Flop count vs. Efficiency

SiScLab-18

Edilbert Christhuraj, Sadulla Aghayev

February 23, 2018

Abstract

This is a short report on the project "Flop count vs. Efficiency". This project is an investigation of the relation between flop count and execution time and a study of sensitivity to perturbations.

Introduction

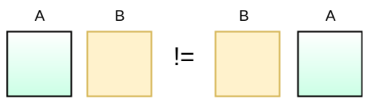
In the recent years there has been tremendous improvements in the development of numerical algorithms. When one constructs an algorithm, one may tend to minimize the number of floating point operations of an algorithm with the intention of minimizing the execution time. The underlying assumption, which unfortunately does not hold in practice, is that all flops cost the same. In this project we investigate relation between flop count and efficiency using an example problem, matrix chain problem and we verify that all flops do not cost the same.

The organization of the report as follow: The first section, preliminaries, gives reader a glance at some basic properties of matrix multiplication and introduction to matrix chain problem. The follow up section, Problem formulation, sets up a ground for discussion of main problem. In the next section, Results and discussion, the results of the studied problem is presented. The section, Perturbation analysis, illuminates the sensitivity of the given problem to perturbations. The final section, Next steps, shows possible directions for further improvement and investigation.

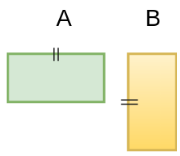
Preliminaries

We recall some of the properties of matrix multiplication.

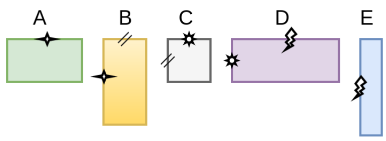
(i) Matrix multiplication is associative but not always commutative. I.e., when we do matrix multiplications the order in which the matrices are multiplied is important.



(ii) For a valid matrix multiplication operation, the matrix dimensions must agree: the number of columns in the first matrix should be equal to number of rows in the second matrix.



(iii) When more than two matrices are multiplied, it is called matrix chain multiplication.



Further we note the following points: In this project we consider a matrix that is filled with double precision real numbers and all the computations were performed using highly optimized library OpenBLAS [1]

Problem formulation

Toy problem

To begin with, we consider a chain with 3 matrices. We have three random matrices A, B, C of sizes $10 * 30, 30 * 5, 5 * 60$ respectively. If we want to multiply these matrices, there are two possible ways. This is illustrated in figure 1 .

