Linear Algebra Review

$N(V) = \{x \in \mathbb{R} \mid Vx = 0\}$

$Im(V) = \{z \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^n s.t. Vx = z\}$

$N(V) = (Im(V))^\perp$

$Im(V) = N(V^T)$

$S^\perp = \{x \in \mathbb{R}^n | x^T z = 0 \forall x \in S\}$ (S: subspace)

$V \in \mathbb{R}^{n\times n} \implies dim(N(V)) + dim(Im(V)) = n$

V non-singular if V^{-1} exists and *implies*

$N(V) = \{0\}$, all rows/columns of V are linearly independent

$det(V) \neq 0$, Eigenvalues non-zero, and $rank(V) = n$

$\chi_V(\lambda) = det(\lambda I - V) = 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Solution to Linear System

$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,\tau)B(\tau)u(\tau)d\tau$

$\Phi(t,\tau)$: state transition matrix

(i)$\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau)$

(ii)$\frac{d}{d\tau}\Phi(t,\tau) = \Phi(t,\tau)A(t)$

(iii)$\Phi(t,t) = I$

(iv)$\Phi(t_1,t_0)^{-1} = \Phi(t_0,t_1)$

(v)$\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0)$

For LTI, $\Phi(t,t_0) = e^{A(t-t_0)}$

Matrix Exponential

$e^{At} = \sum_{k=0}^\infty \frac{A^k t^k}{k!}$

$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$

if A (block) diagonal,

$$A^k = \begin{bmatrix} A_{11}^k & 0 \\ 0 & A_{22}^k \end{bmatrix} \text{ and } e^{At} = \begin{bmatrix} e^{A_{11}t} & 0 \\ 0 & e^{A_{22}t} \end{bmatrix}$$

$$\mu = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \implies e^{\mu t} = \begin{bmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{bmatrix}$$

Stability (Lyapunov)

Stable: $\forall \epsilon > 0 \exists \delta > 0 s.t. \|x(t_0) - x_e\| \leq \delta \implies \|x(t) - x_e\| \leq \epsilon \forall t \geq t_o$

A.S.: $\exists \eta \in R s.t. \|x(t_0) - x_e\| < \eta \implies x(t) \to x_e$ as $t \to \infty$

G.A.S: $x(t) \to x_e$ as $t \to \infty$ if $\forall x_0$ where $x(t_0) = x_0$

Stability Tests (only for LTI)

$\dot{x} = Ax + Bu$, $\lambda \in eig(A)$

	C.T.	C.T.
Unstable	Not Stable	Not Stable
Stable	$Re(\lambda) \leq 0$ unique $Re(\lambda) < 0$ for repeated λ	$\|\lambda\| = 1$ for non-repeated values of $\lambda \pm 1$ $\|\lambda\| < 1$ for all others
GAS	$Re(\lambda) < 0$ for all λ	$\|\lambda\| < 1$ for all eigenvalues
BIBO	All poles in OLHP	All poles in unit circle

Stability Tests (Linearized)

$\delta\dot{x} = \bar{A}\delta x + \bar{B}\delta u$, $\lambda \in eig(\bar{A})$

If $Re(\lambda) = 0$ for any λ, we know nothing.

	C.T.	C.T.
Unstable	$Re(\lambda) > 0$ or Not Stable	$\|\lambda\| > 1$ or Not Stable
Stable		
LAS	$Re(\lambda) < 0$ for all λ	$\|\lambda\| < 1$ for all eigenvalues

Dynamical Systems

state transition function: $g : \tau \times \tau \times X \times U \to X$

output mapping: $h : \tau \times X \times U \to Y$

Is a dynamical system if

$\forall t_0$ and $t_1 > t_0, g(t_0, t_1, x, u_{[t_0 t_1]})$ well defined

and $g(t_0, t_0, x, u) = x$

Four parts of a dynamical system: initial time, initial state, input over desired time, time of interest

Linearization

$\bar{A} = \frac{\partial f}{\partial x}|_{x^{eq},u^{eq}}$ $\quad \bar{B} = \frac{\partial f}{\partial u}|_{x^{eq},u^{eq}}$

$\bar{C} = \frac{\partial g}{\partial x}|_{x^{eq},u^{eq}}$ $\quad \bar{D} = \frac{\partial f}{\partial u}|_{x^{eq},u^{eq}}$

$\delta\dot{x} = \bar{A}\delta x + \bar{B}\delta u$ $\quad \dot{y} = \bar{C}\delta x + \bar{D}\delta u$

$\delta x = x - x^{eq}$ $\quad \delta u = u - u^{eq}$

Equilibrium Point(s)

$f(x^{eq}, u^{eq}) = 0$

$x(t_0) = x^{eq}$ and $u(\tau) = u^{eq}, \tau \geq t_0 \implies x(t) = x^{eq} \forall t > t_0$

Trajectory

Show that solution satisfies dynamics

$\dot{x}^{sol}(t) = f(x^{sol}(t), u^{sol}(t))$ and $y^{sol}(t) = g(x^{sol}(t), u^{sol}(t))$

Typically results in LTV, but can result in LTI

Feedback Linearization

let $u = u_{ff} + \hat{u}$

u_{ff} cancels out non-linearities

$\hat{u}$ is new linear control (requires inversion of sytem)

Lyapunov Equation and Function

If LTI: $A^TP + PA = -Q$ and $V = x^TPx$

Pick a $Q \succ 0$, solve for P.

