10/21/24 | ECE 6320 / HW8 | Hower, Kade

11.3 (*A*-invariance and controllability). Consider the continuous-time LTI system

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k. \qquad \text{(AB-CLTI)}$$

Prove the following two statements:

(a) The controllable subspace $\mathscr{C}$ of the system (AB-CLTI) is A-invariant.

(b) The controllable subspace $\mathscr{C}$ of the system (AB-CLTI) contains Im B. □

Notation. A linear subspace $\mathscr{V}$ is said to be *A-invariant* if for every vector $v \in \mathscr{V}$ we have $Av \in \mathscr{V}$ (Section 12.2).
► p. 150

a) let $x \in \mathcal{C}_{[t_0\, t_1]} \Rightarrow x \in \text{Im}(\Gamma)$ since LTI

$\Rightarrow \exists\, \eta_1$ s.t. $\Gamma \eta_1 = x \Rightarrow [B \;\; AB \; \dots \; A^{n-1}B]\eta_1 = x$

$\Rightarrow A[B \;\; AB \cdots A^{n-1}]\eta_1 = Ax$

$\Rightarrow [AB \;\; A^2B \cdots A^{n-1}B \;\; A^nB]\eta_1 = Ax$

by Cayley Hamilton we know $A^n = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1}A^{n-1}$

$\Rightarrow [AB \;\; A^2B \dots A^{n-1}B \;\; (\alpha_0 B + \alpha_1 AB + \alpha_2 A^2B + \cdots + \alpha_{n-1}A^{n-1}B)]\eta_1 = Ax$

Since $\text{Im}(\alpha_0 B + \alpha_1 AB + \cdots A_{n-1}A^{n-1}B)$ will at least contain $\text{Im}(B)$

we can add B's colums to form Γ, multiplied by a new η_2

$\Rightarrow [B \;\; AB \;\; A^2B \cdots A^{n-1}B]\eta_2 = Ax$

$\Rightarrow \Gamma\eta_2 = Ax \Rightarrow Ax \in \text{Im}(\Gamma)$

b) We know $\mathcal{C}_{[t_0\, t_1]} = \text{Im}(\Gamma) = \text{Im}([B \;\; AB \cdots A^{n-1}B])$

Since adding columns to a matrix can never reduce its image space

$\Rightarrow \text{Im}(B) \in \text{Im}([B \;\; AB \cdots A^{n-1}B])$

$\Rightarrow \text{Im}(B) \in \mathcal{C}_{[t_0\, t_1]}$

12.6 (Satellite). The equations of motion of a satellite linearized around a steady-state solution, are given by

$$\dot{x} = Ax + Bu, \qquad A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 1 \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

where the state vector $x := \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}$ includes the perturbation x_1 in the orbital radius, the perturbation x_2 in the radial velocity, the perturbation x_3 in the angle, and the perturbation x_4 in the angular velocity; and the input vector $u := \begin{bmatrix} u_1 & u_2 \end{bmatrix}$ includes the radial thruster u_1 and a tangential thruster u_2.

(a) Show that the system is controllable from the input vector u.

(b) Can the system still be controlled if the radial thruster does not fire? What if it is the tangential thruster that fails? □

a) $\Gamma := \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$, $\operatorname{rank}(\Gamma) = 4 \Rightarrow$ Completely Controllable

no radial thruster

b) With $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\operatorname{rank}(\Gamma) = 4 \Rightarrow$ Completely Controllable

With $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\operatorname{rank}(\Gamma) = 3 \Rightarrow$ NOT Completely Controllable

no tangential thruster

12.7 (Controllability of local linearization: nonholonomic integrator). Consider the following nonlinear system, known as the *nonholonomic interator*:

$$\begin{cases} \dot{x}_1 = -x_1 + u_1 \\ \dot{x}_2 = -x_2 + u_2 \\ \dot{x}_3 = x_2 u_1 - x_1 u_2, \end{cases} \qquad y = x_1^2 + x_2^2 + x_3^2.$$

(a) Linearize the system around the equilibrium point $x_1 = x_2 = x_3 = 0$. Is the linearized system controllable?