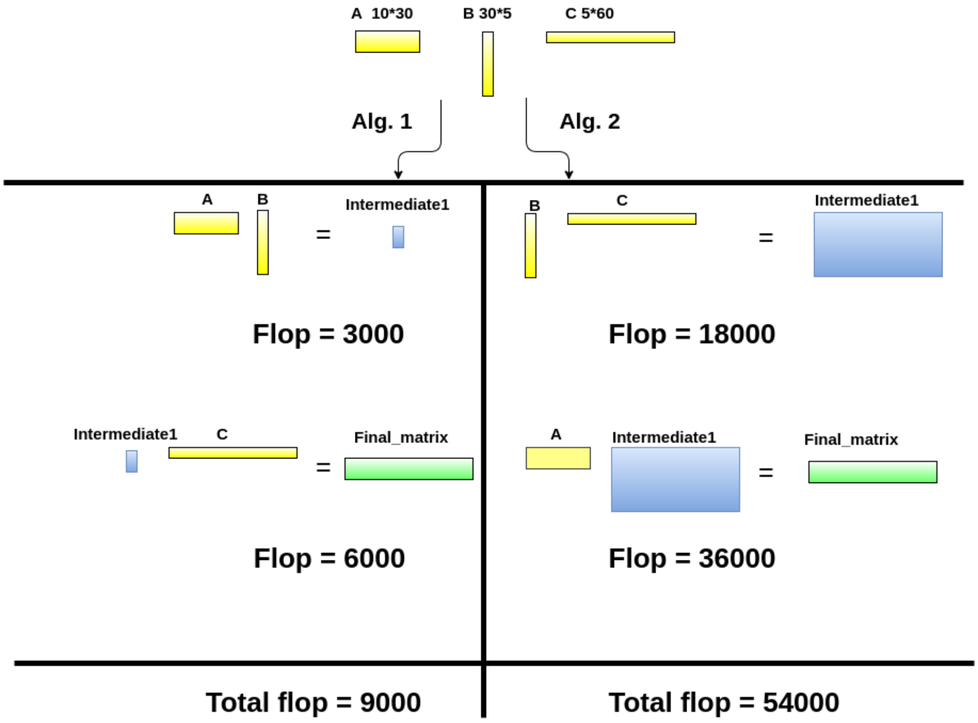


Figure 1: Two possible ways of multiplying 3 matrices

In algorithm 1 we multiply the matrices A and B first. Then with the intermediate result we multiply the third matrix C. Whereas in algorithm 2 we start with matrices B and C first and then we multiply the intermediate result with the matrix A. In both ways we obtain the same end result but the main difference lies in amount of flop[1] performed in order to obtain the end result. From figure 1 we can infer that the algorithm 2 is 6 times costlier than the algorithm 1 but which algorithm executes faster? We defer such delicate question related to execution time to later sections.

Main problem

Now we move from a simple toy example to a sophisticated problem. We consider 5 matrices of random sizes A,B,C,D,E.

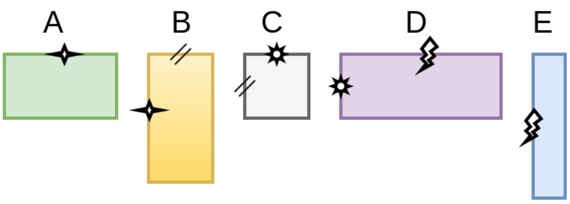


Figure 2: 5 matrices chain

We want to multiply these matrices. To do so, we multiply matrices A,B first. Then with the first intermediate result we multiply C and then with the second intermediate result we multiply matrix D and finally with the third intermediate results we multiply E. This is possibility-1 (((A*B)*C)*D)*E but there is also another possibility. We can start with matrices D, E and do the same but in reverse order. This is

[1]Flop is a quantity which tells how many number of arithmetic operations are performed in computation

possibility-2 A*(B*(C*(D*E))). There are many more such possibilities. There exists a set up which allows to find all the possibilities. It is called Catalan number and parenthesization problem [2].
P_n parenthesization: $P_n = C_{n-1}$
n^{th} Catalan number : $C_n = \frac{1}{n+1}\binom{2n}{n}$

n	2	3	4	5	6	7	8	9
P_n	1	2	5	14	42	132	429	1430

Table 1: Catalan number and paranthesization

The above set up gives us possibility to count all ways to group n factors with parenthesis. Using that, we formed the following 14 possibilities.

Alg.-0 (((A*B)*C)*D)*E
Alg.-1 (A*(B*(C*D)))*E
Alg.-2 A*(B*(C*(D*E)))
Alg.-3 A*(((B*C)*D)*E)
Alg.-4 A*((B*C)*(D*E))
Alg.-5 A*(B*((C*D)*E))
Alg.-6 A*((B*(C*D))*E)
Alg.-7 (A*B)*(C*(D*E))
Alg.-8 (A*B)*((C*D)*E)
Alg.-9 ((A*B)*C)*(D*E)
Alg.-10 (A*(B*C))*(D*E)
Alg.-11 ((A*(B*C))*D)*E
Alg.-12 (A*((B*C)*D))*E
Alg.-13 ((A*B)*(C*D))*E

For better understanding we visualized all the algorithms as block diagrams. See Figure 3.

