Convolution with Singularity Functions

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Abstract—**A simple, nongraphical technique for convolving two functions (when one or both functions can be expressed explicitly in terms of singularity functions) is described in this paper. The technique is based on some properties of singularity functions and can easily be explained to undergraduate students. In fact, the paper is written such that it can serve as class notes to undergraduates.**

I. Introduction

CONVOLUTION is a very important tool to the engineer; it provides another means of viewing and characterizing physical systems. For example, it is used in linear systems and control theory to find the response $y(t)$ of a system to an excitation $x(t)$ knowing the system impulse response $h(t)$. In general, the convolution $y(t)$ of two signals $x_1(t)$ and $x_2(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{-\infty}^{\infty} x_2(\lambda)\, x_1(t-\lambda)\, d\lambda \tag{1a}$$

or

$$y(t) = x_1(t) * x_2(t) = x_2(t) * x_1(t) \tag{1b}$$

where λ is a dummy variable and the star sign denotes convolution.

A traditional method of handling the convolution integral in (1) is by the graphical procedure involving four steps: folding, displacement, multiplication, and integration [1]–[3]. When one or both functions to be convolved can be expressed explicitly in terms of singularity functions, the four steps required by the graphical method can be reduced to two by using a simple, nongraphical method described in this paper.

II. Definition of Singularity Functions

The singularity functions are a special class of functions used extensively in science and engineering. For the purpose of illustration, we define five of these functions: unit square or parabolic function $p(t)$, unit ramp function $r(t)$, unit step function $u(t)$, unit impulse or delta function $\delta(t)$, and unit doublet function $\delta'(t)$, i.e.,

$$\cdots\ \delta'(t) \underset{I}{\overset{D}{\rightleftarrows}} \delta(t) \underset{I}{\overset{D}{\rightleftarrows}} u(t) \underset{I}{\overset{D}{\rightleftarrows}} r(t) \underset{I}{\overset{D}{\rightleftarrows}} p(t) \cdots$$

Fig. 1. Differentiating (D) or integrating (I) singularity functions.

$$\vdots$$

$$p(t) = \begin{bmatrix} \frac{1}{2}t^2, & t \geq 0 \\ 0, & t < 0 \end{bmatrix} \tag{2a}$$

$$r(t) = \begin{bmatrix} t, & t \geq 0 \\ 0, & t < 0 \end{bmatrix} \tag{2b}$$

$$u(t) = \begin{bmatrix} 1, & t \geq 0 \\ 0, & t < 0 \end{bmatrix} \tag{2c}$$

$$\delta(t) = \begin{bmatrix} \infty, & t = 0 \\ 0, & t \neq 0 \end{bmatrix} \tag{2d}$$

$$\delta'(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{t} \delta'(\lambda)\, d\lambda = \delta(t), \quad t \neq 0 \tag{2e}$$

$$\vdots$$

The dots indicate that singularity functions of higher or lower order can be added. One interesting property of the singularity functions to be noticed in (2) is that (loosely speaking) if we differentiate (or integrate) one, we get the next one of lower (or higher) order as shown in Fig. 1. This inherent property is taken advantage of when convolving any function with a singularity function.

III. Convolution with Singularity Functions

It is easily shown [4], [5] that

$$\vdots$$

$$x(t) * \delta'(t) = x'(t) \tag{3a}$$

$$x(t) * \delta(t) = x(t) \tag{3b}$$

$$x(t) * u(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda \tag{3c}$$

$$x(t) * r(t) = \int_{-\infty}^{t} \int_{-\infty}^{\mu} x(\lambda)\, d\lambda\, d\mu \tag{3d}$$

Manuscript received May 12, 1986.

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IEEE Log Number 8611185.

0018-9359/87/0200-0052$01.00

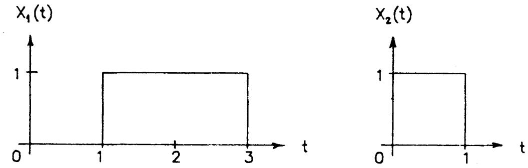

Fig. 2. For example 1.

$$x(t) * p(t) = \int_{-\infty}^{t} \int_{-\infty}^{\mu} \int_{-\infty}^{\nu} x(\lambda)\, d\lambda\, d\nu\, d\mu \quad (3e)$$

$$\vdots$$

Although the properties expressed in (3) are mentioned by a few textbooks [4], [5], the properties have neither been given their due credit nor taken advantage of. In this paper, we shall illustrate with specific examples how they can be applied and make the convolution process a lot easier when one or both functions to be convolved can be expressed in terms of singularity functions. Before doing this, it is advantageous to add that if

$$x_1(t) * x_2(t) = y(t) \quad (4a)$$

then

$$x_1(t + t_1) * x_2(t) = y(t + t_1) \quad (4b)$$

and

$$x_1(t + t_1) * x_2(t + t_2) = y(t + t_1 + t_2) \quad (4c)$$

where t_1 and t_2 are constants. The properties in (4b) and (4c) can be proved directly by using (1) or indirectly by utilizing the "convolution-multiplication" property of the Laplace transform. To illustrate the properties expressed in (3) and (4), we consider the following examples.

