4.5 (Observable canonical form). Given a transfer function $\hat{G}(s)$, let$(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be a realization for its transpose $\bar{G}(s) := \hat{G}(s)'$. Show that (A, B, C, D) with

$$A := \bar{A}', \qquad B := \bar{C}', \qquad C := \bar{B}', \qquad D := \bar{D}'$$

is a realization for $\hat{G}(s)$. □

Given $\bar{G}(s) = \hat{G}(s)^T$

$$\Rightarrow \hat{G}(s) = \bar{G}(s)^T$$

$$= \left(\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}\right)^T$$

$$= \left[\bar{C}(sI - \bar{A})^{-1}\bar{B}\right]^T + \bar{D}^T$$

$$= \bar{B}^T\left[(sI - \bar{A})^T\right]^{-1}\bar{C}^T + \bar{D}^T$$

$$= \bar{B}^T(sI^T - \bar{A}^T)^{-1}\bar{C}^T + \bar{D}^T$$

$$\underline{\hat{G}(s) = C(sI - A)^{-1}B + D}$$

4.6 (SISO realizations). This exercise aims at proving the theorem 4.4. Use the construction outlined in Section 4.3.2 to arrive at results consistent with the Theorem 4.4.

(a) Compute the controllable canonical form realization for the transfer function

$$\hat{g}(s) = \frac{k}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n}.$$

(b) For the realization in (a), compute the transfer function from the input u to the new output $y = x_i$, where x_i is the ith element of the state x.

Hint: You can compute $(sI - A)^{-1}b$ using the technique used in class for MIMO systems, or you may simply invert $(sI - A)^{-1}$ using the adjoint formula for matrix inversion:

$$M^{-1} = \frac{1}{\det M} \operatorname{adj} M, \qquad \operatorname{adj} M := (\operatorname{cof} M)', \qquad \operatorname{cof} M := [\operatorname{cof}_{ij} M],$$

where $\operatorname{cof}_{ij} M$ denotes the ijth cofactor of M. In this problem you actually need only to compute a single entry of $(sI - A)^{-1}$.

(c) Compute the controllable canonical form realization for the transfer function

$$\hat{g}(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n}. \tag{4.10}$$

(d) Compute the observable canonical form realization for the transfer function in equation (4.10).

Hint: See Exercise 4.5. □

a)

$$\lim_{s\to\infty} \hat{g}(s) = 0 \Rightarrow \underline{D = 0}$$

$$\hat{G}_{sp}(s) = \frac{1}{d(s)}(k) \quad \text{where} \quad d(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots \alpha_{n-1} s + \alpha_n$$

$$\hat{G}_2(s) = \frac{1}{d(s)}, \quad \hat{G}_1(s) = k$$

$$u(s) \longrightarrow \boxed{G_2(s)} \xrightarrow{\xi(s)} \boxed{G_1(s)} \longrightarrow v(s)$$

let $\xi(s) = \hat{G}_2(s)\, u(s) = \dfrac{u(s)}{d(s)}$

$$\Rightarrow \overset{(n)}{\xi} = -\alpha_1 \underset{x_1}{\overset{(n-1)}{\xi}} - \alpha_2 \underset{x_2}{\overset{(n-2)}{\xi}} - \cdots - \alpha_{n-1} \underset{x_{n-1}}{\dot{\xi}} - \alpha_n \underset{x_n}{\xi} + u$$

$$x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \overset{(n-1)}{\xi} \\ \overset{(n-2)}{\xi} \\ \vdots \\ \xi \end{bmatrix} \Rightarrow \boxed{\dot{x} = \begin{bmatrix} \overset{(n)}{\xi} \\ \overset{(n-1)}{\xi} \\ \vdots \\ \dot{\xi} \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u}$$

State definition

$$Y(s) = \Xi(s)\, G_1(s) = K\, \Xi(s)$$

$$\Rightarrow y = K \underbrace{\xi}_{x_n}$$

$$\boxed{y = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & K \end{bmatrix}}_{C} x + \underbrace{0}_{D} u}$$

c)