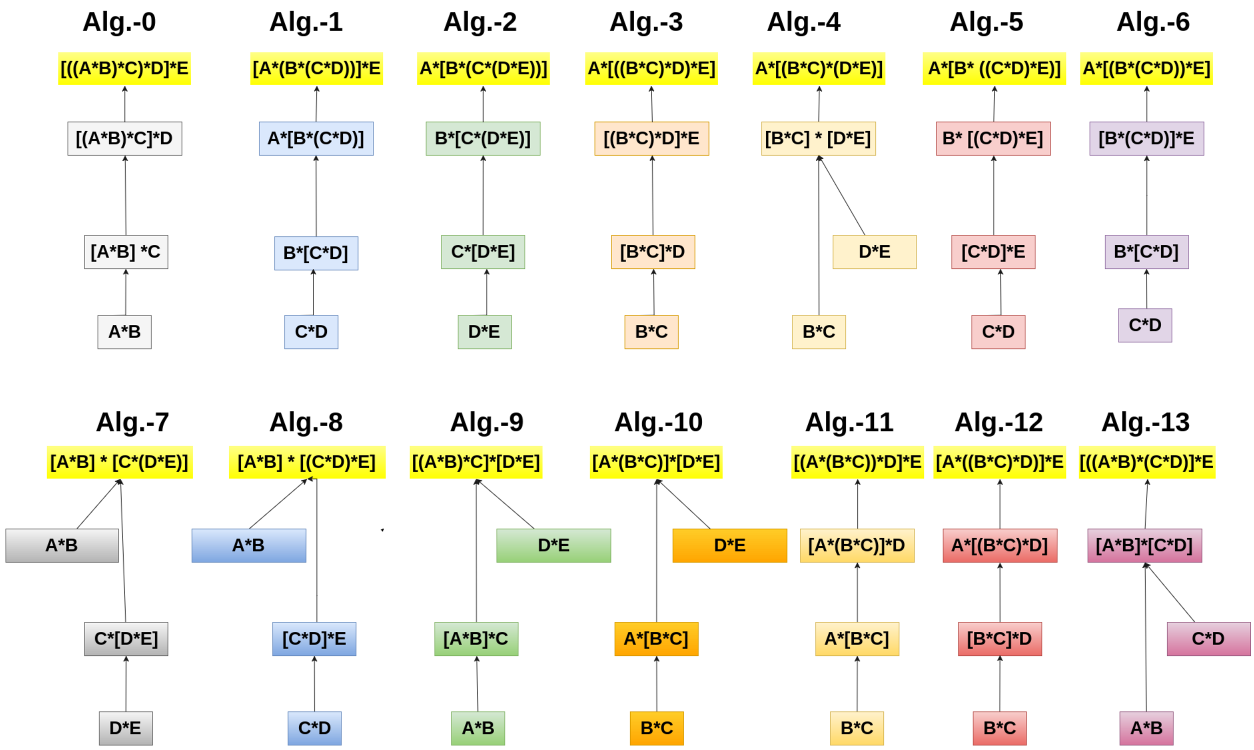


Figure 3: 14 possible algorithms for 5 matrices chain

The two main questions we want to ask ourselves are as follows: out of all algorithms which algorithm performs minimum number of flop?, which algorithm takes minimum time to execute? We are also interested in a quantity "deviation" which is described below.

(i) **Flop** - How many floating point operations does each algorithm perform?
min_flop_alg - Algorithm which performs minimum flop

(ii) **Execution time (in Seconds)** - How much time does each algorithm take to execute?
min_time_alg - Algorithm which takes minimum time to execute

(iii) **Deviation (in %)**
$Deviation = \frac{t_{min\_flop\_alg} - t_{min\_time\_alg}}{t_{min\_time\_alg}} * 100$
How much percentage is the "min_time_alg" faster than the "min_flop_alg" ?
$t_{min\_flop\_alg}$ - time for minimum flop algorithm
$t_{min\_time\_alg}$ - time for minimum time algorithm

Pseudo algorithm

We have written a software application, in C programming language, which gives us the results of questions that are addressed in the previous section. For more details refer [3].

```
main {
Timer start
Alg. 0}
Timer finish
Timer start
Alg. 1
Timer finish
.
.
.
Timer start
Alg. 13
Timer finish
      }
```

Input : Matrix sizes$[1, 2, 3, 4, 5, 6]$
output :
time_0 flop_0
time_1 flop_1
time_2 flop_2
. .
. .
time_12 flop_12
time_13 flop_13
Min_flop_alg
Min_flop_alg
Deviation

Test cases

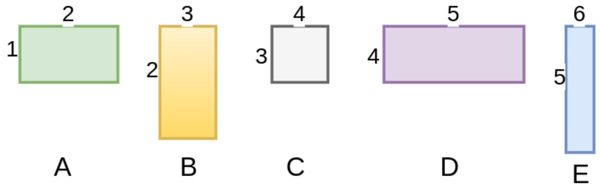


Figure 4: 5 matrices chain

As a test case we consider two chains randomly. First chain consists of matrices of different sizes, size ranges size from 100 to 1300 and second one consists of matrices of more or less same sizes.
1^{st} Chain 130 700 383 1340 193 900
2^{nd} Chain 376 561 477 532 425 590

Results and Discussion

In this section we look at the results of our problem. We use the following notations, throughout the text, chain[2] and run[3] for the sake of simplicity. we conducted 50 runs for each chain. We present only 3 run-results for each chain out of 50 runs.