Example 1: Consider signals $x_1(t)$ and $x_2(t)$ shown in Fig. 2. In this case, both functions can be expressed in terms of singularity functions, i.e.,

$$x_1(t) = u(t - 1) - u(t - 3),$$
$$x_2(t) = u(t) - u(t - 1).$$

Hence,

$$\begin{aligned} y(t) &= x_1(t) * x_2(t) \\ &= u(t - 1) * u(t) - u(t - 3) * u(t) \\ &\quad - u(t - 1) * u(t - 1) + u(t - 3) * u(t - 1). \end{aligned}$$

Applying the properties in (3c) and (4) results in

$$\begin{aligned} y(t) &= r(t - 1) - r(t - 2) - r(t - 3) + r(t - 4) \\ &= \begin{bmatrix} t - 1, & 1 \le t \le 2 \\ 1, & 2 \le t \le 3 \\ 4 - t, & 3 \le t \le 4 \\ 0, & \text{elsewhere} \end{bmatrix} \end{aligned}$$

which is easily verified using graphical convolution.

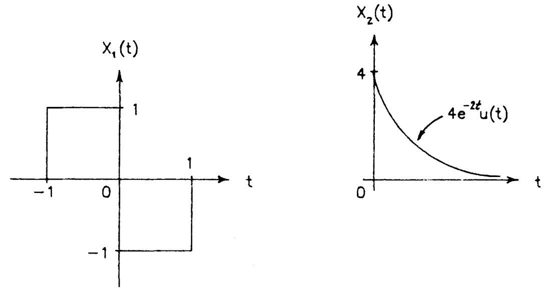

Fig. 3. For example 2.

Example 2: Consider signals $x_1(t)$ and $x_2(t)$ shown in Fig. 3. Only $x_1(t)$ can be explicitly expressed in terms of singularity function $u(t)$, i.e.,

$$x_1(t) = u(t + 1) - 2u(t) + u(t - 1),$$
$$x_2(t) = 4e^{-2t}u(t).$$

Hence,

$$\begin{aligned} y(t) &= x_1(t) * x_2(t) \\ &= u(t + 1) * x_2(t) - 2u(t) * x_2(t) \\ &\quad + u(t - 1) * x_2(t). \end{aligned}$$

Applying (3c) and (4) yields

$$y(t) = y_0(t + 1) - 2y_0(t) + y_0(t - 1)$$

where

$$\begin{aligned} y_0(t) &= u(t) * x_2(t) = \int_{-\infty}^{t} x_2(\lambda)\, d\lambda \\ &= \begin{bmatrix} 0, & t \le 0 \\ 4\int_0^t e^{-2\lambda}\, d\lambda, & t > 0 \end{bmatrix} \\ &= 2(1 - e^{-2t})\, u(t). \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= 2(1 - e^{-2t-2})\, u(t + 1) - 4(1 - e^{-2t})\, u(t) \\ &\quad + 2(1 - e^{-2t+2})u(t - 1) \\ &= \begin{bmatrix} 0, & t < -1 \\ 2(1 - e^{-2t-2}), & -1 \le t \le 0 \\ 2(2e^{-2t} - e^{-2t-2} - 1), & 0 \le t \le 1 \\ 2(2e^{-2t} - e^{-2t-2} - e^{-2t+2}), & t > 1. \end{bmatrix} \end{aligned}$$

Example 3: In this case, $x_1(t)$ and $x_2(t)$ are shown in Fig. 4; only $x_1(t)$ can be expressed explicitly in terms of singularity function $r(t)$, i.e.,

$$x_1(t) = r(t) - 2r(t - 1) + r(t - 2),$$
$$x_2(t) = e^{-t}u(t).$$

$$\begin{aligned} y(t) &= x_1(t) * x_2(t) = x_2(t) * r(t) - 2x_2(t) * r(t - 1) \\ &\quad + x_2(t) * r(t - 2). \end{aligned}$$

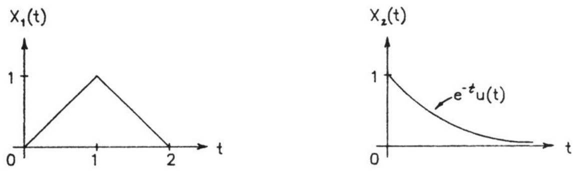

Fig. 4. For example 3.