(b) Linearize the system around the equilibrium point $x_1 = x_2 = x_3 = 1$. Is the linearized system controllable? □

MATLAB 'jacobian()'

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -u_2 & u_1 & 0 \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ x_2 & -x_1 \end{bmatrix}$$

a) $x_{eq_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_{eq_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftarrow f(x_{eq_1}, u_{eq_1}) = 0$

$$A_1 = \left.\frac{\partial f}{\partial x}\right|_{x_{eq_1}, u_{eq_1}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \left.\frac{\partial f}{\partial u}\right|_{x_{eq_1}, u_{eq_1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Gamma_1 = [B_1\ A_1B_1\ A_1^2B_1]$, $\operatorname{rank}(\Gamma_1) = 2 \Rightarrow$ **Not complety controllable**

b) $x_{eq_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow u_{eq_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftarrow f(x_{eq}, u_{eq}) = 0$

$$A_2 = \left.\frac{\partial f}{\partial x}\right|_{x_{eq_2}, u_{eq_2}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad B_2 = \left.\frac{\partial f}{\partial u}\right|_{x_{eq_2}, u_{eq_2}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$\Gamma_2 = [B_2\ A_2B_2\ A_2^2B_2]$, $\operatorname{rank}(\Gamma_2) = 2 \Rightarrow$ **Not complety controllable**

13.1 (Controllable decomposition). Consider an LTI system with realization

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Is this realization controllable? If not, perform a controllable decomposition to obtain a controllable realization of the same transfer function. ☐

$\Gamma = [B \;\; AB]$, $\operatorname{rank}(\Gamma) = 1 \Rightarrow$ not completely controllable.

Choose $T = [v \;\; w]$, where v is basis set of $\operatorname{Im}(\Gamma)$ and w is a basis set for $N(\Gamma^T)$

$$T = \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix} \qquad \begin{array}{l} v = \operatorname{orth}(\Gamma) \\ w = \operatorname{null}(\Gamma') \end{array}$$

$$\Rightarrow \hat{A} = T^{-1}AT = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{B} = T^{-1}B = \begin{bmatrix} 1.4142 \\ 0 \end{bmatrix}.$$

$$\hat{C} = TC = \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}, \quad \hat{D} = D = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

where $T\hat{x} = x$

Original TF: $G = \begin{bmatrix} 2 - \frac{1}{s+1} \\ 1 + \frac{1}{s+1} \end{bmatrix}$

Choose $\bar{A} = [-1]$, $\bar{B} = [1.4142]$, $\bar{C} = [-0.7071]$, $\bar{D} = [2]$

$\Rightarrow$ controllable realization is $\dot{x}_1 = \bar{A}x_1 + \bar{B}u_1$, $y_1 = \bar{C}x_1 + \bar{D}u_1$

TF of controllable realization is

$$\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = -\frac{1}{\sqrt{2}}(s+1)^{-1}\sqrt{2} + 2 = 2 - \frac{1}{s+1}$$

$\Rightarrow$ Same TF for $\frac{Y_1}{X_1} = G_1 = 2 - \frac{1}{s+1}$

Control Design review (20 points)

For each of the following systems, state whether it is completely controllable, stabilizable, or neither. If it is not completely controllable, perform a controllable decomposition and state the transform. Create a stabilizing control law if possible. If not possible, state why.