If $\exists P \succ 0 \implies A$ stability matrix.

Given A stability and $\dot{x} = Ax$,

$\|x(t)\|^2 \leq \frac{1}{\lambda_{min}(P)} e^{\mu(t-t_0)} x^T(t_0)Px(t_0)$

$\mu = \frac{-\lambda_{min}(Q)}{\lambda_{max}(P)}$

Stability via Lyapunov Function

If $\dot{x} = f(x)$ and $f(0) = 0$,

i)$V(x) > 0 \forall x \neq 0$ and $V(0) = 0$

iia) $\dot{V}(x(t)) \leq 0 \forall x$

iib) $\dot{V}(x(t)) < 0 \forall x \neq 0, \dot{V}(0) = 0$

Stable if i) and iia)

GAS if i) and iib)

LAS if i) and iib) around 0

Feedback Control(lability)

Use feedback $u = -Kx$, so $\dot{x} = (A - BK)x$

To set eigenvalues, find char. eqn. of $\bar{A} = (A - BK)$ and set equal to a desired char. eqn. after picking eigenvalues

Let $\Gamma = [B, AB, A^2B, \ldots, A^{n-1}B]$

Controllable if:

a) $rank([\lambda I - A, B]) = n \forall \lambda$

or b) $rank(\Gamma) = n$

Stabilizable if:

$rank([\lambda I - A, B]) = n \forall \lambda s.t. Re(\lambda) \geq 0$

Discretization

$x_{k+1} = \bar{A}x_k + \bar{B}u_k$

Exact: $\bar{A} = e^{AT}$, $\bar{B} = \int_0^T e^{A(T-\tau)}d\tau B$, $\bar{C} = C$

Solving $\bar{B}$, subsitute new $s = T - \tau$ where $T = \frac{1}{f_s}$

Euler: $\bar{A} = I + TA$, $\bar{B} = TB$

Similarity/Cayley-Hamilton

If $V = [v1, v2, \ldots, vn]$ is formed from eigenvectors,

$\hat{A} = V^{-1}AV$ is diagonal matrix with λ_is and $e^{At} = Ve^{\hat{A}t}V^{-1}$

A satifies its own characteristic equation

e^A and A^i are linear combinations of A^i for $i \in [0 \ldots n-1]$

Repeated real eigs, larger than 1x1.

Repeated complex, larger than 2x2 (Jordan Blocks).

Realization Theory

$$G(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1}s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1}s + \alpha_n}$$

SS to TF: $Y(s) = (C(sI - A)^{-1}B + D)U(s)$

Same TF $\implies$ zero-state eq. $\implies$ same zero-state response but not necessarily initial cond. response

If mapping $\bar{A} = V^{-1}AV$,$\bar{B} = V^{-1}$,and $\bar{C} = CV$ exists $\implies$

Algebraic Equivalence $\implies$ same eigenvalues, same dimension, an initial condition in the other system with same trajectory, and zero-state eq (same similarity transform for discrete system).

Controllable Canonical Form

$$\dot{x} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{bmatrix}, x = \begin{bmatrix} \xi^{(n-1)} \\ \xi^{(n-2)} \\ \vdots \\ \dot{\xi} \\ \xi \end{bmatrix}$$

Observable Canonical Form

$\bar{A} = A^T$, $\bar{B} = C^T$, $\bar{C} = B^T$, $\bar{D} = D^T$,

Reachability/Controllability

$\mathcal{R}[t_0,t_1] = \{x_1 \in \mathbb{R}^n | \exists u(.) \in \mathcal{U}_{[t_0,t_1]}, x_1 = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau\}$

$\mathcal{C}[t_0,t_1] = \{x_0 \in \mathbb{R}^n | \exists u(.) \in \mathcal{U}_{[t_0,t_1]}, 0 = \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau\}$

$W_\mathcal{R}(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^T(\tau)\Phi^T(t_1,\tau)d\tau$ and $Im(W_\mathcal{R}(t_0,t_1)) = \mathcal{R}[t_0,t_1]$

$W_\mathcal{C}(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)d\tau$ and $Im(W_\mathcal{C}(t_0,t_1)) = \mathcal{C}[t_0,t_1]$

$W_\mathcal{R}(t_0,t_1) = \Phi(t_1,t_0)W_\mathcal{C}(t_0,t_1)\Phi^T(t_1,t_0)$

Control Inputs

$x_1 \in Im(W_R) \implies \exists \eta_1 s.t. x_1 = W_R\eta_1$

$u_R(t) = B^T(t)\Phi^T(t_f,t)\eta_1$

$x_0 \in Im(W_C) \implies \exists \eta_0 s.t. x_0 = W_C\eta_0$

$u_C(t) = -B^T(t)\Phi^T(t_0,t)\eta_0$

Can move between two points if:

a) $x_1 - \Phi(t_1,t_0)x_0 \in Im(W_R(t_0,t_1))$

or b) $x_0 \in \mathcal{C}$ and $x_1 \in \mathcal{R}$