09/28/24	ECE 6320 / HW5	Howes, Kade

8.6 (Stability of LTV systems). Consider a linear system with a state transition matrix $\Phi(t, t_0)$ for which

$$\Phi(t,0) = \begin{bmatrix} e^t \cos 2t & e^{-2t}\sin 2t \\ -e^t \sin 2t & e^{-2t}\cos 2t \end{bmatrix}.$$

$$\Rightarrow \Phi(t_0, 0) = \begin{bmatrix} e^{t_0}\cos 2t_0 & e^{-2t_0}\sin 2t_0 \\ -e^{t_0}\sin 2t_0 & e^{-2t_0}\cos 2t_0 \end{bmatrix}$$

(a) Compute the state transition matrix $\Phi(t, t_0)$ for an arbitrary time t_0.

(b) Compute a matrix $A(t)$ that corresponds to the given state transition matrix.

(c) Compute the eigenvalues of $A(t)$.

(d) Classify this system in terms of Lyapunov stability.
Hint: In answering part d, do not be mislead by your answer to part c.

$(i)\ \frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau)$

$(ii)\ \frac{d}{d\tau}\Phi(t,\tau) = \Phi(t,\tau)A(\tau)$

$(iii)\ \Phi(t,t) = I$

$(iv)\ \Phi(t_1,t_0)^{-1} = \Phi(t_0,t_1)$

$(v)\ \Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0)$

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau$$

a) $\Phi(t, t_0) = \Phi(t,0)\Phi(0,t_0)$ ← P5.3 book

↓ P5.4

$= \Phi(t,0)\Phi(t_0,0)^{-1}$

MATLAB

$$\Phi(t,t_0) = \begin{bmatrix} e^t \cos 2t & e^{-2t}\sin 2t \\ -e^t \sin 2t & e^{-2t}\cos 2t \end{bmatrix}\begin{bmatrix} e^{t_0}\cos 2t_0 & e^{-2t_0}\sin 2t_0 \\ -e^{t_0}\sin 2t_0 & e^{-2t_0}\cos 2t_0 \end{bmatrix}^{-1}$$

$$\Phi(t,t_0) = \begin{bmatrix} \sin(2t)\sin(2t_0)e^{(2t_0-2t)} + \cos(2t)\cos(2t_0)e^{(t-t_0)} & \cos(2t_0)\sin(2t)e^{(2t_0-2t)} - \cos(2t)\sin(2t_0)e^{(t-t_0)} \\ \cos(2t)\sin(2t_0)e^{(2t_0-2t)} - \cos(2t_0)\sin(2t)e^{(t-t_0)} & \cos(2t)\cos(2t_0)e^{(2t_0-2t)} + \sin(2t)\sin(2t_0)e^{(t-t_0)} \end{bmatrix}$$

b) $\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau) \Rightarrow \frac{d}{dt}\Phi(t,\tau)\Phi(t,\tau)^{-1} = A(t)$

choose $\tau = 0$

$\Rightarrow \frac{d}{dt}\Phi(t,0)\Phi(t,0)^{-1} = A(t)$

↓ using MATLAB

$$\Rightarrow A(t) = \begin{bmatrix} \frac{3\cos(4t)}{2} - \frac{1}{2} & 2 - \frac{3\sin(4t)}{2} \\ -\frac{3\sin(4t)}{2} - 2 & -\frac{3\cos(4t)}{2} - \frac{1}{2} \end{bmatrix}$$

✓

c) $\det(\lambda I - A(t)) = \det \begin{bmatrix} \lambda + \frac{1}{2} - \frac{3}{2}\cos(4t) & -2 + \frac{3}{2}\sin(4t) \\ 2 + \frac{3}{2}\sin(4t) & \lambda + \frac{1}{2} + \frac{3}{2}\cos(4t) \end{bmatrix}$

