## Linear Algebra Review

$N(V) = \{x \in \mathbb{R} \mid Vx = 0\}$

$Im(V) = \{z \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^n s.t. Vx = z\}$

$N(V) = (Im(V))^{\perp}$

$Im(V) = N(V^T)$

$S^{\perp} = \{x \in \mathbb{R}^n | x^T z = 0 \forall x \in S\}$ (S: subspace)

$v \cdot u = 0 \implies v \perp u$

$V \in \mathbb{R}^{n\times n} \implies dim(N(V)) + dim(Im(V)) = n$

$V$ non-singular if $V^{-1}$ exists and *implies*

$N(V) = \{0\}$, all rows/columns of V are linearly independent

$det(V) \neq 0$, Eigenvalues non-zero, and $rank(V) = n$

and $N(V)$ is the zero-element and zero dimensional

$\chi_V(\lambda) = det(\lambda I - V) = 0$ (finding eigenvalues)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## Solution to Linear System

$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$

$\Phi(t,\tau)$: state transition matrix

(i) $\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau)$

(ii) $\frac{d}{d\tau}\Phi(t,\tau) = \Phi(t,\tau)A(\tau)$

(iii) $\Phi(t,t) = I$

(iv) $\Phi(t_1,t_0)^{-1} = \Phi(t_0,t_1)$

(v) $\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0)$

For LTI, $\Phi(t,t_0) = e^{A(t-t_0)}$

*Matrix Exponential*

$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$

$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$

if A (block) diagonal,

$$A^k = \begin{bmatrix} A_{11}^k & 0 \\ 0 & A_{22}^k \end{bmatrix} \text{ and } e^{At} = \begin{bmatrix} e^{A_{11}t} & 0 \\ 0 & e^{A_{22}t} \end{bmatrix}$$

$$\mu = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \implies e^{\mu t} = \begin{bmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{bmatrix}$$

## Stability (Lyapunov)

Stable: $\forall \epsilon > 0 \exists \delta > 0 s.t. \|x(t_0) - x_e\| \leq \delta \implies \|x(t) - x_e\| \leq \epsilon \forall t \geq t_o$

A.S.: $\exists \eta \in R s.t. \|x(t_0) - x_e\| < \eta \implies x(t) \to x_e$ as $t \to \infty$

G.A.S: $x(t) \to x_e$ as $t \to \infty$ if $\forall x_0$ where $x(t_0) = x_0$

**Stability Tests (only for LTI)**

$\dot{x} = Ax + Bu$, $\lambda \in eig(A)$

| | C.T. | C.T. |
|---|---|---|
| Unstable | Not Stable | Not Stable |
| Stable | $Re(\lambda) \leq 0$ unique<br>$Re(\lambda) < 0$ for repeated $\lambda$ | $\lvert\lambda\rvert = 1$ for non-repeated values of $\lambda \pm 1$<br>$\lvert\lambda\rvert < 1$ for all others |
| GAS | $Re(\lambda) < 0$ for all $\lambda$ | $\lvert\lambda\rvert < 1$ for all eigenvalues |
| BIBO | All poles in OLHP | All poles in unit circle |

**Stability Tests (Linearized)**

$\delta\dot{x} = \bar{A}\delta x + \bar{B}\delta u$, $\lambda \in eig(\bar{A})$

If $Re(\lambda) = 0$ for any $\lambda$, we know nothing.

| | C.T. | C.T. |
|---|---|---|
| Unstable | $Re(\lambda) > 0$ or Not Stable | $\lvert\lambda\rvert > 1$ or Not Stable |
| Stable | | |
| LAS | $Re(\lambda) < 0$ for all $\lambda$ | $\lvert\lambda\rvert < 1$ for all eigenvalues |

## Dynamical Systems

state transition function: $g : \tau \times \tau \times X \times U \to X$

output mapping: $h : \tau \times X \times U \to Y$

Is a dynamical system if

$\forall t_0$ and $t_1 > t_0, g(t_0, t_1, x, u_{[t_0 t_1]})$ well defined

and $g(t_0, t_0, x, u) = x$

Four parts of a dynamical system: initial time, initial state, input over desired time, time of interest

## Linearization

$\bar{A} = \frac{\partial f}{\partial x}|_{x^{eq},u^{eq}}$ $\quad \bar{B} = \frac{\partial f}{\partial u}|_{x^{eq},u^{eq}}$

