Singularity in Green's Function and Its Numerical Evaluation

SHUNG-WU LEE, SENIOR MEMBER, IEEE, JOHANNES BOERSMA, CHAK-LAM LAW, STUDENT MEMBER, IEEE, AND GEORGES A. DESCHAMPS, FELLOW, IEEE

Abstract—**The free-space scalar Green's function g has an R^{-1} singularity, where R is the distance between the source and observation points. The second derivatives of g have R^{-3} singularities, which are not generally integrable over a volume. The derivatives of g are treated as generalized functions in the manner described by Gel'fand and Shilov, and a new formula is derived that regularizes a divergent convolution integral involving the second derivatives of g. When the formula is used in the dyadic Green's function formulation for calculating the E field, all previous results are recovered as special cases. Furthermore, it is demonstrated that the formula is particularly suitable for the numerical evaluation of the field at a source point, because it allows the exclusion of an arbitrary finite region around the singular point from the integration volume. This feature is not shared by any of the previous results on the dyadic Green's function.**

I. INTRODUCTION

A FUNDAMENTAL problem in electromagnetic theory is to calculate the field at a source point. It arises in the evaluation of the antenna impedance, the power radiation, the induced current on a scatterer, and other situations. This paper presents a new formula for doing this calculation. We begin with a statement of the problem under consideration.

The free-space Green's function for the scalar wave equation is given by the well-known expression [for the exp $(j\omega t)$ time convention]

$$g(R)=\frac{1}{4\pi R}\, e^{-jkR}, \qquad \text{where } R=|\vec{r}\,'-\vec{r}\,|. \tag{1.1}$$

Here k is the wavenumber and $\vec{r} = (x_1, x_2, x_3)$ and $\vec{r}\,' = (x_1', x_2', x_3')$ are the observation and source points, respectively. In using the Green's function method to calculate the electric field due to a distributed current source, one encounters the following typical integral:

$$I_{mn}(\vec{r})=\frac{\partial^2}{\partial x_m \partial x_n}\int_V J(\vec{r}\,')g(R)\, dv', \qquad \text{for } m, n=1, 2, 3, \tag{1.2}$$

where $J(\vec{r}\,')$ is the density function of the current source, having a finite support in volume V. In this paper we concentrate on the case in which $\vec{r}$ is an interior point of V, i.e., the observation point is inside the source region. Then the function g has an R^{-1} singularity as $R \to 0$. Despite this singularity, the function $I_{mn}(\vec{r})$ is well-defined provided that the current density $J(\vec{r}\,')$ satisfies the so-called Hölder condition at $\vec{r}$ [1]. A function $J(\vec{r}\,')$ is said to satisfy the Hölder condition at $\vec{r}$ if there are three positive constants c, A, and α, such that

$$|J(\vec{r}\,')-J(\vec{r})| \leq AR^{\alpha}, \tag{1.3}$$

for all points $\vec{r}\,'$ for which $R \leqslant c$. Throughout this paper we assume that $J(\vec{r}\,')$ satisfies the Hölder condition at all points inside V. It is worthwhile to mention that mere continuity of $J(\vec{r}\,')$ without satisfying the Hölder condition is not sufficient for the second derivatives I_{mn} to exist. For a counterexample, see [2, pp. 118–124, lemma 44].

For numerical calculations it would be preferable to interchange the differentiation and the integral in (1.2) and calculate I_{mn} from the expression

$$\tilde{I}_{mn}(\vec{r})=\int_V J(\vec{r}\,')\,\frac{\partial^2 g}{\partial x_m'\partial x_n'}\, dv', \qquad \text{for } m, n=1, 2, 3. \tag{1.4}$$

The tilde on $\tilde{I}_{mn}$ emphasizes the possible difference between (1.2) and (1.4). Note that the differentiation in (1.4) is now with respect to the primed coordinates (those of the source points). A formal differentiation of (1.1) yields the following results:

$$\frac{\partial^2 g}{\partial x_m'^2}=k^2\left[-\cos^2\theta_m+\frac{j}{kR}\left(1-\frac{j}{kR}\right)\cdot(3\cos^2\theta_m-1)\right]g(R), \qquad m=1, 2, 3 \tag{1.5a}$$

$$\frac{\partial^2 g}{\partial x_m'\partial x_n'}=k^2\cos\theta_m\cos\theta_n\left[-1+\frac{3j}{kR}\left(1-\frac{j}{kR}\right)\right]g(R), \qquad m\neq n, \tag{1.5b}$$

where

$$\cos\theta_n=(x_n'-x_n)/R, \qquad \text{for } n=1, 2, 3. \tag{1.5c}$$

The second derivatives of $g(R)$ have an R^{-3} singularity at $R = 0$. This singularity is not generally integrable. Thus, the integral in (1.4) is generally divergent and $\tilde{I}_{mn}(\vec{r})$ cannot be identified with the well-defined function $I_{mn}(\vec{r})$ in (1.2).

The purpose of this paper is to apply a regularization to the divergent integral in (1.4) so that, after the regularization, $\tilde{I}_{mn}$ in (1.4) is indeed equivalent to I_{mn} in (1.2). Furthermore, we show that our regularization is a general one, which includes all regularizations reported in the literature as special cases.

II. REGULARIZATION OF $\tilde{I}_{mn}$

Regularization of a divergent integral is not generally a unique process. In the present problem we are interested in

Manuscript received July 5, 1979; revised November 23, 1979. This work was supported by the National Science Foundation under Grant Eng-77-20820.