A. $\dot{x} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

B. $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

C. $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

D. $\dot{x} = \begin{bmatrix} -100 & 0 \\ 0 & -100 \end{bmatrix} x$

A. $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\Gamma = [B \;\; AB]$

$\text{rank}(\Gamma) = 2 \Rightarrow$ completely controllable

choose $u = -Kx \Rightarrow \dot{x} = Ax - BKx = (A - BK)x = \bar{A}x$

where $\bar{A} = (A - BK)$ and $K = [K_1 \;\; K_2]$

$\Rightarrow \bar{A} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \;\; K_2] = \begin{bmatrix} 0 & -2 \\ 2-K_1 & -K_2 \end{bmatrix}$

$\chi_{\bar{A}}(\lambda) = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 2 \\ K_1 - 2 & \lambda + K_2 \end{bmatrix}\right) = \lambda^2 + K_2\lambda - (2K_1 - 4)$

$= \lambda^2 + (K_2)\lambda + (4 - 2K_1)$

desired char. poly.

$\phi_{\bar{A}}(\lambda) = (\lambda + 2)(\lambda + 2) = \lambda^2 + 4\lambda + 4$

choosing $K_2 = 4$, $K_1 = 0$ $\Rightarrow \chi_{\bar{A}}(\lambda) = \lambda^2 + 4\lambda + 4$

$\Rightarrow \bar{A} = \begin{bmatrix} 0 & -2 \\ 2 & -4 \end{bmatrix} \Rightarrow \text{eig}(\bar{A}) = \{-2, -2\}$ and system is stabilized

Control Design review (20 points)

For each of the following systems, state whether it is completely controllable, stabilizable, or neither. If it is not completely controllable, perform a controllable decomposition and state the transform. Create a stabilizing control law if possible. If not possible, state why.

A. $\dot{x} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

B. $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

C. $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

D. $\dot{x} = \begin{bmatrix} -100 & 0 \\ 0 & -100 \end{bmatrix} x$

B. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} B & AB \end{bmatrix}$

$\text{rank}(\Gamma) = 1 \Rightarrow$ not completely controllable

Choose $T = \begin{bmatrix} v & w \end{bmatrix}$ where v is a basis of $\text{Im}(\Gamma)$, w is a basis of $N(\Gamma^T)$

$\Rightarrow T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$\Rightarrow \hat{A} = T^{-1}AT = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\hat{B} = T^{-1}B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\hat{A}_{22}$

Since $\hat{A}_{22} = [1]$ and is uncontrollable, the system is NOT stabilizable

Control Design review (20 points)

For each of the following systems, state whether it is completely controllable, stabilizable, or neither. If it is not completely controllable, perform a controllable decomposition and state the transform. Create a stabilizing control law if possible. If not possible, state why.

A. $\dot{x} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

B. $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

C. $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

D. $\dot{x} = \begin{bmatrix} -100 & 0 \\ 0 & -100 \end{bmatrix} x$

C. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\Gamma = [B \;\; AB] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

$\text{rank}(\Gamma) = 1 \Rightarrow$ not completely controllable

Choose $T = [v \;\; w]$ where v is a basis of $\text{Im}(\Gamma)$, orth(Γ)
w is a basis of $N(\Gamma^T)$

$\Rightarrow T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$\Rightarrow \hat{A} = T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\hat{B} = T^{-1}B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\hat{A}_{11}$ $\hat{B}_1$ $\hat{A}_{22}$

Since $\hat{A}_{22} = [-1]$, the system is stabilizable

To stabilize $\text{eig}(\hat{A}_{11} - \hat{B}_1 K_1)$ must all have real part < 0

Since $\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$ and $u = -K\hat{x}_1 \Rightarrow \dot{\hat{x}} = (\hat{A} - \hat{B}K)\hat{x}$

let $\bar{A} = \hat{A} - \hat{B}K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 1+K_1 & K_2 \\ 0 & -1 \end{bmatrix}$

So, choose $K_1 = -2$, $K_2 = 0$ $\Rightarrow \bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\Rightarrow$ The system $\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$ is now stabilized

where $u = -K\hat{x} = -[-2 \;\; 0]\hat{x}$

$u = -KT^{-1}x = [2 \;\; 0]\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} x$

$u = -[0 \;\; 2]\, x$

Control Design review (20 points)

For each of the following systems, state whether it is completely controllable, stabilizable, or neither. If it is not completely controllable, perform a controllable decomposition and state the transform. Create a stabilizing control law if possible. If not possible, state why.