1^{st} Chain (Run-1)

Algorithm	**Time** (seconds)	**Flop**
0	0.051306	315,546,400
1	0.065909	381,877,520
2	0.247766	2,035,692,000
3	0.180365	1,487,556,000
4	0.361984	3,036,224,000
5	0.118680	977,537,120
6	0.088250	708,569,520
7	0.185151	1,548,640,000
8	0.061688	490,485,120
9	0.122389	982,219,200
10	0.215232	1,741,464,000
11	0.131261	1,074,791,200
12	0.140059	1,160,864,000
13	0.042321	332,189,860

Table 2: 1^{st} Chain (Run-1)

First run result of first chain is presented in table 2, we see that min_flop_alg is 0, min_time_alg is 13 and deviation is 21.2%. That means the algorithm 13 is 21.2% faster than the algorithm 0, even though it performs more flop than algorithm 0.

1^{st} Chain (Run-2)

We repeated the experiment and obtained the results in table 3. We see that this time min_flop_alg is 0, min_time_alg is 0 and deviation is 0.0%. That means the algorithm 0 is the fastest among all.

Algorithm	**Time** (seconds)	**Flop**
0	0.042322	315,546,400
1	0.054053	381,877,520
2	0.255563	2,035,692,000
3	0.187326	1,487,556,000
4	0.381132	3,036,224,000
5	0.120140	977,537,120
6	0.087574	708,569,520
7	0.189211	1,548,640,000
8	0.061826	490,485,120
9	0.151340	982,219,200
10	0.274111	1,741,464,000
11	0.178278	1,074,791,200
12	0.181404	1,160,864,000
13	0.045157	332,189,860

Table 3: 1^{st} Chain (Run-2)

[2]Chain refers to matrix chain and specifies whether it is 1^{st} or 2^{nd}chain
[3]It is n repetition of experiment with the same input

Algorithm	**Time** (seconds)	**Flop**
0	0.055871	315,546,400
1	0.056499	381,877,520
2	0.261406	2,035,692,000
3	0.206040	1,487,556,000
4	0.387012	3,036,224,000
5	0.131321	977,537,120
6	0.092505	708,569,520
7	0.196337	1,548,640,000
8	0.066044	490,485,120
9	0.128545	982,219,200
10	0.226485	1,741,464,000
11	0.137919	1,074,791,200
12	0.146254	1,160,864,000
13	0.044547	332,189,860

Table 4: 1^{st} Chain (Run-3)

1^{st} Chain (Run-3)

We consider one more sample result before moving on to next chain. From table 4, we see that this time again the min_flop_alg is 0, min_time_alg is 13 and deviation is 25.4%. That means the algorithm 13 is 25.4% faster than the algorithm 0, even though it performs more flop than algorithm 0.

2^{nd} Chain (Run-1)

First run results of second chain is presented in table 5. We observe similar phenomenon for the 2^{nd} chain too. For each run we observed different results. We see the result of first run of second chain. The min_flop_alg is 0, min_time_alg is 13 and deviation is 8.44%.

Algorithm	**Time** (seconds)	**Flop**
0	0.101891	750,654,672
1	0.099232	811,016,450
2	0.140206	1,130,908,460
3	0.130707	1,068,653,388
4	0.139825	1,152,599,048
5	0.126232	1,019,583,840
6	0.121548	973,402,830
7	0.121522	979,107,824
8	0.110458	867,783,204
9	0.112590	894,899,232
10	0.124695	1,011,994,872
11	0.106382	867,750,312
12	0.112451	906,267,008
13	0.093955	757,945,544

Table 5: 2^{nd} Chain (Run-1)

2^{nd} Chain (Run-2)

For the second run, from table 6, we observe that this time min_flop_alg is 0, min_time_alg is 0 and deviation is 0.0%. That means the algorithm 0 is the fastest among all.