MATLAB

$= \lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda = -0.5 \pm 1.3229j$

↳ Tells us nothing about stability

d) For this system to be stable, $x(t) = \Phi(t,0)x_0$ must be bounded $\forall x_0 := x(t_0)$. Choosing $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow x(t) = \begin{bmatrix} e^{t}\cos 2t & e^{-2t}\sin 2t \\ -e^{t}\sin 2t & e^{-2t}\cos 2t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{t}\cos 2t \\ -e^{t}\sin 2t \end{bmatrix}$

selecting this column

$\Rightarrow x(t) \to \begin{bmatrix} \infty \\ -\infty \end{bmatrix}$ and grows unbounded as $t \to \infty$

$\Rightarrow$ This system is unstable

8.12 (Stability of nonlinear systems). Investigate whether or not the solutions to the following nonlinear systems converge to the given equilibrium point when they start sufficiently close to it:

(a) The state-space system

$$\dot{x}_1 = -x_1 + x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_2 + x_2(x_1^2 + x_2^2),$$

with equilibrium point $x_1 = x_2 = 0$.

(b) The second-order system

$$\ddot{w} + g(w)\dot{w} + w = 0, \qquad (8.24)$$

with equilibrium point $w = \dot{w} = 0$. Determine for which values of $g(0)$ we can guarantee convergence to the origin based on the local linearization. □

a)

$$f(x) = \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_1(x_1^2 + x_2^2) \\ -x_2 + x_2(x_1^2 + x_2^2) \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Jacobian

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 + (x_1^2 + x_2^2) + x_1(2x_1) & 2x_1 x_2 \\ 2x_1 x_2 & -1 + (x_1^2 + x_2^2) + x_2(2x_2) \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 3x_1^2 + x_2^2 - 1 & 2x_1 x_2 \\ 2x_1 x_2 & 3x_2^2 + x_1^2 - 1 \end{bmatrix}$$

$$\Rightarrow \delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x_{eq}} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{A}, \qquad \delta x = x - x_{eq}$$

$\Rightarrow$ eigenvalues of A: $\lambda_1 = -1$, $\lambda_2 = -1$

$x = \delta x + x_{eq}$

Since $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \mathrm{eig}(A)$, the solution to the system converges locally around equilibrium point $x_1 = x_2 = 0$

$\Rightarrow$ LAS (CLES around small region) b let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ \dot{\omega} \end{bmatrix} \Rightarrow x_{eq} = \begin{bmatrix} \omega_{eq} \\ \dot{\omega}_{eq} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \dot{x} = \begin{bmatrix} \dot{\omega} \\ \ddot{\omega} \end{bmatrix} = \begin{bmatrix} x_2 \\ -g(x_1)x_2 - x_1 \end{bmatrix} = f(x), \quad \ddot{\omega} = -g(\omega)\dot{\omega} - \omega$

$\ddot{\omega} = -g(x_1)x_2 - x_1$

Jacobian

$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dot{g}(x_1)x_2 - 1 & -g(x_{eq}) \end{bmatrix}$

$\Rightarrow \delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x = x_{eq}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -g(0) \end{bmatrix}}_{A}, \quad \delta x = x - x_{eq}$

eig(A): $\det(\lambda I - A) = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda + g(0) \end{bmatrix} = \lambda^2 + g(0)\lambda + 1 = 0$

$\Rightarrow \text{eig}(A) = \lambda = \frac{-g(0) \pm \sqrt{g(0)^2 - 4}}{2}$

When $|g(0)| \le 2$, $g(0)^2 - 4 \le 0 \Rightarrow Re(\lambda) = \frac{-g(0)}{2}$

$\Rightarrow g(0) > 0$ for LAS ⓐ

When $|g(0)| > 2$, $g(0)^2 - 4 > 0 \Rightarrow Re(\lambda) = \frac{-g(0) \pm \sqrt{g(0)^2 - 4}}{2}$

$\Rightarrow -g(0) \pm \sqrt{g(0)^2 - 4} < 0 \Rightarrow \pm\sqrt{g(0)^2 - 4} < g(0)$ (can square since $g(0) > 0$ by ⓐ)

$\Rightarrow g(0)^2 - 4 < g(0)^2 \Rightarrow$ true for all $g(0)$

For the system to converge to the origin based on local linearization

$Re(\lambda) < 0 \;\forall \lambda \in \text{eig}(A) \Rightarrow \frac{-g(0)}{2} < 0 = \boxed{g(0) > 0}$

for L.A.S.

