11/14/24 | ECE 6320 / HW12 | Howes, Kade

**24.4.** Verify that a solution to (24.19) is given by (24.20).

*Hint: Use direct substitution of the "candidate" solution into* (24.19). □

$$\begin{cases} Ax_{\text{eq}} + Bu_{\text{eq}} = 0 \\ r = Gx_{\text{eq}} + Hu_{\text{eq}} \end{cases} \Leftrightarrow \begin{bmatrix} -A & B \\ -G & H \end{bmatrix}_{(n+\ell)\times(n+k)} \begin{bmatrix} -x_{\text{eq}} \\ u_{\text{eq}} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}. \quad (24.19)$$

$$\begin{bmatrix} -x_{\text{eq}} \\ u_{\text{eq}} \end{bmatrix} = P(0)'\big(P(0)P(0)'\big)^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}. \quad (24.20)$$

**Note.** In practice, a solution typically exists if the reference signal $r$ is carefully chosen so that $\left[\begin{smallmatrix} 0 \\ r \end{smallmatrix}\right]$ is in the image of $\left[\begin{smallmatrix} -A & B \\ -G & H \end{smallmatrix}\right]$. For example if $z = \left[\begin{smallmatrix} y \\ y \end{smallmatrix}\right]$ as in (23.8), the set-point vector $r$ should be of the form $\left[\begin{smallmatrix} y_{\text{des}}\mathbf{1} \\ 0 \end{smallmatrix}\right]$ to make sure that the set-point is compatible with some equilibrium point.

Such an equilibrium point $(x_{\text{eq}}, u_{\text{eq}})$ must satisfy the equation

$$\begin{cases} Ax_{\text{eq}} + Bu_{\text{eq}} = 0 \\ r = Gx_{\text{eq}} + Hu_{\text{eq}} \end{cases} \Leftrightarrow \begin{bmatrix} -A & B \\ -G & H \end{bmatrix}_{(n+\ell)\times(n+k)} \begin{bmatrix} -x_{\text{eq}} \\ u_{\text{eq}} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}. \quad (24.19)$$

To understand when these equations have a solution, three distinct cases should be considered.

1. When the number of inputs $k$ is strictly smaller than the number of controlled outputs $\ell$, we have an *underactuated system*. In this case, the system of equations (24.19) may not have a solution, because it presents more equations than unknowns.
2. When the number of inputs $k$ is equal to the number of controlled outputs $\ell$, (24.19) always has a solution as long as Rosenbrock's system matrix

$$P(s) := \begin{bmatrix} sI - A & B \\ -G & H \end{bmatrix}$$

is nonsingular for $s = 0$. This means that $s = 0$ should not be an invariant zero of the system, and therefore it cannot also be a transmission zero of the transfer matrix $G(sI - A)^{-1}B + H$.

Intuitively, one should expect problems when $s = 0$ is an invariant zero of the system, because when the state converges to an equilibrium point, the control input $u(t)$ must converge to a constant. By the zero-blocking property, one should then expect the controlled output $z(t)$ to converge to zero and not to $r$.

3. When the number of inputs $k$ is strictly larger than the number of controlled outputs $\ell$, we have an *overactuated system*, and (24.19) generally has multiple solutions.

When $P(0)$ is full row-rank (i.e., when it has $n + \ell$ linearly independent rows), the $(n + \ell) \times (n + \ell)$ matrix $P(0)P(0)'$ is nonsingular, and one solution to (24.19) is given by

$$\begin{bmatrix} -x_{\text{eq}} \\ u_{\text{eq}} \end{bmatrix} = P(0)'\big(P(0)P(0)'\big)^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}. \quad (24.20)$$

Also in this case, $s = 0$ should not be an invariant zero of the system, because otherwise $P(0)$ cannot be full rank.

**Attention!** This Rosenbrock's matrix is obtained by regarding the controlled output $z$ as the only output of the system.

**Note.** Recall that a transmission zero of a transfer matrix is always an invariant zero of its state-space realizations (see Theorem 19.2).