(c) Compute the controllable canonical form realization for the transfer function

$$\hat{g}(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n}. \qquad (4.10)$$

$$\lim_{s \to \infty} \hat{g}(s) = 0 \Rightarrow \underline{D = 0}$$

$$u(s) \longrightarrow \boxed{G_2(s)} \xrightarrow{\Xi(s)} \boxed{G_1(s)} \longrightarrow y(s)$$

$$\hat{G}_{sp}(s) = \frac{n(s)}{d(s)} \quad \text{where} \quad d(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots \alpha_{n-1} s + \alpha_n$$

and

$$\hat{G}_2(s) = \frac{1}{d(s)}, \quad \hat{G}_1(s) = n(s) \quad n(s) = \beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n$$

let $\Xi(s) = \hat{G}_2(s)\, u(s) = \dfrac{u(s)}{d(s)}$

$$\Rightarrow \overset{(n)}{\xi} = -\alpha_1 \underset{x_1}{\overset{(n-1)}{\xi}} - \alpha_2 \underset{x_2}{\overset{(n-2)}{\xi}} - \cdots - \alpha_{n-1} \underset{x_{n-1}}{\xi} - \alpha_n \underset{x_n}{\xi} + u$$

$$x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \overset{(n-1)}{\xi} \\ \overset{(n-2)}{\xi} \\ \vdots \\ \xi \end{bmatrix} \Rightarrow \boxed{\dot{x} = \begin{bmatrix} \overset{(n)}{\xi} \\ \overset{(n-1)}{\xi} \\ \vdots \\ \xi \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u}$$

State definition

$$Y(s) = \hat{G}_1(s)\,\square(s) = h(s)\,\square(s) = \left(\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n\right)$$

$$\Rightarrow \quad y = \beta_1 \xi^{(n-1)} + \beta_2 \xi^{(n-2)} + \cdots \beta_{n-1} \dot{\xi} + \beta \xi$$

$$\Rightarrow \boxed{y = \underbrace{\begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{bmatrix}}_{C} x + \underbrace{0}_{D} u}$$

d)

$$\hat{g}(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n}. \tag{4.10}$$

(d) Compute the observable canonical form realization for the transfer function in equation (4.10).

Hint: See Exercise 4.5. □

Let A_o, B_o, C_o, D_o be the observable canonical form realization for (4.10)

By Exercise 4.5 and related notation

$$A_o = A_c^T, \quad B_o = C_c^T, \quad C_o = B_c^T, \quad D_o = D_c^T$$

where A_c, B_c, C_c, D_c are the controllable canonical form realization for (4.10), found in part (c).

$$A_o = \underbrace{\begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A_c}^T = \underbrace{\begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & \cdots \\ -\alpha_2 & 0 & 1 & \cdots & \cdots \\ \vdots & & & & \\ -\alpha_{n-1} & 0 & 0 & \cdots & 0 \; 1 \\ -\alpha_n & 0 & 0 & \cdots & 0 \; 0 \end{bmatrix}}_{A_o}$$

$$C_o = \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T}_{B_c} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}}_{C_o}$$

$$B_o = \underbrace{\begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{bmatrix}}_{C_c}^T = \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}}_{B_o}$$

$$D_o = \underbrace{[0]}_{D_c}^T = \underbrace{0}_{D_o}$$

$$\dot{x} = A_o x + B_o, \quad y = C_o x + D_o$$

4.7 (Zero-state equivalence). Show that the following pairs of systems are zero-state equivalent, but not algebraically. equivalent

(a)

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A_1} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_1} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_1} x \end{cases} \qquad \begin{cases} \dot{\bar{x}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{A_2} \bar{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_2} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_2} \bar{x} \end{cases}$$

(b)

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A_3} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_3} u \\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_3} x \end{cases} \qquad \begin{cases} \dot{\bar{x}} = \underbrace{\bar{x}}_{A_4} + \underbrace{u}_{B_4} \\ y = \underbrace{\bar{x}}_{C_4}. \end{cases}$$

Hint: To prove that the systems are not algebraically equivalent, you must show that it is impossible to construct a similarity transformation that transforms one system into the other. □

a)

$$G_1(s) = C_1(sI - A)^{-1}B_1$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s-1)^2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix}}{(s-1)^2} \begin{bmatrix} s-1 \\ 0 \end{bmatrix} = \frac{s-1}{s-1^2} = \underline{\frac{1}{s-1}}$$

$$G_2(s) = C_2(sI - A_2)^{-1}B_2 =$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s-1)(s-2)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & 0 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s-1)(s-2)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s-2 \\ 0 \end{bmatrix} = \frac{\cancel{s-2}}{(s-1)\cancel{(s-2)}} = \underline{\frac{1}{s-1}}$$

As seen above, the two systems realize the same transfer function and are therefore zero-state equivalent.

