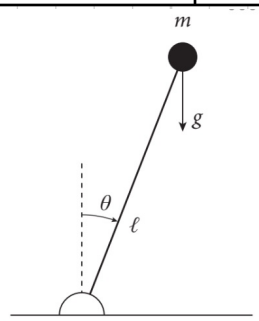
09/07/24 | ECE 6320/HW 2 | Howe, Kade



From Newton's law:

$$m\ell^2\ddot{\theta} = mg\ell \sin\theta - b\dot{\theta} + T,$$

where T denotes a torque applied at the base, and g is the gravitational acceleration.

FIGURE 2.8. Inverted pendulum.

1) Dynamic Equation
2) Find Eq point(s)
3) Take Jacobians
4) Evaluate Jacobians at eq point(s) for A, B, C, D

2.6 EXERCISES

2.3 (Local linearization around equilibrium: saturated inverted pendulum). Consider the inverted pendulum in Figure 2.8 and assume the input and output to the system are the signals u and y defined as

$$T = \text{sat}(u), \quad y = \theta,$$

where "sat" denotes the unit-slope saturation function that truncates u at $+1$ and -1.

(a) Linearize this system around the equilibrium point for which $\theta = 0$.

(b) Linearize this system around the equilibrium point for which $\theta = \pi$ (assume that the pendulum is free to rotate all the way to this configuration without hitting the table).

(c) Linearize this system around the equilibrium point for which $\theta = \frac{\pi}{4}$. Does such an equilibrium point always exist?

(d) Assume that $b = 1/2$ and $mg\ell = 1/4$. Compute the torque $T(t)$ The needed for the pendulum to fall from $\theta(0) = 0$ with constant velocity $\dot{\theta}(t) = 1, \forall t \geq 0$. Linearize the system around this trajectory. □

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta - \frac{b}{m\ell^2}\dot{\theta} + \frac{T}{m\ell^2}$$

$$= \frac{g}{\ell}\sin x_1 - \frac{b}{m\ell^2}x_2 + \frac{\text{sat}(u)}{m\ell^2}$$

$$\Rightarrow \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{\ell}\sin x_1 - \frac{b x_2}{m\ell^2} + \frac{\text{sat}(u)}{m\ell^2} \end{bmatrix} = f(x, u) \qquad y = \theta = \begin{bmatrix} 1 & 0 \end{bmatrix} x = g(x, u)$$

a) Find equilibrium point at $\theta = 0$: $x_1^{eq} = 0$

$$f(x^{eq}, u^{eq}) = 0 \Rightarrow \quad x_2^{eq} = 0$$

$$\frac{g}{\ell}\sin x_1 - \frac{b x_2}{m\ell^2} + \frac{\text{sat}(u)}{m\ell^2} = 0$$

$$\Rightarrow \text{sat}(u^{eq}) = 0$$

$$\Rightarrow u^{eq} = 0$$

$$x^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u^{eq} = 0, \quad y^{eq} = 0$$

a) choose $x^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u^{eq} = 0$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 - \frac{b x_2}{m l^2} + \frac{\tau_{sat}(u)}{m l^2} \end{bmatrix}$$

Jacobians

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos x_1 & -\frac{b}{m l^2} \end{bmatrix}$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\frac{\partial g}{\partial u} = 0$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m l^2} \end{bmatrix} \quad \text{for } u \in (-1, 1)$$

Eval @ equilibrium

$$\Rightarrow \delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{m l^2} \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m l^2} \end{bmatrix}}_{B} \delta u$$

$$y = g(x, u) = x_1$$

$$\Rightarrow \delta y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \delta x + \underbrace{0}_{D=0} \delta u$$

where $\delta x = x - x^{eq} \Rightarrow x = \delta x + x^{eq} \Rightarrow x = \delta x$

$\delta y = y - x^{eq} \Rightarrow y = \delta y + y^{eq} \Rightarrow y = \delta y$

$\delta u = u - u^{eq} \Rightarrow u = \delta u + u^{eq} \Rightarrow u = \delta u$