$\bar{C} = \frac{\partial g}{\partial x}|_{x^{eq},u^{eq}}$ $\quad \bar{D} = \frac{\partial f}{\partial u}|_{x^{eq},u^{eq}}$

$\delta\dot{x} = \bar{A}\delta x + \bar{B}\delta u \quad \dot{y} = \bar{C}\delta x + \bar{D}\delta u$

$\delta x = x - x^{eq} \quad \delta u = u - u^{eq}$

For a control $u = u^{eq} + \delta u$ after linearization:

$u = u^{eq} - K\delta x = u^{eq} - K(x - x^{eq})$

Steps: 1. Choose state $x$

2. Find $\dot{x} = f(x, u)$ in terms of state $x$ and input $u$

3. Find $y = f(x, u)$ in terms of state $x$ and input $u$

4. Find set of equilibrium points $(x^{eq}, u^{eq})$ s.t. $f(x^{eq}, u^{eq}) = 0$

5. Find Jacobians as above (generically, then at a given eq. point

6. Write linearized system as above

(remember definitions for $\delta x, \delta u, \delta y$)

*Equilibrium Point(s)*

$f(x^{eq}, u^{eq}) = 0$

$x(t_0) = x^{eq}$ and $u(\tau) = u^{eq}, \tau \geq t_0 \implies x(t) = x^{eq} \forall t > t_0$

*Trajectory*

Show that solution satisfies dynamics

$\dot{x}^{sol}(t) = f(x^{sol}(t), u^{sol}(t))$ and $y^{sol}(t) = g(x^{sol}(t), u^{sol}(t))$

Typically results in LTV, but can result in LTI

*Feedback Linearization*

let $u = u_{ff} + \hat{u}$

$u_{ff}$ cancels out non-linearities

$\hat{u}$ is new linear control (requires inversion of sytem)

## Feedback Control(lability)

Use feedback $u = -Kx$, so $\dot{x} = (A - BK)x$

To set eigenvalues, find char. eqn. of $\bar{A} = (A - BK)$ and set equal to a desired char. eqn. after picking eigenvalues

Let $\Gamma = [B, AB, A^2B, \ldots, A^{n-1}B]$

Controllable if:

a) $rank([\lambda I - A, B]) = n \forall \lambda$

or b) $rank(\Gamma) = n$

Stabilizable if:

$rank([\lambda I - A, B]) = n \forall \lambda s.t. Re(\lambda) \geq 0$

## Lyapunov Equation and Function

If LTI: $A^T P + PA = -Q$ and $V = x^T Px$

Pick a $Q \succ 0$, solve for P.

If $\exists P \succ 0 \implies A$ stability matrix.

Given $A$ stability and $\dot{x} = Ax$,

$\|x(t)\|^2 \leq \frac{1}{\lambda_{min}(P)} e^{\mu(t-t_0)} x^T(t_0)Px(t_0)$

$\mu = \frac{-\lambda_{min}(Q)}{\lambda_{max}(P)}$

*Stability via Lyapunov Function*

If $\dot{x} = f(x)$ and $f(0) = 0$,

i) $V(x) > 0 \forall x \neq 0$ and $V(0) = 0$

iia) $\dot{V}(x(t)) \leq 0 \forall x$

iib) $\dot{V}(x(t)) < 0 \forall x \neq 0, \dot{V}(0) = 0$

Stable if i) and iia)

GAS if i) and iib)

LAS if i) and iib) around 0

Linearization around an eq. point for nonlinear system $\dot{x} = f(x, u)$ will always result in LTI system (also uses Jacobian and Taylor expansion)

Linearization around a traj. can possilby produce LTI

Feedback linearization is not always possible (or recommended)

You cannot always find a similarity transform to diagonalize the system.

