{1}{2S}m^2\right)\right\} \\
&= \exp\{\underbrace{\begin{pmatrix} m/S \\ -\frac{1}{2}/S \end{pmatrix}^\top}_{\theta} \underbrace{\begin{pmatrix} x \\ x^2 \end{pmatrix}}_{\psi(x)} - A(\theta)\}
\end{aligned}$$

Hence by matching coefficients we have

$$\exp\left\{-\tfrac{1}{2}Kx^2 + hx + g\right\} \Leftrightarrow S = K^{-1} \qquad m = K^{-1}h$$



## Example, Gaussian





## Example, Inverse Gamma

The inverse Gamma distribution with shape $a$ and scale $b$

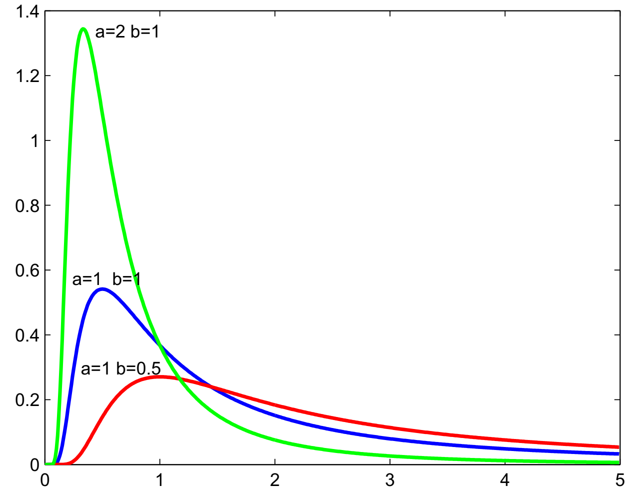
$$\begin{aligned}
\mathcal{IG}(r; a, b) &= \frac{1}{\Gamma(a)} \frac{r^{-(a+1)}}{b^a} \exp(-\frac{1}{br}) \\
&= \exp\left(-(a+1)\log r - \frac{1}{br} - \log\Gamma(a) - a\log b\right) \\
&= \exp\left(\begin{pmatrix} -(a+1) \\ -1/b \end{pmatrix}^\top \begin{pmatrix} \log r \\ 1/r \end{pmatrix} - \log\Gamma(a) - a\log b\right)
\end{aligned}$$

Hence by matching coefficients, we have

$$\exp\left\{\alpha\log r + \beta\frac{1}{r} + c\right\} \Leftrightarrow a = -\alpha - 1 \qquad b = -1/\beta$$



## Example, Inverse Gamma



## Example, Beta

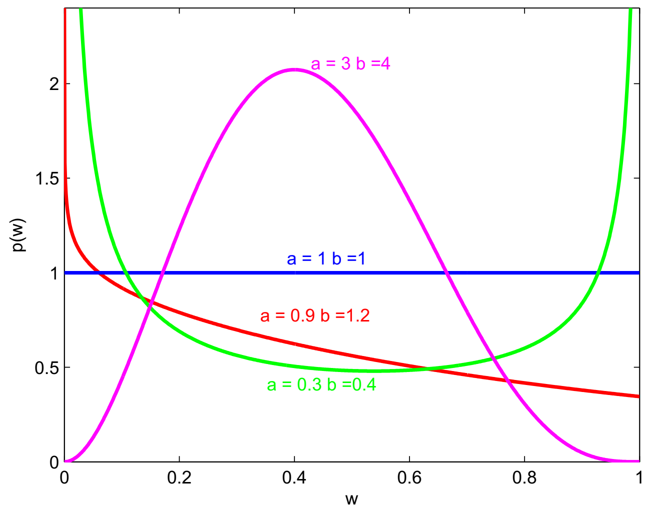
$$\begin{aligned}\mathcal{B}(w;a,b) &\equiv \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} w^{a-1}(1-w)^{b-1} \\ &= \exp\left((a-1)\log w + (b-1)\log(1-w) - A(a,b)\right) \\ &= \exp\left(\begin{pmatrix} a-1 & b-1 \end{pmatrix}\begin{pmatrix} \log w \\ \log(1-w)\end{pmatrix} - A(a,b)\right) \\ A(a,b) &= \log\Gamma(a) + \log\Gamma(b) - \log\Gamma(a+b)\end{aligned}$$

Mean :

$$\langle w \rangle_{\mathcal{B}} = a/(a+b)$$



## Example, Beta






## Conjugate priors: Posterior is in the same family as the prior.

Example: posterior inference for the probability of sucsess $w$ of a binary (Bernoulli) random variable $c$

$$\begin{aligned} p(c|w) &= \mathcal{C}(c;w) = \exp\left(c\log w + (1-c)\log(1-w)\right) \\ p(w) &= \mathcal{B}(w;a,b)\end{aligned}$$

$$\begin{aligned} p(w|c) &\propto p(c|w)p(w) \\ &\propto \exp\left(c\log w + (1-c)\log(1-w)\right) \\ &\quad \times \exp\left((a-1)\log w + (b-1)\log(1-w)\right) \\ &\propto \mathcal{B}(w;a+c,b+(1-c))\end{aligned}$$

$$p(w|c) = \begin{cases} \mathcal{B}(w;a+1,b) & c=1 \\ \mathcal{B}(w;a,b+1) & c=0\end{cases}$$



## Conjugate priors: Posterior is in the same family as the prior.

Example: posterior inference for the variance $R$ of a zero mean Gaussian.

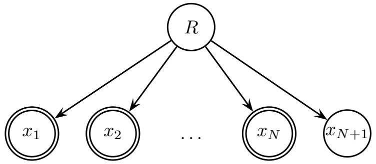
$$\begin{aligned} p(x|R) &= \mathcal{N}(x;0,R) \\ p(R) &= \mathcal{IG}(R;a,b)\end{aligned}$$

$$\begin{aligned} p(R|x) &\propto p(R)p(x|R) \\ &\propto \exp\left(-(a+1)\log R - (1/b)\frac{1}{R}\right)\exp\left(-(x^2/2)\frac{1}{R} - \frac{1}{2}\log R\right) \\ &= \exp\left(\begin{pmatrix} -(a+1+\frac{1}{2}) \\ -(1/b + x^2/2)\end{pmatrix}^\top \begin{pmatrix}\log R \\ 1/R\end{pmatrix}\right) \\ &\propto \mathcal{IG}(R; a+\frac{1}{2}, \frac{2}{x^2+2/b})\end{aligned}$$

Like the prior, this is an inverse-Gamma distribution.

## Conjugate priors: Posterior is in the same family as the prior.

Example: posterior inference of variance $R$ from $x_1, \dots, x_N$.

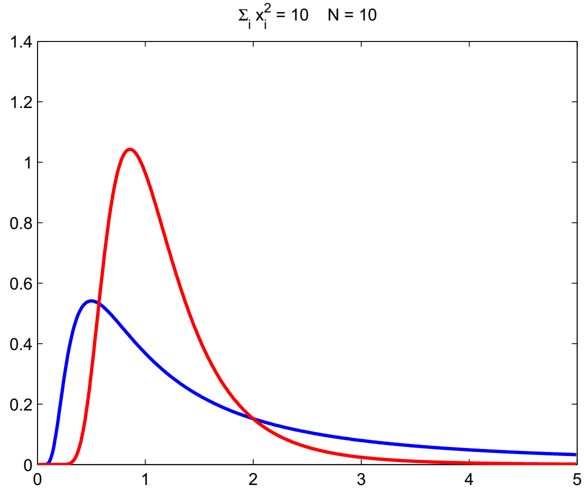



$$
\begin{aligned}
p(R|x) &\propto p(R) \prod_{i=1}^{N} p(x_i|R) \\
&\propto \exp\left(-(a+1)\log R - (1/b)\frac{1}{R}\right) \exp\left(-\left(\frac{1}{2}\sum_i x_i^2\right)\frac{1}{R} - \frac{N}{2}\log R\right) \\
&= \exp\left(\left(\begin{array}{c} -(a+1+\frac{N}{2}) \\ -(1/b+\frac{1}{2}\sum_i x_i^2) \end{array}\right)^\top \left(\begin{array}{c} \log R \\ 1/R \end{array}\right)\right) \propto \mathcal{IG}(R; a+\frac{N}{2}, \frac{2}{\sum_i x_i^2 + 2/b})
\end{aligned}
$$

Sufficient statistics are **additive**

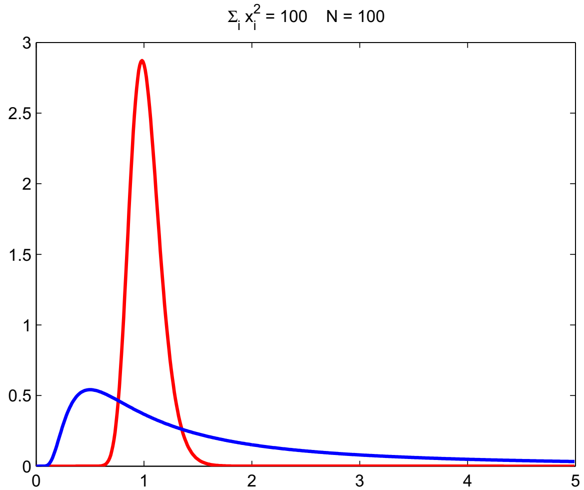


## Inverse Gamma, $\sum_i x_i^2 = 10 \quad N = 10$



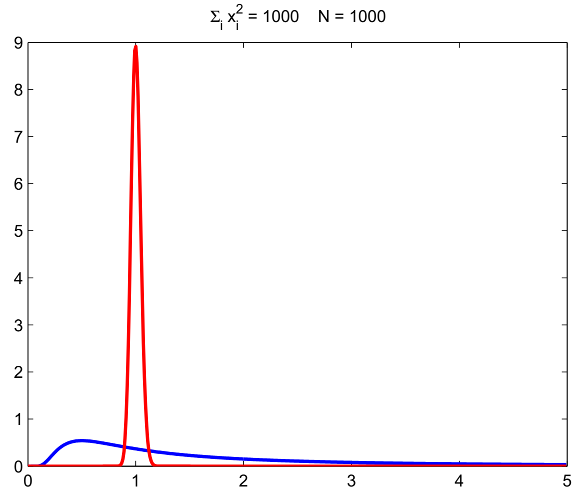



## Inverse Gamma, $\sum_i x_i^2 = 100 \quad N = 100$

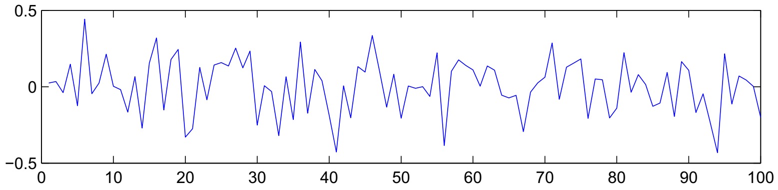





## Inverse Gamma, $\sum_i x_i^2 = 1000 \quad N = 1000$



# Example: AR(1) model




$$x_k = Ax_{k-1} + \epsilon_k \qquad k = 1 \dots K$$

$\epsilon_k$ is i.i.d., zero mean and normal with variance $R$.

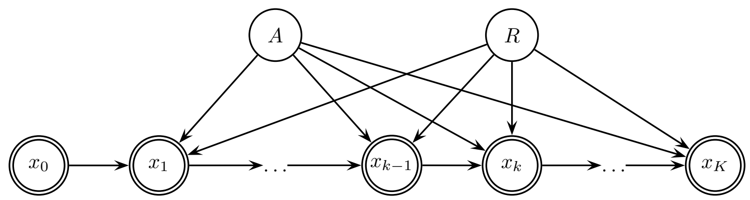
**Estimation problem**:

Given $x_0, \dots, x_K$, determine coefficient $A$ and variance $R$ (both scalars).



# AR(1) model, Generative Model notation

$$\begin{aligned} A &\sim \mathcal{N}(A; 0, P) \\ R &\sim \mathcal{IG}(R; \nu, \beta/\nu) \\ x_k | x_{k-1}, A, R &\sim \mathcal{N}(x_k; Ax_{k-1}, R) \qquad x_0 = \hat{x}_0 \end{aligned}$$




Observed variables are shown with double circles

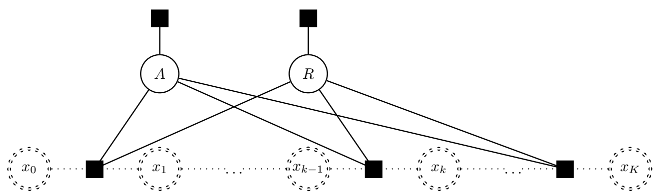


# AR(1) Model. Bayesian Posterior Inference

$$\begin{aligned} p(A, R | x_0, x_1, \dots, x_K) &\propto p(x_1, \dots, x_K | x_0, A, R) p(A, R) \\ \text{Posterior} &\propto \text{Likelihood} \times \text{Prior} \end{aligned}$$

Using the Markovian (conditional independence) structure we have

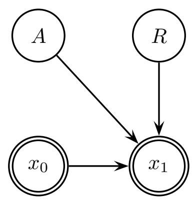
$$p(A, R | x_0, x_1, \dots, x_K) \quad \propto \quad \left( \prod_{k=1}^{K} p(x_k | x_{k-1}, A, R) \right) p(A) p(R)$$

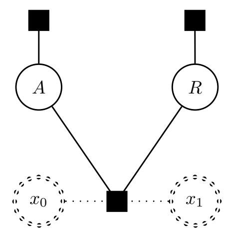





# Numerical Example

Suppose $K = 1$,




By Bayes' Theorem and the structure of AR(1) model

$$\begin{aligned} p(A, R | x_0, x_1) &\propto p(x_1 | x_0, A, R) p(A) p(R) \\ &= \mathcal{N}(x_1; Ax_0, R) \mathcal{N}(A; 0, P) \mathcal{IG}(R; \nu, \beta/\nu) \end{aligned}$$

## Numerical Example

$$
\begin{aligned}
p(A, R|x_0, x_1) &\propto p(x_1|x_0, A, R)p(A)p(R) \\
&= \mathcal{N}(x_1; Ax_0, R)\mathcal{N}(A; 0, P)\mathcal{IG}(R; \nu, \beta/\nu) \\
&\propto \exp\left(-\frac{1}{2}\frac{x_1^2}{R} + x_0 x_1 \frac{A}{R} - \frac{1}{2}\frac{x_0^2 A^2}{R} - \frac{1}{2}\log 2\pi R\right) \\
&\quad \exp\left(-\frac{1}{2}\frac{A^2}{P}\right)\exp\left(-(\nu+1)\log R - \frac{\nu}{\beta}\frac{1}{R}\right)
\end{aligned}
$$

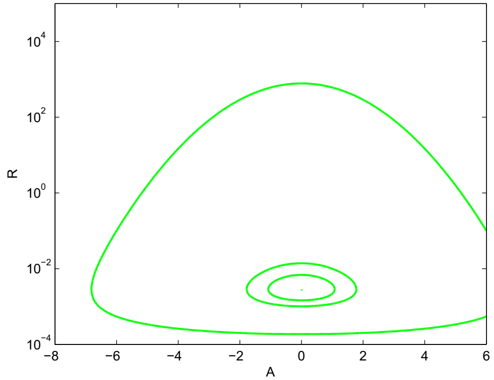
This posterior has a nonstandard form

$$
\exp\left(\alpha_1 \frac{1}{R} + \alpha_2 \frac{A}{R} + \alpha_3 \frac{A^2}{R} + \alpha_4 \log R + \alpha_5 A^2\right)
$$



## Numerical Example, the prior $p(A, R)$

Equiprobability contour of $p(A)p(R)$

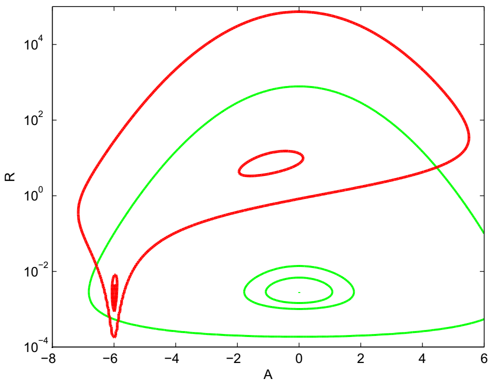



$A \sim \mathcal{N}(A; 0, 1.2) \qquad R \sim \mathcal{IG}(R; 0.4, 250)$

Suppose: $x_0 = 1 \qquad x_1 = -6 \qquad x_1 \sim \mathcal{N}(x_1; Ax_0, R)$



## Numerical Example, the posterior $p(A, R|x)$




Note the bimodal posterior with $x_0 = 1, x_1 = -6$

- $A \approx -6 \Leftrightarrow$ low noise variance $R$.
- $A \approx 0 \Leftrightarrow$ high noise variance $R$.