A. $\dot{x} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

B. $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

C. $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

D. $\dot{x} = \begin{bmatrix} -100 & 0 \\ 0 & -100 \end{bmatrix} x$

D. $A = \begin{bmatrix} -100 & 0 \\ 0 & -100 \end{bmatrix}$, the system is not completely controllable but is stabilizable since A is a stability matrix.

Any control law results in a stable system

Segway-like Robot Control (20 points)

Consider the Segway-like robot described in InvertedPendulumWheeledVehicle.pdf and shell Matlab function in CreateSegwayControl.m. Do the following:

1 - Show that the point of zero-state and zero-input is an equilibrium point
2 - Evaluate the controllability of the linearized system about of the zero-state, zero-input equilibrium
3 - Show that the linearized formulation for $z = \begin{bmatrix} \omega & v & \phi & \dot{\phi} \end{bmatrix}^T$ about $z = 0, u = 0$ is completely controllable.
4 - Explain why z can be controlled in isolation from other states.
    - Hint: a controllability decomposition may be helpful (use the identity matrix with the columns shifted around)
5 - Create a feedback control law that will render the z LAS (explicitly state the control law)

1 Using MATLAB: subs(f, [x' u'], zeros([1 length(x')+length(u')]))

returns all zeros => $f(x_{eq}, u_{eq}) = 0$ for zero-state, zero-input

=> equilibrium point ✓

2 When $A = \left.\frac{\partial f}{\partial x}\right|_{x_{eq}, u_{eq}}$ and $B = \left.\frac{\partial f}{\partial u}\right|_{x_{eq}, u_{eq}}$, $\Gamma = \text{ctrb}(A, B)$, $\text{rank}(\Gamma) = 6$

=> not completely controllable

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 72.49 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.029 \\ -1.67 & 0 \\ 0 & 0 \\ -24.15 & 0 \end{bmatrix}$$

3 New $\Gamma = \text{ctrb}(A,B)$, $\text{rank}(\Gamma) = 4 \Rightarrow$ completely controllable

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2.16 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 72.49 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0.029 \\ -1.67 & 0 \\ 0 & 0 \\ -24.15 & 0 \end{bmatrix}$$

4. Show why we can control z in isolation

choosing

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

results in

$$\hat{A} = T^{-1}AT = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 72.49 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} 0 & 0.029 \\ -1.67 & 0 \\ 0 & 0 \\ -24.15 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\dot{\hat{x}} = \begin{bmatrix} \dot{\hat{x}}_c \\ \dot{\hat{x}}_u \end{bmatrix} = \hat{A} \begin{bmatrix} \hat{x}_c \\ \hat{u} \end{bmatrix} + \hat{B} u$$

$$T\hat{x} = x \Rightarrow \hat{x} = T^{-1} x = \begin{bmatrix} \omega \\ v \\ \phi \\ \dot{\phi} \\ x \\ \psi \\ y \end{bmatrix}$$

$$\Rightarrow \hat{x}_c = \begin{bmatrix} \left.\begin{matrix} \omega \\ v \\ \phi \\ \dot{\phi} \end{matrix}\right\} z \\ x \\ \psi \end{bmatrix}$$

So, since z is contained directly in the controlled portion $\hat{x}_c \Rightarrow z$ can be controlled in isolation

5\. Using place(A, B, [-1, -2, -3, -4]) results in

$$K = \begin{bmatrix} 0.002 & 0.1177 & -3.5847 & -0.2984 \\ 103.0394 & 43.1557 & -148.5785 & -37.2643 \end{bmatrix}$$

where $u = -Kz$

Controllability proofs (20 points)

- Prove that, for LTI systems, the image of the reachability matrix is equal to the image of the Controllability Grammian (i.e. $Im(\Gamma) = Im\Big(W_C(t_0, t_1)\Big)$, where $\Gamma = [B \quad AB \quad \dots \quad A^{n-1}B]$)
  - *Hint: see Lecture S03_L02*