Characterize the stability of the stated equilibrium points for each of the following problems (10 points each problem) Problem 2.3 parts (a), (b), and (c) (assume g = 9.8, m = 1/9.8, l = 0.25, and b = 1)

2.3 (Local linearization around equilibrium: saturated inverted pendulum). Consider the inverted pendulum in Figure 2.8 and assume the input and output to the system are the signals u and y defined as

$$T = \text{sat}(u), \quad y = \theta,$$

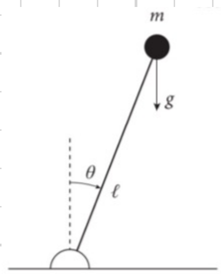
where "sat" denotes the unit-slope saturation function that truncates u at $+1$ and -1.

(a) Linearize this system around the equilibrium point for which $\theta = 0$.

(b) Linearize this system around the equilibrium point for which $\theta = \pi$ (assume that the pendulum is free to rotate all the way to this configuration without hitting the table).

(c) Linearize this system around the equilibrium point for which $\theta = \frac{\pi}{4}$. Does such an equilibrium point always exist?

~~(d)~~ Assume that $b = 1/2$ and $mg\ell = 1/4$. Compute the torque $T(t)$ The needed for the pendulum to fall from $\theta(0) = 0$ with constant velocity $\dot\theta(t) = 1, \forall t \geq 0$. Linearize the system around this trajectory. □



From Newton's law:

$$m\ell^2\ddot\theta = mg\ell\sin\theta - b\dot\theta + T,$$

where T denotes a torque applied at the base, and g is the gravitational acceleration.

FIGURE 2.8. Inverted pendulum.

a) Linearization from Homework 2:

$$\Rightarrow \delta\dot x = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & \frac{-b}{m\ell^2} \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}}_{B} \delta u$$

$$y = g(x, u) = x_1$$

$$\Rightarrow \delta y = \underbrace{[1 \;\; 0]}_{C} \delta x + \underbrace{0}_{D=0} \delta u$$

Where $\delta x = x - x^{eq}$, $\delta y = y - x^{eq}$, $\delta u = u - u^{eq}$

Finding Eigenvalues

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & \frac{-b}{m\ell^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 39.2 & -156.8 \end{bmatrix}$$

$$\det(A - \lambda I) = \left| \begin{bmatrix} -\lambda & 1 \\ 39.2 & -156.8 - \lambda \end{bmatrix} \right| = -\lambda(-156.8 - \lambda) - 39.2$$

$$= \lambda^2 + 156.8\lambda - 39.2$$

$$\Rightarrow \underline{\lambda_1 = 0.2496, \; \lambda_2 = -157.0496} = (\lambda - 0.2496)(\lambda + 157.0496)$$

Unstable around equilibrium point $\theta = 0$ since $Re(\lambda) > 0$

b) Linearization from Homework 2

$$\Rightarrow \delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}}_{B} \delta u$$

$$y = g(x, u) = x_1$$

$$\Rightarrow \delta y = \overset{C}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \delta x$$

where $\delta x = x - x^{eq}$, $\delta y = y - x^{eq}_1$, $\delta u = u - u^{eq}$

Finding Eigenvalues

($g = 9.8$, $m = 1/2P$, $\ell = 0.25$, $L = 1$)

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -39.2 & -156.8 \end{bmatrix}$$

$$\det(A - \lambda I) = \left| \begin{bmatrix} -\lambda & 1 \\ -39.2 & -156.8 - \lambda \end{bmatrix} \right| = -\lambda(-156.8 - \lambda) + 39.2$$

$$= \lambda^2 + 156.8\lambda + 39.2$$

$$\Rightarrow \lambda_1 = -0.2504,\ \lambda_2 = -156.5496 = (\lambda + 0.2504)(\lambda + 156.5496)$$