$(g = 9.8,\ m = 1/4.9,\ l = 0.25,\ b = 1)$

Finding Eigenvalues

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{m l^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 39.2 & -156.8 \end{bmatrix}$$

$$\det(A - \lambda I) = \left| \begin{bmatrix} -\lambda & 1 \\ 39.2 & -156.8 - \lambda \end{bmatrix} \right| = -\lambda(-156.8 - \lambda) - 39.2$$

$$= \lambda^2 + 156.8\lambda - 39.2$$

$$\Rightarrow \underline{\lambda_1 = 0.2496,\ \lambda_2 = -157.0496} = (\lambda - 0.2496)(\lambda + 157.0496)$$

b) Find equilibrium point at $\theta = \pi \Rightarrow x_1^{eq} = \pi$

$x_2 = 0$

$\Rightarrow f(x^{eq}, u^{eq}) = 0 \Rightarrow$

$$\frac{g}{l}\sin x_1^{eq} - \frac{b x_2^{eq}}{ml^2} + \frac{sat(u^{eq})}{ml^2} = 0$$

$$\Rightarrow sat(u^{eq}) = 0$$

$$\Rightarrow \underline{u^{eq} = 0}$$

Linearize around $x^{eq} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$, $u^{eq} = 0$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 - \frac{b x_2}{ml^2} + \frac{sat(u)}{ml^2} \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(x_1) & -\frac{b}{ml^2} \end{bmatrix} \quad \text{for } u \in [-1, 1]$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \quad \text{for } u \in [-1, 1]$$

$$\Rightarrow \underline{\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \delta u}$$

$$y = g(x, u) = x_1$$

$$\Rightarrow \underline{\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x}$$

where $\delta x = x - x^{eq} \Rightarrow x = \delta x + x^{eq}$

$\delta y = y - x^{eq} \Rightarrow y = \delta y + y^{eq}$

$\delta u = u - u^{eq} \Rightarrow u = \delta u + u^{eq}$

Finding Eigenvalues ($g = 9.8$, $m = 1/4.8$, $l = 0.25$, $b = 1$)

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -39.2 & -156.8 \end{bmatrix}$$

$$\det(A - \lambda I) = \left| \begin{bmatrix} -\lambda & 1 \\ -39.2 & -156.8 - \lambda \end{bmatrix} \right| = -\lambda(-156.8 - \lambda) + 39.2$$

$$= \lambda^2 + 156.8\lambda + 39.2$$

$$\Rightarrow \underline{\lambda_1 = -0.2504,\ \lambda_2 = -156.5496} = (\lambda + 0.2504)(\lambda + 156.5496)$$

c) Find equilibrium point at $\theta = \frac{\pi}{4} \Rightarrow x_1^{eq} = \frac{\pi}{4}$

$$x_2^{eq} = 0$$

$$\Rightarrow f(x^{eq}, u^{eq}) = 0 \Rightarrow \frac{g}{l}\sin x_1^{eq} - \frac{b x_2^{eq}}{ml^2} + \frac{\mathrm{sat}(u^{eq})}{ml^2} = 0$$

$$\Rightarrow \frac{g}{l}\frac{1}{\sqrt{2}} + \frac{\mathrm{sat}(u^{eq})}{ml^2} = 0$$

$$\Rightarrow \mathrm{sat}(u^{eq}) = -\frac{glm}{\sqrt{2}}$$

$$\underline{u^{eq} = \frac{-glm}{\sqrt{2}},\ |glm| \le \sqrt{2}}$$

Note: such an equilibrium point does not always exist. First, if there is every a combination of g,l, and m such that (-glm/(sqrt2)) is greater than 1, the equilibrium input control does not exist for this equilibrium point. This is due to u = sat(T).