However, the eigenvalue of A1 are 1 and 1 while the eigenvalues of A2 are 1 and 2. Since the two system do not share eigenvalues, they cannot be algebraically equivalent.

b)

$$G_3(s) = C_3(sI - A_3)^{-1}B_3$$

$$= \underline{\frac{1}{s-1}}, \quad \text{since } A_3 = A_1,\ B_3 = A_3,\ C_3 = C, \quad (\text{see } a)$$

$$G_4(s) = C_4(sI - A_4)^{-1}B_4$$

$$= 1[s-1]^{-1}1 = \underline{\frac{1}{s-1}}$$

As seen above, the two systems realize the same transfer function and are therefore zero-state equivalent.

However, the dimensions of A3 and A4 are not equal. Therefore, they cannot be algebraically equivalent.

4.9 (Model matching). Consider the following system

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

and the following state-feedback control

$$u = -\begin{bmatrix} f_1 & f_2 \end{bmatrix} x + v,$$

where v denotes a disturbance input and f_1 and f_2 two scalar constants to be specified later.

(a) Compute the close-loop state-space model with input v and output y, leaving your answer as a function of the constants f_1 and f_2.

(b) Compute the closed loop transfer function from v to y, leaving your answer as a function of the constants f_1 and f_2.

(c) Determine values for f_1 and f_2 so as to obtain the following transfer function from v to y:

er

$$\frac{1}{s+1}.$$

Hint: To obtain this transfer function you will need to cancel one zero with one pole in the transfer function that you determined in b. □

a)

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} f_1 & f_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ -f_1 & -f_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$\boxed{\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ -f_1 & -f_2 \end{bmatrix}}_{A_1} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_1} v \\ y &= Cx = \begin{bmatrix} 1 & 1 \end{bmatrix} x \end{aligned}}$$

b) Let $G_{CL}(s)$ be closed loop transfer function

$$G_{CL}(s) = C(sI - A_1)^{-1} B_1$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ f_1 & s+f_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ f_1 & s+f_2 \end{bmatrix}^{-1} = \frac{1}{s^2 + sf_2 + f_1} \begin{bmatrix} s+f_2 & 1 \\ -f_1 & s \end{bmatrix}$$

$$\Rightarrow G_{CL}(s) = \frac{1}{s^2 + sf_2 + f_1} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+f_2 & 1 \\ -f_1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2 + sf_2 + f_1} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix}$$

$$= \frac{s+1}{s^2 + sf_2 + f_1}$$

$$\boxed{G_{CL}(s) = \frac{s+1}{s^2 + sf_2 + f_1}}$$

c) We want $G_{CL}(s) = \frac{1}{s+1} = \frac{s+1}{(s+1)(s+1)}$

$$\Rightarrow s^2 + sf_2 + f_1 = (s+1)(s+1) = s^2 + 2s + 1$$

$$\Rightarrow \boxed{\begin{matrix} f_2 = 2 \\ f_1 = 1 \end{matrix}}$$

Consider the following systems

$$S1: \dot{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u; y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x$$

$$S2: \dot{x} = \begin{bmatrix} 5 & -6 & -4 \\ -8 & 9 & 7 \\ -2 & 3 & 1 \end{bmatrix} x + \begin{bmatrix} -1 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} u; y = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix} x$$

$$S3: \dot{x} = \begin{bmatrix} 9 & -7 & 8 \\ -3 & 1 & -2 \\ 6 & -4 & 5 \end{bmatrix} x + \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} u; y = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix} x$$

a) Which pair(s) of these systems are zero-state equivalent?

b) Based on simple tests, which systems have the potential of being algebraically equivalent? Justify your answer.

c) Verify that the algebraically equivalent systems are equivalent using the transform $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$

Hint: The following function may be very helpful

```
function G = transfer_function(A, B, C)
    syms s real
    n = size(A,1);
    G = simplify(...); % Insert something here for the ...
end
```

Show transfer functions are equivalent

Show eigenvalue are the same and dimensions are equal (check notes for all necessary conditions)