S. W. Lee, C. L. Law, and G. A. Deschamps are with the Department of Electrical Engineering, University of Illinois, Urbana, IL 61801.

J. Boersma is with the Department of Mathematics, Technological University, Eindhoven, The Netherlands.

0018-926X/80/0500-0311$00.75 © 1980 IEEE

the particular regularization that redefines $\tilde{I}_{mn}$ in (1.4) so that it becomes equivalent to I_{mn} in (1.2).

We interpret the function $\partial^2 g/\partial x_m' \partial x_n'$ as a generalized function $\{\partial^2 g/\partial x_m' \partial x_n'\}$ (with curly brackets), whose definition[1] is given by

$$I_{mn}(\vec{r}) = \int_V J(\vec{r}') \left\{ \frac{\partial^2 g}{\partial x_m' \partial x_n'} \right\} dv', \qquad \text{for } m, n = 1, 2, 3 \tag{2.1a}$$

$$= A_{mn} + B_{mn} + C_{mn}, \tag{2.1b}$$

where

$$A_{mn} = \int_{V - V_\epsilon} J(\vec{r}') \frac{\partial^2 g}{\partial x_m' \partial x_n'} dv' \tag{2.1c}$$

$$B_{mn} = \int_{V_\epsilon} \left[J(\vec{r}') \frac{\partial^2 g}{\partial x_m' \partial x_n'} - J(\vec{r}) \frac{\partial^2 g_0}{\partial x_m' \partial x_n'} \right] dv' \tag{2.1d}$$

$$C_{mn} = J(\vec{r}) \frac{-1}{4\pi} \int_{\partial V_\epsilon} \frac{(\hat{x}_m \cdot \hat{N})(\hat{x}_n \cdot \hat{R})}{R^2} d\sigma' \tag{2.1e}$$

$$= J(\vec{r}) \frac{\partial^2}{\partial x_m \partial x_n} \int_{V_\epsilon} \frac{1}{4\pi R} dv'. \tag{2.1f}$$

The tilde on I_{mn} in (2.1a) has been removed because in the next section it is shown that (2.1) is indeed equivalent to (1.2). We believe that (2.1) is new. The various notations in it are explained below. As sketched in Fig. 1, volume V_ϵ is an arbitrary volume inside V and contains the observation point $\vec{r}$. We emphasize that

1) V_ϵ need not be small, and
2) the value of I_{mn} is independent of the choice of V_ϵ.

Term A_{mn} in (2.1c) is convergent because the region V_ϵ is excluded from the domain of integration. The static Green's function g_0 in (2.1d) is given by

$$g_0(R) = \frac{1}{4\pi R}, \tag{2.2}$$

which is obtained by setting $k = 0$ in the (dynamic) Green's function in (1.1). Note that g and g_0 have the same R^{-1} singularity at $R = 0$. This fact and the Hölder condition in (1.3) ensure the convergence of term B_{mn} in (2.1d). In term C_{mn} in (2.1e) there is a surface integral over the boundary surface of V_ϵ, which is denoted by ∂V_ϵ (Fig. 2). $\hat{N}$ is the unit outward normal of ∂V_ϵ at a point $\vec{r}'$, and $\hat{R}$ is the unit vector along $\vec{R} = \vec{r}' - \vec{r}$. An alternative expression of C_{mn} is given in (2.1f), which contains a volume integral. We note that this volume integral

$$\psi(\vec{r}) = \int_{V_\epsilon} \frac{1}{4\pi R} dv' \tag{2.3}$$

[1] A generalized function (distribution) is defined by describing its effect on test functions that have derivatives of all orders and a compact support. Equation (2.1) constitutes a definition of the generalized function $\{\partial^2 g/\partial x_m' \partial x_n'\}$ if the current density J is such a test function. The convolution in (2.1a) is, however, meaningful under more general conditions, for example, when J is any distribution with a compact support. In the present paper we restrict J to be an ordinary function satisfying the Hölder condition so that the integral in (2.1d) is convergent.

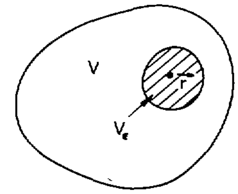


Fig. 1. V_ϵ is an arbitrary volume inside V and containing observation point $\vec{r}$.

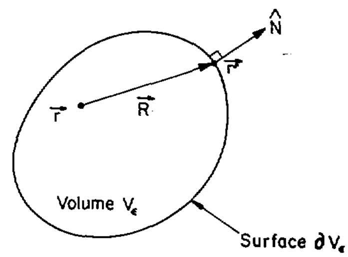


Fig. 2. Symbols for surface integral in (2.1e). $\vec{r}$ is observation point and $\vec{r}'$ is integration point on surface ∂V_ϵ.

can be interpreted to be the electrostatic potential due to a charge distribution of unit density with support in V_ϵ.