Linearize around $x^{eq} = \begin{bmatrix} \pi/4 \\ 0 \end{bmatrix}$, $u^{eq} = \frac{-gl m}{\sqrt{2}}$, $|gl m| \leq \sqrt{2}$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{l}\sin x_1 - \frac{b x_2}{m l^2} + \frac{\text{sat}(u)}{m l^2} \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(x_1) & -\frac{b}{m l^2} \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m l^2} \end{bmatrix} \quad \text{for } u \in [-1, 1]$$

$$\Rightarrow \underline{\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l\sqrt{2}} & -\frac{b}{m l^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{m l^2} \end{bmatrix} \delta u}$$

$$y = g(x, u) = x_1$$

$$\Rightarrow \underline{\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x}$$

where $\delta x = x - x^{eq} \Rightarrow x = \delta x + x^{eq}$

$\delta y = y - x^{eq} \Rightarrow y = \delta y + y^{eq}$

$\delta u = u - u^{eq} \Rightarrow u = \delta u + u^{eq}$

Finding Eigenvalues ($g = 9.8$, $m = \frac{1}{9.8}$, $l = 0.25$, $b = 1$)

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l\sqrt{2}} & \frac{-b}{m l^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 27.72 & -156.8 \end{bmatrix}$$

$$\det(A - \lambda I) = \left| \begin{bmatrix} -\lambda & 1 \\ 27.72 & -156.8 - \lambda \end{bmatrix} \right| = -\lambda(-156.8 - \lambda) - 27.72$$

$$= \lambda^2 + 156.8\lambda - 27.72$$

$$\Rightarrow \underline{\lambda_1 = 0.1766, \ \lambda_2 = -156.9766} = (\lambda - 0.1766)(\lambda + 156.9766)$$

d) $b = \frac{1}{2}$, $mgl = 1/4$, $\theta(0) = 0$, $\dot{\theta}(t) = 1$, $\forall t \geq 0$

$\Rightarrow \theta(t) = \int \dot{\theta}(t)dt = t + \underline{k} = t$

$\theta(0) = 0 + k$
$\Rightarrow \underline{k = 0}$

$\Rightarrow \ddot{\theta}(t) = 0$

into differential equation: $ml^2\ddot{\theta}(t) = mgl\sin\theta(t) - b\dot{\theta}(t) + T(t)$

$\Rightarrow 0 = mgl\sin(t) - b + T(t)$

$\Rightarrow \underline{T(t) = b - mgl\sin(t)}$

$x := x^{sol} + \delta x$
$\Rightarrow \delta x = x - x^{sol}$
$u := u^{sol} + \delta x$
$\Rightarrow \delta u = u - u^{sol}$
$y := y^{sol} + \delta x$
$\Rightarrow \delta y = y - y^{sol}$

$\max(T(t)) = \frac{1}{2} - \frac{1}{4}\sin(\frac{3\pi}{2})$
$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$\min(T(t)) = \frac{1}{2} - \frac{1}{4}\sin(\frac{\pi}{2})$
$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

since $\frac{1}{4} \leq T(t) \leq \frac{1}{4} \Rightarrow \underline{u^{sol}(t) = T(t)}$

$x^{sol}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$ $\quad u^{sol} = b - mgl\sin(t)$ $\quad y^{sol} = t$

$u^{sol} = \frac{1}{2} - \frac{1}{4}\sin(t)$

$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{g}{l}\sin(x_1(t)) \quad \frac{-b\,x_2(t)}{ml^2} + \frac{sat(u(t))}{ml^2} \end{bmatrix} = f(x, u, t)$

$\Rightarrow \frac{df}{dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(x_1(t)) & -\frac{b}{ml^2} \end{bmatrix} \Big|_{\begin{bmatrix} t \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(t) & \frac{-b}{ml^2} \end{bmatrix}$

$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$

$y = g(x, u, t) = x_1(t)$

$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$\Rightarrow \quad \delta \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos(t) & \frac{-b}{ml^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta \dot{x} + 0\, \delta u$$

2.4 (Local linearization around equilibrium: pendulum). The following equation models the motion of a frictionless pendulum:

$$\ddot{\theta} + k \sin\theta = \tau$$

where $\theta \in \mathbb{R}$ is the angle of the pendulum with the vertical, $\tau \in \mathbb{R}$ an applied torque, and k a positive constant.

(a) Compute a state-space model for the system when $u := \tau$ is viewed as the input and $y := \theta$ as the output. Write the model in the form

$$\dot{x} = f(x, u) \qquad\qquad y = g(x, u)$$

for appropriate functions f and g.

