10/26/24 | ECE 6320 / HW 9 | Howes, Kade

Control design using the controllability decomposition (30 points)

For the following systems $\dot{x} = Ax + Bu$:

- Compute a control design **using LQR**. For sake of simplicity, always choose Q = diag([1, 2, 3, ...]) and R = diag([1, 2, ...]) for whatever size you need.
- If the system is stabilizable, turn in the feedback control in terms of $u = -Kx$ (i.e., do not leave it in terms of $\hat{x}$). Otherwise, indicate that the system is not stabilizable and show that it is not stabilizable with the controllability decomposition.
- Calculate and turn in the resulting eigenvalues for the closed loop system.

1) $A = \begin{bmatrix} -2.2 & 0 & -0.4 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0.6 & 0 & -0.8 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$

using MATLAB (see Homework9.m for all details, including decomposition)

- System is stabilizable: $u = -Kx = -\begin{bmatrix} 0 & 8.4075 & 0 & 1.8855 & 7.1582 \\ 0 & 7.9588 & 0 & 7.0166 & 5.9493 \end{bmatrix} x$

- Closed loop eigenvalues $= \{-2, -1, -0.8936, -3.4949, -3.1445\}$

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- Calculate and turn in the resulting eigenvalues for the closed loop system.

$$2)\ A = \begin{bmatrix} 2.2 & 0 & 0.4 & 0 & 0 \\ 0 & 2 & 0 & -1 & 3 \\ -0.6 & 0 & 0.8 & 0 & 0 \\ 0 & 2 & 0 & 5 & -2 \\ 0 & 2 & 0 & 2 & 2 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 & -.2 \\ 0 & -1 & 0 \\ 0 & 0 & 0.6 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

- System is not stablizable.

Using transform $T = [\mathrm{orth}(\Gamma),\ \mathrm{null}(\Gamma')]$

$$\Rightarrow \hat{A} = T^{-1}AT = \begin{bmatrix} 4.7527 & -0.1462 & -1.7845 & 0 & 0 \\ 0.3345 & 3.0228 & 4.7385 & 0 & 0 \\ -0.0102 & -0.0171 & 1.2244 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

with the lower right block labelled $\hat{A}_{22}$.

$$\Rightarrow \hat{A}_{22} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$ which is uncontrolled

and $\mathrm{eig}(\hat{A}_{22}) = \{1, 2\} \Rightarrow$ not stabilizable

Control design using the controllability decomposition (30 points)

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- If the system is stabilizable, turn in the feedback control in terms of $u = -Kx$ (i.e., do not leave it in terms of $\hat{x}$). Otherwise, indicate that the system is not stabilizable and show that it is not stabilizable with the controllability decomposition.
- Calculate and turn in the resulting eigenvalues for the closed loop system.

3) $A = \begin{bmatrix} 5 & 0 & -2 & 2 & 0 \\ 0.4 & 1.6 & 0.8 & 0.4 & 1.8 \\ 2 & 0 & 2 & 2 & 0 \\ -1 & 0 & 3 & 2 & 0 \\ -0.8 & -1.2 & -1.6 & -0.8 & -2.6 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 0 \\ 0.8 & 0.2 & 0.6 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \\ -0.6 & -0.4 & -0.2 \end{bmatrix}$

using MATLAB (see Homework4.m for all details, including decomposition)

- System is stabilizable:

$$u = -Kx = -\begin{bmatrix} 3.4564 & -1.2883 & 9.4394 & 13.0581 & -2.7170 \\ 9.5134 & 0.4328 & 1.3574 & 10.5340 & -0.5967 \\ 2.2275 & 3.5316 & -3.0063 & 1.9722 & -0.1749 \end{bmatrix} x$$

- Closed loop eigenvalue $= \{-4.7512, -1.2631, -3.1768 \pm 0.2893j, -2\}$

Section 2.6 - Control Design (40 points)

