08/27/24 | ECE6320 / HW1 | Howes, Kade

Linear Algebra Review

Matrix multiplication (5 points): Practice matrix multiplication by hand for the following problems:

P1: $\begin{bmatrix} -3 & 5 \\ 7 & -10 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, P2: $\begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 10 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

P1: $\begin{bmatrix} -3 & 5 \\ 7 & -10 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3+15 \\ -7-30 \end{bmatrix} = \begin{bmatrix} 18 \\ -37 \end{bmatrix}$

P2: $\begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 10 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4+10+3 \\ 3+14+30 \\ 1+0+3 \end{bmatrix} = \begin{bmatrix} 17 \\ 47 \\ 4 \end{bmatrix}$

Orthogonality (10 points): Consider the vectors $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, y = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}, z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

P3: Are any of these vectors orthogonal to each other?

P4: Find a non-zero orthogonal vector for each of these vectors without using any of the other vectors.

P3: $\langle x, z \rangle = 1 + (-1) = 0 \Rightarrow$ x and z are orthogonal

Neither x or z have the same dimension as y, so neither are orthogonal with y.

$\Rightarrow$ Yes, x and z are orthogonal

P4: x: $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is orthogonal to x

y: $-3a + 2b + 4c = 0 \Rightarrow$ choose $b = c = 1$

$\Rightarrow -3a = -6$

$\Rightarrow a = 2$

$\Rightarrow \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is orthogonal to y

z: $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$ is orthogonal to z

Span and basis set (10 points): Consider the vectors $x_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $x_2 = \begin{bmatrix}4\\0\\0\end{bmatrix}$, $x_3 = \begin{bmatrix}2\\0\\9\end{bmatrix}$

P5: Express a basis set of unit vectors for the subspace defined by $span\{x_1, x_2, x_3\}$

P5: check independence:

$\{x_1, x_2, x_3\}$: $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \begin{bmatrix}\alpha_1 + 4\alpha_2 + 2\alpha_3\\0\\\alpha_1 + 9\alpha_3\end{bmatrix}$

$\Rightarrow$ $\alpha_1 + 4\alpha_2 + 2\alpha_3 = 0$ ①

$\alpha_1 = -9\alpha_3$ ②

② into ①: $4\alpha_2 - 7\alpha_3 = 0 \Rightarrow \alpha_2 = \frac{7}{4}\alpha_3$

Choose $\alpha_3 = 1 \Rightarrow \alpha_2 = \frac{7}{4}$, $\alpha_1 = -9$

$\Rightarrow$ Since there exists some $\alpha_1, \alpha_2, \alpha_3$ s.t. $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$

x_1, x_2, x_3 are not linearly independent

try: $\{x_1, x_2\}$ $\alpha_1 x_1 + \alpha_2 x_2 = \begin{bmatrix}\alpha_1 + 4\alpha_2\\0\\\alpha_1\end{bmatrix} \Rightarrow$ $\alpha_1 + 4\alpha_2 = 0$ ①

$\alpha_1 = 0$ ②

① + ② $\Rightarrow \alpha_2 = 0$

Only α_1, α_2 s.t. $\alpha_1 x_1 + \alpha_2 x_2 = 0$ is $\alpha_1 = \alpha_2 = 0$

$\Rightarrow$ x_1, x_2 are linearly independent

$\Rightarrow$ span$\{x_1, x_2, x_3\}$ = span$\{x_1, x_2\}$

$\Rightarrow$ A basis set of vectors for subspace of span$\{x_1, x_2, x_3\}$

is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}4\\0\\0\end{bmatrix}\right\}$

$\Rightarrow$ A basis set of <u>unit</u> vectors for subspace span$\{x_1, x_2, x_3\}$

is $\left\{\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

Null and Image subspaces (20 points): Consider the matrices $A_1 = \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

P6: Find the null space for each of the matrices $N(A) = \{x \mid Ax = 0\}$

P7: Find the Image of each of the matrices $Im(A) = \{y \mid \exists x \text{ where } y = Ax\}$

P8: For A_2 and A_3, show that the image space is the orthogonal complement of the null space $V^{\perp} = \{z \mid v^T z = 0 \ \forall v \in V\}$

P6: A_1: $A_1 x = 0 \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ ① $2x_1 + 3x_2 + 5x_3 = 0$

② $-4x_1 + 2x_2 + 3x_3 = 0$

① $x_1 = -\frac{3}{2}x_2 - \frac{5}{2}x_3$

① into ②: $-4(-\frac{3}{2}x_2 - \frac{5}{2}x_3) + 2x_2 + 3x_3 = 0$

$8x_2 + 13x_3 = 0 \Rightarrow x_2 = -\frac{13}{8}x_3$ ③

③ into ①

$x_1 = -\frac{3}{2}(-\frac{13}{8}x_3) - \frac{5}{2}x_3$

$x_1 = (\frac{39}{16} - \frac{5}{2})x_3 = -\frac{1}{16}x_3$

$\Rightarrow$ any vector $\begin{bmatrix} -\frac{1}{16}\alpha \\ -\frac{13}{8}\alpha \\ \alpha \end{bmatrix}$ s.t. $\alpha \in \mathbb{R}$ is in $N(A) \Rightarrow N(A) = \text{span}\left\{\begin{bmatrix} -1/16 \\ 13/8 \\ 1 \end{bmatrix}\right\}$