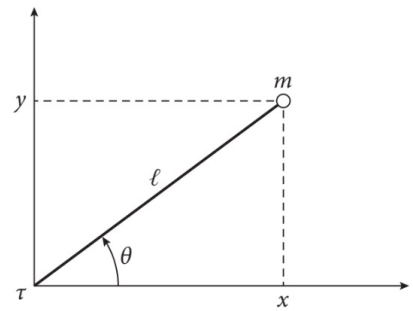


FIGURE 2.9. One-link robotic manipulator.

(b) Find the equilibrium points of this system corresponding to the constant input $\tau(t) = 0, t \geq 0$.

Hint: There are many.

(c) Compute the linearization of the system around the solution $\tau(t) = \theta(t) = \dot{\theta}(t) = 0, t \geq 0$.

Hint: Do not forget the output equations. □

a) $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, $u := \tau$, $y := \theta = g(x,u)$

where $g(x,u) = x_1$

$$\Rightarrow \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \tau - k\sin\theta \end{bmatrix} = f(x,u)$$

$$\Rightarrow \dot{x} = f(x,u) = \begin{bmatrix} x_2 \\ u - k\sin(x_1) \end{bmatrix}$$

$$y := \theta = g(x,u) = x_1$$

$$\Rightarrow y = g(x,u) = x_1$$

b) Finding equilibrium points

given: $\tau(t) = 0,\ t \geq 0$

$\Rightarrow u^{eq} = 0$

set $f(x^{eq}, u^{eq}) = 0$

$$\Rightarrow \begin{bmatrix} x_2^{eq} \\ u^{eq} - k\sin(x_1^{eq}) \end{bmatrix} = 0 \Rightarrow$$

$x_2^{eq} = 0$

$\sin(x_1^{eq}) = 0$

$\Rightarrow x_1^{eq} = n\pi,\ n \in \mathbb{Z}$

$$x^{eq} = \begin{bmatrix} n\pi \\ 0 \end{bmatrix},\quad u^{eq} := 0,\quad n \in \mathbb{Z}$$

c) Compute Linearization

given: $\tau(t) = \theta(t) = \dot{\theta}(t) = 0$

$\Rightarrow x^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\ u^{eq} = 0$

Jacobians

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k\cos(x_1) & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\frac{\partial g}{\partial u} = 0$$

$$\Rightarrow \quad \delta\dot{x} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

where $\delta x = x - x^{eq}$, $\delta y = y - y^{eq}$, $\delta u = u - u^{eq}$

2.6 (Local linearization around equilibrium: one-link robot). Consider the one-link robot in Figure 2.9, where θ denotes the angle of the link with the horizontal, τ the torque applied at the base, (x, y) the position of the tip, ℓ the length of the link, I its moment of inertia, m the mass at the tip, g gravity's acceleration, and b a friction coefficient. This system evolves according to the following equation:

$$I\ddot{\theta} = -b\dot{\theta} - gm\cos\theta + \tau.$$

(a) Compute the state-space model for the system when $u = \tau$ is regarded as the input and the vertical position of the tip y is regarded as the output.

Please denote the state vector by z to avoid confusion with the horizontal position of the tip x, and write the model in the form

$$\dot{z} = f(z, u) \quad y = g(z, u)$$

for appropriate functions f and g.

Hint: Do not forget the output equation!

(b) Show that $\theta(t) = \pi/2$, $\tau(t) = 0$, $\forall t \geq 0$ is a solution to the system and compute its linearization around this solution.

From your answer, can you predict if there will be problems when one wants to control the tip position close to this configuration just using feedback from y? □

a) let $z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, $u := \tau$

$\Rightarrow \dot{z} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{b}{I} z_2 - \frac{gm}{I}\cos(z_1) + \frac{u}{I} \end{bmatrix}$ $\quad \ddot{\theta} = -\frac{b}{I}\dot{\theta} - \frac{gm}{I}\cos\theta + \tau$

$\ddot{\theta} = -\frac{b}{I} z_2 - \frac{gm}{I}\cos z_1 + \frac{u}{I}$

$\Rightarrow \dot{z} = f(z,u) = \begin{bmatrix} z_2 \\ -\frac{b}{I} z_2 - \frac{gm}{I}\cos(z_1) + \frac{u}{I} \end{bmatrix}$