- For problem 2.3 (a) and 2.3 (c) do the following (assume g = 9.8, m = 1/9.8, l = 0.25, and b = 1) (20 points)
  - Create a feedback controller that will keep the system at the equilibrium point (or state that it is not possible using LTI control techniques). If possible:
    - Explicitly write the full control input $u(t)$
    - Evaluate the stability of the equilibrium point given the feedback control (i.e., characterize the stability of the system $\delta\dot{x} = (A - BK)\delta x$)
    - Show plots vs time for $x_1, x_2,$ and T for $t \in [0, 10]$ with the following initial states for each portion:
      - (a) $x_0 = [0.1, 0.0]^T$, and $x_0 = [\pi - 0.1, 0.0]^T$
      - (c) $x_0 = [\frac{\pi}{4} - 0.1, 0.0]^T$, and $x_0 = [\pi - 0.1, 0]^T$
  - Hints:
    - All the code you need for plotting is in S02_L03_PendulumEnergy.m. You need to create functions for the feedback control and replace "@zeroControl" with your functions on line 13.
    - Be careful to include your feed-forward term in the control.
    - Ensure all of your parameters are defined correctly (i.e., g, m, b, and l need to be defined in the code as they are in the problem statement)

2.3 a) Linearization from Homework 2: $x^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $u^{eq} = 0$, $y^{eq} = 0$

0) $\Rightarrow \delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}}_{B} \delta u$

$y = g(x, u) = x_1$

$\Rightarrow \delta y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \delta x + \underbrace{0}_{D=0} \delta u$

1-3) where $\delta x = x - x^{eq}$, $\delta y = y - x^{eq}$, $\delta u = u - u^{eq}$

$x_d = x^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \dot{x}_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4) Check controllability: $\Gamma = [B \;\; AB] \Rightarrow \text{rank}(\Gamma) = 2 \Rightarrow$ C.C.

5) Place poles: $\delta u = -K\delta x$, $\delta\dot{x} = (A - BK)\delta x = \bar{A}\delta x$

