10/07/24	ECE 6320/HW6	Hower, Kade

Answer the following for each of the linearized systems above:

- Is the system controllable?
- Is the system stabilizable?

2.3 (Local linearization around equilibrium: saturated inverted pendulum). Consider the inverted pendulum in Figure 2.8 and assume the input and output to the system are the signals u and y defined as

$$T = \mathrm{sat}(u), \qquad y = \theta,$$

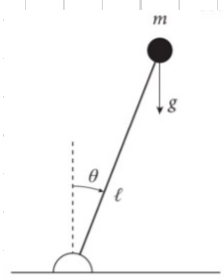
where "sat" denotes the unit-slope saturation function that truncates u at $+1$ and -1.

(a) Linearize this system around the equilibrium point for which $\theta = 0$.

(b) Linearize this system around the equilibrium point for which $\theta = \pi$ (assume that the pendulum is free to rotate all the way to this configuration without hitting the table).

(c) Linearize this system around the equilibrium point for which $\theta = \frac{\pi}{4}$. Does such an equilibrium point always exist?

~~(d)~~ Assume that $b = 1/2$ and $mg\ell = 1/4$. Compute the torque $T(t)$ The needed for the pendulum to fall from $\theta(0) = 0$ with constant velocity $\dot{\theta}(t) = 1, \forall t \geq 0$. Linearize the system around this trajectory. ☐



From Newton's law:

$$m\ell^2\ddot{\theta} = mg\ell \sin\theta - b\dot{\theta} + T,$$

where T denotes a torque applied at the base, and g is the gravitational acceleration.

FIGURE 2.8. Inverted pendulum.

a) Linearization from Homework 2:

$$\Rightarrow \delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}}_{B} \delta u$$

$$y = g(x, u) = x_1$$

$$\Rightarrow \delta y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \delta x + 0\, \delta u$$

$$D = 0$$

where $\delta x = x - x^{eq}$, $\delta y = y - x^{eq}$, $\delta u = u - u^{eq}$

Finding Eigenvalues

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 39.2 & -156.8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 156.8 \end{bmatrix}$$

Using MATLAB, $\mathrm{rank}(\Gamma) = 2 = n \Rightarrow$ Controllable and stabilizable

where $\Gamma = [B \;\; AB]$

b) Linearization from Homework 2

$$\Rightarrow \delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}}_{B} \delta u$$

$$y = g(x, u) = x_1$$

$$\Rightarrow \delta y = \overset{C}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \delta x$$

where $\delta x = x - x^{eq}$, $\delta y = y - x^{eq}$, $\delta u = u - u^{eq}$

Finding Eigenvalues

($g = 9.8$, $m = 1/9.8$, $\ell = 0.25$, $L = 1$)

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -39.2 & -156.8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 156.8 \end{bmatrix}$$

Using MATLAB, $\text{rank}(\Gamma) = 2 = n \Rightarrow$ Controllable and stabilizable

where $\Gamma = [B \;\; AB]$

C) Linearization from Homework 2

$$\delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{g}{\ell\sqrt{2}} & -\frac{b}{m\ell^2} \end{bmatrix}}_{A} \delta x + \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix} \delta u$$

$y = g(x, u) = x_1$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

where $\delta x = x - x^{eq}$, $\delta y = y - x^{eq}$, $\delta u = u - u^{eq}$

Finding Eigenvalues

$(g = 9.8,\ m = \frac{1}{9.8},\ \ell = 0.25,\ b = 1)$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell\sqrt{2}} & \frac{-b}{m\ell^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 27.72 & -156.8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 156.8 \end{bmatrix}$$

Using MATLAB, $\text{rank}(\Gamma) = 2 = n \Rightarrow$ Controllable and Stabilizable

where $\Gamma = \begin{bmatrix} B & AB \end{bmatrix}$

2.4 (Local linearization around equilibrium: pendulum). The following equation models the motion of a frictionless pendulum:

$$\ddot{\theta} + k \sin\theta = \tau$$

where $\theta \in \mathbb{R}$ is the angle of the pendulum with the vertical, $\tau \in \mathbb{R}$ an applied torque, and k a positive constant.

(a) Compute a state-space model for the system when $u := \tau$ is viewed as the input and $y := \theta$ as the output. Write the model in the form

$$\dot{x} = f(x, u) \qquad y = g(x, u)$$

for appropriate functions f and g.

(b) Find the equilibrium points of this system corresponding to the constant input $\tau(t) = 0$, $t \geq 0$.

Hint: There are many.

(c) Compute the linearization of the system around the solution $\tau(t) = \theta(t) = \dot{\theta}(t) = 0$, $t \geq 0$.

Hint: Do not forget the output equations. ☐

c) Linearization from Homework 2

$$X = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad u_{eq} = 0$$

$$\delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}}_{A} \delta x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

where $\delta x = x - x^{eq}$, $\delta y = y - y^{eq}$, $\delta u = u - u^{eq}$

Finding eigenvalues

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using MATLAB, $\text{rank}(\Gamma) = 2 = n \Rightarrow$ Controllable and stabilizable

where $\Gamma = [B \ \ AB]$

2.6 (Local linearization around equilibrium: one-link robot). Consider the one-link robot in Figure 2.9, where θ denotes the angle of the link with the horizontal, τ the torque applied at the base, (x, y) the position of the tip, ℓ the length of the link, I its moment of inertia, m the mass at the tip, g gravity's acceleration, and b a friction coefficient. This system evolves according to the following equation:

$$I\ddot{\theta} = -b\dot{\theta} - gm\cos\theta + \tau.$$

(a) Compute the state-space model for the system when $u = \tau$ is regarded as the input and the vertical position of the tip y is regarded as the output.