A2: $A_2 x = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ ① $x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

② $5x_1 + 2x_2 + x_3 = 0$

③ $x_1 + 2x_2 + 2x_3 = 0 \Rightarrow x_2 = -\frac{1}{2}x_1 - x_3$

① + ③ into ②:

$5x_1 + x_1 - x_1 = 0$

$x_1 = 0 \Rightarrow x_2 = 0, x_3 = 0$

① into ③: $x_2 = -\frac{1}{2}x_3$

$x_2 = \frac{1}{2}x_1$ ④

$\Rightarrow N(A_2) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

A3: $A_3 x = 0 \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ ① $2x_1 + x_2 + x_3 = 0$
② $x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$
③ $x_1 + x_3 = 0 \Rightarrow x_3 = -x_1$

② + ③ into ①: $2x_1 - x_1 - x_1 = 0$, $0 = 0$ x_1 is free

$\Rightarrow$ any vector $\begin{bmatrix} \alpha \\ -\alpha \\ -\alpha \end{bmatrix}$ s.t. $\alpha \in \mathbb{R}$ is in $N(A_3)$ $\Rightarrow N(A_3) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right\}$

P7: A_1:

Row Echelon Form:

$$\begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 + 2R_1 \to R_2} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 8 & 13 \end{bmatrix}$$

pivots

$\Rightarrow \text{Im}(A_1) = \text{span}\left\{\begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$

A_2:

Row Echelon:

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 5R_1 \to R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & -3 \end{bmatrix}$$

pivots

$\Rightarrow \text{Im}(A_1) = \text{span}\left\{\begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right\}$

A_3:

Row Echelon:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - \frac{1}{2}R_1 \to R_3]{R_2 - \frac{1}{2}R_1 \to R_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

pivots

$\Rightarrow \text{Im}(A_3) = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$

P8: A_2: $N(A_2) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Trivially, all vectors are orthogonal to the zero vector ✓

$\Rightarrow N(A_2) \perp \mathrm{Im}(A_2)$

A_3:

We know $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is a basis for $N(A_3)$ and

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are the basis for $\mathrm{Im}(A_3)$.

For $N(A_3) \perp \mathrm{Im}(A_3)$, their bases must be orthogonal. Show:

$$\begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 2 - 1 - 1 = 0$$

$$\begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 - 1 + 0 = 0$$

This is sufficient to show the two spaces are orthogonal since all vectors in each space can be made from the span of their bases.

Eigen vectors and values (10 points): Consider A_2 thruough A_6 below

$$A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A_4 = \begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 10 \\ 1 & 0 & 1 \end{bmatrix} A_5 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_6 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & 10 \\ 0 & 0 & 8 \end{bmatrix}$$

P9: Find the eigenvalues and eigenvectors

P10: Which (if any) of the matrices are full rank?

P11: What pattern do you notice for matrices A_5 and A_6?

A_2:

$$\det(A_2 - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 1 \\ 5 & 2-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)\left[(2-\lambda)(2-\lambda) - 2\right] - 0 + 1\left[10 - (2-\lambda)\right]$$

$$= (1-\lambda)\left[4 - 4\lambda + \lambda^2 - 2\right] + 10 - 2 + \lambda$$

$$= 4 - 4\lambda + \lambda^2 - 2 - 4\lambda + 4\lambda^2 - \lambda^3 + 2\lambda + 10 - 2 + \lambda$$

$$= -\lambda^3 + 5\lambda^2 - 5\lambda + 10$$

↳ using MATLAB roots()

$$= (\lambda - 4.3797)(\lambda - 0.3102 - 1.4789i)(\lambda - 0.3102 + 1.4789i)$$

$$\Rightarrow \underline{\lambda_1 = 4.3797, \ \lambda_2 = 0.3102 + 1.4789i, \ \lambda_3 = 0.3102 - 1.4789i}$$

Eigenvectors

$$A_2 v = \lambda v \Rightarrow (A_2 - \lambda) v = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & 1 \\ 5 & 2-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

For λ_1,

$$\begin{bmatrix} -3.3797 & 0 & 1 \\ 5 & -2.3797 & 1 \\ 1 & 2 & -2.3797 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0$$

Using MATLAB eig

$$\Rightarrow \underline{v_1 = \left\{ \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \right\} = \left\{ \alpha \begin{bmatrix} -0.2007 \\ -0.2068 \\ -0.6784 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}}$$