**Note.** We shall confirm in Exercise 24.4 that (24.20) is indeed a solution to (24.19). ► p. 304

**Note.** $P(0)'\big(P(0)P(0)'\big)^{-1}$ is called the *pseudoinverse* of $P(0)$ (see Definition 17.2).

(Knowing that $P(s) := \begin{bmatrix} sI - A & B \\ -G & H \end{bmatrix}$ from the book $\Rightarrow P(0) = \begin{bmatrix} -A & B \\ -G & H \end{bmatrix}$ (*)

Then, substituting (*) into 24.19

$\Rightarrow P(0)\begin{bmatrix} -x_{eq} \\ u_{eq} \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$ (**)

Then, substituting in 24.20 into (**)

$\Rightarrow P(0)P(0)^T(P(0)P(0)')^{-1}\begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$

let $M = P(0)\cdot P(0)^T$

$\Rightarrow \cancel{M}\cancel{M^{-1}}\begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \Rightarrow$ (24.20) is a solution to (24.19)

Modified 2.1 Problem from Lavretksy (20 points)

Consider the following problem:

$$\begin{aligned} \dot{x}_1 &= x_2 & t_0 &= 0 \\ \dot{x}_2 &= -2x_1 - 3x_2 + u & T &= 10 \end{aligned} \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$J = \int_0^T \left(x_1^2 + u^2\right) d\tau + x_1^2(T) + x_2^2(T)$$

Do the following

- Express the system matrices
- Express the LQR matrices
- Plot the solution to the Riccati equation using ode45 as well as the numeric solution to the Riccati equation from Section 5 Lecture 2
  - Each element of P should be in its own subplot
  - Each subplot should have two plots, one for the ode45 solution and one for the numeric solution from lecture
  - Use a time step of 0.1
- Plot the difference of the solutions to the Ricatti equation, one subplot for each element of P
- Simulate the system twice using the optimal control and overlay the state plots (both solutions should use the exponential numeric solution to the Riccati equation from lecture)
  - Use ode45 to simulate the dynamics
  - Use Euler integration to simulate the dynamics

Exercise 2.1. Consider

$$\begin{aligned} \dot{x}_1 &= x_2 & t_0 &= 0 \\ \dot{x}_2 &= -2x_1 - 3x_2 + u & T &= 1 \end{aligned} \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$J = \int_0^T (x_1^4 + u^2) d\tau + x_1^2(T) + x_2^2(T)$$

Set up (but not solve) the HJB equation with the corresponding boundary conditions.

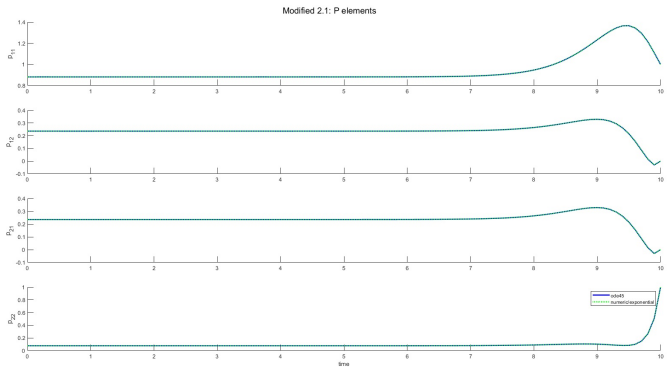
## System Matrices

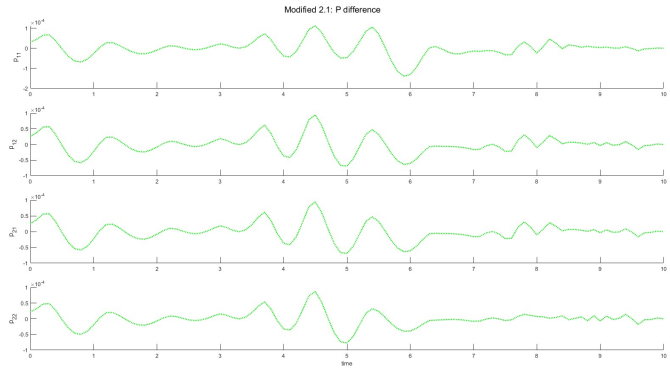
$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