Apply TAT^-1 and see if A1 turns into A3

$$Y(s) = C(sI - A)^{-1}BU(s) = G(s)U(s)$$

a) <u>zero-state equivalence</u>

To be zero-stat equivalent, transfer functions found by:

$$G(s) = \frac{Y(s)}{U(s)} = C_1(sI - A_1)^{-1}B_1 U(s)$$

must be equivalent

let $G_1(s)$ be transfer function of S1, S2 $\rightarrow G_2(s)$, S3 $\rightarrow G_3(s)$

using:

```
function G = transfer_function(A, B, C)
    syms s real
    n = size(A,1);
    G = simplify(C*(s*eye(n) - A)^-1 * B);
end
```

$$G_1(s) = \frac{1}{s(-s^2+15s+18)} \begin{bmatrix} -(s^2-s+6) & -(s^2+s-3) \\ -(s^2+11s+6) & (-2s^2+2s+3) \end{bmatrix}$$

$$G_2(s) = \frac{1}{s(-s^2+15s+18)} \begin{bmatrix} (s-15)s & -(s-15)s \\ (-s^2+15s+27) & (-2s^2+30s+27) \end{bmatrix}$$

$$G_3(s) = \frac{1}{s(-s^2+15s+18)} \begin{bmatrix} -(s^2-s+6) & -(s^2+s-3 \\ -(s^2+11s+6) & (-2s^2+2s+3) \end{bmatrix}$$

As seen above, $S1$ and $S3$ have the same transfer function, and are therefore zero-state equivalent.

However, $S2$ does not share the same transfer function and is therefore NOT zero-state equivalent with S_1 or S_3.

$\boxed{S_1 \text{ and } S_2 \text{ \underline{are} zero state equivalent}}$

$\boxed{S_2 \text{ is NOT zero-state equivalent with } S_1 \text{ or } S_3.}$

b) <u>Test 1: zero state equivalence.</u>

As seen above S_2 is not zero state equivalent with S_1 or S_3, and thus <u>cannot</u> be algebraically equivalent to S_1 or S_3.

<u>Test 2: eigenvalues</u>

S_1 eigenvalues (using MATLAB eig): $\lambda_1 = 16.1168$, $\lambda_2 = -1.1168$, $\lambda_3 = 0$

S_2 eigenvalues (using MATLAB eig): $\lambda_1 = 16.1168$, $\lambda_2 = -1.1168$, $\lambda_3 = 0$

S_3 eigenvalues (using MATLAB eig): $\lambda_1 = 16.1168$, $\lambda_2 = -1.1168$, $\lambda_3 = 0$

<u>Test 3: dimension</u>

Clearly, all 3 systems have the same dimensions

So, by the 3 simple tests above, S_1 and S_3 <u>could</u> be algebraically equivalent since they are zero state equivalent, they have the same eigenvalues, and they are of the <u>same dimension</u>.

c) To verify S_1 and S_3 are algebraically equivalent, must show a matrix T s.t.

$$A_3 = T^{-1}A_1T \quad (1)$$
$$B_3 = T^{-1}B_1 \quad (2)$$
$$C_3 = TC \quad (3)$$

Given $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$

$$\Rightarrow T^{-1} = \frac{1}{-1(-1)}\begin{bmatrix} 0 & -(0-1) & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

(1)
$$T^{-1}A_1T = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & 1 & -2 \\ -6 & 4 & -5 \\ -9 & 7 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -7 & 8 \\ 3 & 1 & -2 \\ 6 & -4 & 5 \end{bmatrix} = A_3 \checkmark$$

(2)
$$T^{-1}B_1 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = B_3 \checkmark$$

(3)
$$C_1T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix} = C_3 \checkmark$$

$\Rightarrow$ $\boxed{S_1 \text{ and } S_3 \text{ are algebraically equivalent}}$

6.4 (Matrix powers and exponential). Compute A^t and e^{At} for the following matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}. \qquad \square$$

$A_1)$

$$A_1^t = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t$$

$$A_1^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_1^3 = A_1^2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vdots$$

$$\boxed{A_1^t = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$e^{A_1 t} = \sum_{k=0}^{\infty} A_1^k \frac{t^k}{k!}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{t^k}{k!}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} \frac{t^k}{k!} & \frac{k t^k}{k!} & 0 \\ 0 & \frac{t^k}{k!} & 0 \\ 0 & 0 & \frac{t^k}{k!} \end{bmatrix}$$

$$= \begin{bmatrix} \sum \frac{t^k}{k!} & \sum \frac{k t^k}{k!} & 0 \\ 0 & \sum \frac{t^k}{k!} & 0 \\ 0 & 0 & \sum \frac{t^k}{k!} \end{bmatrix}$$

$$\boxed{e^{A_1 t} = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}}$$

where: $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$

$$\sum_{k=0}^{\infty} \frac{k t^k}{k(k-1)!} \qquad \text{let } k = k-1, \quad k = l+1$$