For $\lambda_2 = 0.3102 + 1.4789j$

$$\begin{bmatrix} 0.6898 - 1.4789j & 0 & 1 \\ 5 & 1.6898 - 1.4789j & 1 \\ 1 & 2 & 1.6898 - 1.4789j \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$$

$$\Rightarrow V_2 = \left\{ \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} \right\} = \left\{ \alpha \begin{bmatrix} -0.1786 + 0.3109j \\ 0.7275 \\ -0.3366 - 0.4785j \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

For $\lambda_3 = 0.3102 - 1.4789j$

$$\begin{bmatrix} 0.6898 + 1.4789j & 0 & 1 \\ 5 & 1.6898 + 1.4789j & 1 \\ 1 & 2 & 1.6898 + 1.4789 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = 0$$

$$\Rightarrow V_3 = \left\{ \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} \right\} = \left\{ \alpha \begin{bmatrix} -0.1786 - 0.3109j \\ 0.7275 \\ -0.3366 + 0.4785j \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

A_3: eigenvalues $\det(A_3 - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right)$

$$= \det\left(\begin{bmatrix} 1 & 0 & 1-\lambda \\ 1 & 1-\lambda & 0 \\ 2-\lambda & 1 & 1 \end{bmatrix}\right)$$

$$= 1(1-\lambda-0) - 0 + (1-\lambda)\left[1-(1-\lambda)(2-\lambda)\right]$$

$$= 1-\lambda + (1-\lambda)\left[1-(\lambda^2-3\lambda+2)\right] = 1-\lambda + (1-\lambda)\left[-\lambda^2+3\lambda-1\right]$$

$$= 1-\lambda-\lambda^2+3\lambda-1+\lambda^3-3\lambda^2+\lambda = \lambda^3-4\lambda^2+3\lambda$$

$$= \lambda(\lambda^2-4\lambda+3) = \lambda(\lambda-1)(\lambda-3)$$

$$\Rightarrow \underline{\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 3}$$

A3: eigenvector: $A_3 v = \lambda v \Rightarrow (A_3 - \lambda I)v = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

For $\lambda_1 = 0$,

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0 \Rightarrow \begin{aligned} 2v_{11} + v_{12} + v_{13} &= 0 \quad (1) \\ v_{11} + v_{12} &= 0 \quad (2) \\ v_{11} + v_{13} &= 0 \quad (3) \end{aligned}$$

$$\Rightarrow \begin{aligned} (2)\ v_{12} &= -v_{11} \\ (3)\ v_{13} &= -v_{11} \end{aligned}$$

(2) + (3) into (1): $2v_{11} - v_{11} - v_{11} = 0$

$0 = 0$

v_{11} is free

$$\Rightarrow \underline{V_1 = \left\{\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix}\right\} = \left\{\begin{bmatrix} \alpha \\ -\alpha \\ -\alpha \end{bmatrix} \middle| \ \alpha \in \mathbb{R}\right\} \setminus \{0\}}$$

For $\lambda_2 = 1$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0 \Rightarrow \begin{array}{ll} v_{21} + v_{22} + v_{23} = 0 & (1) \\ v_{21} = 0 & (2) \\ v_{21} = 0 & (3) \end{array}$$

(2) into (1): $v_{22} = -v_{23}$

$$\Rightarrow \underline{V_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ -\alpha \end{bmatrix} \text{ s.t. } \alpha \in \mathbb{R}}$$

For $\lambda_3 = 3$,

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = 0 \Rightarrow \begin{array}{ll} -v_{31} + v_{32} + v_{33} = 0 & (1) \\ v_{31} - 2v_{32} = 0 & (2) \\ v_{31} - 2v_{33} = 0 & (3) \end{array}$$

$\Rightarrow$ (2) $v_{32} = \frac{1}{2} v_{31}$

(3) $v_{33} = \frac{1}{2} v_{31}$

(2) & (3) into (1): $-v_{31} + \frac{1}{2} v_{31} + \frac{1}{2} v_{31} = 0$

$0 = 0$

v_{31} is free

$$\Rightarrow \underline{V_3 = \left\{\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix}\right\} = \left\{\begin{bmatrix} \alpha \\ \alpha/2 \\ \alpha/2 \end{bmatrix} \middle| \alpha \in \mathbb{R}\right\} \setminus \{0\}}$$

A_4: eigenvalues: $\det(A_4 - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 5 & 1 \\ 3 & 7-\lambda & 10 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right)$

$$= -\det\left(\begin{bmatrix} 1 & 0 & 1-\lambda \\ 3 & 7-\lambda & 10 \\ 4-\lambda & 5 & 1 \end{bmatrix}\right) = -\left[(7-\lambda-50) - 0 + (1-\lambda)\left[15 - (\lambda^2 - 11\lambda + 28)\right]\right]$$

$$= -\left[-\lambda - 43 - \lambda^2 + 11\lambda - 13 + \lambda^3 - 11\lambda^2 + 13\lambda\right]$$

$$= -\left[\lambda^3 - 12\lambda^2 + 23\lambda - 56\right] = -\lambda^3 + 12\lambda^2 - 23\lambda + 56$$

roots() in MATLAB

$$= (\lambda - 10.2942)(\lambda - 0.8529 - 2.1708j)(\lambda - 0.8529 + 2.1708j)$$

$$\Rightarrow \underline{\lambda_1 = 10.2942,\ \lambda_2 = 0.8529 + 2.1708j,\ \lambda_3 = 0.8529 - 2.1708j}$$