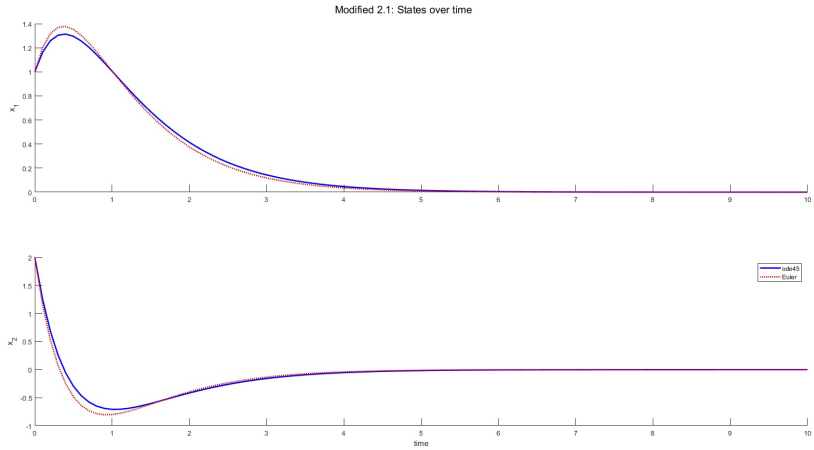
## LQR Matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = [1] \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

See Modified 2_1.m for code, plots below



LQR using Bryson's method (20 points)

Consider the following problem

$$\begin{aligned} \dot{x}_1 &= 3x_1 + 6x_2 + 4x_3 - \tfrac{2}{3}u_1 + \tfrac{1}{3}u_2 \\ \dot{x}_2 &= 9x_1 + 6x_2 + 10x_3 + \tfrac{1}{3}u_1 - \tfrac{2}{3}u_2 \\ \dot{x}_3 &= -7x_1 - 7x_2 - 9x_3 + \tfrac{1}{3}u_1 + \tfrac{1}{3}u_2 \end{aligned} \qquad t_0 = 0 \quad t_f = 1\,, x_0 = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

Do the following

- Express the system matrices
- Is this system controllable or stabilizable? Justify your answer.
- Express the LQR matrices using Bryson's method with the following bounds
  - Transient bounds:
    - $x_{1,max} = 1 \quad u_{1,max} = 5$
    - $x_{2,max} = 100 \quad u_{2,max} = 10$
    - $x_{3,max} = 100$
  - Terminal bounds:
    - $x_{1,max} = 10$
    - $x_{2,max} = 1$
    - $x_{3,max} = 2$
- Plot the solution to the Riccati equation using ode45 as well as the numeric solution to the Riccati equation from Section 5 Lecture 2
  - Each element of P should be in its own subplot
  - Each subplot should have two plots, one for the ode45 solution and one for the numeric solution from lecture
  - Use a time step of 0.01
- Plot the difference of the solutions to the Ricatti equation, one subplot for each element of P
- Simulate the system twice using the optimal control and overlay the state plots (both solutions should use the exponential numeric solution to the Riccati equation from lecture)
  - Use ode45 to simulate the dynamics
  - Use Euler integration to simulate the dynamics
- Repeat the state simulation with everything the same, except $S = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$
  - What is different in the state at the final time using this new value of S?

## System Matrices

$$\dot{x} = \underbrace{\begin{bmatrix} 3 & 6 & 4 \\ 9 & 6 & 10 \\ -7 & -7 & -9 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}}_{B} u$$

## System controlable/stabilizable?

$\Gamma = \text{ctrb}(A, B) \qquad T = [\text{orth}(\Gamma) \;\; \text{null}(\Gamma)]$

$$\hat{A} = T^{-1}AT = \begin{bmatrix} -3.1820 & -1.5473 & -7.1095 \\ 0.1347 & -1.8180 & -19.1169 \\ 0 & 0 & 5 \end{bmatrix} \qquad \hat{B} = T^{-1}B = \begin{bmatrix} -0.6563 & 0.3488 \\ 0.4857 & 0.3256 \\ 0 & 0 \end{bmatrix}$$