$$=$$

$$te^t = \sum_{k=0}^{\infty} \frac{t t^k}{k!}$$

let $\sum := \sum_{k=0}^{\infty}$

$$\sum_{k=0}^{\infty} \frac{k t^k}{k!} = \sum_{k=0}^{\infty} \frac{k t t^{k-1}}{k(k-1)!}$$

let $l = k-1$

where $\sum_{k=1}^{\infty} \frac{k t^k}{k(k-1)!} = \sum_{l=0}^{\infty} \frac{t t^l}{l!}$

$$= te^t$$

and $\sum \frac{t^k}{k!} = e^t$

A2)

$$A_2^t = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}^t \qquad A_2^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2^3 = A_2^2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2^4 = A_2^3 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

using $0^0 := 1$

$\vdots$

$$\boxed{A_2^t = \begin{bmatrix} 1 & 1 & t-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -(0^t) & 0^t \\ 0 & 0^t & -(0^t) \\ 0 & 0 & 0 \end{bmatrix}}$$

For $t=0$

$A^0 = I$

$$e^{A_2 t} = \sum_{k=0}^{\infty} A_2^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} \begin{bmatrix} 1 & 1 & k-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \frac{t^k}{k!} + \sum_{k=0}^{\infty} \begin{bmatrix} 0 & -(0^k) & 0^k \\ 0 & 0^k & -(0^k) \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sum \frac{t^k}{k!} & \sum \frac{t^k}{k!} & \sum \frac{(k-1)t^k}{k!} \\ 0 & 0 & \sum \frac{t^k}{k!} \\ 0 & 0 & \sum \frac{t^k}{k!} \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^t & (t-1)e^t \\ 0 & 0 & e^t \\ 0 & 0 & e^t \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\sum_{k=0}^{\infty} \frac{(k-1)t^k}{k!} = \sum_{k=0}^{\infty} \frac{k t^k}{k!} - \sum_{k=0}^{\infty} \frac{t^k}{k!}$

$= te^t - e^t$

$= (t-1)e^t$

$$\boxed{e^{A_2 t} = \begin{bmatrix} e^t & e^t - 1 & (t-1)e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}}$$

in lecture

A3)

$$A_3^t = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix}^t = \begin{bmatrix} A_{31}^t & 0 & 0 \\ & 0 & 0 \\ 0 & 0 & A_{32}^t \\ 0 & 0 & \end{bmatrix}$$

where $A_{31} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$, $A_{32} = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$

$$\boxed{A_3^t = \begin{bmatrix} 2^t & 0 & 0 & 0 \\ t2^t & 2^t & 0 & 0 \\ 0 & 0 & 3^t & t3^t \\ 0 & 0 & 0 & 3^t \end{bmatrix}}$$

and

$$A_{31}^2 = \begin{bmatrix} 4 & 0 \\ 8 & 4 \end{bmatrix} \quad A_{32}^2 = \begin{bmatrix} 9 & 18 \\ 0 & 9 \end{bmatrix}$$

$$A_{31}^3 = \begin{bmatrix} 8 & 8 \\ 24 & 8 \end{bmatrix} \quad A_{32}^3 = \begin{bmatrix} 27 & 81 \\ 0 & 27 \end{bmatrix}$$

$$\vdots \qquad \vdots$$

$$A_{31}^t = \begin{bmatrix} 2^t & 0 \\ t2^t & 2^t \end{bmatrix} \quad A_{32}^t = \begin{bmatrix} 3^t & t3^t \\ 0 & 3^t \end{bmatrix}$$

$$e^{A_3 t} = \begin{bmatrix} e^{A_{31}t} & 0 & 0 \\ & 0 & 0 \\ 0 & 0 & e^{A_{32}t} \\ 0 & 0 & \end{bmatrix}$$

where $e^{A_{31}t} = \sum_{k=0}^{\infty} A_{31}^k \frac{t^k}{k!}$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} 2^k & 0 \\ k2^k & 2^k \end{bmatrix} \frac{t^k}{k!}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{2^k t^k}{k!} & 0 \\ \sum_{k=0}^{\infty} \frac{k2^k t^k}{k!} & \sum_{k=0}^{\infty} \frac{2^k t^k}{k!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 0 \\ 2te^{2t} & e^{2t} \end{bmatrix}$$

and $e^{A_{32}t} = \sum_{k=0}^{\infty} A_{32}^k \frac{t^k}{k!}$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} 3^k & k3^k \\ 0 & 3^k \end{bmatrix} \frac{t^k}{k!}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{3^k t^k}{k!} & \sum_{k=0}^{\infty} \frac{k3^k t^k}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{3^k t^k}{k!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t} & 3te^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