Locally asymptotically stable equilibrium $\theta = \pi$

Also LES around very small region

C) Linearization from Homework 2

$$\delta \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{\ell\sqrt{2}} & -\frac{b}{m\ell^2} \end{bmatrix}}_{A} \delta x + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} \delta u$$

$$y = g(x,u) = x_1$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

where $\delta x = x - x^{eq}$, $\delta y = y - x^{eq}$, $\delta u = u - u^{eq}$

Finding Eigenvalues

$(g = 9.8,\ m = \frac{1}{9.8},\ \ell = 0.25,\ b = 1)$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell\sqrt{2}} & \frac{-b}{m\ell^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 27.72 & -156.8 \end{bmatrix}$$

$$\det(A - \lambda I) = \left| \begin{bmatrix} -\lambda & 1 \\ 27.72 & -156.8 - \lambda \end{bmatrix} \right| = -\lambda(-156.8 - \lambda) - 27.72$$

$$= \lambda^2 + 156.8\lambda - 27.72$$

$$\Rightarrow \lambda_1 = 0.1766,\ \lambda_2 = -156.9766 = (\lambda_1 - 0.1766)(\lambda_2 + 156.9766)$$

Unstable at equilibrium $\theta = \frac{\pi}{4}$ since $Re(\lambda) > 0$

Characterize the stability of the stated equilibrium points for each of the following problems (10 points each problem) Problem 2.4 part (c) (assume k = 1)

2.4 (Local linearization around equilibrium: pendulum). The following equation models the motion of a frictionless pendulum:

$$\ddot{\theta} + k \sin\theta = \tau$$

where $\theta \in \mathbb{R}$ is the angle of the pendulum with the vertical, $\tau \in \mathbb{R}$ an applied torque, and k a positive constant.

~~(a)~~ Compute a state-space model for the system when $u := \tau$ is viewed as the input and $y := \theta$ as the output. Write the model in the form

$$\dot{x} = f(x, u) \qquad y = g(x, u)$$

for appropriate functions f and g.

~~(b)~~ Find the equilibrium points of this system corresponding to the constant input $\tau(t) = 0, t \geq 0$.

Hint: There are many.

(c) Compute the linearization of the system around the solution $\tau(t) = \theta(t) = \dot{\theta}(t) = 0, t \geq 0$.

Hint: Do not forget the output equations. □

c) Linearization from Homework 2

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_{eq} = 0$$

$$\delta \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}}_{A} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

Where $\delta x = x - x^{eq}$, $\delta y = y - y^{eq}$, $\delta u = u - u^{eq}$

Finding eigenvalues

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}\right)$$

$$= \lambda^2 + 1 \Rightarrow \lambda = \pm j$$

Unknown stability from eigenvalues

Try using $AP + PA^T + Q = 0$ for any

$Q \succ 0$, see if exists $P \succ 0$.

Using Matlab $\quad Q = eye(2)$;

$\quad P = lyap(A^T, Q)$;

$P \succ 0$ does not exist, unknown whether stable or not

Characterize the stability of the stated equilibrium points for each of the following problems (10 points each problem) Problem 2.6 parts (b) (assume g = 9.8, m = 1/9.8, I = 1, b = 2)

2.6 (Local linearization around equilibrium: one-link robot). Consider the one-link robot in Figure 2.9, where θ denotes the angle of the link with the horizontal, τ the torque applied at the base, (x, y) the position of the tip, ℓ the length of the link, I its moment of inertia, m the mass at the tip, g gravity's acceleration, and b a friction coefficient. This system evolves according to the following equation:

$$I\ddot{\theta} = -b\dot{\theta} - gm\cos\theta + \tau.$$

(a) Compute the state-space model for the system when $u = \tau$ is regarded as the input and the vertical position of the tip y is regarded as the output.

Please denote the state vector by z to avoid confusion with the horizontal position of the tip x, and write the model in the form

$$\dot{z} = f(z, u) \quad y = g(z, u)$$

for appropriate functions f and g.

Hint: Do not forget the output equation!

(b) Show that $\theta(t) = \pi/2$, $\tau(t) = 0$, $\forall t \geq 0$ is a solution to the system and compute its linearization around this solution.