$y = g(z,u) = \ell \sin(z_1)$

b) Showing $\theta(t) = \frac{\pi}{2}$, $\tau(t) = 0$ $\forall t \geq 0$ is a solution

$\hookrightarrow u^{sol} = 0$

$\Rightarrow \dot{\theta}(t) = 0 \Rightarrow \ddot{\theta}(t) = 0$

Into diff. eq: $I(0) = -b(0) - gm\cos(\frac{\pi}{2}) + 0$

$0 = 0$ ✓

$\Rightarrow z^{sol} = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}$ OR

$\Rightarrow \dot{z}^{sol} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = f(z^{sol}, u^{sol}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓ $\quad y^{sol} = \ell = g(z^{sol}, u^{sol})$ ✓

Linearization

$z^{sol} = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}, \; u^{sol} = 0 \Rightarrow y^{sol} = l$

Jacobians

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{gm}{I}\sin(z_1) & -\frac{b}{I} \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix}$$

$$\frac{\partial g}{\partial z} = \begin{bmatrix} \frac{\partial g}{\partial z_1} & \frac{\partial g}{\partial z_2} \end{bmatrix} = \begin{bmatrix} l\cos(z_1) & 0 \end{bmatrix}$$

$$\frac{\partial g}{\partial u} = 0$$

At z^{sol}, u^{sol}

$$\Rightarrow \delta\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{gm}{I} & -\frac{b}{I} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1/I \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 0 & 0 \end{bmatrix} \delta x + 0 \, \delta u$$

$x = x^{eq} + \delta x$

where $\delta x = x - x^{eq}$, $\delta u = u - u^{eq}$, $\delta y = y - y^{eq}$

Note: In the linearized model, delta y will always be zero. So, using only the y tip position at this point will cause difficulties in controlling the system and estimating the system using this linearized model. Additionally, the value of y (y location of the tip position) will follow a sine wave (since y = l*sin(theta)). Additionally, the solution we have solved around is when theta is close to pi/2 and in the small-angle approximation the sine of anything “close” to pi/2 is 1. So, if we control the tip position only around the angle theta = pi/2, the y position will only have very small changes, potentially leading to sensor resolution difficulties.

2.7 (Local linearization around trajectory: unicycle). A single-wheel cart (unicycle) moving on the plane with linear velocity v and angular velocity ω can be modeled by the nonlinear system

$$\dot{p}_x = v\cos\theta, \quad \dot{p}_y = v\sin\theta, \quad \dot{\theta} = \omega, \tag{2.13}$$

where (p_x, p_y) denote the Cartesian coordinates of the wheel and θ its orientation. Regard this as a system with input $u := \begin{bmatrix} v & \omega \end{bmatrix}' \in \mathbb{R}^2$.

(a) Construct a state-space model for this system with state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} p_x\cos\theta + (p_y - 1)\sin\theta \\ -p_x\sin\theta + (p_y - 1)\cos\theta \\ \theta \end{bmatrix}$$

and output $y := \begin{bmatrix} x_1 & x_2 \end{bmatrix}' \in \mathbb{R}^2$.

(b) Compute a local linearization for this system around the equilibrium point $x^{\text{eq}} = 0$, $u^{\text{eq}} = 0$.

(c) Show that $\omega(t) = v(t) = 1$, $p_x(t) = \sin t$, $p_y(t) = 1 - \cos t$, $\theta(t) = t$, $\forall t \geq 0$ is a solution to the system.