A_4 eigenvector: $A_4 v = \lambda v \Rightarrow (A_4 - \lambda I)v = 0$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 5 & 1 \\ 3 & 7-\lambda & 10 \\ 1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

For each $\lambda_1, \lambda_2, \lambda_3$, solve

$$(4-\lambda)v_1 + 5v_2 + 1v_3 = 0$$
$$3v_1 + (7-\lambda)v_2 + 10v_3 = 0$$
$$v_1 + (1-\lambda)v_3 = 0$$

Using Matlab eig( ) function:

$$V_1 = \left\{ \alpha \begin{bmatrix} 0.6271 \\ 0.7700 \\ 0.0675 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

$$V_2 = \left\{ \alpha \begin{bmatrix} -0.7301 \\ 0.455 - 0.3839j \\ 0.0227 + 0.3348j \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

$$V_3 = \left\{ \alpha \begin{bmatrix} -0.7301 \\ 0.455 + 0.3839j \\ 0.0227 - 0.3348j \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

A_5: eigenvalues: $\det(A_5 - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & 7-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right)$

$= (4-\lambda)(7-\lambda)(1-\lambda)$

$\Rightarrow \underline{\lambda_1 = 4,\ \lambda_2 = 7,\ \lambda_3 = 1}$

A_5: eigenvectors: $A_5 v = \lambda v \Rightarrow (A_5 - \lambda I)v = 0$

For $\lambda_1 = 4$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0 \Rightarrow \begin{matrix} 3v_{12} = 0 \\ -3v_{13} = 0 \end{matrix}$$

$$\Rightarrow V_1 = \left\{ \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

For $\lambda_2 = 7$,

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0 \Rightarrow \begin{matrix} -3v_{11} = 0 \\ -6v_{13} = 0 \end{matrix}$$

$$V_2 = \left\{ \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} \middle| \ \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

For $\lambda_3 = 1$,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = 0 \Rightarrow \begin{matrix} 3v_{31} = 0 \\ 6v_{32} = 0 \end{matrix}$$

$$V_3 = \left\{ \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \middle| \ \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

A_6: eigenvalues: $\det(A_6 - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 5-\lambda & 10 \\ 0 & 0 & 8-\lambda \end{bmatrix}\right)$

$$= -\det\left(\begin{bmatrix} 0 & 0 & 8-\lambda \\ 0 & 5-\lambda & 10 \\ 2-\lambda & 1 & 1 \end{bmatrix}\right) = (8-\lambda)\left[0 - (2-\lambda)(5-\lambda)\right]$$

$$= (8-\lambda)\left[-\lambda^2 + 7\lambda - 10\right] = -8\lambda^2 + 56\lambda - 80 + \lambda^3 - 7\lambda^2 + 10\lambda$$

MATLAB roots()

$$= \lambda^3 - 15\lambda^2 + 66\lambda - 80$$

$$= (\lambda - 2)(\lambda - 5)(\lambda - 8)$$

$$\Rightarrow \underline{\lambda_1 = 2, \ \lambda_2 = 5, \ \lambda_3 = 8}$$

A_6: eigenvectors: $A_6 v = \lambda v \Rightarrow (A_6 - \lambda I) v = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 5-\lambda & 10 \\ 0 & 0 & 8-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$$

For $\lambda_1 = 2$,

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 10 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0 \Rightarrow$$

$v_{12} + v_{13} = 0$

$3v_{12} + 10v_{13} = 0$

$6v_{13} = 0$

$\Rightarrow v_{13} = 0,\ v_{12} = 0$

v_{11} is free

$$\Rightarrow V_1 = \left\{ \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

For $\lambda_2 = 5$

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0 \Rightarrow$$

$-3v_{21} + v_{22} + v_{23} = 0$ ①

$5v_{23} = 0$ ②

$3v_{23} = 0$ ③ $\Rightarrow v_{23} = 0$

③ into ①: $-3v_{21} + v_{22} = 0$

$v_{22} = 3v_{21}$

$$\Rightarrow V_2 = \left\{ \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \alpha \\ 3\alpha \\ 0 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

For $\lambda_3 = 8$,

$$\begin{bmatrix} -6 & 1 & 1 \\ 0 & -3 & 10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = 0 \Rightarrow$$