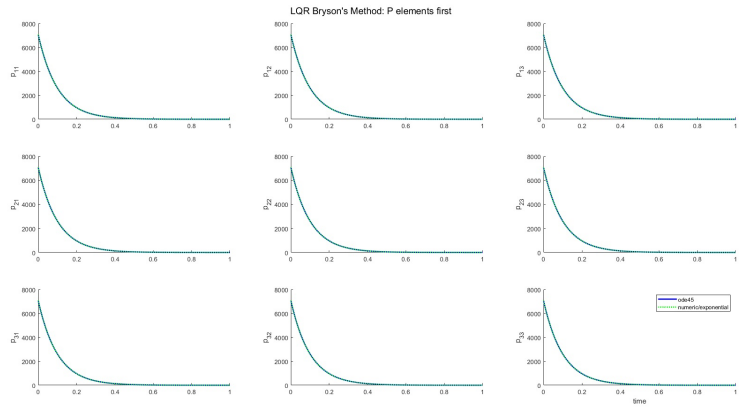
$\hookrightarrow \hat{A}_{22}$

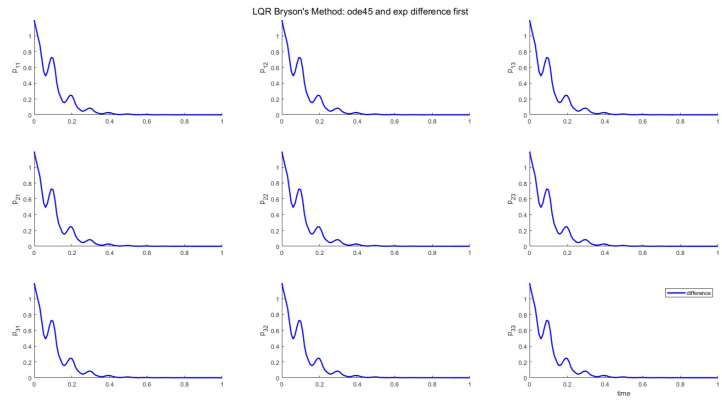
Since $\exists\, \lambda \in \text{eig}(\hat{A}_{22})$ s.t. $\text{Re}(\lambda) > 0$, the system is NOT controllable and NOT stabilizable

$\hookrightarrow$ Still can proceed with LQR control

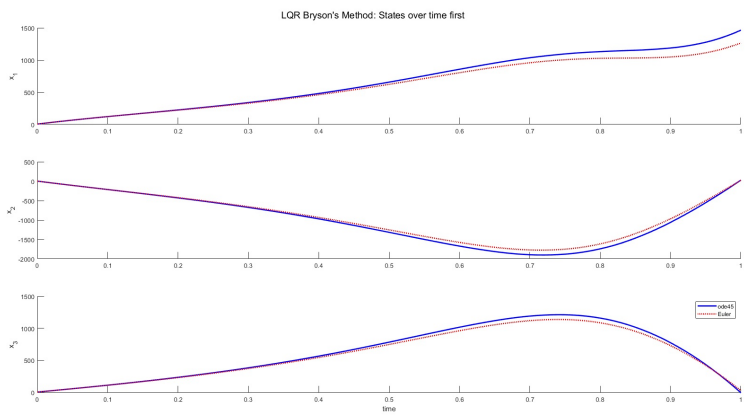
Choose $Q = \begin{bmatrix} 1/1 & 0 & 0 \\ 0 & 1/100^2 & 0 \\ 0 & 0 & 1/100^2 \end{bmatrix}$, $R = \begin{bmatrix} 1/5^2 & 0 \\ 0 & 1/10^2 \end{bmatrix}$, $S = \begin{bmatrix} 1/10^2 & 0 & 0 \\ 0 & 1/1^2 & 0 \\ 0 & 0 & 1/2^2 \end{bmatrix}$

## Plotting ODE45 and numeric Riccati solution

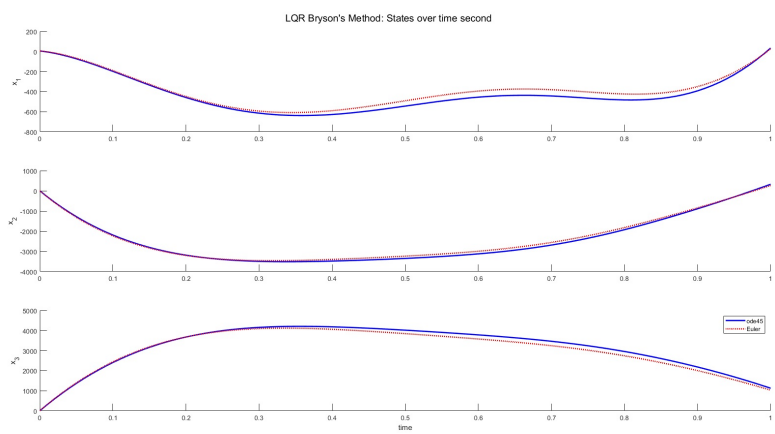



$S = \begin{bmatrix} \frac{1}{10^2} & 0 & 0 \\ 0 & \frac{1}{1^2} & 0 \\ 0 & 0 & \frac{1}{2^2} \end{bmatrix}$:




$S = \begin{bmatrix} 10 & 0 & 0 \\ 0 & \frac{1}{1^2} & 0 \\ 0 & 0 & \frac{1}{2^2} \end{bmatrix}$:




Note how the first state converges closer to 0 by the end of the finite time LQR simulation. This makes sense since changing S so that the top left value is much larger makes the “importance” of that corresponding state not going to zero by the final time relatively more punishing in the cost function. Therefore, we would expect that state to get closer to zero at the finite time, which we see in two plots above.