From your answer, can you predict if there will be problems when one wants to control the tip position close to this configuration just using feedback from y? □

b) Linearization from Homework 2 $\quad z^{sol} = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}, \; u^{sol} = 0 \Rightarrow y^{sol} = \ell$

$$\delta\dot{z} = \begin{bmatrix} 0 & 1 \\ \frac{gm}{I} & -\frac{b}{I} \end{bmatrix} \delta z + \begin{bmatrix} 0 \\ 1/I \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 0 & 0 \end{bmatrix} \delta + 0 \, \delta u$$

where $\delta z = z - z^{eq}$, $\delta u = u - u^{eq}$, $\delta y = y - y^{eq}$

Finding Eigenvalues $\quad g = 9.8 \quad m = \frac{1}{9.8} \quad I = 1 \quad b = 2$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{gm}{I} & -\frac{b}{I} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda+2 \end{bmatrix}\right) = \lambda(\lambda+2) - 1 = \lambda^2 + 2\lambda - 1$$

$$\lambda = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \frac{\sqrt{8}}{2} \Rightarrow eig(A) = \{0.4142, -2.4142\}$$

$\Rightarrow$ Unstable at solution

$\theta_{sol}(t) = \frac{\pi}{2}$, $u_{sol} = 0$ $\quad$ since $\exists \lambda \in eig(A)$ s.t. $Re(\lambda) > 0$

Characterize the stability of the stated equilibrium points for each of the following problems (10 points each problem) Problem 2.7 parts (b) and (d)

2.7 (Local linearization around trajectory: unicycle). A single-wheel cart (unicycle) moving on the plane with linear velocity v and angular velocity ω can be modeled by the nonlinear system

$$\dot{p}_x = v\cos\theta, \quad \dot{p}_y = v\sin\theta, \quad \dot{\theta} = \omega, \tag{2.13}$$

where (p_x, p_y) denote the Cartesian coordinates of the wheel and θ its orientation. Regard this as a system with input $u := \begin{bmatrix} v & \omega \end{bmatrix}' \in \mathbb{R}^2$.

~~(a)~~ Construct a state-space model for this system with state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} p_x\cos\theta + (p_y - 1)\sin\theta \\ -p_x\sin\theta + (p_y - 1)\cos\theta \\ \theta \end{bmatrix}$$

and output $y := \begin{bmatrix} x_1 & x_2 \end{bmatrix}' \in \mathbb{R}^2$.

(b) Compute a local linearization for this system around the equilibrium point $x^{eq} = 0$, $u^{eq} = 0$.

~~(c)~~ Show that $\omega(t) = v(t) = 1$, $p_x(t) = \sin t$, $p_y(t) = 1 - \cos t$, $\theta(t) = t$, $\forall t \geq 0$ is a solution to the system.

(d) Show that a local linearization of the system around this trajectory results in an LTI system. □

b) Linearization from Homework 2

$$\delta\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x + 0\,\delta u$$

Where $\delta x = x - x_{eq}$, $\delta y = y - y_{eq}$, $\delta u = u - u_{eq}$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$ $\quad x^{eq} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $u^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $y^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ Unknown stability from non-unique eigenvalue with $Re(\lambda) = 0$ according to eigenvalue test

d) Linearization from Homework 2

$$\delta \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \delta u$$

where $\delta x = x - x_{sol}$, $\delta y = y - y_{sol}$, $\delta u = u - u_{sol}$

$x_{sol} = \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix}$, $y_{sol} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $u_{sol} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Finding eigenvalues

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \lambda(\lambda^2) - (-1)(\lambda) + 0$$

$$= \lambda^3 + \lambda = 0$$

$$= \lambda(\lambda^2 + 1) = 0$$

$\Rightarrow \text{eig}(A) = \{0, \pm j\}$

$\Rightarrow$ No conclusion can be drawn since unique eigenvalue with $Re(\lambda) = 0$, (Optional - +20 pnts extra credit) - For each problem above (2.3 - 2.7) that is A.S.:

a) - Find a bound on the exponential convergence rate
    - +10 additional points if you find the best (tightest) exponential rate

b) - Choose an initial condition, simulate the nonlinear system forward in time, and plot both $||\delta x(t)||$ and the theoretical bound

c) - For the same initial condition, simulate the linearized system forward in time, and plot both $||\delta x(t)||$ and the theoretical bound

d) - Note any observations

Only 2.3b is A.S. (see MATLAB % extra credit) for computations

2.3

a) $A = \begin{bmatrix} 0 & 1 \\ -39.2 & -156.8 \end{bmatrix}$, choose Q using by just trying out some values