using lqr() in MATLAB with $Q = \text{diag}([1 \; 2])$, $R = 1$

$\Rightarrow K = [1.2808 \quad 0.7368]$

6) Express Full Control: $u = u^{eq} + \delta u = u^{eq} - K\delta x$

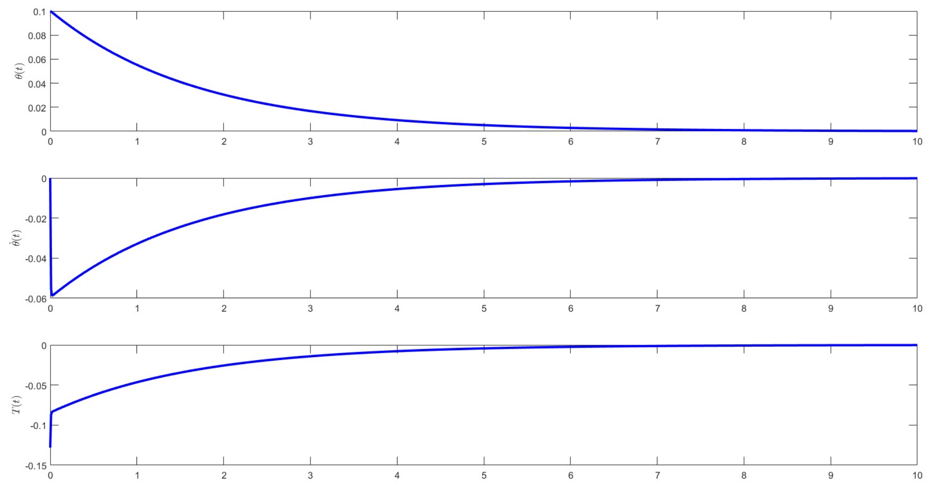
$u = u^{eq} - K(x - x^{eq})$

$u = -[1.2808 \;\; 0.7368] x$

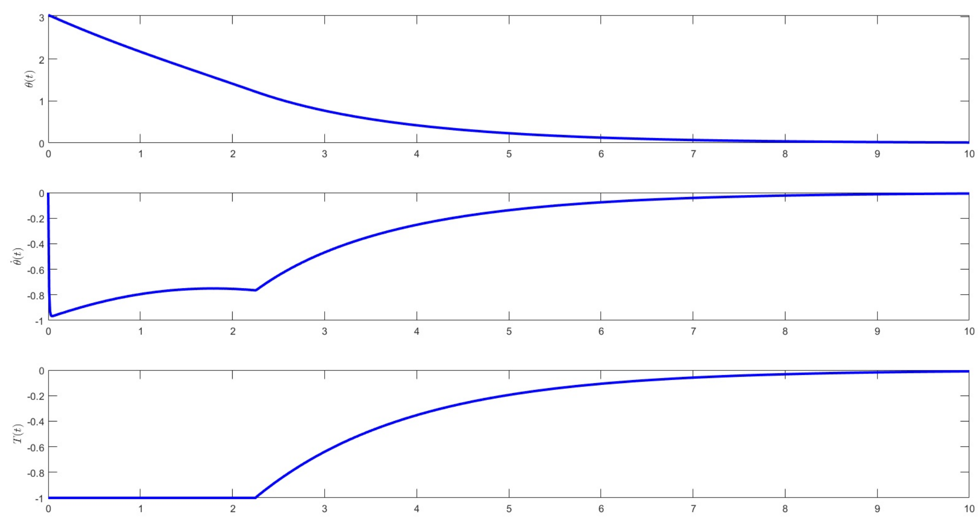
Eigenvalues of $\bar{A} = A - BK$: $\{-0.5948, -271.7292\}$

$\hookrightarrow$ Stable since $Re(\lambda) < 0 \;\; \forall \lambda \in eig(\bar{A})$

$X_0 = [0.1,\ 0]^T$



$X_0 = [\pi - 0.1,\ 0]^T$



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2.3 c) Linearization from Homework 2

$$\delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{l\sqrt{2}} & -\frac{b}{ml^2} \end{bmatrix}}_{A} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \delta u$$

$y = g(x, u) = x_1$

$$x^{eq} = \begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}, \quad u^{eq} = \frac{-glm}{\sqrt{2}}$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

where $\delta x = x - x^{eq}$, $\delta y = y - x^{eq}$, $\delta u = u - u^{eq}$

where $x_d = x^{eq} = \begin{bmatrix} \pi/4 \\ 0 \end{bmatrix} \Rightarrow \dot{x}_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4) Check controllability: $\Gamma = [B \ AB] \Rightarrow \text{rank}(\Gamma) = 2 \Rightarrow$ C.C.

5) Place poles: $\delta u = -K\delta x$, $\delta\dot{x} = (A - BK)\delta x = \bar{A}\delta x$

using lqr() in MATLAB with Q = diag([1 2]), R = 1

$\Rightarrow K = [1.1923 \ \ 0.7364]$

6) Express Full control: $u = u^{eq} + \delta u = u^{eq} - K\delta x$

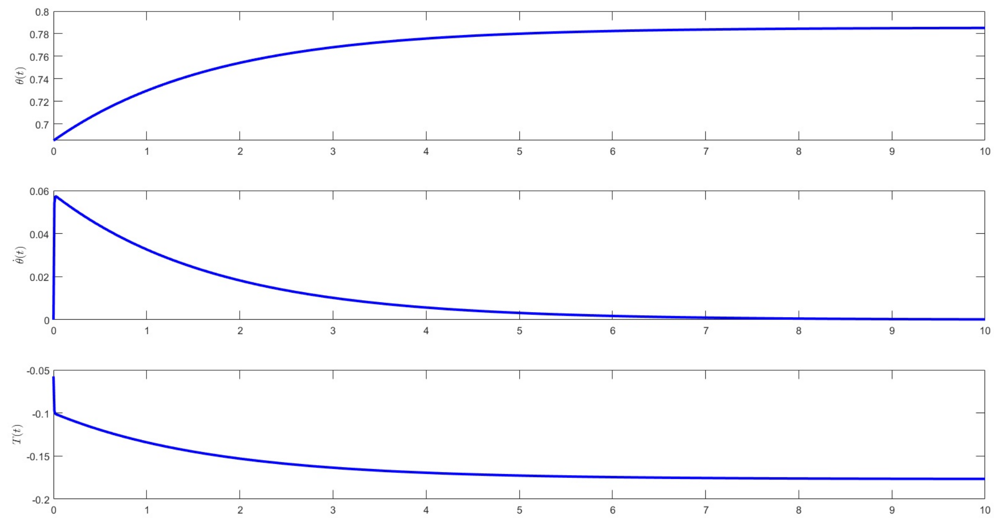
$u = u^{eq} - K(x - x^{eq})$

$$u = -0.1768 - [1.1923 \ \ 0.7364]\left(x - \begin{bmatrix} \pi/4 \\ 0 \end{bmatrix}\right)$$