Exercise 2.3 (20 points)

Solve by hand, do not use any computational tools such as Matlab

Exercise 2.3. For

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 + u$$

$$J = \int_0^\infty (x_1^2 + u^2)\,d\tau$$

find the LQR-optimal control policy $u$ to minimize the cost $J$.

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad R = 1$$

LQR - optimal control

$$u = -R^{-1} B^T P x$$

ARE

Solving for $P$ → For optimal control: $0 = A^T P + PA + Q - PBR^{-1}B^T P$

$$0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T P + P \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - P \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T P$$

$$\Rightarrow 0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

$$\Rightarrow 0 = \begin{bmatrix} P_{12} & P_{22} \\ P_{11} & P_{12} \end{bmatrix} + \begin{bmatrix} P_{12} & P_{11} \\ P_{22} & P_{12} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ P_{12} & P_{22} \end{bmatrix}$$

$$\Rightarrow 0 = \begin{bmatrix} 2P_{12} + 1 & P_{22} + P_{11} \\ P_{11} + P_{22} & 2P_{12} \end{bmatrix} - \begin{bmatrix} P_{12}^2 & P_{12}P_{22} \\ P_{12}P_{22} & P_{22}^2 \end{bmatrix}$$

$$\Rightarrow 0 = \begin{bmatrix} \underbrace{-P_{12}^2 + 2P_{12} + 1}_{1} & \underbrace{P_{22} + P_{11} - P_{12}P_{22}}_{2} \\ \underbrace{P_{22} + P_{11} - P_{12}P_{22}}_{2} & \underbrace{2P_{12} - P_{22}^2}_{3} \end{bmatrix}$$

$$\Rightarrow \underbrace{0 = -P_{12}^2 + 2P_{12} + 1}_{1} \qquad \underbrace{0 = P_{22} + P_{11} - P_{12}P_{22}}_{2} \qquad \underbrace{0 = 2P_{12} - P_{22}^2}_{3}$$

① : $P_{12} = \dfrac{-2 \pm \sqrt{4+4}}{-2} = 1 \pm \sqrt{2}$ Choose $P_{12} = 1+\sqrt{2} = 2.4142$

③ : $0 = 2(1+\sqrt{2}) - P_{22}^2 \Rightarrow P_{22} = \dfrac{\pm\sqrt{-4(2+2\sqrt{2})(-1)}}{-2}$

$\Rightarrow P_{22} = \dfrac{\pm\sqrt{8+8\sqrt{2}}}{-2} = \dfrac{\pm 2\sqrt{2+2\sqrt{2}}}{-2}$ Choose $P_{22} = \sqrt{2+2\sqrt{2}} = 2.1974$

② : $0 = \sqrt{2+2\sqrt{2}} + P_{11} - (1+\sqrt{2})(\sqrt{2+2\sqrt{2}})$

$\Rightarrow 0 = \cancel{\sqrt{2+2\sqrt{2}}} - \cancel{\sqrt{2+2\sqrt{2}}} - \sqrt{2}\sqrt{2+2\sqrt{2}} + P_{11}$

$\Rightarrow P_{11} = \sqrt{2}\sqrt{2+2\sqrt{2}} = \sqrt{2}\sqrt{2}\sqrt{1+\sqrt{2}} = 2\sqrt{1+\sqrt{2}} = 3.1075$

$$\Rightarrow P = \begin{bmatrix} 2\sqrt{1+\sqrt{2}} & 1+\sqrt{2} \\ 1+\sqrt{2} & \sqrt{2+2\sqrt{2}} \end{bmatrix}$$

Full control

$$u = -R^{-1}B^T P x = -1\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 2\sqrt{1+\sqrt{2}} & 1+\sqrt{2} \\ 1+\sqrt{2} & \sqrt{2+2\sqrt{2}} \end{bmatrix} x$$

$$\Rightarrow u = -\underbrace{\begin{bmatrix} 1+\sqrt{2} & \sqrt{2+2\sqrt{2}} \end{bmatrix}}_{K} x$$