Algorithm	**Time** (seconds)	**Flop**
0	0.090527	750,654,672
1	0.096886	811,016,450
2	0.133695	1,130,908,460
3	0.126657	1,068,653,388
4	0.137426	1,152,599,048
5	0.122484	1,019,583,840
6	0.116337	973,402,830
7	0.122186	979,107,824
8	0.113717	867,783,204
9	0.109312	894,899,232
10	0.123241	1,011,994,872
11	0.105787	867,750,312
12	0.110508	906,267,008
13	0.091250	757,945,544

Table 6: 2^{nd} Chain (Run-2)

2^{nd} Chain (Run-3)

For the third run,from table 7, we see that, again the min_flop_alg is 0, min_time_alg is 13 and deviation is 9.12%.

Algorithm	**Time** (seconds)	**Flop**
0	0.102252	750,654,672
1	0.114649	811,016,450
2	0.145311	1,130,908,460
3	0.129112	1,068,653,388
4	0.139820	1,152,599,048
5	0.129094	1,019,583,840
6	0.120107	973,402,830
7	0.120611	979,107,824
8	0.107226	867,783,204
9	0.110580	894,899,232
10	0.123456	1,011,994,872
11	0.105604	867,750,312
12	0.110942	906,267,008
13	0.093701	757,945,544

Table 7: 2^{nd} Chain (Run-3)

Considering all the run results that we have seen so far, we need no explanation when the min_flop_alg is the fastest but we want to understand what happens in the other cases. i.e, the cases when there is an algorithm with more flop executes faster. This happens because all flop do not cost the same. This leads us to the next section which verifies this statement.

All flops do not cost the same

Let us consider multiplication of two square matrices. We recorded Flops for different matrix sizes from 100 to 3000.

We plotted a graph for number of flops against matrix sizes. Refer figure 5. What we can observe from figure 5 is that when we give large matrices, more flops are performed in opposition to small matrix sizes. To emphasize, when we multiply two matrices of size $100 * 100$ the machine performs approximately $4.6 * 10^9$ Flop per second but when we multiply, for instance $3000 * 3000$ matrices the machine does not restrict itself to the value $4.6 * 10^9$ rather it performs $7.4 * 10^9$ Flop per second. We conclude this section by verifying

Size of A,B	Flop	FLOPS = $\frac{Flop}{Second}$
$(100 * 100)$	2,000,000	4,672,897,196
$(200 * 200)$	16,000,000	4,705,882,352
$(300 * 300)$	54,000,000	5,819,592,628
$(400 * 400)$	128,000,000	6,994,535,519
..	...	...
..	...	...
..	...	...
$(1300 * 1300)$	4,394,000,000	7,036,896,462
$(1400 * 1400)$	5,488,000,000	7,109,296,363
$(1500 * 1500)$	6,750,000,000	7,048,975,235
...	...	...
$(3000 * 3000)$	54,000,000,000	7,404,131,916