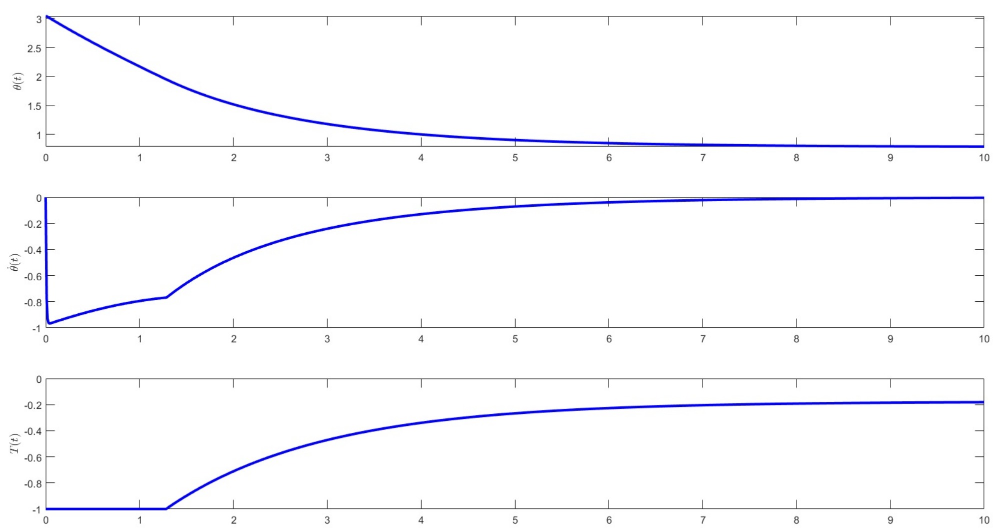
Eigenvalues of $\bar{A} = A - BK$: $\{-0.5861 \ \ -271.6870\}$

$\hookrightarrow$ stable since $Re(\lambda) < 0 \ \forall \lambda \in es(\bar{A})$

$X_0 = [\pi/4 - 0.1,\ 0]^T$



$X_0 = [\pi - 0.1,\ 0]^T$



- 2.8 (20 points)
  - Ignore the Simulink simulation
  - Assume that "using LTI control techniques" corresponds to the control techniques taught in class (i.e., placing the eigenvalues)

2.8 (Feedback linearization controller). Consider the inverted pendulum in Figure 2.8.

(a) Assume that you can directly control the system in torque (i.e., that the control input is $u = T$).
Design a feedback linearization controller to drive the pendulum to the upright position. Use the following values for the parameters: $\ell = 1\,\mathrm{m}$, $m = 1\,\mathrm{kg}$, $b = 0.1\,\mathrm{N\,m^{-1}\,s^{-1}}$, and $g = 9.8\,\mathrm{m\,s^{-2}}$. Verify the performance of your system in the presence of measurement noise using Simulink.

(b) Assume now that the pendulum is mounted on a cart and that you can control the cart's jerk, which is the derivative of its acceleration a. In this case,

$$T = -m\,\ell\,a\cos\theta, \quad \dot{a} = u.$$

Design a feedback linearization controller for the new system.

What happens around $\theta = \pm\pi/2$? Note that, unfortunately, the pendulum needs to pass by one of these points for a swing-up, which is the motion from $\theta = \pi$ (pendulum down) to $\theta = 0$ (pendulum upright). □

a)

$$x = \begin{bmatrix}\theta \\ \dot\theta\end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix}\dot\theta \\ \ddot\theta\end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{\ell}\sin x_1 - \frac{b}{m\ell^2}x_2 + \frac{1}{m\ell^2}u \end{bmatrix}$$

let $u = u_{ff} + \hat{u}$

$$\Rightarrow \dot{x} = \begin{bmatrix} x_2 \\ \frac{g}{\ell}\sin x_1 - \frac{b}{m\ell^2}x_2 + \frac{1}{m\ell^2}u_{ff} + \frac{1}{m\ell^2}\hat{u} \end{bmatrix}$$

Setting $\frac{g}{\ell}\sin(x_1) + \frac{1}{m\ell^2}u_{ff} = 0 \Rightarrow u_{ff} = -gm\ell\sin x_1$

$$\Rightarrow \dot{x} = \begin{bmatrix} x_2 \\ -\frac{b}{m\ell^2}x_2 + \frac{1}{m\ell^2}\hat{u} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m\ell^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}\hat{u}$$