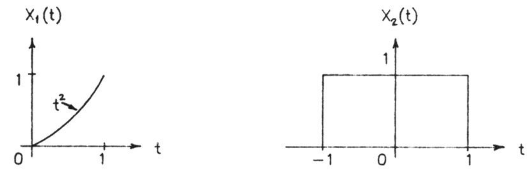

Fig. 5. For example 4.

Applying (3d) and (4b),

$$y(t) = y_0(t) - 2y_0(t-1) + y_0(t-2)$$

where

$$y_0(t) = x_2(t) * r(t) = \int_{-\infty}^{t} \int_{-\infty}^{\mu} x_2(\lambda)\, d\lambda\, d\mu$$

$$= \begin{bmatrix} 0, & t < 0 \\ \displaystyle\int_0^t (1 - e^{-\mu})\, d\mu, & t \geq 0 \end{bmatrix}$$

$$= (e^{-t} + t - 1)\, u(t).$$

Hence,

$$y(t) = (e^{-t} + t - 1)\, u(t) - 2(e^{-t+1} + t - 2)\, u(t-1) + (e^{-t+2} + t - 3)\, u(t-2)$$

$$= \begin{bmatrix} 0, & t < 0 \\ e^{-t} + t - 1, & 0 \leq t < 1 \\ e^{-t} - 2e^{-t+1} - t + 3, & 1 \leq t \leq 2 \\ e^{-t} - 2e^{-t+1} + e^{-t+2}, & t > 2. \end{bmatrix}$$

Example 4: Consider $x_1(t)$ and $x_2(t)$ as shown in Fig. 5. We can find $y(t) = x_1(t) * x_2(t)$ by expressing $x_2(t)$ in terms of $u(t)$ and following the procedure taken in Example 2. However, we choose to express $x_1(t)$ in terms of singularity functions so that we can apply (3e). Now

$$x_1(t) = 2p(t) - 2p(t-1) - 2r(t-1) - u(t-1),$$

$$x_2(t) = u(t+1) - u(t-1).$$

Hence,

$$\begin{aligned} y(t) &= 2p(t) * u(t+1) - 2p(t-1) * u(t+1) \\ &\quad - 2r(t-1) * u(t+1) - u(t-1) * u(t+1) \\ &\quad - 2p(t) * u(t-1) + 2p(t-1) * u(t-1) \\ &\quad + 2r(t-1) * u(t-1) + u(t-1) * u(t-1) \\ &= 2y_0(t+1) - 2y_0(t) - 2p(t) - r(t) - 2y_0(t-1) \\ &\quad + 2y_0(t-2) + 2p(t-2) + r(t-2) \end{aligned}$$

where

$$y_0(t) = p(t) * u(t) = \int_{-\infty}^{t} \int_{-\infty}^{\mu} \int_{-\infty}^{\nu} u(\lambda)\, d\lambda\, d\nu\, d\mu$$

$$= \begin{bmatrix} 0, & t < 0 \\ \frac{1}{6} t^3, & t > 0. \end{bmatrix}$$

Thus,

$$y(t) = \begin{bmatrix} \frac{1}{3}(t+1)^3, & -1 \leq t \leq 0 \\ \frac{1}{3}, & 0 \leq t \leq 1 \\ \frac{1}{3} - \frac{1}{3}(t-1)^3, & 1 \leq t \leq 2 \\ 0, & \text{elsewhere.} \end{bmatrix}$$

IV. Conclusions

From these examples, it is evident that using the technique discussed in this paper involves two steps: expressing functions explicitly in terms of singularity functions and applying the properties in (3) and (4). This procedure for convolving two functions is shorter and perhaps easier when one or both functions can be expressed explicitly in terms of singularity functions. The technique can be taught (and has been taught by the author) at the undergraduate level. All the student needs to know are the properties of singularity functions and signal manipulation. Also the technique can be extended to evaluating similar integrals such as the cross correlation function of two given signals.

Acknowledgment

The author would like to acknowledge R. Swan, A. Breese, and R. Sudhakar for their contribution.

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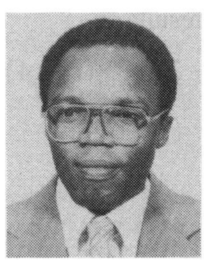

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