Table 8: Matrix size vs FLOPS

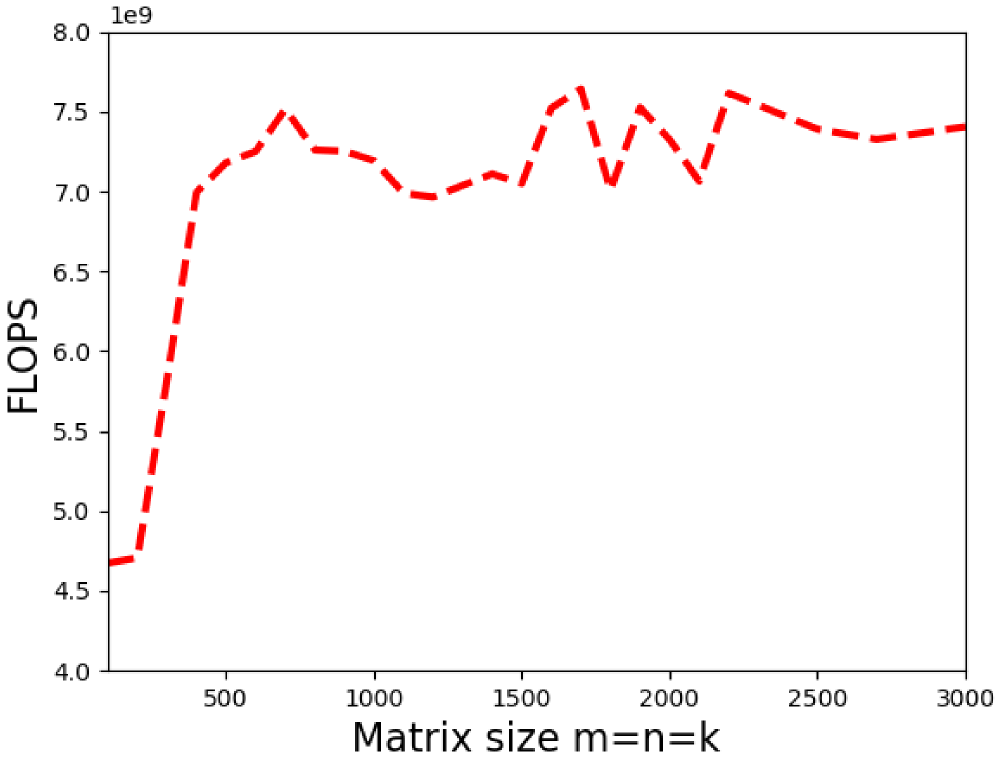


Figure 5: All flops do not cost the same

that, indeed all flops do not cost the same.

Perturbation analysis

Goal

From previous sections we saw that some of the quantities that we are interested in such as execution time and deviation are not reproducible. So we conducted several runs for the same input. i.e., we gave 1^{st} chain as input and repeated the experiment 50 times for same input and recorded following quantities:

(i) **Deviation frequency**: It shows us, out of 50 iterations how often does deviation occur? In others words, how often the *min_flop_alg* and *min_time_alg* are not the same?

(ii) **Deviation range**: It reflects, out of 50 iterations, if deviation occurs, then what is the minimum deviation and the maximum deviation?

(iii) **Flop difference**: It shows the flop difference between the *min_flop_alg* and the *min_time_alg*.

Now we want to perturb the matrix chain i.e., we consider a new chain in the neighborhood of unperturbed matrix chain. If we look carefully, we see that each matrix chain has 6 elements in it. We want to perturb each element by same amount and find a new chain. For example, if our unperturbed chain was 130 700 383 1340 193 900 and we perturbed each element by 1% , then we would get a new chain as 131 707 386 1353 198 909. We consider range from -15% to +15%. For each new chain we conducted equal number of runs as we did for our original chain and we want to study what happens to the quantities that we defined earlier in the section.

1^{st} Chain ***All elements perturbation***

Perturbation percentage	Matrix chain	Deviation range	Deviation frequency	Flop difference
+15%	149 805 440 1541 221 1035	0.01-29 %	25%	25,128,102
+10%	143 770 421 1474 152 990	3.5-15 %	8%	21,006,788
+5%	136 735 402 1407 202 945	0.01-68 %	30%	19,442,328
+3%	133 721 394 1380 198 927	0.01-40 %	40%	18,752,952
+1%	131 707 386 1353 194 909	0.01-18.5%	40%	16,653,820
Unperturbed	130 700 383 1340 193 900	0.01-36 %	27%	16,643,460
-1%	128 693 379 1326 191 891	0.01-21 %	26%	17,017,292
-3%	126 679 371 1299 187 873	0.01-42 %	30%	15,064,266
-5%	123 665 363 1273 183 855	0.01-80 %	34%	14,485,500
-10%	117 630 344 1206 173 810	0.01-22 %	30%	11,569,284
-15%	110 595 325 1139 164 765	0.01-18 %	28%	10,609,780

Table 9: 1^{st} Chain - All elements perturbation

The table 9 shows results of perturbation of 1^{st} chain. The blue colored row represents unperturbed chain. The rows which are placed above and below the unperturbed chain, are perturbed in an increasing and decreasing manner respectively. For the unperturbed chain deviation frequency is around 27% out of 50 runs and the deviation range is from 0.01 % to 36%. We see that, for all perturbed chains there is corresponding deviation frequency and the maximum value is limited to 40%. Particularly, in the case of +10% perturbation deviation occurrence is rare and it is only about 8% out of 50 runs.