Checking controllability: $\Gamma = [B \; AB] \Rightarrow \mathrm{rank}(\Gamma) = 2 \Rightarrow$ C.C. ✓

let $\hat{u} = -Kx$,

pole placement: $lqr(A, B, Q, R)$ in MATLAB with $Q = diag(1{:}n)$ $R = 1$

$$\Rightarrow K = [1 \quad 1.9025]$$

$$\Rightarrow u = u_{ff} + \hat{u} = -gm\ell\sin x_1 - [1 \quad 1.9025]x$$

b) $T = -mla\cos\theta \Rightarrow \ddot{\theta} = \frac{g}{l}\sin\theta - \frac{b}{ml^2}\dot{\theta} - \frac{a}{l}\cos\theta$

$\dot{a} = u$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \\ a \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 - mlx_3\cos x_1 \\ u \end{bmatrix}$$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \frac{g}{l}x_2\cos x_1 - \frac{b}{ml^2}x_3 - \frac{u}{l}\cos x_1 + \frac{a}{l}x_2\sin x_1 \end{bmatrix}$$

$\dot{a}$

choose $u = \frac{l}{\cos x_1}(\hat{u} + u_{ff})$

and $u_{ff} = \frac{g}{l}x_2\cos x_1 + \frac{a}{l}x_2\sin x_1$

$$\Rightarrow \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{b}{ml^2} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{B} \hat{u}$$

Check controllability: rank([B AB A²B]) = 3 ⇒ C.C.

choose $\hat{u} = -Kx$, choose K = lqr(A, B, diag(1:n), diag(1:m))

$\Rightarrow K = [-1 \quad -2.8058 \quad -2.8363]$

$$u = \frac{l}{\cos x_1}(\hat{u} + u_{ff}) = \frac{l}{\cos x_1}\left[-[-1 \ -2.8058 \ -2.8363]x + \frac{g}{l}x_2\cos x_1 + \frac{a}{l}x_2\sin x_1\right]$$

$$u = -\frac{l}{\cos x_1}[-1 \ -2.8058 \ -2.8363]x + gx_2 + ax_2\tan x_1$$

If theta = +/- pi/4, it can be seen that there are at least two effects. First, the control designed above is undefined at these points since cos(pi/4)=0. However, this is expected since the other effect is that our control input has no effect on the state dynamics at these points since our input exists only coinciding with a cosine term. This intuitively makes sense since when the pendulum is perfectly horizontal, any change we make to the horizontal movement of the car will have no effect on the angle of the pendulum or our states.

Cart-Pendulum (30 points)

Recall the cart-pendulum from homework 1 (described here).

1. Evaluate the stability of the equilibrium point corresponding to ($\theta = \pi$)
2. Create a controller to keep the pendulum in the upright configuration ($\theta = \pi$) (or indicate that this is not possible using LTI control techniques). If possible:
   - Explicitly write the full control input
   - Evaluate the stability of the equilibrium point given the feedback control
   - Plot the results for states and inputs for $t \in [0, 10]$ assuming $\theta(0) = \pi - 0.25$ and the remainder of the states are zero
3. Create a controller to have the pendulum stay at $\theta = \frac{3\pi}{4}$ (or indicate that this is not possible using LTI control techniques). If possible:
   - Explicitly write the full control input
   - Plot the results for states and inputs for $t \in [0, 10]$ assuming $\theta(0) = \pi - 0.25$ and the remainder of the states are zero

Hints

- Week 02 homework solutions have all of the simulation code needed for the Cart-Pendulum problem

x: cart horizontal position

θ: pendulum angle

Two equations from document

(1) $(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F$

(2) $(I+ml^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta$

let $z = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$

1. Finding A in MATLAB

$\Rightarrow \text{eig}(A) = \{0, -0.1428, -5.6041, 5.5651\}$

$\Rightarrow$ unstable since $\text{Re}(\text{eig}(A)) > 0$

2. $\text{rank}(\Gamma) = 4 \Rightarrow$ C.C.