## Remarks

- The point estimates such as ML or MAP are not always representative about the solution
- (Unfortunately), exact posterior inference is only possible for few special cases
- Even very simple models can lead easily to complicated posterior distributions
- Ambiguous data usually leads to a multimodal posterior, each mode corresponding to one possible explanation

## Remarks

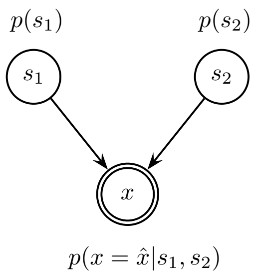
- *A-priori* independent variables often become dependent *a-posteriori* (“Explaining away”)
- The difficulty of an inference problem depends, among others, upon the particular “parameter regime” and observed data sequence

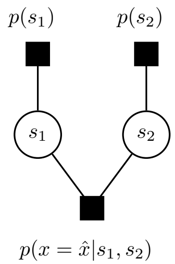


## Approximate Inference



## A Toy Model

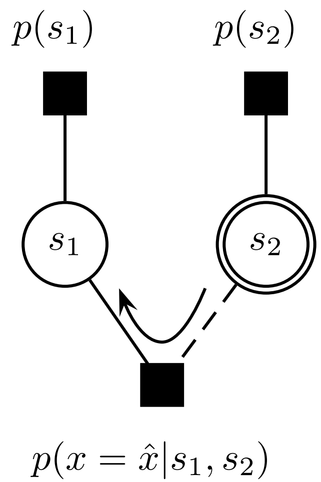


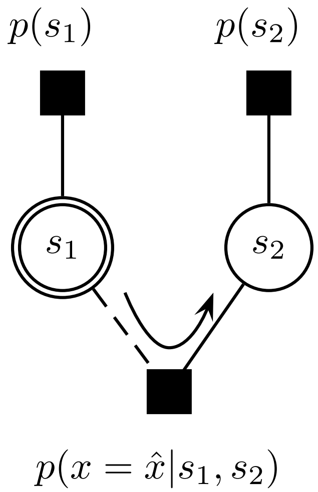

$$
\begin{aligned}
s_1 &\sim p(s_1) = \mathcal{N}(s_1; \mu_1, P_1) \\
s_2 &\sim p(s_2) = \mathcal{N}(s_2; \mu_2, P_2) \\
x|s_1, s_2 &\sim p(x|s_1, s_2) = \mathcal{N}(x; s_1 + s_2, R)
\end{aligned}
$$



## Gibbs Sampling

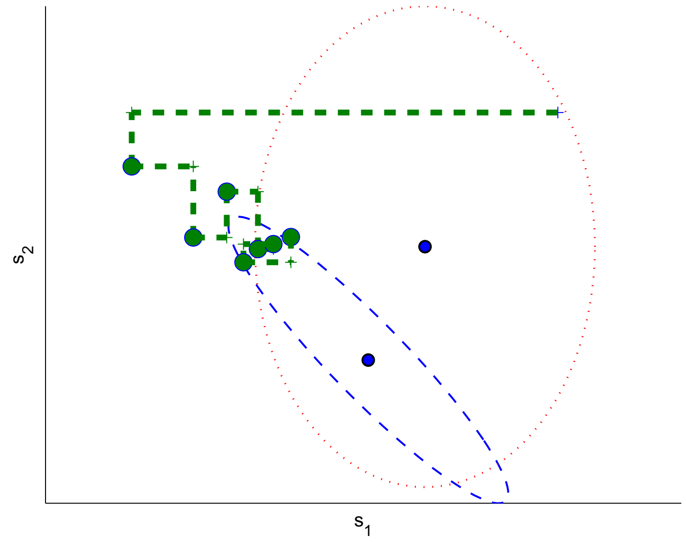


## Gibbs Sampling



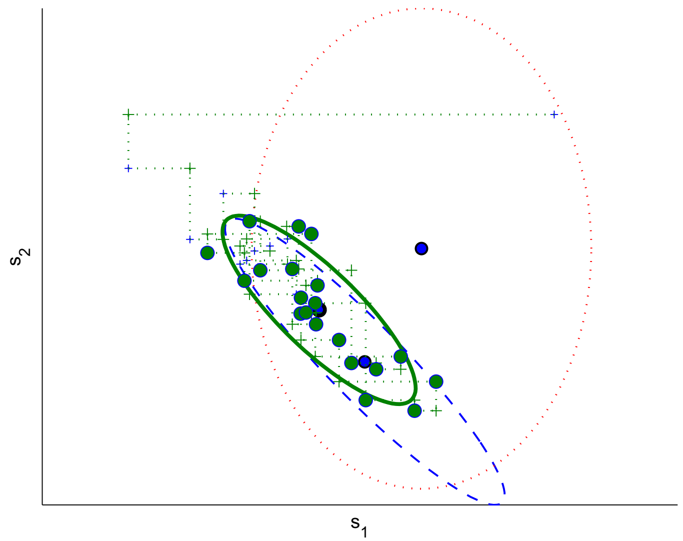



## Gibbs Sampling



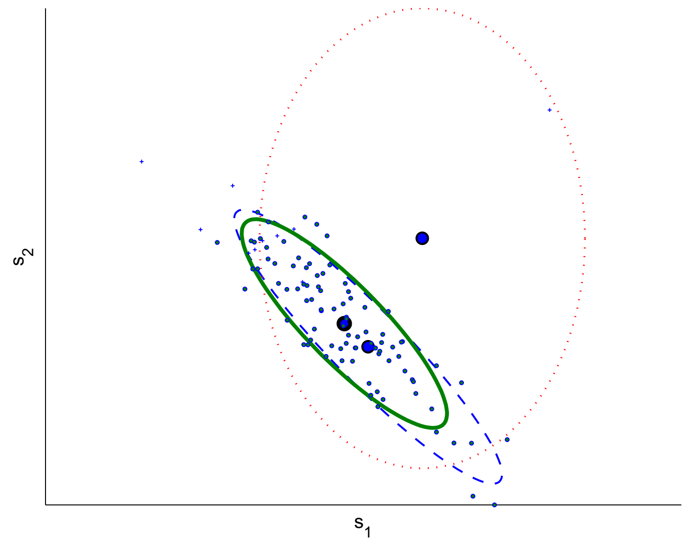



## Gibbs Sampling, $t = 20$

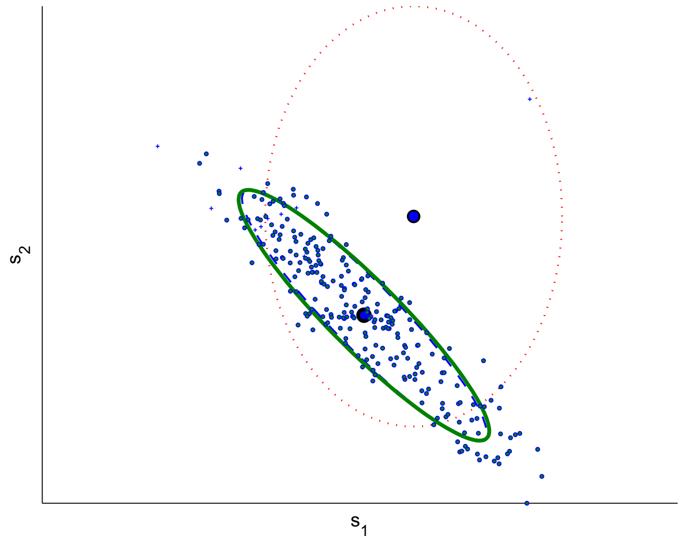





## Gibbs Sampling, $t = 100$



## Gibbs Sampling, $t = 250$






## Gibbs Sampling

- A remarkable fact is that we can estimate any desired expectation by ergodic averages

$$\langle f(\mathbf{s}) \rangle_{\mathcal{P}} \approx \frac{1}{t - t_0} \sum_{n=t_0}^{t} f(\mathbf{s}^{(n)})$$

- Consecutive samples $\mathbf{s}^{(t)}$ are dependent but we can "pretend" as if they are independent!
- The sequence of samples are obtained from a Markov chain, hence the name MCMC



## Variational Bayes (VB), mean field

We will approximate the posterior $\mathcal{P}$ with a simpler distribution $\mathcal{Q}$.

$$\mathcal{P} = \frac{1}{Z_x} p(x = \hat{x} | s_1, s_2) p(s_1) p(s_2)$$
$$\mathcal{Q} = q(s_1) q(s_2)$$

Here, we choose

$$q(s_1) = \mathcal{N}(s_1; m_1, S_1) \qquad q(s_2) = \mathcal{N}(s_2; m_2, S_2)$$

A "measure of fit" between distributions is the KL divergence



## Kullback-Leibler (KL) Divergence

- A "quasi-distance" between two distributions $\mathcal{P} = p(x)$ and $\mathcal{Q} = q(x)$.

$$KL(\mathcal{P}||\mathcal{Q}) \equiv \int_{\mathcal{X}} dx p(x) \log \frac{p(x)}{q(x)} = \langle \log \mathcal{P} \rangle_{\mathcal{P}} - \langle \log \mathcal{Q} \rangle_{\mathcal{P}}$$

- Unlike a metric, (in general) it is not symmetric,

$$KL(\mathcal{P}||\mathcal{Q}) \neq KL(\mathcal{Q}||\mathcal{P})$$

- But it is non-negative (by Jensen's Inequality)

$$\begin{aligned} KL(\mathcal{P}||\mathcal{Q}) &= -\int_{\mathcal{X}} dx p(x) \log \frac{q(x)}{p(x)} \\ &\geq -\log \int_{\mathcal{X}} dx p(x) \frac{q(x)}{p(x)} = -\log \int_{\mathcal{X}} dx q(x) = -\log 1 = 0 \end{aligned}$$

## OSSS example, cont.

Let the approximating distribution be factorized as

$$\mathcal{Q} = q(s_1)q(s_2)$$

$$q(s_1) = \mathcal{N}(s_1; m_1, S_1) \qquad q(s_2) = \mathcal{N}(s_2; m_2, S_2)$$

The $m_i$ and $S_j$ are the *variational* parameters to be optimized to minimize

$$KL(\mathcal{Q}||\mathcal{P}) = \langle \log \mathcal{Q} \rangle_{\mathcal{Q}} - \left\langle \log \underbrace{\frac{1}{Z_x}\phi(s_1, s_2)}_{=\mathcal{P}} \right\rangle_{\mathcal{Q}} \tag{1}$$



## The form of the mean field solution

$$\begin{aligned} 0 &\leq \langle \log q(s_1)q(s_2) \rangle_{q(s_1)q(s_2)} + \log Z_x - \langle \log \phi(s_1, s_2) \rangle_{q(s_1)q(s_2)} \\ \log Z_x &\geq \langle \log \phi(s_1, s_2) \rangle_{q(s_1)q(s_2)} - \langle \log q(s_1)q(s_2) \rangle_{q(s_1)q(s_2)} \\ &\equiv -F(p; q) + H(q) \end{aligned} \tag{2}$$

Here, $F$ is the *energy* and $H$ is the *entropy*. We need to maximize the right hand side.

$$\text{Evidence} \geq -\text{Energy} + \text{Entropy}$$

Note r.h.s. is a **lower bound** [16]. The mean field equations **monotonically** increase this bound. Good for assessing convergence and debugging computer code.



## The form of the solution

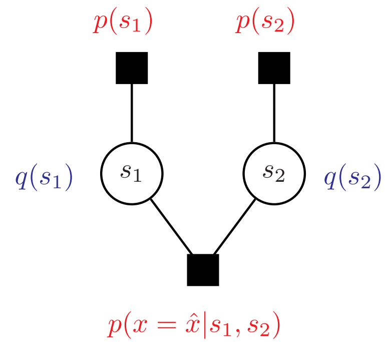
- No direct analytical solution
- We obtain fixed point equations in closed form

$$\begin{aligned} q(s_1) &\propto \exp(\langle \log \phi(s_1, s_2) \rangle_{q(s_2)}) \\ q(s_2) &\propto \exp(\langle \log \phi(s_1, s_2) \rangle_{q(s_1)}) \end{aligned}$$

Note the nice symmetry

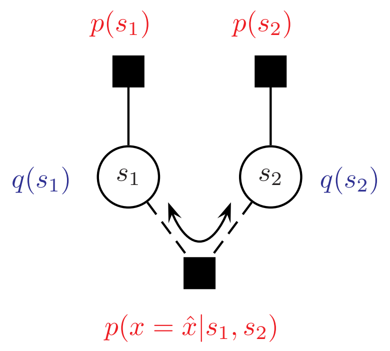


## Variational Message Passing on a Factor Graph




- **Factor nodes**: Factor potentials (local functions) defining the posterior $\mathcal{P}$.
- **Variable nodes**: Now, think of them as “factors” of the approximating distribution $\mathcal{Q}$. (Caution – non standard interpretation!)

## Fixed Point Iteration

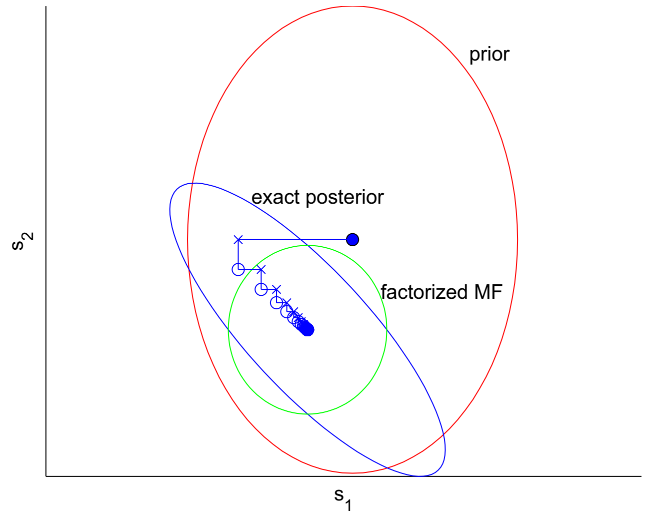



$$\log q(s_1) \quad \leftarrow \quad \log p(s_1) + \langle \log p(x = \hat{x}|s_1, s_2) \rangle_{q(s_2)}$$

$$\log q(s_2) \quad \leftarrow \quad \log p(s_2) + \langle \log p(x = \hat{x}|s_1, s_2) \rangle_{q(s_1)}$$



## VB Convergence






## Direct Link to Expectation-Maximisation (EM) [12]

Suppose we choose one of the distributions degenerate, i.e.

$$\tilde{q}(s_2) \quad = \quad \delta(s_2 - \tilde{m})$$

where $\tilde{m}$ corresponds to the "location parameter" of $\tilde{q}(s_2)$. We need to find the closest degenerate distribution to the actual mean field solution $q(s_2)$, hence we take one more KL and minimize

$$\tilde{m} \quad = \quad \operatorname*{argmin}_{\xi} KL(\delta(s_2 - \xi)||q(s_2))$$

It can be shown that this leads exactly to the EM fixed point iterations.



## Iterated Conditional Modes (ICM) [2, 11]

If we choose both distributions degenerate, i.e.

$$\tilde{q}(s_1) \quad = \quad \delta(s_1 - \tilde{m}_1)$$
$$\tilde{q}(s_2) \quad = \quad \delta(s_2 - \tilde{m}_2)$$

It can be shown that this leads exactly to the ICM fixed point iterations. This algorithm is equivalent to coordinate ascent in the original posterior surface $\phi(s_1, s_2)$.

$$\tilde{m}_1 \quad = \quad \operatorname*{argmax}_{s_1} \phi(s_1, s_2 = \tilde{m}_2)$$
$$\tilde{m}_2 \quad = \quad \operatorname*{argmax}_{s_2} \phi(s_1 = \tilde{m}_1, s_2)$$

## ICM, EM, VB ...

For OSSS, all algorithms are identical. This is in general not true.

While algorithmic details are very similar, there can be big qualitative differences in terms of fixed points.

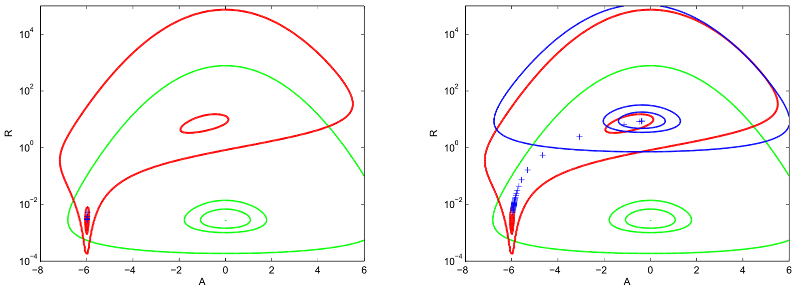



Figure 1: Left, ICM, Right VB. EM is similar to ICM in this AR(1) example.