We now give the derivation of the formula in (2.1). According to Gel'fand and Shilov's definition [3, p. 295] the generalized function

$$f(\vec{r}') = \left\{ \frac{\partial g_0}{\partial x_n'} \right\}$$

is a homogenous function of degree (−2). Its derivative is defined by [2, p. 306, eq. (5)], namely,

$$\int_V J(\vec{r}') \left\{ \frac{\partial^2 g_0}{\partial x_m' \partial x_n'} \right\} dv'$$

$$= \int_{V - V_\epsilon} J(\vec{r}') \frac{\partial^2 g_0}{\partial x_m' \partial x_n'} dv'$$

$$+ \int_{V_\epsilon} [J(\vec{r}') - J(\vec{r})] \frac{\partial^2 g_0}{\partial x_m' \partial x_n'} dv' + C_{mn}. \tag{2.4}$$

Term C_{mn} in (2.4) is given by

$$C_{mn} = J(\vec{r}) \int_{\partial V_\epsilon} \frac{\partial g_0}{\partial x_n'} dx_p' \, dx_q', \qquad \text{where } p, q \neq m. \tag{2.5}$$

To simplify C_{mn}, note the relations (Fig. 2)

$$\frac{\partial g_0}{\partial x_n'} = \frac{-\cos\theta_n}{4\pi R^2} = \frac{-\hat{x}_n \cdot \hat{R}}{4\pi R^2} \tag{2.6a}$$

$$dx_p' \, dx_q' = (\hat{x}_m \cdot \hat{N}) \, d\sigma'. \tag{2.6b}$$

Substitution of (2.6) into (2.5) leads to the final form of C_{mn} in (2.1e). To relate the derivative of g_0 to that of g, we note the decomposition

$$\left\{\frac{\partial^2 g}{\partial x_m' \partial x_n'}\right\} = \frac{\partial^2 (g-g_0)}{\partial x_m' \partial x_n'} + \left\{\frac{\partial^2 g_0}{\partial x_m' \partial x_n'}\right\}. \tag{2.7}$$

The first term on the right side of (2.7) is well-defined because in the neighborhood of $R=0$

$$g - g_0 = \frac{1}{4\pi R}\left[(-jkR) + \tfrac{1}{2}(-jkR)^2 + \cdots\right]. \tag{2.8}$$

Substituting (2.7) into (2.1a) and making use of (2.4), we obtain the desired formula in (2.1b) after some straightforward manipulations.

III. CORRECT FINAL RESULT

Starting with the divergent integral in (1.4), the application of Gel'fand and Shilov's regularization leads to the convergent result in (2.1). It remains to show that I_{mn} in (2.1) is indeed equivalent to that in (1.2). This is done below.

It has been shown in [3, pp. 303–304] that the right side of (2.1) is independent of V_ϵ. Because of this property, let us choose V_ϵ to be a sphere centered at $\vec{r}$ and with radius a. Term B_{mn} in (2.1d) can be rewritten as

$$B_{mn} = \int_{V_\epsilon} [J(\vec{r}\,') - J(\vec{r})] \frac{\partial^2 g}{\partial x_m' \partial x_n'}\, dv' + J(\vec{r}) \int_{V_\epsilon} \frac{\partial^2 (g-g_0)}{\partial x_m' \partial x_n'}\, dv'. \tag{3.1}$$

Using the results in (1.5), we can evaluate the second integral in (3.1) explicitly, which leads to

$$B_{mn} = \int_{V_\epsilon} [J(\vec{r}\,') - J(\vec{r})] \frac{\partial^2 g}{\partial x_m' \partial x_n'}\, dv' + \delta_{mn} J(\vec{r}) \tfrac{1}{3}[1 - e^{-jka}(1+jka)], \tag{3.2}$$

where $\delta_{mn} = 1$ if $m=n$ and $\delta_{mn} = 0$ if $m \neq n$. The evaluation of term C_{mn} in (2.1d) over the spherical V_ϵ gives

$$C_{mn} = -\tfrac{1}{3}\, \delta_{mn} J(\vec{r}). \tag{3.3}$$

Substituting (3.2) and (3.3) into (2.1b), we obtain I_{mn} calculated from the generalized function in (2.1), namely,

$$I_{mn}(\vec{r}) = \int_{V - V_\epsilon} J(\vec{r}\,') \frac{\partial^2 g}{\partial x_m' \partial x_n'}\, dv' + \int_{V_\epsilon} [J(\vec{r}\,') - J(\vec{r})] \frac{\partial^2 g}{\partial x_m' \partial x_n'}\, dv' + \delta_{mn} J(\vec{r}) \left(\frac{-1}{3}\right)(1+jka)e^{-jka}, \tag{3.4}$$

where V_ϵ is a sphere centered at $\vec{r}$ and with radius a.

Next we show that I_{mn} in (1.2) also gives the same result (3.4) for the spherical V_ϵ. Equation (2.1) can be rewritten as

$$I_{mn}(\vec{r}) = \int_{V - V_\epsilon} J(\vec{r}\,') \frac{\partial^2 g}{\partial x_m' \partial x_n'}\, dv' + \int_{V_\epsilon} [J(\vec{r}\,') - J(\vec{r})] \frac{\partial^2 g}{\partial x_m' \partial x_n'}\, dv' + D_{mn}, \tag{3.5}$$

where

$$D_{mn} = J(\vec{r}) \frac{\partial^2}{\partial x_m \partial x_n} \int_{V_\epsilon} g(R)\, dv'. \tag{3.6}$$

The integral in (3.6) has been evaluated by Fikioris [5] with the result

$$D_{mn} = J(\vec{r}) \frac{1}{k^2} \left\{ \frac{\partial^2}{\partial y_m \partial y_n} \left[\frac{\sin k\sqrt{y_1^2 + y_2^2 + y_3^2}}{k\sqrt{y_1^2 + y_2^2 + y_3^2}} \cdot (1+jka)e^{-jka} - 1 \right] \right\}_{y_1 = y_2 = y_3 = 0},$$

$$= \delta_{mn} J(\vec{r}) \left(\frac{-1}{3}\right)(1+jka)e^{-jka}. \tag{3.7}$$

Substituting (3.7) into (3.5) we recover (3.4). This completes the proof that (1.2) and (2.1) are equivalent for a spherical and therefore arbitrary V_ϵ. By using generalized functions, a more direct proof that (1.2) and (2.1) are equivalent is, in fact, given in [3, p. 306].