Choose $K = \text{lqr}(A, B, \text{diag}(1{:}n), \text{diag}(1{:}m))$

$= [-1 \quad -2.3752 \quad 22.3744 \quad 4.6505]$

let $u = u^{eq} + \delta u$, $z = z^{eq} + \delta z$ $\qquad z^{eq} = \begin{bmatrix} 0 \\ 0 \\ \pi \\ 0 \end{bmatrix}$

and $\delta u = -K\delta z$

$\Rightarrow u = -K(z - z^{eq}) + u^{eq} \leftarrow$ full control

$u^{eq} = 0$

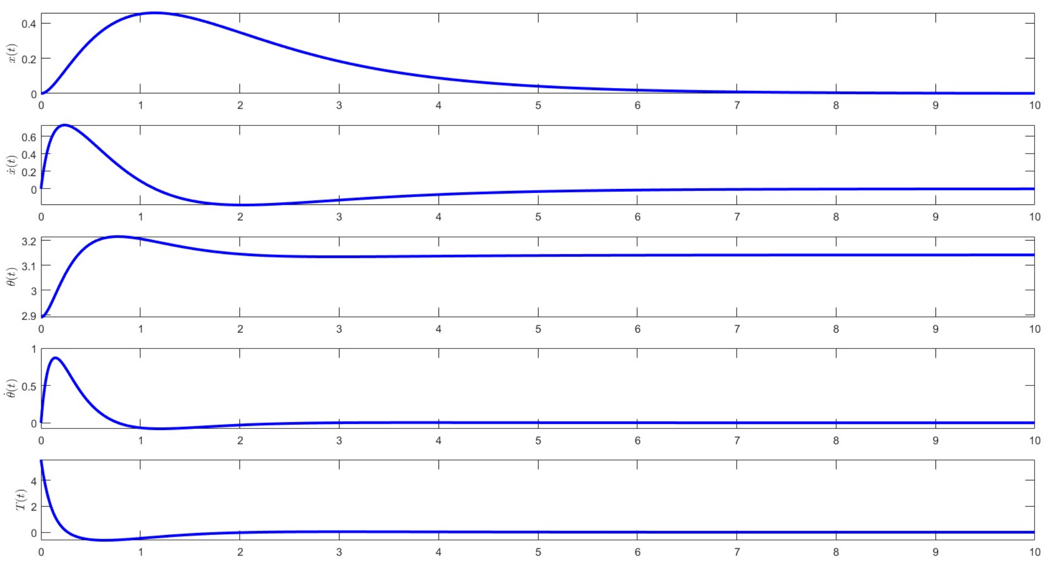
Evaluate stability of controlled system

$\tilde{A} = A - BK \Rightarrow \text{eig}(\tilde{A}) = \{-11.9279, -2.1554 \pm 0.4984j, -0.7631\}$

$\Rightarrow$ LAS since $\text{Re}(\text{eig}(\tilde{A})) < 0$

Plot simulation results

$$z(0) = \begin{bmatrix} 0 \\ 0 \\ \pi - 0.25 \\ 0 \end{bmatrix}$$



3. Using MATLAB:

$$z = z^{sol} + \delta z$$
$$u = u^{sol} + \delta u$$

$$z_{sol} = \begin{bmatrix} \frac{49}{10} t^2 \\ \frac{49}{5} t \\ \frac{3\pi}{4} \\ 0 \end{bmatrix} \Rightarrow u_{sol} = \frac{\dot{x}}{10} + \frac{343}{50}$$

$$u_{sol} = \frac{z_2}{10} + \frac{343}{50}$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.16 & 2.352 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.2828 & 38.806 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1.6 \\ 0 \\ 2.8284 \end{bmatrix}$$