First, to show if $x \in Im(W_c(t_0, t_1)) \Rightarrow x \in Im(\Gamma)$

$$x \in Im(W_c(t_0,t_1)) \Rightarrow \exists\, u(\cdot) \text{ s.t. } 0 = \Phi(t_1,t_0)x + \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)d\tau$$

$$\Rightarrow -\Phi(t_1,t_0)x = \int_{t_0}^{t_1} \Phi(t_1,\tau)Bu(\tau)d\tau$$

$$\Rightarrow x = -\Phi(t_1,t_0)^{-1}\int_{t_0}^{t_1} \Phi(t_1,\tau)Bu(\tau)d\tau = \int_{t_0}^{t_1} -\Phi(t_0,t_1)\Phi(t_1,\tau)Bu(\tau)d\tau$$

$$\Rightarrow x = \int_{t_0}^{t_1} -\Phi(t_0,\tau)Bu(\tau)d\tau = \int_{t_0}^{t_1} -e^{A(t_0-\tau)}Bu(\tau)d\tau$$

$$\Rightarrow x = \int_{t_0}^{t_1} \sum_{i=0}^{n-1} \alpha_i(t_0-\tau)A^i Bu(\tau)d\tau = \int_{t_0}^{t_1} \sum_{i=0}^{n-1} A^i Bu(\tau)\alpha_i(t_0-\tau)d\tau$$

$$= \sum_{i=0}^{\infty} A^i B \int_{t_0}^{t_1} u(\tau)\alpha_i(t_0-\tau)d\tau$$

$$= \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{\Gamma} \underbrace{\begin{bmatrix} \int_{t_0}^{t_1} u(\tau)\alpha_1(t_0-\tau)d\tau \\ \vdots \\ \int_{t_0}^{t_1} u(\tau)\alpha_{n-1}(t_0-\tau)d\tau \end{bmatrix}}_{\eta}$$

$$x = \Gamma\eta \Rightarrow x \in Im(\Gamma)$$

$$\Rightarrow \text{if } x \in (W_c(t_0,t_1)) \Rightarrow x \in Im(\Gamma)$$

Second, to show if $x \in Im(\Gamma) \Rightarrow x \in Im(W_c(t_0, t_1))$

$x \in Im(\Gamma) \Rightarrow \exists\ v$ s.t. $x = \Gamma v$

Let $\eta_1 \in N(W_c(t_0, t_1))$. From before, we know $B^T(\tau)\Phi^T(t_0, \tau)\eta_1 = 0$

$$\Rightarrow B^T(e^{(t_0-\tau)A})^T \eta_1 = 0$$
$$\Rightarrow \eta_1^T e^{(t_0-\tau)A} B = 0 \quad \forall\ \tau$$

take derivatives

$\Rightarrow$

$$\frac{d}{d\tau}(\eta_1^T e^{(t_0-\tau)A} B) = -\eta_1^T A e^{(t_0-\tau)A} B = 0$$

$$\frac{d^2}{d\tau^2}(\quad '' \quad) = \eta_1^T A^2 e^{(t_0-\tau)A} B = 0$$

$$\vdots$$

$$\frac{d^k}{d\tau^k}(\quad '' \quad) = (-1)^k \eta_1^T A^k e^{(t_0-\tau)} B = 0$$

When $\tau = t_0 \Rightarrow (-1)^k \eta_1^T A^k \underbrace{e^{(t_0-t_0)A}}_{I} B = 0$

$$\Rightarrow (-1)^k \eta_1^T A^k B = 0$$

For $k=0$	For $k=1$	For $k=2$	$\cdots$	For $k=n-1$
$\eta_1^T B = 0$	$\eta_1^T AB = 0$	$\eta_1^T A^2 B = 0$	$\cdots$	$\eta_1^T A^{n-1} B = 0$