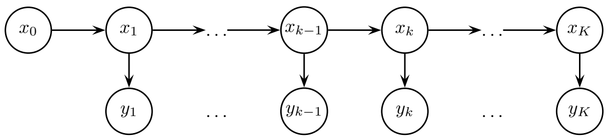


## Models and Applications



## Time series models and Inference, Terminology

In music signal processing and machine learning many phenomena are modelled by dynamical models



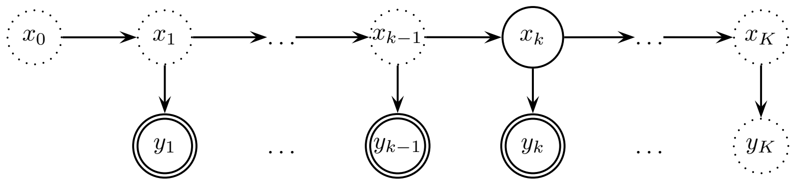

$$
\begin{aligned}
x_k &\sim p(x_k|x_{k-1}) && \text{Transition Model} \\
y_k &\sim p(y_k|x_k) && \text{Observation Model}
\end{aligned}
$$

- $x$ is the latent state (tempo, pitch, velocity, attitude, class label, ...)
- $y$ are observations (samples, onsets, sensor reading, pixels, features, ... )
- In a full Bayesian setting, $x$ includes unknown model parameters



## Online Inference, Terminology

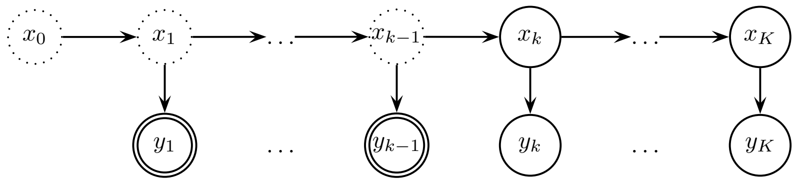
- **Filtering:** $p(x_k|y_{1:k})$
  - Distribution of current state given all past information
  - Realtime/Online/Sequential Processing




- Potentially confusing misnomer:
  - More general than “digital filtering” (convolution) in DSP – but algoritmically related for some models (KFM)

## Online Inference, Terminology

- **Prediction** $p(y_{k:K}, x_{k:K}|y_{1:k-1})$
  - evaluation of possible future outcomes; like filtering without observations

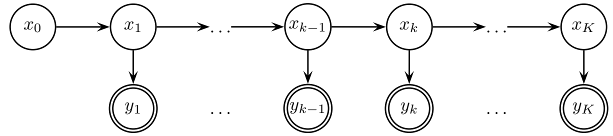


- Accompaniment, Tracking, Restoration

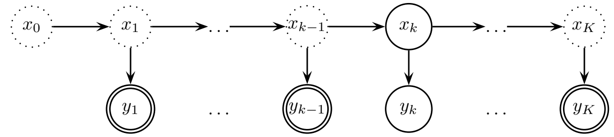


## Offline Inference, Terminology

- **Smoothing** $p(x_{0:K}|y_{1:K})$,
  **Most likely trajectory – Viterbi path** $\arg\max_{x_{0:K}} p(x_{0:K}|y_{1:K})$
  better estimate of past states, essential for learning



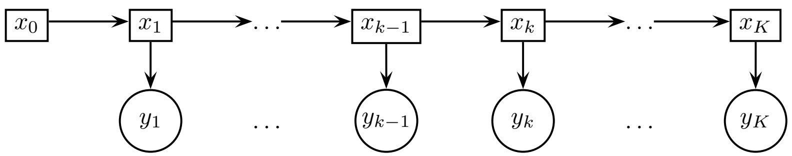
- **Interpolation** $p(y_k, x_k|y_{1:k-1}, y_{k+1:K})$
  fill in lost observations given past and future





## Hidden Markov Model [17]

- Mixture model evolving in time

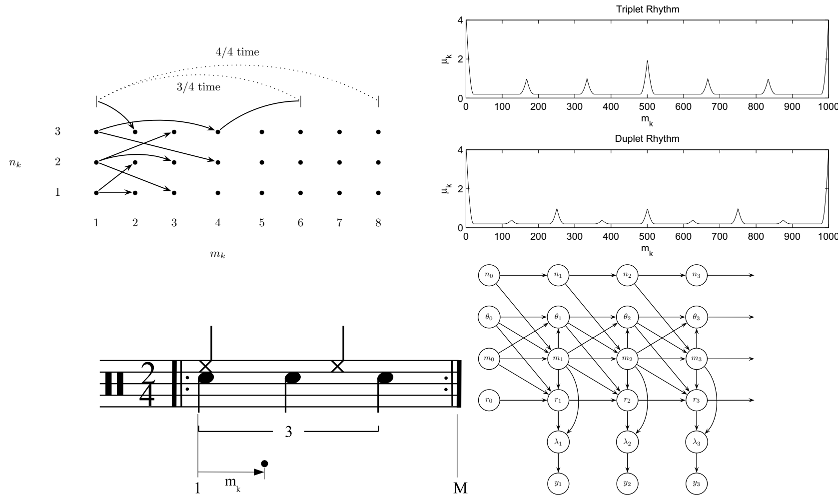


- Observations $y_k$ are continuous or discrete
- Latent variables $x_k$ are discrete
  - Represents the fading memory of the process
- Exact inference possible if $x_k$ has a “small” number of states

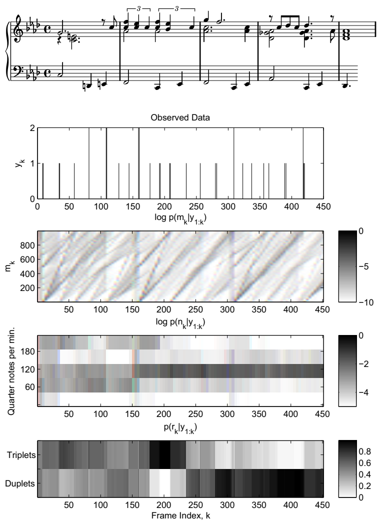


## Tempo, Rhythm, Meter analysis

Bar Pointer Model (Whiteley, Cemgil, Godsill 2006)

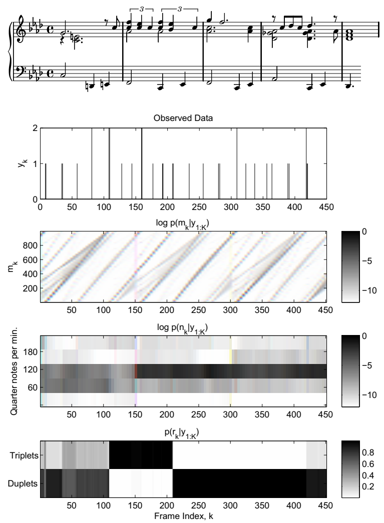


# Filtering





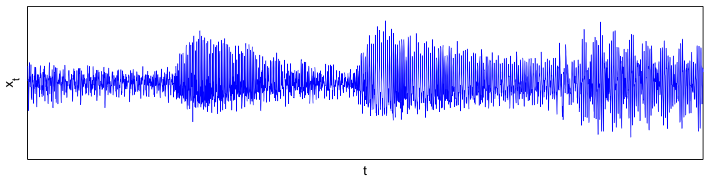

# Smoothing





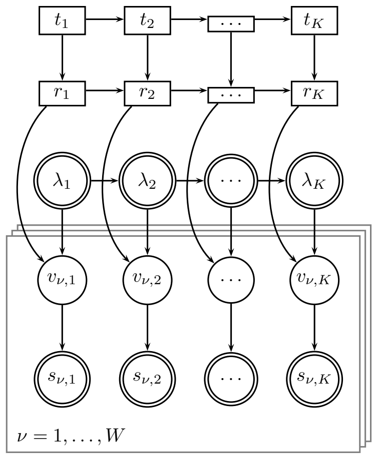

# Score-Performance matching (Peeling, Cemgil, Godsill)

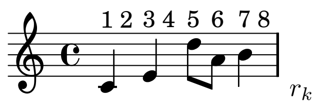
- Given a musical score, associate note events with the audio

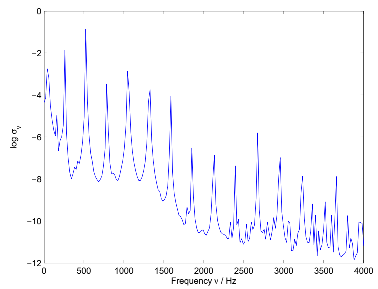





# Score-Performance matching - Graphical Model

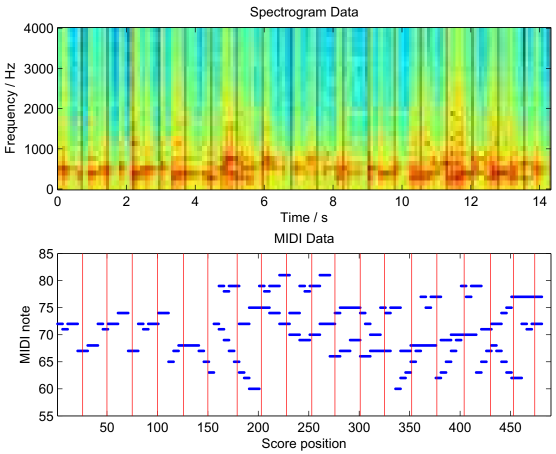


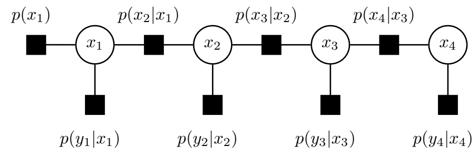

$$v_{\nu,\tau} \sim \mathcal{IG}(v_{\nu,\tau}; a, 1/(a\lambda\sigma_\nu(r_\tau)))$$

# Score-Performance matching






# Exact Inference in HMM, Forward/Backward Algorithm




- Forward Pass

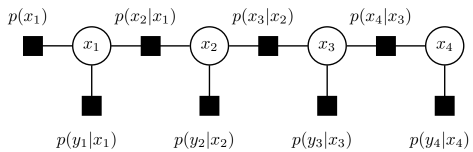
$$\begin{aligned} p(y_{1:K}) &= \sum_{x_{1:K}} p(y_{1:K}|x_{1:K})p(x_{1:K}) \\ &= \underbrace{\sum_{x_K} p(y_T|x_K) \sum_{x_{K-1}} p(x_K|x_{K-1})}_{\alpha_K} \cdots \sum_{x_2} p(x_3|x_2) \underbrace{p(y_2|x_2) \overbrace{\sum_{x_1} p(x_2|x_1)}^{\alpha_{2|1}}}_{\alpha_2} \underbrace{p(y_1|x_1) \overbrace{p(x_1)}^{\alpha_{1|0}}}_{\alpha_1} \end{aligned}$$

- Backward Pass

$$p(y_{1:K}) = \sum_{x_1} p(x_1)p(y_1|x_1) \dots \underbrace{\sum_{x_{K-1}} p(x_{K-1}|x_{K-2})p(y_{K-1}|x_{K-1})}_{\beta_{K-2}} \underbrace{\sum_{x_K} p(x_K|x_{K-1})p(y_K|x_K)}_{\beta_{K-1}} \underbrace{\mathbf{1}}_{\beta_K}$$



# Exact Inference in HMM, Viterbi Algorithm




- Merely replace sum by max, equivalent to dynamic programming
- Forward Pass

$$\begin{aligned} p(y_{1:K}|x^*_{1:K}) &= \max_{x_{1:K}} p(y_{1:K}|x_{1:K})p(x_{1:K}) \\ &= \underbrace{\max_{x_K} p(y_T|x_K) \max_{x_{K-1}} p(x_K|x_{K-1})}_{\alpha_K} \dots \max_{x_2} p(x_3|x_2) \underbrace{p(y_2|x_2) \overbrace{\max_{x_1} p(x_2|x_1)}^{\alpha_{2|1}}}_{\alpha_2} \underbrace{p(y_1|x_1) \overbrace{p(x_1)}^{\alpha_{1|0}}}_{\alpha_1} \end{aligned}$$

- Backward Pass

$$p(y_{1:K}|x^*_{1:K}) = \max_{x_1} p(x_1)p(y_1|x_1) \dots \underbrace{\max_{x_{K-1}} p(x_{K-1}|x_{K-2})p(y_{K-1}|x_{K-1})}_{\beta_{K-2}} \underbrace{\max_{x_K} p(x_K|x_{K-1})p(y_K|x_K)}_{\beta_{K-1}} \underbrace{\mathbf{1}}_{\beta_K}$$

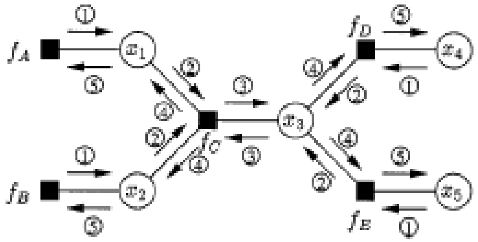


# Exact Inference on general factor graphs

- When the factor graph is a tree, one can define a local message propagation
  - If factor graph is not a tree, one can always do this by clustering nodes together
- Sum-product
  - Generalises Forward/Backward
  - Rule:
    "The message sent from a node $v$ on an edge $e$ is the product of the local function at $v$ (or the unit function if is a variable node) with all messages received at $v$ on edges other than $e$, summarized for the variable associated with $e$."
- Max-product
  - Generalises Viterbi

Look at the seminal tutorial paper by Kschischang, Frey and Loeliger [14] on factor graphs.

## Exact Inference on general factor graphs




**variable to local function:**

$$\mu_{x \to f}(x) = \prod_{h \in n(x) \setminus \{f\}} \mu_{h \to x}(x)$$

**local function to variable:**

$$\mu_{f \to x}(x) = \sum_{\sim\{x\}} \left( f(X) \prod_{y \in n(f) \setminus \{x\}} \mu_{y \to f}(y) \right)$$



## Kalman Filter Models, Linear Dynamical Systems

- The latent variables $s_k$ and observations $y_k$ are continuous
- The transition and observations models are linear
  - Example: a point moving on the real line
  - A deterministic dynamical system with two state variables

$$\mathbf{s}_k = \begin{pmatrix} \text{position} \\ \text{velocity} \end{pmatrix}_k = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{s}_{k-1} = \mathbf{A}\mathbf{s}_{k-1}$$

$$y_k = \text{position}_k = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{s}_k = \mathbf{C}\mathbf{s}_k$$



## Tracking

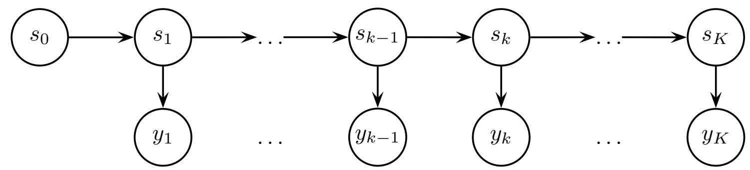
- We allow random (unknown) accelerations

$$\begin{aligned} \mathbf{s}_k &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{s}_{k-1} + \epsilon_k \\ &= \mathbf{A}\mathbf{s}_{k-1} + \epsilon_k \end{aligned}$$

$$\begin{aligned} y_k &= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{s}_k + \nu_k \\ &= \mathbf{C}\mathbf{s}_k + \nu_k \end{aligned}$$



## Tracking

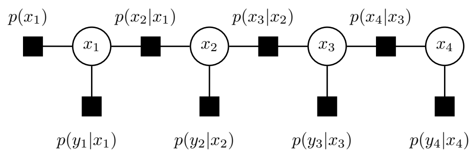



- In generative model notation

$$\begin{aligned} \mathbf{s}_k &\sim \mathcal{N}(\mathbf{s}_k; \mathbf{A}\mathbf{s}_{k-1}, Q) \\ y_k &\sim \mathcal{N}(y_k; \mathbf{C}\mathbf{s}_k, R) \end{aligned}$$

- Tracking = estimating the latent state of the system = Kalman filtering

## Kalman Filtering and Smoothing (two filter formulation)




- Forward Pass

$$p(y_{1:K}) = \underbrace{\int_{x_K} p(y_T|x_K) \int_{x_{K-1}} p(x_K|x_{K-1})}_{\alpha_K} \dots \int_{x_2} p(x_3|x_2) \underbrace{p(y_2|x_2) \overbrace{\int_{x_1} p(x_2|x_1)}^{\alpha_{2|1}}}_{\alpha_2} \underbrace{p(y_1|x_1) \overbrace{p(x_1)}^{\alpha_{1|0}}}_{\alpha_1}$$