(d) Show that a local linearization of the system around this trajectory results in an LTI system. □

$$\dot{x}_1 = \dot{p}_x\cos\theta - p_x\sin(\theta)\dot{\theta} + \dot{p}_y\sin\theta + (p_y - 1)\cos(\theta)\dot{\theta}$$
$$\dot{x}_2 = -\dot{p}_x\sin\theta - p_x\cos(\theta)\dot{\theta} + \dot{p}_y\cos\theta - (p_y - 1)\sin(\theta)\dot{\theta}$$
$$\dot{x}_3 = \dot{\theta}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} v\cancel{\cos^2\theta} - p_x\sin(\theta)\dot{\theta} + v\cancel{\sin^2\theta} + (p_y - 1)\cos(\theta)\dot{\theta} \\ \cancel{-v\cos\theta\sin\theta} - p_x\cos(\theta)\dot{\theta} + \cancel{v\sin\theta\cos\theta} - (p_y - 1)\sin(\theta)\dot{\theta} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} v + \dot{\theta}(-p_x\sin\theta + (p_y - 1)\cos\theta) \\ -\dot{\theta}(p_x\cos\theta + (p_y - 1)\sin\theta) \\ \dot{\theta} \end{bmatrix}$$

a)

$$\dot{x} = \begin{bmatrix} u_1 + u_2 x_2 \\ -u_2 x_1 \\ u_2 \end{bmatrix} = f(x, u)$$

$$y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = g(x, u)$$

b) For equilibrium point $x^{eq} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $u^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $y^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & u_2 & 0 \\ -u_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 1 & x_2 \\ 0 & -x_1 \\ 0 & 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\frac{\partial g}{\partial u} = 0$$

$$\Rightarrow \dot{\delta x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x + 0\, \delta u$$

c) First showing $\dot{x}^{sol} = f(x^{sol}, u^{sol})$

$$u^{sol} = \begin{bmatrix} u^{sol}(t) \\ w^{sol}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x^{sol} = \begin{bmatrix} \sin t \cos t + (1 - \cos t - 1)\sin t \\ -\sin t \sin t + (1 - \cos t - 1)\cos t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix}$$

$$\dot{x}^{sol} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad f(x^{sol}, u^{sol}) = \begin{bmatrix} u_1 + u_2 x_2 \\ -u_2 x_1 \\ u_2 \end{bmatrix}\Bigg|_{x^{sol}, u^{sol}}$$

$$\dot{x}^{sol} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = f(x^{sol}, u^{sol}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Second, showing $y^{sol} = g(x^{sol}, u^{sol})$

$$y^{sol} = \begin{bmatrix} \sin t \cos t + (1 - \cos t - 1)\sin t \\ -\sin t \sin t + (1 - \cos t - 1)\cos t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$g(x^{sol}, u^{sol}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Bigg|_{x = x^{sol},\, u = u^{sol}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$y^{sol} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = g(x^{sol}, u^{sol})$$

d) $\delta x := x - x^{sol}$ $\quad \delta u := u - u^{sol}$ $\quad \delta y := y - y^{sol}$

where $x^{sol} = \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix}$, $u^{sol} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $y^{sol} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\Rightarrow A = \left.\frac{\partial f}{\partial x}\right|_{u=u^{sol}, x=x^{sol}} = \left.\begin{bmatrix} 0 & u_2 & 0 \\ -u_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right|_{x=x^{sol}, u=u^{sol}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{x=x^{sol}, u=u^{sol}} = \left.\begin{bmatrix} 1 & x_2 \\ 0 & -x_1 \\ 0 & 1 \end{bmatrix}\right|_{x=x^{sol}, u=u^{sol}} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \left.\frac{\partial g}{\partial x}\right|_{x^{sol}, u^{sol}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$D = 0$$

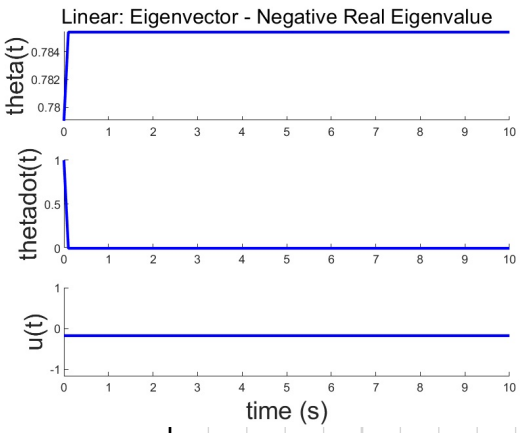
$$\Rightarrow \delta \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u$$