where $\delta \dot{z} = A \delta z + B \delta u$

Using $K = lqr(A, B, diag(1{:}n), diag(1{:}m))$

$$\Rightarrow K = [-1 \quad -2.2913 \quad 41.0791 \quad 7.0094]$$

$$u = u^{sol} + \delta u \qquad z = z^{sol} + \delta z$$

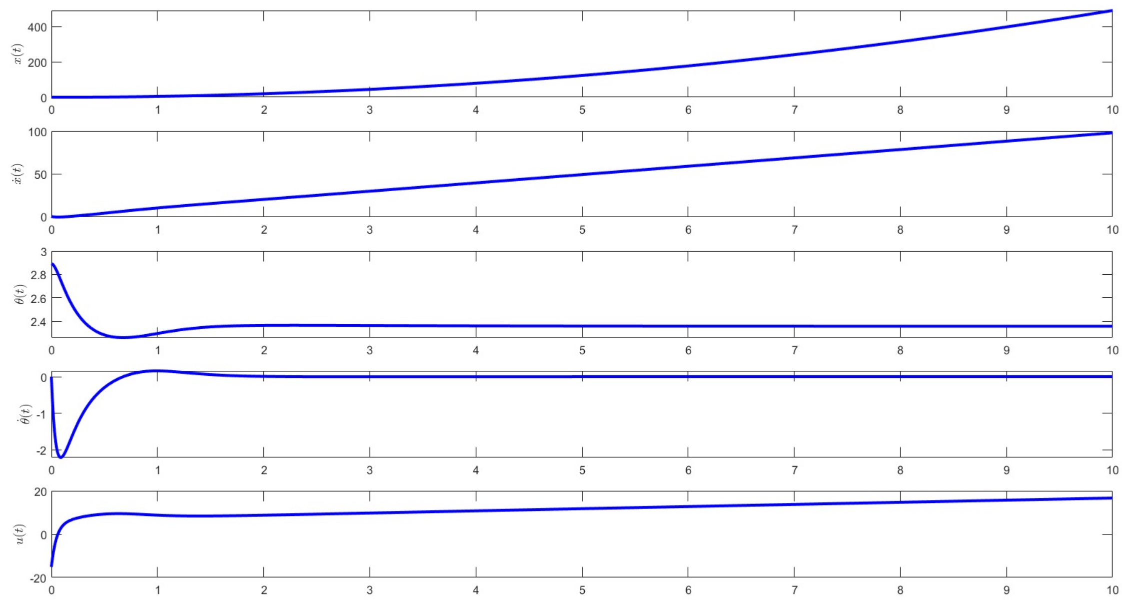
Choose $\delta u = -K \delta z$

$$\Rightarrow u = u^{sol} - K \delta z = u^{sol} - K(z - z^{sol})$$

Plot Simulation Results

$$z(0) = \begin{bmatrix} 0 \\ 0 \\ \pi - 0.25 \\ 0 \end{bmatrix}$$

$$\theta_{sol} = \frac{3\pi}{4}$$



1.1 Feedback Linearize

Consider the point, y_ϵ

$$y_\epsilon = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \epsilon \begin{bmatrix} \cos(\psi) \\ \sin(\psi) \end{bmatrix}, \tag{2}$$

where $\epsilon > 0$ is some pre-defined parameter as shown in Figure 1.

1.1.1 Derive the dynamics

Determine equations for $\dot{y}_\epsilon$ and $\ddot{y}_\epsilon$.

Hint 1. $\frac{d}{dt}\sin\psi = \omega\cos\psi$

Hint 2. *You should be able to use the following matrices and vectors in expressing your solution:*

$$R_\epsilon = \begin{bmatrix} \cos(\psi) & -\epsilon\sin(\psi) \\ \sin(\psi) & \epsilon\cos(\psi) \end{bmatrix}, \; \hat{\omega}_\epsilon = \begin{bmatrix} 0 & -\epsilon\omega \\ \frac{\omega}{\epsilon} & 0 \end{bmatrix}, \; \bar{a} = \begin{bmatrix} a \\ \alpha \end{bmatrix}, \; \bar{v} = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Hint 3. *The time derivative of the matrix R_ϵ can be written as:*

$$\dot{R}_\epsilon = R_\epsilon \hat{\omega}_\epsilon.$$

Thus, if you have an equation, $R_\epsilon q$ then you could express the time derivative as $R_\epsilon \dot{q} + R_\epsilon \hat{\omega}_\epsilon q$

Extra credit: Approximate control of unicycle robot (+15 points)

This problem will create a link between the point control discussed in S03_L05 Control Examples - Pre.pptx and vehicle control. Complete the problem detailed in Approximate_diffeomorphism.pdf in the homework 09 folder