IV. DISCUSSIONS

Our final result for the regularization of the divergent integral $\tilde{I}_{mn}$ in (1.4) is given in (2.1), which is valid for an arbitrary volume V_ϵ. It is also shown in Section III that the two expressions of I_{mn} in (1.2) and (2.1) are equivalent. For numerical evaluations, (2.1) is preferred to (1.2) because no numerical derivatives are necessary in using (2.1). Several discussions of our final result in (2.1) are presented in this section.

A. Explicit Expressions for C_{mn}

The term C_{mn} defined in (2.1e) or (2.1f) can be explicitly evaluated for several special V_ϵ.

1) V_ϵ is a sphere of radius a with the observation point $\vec{r}$ at an arbitrary point inside V_ϵ:

$$C_{mn} = -\tfrac{1}{3} J(\vec{r}) \delta_{mn}. \tag{4.1}$$

2) V_ϵ is a cube with center at $\vec{r}$ but is arbitrarily oriented, i.e., the faces of the cube need not be orthogonal to the x_1, x_2, x_3 axes:

$$C_{mn} = -\tfrac{1}{3} J(\vec{r}) \delta_{mn}. \tag{4.2}$$

3) V_ϵ is a cylinder with center at $\vec{r}$, radius a, and height $2b$, and its axis coincides with the x_3 axis:

$$C_{11} = C_{22} = -\frac{b}{2\sqrt{a^2 + b^2}} J(\vec{r}) \tag{4.3a}$$

$$C_{33} = \left(\frac{b}{\sqrt{a^2 + b^2}} - 1\right) J(\vec{r}) \tag{4.3b}$$

$$C_{mn} = 0, \qquad \text{if } m \neq n. \tag{4.3c}$$

An interesting property of C_{mn}, valid for arbitrary V_ϵ, is that

$$C_{11} + C_{22} + C_{33} = J(\vec{r}) \nabla^2 \psi = -J(\vec{r}) \tag{4.4}$$

where ψ was defined in (2.3). This property was first discovered by Yaghjian [4].

B. Direct Derivation of (2.1) by Classical Analysis

Without recourse to the theory of generalized functions, our result in (2.1) can also be derived by using a classical analysis as is briefly outlined below. We first rewrite (1.2) as

$$I_{mn} = A_{mn} + E_{mn}, \tag{4.5}$$

where

$$E_{mn} = \frac{\partial^2}{\partial x_m \partial x_n} \int_{V_\epsilon} J(\vec{r}')g(R)\, dv'. \tag{4.6}$$

Taking the first derivative of the integral in (4.6) leads to

$$\frac{\partial}{\partial x_n} \int_{V_\epsilon} J(\vec{r}')g(R)\, dv' = -\int_{V_\epsilon} J(\vec{r}')\frac{\partial g}{\partial x_n'}\, dv' \tag{4.7a}$$

$$= -\int_{\partial V_\epsilon} J(\vec{r}')g(R)(\hat{x}_n \cdot \hat{N})\, d\sigma' + \int_{V_\epsilon} \frac{\partial J}{\partial x_n'} g(R)\, dv', \tag{4.7b}$$

where Green's theorem is applied in going from (4.7a) to (4.7b). Taking another derivative of the quantity in (4.7) leads to

$$E_{mn} = \int_{\partial V_\epsilon} J(\vec{r}') \frac{\partial g}{\partial x_m'} (\hat{x}_n \cdot \hat{N})\, d\sigma' - \int_{V_\epsilon} \frac{\partial}{\partial x_n'} [J(\vec{r}') - J(\vec{r})] \frac{\partial g}{\partial x_m'}\, dv', \tag{4.8}$$

where we have used the fact that $\partial J(\vec{r})/\partial x_n' = 0$. Applying Green's theorem to the last integral in (4.8) gives

$$\begin{aligned} E_m &= J(\vec{r}) \int_{\partial V_\epsilon} \frac{\partial g}{\partial x_m'} (\hat{x}_n \cdot \hat{N})\, d\sigma' + \int_{V_\epsilon} [J(\vec{r}') - J(\vec{r})] \frac{\partial^2 g}{\partial x_m' \partial x_n'}\, dv' \\ &= J(\vec{r}) \int_{\partial V_\epsilon} \frac{\partial g}{\partial x_m'} (\hat{x}_n \cdot \hat{N})\, d\sigma' + \int_{V_\epsilon} \left[J(\vec{r}') \frac{\partial^2 g}{\partial x_m' \partial x_n'} - J(\vec{r}) \frac{\partial^2 g_0}{\partial x_m' \partial x_n'} \right] dv' \\ &\quad - J(\vec{r}) \int_{V_\epsilon} \frac{\partial^2 (g - g_0)}{\partial x_m' \partial x_n'}\, dv'. \end{aligned} \tag{4.9}$$

Applying Green's theorem to the last integral again, we obtain

$$E_{mn} = B_{mn} + C_{mn}. \tag{4.10}$$

Substitution of (4.10) into (4.5) recovers the desired formula in (2.1). The proof by classical analysis is completed.