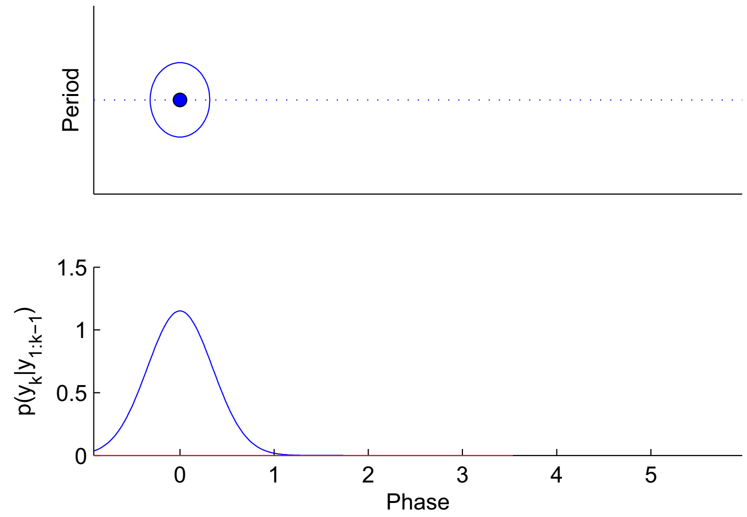
- Backward Pass

$$p(y_{1:K}) = \int_{x_1} p(x_1)p(y_1|x_1) \dots \underbrace{\int_{x_{K-1}} p(x_{K-1}|x_{K-2})p(y_{K-1}|x_{K-1})}_{\beta_{K-2}} \underbrace{\int_{x_K} p(x_K|x_{K-1})p(y_K|x_K)}_{\beta_{K-1}} \underbrace{\mathbf{1}}_{\beta_K}$$

- Replace summation by integration

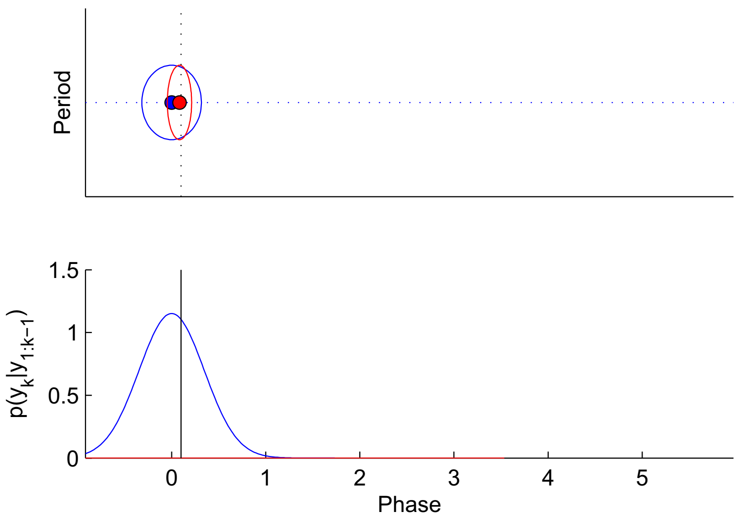


$$p(s_1)$$



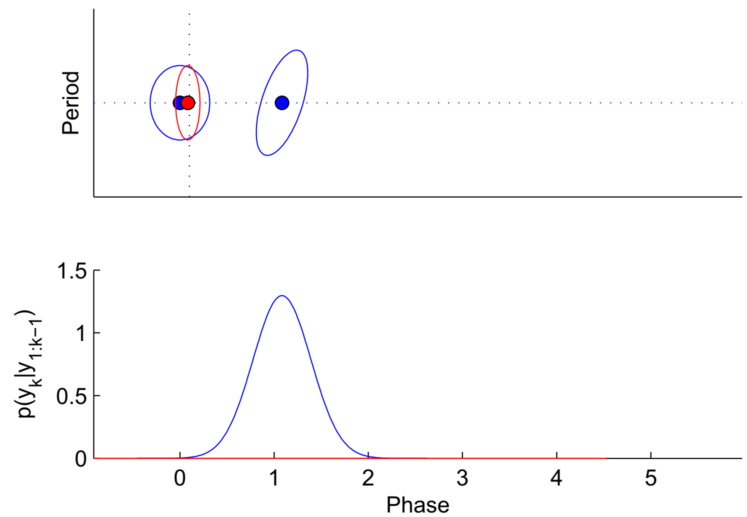



$$p(y_1|s_1)p(s_1)$$

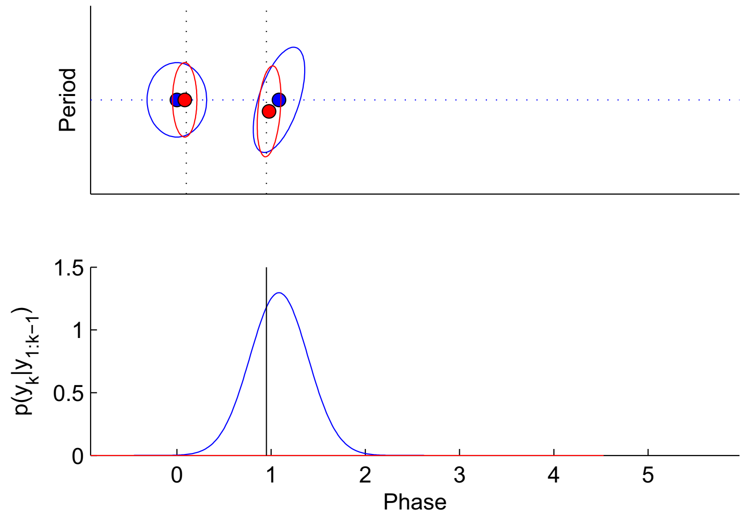





$$p(s_2|y_1) \propto \int ds_1 p(s_2|s_1)p(y_1|s_1)p(s_1)$$

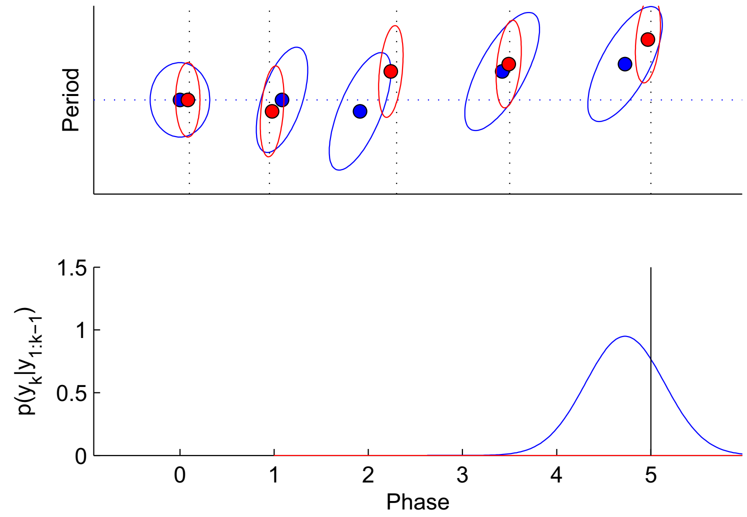


$p(y_2|s_2)p(s_2|y_1)$





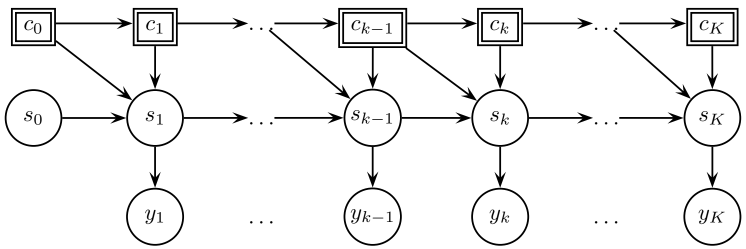

$p(s_5|y_{1:5})$






## Computer Accompaniment

(Music Plus One, Raphael 2000 [18], Dannenberg and Raphael 2006)




- $c_k$ are score positions of notes of the soloist and $l_k = c_k - c_{k-1}$

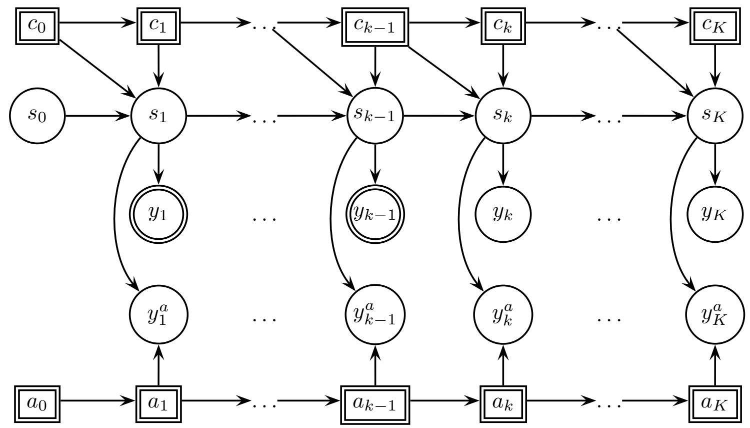
$$\mathbf{s}_k = \begin{pmatrix} 1 & l_k \\ 0 & 1 \end{pmatrix} \mathbf{s}_{k-1} + \epsilon_k = \mathbf{A}_k \mathbf{s}_{k-1} + \epsilon_k \qquad y_k = C\mathbf{s}_k + \nu_k$$

$$\epsilon_k \sim \mathcal{N}(\epsilon; 0, Q_k)$$

$$\nu_k \sim \mathcal{N}(\nu; m_k, R_k)$$ (note $k$ dependent mean and variance!)

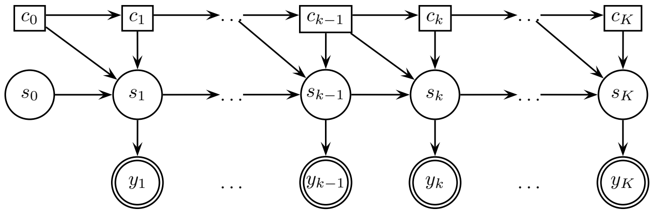


## Music Plus One




- Note that this is ruthless simplification, see Chris Raphaels' papers...

# Switching State Space models

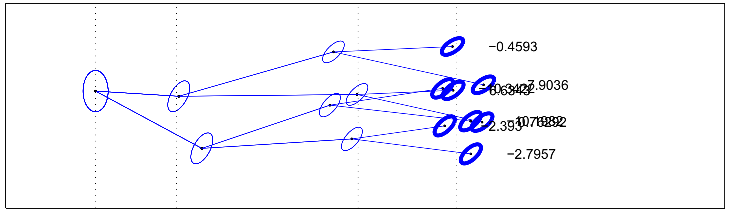


- We introduce latent switch variables to switch between different transition and observation models
- Powerful framework for modelling nonstationary processes and nonlinear dynamical systems



# Inference in Switching State Space models

- Unlike HMM's or KFM's, summing over $c_k$ does not simplify the filtering density.
- Number of Gaussian kernels to represent exact filtering density $p(c_k, s_k|y_{1:k})$ increases exponentially

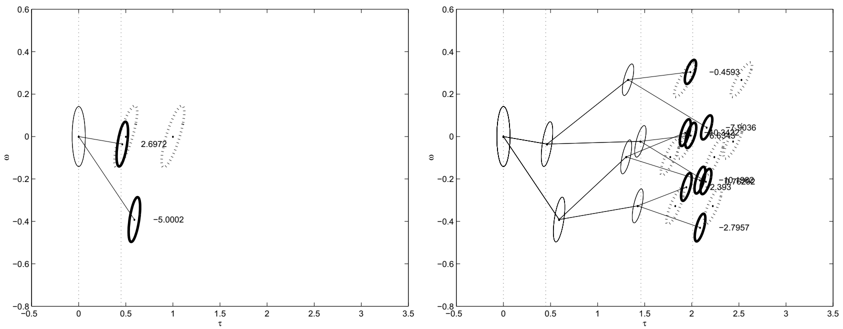



- Bad news: exact inference problem is shown to be NP hard



# Example

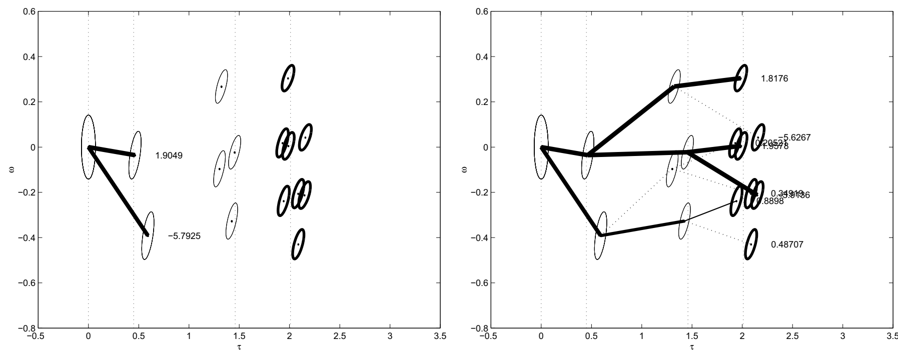
Suppose that a score can consist of only two notes:



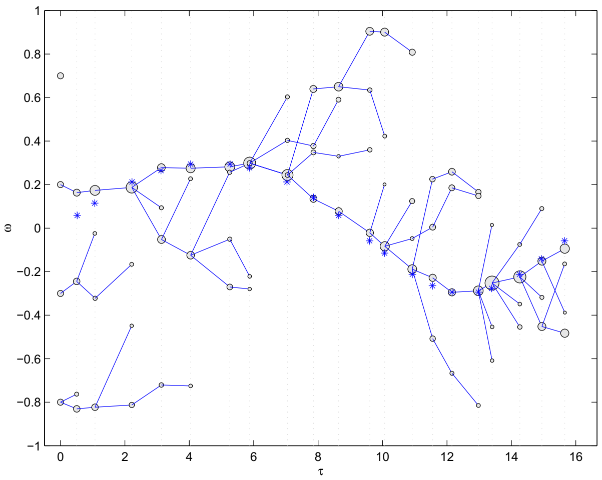



# Sequential Monte Carlo (Particle Filtering)

- Main idea: Select a branch to expand with a probability propotional to the evidence



# Particle Filtering for tracking






# Sequential Monte Carlo

- This variant is known as Mixture Kalman Filter or Rao-Blackwellized Particle filter (Chen and Liu 2001 [9], Cemgil 2002 [6], Hainsworth and MacLeod 2003)
- (For this model) algorithmically similar to Breadth first search/Multi Hypothesis Tracking/Genetic algorithms
- Generic tool for inference with a rich background theory (Doucet, et. al. 2001, Del Moral, "Feynman-Kac Formulae", 2005)
- Many applications in various fields
  - Robotics, Navigation, Econometrics,...

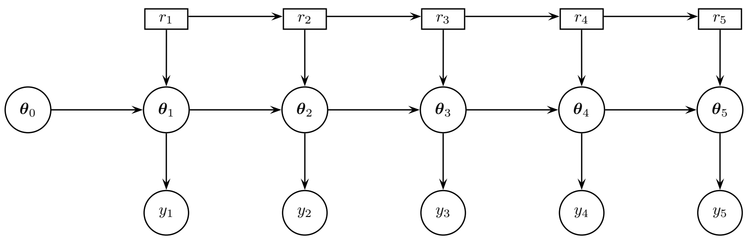


# Changepoint models

$r_k \sim p(r_k|r_{k-1})$ Indicators $\in \{\text{new}, \text{reg}\}$

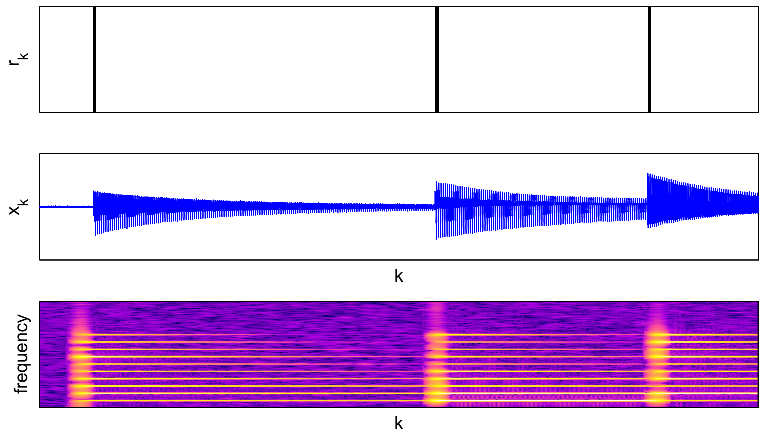
$\theta_k \sim [r_k = \text{reg}] \underbrace{f(\theta_k|\theta_{k-1})}_{\text{Transition}} + [r_k = \text{new}] \underbrace{\pi(\theta_k)}_{\text{Reinitialization}}$ Latent State

$y_k \sim p(y_k|\theta_k)$ Observations





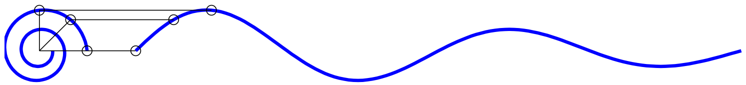

# Example: Single Key, Onsets




- Each changepoint denotes the onset of a new audio event

## Dynamic Harmonic Model (Cemgil et. al. 2005, 2006) [4, 7]

$$
\begin{aligned}
r_k | r_{k-1} &\sim p(r_k | r_{k-1}) \\
s_k | s_{k-1}, r_k &\sim \underbrace{[r_k = 0]\mathcal{N}(A s_{k-1}, Q)}_{\text{reg}} + \underbrace{[r_k = 1]\mathcal{N}(0, S)}_{\text{new}} \\
y_k | s_k &\sim \mathcal{N}(C s_k, R)
\end{aligned}
$$



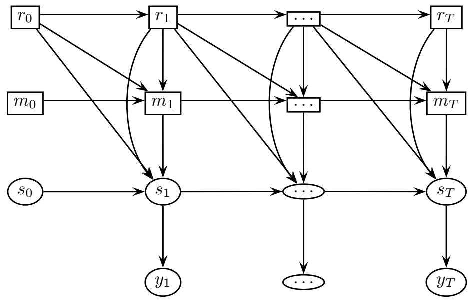
$$
A = \begin{pmatrix} G_\omega & & & \\ & G_\omega^2 & & \\ & & \ddots & \\ & & & G_\omega^H \end{pmatrix}^N \qquad G_\omega = \rho_k \begin{pmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{pmatrix}
$$

damping factor $0 < \rho_k < 1$, framelength $N$ and damped sinusoidal basis matrix $C$ of size $N \times 2H$



## Monophonic model [7]

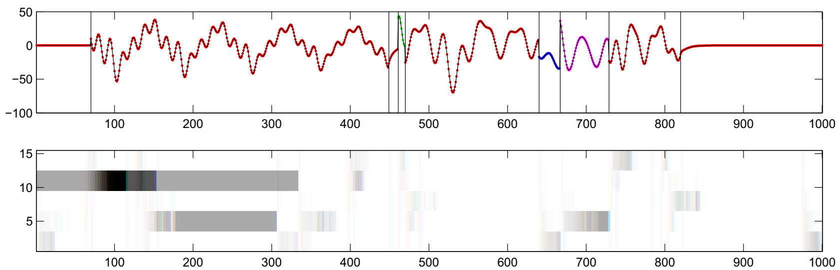
- We introduce a pitch label indicator $m$
- At each time $k$, the process can be in one of the {"mute", "sound"} $\times M$ states.