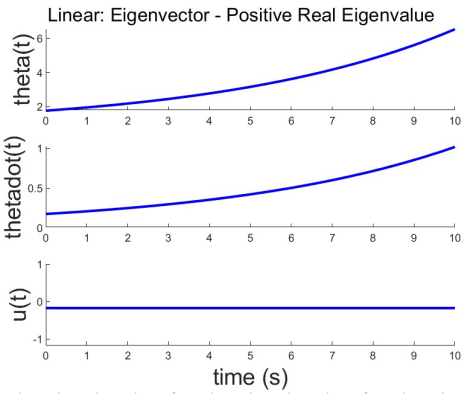
$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \delta u$$

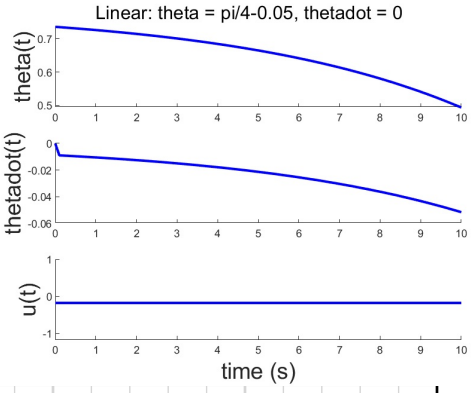
This system is LTI since A, B, C, D are constant over all time.

Simulation Results

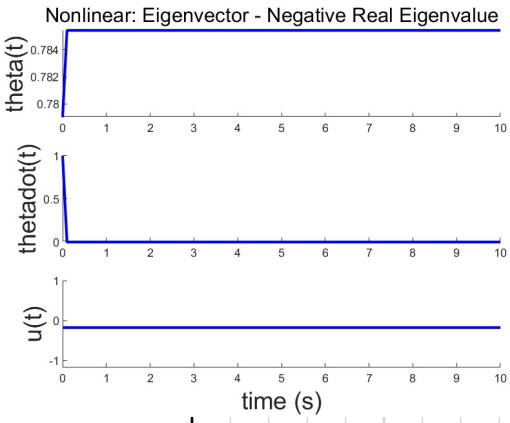
Linearized Simulations

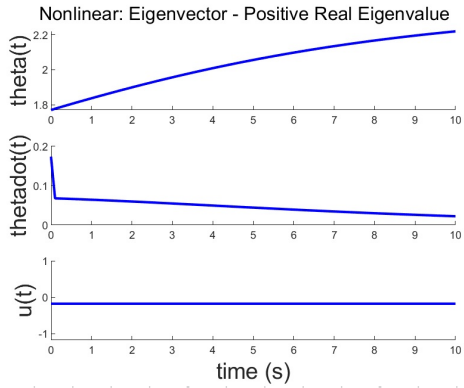


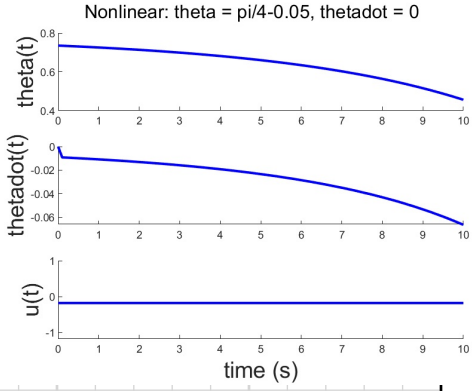




Nonlinear Simulations







Description of similarities/differences: As can be seen in the plots above, the linearized simulation mostly match the nonlinear simulations. This can be seen in the first plots of both types where the state converges to the [pi/4;0] equilibrium point after some time. Also, in the third plots where the state starts near the equilibrium point and slowly diverges. (Note: the nonlinear simulation for the case where the state starts near the equilibrium point will converge to a different equilibrium where theta = 3*pi/4, but the linear simulation will not)

Constrastingly, the simulations performed very differently for the positive real eigenvalue initial condition. I believe this to be true because the initial starting state is not near the equilibrium point from which our linearized system was obtained. So, the linearized system purely diverges after a long time. However, the nonlinear simulation actually converges to a stable equilibrium for the corresponding initial state of a positive real eigenvalued eigenvector plus the equilibrium point vector.