C. Dyadic Green's Function in Free Space

Due to a current density $\vec{J}(\vec{r}')$ in the free space, the electric field $\vec{E}$ can be calculated by the convolution of $\vec{J}$ and the dyadic Green's function $\bar{\bar{G}}$. In matrix notation, the formula reads [6]

$$\vec{E}(\vec{r}) = \int_V \bar{\bar{G}}(\vec{r}, \vec{r}')\vec{J}(\vec{r}')\, dv', \tag{4.11}$$

where

$$\bar{\bar{G}}(\vec{r}, \vec{r}') = \begin{bmatrix} k^2 + \left(\frac{\partial}{\partial x_1'}\right)^2 & \frac{\partial^2}{\partial x_1' \partial x_2'} & \frac{\partial^2}{\partial x_1' \partial x_3'} \\ \frac{\partial^2}{\partial x_2' \partial x_1'} & k^2 + \left(\frac{\partial}{\partial x_2'}\right)^2 & \frac{\partial^2}{\partial x_2' \partial x_3'} \\ \frac{\partial^2}{\partial x_3' \partial x_1'} & \frac{\partial^2}{\partial x_3' \partial x_2'} & k^2 + \left(\frac{\partial}{\partial x_3'}\right)^2 \end{bmatrix} \frac{g(R)}{j\omega\epsilon_0}. \tag{4.12}$$

In order to avoid divergent integrals and to compute $\vec{E}$ correctly, the second-order derivatives in (4.12) must be treated as generalized functions in the manner defined in (2.1). For example, if $\vec{J}$ is in the x_1 direction and $\vec{J} = \hat{x}_1 J$, (4.11) and (4.12) become

$$E_1(\vec{r}) = -j\omega\mu_0 \int_V J(\vec{r}')g\, dv' + \frac{1}{j\omega\epsilon_0} I_{11}(\vec{r}) \tag{4.13a}$$

$$E_2(\vec{r}) = \frac{1}{j\omega\epsilon_0} I_{21}(\vec{r}) \tag{4.13b}$$

$$E_3(\vec{r}) = \frac{1}{j\omega\epsilon_0} I_{31}(\vec{r}) \tag{4.13c}$$

where $I_{mn}(\vec{r})$ is given in (2.1) or (3.4). For the special case that V_ϵ is a sphere, (4.13) is equivalent to [5, eq. (20)].

D. Vanishing Volume V_ϵ

Let $\vec{r}'$ be a point on the boundary surface of volume V_ϵ, and let the surface be described by (Fig. 2)

$$\text{surface } \partial V_\epsilon: \qquad \vec{r}' = \vec{r} + \epsilon \vec{f}(u, v), \tag{4.14}$$

where (u, v) are the parameters of the surface; for example, (u, v) may be the spherical coordinates (θ, ϕ) centered at $\vec{r}$. As $\epsilon \to 0$, the shape of V_ϵ is maintained while its volume is shrunk to zero (V_ϵ is undergoing a "similarity transformation" as defined in [3, p. 9]). Then $I_{mn}(\vec{r})$ in (2.1) is reduced to

$$I_{mn}(\vec{r}) = \lim_{\epsilon \to 0} [A_{mn} + C_{mn}], \tag{4.15}$$

where term B_{mn} is absent because it goes to zero as does ϵ^{α} because of the Hölder condition in (1.3). Under the limit $\epsilon \to 0$, term A_{mn} in (4.15) can be interpreted as a principal value integral. Many authors, including Wilcox [7], Van Bladel [8], Chen [9], and Yaghjian [4], have previously derived this special result in (4.15). A summary and critical review can be found in [10]. We must point out that while (4.15) is useful for theoretical consideration, it is not suitable for practical numerical calculations. This is due to the fact that the integrand of A_{mn} becomes singular as V_ϵ shrinks to zero; one is never sure how small V_ϵ should be in order to obtain a preselected accuracy in I_{mn}.

E. Green's Function in Waveguides or Cavities

So far we have considered only the singularity associated with the free-space Green's function g defined in (1.1). In a bounded region, such as a waveguide or a cavity, the Green's function $\bar{g}(\vec{r}, \vec{r}')$ (with a bar) is also singular when $\vec{r} = \vec{r}'$. This type of singularity has been subjected to discussions by several authors, including Tai [11], Collin [12], Rahmat-Samii [13], Howard and Seidel [14], and Yaghjian [4], [10]. We now show that $\bar{g}$ has exactly the *same* singularity as g. Therefore, formula (2.1) can also be used for its calculation. Note that $\bar{g}$ satisfies the inhomogeneous wave equation

$$(\nabla^2 + k^2)\bar{g}(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}'), \tag{4.16}$$

subject to the appropriate boundary conditions. The difference between g and $\bar{g}$ satisfies the homogeneous wave equation. All solutions of the latter equation are known to be regular, that is, infinitely differentiable. Hence, $\bar{g}$ and g must have the same singularity, regardless of the shape of the waveguide (cavity) boundary. This conclusion was previously arrived at by Yaghjian [4] and Howard and Seidel [14].