$m$ repeated, real eigenvalues will have $m \times m$ Jordan block

Poles are a subset of the eigenvalues (GAS implies BIBO, not vice versa)

A GAS LTI system will also be GES

Quadratic formula in case: $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

When doing a derivative, ensure chain rule followed if function of time

(ex: $\frac{d}{dt}(p_x \cos\theta + (p_y - 1)\sin\theta) = \dot{p_x}\cos\theta - p_x \sin\theta\dot{\theta} + \dot{p_y}\sin\theta + (p_y - 1)\cos\theta\dot{\theta}$)

$$A_1^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1^3 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1^t = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{A_1 t} = \sum_{k=0}^{\infty} \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{t^k}{k!} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{t^k}{k!} & \frac{kt^k}{k!} & 0 \\ 0 & \frac{t^k}{k!} & 0 \\ 0 & 0 & \frac{t^k}{k!} \end{bmatrix} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

(note $\sum_{k=0}^{\infty} k \frac{t^k}{k!} = te^t$)

**Finding the Determinant of a Three-By-Three Matrix**

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\det(A) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

## Discretization

$x_{k+1} = \bar{A}x_k + \bar{B}u_k$

Exact: $\bar{A} = e^{AT}$, $\bar{B} = \int_0^T e^{A(T-\tau)}d\tau B$, $\bar{C} = C$

(Exact assumes constant control over discretization time interval)

Solving $\bar{B}$, subsitute new $s = T - \tau$ where $T = \frac{1}{f_s}$

Euler: $\bar{A} = I + TA$, $\bar{B} = TB$

## Similarity/Cayley-Hamilton

If $V = [v1, v2, \ldots, vn]$ is formed from eigenvectors,

$\hat{A} = V^{-1}AV$ is diagonal matrix with $\lambda_i$s and $e^{At} = Ve^{\hat{A}t}V^{-1}$

$A$ satifies its own characteristic equation

$e^A$ and $A^i$ are linear combinations of $A^i$ for $i \in [0 \ldots n-1]$

Repeated real eigs, larger than 1x1.

Repeated complex, larger than 2x2 (Jordan Blocks).

## Realization Theory

$$G(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1}s + \alpha_n}$$

SS to TF: $Y(s) = (C(sI - A)^{-1}B + D)U(s)$

Same TF $\implies$ zero-state eq. $\implies$ same zero-state response but not necessarily initial cond. response

If mapping $\bar{A} = V^{-1}AV, \bar{B} = V^{-1}$, and $\bar{C} = CV$ exists $\implies$

Algebraic Equivalence $\implies$ same eigenvalues, same dimension, an initial condition in the other system with same trajectory, and zero-state eq (same similarity transform for discrete system).

*Controllable Canonical Form*

$$\dot{x} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{bmatrix}, \quad x = \begin{bmatrix} \xi^{(n-1)} \\ \xi^{(n-2)} \\ \vdots \\ \dot{\xi} \\ \xi \end{bmatrix}$$

*Observable Canonical Form*

$\bar{A} = A^T$, $\bar{B} = C^T$, $\bar{C} = B^T$, $\bar{D} = D^T$,

## Reachability/Controllability

$\mathcal{R}[t_0, t_1] = \{x_1 \in \mathbb{R}^n | \exists u(.) \in \mathcal{U}_{[t_0,t_1]}, x_1 = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau\}$

$\mathcal{C}[t_0, t_1] = \{x_0 \in \mathbb{R}^n | \exists u(.) \in \mathcal{U}_{[t_0,t_1]}, 0 = \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau\}$

$W_{\mathcal{R}}(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^T(\tau)\Phi^T(t_1,\tau)d\tau$ and $Im(W_{\mathcal{R}}(t_0,t_1)) = \mathcal{R}[t_0, t_1]$

$W_{\mathcal{C}}(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)d\tau$ and $Im(W_{\mathcal{C}}(t_0,t_1)) = \mathcal{C}[t_0, t_1]$

$W_{\mathcal{R}}(t_0, t_1) = \Phi(t_1, t_0)W_{\mathcal{C}}(t_0, t_1)\Phi^T(t_1, t_0)$

*Control Inputs*

$x_1 \in Im(W_R) \implies \exists \eta_1 s.t. x_1 = W_R\eta_1$

$u_R(t) = B^T(t)\Phi^T(t_f, t)\eta_1$

$x_0 \in Im(W_C) \implies \exists \eta_0 s.t. x_0 = W_C\eta_0$

$u_C(t) = -B^T(t)\Phi^T(t_0, t)\eta_0$

Can move between two points if:

a) $x_1 - \Phi(t_1, t_0)x_0 \in Im(W_R(t_0, t_1))$

or b) $x_0 \in \mathcal{C}$ and $x_1 \in \mathcal{R}$