$-6v_{31} + v_{32} + v_{33} = 0$ ①

$-3v_{32} + 10v_{33} = 0$ ②

$\Rightarrow v_{33} = \frac{3}{10} v_{32}$ ③

③ into ①: $-6v_{31} + v_{32} + \frac{3}{10} v_{32} = 0$

$-6v_{31} + \frac{13}{10} v_{32} = 0$

$\Rightarrow v_{32} = \frac{60}{13} v_{31}$ ④

$v_{33} = \frac{3}{10}\left(\frac{60}{13} v_{32}\right)$

$= \frac{180}{130} v_{32}$

$v_{33} = \frac{18}{13} v_{31}$ ⑤

$$\Rightarrow V_3 = \left\{ \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \alpha \\ \frac{60}{13}\alpha \\ \frac{18}{13}\alpha \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \setminus \{0\}$$

P10:

Row Echelon Form

$$A_2: \begin{bmatrix} 1 & 0 & 1 \\ 5 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow[R_2 - 5 \cdot R_1 \to R_2]{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 - 2R_2 \to R_3]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 7 \end{bmatrix}$$

A_2 is full rank

$$A_3: \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - \frac{1}{2} R_1 \to R_2]{R_3 - \frac{1}{2} R_1 \to R_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

A_3 is NOT full rank

$$A_4: \begin{bmatrix} 4 & 5 & 1 \\ 3 & 7 & 10 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_3]{R_3 \to R_1} \begin{bmatrix} 1 & 0 & 1 \\ 3 & 7 & 10 \\ 4 & 5 & 1 \end{bmatrix} \xrightarrow[R_3 - 4R_1 \to R_3]{R_2 - 3R_1 \to R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & 7 \\ 0 & 5 & -3 \end{bmatrix} \xrightarrow{R_3 - \frac{7}{5} R_2 \to R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 7 & 7 \\ 0 & 0 & -\frac{64}{5} \end{bmatrix}$$

A_4 is full rank

$$A_5: \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A_5 is full rank

$$A_6: \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & 10 \\ 0 & 0 & 8 \end{bmatrix}$$

A_6 is full rank

P11:

I notice that A_5 and A_6 both have eigenvalues equal to their diagonals. Also, they are both given in row Echelon form.

Fundamental theorem of linear equations (10 points):

Prove Lemma 11.1 (i.e. show that $Im(V) = N(V^T)^\perp$)

- Recall that kernel and null space are synonymous
- The proof is in the book, just explain it in your own words

Lemma 11.1 (Range versus null space). *For every $m \times n$ matrix W,*

$$\operatorname{Im} W = (\ker W')^\perp, \qquad \ker W = (\operatorname{Im} W')^\perp.$$

Proof of Lemma 11.1. Assuming that $x \in \operatorname{Im} W$,

$$\left.\begin{array}{lll} 1\ x \in \operatorname{Im} W & \Rightarrow & \exists \eta : x = W\eta \\ 2\ z \in \ker W' & \Rightarrow & W'z = 0 \end{array}\right\} \quad \Rightarrow \quad 3\ z'x = z'W\eta = 0.$$

4 Therefore x is orthogonal to every vector in $\ker W'$, which means that $x \in (\ker W')^\perp$. We thus conclude that $\operatorname{Im} W \subset (\ker W')^\perp$.

However,

$$\left.\begin{array}{r} \dim \ker W' + \dim(\ker W')^\perp = m \\ \dim \ker W' + \dim \operatorname{Im} W' = m \end{array}\right\} 5$$

$$\Rightarrow \quad \dim(\ker W')^\perp = \dim \operatorname{Im} W' = \operatorname{rank} W = \dim \operatorname{Im} W \quad 6$$

and therefore $\operatorname{Im} W$ cannot be a strict subset of $(\ker W')^\perp$. 7 ■

1. First, by assuming that some vector x is in the image of W, we know that there is some vector N (eta) that when W is applied to N, we get x
2. Also, by assuming that some vector z is in the kernel of W', we know that applying W' to z will result in the zero vector
3. Then, we can combine the previous two steps by doing the inner product of z and x. However, by step 1, we know that x is the same as W applied to N. Additionally, we know that z'W = 0 by taking the transpose of z'W, resulting in (W'z)'=0 by step 2. So, z'x = 0.
4. Since z is representing any vector in the kernel of W', we now know that x is orthogonal to every vector in W'. Therefore, x is in the orthogonal complement of the kernel of W'. Since x is in the image of W, we also know that the image of W is a subset of the orthogonal transpose of the kernel of W'
5. Also, we know that the dimension of the kernel of W' and the dimension of the orthogonal transpose of the kernel of W' must add up to equal the number of rows of W'=m by the fundamental theorem of linear algebra. Second, we know that the dimension of the kernel of W' and the dimension of the image of W' must also add up to equal the number of rows of W'=m.
6. Because of both statements in step 5, the dimension of the orthogonal complement of the kernel of W' and the dimension of the image of W' must be equal. Since the dimension of the image of W' is the same as the rank of W, that also shows that the dimension of the orthogonal complement of the kernel of W' is equal to the dimension of the image of W.
7. Since the dimension of the orthogonal complement of kernel W' is equal to the dimension of the image of W, the image of W is not a strict subset of the orthogonal complement of the kernel W'. So, by step 4 and step 7, we know the image of W is equal to the orthogonal complement of the kernel of W'