So, $\eta_1^T [B\ AB \cdots A^{n-1}B] = 0 \Rightarrow \eta_1^T \Gamma = 0$

Since $\eta_1 \in N(W_c(t_0, t_1))$ and $x = \Gamma v \Rightarrow \eta_1^T x = \eta_1^T \Gamma v = 0$

$\Rightarrow x \in (N(W_c(t_0, t_1)))^{\perp} = x \in Im(W_c(t_0, t_1))$ $\square$

So, Since $x \in Im(W_c(t_0, t_1)) \Rightarrow x \in Im(\Gamma)$ ①

AND

$x \in Im(\Gamma) \Rightarrow x \in Im(W_c(t_0, t_1))$ ②

We know, $\underline{Im(\Gamma) = Im(W_c(t_0, t_1))}$ $\square$

Extra Credit

14.2 (Transformation to controllable canonical form). Consider the following third-order SISO LTI system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^3, \; u \in \mathbb{R}^1. \tag{AB-CLTI}$$

Assume that the characteristic polynomial of A is given by

$$\det(sI - A) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$$

and consider the 3×3 matrix

$$T := \mathcal{C} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{14.5}$$

where $\mathcal{C}$ is the system's controllability matrix.

(a) Show that the following equality holds:

$$B = T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Show that the following equality holds:

$$AT = T \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hint: Compute separately the left- and right-hand side of the equation above and then show that the two matrices are equal with the help of the Cayley-Hamilton theorem.

(c) Show that if the system (AB-CLTI) is controllable, then T is a nonsingular matrix.

(d) Combining parts (a)–(c), you showed that, if the system (AB-CLTI) is controllable, then the matrix T given by equation (14.5) can be viewed as a similarity transformation that transforms the system into the controllable canonical form

$$T^{-1}AT = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Use this to find the similarity transformation that transforms the following pair into the controllable canonical form

$$A := \begin{bmatrix} 6 & 4 & 1 \\ -5 & -4 & 0 \\ -4 & -3 & -1 \end{bmatrix}, \quad B := \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Hint: You may use the MATLAB® functions `poly(A)` to compute the characteristic polynomial of `A` and `ctrb(A,B)` to compute the controllability matrix of the pair `(A,B)`. □

a) $$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = C\begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [B \; AB \; A^2B]$$

$$= [B \; AB \; A^2B]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = B$$

b) Left Hand: $AT = AC\begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow AT = A[B \; AB \; A^2B]\begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix} = A[B \quad \alpha_1 B + AB \quad \alpha_2 B + \alpha_1 AB + A^2B]$$

$$= [AB \quad \alpha_1 AB + A^2B \quad \alpha_2 AB + \alpha_1 A^2B + A^3B]$$

By Cayley Hamilton $A^3 = -\alpha_1 A^2 - \alpha_2 A - \alpha_3$

$$= [AB \quad \alpha_1 AB + A^2B \quad \cancel{\alpha_2 AB} + \cancel{\alpha_1 A^2B} + (-\cancel{\alpha_1 A^2} - \cancel{\alpha_2 A} - \alpha_3)B]$$

$$= [AB \quad \alpha_1 AB + A^2B \quad -\alpha_3 B]$$ 1

Right Hand: $T\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = C\begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} B & AB & A^2B \end{bmatrix}\begin{bmatrix} 0 & 0 & -\alpha_3 \\ 1 & \alpha_1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} AB & \alpha_1 AB + A^2B & -\alpha_3 B \end{bmatrix}$ ②

① = ② $\Rightarrow AT = T\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ✓

c) System controllable $\Rightarrow$ C is full rank

since $\begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix}$ is clearly full rank

and a full rank matrix multiplied by another full rank matrix, $T = C\begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix}$ is full rank

d) Using poly(A) and ctrb(A,B) to compute $T = C\begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_2 \\ 0 & 0 & 1 \end{bmatrix}$,

the similarity transform $T = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

checking:

$$T^{-1}AT = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \checkmark$$

$$T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$