V. NUMERICAL RESULTS

Example A

Consider a current density described by

$$J(\vec{r}) = 1 - (r/\lambda) \sin^2\theta \sin 2\phi + \tfrac{3}{2}(r/\lambda)^2 \sin^2\theta \cos\theta \sin 2\phi \tag{5.1}$$

in spherical coordinates, where $\lambda = 2\pi/k$ = wavelength. The support of J is a sphere centered at $\vec{r} = 0$ with radius equal to 1 λ. We wish to evaluate I_{11} and I_{12} at the observation point $\vec{r} = 0$ (center of the spherical current). For the particular choice of J in (5.1), the expression given in (1.2) has been evaluated *exactly* by Boersma and de Doelder [15] with the result

$$I_{11}(\vec{r}=0) = -\tfrac{1}{3}(1 + j2\pi) \sim 2.1208 \exp(-j99.04°) \tag{5.2}$$

$$I_{12}(\vec{r}=0) = \tfrac{1}{15}(10 + j4\pi) \sim 1.0706 \exp(j51.49°). \tag{5.3}$$

Next, we employ the alternative formula in (2.1) to evaluate I_{11} and I_{12} numerically by using a spherical, cubical, and cylindrical V_ϵ centered at $\vec{r} = 0$. The size of V_ϵ is varied through a parameter a, which is the radius of a sphere, the side of a cube, or the radius of a cylinder (the height of the cylinder is $2a$). Numerical values for I_{11} and I_{12} at $\vec{r} = 0$ for several different values of a are presented in Table I. Within the tolerance of numerical integrations, those values

TABLE I
I_{mn} EVALUATED FROM (2.1)

a/λ	$I_{11}(\vec{r}=0)$			$I_{12}(\vec{r}=0)$		
	sphere	cube	cylinder	sphere	cube	cylinder
0.1	2.1312 -99.91°	2.1306 -99.81°	2.1310 -99.87°	1.0740 50.73°	1.0740 50.73°	1.0740 50.73°
0.3	2.1319 -100.02°	2.1313 -99.91°	2.1248 -98.86°	1.0740 50.73°	1.0740 50.73°	1.0740 50.73°
0.5	2.1322 -100.06°	2.1296 -99.66°	2.1222 -98.39°	1.0740 50.73°	1.0740 50.73°	1.0740 50.73°
0.7	2.1325 -100.10°	-	2.1210 -98.17°	1.0740 50.73°	1.0740 50.73°	1.0740 50.73°

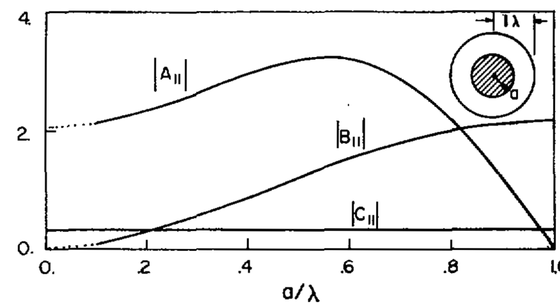


Fig. 3. Magnitudes of A_{11}, B_{11}, and C_{11} defined in (2.1) where V_ϵ is a sphere of radius a.

are in reasonable agreement with the exact ones in (5.2) and (5.3). For the case of spherical V_ϵ, we plot the magnitudes of A_{11}, B_{11}, and C_{11} as a function of a/λ in Fig. 3. As expected, $B_{11} = 0$ at $a = 0$, while $A_{11} = 0$ at $a = 1$ λ (when V_ϵ coincides with V). The value of C_{11} is a constant, $C_{11} = -1/3$, independent of a. It can be added that when V_ϵ is too small (say a/λ less than 0.01), the numerical evaluation of A_{mn} in (2.1c) becomes more difficult, because the integrand is nearly singular. This is why the formula given in (4.15), valid only for vanishing small V_ϵ, has little value in numerical computations.

Example B

Our particular choice of current in (5.1) allows us to evaluate $I_{11}(\vec{r} = 0)$ from (1.2) analytically. For a general current, the differentiation of the integral in (1.2) usually cannot be carried out in a closed form. Thus it is instructive to use the same current in (5.1) and evaluate (1.2) numerically. For numerical evaluation, we replace the differential operators by difference operators, namely,

$$I_{11}(\vec{r}=0) \sim \frac{1}{\Delta^2}\,[P(\vec{r}=\hat{x}\Delta) - 2P(\vec{r}=0) + P(\vec{r}=-\hat{x}\Delta)], \tag{5.4a}$$

where

$$P(\vec{r}) = \int_V J(\vec{r}')g(R)\,dv'. \tag{5.4b}$$

Theoretically, (5.4) becomes exact in the limit $\Delta \to 0$. We evaluate (5.4b) by use of spherical coordinates, and approximate integrations by summations. The numbers of subdivisions in (r, θ, ϕ) directions are (200, 15, 15), respectively. For various values of Δ, the result of I_{11} calculated from (5.4) is presented in Table II. In light of the exact value of I_{11} in

TABLE II
I_{11} EVALUATED FROM (5.4)

Δ/λ	$I_{11}(\vec{r}=0)$	
10^{-1}	2.0896 ,	-99.95°
10^{-2}	2.1194 ,	-97.96°
10^{-3}	2.0995 ,	-90.56°
10^{-4}	2.0995 ,	-90.55°
10^{-5}	2.1000 ,	-90.54°
10^{-6}	2.1197 ,	-88.72°
10^{-7}	10.1077 ,	-63.67°

(5.2), we note that the present difference method does not yield good numerical results. For $\Delta = 10^{-7}\lambda$, its solution for I_{11} is grossly incorrect. If $P(r)$ is calculated more accurately, the solution for very small Δ would be better. Then the computation becomes more laborious. From this example we clearly see the numerical superiority of the formula in (2.1) over that in (1.2).