$\psi = \omega t$?
$\dot{\psi} = \omega$

1.1.1

$$\dot{y}_\epsilon = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \epsilon \begin{bmatrix} -\omega\sin\psi \\ \omega\cos\psi \end{bmatrix} = \begin{bmatrix} v\cos\psi \\ v\sin\psi \end{bmatrix} + \epsilon \begin{bmatrix} -\omega\sin\psi \\ \omega\cos\psi \end{bmatrix}$$

$$\dot{y}_\epsilon = \begin{bmatrix} v\cos\psi - \epsilon\omega\sin\psi \\ v\sin\psi + \epsilon\omega\cos\psi \end{bmatrix} = R_\epsilon \bar{v}$$

$$\Rightarrow \ddot{y}_\epsilon = R_\epsilon \dot{\bar{v}} + R_\epsilon \hat{\omega} \bar{v}$$

$$\ddot{y}_\epsilon = R_\epsilon \bar{a} + R_\epsilon \hat{\omega} \bar{v}$$

1.1.2 Feedback Linearize

You should have found that $\ddot{y}_\epsilon$ is a function of $\bar{a}$, which is the input to the system. Find an equation for $\bar{a}$ (in terms a new variable u) which will render the dynamics to be of the form:

$$\ddot{y}_\epsilon = u. \tag{3}$$

Write the feedback linearized system in the form:

$$\begin{bmatrix} \dot{y}_\epsilon \\ \ddot{y}_\epsilon \end{bmatrix} = Ay + Bu, \text{ where } y = \begin{bmatrix} y_\epsilon \\ \dot{y}_\epsilon \end{bmatrix}$$

Hint 4. *$\bar{a}$ should be a function of*

$$R_\epsilon^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\epsilon} \end{bmatrix} \begin{bmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{bmatrix}$$

$$R_\epsilon \bar{a} + R_\epsilon \hat{\omega} \bar{v} = u \Rightarrow R_\epsilon \bar{a} = u - R_\epsilon \hat{\omega} \bar{v}$$

$$\Rightarrow \bar{a} = R_\epsilon^{-1}[u - R_\epsilon \hat{\omega} \bar{v}] = R_\epsilon^{-1} u - \hat{\omega}_\epsilon \bar{v}$$

$$\dot{y} = \begin{bmatrix} \dot{y}_\epsilon \\ \ddot{y}_\epsilon \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

1.2 Trajectory Tracking

Given a desired trajectory (y_d, $\dot{y}_d$, and $\ddot{y}_d$), design a control law which will have y_ϵ converge to and track the desired trajectory. Design it in terms of u and then state both the equations for u and $\bar{a}$.

Hint 5. *The control, u, should have three parts:*

1. *Feedforward term ($\ddot{y}_d$)*
2. *Error term on position*
3. *Error term on velocities*

$$y_{\epsilon d} = y_d$$
$$\dot{y}_{\epsilon d} = \dot{y}_d$$
$$\ddot{y}_{\epsilon d} = \ddot{y}_d$$

$$y_d = \begin{bmatrix} y_{\epsilon d} \\ \dot{y}_{\epsilon d} \end{bmatrix} \Rightarrow \dot{y}_d = \begin{bmatrix} \dot{y}_{\epsilon d} \\ \ddot{y}_{\epsilon d} \end{bmatrix}$$

Change of State

$$z = y - y_d$$
$$\dot{z} = \dot{y} - \dot{y}_d$$
$$= Ay + Bu - \dot{y}_d$$
$$= A(z + y_d) + Bu - \dot{y}_d$$
$$= Az + Bu + Ay_d - \dot{y}_d$$

let $u = \hat{u} + u_{ff}$

set $Bu_{ff} + Ay_d - \dot{y}_d = 0$

$$\begin{bmatrix} 0 \\ I \end{bmatrix} u_{ff} + \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{\epsilon d} \\ \dot{y}_{\epsilon d} \end{bmatrix} - \begin{bmatrix} \dot{y}_{\epsilon d} \\ \ddot{y}_{\epsilon d} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_{ff} = \ddot{y}_{\epsilon d}$$

let $\hat{u} = -Kz$

$$u = -Kz + \ddot{y}_{\epsilon d} = \underline{\ddot{y}_{\epsilon d} - K\left(y - \begin{bmatrix} y_{\epsilon d} \\ \dot{y}_{\epsilon d} \end{bmatrix}\right)}$$

$$\bar{a} = R_\epsilon^{-1} u + \hat{\omega}\bar{v} = \underline{R_\epsilon^{-1}\left(\ddot{y}_{\epsilon d} - K\left(y - \begin{bmatrix} y_{\epsilon d} \\ \dot{y}_{\epsilon d} \end{bmatrix}\right)\right) - \hat{\omega}\bar{v}}$$