State Representation (10 points)

Consider the systems:

1.
$$\begin{aligned} y_1^{(3)} + 2\ddot{y}_1 + 3\dot{y}_1 - 4\dot{y}_3 + 5\dot{y}_2 &= 4u_1 - u_3 \\ \ddot{y}_2 + 4y_1 - 3\dot{y}_3 &= u_2 \\ \ddot{y}_3 + \ddot{y}_1 &= 2u_1 + 4u_3 \end{aligned}$$

2.
$$\begin{aligned} 10\ddot{y}_1 + 4\dot{y}_1 - 5\dot{y}_3 &= 4u_1 \\ \ddot{y}_2 + 9y_1 - 3\dot{y}_3 &= u_1 + 7u_2 \\ \ddot{y}_3 + 8\dot{y}_1 + 5\dot{y}_2 &= u_1 + 3u_2 \end{aligned}$$

Write each system in state space form as $\dot{x} = Ax + Bu$

let $x := \begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

a)

$$y_1^{(3)} = -3\dot{y}_1 - 2\ddot{y}_1 - 5\dot{y}_2 + 4\dot{y}_3 + 4u_1 - u_3$$

$$\ddot{y}_2 = -4y_1 + 3\dot{y}_3 + u_2$$

$$\ddot{y}_3 = -\ddot{y}_1 + 2u_1 + 4u_3$$

1.

let $x := \begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix}$

$x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = \ddot{y}_1$

$x_4 = y_2$, $x_5 = \dot{y}_2$

$x_6 = y_3$, $x_7 = \dot{y}_3$

$$\Rightarrow \dot{x} = \begin{bmatrix} \dot{y}_1 \\ \ddot{y}_1 \\ y_1^{(3)} \\ \dot{y}_2 \\ \ddot{y}_2 \\ \dot{y}_3 \\ \ddot{y}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ -3x_2 - 2x_3 - 5x_5 + 4x_7 + 4u_1 - u_3 \\ x_5 \\ -4x_1 + 3x_7 + u_2 \\ x_7 \\ -x_3 + 2u_1 + 4u_3 \end{bmatrix}$$

$$\Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 0 & -5 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} u$$

2. $x := \begin{bmatrix} x_1 \\ \vdots \\ x_6 \end{bmatrix}$ $\quad u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$x_1 = y_1, \quad x_2 = \dot{y}_1$

$x_3 = y_2, \quad x_4 = \dot{y}_2$

$x_5 = y_3, \quad x_6 = \dot{y}_3$

$\ddot{y}_1 = -\frac{4}{10}\dot{y}_1 + \frac{5}{10}\dot{y}_3 + \frac{4}{10}u_1$

$\ddot{y}_2 = -9y_1 + 3\dot{y}_2 + u_1 + 7u_2$

$\ddot{y}_3 = -8\dot{y}_1 - 5\dot{y}_2 + u_1 + 3u_2$

$$\Rightarrow \dot{x} = \begin{bmatrix} \dot{y}_1 \\ \ddot{y}_1 \\ \dot{y}_2 \\ \ddot{y}_2 \\ \dot{y}_3 \\ \ddot{y}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{2}{5}x_2 + \frac{1}{2}x_6 + \frac{2}{5}u_1 \\ x_4 \\ -9x_1 + 3x_6 + u_1 + 7u_2 \\ x_6 \\ -8x_2 - 5x_4 + u_1 + 3u_2 \end{bmatrix}$$

$$\Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{5} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -9 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -8 & 0 & -5 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 2/5 & 0 \\ 0 & 0 \\ 1 & 7 \\ 0 & 0 \\ 1 & 3 \end{bmatrix} u$$

Simulation of unstable System (25 points)

Use SimulationScript.m to simulate an unstable system. Do the following:

- Finish implementing the "eulerIntegration" function
  - Hint: 2017 - Mayne - MPC Theory Computation and Design.pdf ch 8.2 will be very useful
- Change the initial state on line 11 and show the state plot for each of the following (using eigenvectors and eigenvalues from the matrix "*A" defined in the f(t,z,u) function*):
  - Start on an eigenvector of magnitude 5 corresponding to a negative eigenvalue with zero control input (u = 0)
  - Start on an eigenvector of magnitude 5 corresponding to a negative eigenvalue and command a constant control (u = 1)
  - Start on an eigenvector of magnitude 5 corresponding to a positive eigenvalue with zero control input (u = 0)
  - Start on a random initial state with zero control input (u = 0)
- Answer the following
  - What is the behavior of the system for each of the described initial conditions?
  - How does Matlab's ode45 solution compare with the Euler integration for each of the described initial conditions?
  - Which solution do you trust more (ode45 or Euler) and why?
- Turn in .m files