$\Rightarrow Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Using lyap(A', Q) solve $A'P + PA = -Q$ to find $P > 0$

$\Rightarrow P = \begin{bmatrix} 1.6282 & 0.0159 \\ 0.0096 & 0.0033 \end{bmatrix}$

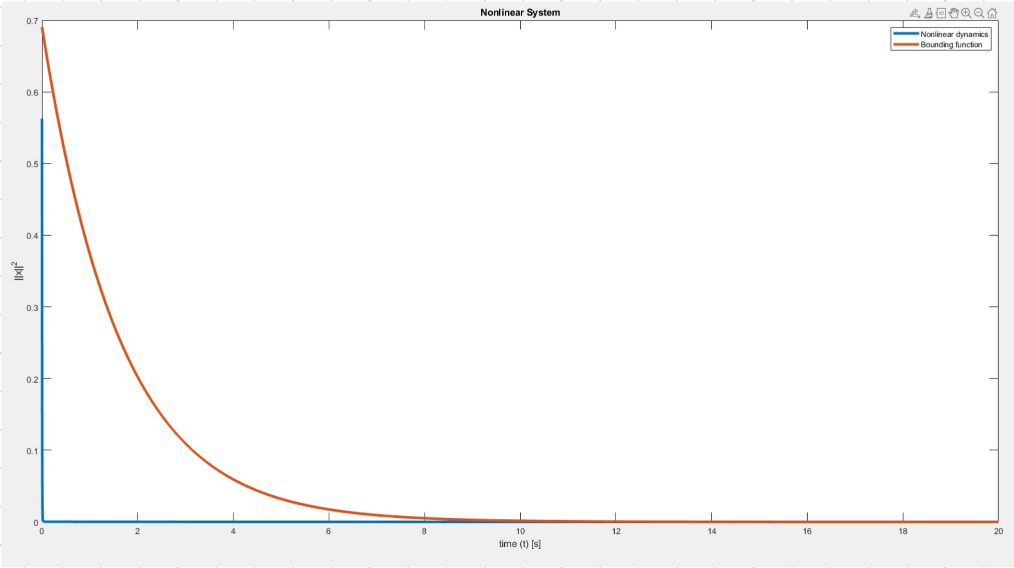
$eig(P) = \{1.6283, \ 0.0032\} \Rightarrow P > 0$

$\Rightarrow ||x(t)||^2 \leq \frac{1}{\lambda_{min}[P]} e^{\mu(t-t_0)} x^T(t_0) P x(t)$