Exercise 2.4 (20 points)

Feel free to use Matlab to help in the control design

Explicitly state your resulting control and the cost matrices you used to generate the control (i.e., Q, and R)

Turn in plots of the simulation (make sure the simulation time is sufficient to show convergence)

Exercise 2.4. Consider the longitudinal aircraft dynamics given in Chap. 1, Exercise 1.2. This linear model represents the aircraft incremental dynamics, with respect to a trim condition. Design an infinite-time LQR to regulate the state vector to zero. Simulate the design with an initial state vector $x(0) = [\,10 \text{ ft/s} \quad 0.1 \text{ rad} \quad 0.1 \text{ rad/s} \quad 0 \text{ rad}\,]$.

let $x = \begin{bmatrix} V \\ \alpha \\ q \\ \theta \end{bmatrix}$, $u = \begin{bmatrix} \delta_{th} \\ \delta_e \end{bmatrix}$

Exercise 1.2. Consider the longitudinal dynamics

$$\begin{pmatrix} \dot{V} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} -0.038 & 18.984 & 0 & -32.174 \\ -0.001 & -0.632 & 1 & 0 \\ 0 & -0.759 & -0.518 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} V \\ \alpha \\ q \\ \theta \end{pmatrix} + \underbrace{\begin{pmatrix} 10.1 & 0 \\ 0 & -0.0086 \\ 0.025 & -0.011 \\ 0 & 0 \end{pmatrix}}_{B} \begin{pmatrix} \delta_{th} \\ \delta_e \end{pmatrix}$$

Choose $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

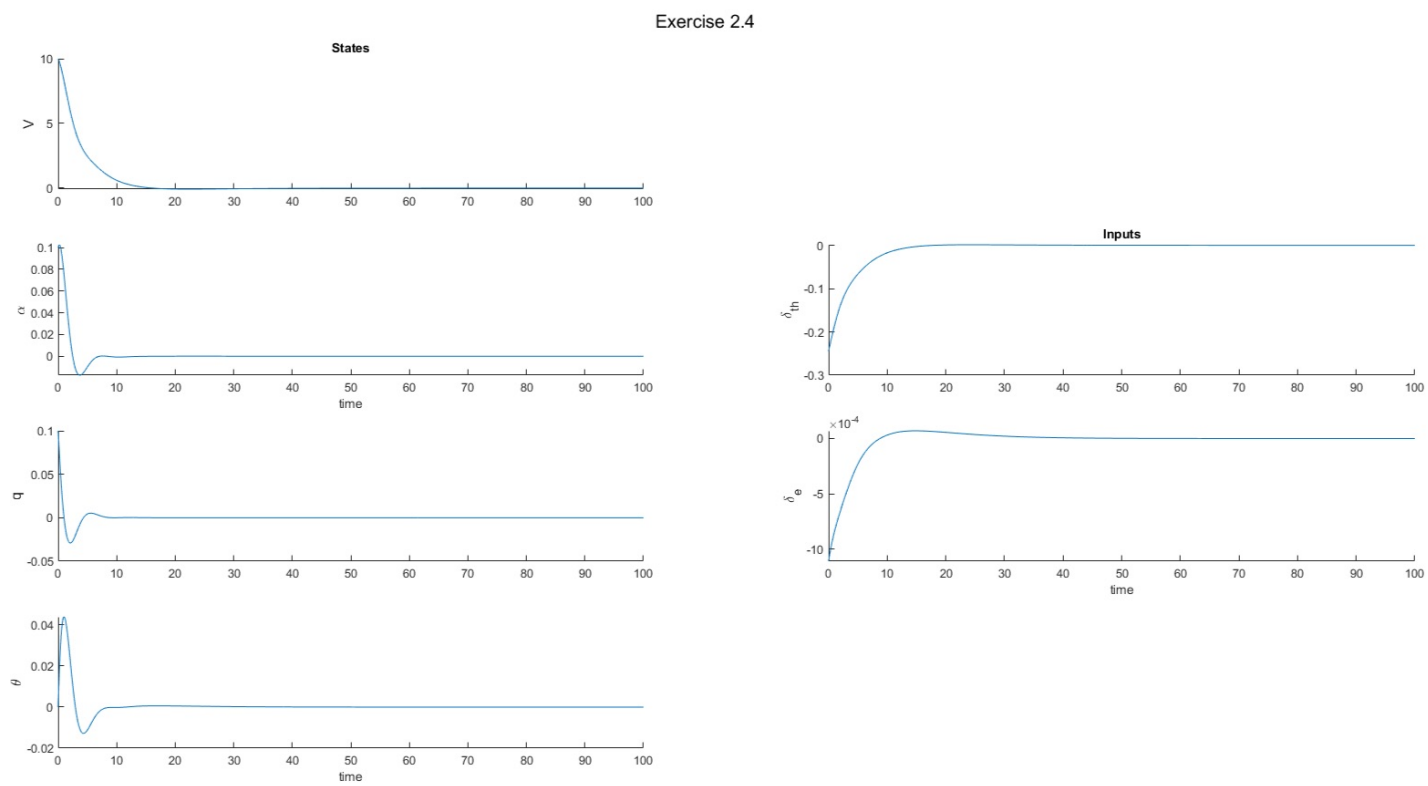
$R = \begin{bmatrix} 1000 & 0 \\ 0 & 2000 \end{bmatrix}$ Try prioritizing small control