Example C

In all of the above examples, the observation point is always located at the "center" of volume V_ϵ. Now let us consider a different situation. Volumes V and V_ϵ are again chosen to be concentric spheres centered at $\vec{r} = 0$, and with radius 1 λ and a, respectively (Fig. 4). The observation point is shifted along the x_3 axis to

$$\vec{r} = \hat{x}_3 d. \tag{5.5}$$

The current density is assumed to be

$$J(\vec{r}) = 1 - 2(r/\lambda) + 3(r/\lambda)^2. \tag{5.6}$$

For this particular choice, Boersma and de Doelder [15] have shown that I_{11} can be exactly evaluated from (1.2) with the result

$$\begin{aligned} I_{11}(0,0,d) = &\frac{2}{\pi D^3}\left(1+\tfrac{1}{2}D^2-\frac{3}{4\pi}D^3\right) \\ &+\frac{j\pi}{D^2}\left(4-\frac{6j}{\pi}-\frac{9}{\pi^2}+\frac{9j}{2\pi^3}+\frac{2j}{\pi^2 D}\right)\cos D \\ &-\frac{j\pi}{D^3}\left(4-\frac{6j}{\pi}-\frac{9}{\pi^2}+\frac{9j}{2\pi^3}-\frac{2j}{\pi^2}D\right)\sin D, \end{aligned} \tag{5.7}$$

where $D = kd$. For the case in which $d = 0.4\,\lambda$, the exact solution in (5.7) yields

$$I_{11}(0,0,d) \sim 1.8509 \exp(-j120.07°). \tag{5.8}$$

Next, we evaluate $I_{11}(0, 0, d = 0.4\,\lambda)$ from (2.1). The term C_{11} is given in (4.1), while terms A_{11} and B_{11} are, as usual, calculated by numerical integrations from (2.1c) and (2.1d). Results of I_{11} for various values of a are presented in Table III. Within the tolerance of numerical integrations, those values are in reasonable agreement with the exact one in (5.8).

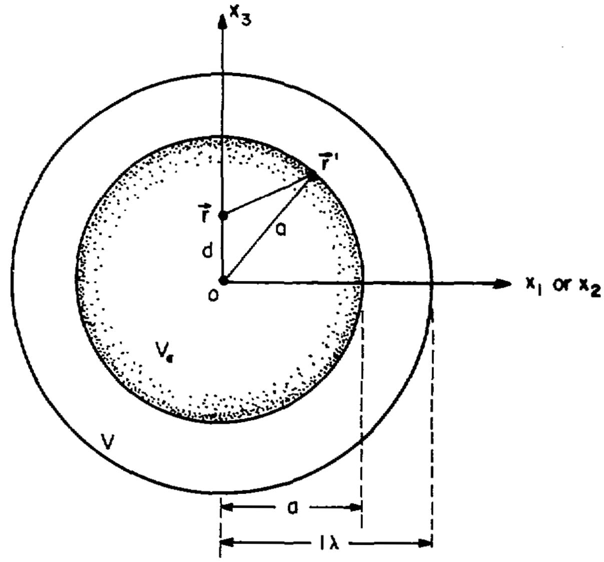


Fig. 4. Geometry for Example C. Observation point $\vec{r}$ is at (0, 0, d).

TABLE III
I_{11} EVALUATED FROM (2.1) WITH SPHERICAL V_ϵ

a/λ	$I_{11}(x_1=0,\ x_2=0,\ x_3=0.4\lambda)$	
.5	1.8656 ,	-119.82°
.6	1.8697 ,	-120.04°
.7	1.8708 ,	-120.10°
.8	1.8713 ,	-120.12°
.9	1.8716 ,	-120.14°
1.0	1.8718 ,	-120.15°

VI. CONCLUSION

The integral I_{mn} defined in (1.2) appears in the calculation of the field due to a current source by the Green's function method. For numerical calculations, the alternative expression in (1.4) is preferred, but it has the problem of being divergent whenever the observation point falls inside the source region. Following the theory of Gel'fand and Shilov, we derive a new formula in (2.1), which regularizes (1.4) so that it is indeed equivalent to the original integral in (1.2). The formula in (2.1) has a unique feature. The volume V_ϵ surrounding the observation point can be of finite size and of arbitrary shape, which are crucial for evaluating the integrals in (2.1) accurately and conveniently. Among all previous works [4]-[14], only Fikioris' formula [5] allows V_ϵ to be a finite sphere but not of arbitrary shape as our formula does.

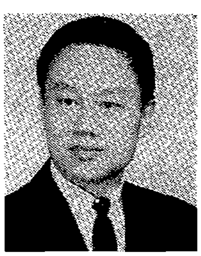
As a final remark, we have shown so far that the formula (2.1) is valid when the current density $J(\vec{r}')$ satisfies the Hölder condition. By using the theory of distribution, the validity of (2.1) can be extended for a more general $J(\vec{r}')$. Details of this extension will be given in a forthcoming paper.