Finding eigenvalue/eigenvector for A:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 2 & 2-\lambda & 2 \\ 3 & 0 & -1-\lambda \end{bmatrix}\right)$$

$$= (2-\lambda)\left[(2-\lambda)(-1-\lambda)\right]$$

$$= (2-\lambda)(\lambda^2 - \lambda - 2)$$

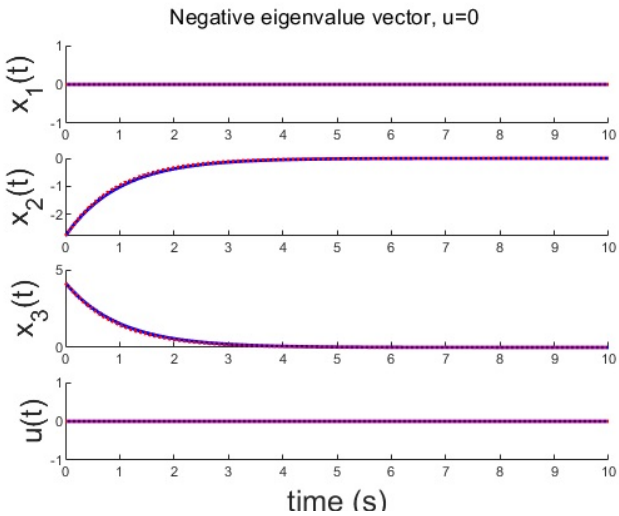
$$= (2-\lambda)(\lambda-2)(\lambda+1)$$

$$\Rightarrow \lambda_1 = 2, \quad \lambda_2 = 2, \quad \lambda_3 = -1$$

Eigenvectors from matlab eig

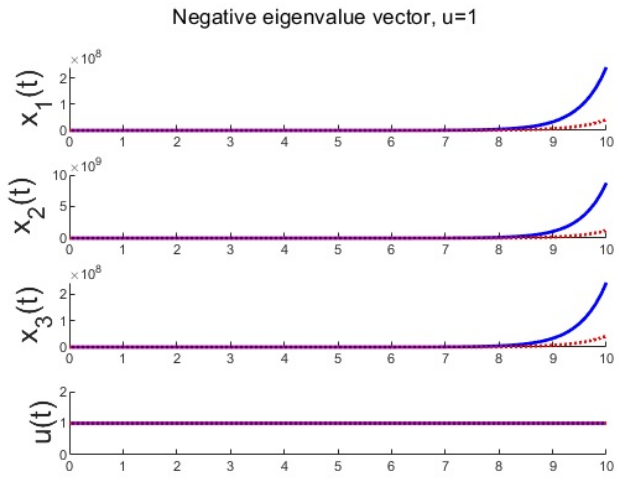
for $\lambda = 2$: $\left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mid \alpha \in \mathbb{R} \right\} \setminus \{0\}$

for $\lambda = -1$: $\left\{ \alpha \begin{bmatrix} 0 \\ -0.5547 \\ 0.8321 \end{bmatrix} \mid \alpha \in \mathbb{R} \right\} \setminus \{0\}$



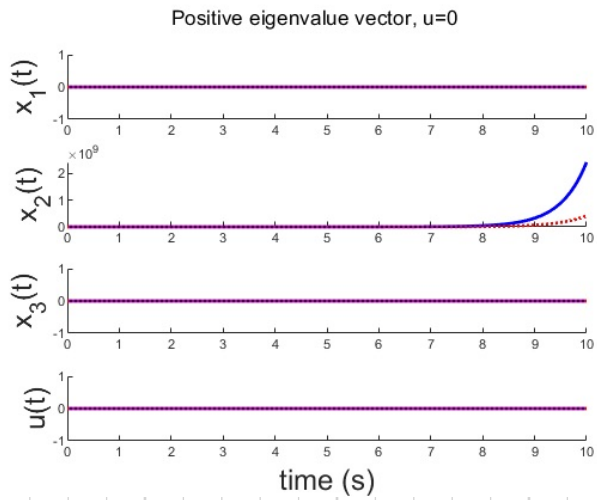
As seen in the figure on the left, when starting at an eigenvector corresponding to a negative eigenvalue with a magnitude 5, state 1 stayed at 0, and states 2 and 3 converge to 0.

MATLAB's ode45 and my Euler integration appear to be near identical for this case.



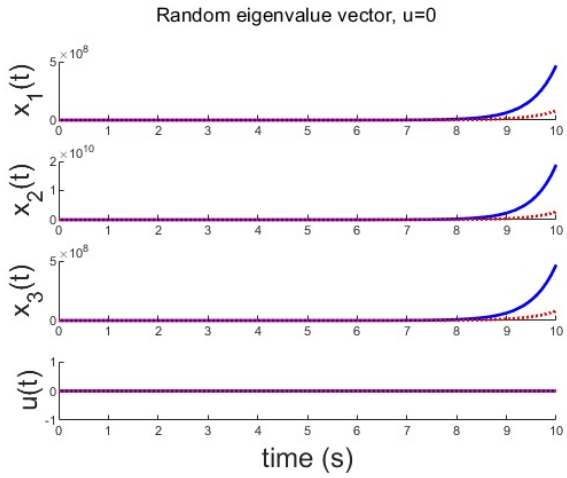
When starting at the same eigenvector as the first figure but with a constant input, all three states eventually diverge positively.