since $\sum_{k=1}^{\infty} \frac{k\alpha^k t^k}{k!}$

$$\sum_{k=1}^{\infty} \frac{\alpha t\, k\, \alpha^{k-1} t^{(k-1)}}{k(k-1)!}$$ let $l := k-1$

$$\alpha t \sum_{l=0}^{\infty} \frac{\alpha^l t^l}{l!} = \underline{\alpha t e^{\alpha t}}$$

$$\Rightarrow \boxed{e^{A_3 t} = \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 2te^{2t} & e^{2t} & 0 & 0 \\ 0 & 0 & e^{3t} & 3te^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}}$$

6.5 (Exponential of an upper triangular matrix). Consider an upper triangular matrix A.

(a) Show that e^{tA} is also upper triangular.

(b) Relate the diagonal elements of A with those of e^{tA}.

Hint: Use the definition of matrix exponential. □

let $A \in \mathbb{R}^{n \times n}$

a) Showing for 3×3 case (see proof below)

Let $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$

$$e^{tA} = \sum_{k=0}^{\infty} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}^k \frac{t^k}{k!}$$

, showing that A^k is upper triangular since $\frac{t^k}{k!}$ is a scalar

$$A \cdot A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$$

$$A \cdot A = \begin{bmatrix} A_{11} \cdot A_{11} & A_{11} A_{12} + A_{12} A_{22} & A_{11} A_{13} + A_{12} A_{23} + A_{13} A_{33} \\ 0 & A_{22} A_{22} & A_{12} A_{23} + A_{23} A_{33} \\ 0 & 0 & A_{33} A_{33} \end{bmatrix}$$

$\Rightarrow A^k$ is upper triangular given A is upper triangular.

$\Rightarrow e^{tA}$ is upper triangular given A is upper triangular.

Proof Attempt for 6.5

$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$, so, if A^k is upper triangular for $0 \le k < \infty$, e^{tA} is upper triangular since $\sum_{k=0}^{\infty} \frac{t^k}{k!}$ is a scalar and any scalar multiplied by an upper triangular matrix is upper triangular.

Since A is upper triangular,

$A_{ij} = 0$ if $i > j$ Let $B \in \mathbb{R}^{n \times n}$ and upper triangular

let $C = AB$, $\Rightarrow$ $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$ Note: if $k > i$,

for all C_{ij} where $(i > j)$, $C_{ij} = 0$ since

see Note $\rightarrow$ $i > k$ or $k > j$ for $1 \le k \le n$. If $i > k$ or $k > j$,

then $A_{ik} = 0$ or $B_{kj} = 0$ for all $1 \le k \le n$.

Note: let $i > j$ $\begin{cases} \text{If } k > i, \text{ then } k > j \\ \text{If } k < j, \text{ then } i > k \end{cases}$

So, if $C_{ij} = 0$ for all $i > j$, then C is upper triangular

$\Rightarrow e^{tA}$ is upper triangular if A is upper triangular $\square$

b) let $C = A \cdot A \Rightarrow$ when $i=j$, $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$

$= A_{ii} B_{ii}$ since all other values are zero (see part (a))

so $A^k = \begin{bmatrix} A_{11}^k & \cdots & \cdots \\ 0 & A_{22}^k & \cdots \\ \vdots & & \ddots \\ 0 & \cdots & A_{nn}^k \end{bmatrix}$

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \begin{bmatrix} A_{11}^k & \cdots & \cdots \\ 0 & A_{22}^k & \cdots \\ \vdots & & \ddots \\ 0 & \cdots & A_{nn}^k \end{bmatrix} \frac{t^k}{k!}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} A_{11}^k \frac{t^k}{k!} & \cdots & \cdots \\ 0 & \sum_{k=0}^{\infty} A_{22}^k \frac{t^k}{k!} & \cdots \\ \vdots & & \ddots \\ 0 & \cdots & \sum_{k=0}^{\infty} A_{nn}^k \frac{t^k}{k!} \end{bmatrix}$$

$$\boxed{e^{tA} = \begin{bmatrix} e^{tA_{11}} & \cdots & \cdots \\ 0 & e^{tA_{22}} & \cdots \\ \vdots & & \ddots \\ 0 & \cdots & 0 \quad e^{tA_{nn}} \end{bmatrix}}$$

where $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & A_{nn} \end{bmatrix}$

As seen above, the diagonal elements of e^tA become e^tAii where Aii is the corresponding diagonal element of A.