## Monophonic Pitch Tracking

Monophonic Pitch Tracking = Online estimation (filtering) of $p(r_k, m_k | y_{1:k})$.

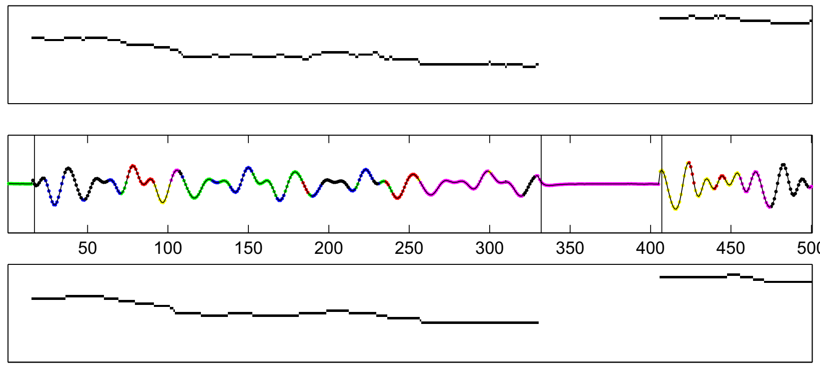



- If pitch is constant exact inference is possible



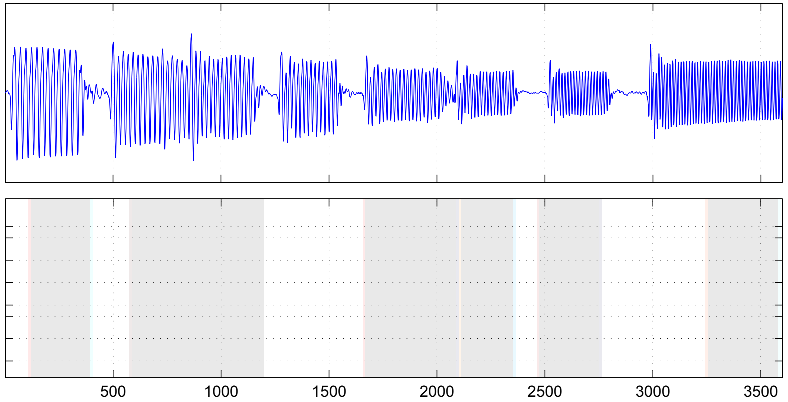
## Tracking Pitch Variations

- Allow $m$ to change with $k$. We take a fine grid Piano-roll, e.g. $\mathcal{Q} = 2^{1/128}$




- Intractable, need to run a particle filter

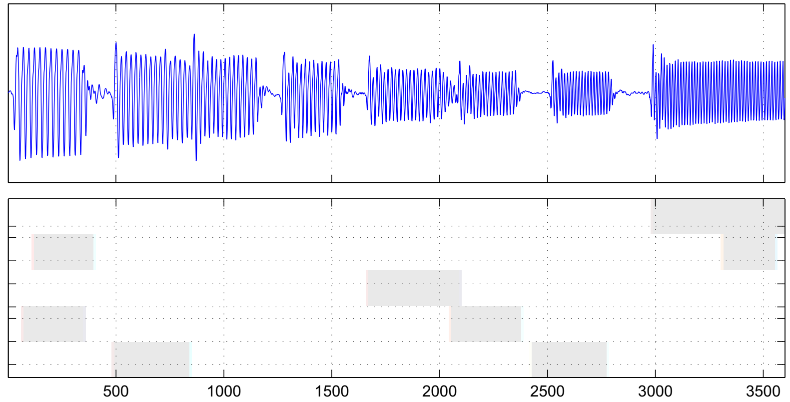
## Real Data Results




Top: F major scale played on an electric bass.
Bottom: Estimated MAP configuration $(r, m)_{1:T}$.



## Real Data Results

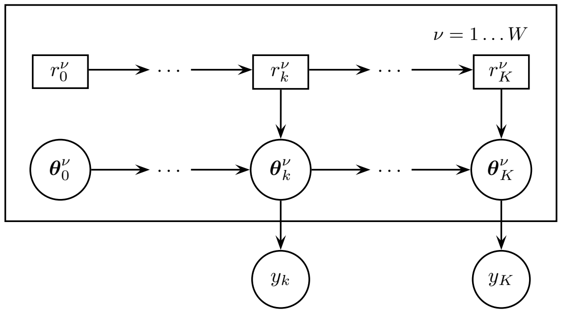



A finer analysis with $\mathcal{Q} = 2^{1/48}$ reveals that the 5'th and 7'th degree of the scale are intonated slightly low.



## Polyphony: Factorial Dynamic Harmonic Model [4]

$$
\begin{aligned}
r_{0,\nu} &\sim \mathcal{C}(r_{0,\nu}; \pi_{0,\nu}) \\
\theta_{0,\nu} &\sim \mathcal{N}(\theta_{0,\nu}; \mu_\nu, P_\nu) \\
r_{k,\nu}|r_{k-1,\nu} &\sim \mathcal{C}(r_{k,\nu}; \pi_\nu(r_{t-1,\nu})) && \text{Changepoint indicator} \\
\theta_{k,\nu}|\theta_{k-1,\nu} &\sim \mathcal{N}(\theta_{k,\nu}; A_\nu(r_k)\theta_{k-1,\nu}, Q_\nu(r_k)) && \text{Latent state} \\
y_k|\theta_{k,1:W} &\sim \mathcal{N}(y_k; C_k\theta_{k,1:W}, R) && \text{Observation}
\end{aligned}
$$

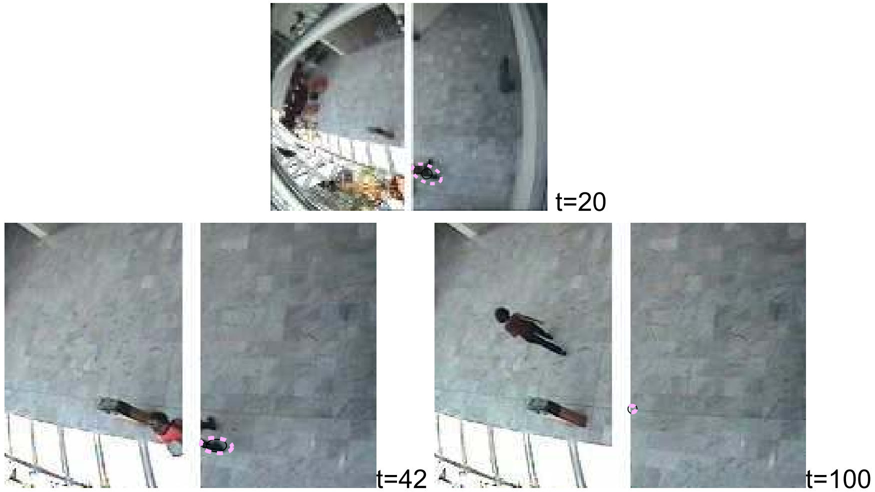





## Visual Tracking

(Video1) (Video2) (Video3)

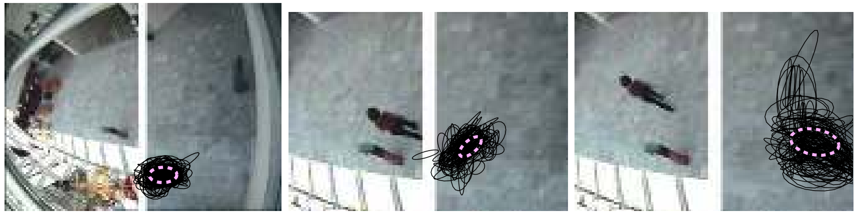
## Visual Tracking – Multimodal Posteriors



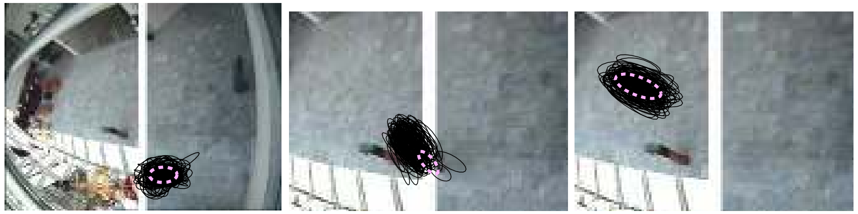

The Kalman Filter looses track due to occlusion



## Visual Tracking – Multimodal Posteriors



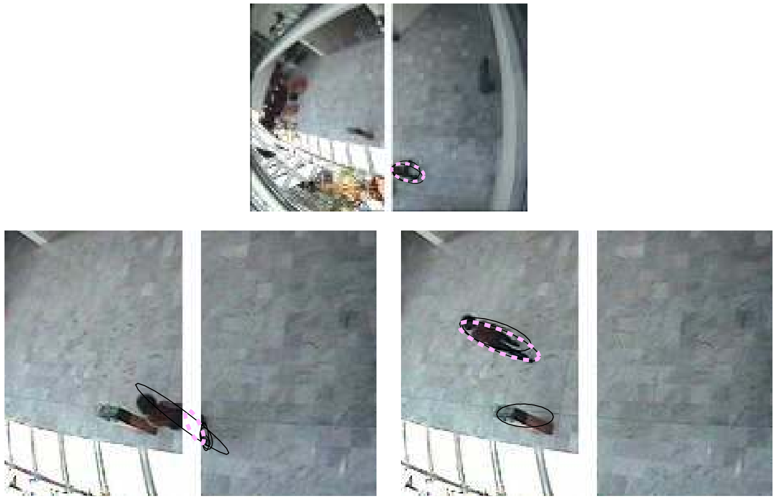
Particle Filter with poorly designed proposal



Particle Filter with better proposal



## Visual Tracking – Multimodal Posteriors

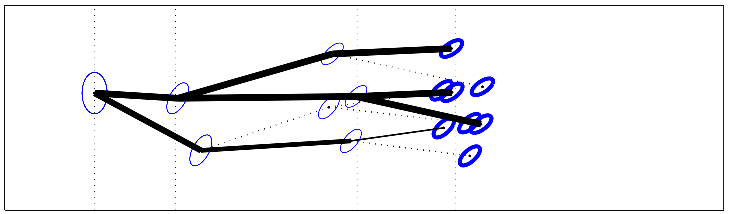


Mixture Kalman Filter



## Sequential Monte Carlo - Particle Filtering

- We try to approximate the so-called filtering density with a set of points/Gaussians $\equiv$ particles
- Algorithms are intuitively similar to randomised search algorithms but are best understood in terms of sequential importance sampling and resampling techniques



## Importance Sampling (IS)

Consider a probability distribution with (possibly unknown) normalisation constant

$$p(\mathbf{x}) = \frac{1}{Z}\phi(\mathbf{x}) \qquad\qquad Z = \int d\mathbf{x}\phi(\mathbf{x}).$$

IS: Estimate expectations (or features) of $p(\mathbf{x})$ by a weighted sample

$$\langle f(\mathbf{x})\rangle_{p(\mathbf{x})} = \int dx f(\mathbf{x})p(\mathbf{x})$$

$$\langle f(\mathbf{x})\rangle_{p(\mathbf{x})} \approx \sum_{i=1}^{N} \tilde{w}^{(i)} f(\mathbf{x}^{(i)})$$



## Importance Sampling (cont.)

- Change of measure with **weight function** $W(\mathbf{x}) \equiv \phi(x)/q(x)$

$$\langle f(\mathbf{x})\rangle_{p(\mathbf{x})} = \frac{1}{Z}\int d\mathbf{x} f(\mathbf{x})\frac{\phi(\mathbf{x})}{q(\mathbf{x})}q(\mathbf{x}) = \frac{1}{Z}\left\langle f(\mathbf{x})\frac{\phi(\mathbf{x})}{q(\mathbf{x})}\right\rangle_{q(\mathbf{x})} \equiv \frac{1}{Z}\langle f(\mathbf{x})W(\mathbf{x})\rangle_{q(\mathbf{x})}$$

- If $Z$ is unknown, as is often the case in Bayesian inference

$$Z = \int d\mathbf{x}\phi(\mathbf{x}) = \int d\mathbf{x}\frac{\phi(\mathbf{x})}{q(\mathbf{x})}q(\mathbf{x}) = \langle W(\mathbf{x})\rangle_{q(\mathbf{x})}$$

$$\langle f(\mathbf{x})\rangle_{p(\mathbf{x})} = \frac{\langle f(\mathbf{x})W(\mathbf{x})\rangle_{q(\mathbf{x})}}{\langle W(\mathbf{x})\rangle_{q(\mathbf{x})}}$$



## Importance Sampling (cont.)

- Draw $i = 1, \dots N$ independent samples from $q$

$$\mathbf{x}^{(i)} \sim q(\mathbf{x})$$

- We calculate the **importance weights**

$$W^{(i)} = W(\mathbf{x}^{(i)}) = \phi(\mathbf{x}^{(i)})/q(\mathbf{x}^{(i)})$$

- Approximate the normalizing constant

$$Z = \langle W(\mathbf{x})\rangle_{q(\mathbf{x})} \approx \sum_{i=1}^{N} W^{(i)}$$

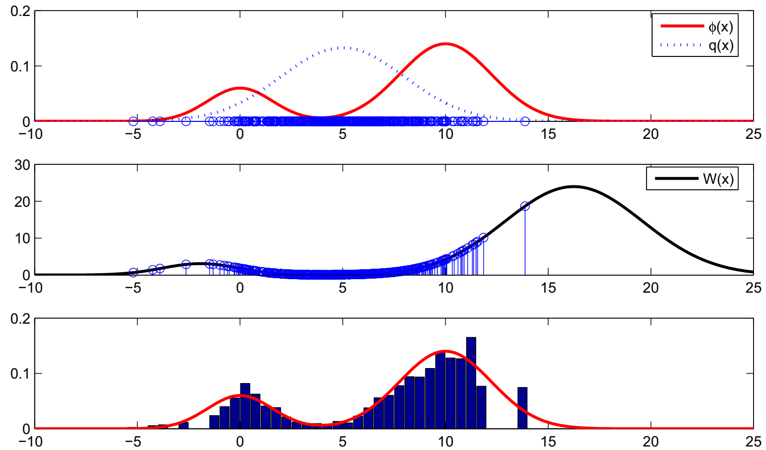
- Desired expectation is approximated by

$$\langle f(\mathbf{x})\rangle_{p(\mathbf{x})} = \frac{\langle f(\mathbf{x})W(\mathbf{x})\rangle_{q(\mathbf{x})}}{\langle W(\mathbf{x})\rangle_{q(\mathbf{x})}} \approx \frac{\sum_{i=1}^{N} W^{(i)} f(\mathbf{x}^{(i)})}{\sum_{i=1}^{N} W^{(i)}} \equiv \sum_{i=1}^{N} \tilde{w}^{(i)} f(\mathbf{x}^{(i)})$$

Here $\tilde{w}^{(i)} = W^{(i)}/\sum_{j=1}^{N} W^{(j)}$ are *normalized importance weights*.