After a closer examination of the deviation range, we find a pattern in behavior of deviation range with respect to perturbation percentage. For small perturbation percentage, this parameter drops and then it increases rapidly. Once it reaches maximum value, it drops again. It is the case for both, positive and negative perturbations.

One interesting point to consider is, perturbation with 5% in both positive and negative directions gives us the highest deviation range which is about 68% and 80% respectively and approximately 2 times greater than the original chain's value.

2^{nd} Chain ***All elements perturbation***

We carried out a similar study for the second chain. The results of perturbation of 2^{nd} chain is shown in table 10. The values are placed like in the previous case.

Interestingly, we observe similar patterns for this chain as well. We see that flop difference increases with increasing matrix sizes. The difference in deviation frequency among all chains, is not very much, values ranging from 40% up to 65%. In the case of deviation range, we see the similar behavior like for the 1^{st} chain, For small perturbation percentage, this parameter drops and then it increases rapidly. Once it reaches the maximum, it drops again. It is the case for both, positive and negative perturbations.

We saw that significant changes occurred in deviation range and deviation frequency when the chains are perturbed by -5% and +5%. It stimulated our curiosity to explore it further. We want to understand what happens when perturbing the chain 5% and which element of chain contributes to the changes. In other words, we want to find out which elements in a chain are sensitive to perturbation.

1^{st} Chain ***Single element perturbation***

Unlike previous perturbations, we perturbed the original chain one element at a time by +5%. We did the same for negative case also. Results of single element perturbation of 1^{st}chain is shown in table 11.

Perturbation percentage	**Matrix chain**	**Deviation range**	**Deviation frequency**	**Flop difference**
+15%	432 645 548 611 488 678	0.01-15 %	46.5%	10,937,920
+10%	413 617 524 585 467 649	3.5-6%	48%	9,576,058
+5%	394 589 500 558 446 619	0.01-67 %	52.5%	8,632,016
+3%	387 577 491 547 437 607	0.01-20 %	43%	7,916,372
+1%	379 566 481 537 429 595	0.01-19 %	63%	7,619,508
Unperturbed	376 561 477 532 425 590	0.01-22 %	40%	7,290,872
-1%	372 555 472 526 420 584	0.01-13 %	65%	6,960,192
-3%	364 544 462 516 412 572	0.01-20 %	47.5%	6,89,088
-5%	357 532 453 505 403 560	0.01-73 %	57%	6,083,796
-10%	338 504 429 478 382 531	0.01-34 %	43%	5,392,088
-15%	319 476 405 452 361 501	0.01-19 %	40%	4,552,094

Table 10: 2^{nd} Chain - All elements perturbation

Perturbation percentage	**Matrix chain**	**Deviation range**	**Deviation frequency**	**Flop difference**
+5%	136 700 383 1340 193 900	0.01-25 %	53%	8,628,408
+5%	130 735 383 1340 193 900	0.01-67 %	33%	16,643,460
+5%	130 700 402 1340 193 900	0.01-24 %	30%	20,804,840
+5%	130 700 383 1407 193 900	0.01-74 %	46%	16,514,686
+5%	130 700 383 1340 202 900	2 - 21 %	26%	23,642,040
+5%	130 700 383 1340 193 945	1 - 40 %	36%	16,643,460
Unperturbed	130 700 383 1340 193 900	0.01-36 %	27%	16,643,460
-5%	123 700 383 1340 193 900	0.01-50 %	23%	26,414,454
-5%	130 665 383 1340 193 900	0.01-85 %	56%	16,643,460
-5%	130 700 363 1340 193 900	0.01-13 %	43%	9,263,060
-5%	130 700 383 1273 193 900	4 - 84 %	30%	16,772,234
-5%	130 700 383 1340 183 900	0.01-26%	56%	8,867,260
-5%	130 700 383 1340 193 855	0.01-39 %	36%	16,643,460

Table 11: 1^{st} Chain - Single element perturbation

Blue colored row represents the unperturbed chain. The first column indicates how much percentage each element is perturbed. In our case it is 5%. The rows which are above original chain are incremented by 5% whereas rows below the original chain are decremented by 5%. It is shown by corresponding symbol + or -. The second column shows which element of the chain is perturbed and rest of quantities are same as we discussed earlier. Overall we can see that this chain is sensitive to single element perturbation.