ACKNOWLEDGMENT

Useful comments from Dr. A. D. Yaghjian are appreciated.

REFERENCES

[1] O.D. Kellogg, *Foundations of Potential Theory*. New York: Dover, 1953, p. 152.
[2] C. Muller, *Foundations of the Mathematical Theory of Electromagnetic Waves*. Berlin: Springer-Verlag, 1969.
[3] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. 1. New York: Academic, 1964.
[4] A. D. Yaghjian, "A direct approach to the derivation of electric dyadic Green's functions," Nat. Bur. Stand., Boulder, CO, NBS Tech. Note 1000, 1978.
[5] J. G. Fikioris, "Electromagnetic field inside a current-carrying region," *J. Math. Phys.*, vol. 6, pp. 1617–1620, 1965.
[6] C. T. Tai, *Dyadic Green's Functions in Electromagnetic Theory*. Scranton, PA: Intext Educational, 1971.
[7] C. H. Wilcox, "Debye potentials," *J. Math. Mech.*, vol. 6, pp. 167–201, 1957.
[8] J. Van Bladel, "Some remarks on Green's dyadic for infinite space," *IRE Trans. Antennas Propagat.*, vol. AP-9, pp. 563–566, Nov. 1961.
[9] K. M. Chen, "A simple physical picture of tensor Green's function in source region," *Proc. IEEE*, vol. 65, pp. 1202–1204, 1977.
[10] A. D. Yaghjian, "Electric dyadic Green's functions in the source region," *Proc. IEEE*, vol. 68, pp. 248–263, Feb. 1980.
[11] C. T. Tai, "On the eigenfunction expansion of dyadic Green's functions," *Proc. IEEE*, vol. 61, pp. 480–481, 1973.
[12] R. E. Collin, "On the incompleteness of E and H modes in wave guides," *Canadian J. Phys.*, vol. 51, pp. 1135–1140, 1973.
[13] Y. Rahmat-Samii, "On the question of computation of dyadic Green's function at the source region in waveguides and cavities," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 762–765, 1975.
[14] A. Q. Howard and D. B. Seidel, "Singularity extraction in kernel functions in a closed region problem," *Radio Sci.*, vol. 13, pp. 425–429, 1978.
[15] J. Boersma and P. J. de Doelder, "Closed-form evaluation of the wave potential due to a spherical current source distribution," Dept. Math., Eindhoven Univ. Technol., Memo. 1979-11, Oct. 1979.



Shung-Wu Lee (S'63–M'66–SM'73) was born in China. He received the Ph.D. degree from the University of Illinois, Urbana, in 1966.

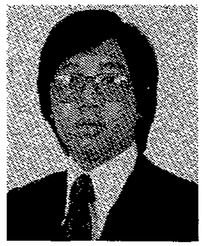
He has been a Professor of Electrical Engineering at the University of Illinois, Urbana, since 1974. He is currently a consultant to JPL, Lockheed, and Ford Aerospace.

Dr. Lee received awards and citations for his undergraduate teaching.

Johannes Boersma was born in Marrum, The Netherlands, on December 5, 1937. He received the Ph.D. degree in applied mathematics from the University of Groningen, Groningen, The Netherlands, in 1964.

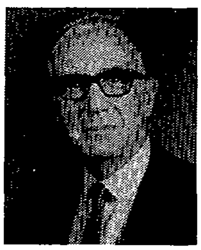
In 1965 he worked as a Research Associate at the Courant Institute of Mathematical Sciences, New York University, New York, NY, in the Division of Electromagnetic Research. In 1966 he joined the Department of Mathematics of the Technological University, Eindhoven, The Netherlands, where he has been a Professor since 1967. His research interests are in applied mathematics, particularly in diffraction theory.

Dr. Boersma is a member of the Dutch National URSI Committee, the American Mathematical Society, and SIAM.



Chak-Lam Law (S'79) was born in Canton, China, on June 6, 1954. He received the B.S. degree from the University of Illinois at Champaign-Urbana in electrical engineering in 1978.

From 1978 to 1979 he was a Research Assistant with the Electromagnetics Laboratory at the University of Illinois at Champaign-Urbana. He is presently completing the M.S. degree in the Department of Electrical and Computer Engineering at the University of Illinois, at Champaign-Urbana.



Georges A. Deschamps (SM'51–F'60) was born and educated in France. He graduated from the Ecole Normale Superieure, Paris, France, in 1934, and received advanced degrees in mathematics and physics from the Sorbonne, Paris, France.

He taught mathematics and physics for about ten years at the Lycee Francais de New York. In 1947 he joined the Federal Telecommunication Laboratories, where he worked as a Project Engineer on direction-finding systems, design of high-frequency and microwave antennas, microstrip development, and radio and inertial navigation. In 1956 he was appointed a Senior Scientist of the ITT Laboratories. He joined the University of Illinois at Urbana-Champaign, in 1958 as a Professor of Electrical Engineering and Director of the Antenna Laboratory, presently renamed the Electromagnetics Laboratory.

Mr. Deschamps is a member of the American Physical Society and the International Scientific Radio Union (URSI), presently Chairman of the US Commission B on "Fields and Waves." He was Chairman of the IRE Committee on Antennas and Waveguides from 1957 to 1958 and Editor of the IRE TRANSACTIONS ON INFORMATION THEORY from 1958 to 1960. He is a member of the National Academy of Engineering.