where rate $\mu = \frac{-\lambda_{min}[Q]}{\lambda_{max}[P]} = -0.6141$

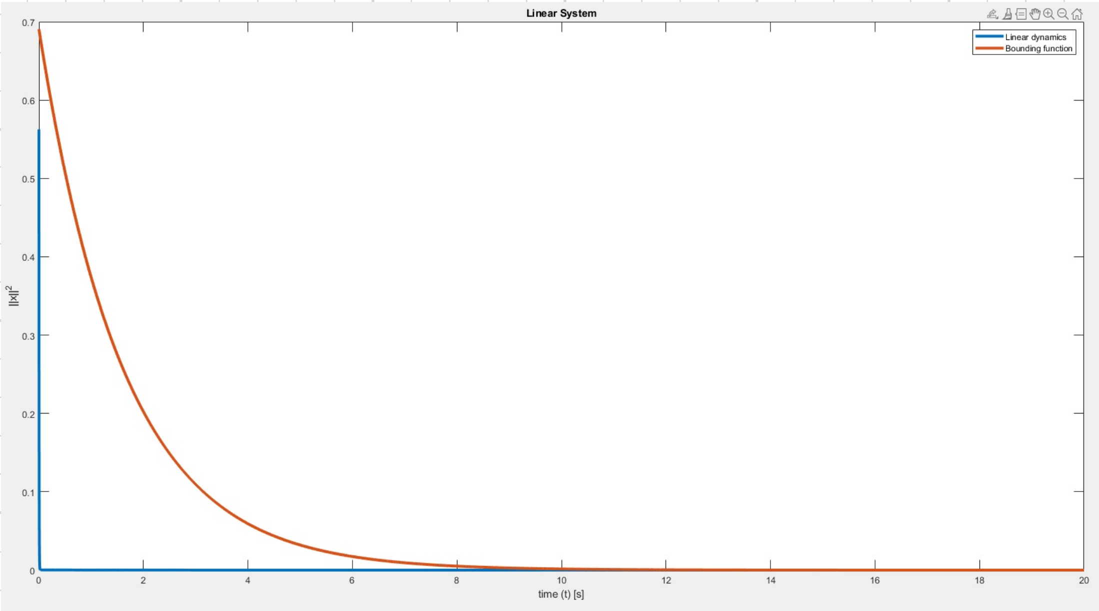
b) Initial Condition: $\delta x(0) = [0.01 ; 0.75]$

MATLAB Simulation Results



c) Initial Condition: $x(0) = dx(0) + [\pi ; 0];$ x_{eq}

MATLAB Simulation Results



d) It is interesting to observe a few things during these simulation creations. First, the linear and nonlinear systems behave nearly the same for this initial dx_0 (and a few other initial states I tried). Also, the factor that had the biggest impact in the results of the bounding function was of course the choice of the Q matrix. Some Q's produce a mu that is bounding, but very far above the actual simulated results. I was able to improve my choice of Q from my starting point of the identity matrix, but didn't find what I believe to be the optimal convergence bound.

Lyapunov equation (+15 extra credit points)

Say that you are using the Lyapunov equation (LE) ($A^T P + PA = -Q$) to evaluate the stability characteristics of an LTI system. For both problems below, assume that all possible values of P are given for the state value of Q. Answer the following questions:

- a) What are the stability characteristics of the equilibrium point being evaluated?
- b) What value of P satisfies all the requirements for showing A.S.?
- c) Does an exponential convergence rate bound exist using these matrices? If so, state one.

See Homework 5.m for computations

Problem 1 (+10 extra credit):

$$P_1 = \begin{bmatrix} 4.5 & -0.866 \\ -0.866 & 3.5 \end{bmatrix}, P_2 = \begin{bmatrix} 4.5 & -0.866 \\ 0.866 & 3.5 \end{bmatrix}, Q = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

Problem 2 (+5 extra credit):

$$P_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}, P_2 = \begin{bmatrix} -1 & 7 \\ 0 & 3 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Problem 1)

a) Note that Q is positive definite since $eig(Q) = \{3, 5\}$
can proceed to check P

$eig(P_1) = \{3, 5\} \Rightarrow P_1 \succ 0 \Rightarrow$ the equilibrium point is GAS since LTI system

$eig(P_2) = \{4 \pm 0.7071j\}$
↳ not needed

b) $P_1 = \begin{bmatrix} 4.5 & -0.866 \\ -0.866 & 3.5 \end{bmatrix}$ satisfies all requirements for showing A.S.

c) Yes, there exists an exponential convergence rate by this equation $\|x(t)\|^2 \le \frac{1}{\lambda_{min}[P]} e^{\mu(t-t_0)} x^T(t_0) P x(t_0)$

where exponential rate $\mu = \frac{-\lambda_{min}[Q]}{\lambda_{max}[P]} = -\frac{1}{5}$

Problem 2

a) Q is positive definite since $eig(Q) = \{0.2679, 3.7321\}$
$\Rightarrow$ can check if $\exists P \succ 0$

$eig(P_1) = \{7.2749, -0.2749\} \Rightarrow$ no conclusion about stability

$eig(P_2) = \{-1, 3\} \Rightarrow$ no conclusion about stability

$\Rightarrow$ can not conclude anything about stability of eq. point

b) No P given satisfies all requirements for A.S.

c) No, not enough information to find exponential convergence rate