## Importance Sampling (cont.)



# Resampling

- Importance sampling computes an approximation with weighted delta functions

$$p(x) \approx \sum_i \tilde{W}^{(i)} \delta(x - x^{(i)})$$

- In this representation, most of $\tilde{W}^{(i)}$ will be very close to zero and the representation may be dominated by few large weights.
- Resampling samples a set of new "particles"

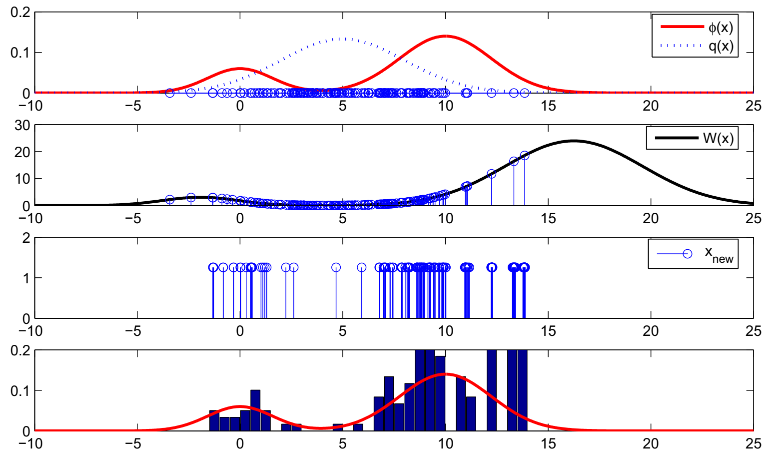
$$x_{\text{new}}^{(j)} \sim \sum_i \tilde{W}^{(i)} \delta(x - x^{(i)})$$

$$p(x) \approx \frac{1}{N} \sum_j \delta(x - x_{\text{new}}^{(j)})$$

- Since we sample from a degenerate distribution, particle locations stay unchanged. We merely dublicate (, triplicate, ...) or discard particles according to their weight.
- This process is also named "selection", "survival of the fittest", e.t.c., in various fields (Genetic algorithms, AI..).



# Resampling




$$x_{\text{new}}^{(j)} \sim \sum_i \tilde{W}^{(i)} \delta(x - x^{(i)})$$



# Sequential Importance Sampling, Particle Filtering

Apply importance sampling to the SSM to obtain some samples from the posterior $p(x_{0:K}|y_{1:K})$.

$$p(x_{0:K}|y_{1:K}) = \frac{1}{p(y_{1:K})} p(y_{1:K}|x_{0:K}) p(x_{0:K}) \equiv \frac{1}{Z_y} \phi(x_{0:K}) \quad (3)$$

Key idea: sequential construction of the proposal distribution $q$, possibly using the available observations $y_{1:k}$, i.e.

$$q(x_{0:K}|y_{1:K}) = q(x_0) \prod_{k=1}^{K} q(x_k|x_{1:k-1} y_{1:k})$$



# Markov Random Fields

# Markov Random Fields

- A set of random variables $\boldsymbol{\xi} = \{\xi_i\}_{i \in \mathcal{V}}$, Given
  - an undirected graph with vertex set $\mathcal{V}$ and undirected edge set $\mathcal{E}$
  - A set of local potential functions (with parameters **a**)

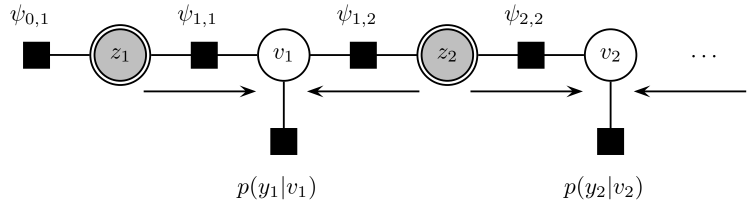
$$p(\boldsymbol{\xi}; \mathbf{a}) = \frac{1}{Z_{\mathbf{a}}} \prod_{i \in \mathcal{V}} \phi_i(\xi_i) \prod_{(i,j) \in \mathcal{E}} \psi_{i,j}(\xi_i, \xi_j)$$

$\phi_i(\xi_i; \mathbf{a})$ : (Singleton)

$\psi_{i,j}(\xi_i, \xi_j; \mathbf{a})$ : (Pairwise)



# VB or Gibbs




- VB

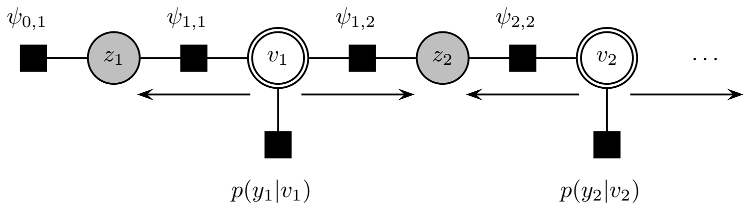
$$q^{(\tau)}(v_k) \quad \leftarrow \quad \exp(\phi_k + \langle \log \psi_{k,k} + \log \psi_{k,k+1} \rangle_{q^{(\tau)}(z_k) q^{(\tau)}(z_{k+1})})$$

- Gibbs

$$v_k^{(\tau)} \quad \sim \quad p(v_k | z_{k-1}, z_k, y_k) \propto p(y_k|v_k) \psi_{k,k}(z_k^{(\tau)}) \psi_{k,k+1}(z_{k+1}^{(\tau)})$$



# VB or Gibbs




- VB

$$q^{(\tau)}(z_k) \quad \leftarrow \quad \exp(\phi_k + \langle \log \psi_{k,k-1} + \log \psi_{k,k} \rangle_{q^{(\tau)}(v_k) q^{(\tau)}(v_{k+1})})$$

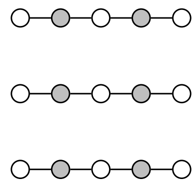
- Gibbs

$$z_k^{(\tau)} \quad \sim \quad p(z_k | v_{k-1}, v_k) \propto \psi_{k,k-1}(v_{k-1}^{(\tau)}) \psi_{k,k}(v_k^{(\tau)})$$

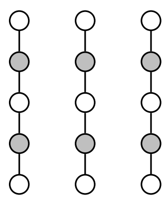


# Harmonic-Transient Decomposition

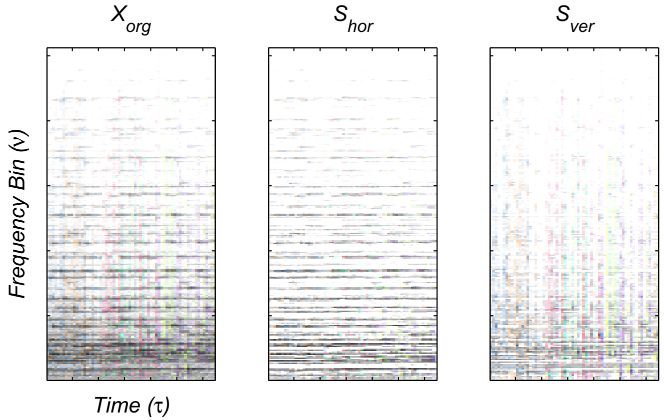
- Source 1: Horizontal : Tie across time : harmonic continuity



- Source 2: Vertical : Tie across frequency : transients, pulse like sounds



## Harmonic-Transient Decomposition

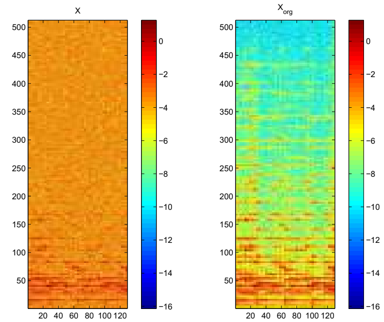



(Original) (Hor) (Vert)

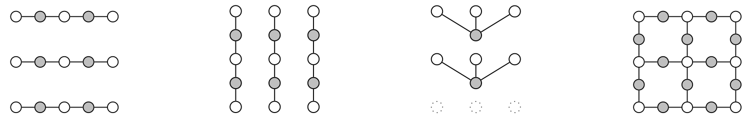
Superstition, (thanks to Tuomas Virtanen)
(Original) (Hor) (Vert)



## Denoising - Piano



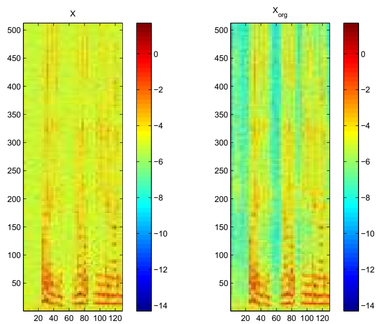

(Noisy) (Original)



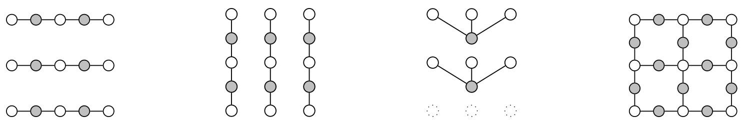
(Hor) (Ver) (Band) (Grid)



## Denoising - Speech




(Noisy) (Original)



(Hor) (Ver) (Band) (Grid)



# Topic-Term-Document Models

# Text Processing, Latent Semantic Indexing

Deerwester et al. (1990), Berry et al. (1995), Manning, Schuetze, Raghavan (2007)

- We are given a database of *documents* $D = \{d_1, \dots, d_j, \dots, d_N\}$
- Each document contains several terms from a codebook of terms $T = \{t_1, \dots, t_i, \dots, t_M\}$
- Retrieval,
  - Given a query $q$ (for example a set of few terms $T_q \subset T$) retrieve a set of documents $D^q_{\text{Retrieved}}$
  - Assume we know the set of relevant documents $D^q_{\text{Relevant}} \subset D$ (with respect to the query q)
  - Quality Measures:

$$\text{Precision}^q = \frac{|D^q_{\text{Relevant}} \cap D^q_{\text{Retrieved}}|}{|D^q_{\text{Retrieved}}|} \qquad \text{Recall}^q = \frac{|D^q_{\text{Relevant}} \cap D^q_{\text{Retrieved}}|}{|D^q_{\text{Relevant}}|}$$



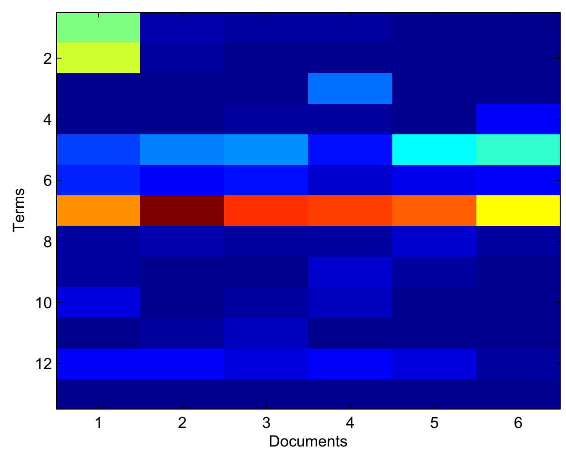
# Representation: Term-Document matrix $A \in \mathbb{R}^{M \times N}$

- Rows : terms $t_i, i = 1 \dots M$
- Columns: documents $d_j = 1 \dots N$

| | j.caesar | hamlet | othello | macbeth | rom&jul | sonnets |
|---|---|---|---|---|---|---|
| caesar | 270 | 2 | 1 | 1 | 0 | 0 |
| brutus | 379 | 1 | 0 | 0 | 0 | 0 |
| malcolm | 0 | 0 | 0 | 60 | 0 | 0 |
| muse | 0 | 0 | 1 | 1 | 0 | 16 |
| ⋮ | | | | | | |
| love | 34 | 68 | 80 | 19 | 150 | 195 |
| friend | 23 | 14 | 18 | 5 | 13 | 16 |
| the | 610 | 1148 | 759 | 733 | 682 | 446 |
| traitor | 1 | 0 | 0 | 5 | 1 | 0 |
| traitors | 9 | 0 | 1 | 3 | 0 | 0 |
| ⋮ | | | | | | |
| napkin | 0 | 1 | 3 | 0 | 0 | 0 |
| sword | 15 | 16 | 10 | 14 | 8 | 1 |
| laptop | 0 | 0 | 0 | 0 | 0 | 0 |



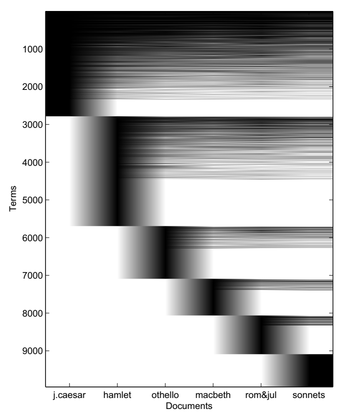
# Term-Document matrix




- Counts



# Term-Document matrix




- Incidence (zero-one) matrix

# Singular Value Decomposition (SVD)

For any $A \in \mathbb{R}^{M \times N}$, there exist **orthogonal** matrices $U \in \mathbb{R}^{M \times M}$ and $V \in \mathbb{R}^{N \times N}$ such that

$$U = [u_1, \dots, u_M] \qquad V = [v_1, \dots, v_N]$$

such that

$$U^\top A V = \mathbf{diag}(\sigma_1, \dots, \sigma_p) \in \mathbb{R}^{M \times N}$$

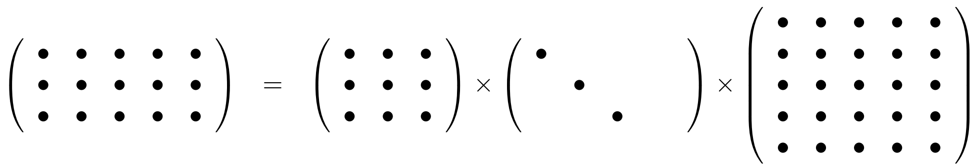
with $p = \min\{M, N\}$. We have

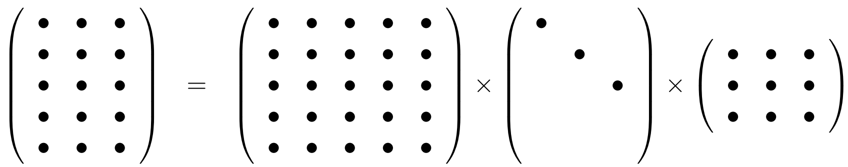
$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$$



# Singular Value Decomposition (SVD)

$$A = U \times \Sigma \times V^\top$$







# Singular Value Decomposition (SVD)

```
>> A =
     1     2
     3     5
     1     0

>> [U S V] = svd(A)          % A == U*S*V'
U =
   -0.3559   -0.2000   -0.9129
   -0.9309   -0.0102    0.3651
   -0.0823    0.9797   -0.1826

S =
    6.2638         0
         0    0.8744
         0         0

V =
   -0.5158    0.8567
   -0.8567   -0.5158
```



# Singular Value Decomposition (SVD)

```
>> U(:,1)*S(1,1)*V(:,1)'
ans =
    1.1498    1.9098
    3.0076    4.9954
    0.2661    0.4419
>> U(:,2)*S(2,2)*V(:,2)'
ans =
   -0.1498    0.0902
   -0.0076    0.0046
    0.7339   -0.4419
>> U(:,1)*S(1,1)*V(:,1)' + U(:,2)*S(2,2)*V(:,2)'   %% ==  U*S*V' == A
ans =
    1.0000    2.0000
    3.0000    5.0000
    1.0000    0.0000
>> A =
     1     2
     3     5
     1     0
```

# Singular Value Decomposition (SVD)

SVD expansion

$$
\begin{aligned}
A &= \sum_{r=1}^{P} \sigma_r u_r v_r^\top \\
&= U\, \mathbf{diag}(\sigma_1, \dots, \sigma_P) V^\top
\end{aligned}
$$

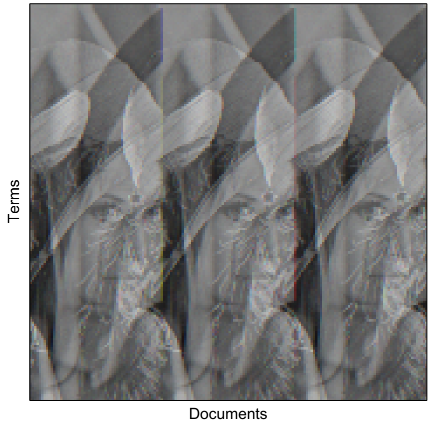
The norm relations for $A \in \mathbb{R}^{M \times N}$, $P = \min\{M, N\}$

$$
\begin{aligned}
\|A\|_F^2 &= \sigma_1^2 + \dots + \sigma_P^2 \\
\|A\|_2^2 &= \sigma_1^2
\end{aligned}
$$