choose K= lqr(A,B,Q,R);

$\Rightarrow K = \begin{bmatrix} 0.0271 & -1.8755 & 1.6048 & 2.3006 \\ 0 & 0.0747 & -0.0645 & -0.1097 \end{bmatrix}$

Choose $u = -Kx$

Simulation Results (See Ex2-4.m)



Exercise 2.5. Consider a second-order system modeled by the input-output equation $\ddot{y} = u$. A feedback controller $u = -k_1 y - k_2 \dot{y}$ is to be designed such that the performance index

$$J = \int_0^\infty (4y^2 + u^2) d\tau$$

is minimized.

1. Find $k_1$ and $k_2$.
2. What are the closed-loop eigenvalues?

let $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

1. $\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Choose $Q = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ $R = [1]$

let $K = [k_1 \quad k_2] = \text{lqr}(A, B, Q, R)$

$\Rightarrow K = [2 \quad 2] \Rightarrow \underline{k_1 = 2, \; k_2 = 2}$

2. $\text{eig}(A - BK) = \underline{\{-1 \pm j\}}$

Output Feedback Control Design using Decompositions (30 points)

For the following system $\dot{x} = Ax + Bu, y = Cx$, compute a control design using output feedback control (note that this is possible) to converge to $x_d = \begin{bmatrix} -\frac{10}{3} & 3 & -3 & \frac{43}{12} & 0 \end{bmatrix}^T$. Use LQR. For sake of simplicity, always choose Q = diag([1, 2, 3, ...]) and R = diag([1, 2, ...]) for whatever size you need.

To avoid mistakes with entering in the A, B, and C matrices, a preliminary matlab script is provided for you: decomposition_problem.m

The system matrices are defined by

$$A = \begin{bmatrix} 3 & 0 & -4 & 0 & 0 \\ 0 & 2.2 & 0.8 & 0 & 3.6 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 4 & 4 & 0 \\ 0 & -1.4 & -0.6 & 0 & -3.2 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 0 \\ 0.8 & 0.2 & 0.6 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \\ -0.6 & -0.4 & -0.2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

## Change of state

let $z = x - x_d$

$Bu_{ff} + Ax_d - \dot{x}_d = 0 \Rightarrow u_{ff} = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}$

### Control

$u = \hat{u} + u_{ff}$

## Check Controllability

$\text{rank}(\Gamma) = 4 > 5 \Rightarrow$ not completely controllable

### Do controllable decomposition

(See decomposition_problem.m for full work)

$T = [\text{orth}(\Gamma) \ \text{null}(\Gamma^T)]$

$\hat{A} = T^{-1}AT = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & -2 \end{bmatrix}$, $\hat{B} = T^{-1}B = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}$

Stabilizable

$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\hat{Q} = T^T Q T = \begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{bmatrix}$, use $\hat{Q}_{11}$ in lqr

### Pole placement

$\hat{u} = -Kz$

$\hat{K}_1 = \text{lqr}(\hat{A}_{11}, \hat{B}_1, \hat{Q}_{11}, R) \Rightarrow \hat{K}_1 = \begin{bmatrix} -14.50 & 0.66 & -2.87 & 5.68 \\ -14.36 & -6.16 & -1.06 & 0.48 \\ 1.89 & -0.69 & 1.65 & -3.92 \end{bmatrix}$

$$\Rightarrow \hat{K} = \begin{bmatrix} \hat{K}_1 & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{bmatrix} \Rightarrow K = \hat{K} T^{-1}$$

$$\Rightarrow K = \begin{bmatrix} 3.57 & -2.87 & 9.10 & 14.03 & -2.08 \\ 10.05 & -0.15 & 1.16 & 11.96 & -0.33 \\ 0.13 & 3.08 & -2.93 & -2.01 & -0.05 \end{bmatrix}$$