MATLAB's ode45 integration causes the solution to diverge quicker. I believe this to be because my Euler integration is only using the current time step's derivative, which will not increase as quickly as other types of integration.



When starting at an eigenvector corresponding to a positive eigenvalue with 0 input, only state 2 (x2) diverges after a period of time. State 1 and 3 stay at 0.

MATLAB's ode45 integration causes the solution of state 2 to diverge quicker. I believe this to be because my Euler integration is only using the current time step's derivative, which will not increase as quickly as other types of integration. Since states 1 and 3 stay at 0 for this time period, there is no noticeable difference.



When starting at a random initial starting state with 0 input, all states diverge after a period of time (assuming an eigenvector isn't randomly selected).

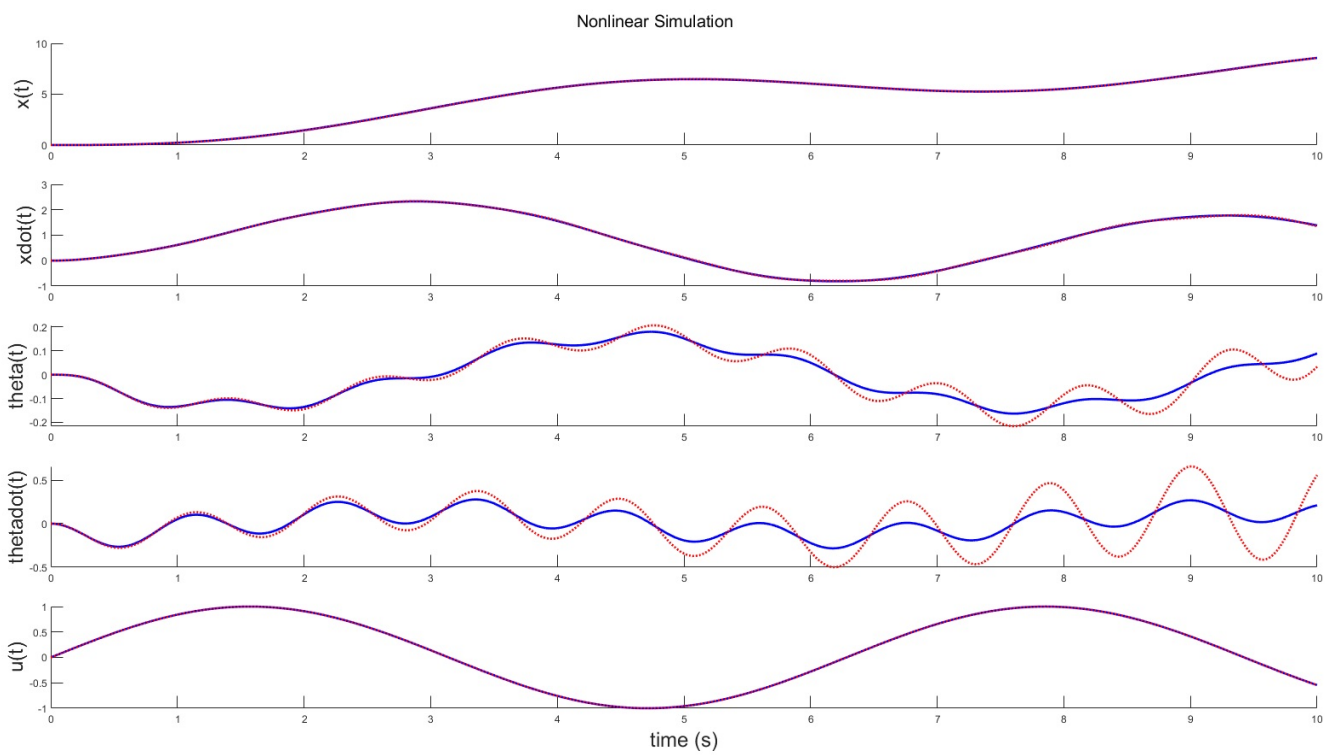
MATLAB's ode45 integration again causes the solution to diverge quicker. I believe this to be because my Euler integration is only using the current time step's derivative, which will not increase as quickly as other types of integration.

<u>Extra credit</u>: Simulation of a nonlinear system (20 points)

Create a simple simulation for the nonlinear inverted pendulum described here.

- Use the states $\begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}$
- Plot the state and input vs time with input u = sin(t).
- Have the simulation start at time zero with zero initial state.
- Turn in .m files

Hints:

- You can use the SimulationScript.m as a shell and update the following:
  - The dynamics function
  - The plotting (to plot all four states)
  - The input function
- To get values for $\ddot{x}$ and $\ddot{\theta}$ run PendulumCart.m

Changed time step (dt = 0.01)