# Singular Value Decomposition of Term-Document Matrices

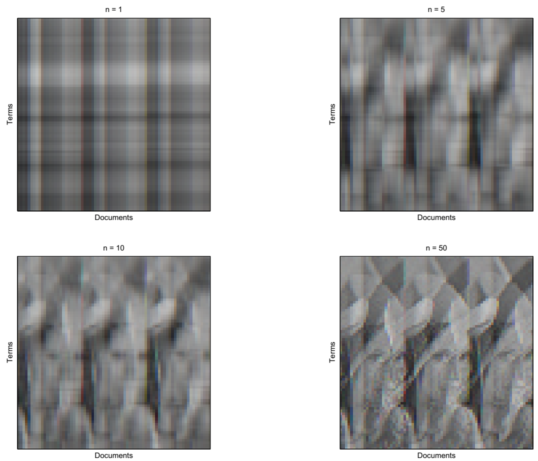
Another "term-document" matrix






# Singular Value Decomposition of Term-Document Matrices

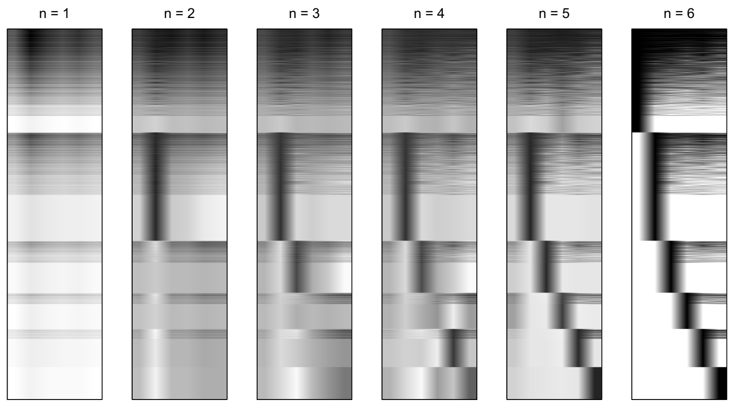
$$A \approx U(:, 1{:}n) S(1{:}n, 1{:}n) V(:, 1{:}n)^\top$$





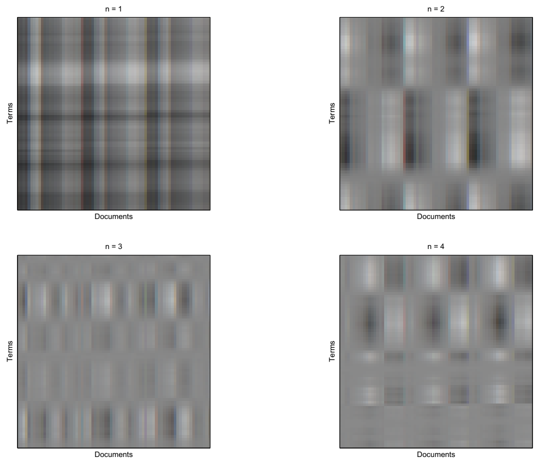

# Singular Value Decomposition of Term-Document Matrices

$$A \approx U(:, 1{:}n) S(1{:}n, 1{:}n) V(:, 1{:}n)^\top$$



# Rank-1 Matrices

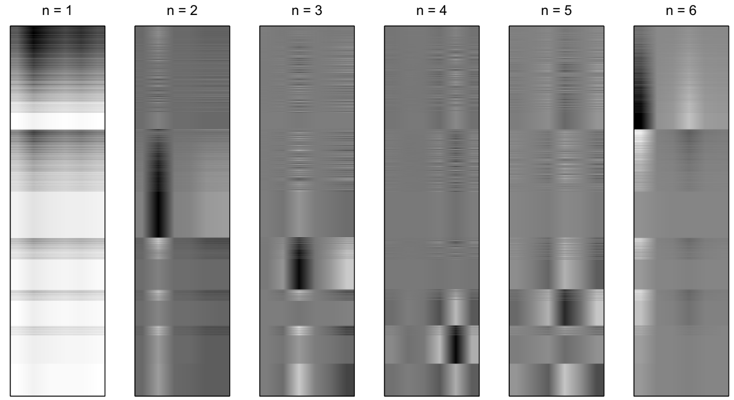
$$U(:, n)V(:, n)^\top$$






# Rank-1 Matrices

$$U(:, n)V(:, n)^\top$$






# LSI: Summary and Remarks

- Low rank approximation to a term-document matrix by keeping the latent dimensions corresponding to $n$ largest singular values of SVD

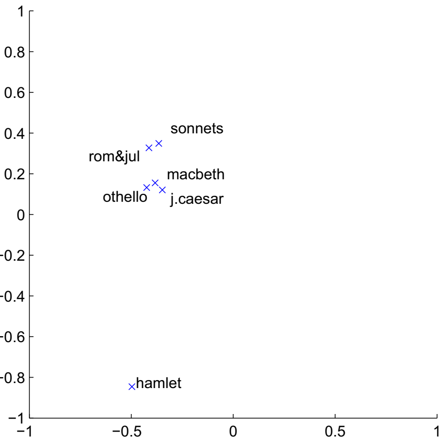
$$A \approx \sum_{r=1}^{n} \sigma_r U(:, r)V(:, r)^\top$$

- No direct statistical interpretation, but loosely
  - Each $r = 1 \dots n$ denotes a *latent topic* ($n$ is the total number of topics)
  - $U(i, r)$ corresponds to *weight* of the $i$'th term given the topic $r$
  - $V(j, r)$ corresponds to *emphasis* of topic $r$ in document $j$
    We can think $V(j, 1{:}n)^\top$ as the coordinates of $j$'th document in an $n$ dimensional *latent topic space*
  - The coordinates of a new document are computed by

$$v_{\text{new}} = \Sigma^{-1} U^\top a_{\text{new}}$$



# Latent Semantic Space



## LSI: Summary and Remarks

- Clustering, assessing similarity, visualisation ...
- Rationale: documents that share frequent co-occurring terms will be close in the latent space
- May deal with synonymy and polysemy
  - different words - same meaning
    `baggage-lugagge`
  - same word - different meaning
    `spider (the animal - the web crawler)`



## Probabilistic Latent Sematic Indexing, the Aspect Model

Hofmann, 1999

| | |
|---|---|
| $d \sim p(d)$ | Document |
| $z \mid d \sim p(z \mid d)$ | Latent Topic |
| $t \mid z \sim p(t \mid z)$ | Term |

More to come ...



# Factorial Models

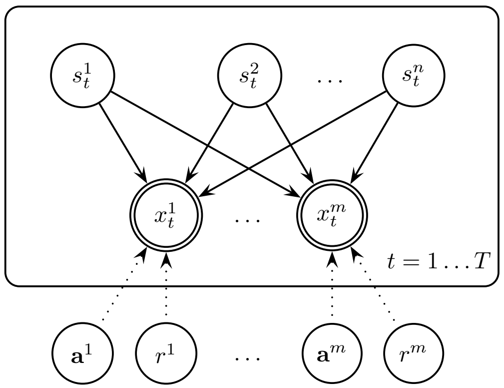
# Source Separation

# Bayesian Model selection



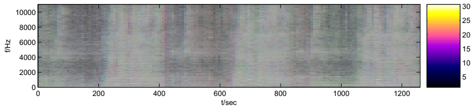
## Audio Source Separation

Estimate $n$ hidden signals $\mathbf{s}_t$ from $m$ observed signals $\mathbf{x}_t$.

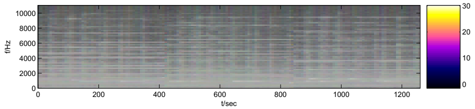



$$
\begin{aligned}
s_t^i &\sim p(s_t^i) \\
x_t^j &\sim \mathcal{N}(x; \mathbf{a}^j s_t^{1:n}, r^j)
\end{aligned}
$$

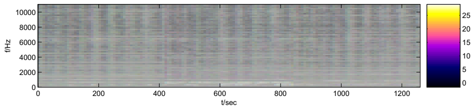
# Audio Source Separation



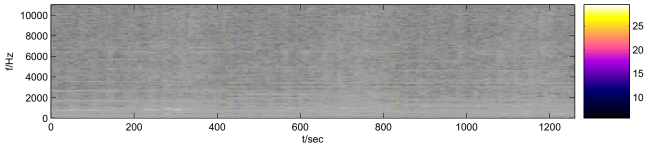

(Speech)




(Piano)




(Guitar)

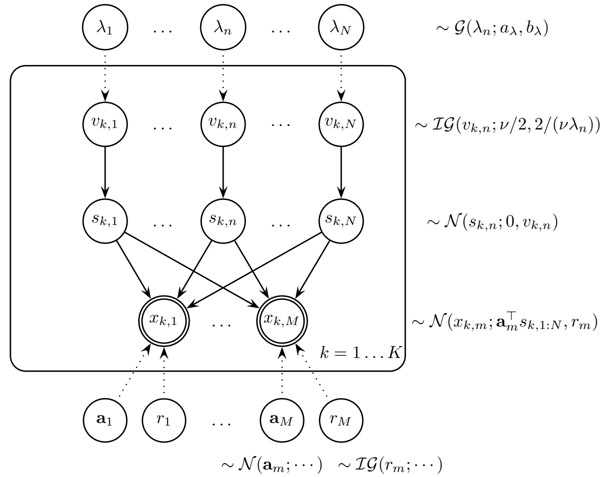



(Mix)



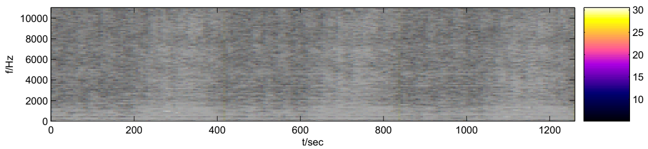
# Audio Source Separation

- Hierarchical Prior Model (Fevotte and Godsill 2005 [10], Cemgil et. al. 2006 [3])

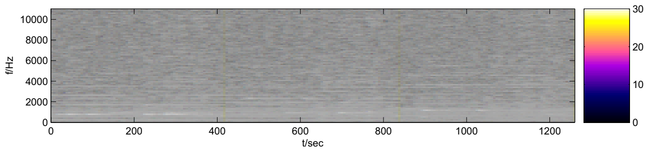




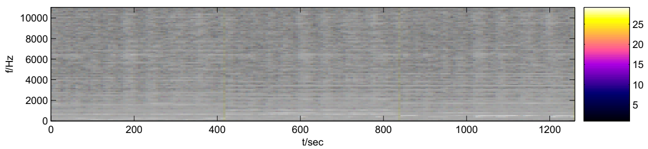

# Reconstructions




(Speech)



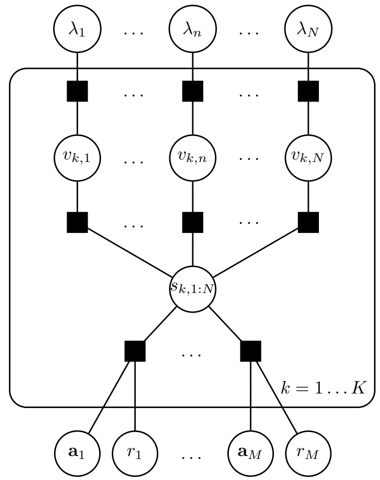

(Piano)




(Guitar)

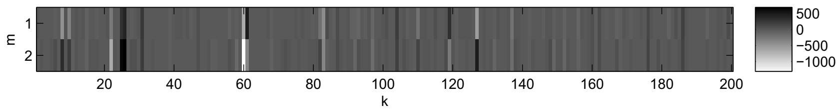


# Audio Source Separation, Inference



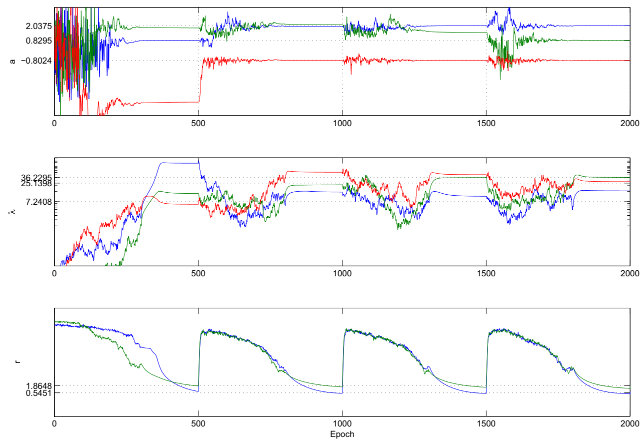

- Exact inference is not possible

# Observations



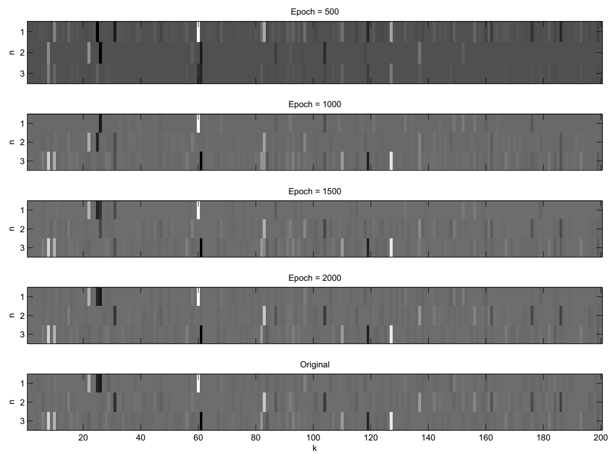



# A typical run, $250/250$ Gibbs/VB with tempering





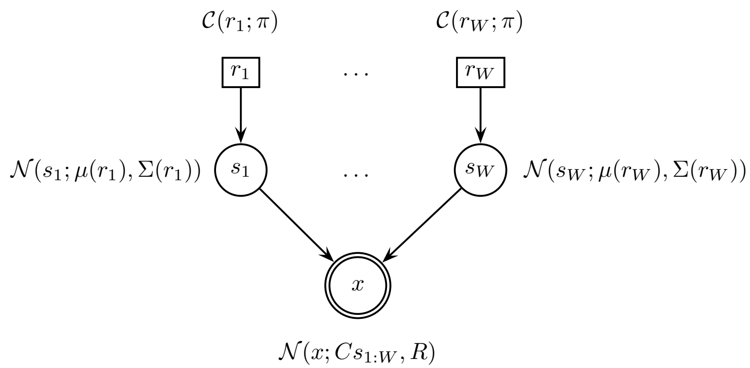

# Reconstructions




Posterior surface is multimodal, each mode corresponding to a viable separation



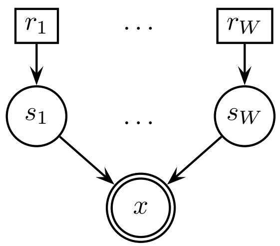
# Bayesian Variable Selection




- Generalized Linear Model – Column's of $C$ are the basis vectors
- The exact posterior is a mixture of $2^W$ Gaussians
- When $W$ is large, computation of posterior features becomes intractable.