## Full control (use $\hat{x}$ as $x$)

$$u = \hat{u} + u_{ff} = -Kz + u_{ff}$$

$$\Rightarrow u = -K(\hat{x} - x_d) + u_{ff}$$

## Observer design

let $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

Observability $\text{rank}(\Omega) = 4, \Rightarrow$ not C.O.

Observable Decomposition

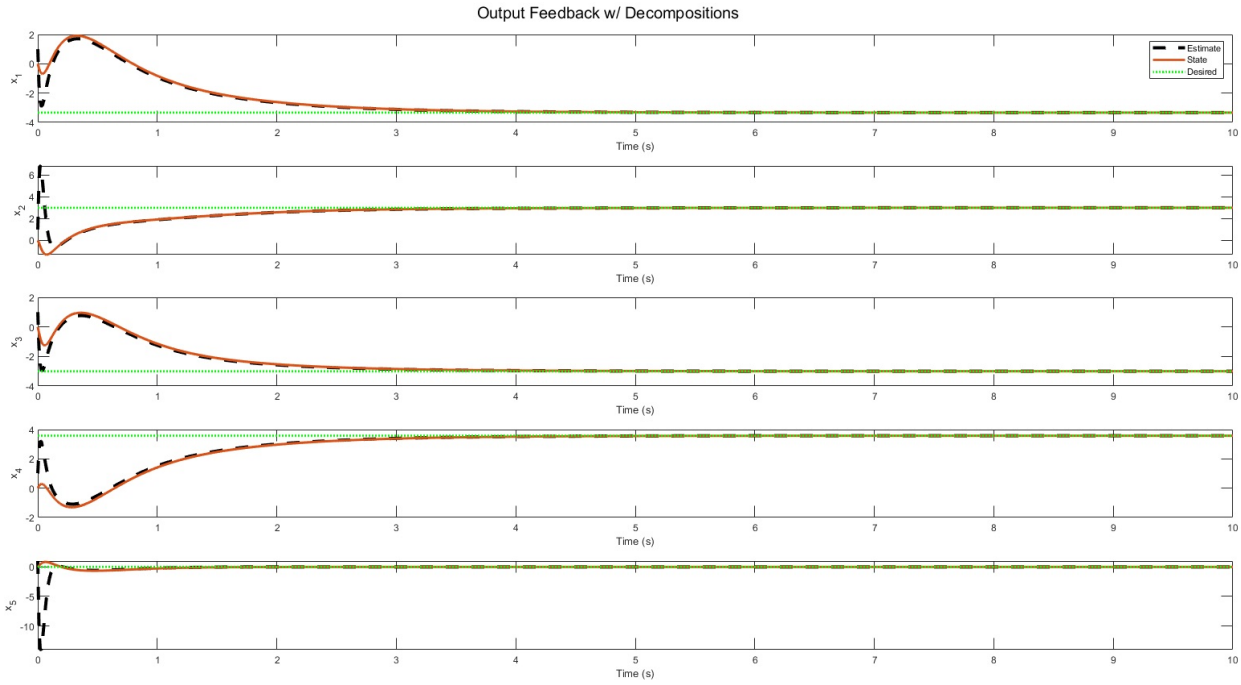
$$T = [\text{orth}(\Omega') \quad \text{null}(\Omega)]$$

$$\hat{A} = T^{-1}AT = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & -1 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}$$

detectable

10 times poles of A-BK (see code)

Choose $\hat{L}_1$ >> $\hat{L}_1$ = place($\hat{A}_{11}^T$, $\hat{C}_{11}^T$, poles);

$$\Rightarrow L = T \begin{bmatrix} \hat{L}_1 \\ [0 \ 0 \ 0] \end{bmatrix} = \begin{bmatrix} 19.73 & -3.11 & 88.97 \\ -0.45 & 1.02 & -236.70 \\ -2.43 & 9.66 & 88.97 \\ 16.99 & 7.29 & -88.97 \\ 0.56 & -1.37 & 548.89 \end{bmatrix}$$

# Simulation Results