For unperturbed chain the deviation range is 0.01%-36% and deviation frequency is 27%. When we perturb the 2^{nd} and 4^{th} elements by +5% the deviation range is increased twice in comparison with original deviation range and deviation frequency is considerably higher. Similarly when we perturb the 2^{nd} and 4^{th} elements by -5% the deviation range is increased almost thrice compared to original deviation range and deviation frequency is considerably higher. On the other hand when we perturb the 3^{rd} element by +5% and -5% the deviation range is lower than the original values.

If we look at flop difference of unperturbed chain, it is 16,643,460. When we perturb 1^{st} element by -5% , flop difference is increased by considerable amount. In fact, it is the highest flop difference among all chain's flop differences. On the contrary when we perturb 1^{st} element by +5% flop difference reaches the lowest value among all other chain's values.

2^{nd} Chain *Single element perturbation*

We carried out similar study for our second chain also. The results are presented in table 12.

Surprisingly, we can see somewhat similar phenomenon for this chain as well but main difference lies in the position of element that is sensitive to perturbation. Flop difference of unperturbed chain is 7,290,872. When we perturb 1^{st} element by -5% , flop difference is increased by more than twice. In fact, it is the highest flop difference among all chain's flop differences. On the contrary when we perturb 4^{st} element by +5% flop difference reaches the lowest value among all other chain's values.

Deviation range of unperturbed chain is 0.01%-22% and deviation frequency is 40%. When we perturb the 3^{rd} elements by +5% the deviation range is increased twice than the original deviation range and deviation frequency is considerably higher. On the other hand when we perturb the 4^{th} and 5^{th} element by +5% and

Perturbation percentage	**Matrix chain**	**Deviation range**	**Deviation frequency**	**Flop difference**
+5%	394 561 477 532 425 590	0.01-11 %	23%	2,686,132
+5%	376 589 477 532 425 590	0.01-17 %	40%	7,290,872
+5%	376 561 500 532 425 590	0.01-49 %	50%	15,840,800
+5%	376 561 477 558 425 590	0.01- 5 %	23%	196,668
+5%	376 561 477 532 446 590	0.01-12%	30%	17,080,400
+5%	376 561 477 532 425 619	0.01-16 %	43%	7,290,872
Unperturbed	376 561 477 532 425 590	0.01-22 %	40%	7,290,872
-5%	357 561 477 532 425 590	0.01-24 %	26%	17,822,154
-5%	376 532 477 532 425 590	0.01-32 %	41%	7,290,872
-5%	376 561 453 532 425 590	0.01-11 %	26%	1,630,792
-5%	376 561 477 505 425 590	0.01-44 %	43%	14,657,930
-5%	376 561 477 532 403 590	0.01- 9 %	26%	2,964,824
-5%	376 561 477 532 425 560	0.01-31 %	43%	7,290,872

Table 12: 2^{nd} Chain - Single element perturbation

-5% the deviation range is much lower than the original value.

Conclusion

Firstly, we studied the relation between flop count and execution time using matrix chain problem as an example problem. Secondly, we also verified that all flops do not cost the same. So when constructing a new algorithm, we should not try to minimize flop of the algorithm for the sake of minimizing execution time. Instead while constructing an algorithm, we should keep the todays computer architecture in our mind and design algorithm accordingly. Finally, we studied the sensitivity of the problem to the perturbation. We also studied crucial elements of the matrix chains.

Future work

We conclude this report by giving some pointers to further improvement of this work. In our project we wrote a software application for the problem and tested it using only one node but with the rapid improvement in parallel programming paradigm, we could do even better. We should try to utilize the system capacity as much as possible. There are lot of scope for parallelization in this project. One could parallelize wherever possible and check for timings, flops and deviation.

References

[1] OpenBLAS: An optimized BLAS library. GitHub: `http://www.openblas.net/`

[2] Link to Catalan number and parenthesization : `https://github.com/edilbert24/Sisc-Lab`

[3] Link to the source code : `https://github.com/edilbert24/Sisc-Lab`

[4] Elmar Peise. *Performance Modeling and Prediction for Dense Linear Algebra* `https://arxiv.org/pdf/1706.01341.pdf`