## Generative model

$$\begin{aligned} r_i &\sim \mathcal{C}(r_i;\pi) \\ s_i|r_i &\sim \mathcal{N}(s_i;\mu(r_i),\Sigma(r_i)) \\ \mathbf{x}|s_{1:W} &\sim \mathcal{N}(\mathbf{x};Cs_{1:W},R) \\ C &\equiv [\ C_1\ \ \dots\ \ C_i\ \ \dots\ \ C_W\ ] \end{aligned}$$

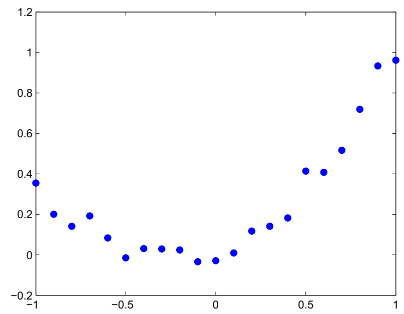



$$p(\mathbf{x},s_{1:W},r_{1:W}) = p(\mathbf{x}|s_{1:W},r_{1:W})\prod_{i=1}^{W} p(s_i|r_i)p(r_i)$$



## Example 1: Variable selection in Polynomial Regression

Given $\{t_j, x(t_j)\}_{j=1\dots J}$, what is the order $N$ of the polynomial?




$$x(t) = \sum_{i=0}^{N} s_{i+1}t^i + \epsilon(t)$$



## Ex1: Regression

$$\begin{aligned} \mathbf{t} &= \begin{pmatrix} t_1 & t_2 & \dots & t_J \end{pmatrix}^\top \\ C &\equiv \begin{pmatrix} \mathbf{t}^0 & \mathbf{t}^1 & \dots & \mathbf{t}^{W-1} \end{pmatrix} \end{aligned}$$

```
>> C = fliplr(vander(0:4))  % Van der Monde matrix
     1     0     0     0     0
     1     1     1     1     1
     1     2     4     8    16
     1     3     9    27    81
     1     4    16    64   256
```

$$\begin{aligned} r_i &\sim \mathcal{C}(r_i;0.5,0.5) \qquad r_i \in \{\text{on},\text{off}\} \\ s_i|r_i &\sim \mathcal{N}(s_i;0,\Sigma(r_i)) \\ \mathbf{x}|s_{1:W} &\sim \mathcal{N}(\mathbf{x};Cs_{1:W},R) \end{aligned}$$

$$\Sigma(r_i=\text{on}) \gg \Sigma(r_i=\text{off})$$



## Ex1: Regression

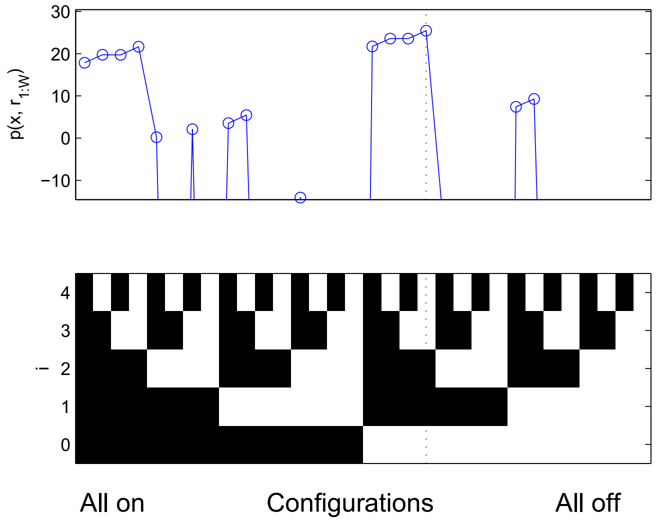
To find the "active" basis functions we need to calculate

$$r^*_{1:W} \equiv \operatorname*{argmax}_{r_{1:W}} p(r_{1:W}|\mathbf{x}) = \operatorname*{argmax}_{r_{1:W}} \int ds_{1:W}\, p(\mathbf{x}|s_{1:W})p(s_{1:W}|r_{1:W})p(r_{1:W})$$

Then, the reconstruction is given by

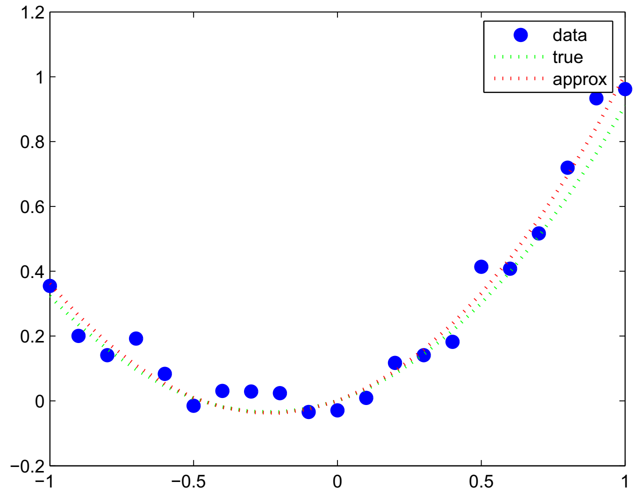
$$\begin{aligned} \hat{x}(t) &= \left\langle \sum_{i=0}^{W-1} s_{i+1}t^i \right\rangle_{p(s_{1:W}|\mathbf{x},r^*_{1:W})} \\ &= \sum_{i=0}^{W-1} \langle s_{i+1}\rangle_{p(s_{i+1}|\mathbf{x},r^*_{1:W})}\, t^i \end{aligned}$$

# Ex1: Regression



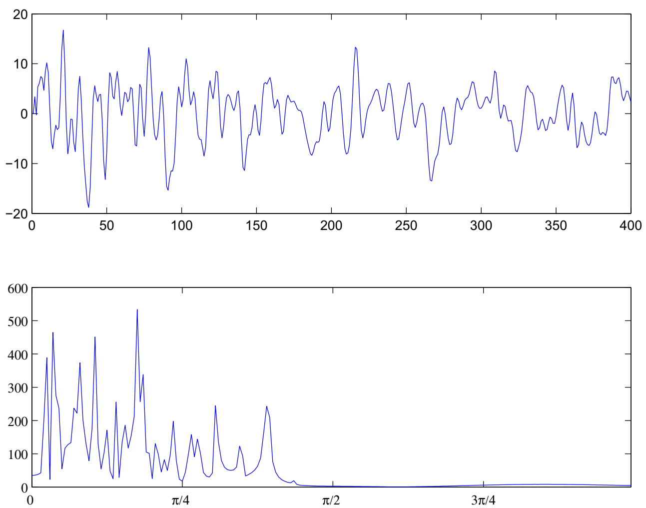



# Ex1: Regression






# Example 2: Chord Recognition





# (Damped) Sinusoidal Basis

- $h = 1 \dots H$, number of harmonics, $t = 0 \dots T-1$, sample index
- $\omega$ : fundamental frequency in rad, $\rho$ damping coefficient

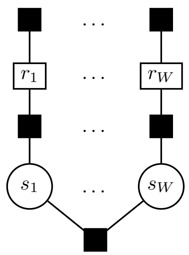
$$C(\omega) \equiv \begin{pmatrix} C_0^1 & \dots & C_0^H \\ \vdots & C_t^h & \vdots \\ C_{T-1}^1 & \dots & C_{T-1}^H \end{pmatrix}$$

$$\begin{aligned} C_t^h &\equiv \rho^t \begin{pmatrix} \cos(th\omega) & \sin(th\omega) \end{pmatrix} \\ \mathbf{C} &= [C(\omega_1) \dots C(\omega_\nu) \dots C(\omega_W)] \end{aligned}$$

- See also Badeau, Boyer, David. Eds parametric modelling and tracking of audio signals. In DAFx 2002

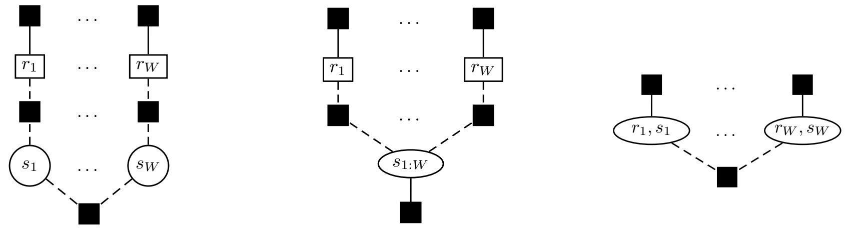
# Factor graph

$$
\begin{aligned}
\log \phi(r_{1:W}, s_{1:W}) &= \sum_{i=1}^{W} (\log \pi(r_i)) \\
&+ \sum_{i=1}^{W} \Big( -\frac{1}{2} s_i^\top \Sigma(r_i)^{-1} s_i + \mu(r_i)^\top \Sigma(r_i)^{-1} s_i \\
&\quad -\frac{1}{2} \mu(r_i)^\top \Sigma(r_i)^{-1} \mu(r_i) - \frac{1}{2} \log |2\pi \Sigma(r_i)| \Big) \\
&-\frac{1}{2} \mathbf{x}^\top R^{-1} \mathbf{x} + s_{1:W}^\top C^\top R^{-1} \mathbf{x} - \frac{1}{2} s_{1:W}^\top C^\top R^{-1} C s_{1:W} - \frac{1}{2} \log |2\pi R|
\end{aligned}
$$






# Approximating Structures

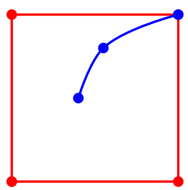



$\mathcal{Q}_1 = \prod_{i=1}^{W} \mathcal{Q}(s_i)\mathcal{Q}(r_i)$ $\quad$ $\mathcal{Q}_2 = \mathcal{Q}(s_{1:W}) \prod_{i=1}^{W} \mathcal{Q}(r_i)$ $\quad$ $\mathcal{Q}_3 = \prod_{i=1}^{W} \mathcal{Q}(s_i, r_i)$



# MCMC versus Variational Bayes (VB)

- Each configuration of $r_{1:W}$ corresponds to a corner of a $W$ dimensional hypercube



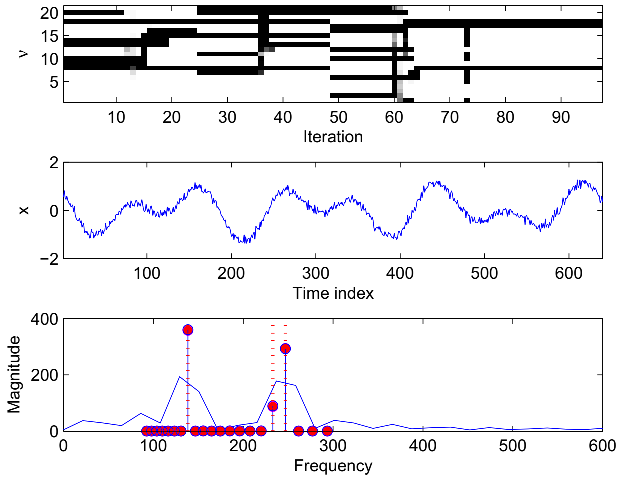
- MCMC moves along the edges stochastically
- Iterative Improvement moves along the edges greedily
- VB moves inside the hypercube deterministically



# Iterative Improvement

| iteration | $r_1$ | | | | | | | | | | | | | | | | | | | | | | | $r_M$ | $\log p(y_{1:T}, r_{1:M})$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | −1220638254 |
| 2 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | −665073975 |
| 3 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | −311983860 |
| 4 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ● | −162334351 |
| 5 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ● | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ● | −43419569 |
| 6 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ● | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ○ | ○ | ● | ○ | ● | ○ | ● | −1633593 |
| 7 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ● | ○ | ○ | ● | ○ | ○ | ● | ○ | ○ | ○ | ○ | ● | ○ | ● | ○ | ● | −14336 |
| 8 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ● | ○ | ● | ● | ○ | ○ | ● | ○ | ○ | ○ | ○ | ● | ○ | ● | ○ | ● | −5766 |
| 9 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ○ | ○ | ● | ● | ○ | ○ | ● | ○ | ○ | ○ | ○ | ● | ○ | ● | ○ | ● | −5210 |
| 10 | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ● | ○ | ○ | ● | ○ | ○ | ○ | ○ | ● | ○ | ● | ○ | ● | −4664 |
| True | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ○ | ● | ● | ○ | ○ | ● | ○ | ○ | ○ | ○ | ● | ○ | ● | ○ | ● | −4664 |

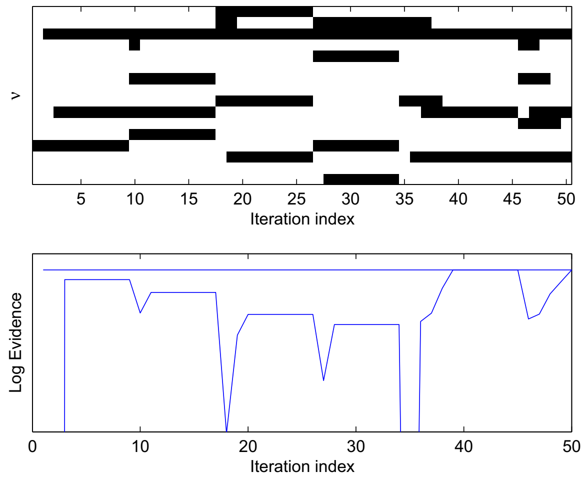
## Results, VB with tempering and reinitialisation




$F_s = 22050$ Hz, $N = 29$ msec, $H = 1$, Midinotes $= 30 \ldots 50$



## Results, MCMC with tempering and reinitialisation




$F_s = 22050$ Hz, $N = 29$ msec, $H = 1$, Midinotes $= 30 \ldots 50$



## Bayesian/Generative/Probabilistic approaches to Polyphonic Transcription

(Walmsley 2000, Davy and Godsill 2002, Raphael 2001, Abdallah 2002, Cemgil et. al. 2003-2006, Vincent 2003, Vincent and Plumbley 2005, Vogel, Jordan and Wessel 2005, Thornburg, Leitsnikov and Berger 2004, Blumensath and Davies 2006, Dubois and Davy 2005)

- Various related but different models
- Inference schemata
  - Reversible Jump MCMC
  - Iterative Improvement
  - Laplace approximation
  - Particle filtering
  - Variational Bayes, MCMC



## Summary

- Bayesian Inference
- Graphical models
- Exact Inference
- Approximate inference

## Summary, Attributes of Probabilistic Inference

- **Exact** $\leftrightarrow$ **Approximate**
- **Deterministic** $\leftrightarrow$ **Stochastic**
- **Online** $\leftrightarrow$ **Offline**
- **Centralized** $\leftrightarrow$ Distributed



## Summary of what we have mentioned

- Exact inference, Belief Propagation
- Approximate inference
  - Deterministic
    - Variational Bayes,
    - Expectation/Maximization (EM), Iterative Conditional Modes (ICM)
  - Stochastic
    - Markov Chain Monte Carlo
    - Importance Sampling,
    - Particle filtering



## Summary of what we have not mentioned

- Exact Inference (Junction Tree ...)
- Deterministic Inference
  - Assumed Density Filter (ADF), Extended Kalman Filter (EKF), Unscented Particle Filter
  - Structured Mean field
  - Loopy Belief Propagation, Expectation Propagation, Generalized Belief Propagation
  - Fractional Belief propagation, Bound Propagation, <your favorite name> Propagation
  - Graph cuts ...
- Stochastic
  - Unscented Particle Filter, Nonparametric Belief Propagation
  - Annealed Importance Sampling, Adaptive Importance Sampling
  - Hybrid Monte Carlo, Exact sampling, Coupling from the past



## Bibliography

- General background about probability theory
- Graphical models
- Exact inference
- Variational Methods
- Markov Chain Monte Carlo
- Sequential Monte Carlo
- Applications

## General background about probability theory

- Dimitri P. Bertsekas and John N. Tsitsiklis. Introduction to Probability. Athena Scientific, 2002
- Geoffrey Grimmet and David Stirzaker, Probability and Random Processes, (3rd Ed), Oxford, 2006



## "Instant Classics" of Bayesian Machine Learning and Graphical Models

- Michael I. Jordan, Learning in Graphical Models, 1998
- David MacKay Information Theory, Learning and Inference Algorithms, 2003, Cambridge
- Chris Bishop, Machine Learning and Pattern Recognition, 2006, Springer



## Further Reading, Variational Methods

- Jaakkola "Tutorial on variational approximation methods", 2000
  `http://people.csail.mit.edu/tommi/papers/Jaa-var-tutorial.ps`
- Wainwright and Jordan 2003 [19] Berkeley EECS Tech. Rep.
- Frey and Jojic, PAMI 2005 [11]
- Winn and Bishop "Variational Message Passing" 2005 JMLR [20]



## Further Reading, MCMC and SMC tutorials and overviews

- Andrieu, de Freitas, Doucet, Jordan. *An Introduction to MCMC for Machine Learning*, 2001
- Andrieu. *Monte Carlo Methods for Absolute beginners*, 2004
- Doucet, Godsill, Andrieu. ”On Sequential Monte Carlo Sampling Methods for Bayesian Filtering”, Statistics and Computing, vol. 10, no. 3, pp. 197-208, 2000
- Gilks, Richardson, Spiegelhalter, *Markov Chain Monte Carlo in Practice*, Chapman Hall, 1996
- Doucet, de Freitas, Gordon, *Sequential Monte Carlo Methods in Practice*, Springer, 2001

## Text Processing and Information Retrieval

- *Information Retrieval*, Manning, Schuetze, and Raghavan, Cambridge University Press, 2007 (Draft)
  `http://www-csli.stanford.edu/ schuetze/information-retrieval-book.html`
- *Modeling the Internet and the Web: Probabilistic Methods and Algorithms*. Pierre Baldi, Paolo Frasconi, Padhraic Smyth, Wiley, 2003
  `http://ibook.ics.uci.edu`

Compression, Efficient Data Structures,

- *Managing Gigabytes: Compressing and Indexing Documents and Images*. Witten, Moffat and Bell. Morgan Kaufmann 1999
- *Introduction to Data Compression*. Khalid Sayood, Morgan Kaufmann (3rd Ed), 2005



## Some Generic Software Packages

- Kevin Murphy's Matlab Bayesian Networks toolkit (BNT)
- Gilks, et. al. BUGS, WinBUGS – (Bayesian analysis Using Gibbs Sampling) A powerful program that compiles Gibbs Samplers from
- Winn, et. al, VIBES – Similar to BUGS but for variational inference

For source separation, there are some specialised libraries

- Petersen and Winther (DTU, Kopenhagen)
- Harva, Raiko, Honkela, Valpola "Bayes Blocks" (HUT, Helsinki)



## Music Applications

- Klapuri and Davy (Eds) Signal processing for Music Transcription, Springer, 2006
- Temperley, Probability and Music, MIT Press, 2007



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## Thank you for your patience and attention!

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`http://www-sigproc.eng.cam.ac.uk/~atc27/acm-tutorial/`