Please denote the state vector by z to avoid confusion with the horizontal position of the tip x, and write the model in the form

$$\dot{z} = f(z, u) \quad y = g(z, u)$$

for appropriate functions f and g.

Hint: Do not forget the output equation!

(b) Show that $\theta(t) = \pi/2$, $\tau(t) = 0$, $\forall t \geq 0$ is a solution to the system and compute its linearization around this solution.

From your answer, can you predict if there will be problems when one wants to control the tip position close to this configuration just using feedback from y? □

b) Linearization from Homework 2 $z^{sol} = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}$, $u^{sol} = 0 \Rightarrow y^{sol} = \ell$

$$\delta\dot{z} = \begin{bmatrix} 0 & 1 \\ \frac{gm}{I} & -\frac{b}{I} \end{bmatrix} \delta z + \begin{bmatrix} 0 \\ 1/I \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 0 & 0 \end{bmatrix} \delta + 0 \, \delta u$$

where $\delta z = z - z^{eq}$, $\delta u = u - u^{eq}$, $\delta y = y - y^{eq}$

Finding Eigenvalues

$g = 9.8$ $m = \frac{1}{9.8}$ $I = 1$ $b = 2$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{gm}{I} & -\frac{b}{I} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/I \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using MATLAB, $\text{rank}(\Gamma) = 2 = n \Rightarrow$ Controllable and Stabilizable

where $\Gamma = [B \;\; AB]$

2.7 (Local linearization around trajectory: unicycle). A single-wheel cart (unicycle) moving on the plane with linear velocity v and angular velocity ω can be modeled by the nonlinear system

$$\dot{p}_x = v\cos\theta, \quad \dot{p}_y = v\sin\theta, \quad \dot{\theta} = \omega, \tag{2.13}$$

where (p_x, p_y) denote the Cartesian coordinates of the wheel and θ its orientation. Regard this as a system with input $u := \begin{bmatrix} v & \omega \end{bmatrix}' \in \mathbb{R}^2$.

(a) Construct a state-space model for this system with state

✗

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} p_x\cos\theta + (p_y - 1)\sin\theta \\ -p_x\sin\theta + (p_y - 1)\cos\theta \\ \theta \end{bmatrix}$$

and output $y := \begin{bmatrix} x_1 & x_2 \end{bmatrix}' \in \mathbb{R}^2$.

(b) Compute a local linearization for this system around the equilibrium point $x^{\text{eq}} = 0$, $u^{\text{eq}} = 0$.

(c) Show that $\omega(t) = v(t) = 1$, $p_x(t) = \sin t$, $p_y(t) = 1 - \cos t$, $\theta(t) = t$, $\forall t \geq 0$ is a solution to the system.

(d) Show that a local linearization of the system around this trajectory results in ✗ an LTI system. □

b) Linearization from Homework 2

$$\dot{\delta x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \delta x + 0\,\delta u$$

Where $\delta x = x - x_{eq}$, $\delta y = y - x_{eq}$, $\delta u = u - u_{eq}$

Using MATLAB, $\text{rank}(\Gamma) = 2$ $\Rightarrow$ not controllable

for $\lambda = 0$, $\text{rank}(\lambda I - A, B) = 2$ $\Rightarrow$ not stabilizable

d) Linearization from Homework 2

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\delta u$$

where $\delta x = x - x_{sol}$, $\delta y = y - y_{sol}$, $\delta u = u - u_{sol}$

$$x_{sol} = \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix}, \quad y_{sol} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad u_{sol} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Using MATLAB, $\text{rank}(\Gamma) = 3 = n \Rightarrow$ controllable and stabilizable

where $\Gamma = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$

Control Design (60 points)

Simple system design (30)

Design a control law that will render the controlled system GAS for each of the three systems below. If it is not possible, state why.

$$S1 : \dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$S2 : \dot{x} = \begin{bmatrix} -1 & -2 \\ 6 & 7 \end{bmatrix} x + \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} u$$

$$S3 : \dot{x} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u$$

S1:

$$\bar{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [K_1 \quad K_2] = \begin{bmatrix} 1 & 2 \\ 3-K_1 & 4-K_2 \end{bmatrix}$$ where $K = [K_1 \; K_2]$

$\text{rank}([B \quad AB]) = 2 \Rightarrow$ controllable and stabilizable

$$\chi_{\bar{A}} = \det[\lambda I - \bar{A}] = \det \begin{bmatrix} \lambda-1 & -2 \\ K_1-3 & \lambda-4+K_2 \end{bmatrix} = (\lambda-1)(\lambda-4+K_2) - (-2)(K_1-3)$$

$$= \lambda^2 - 4\lambda + K_2\lambda - \lambda + 4 - K_2 + 2K_1 - 6$$

$$= \lambda^2 + (K_2-5)\lambda + (2K_1 - K_2 - 2) = 0$$

Choose desired $\phi(\lambda) = (\lambda+1)(\lambda+1)$

$$= \lambda^2 + 2\lambda + 1$$

Set $\chi_{\bar{A}}(\lambda) = \phi(\lambda) \Rightarrow \lambda^2 + (K_2-5)\lambda + (2K_1 - K_2 - 2) = \lambda^2 + 2\lambda + 1$

$$\Rightarrow K_2 - 5 = 2, \quad 2K_1 - K_2 - 2 = 1$$

$$\Rightarrow K_2 = 7 \quad \Rightarrow \quad K_1 = 5$$

check ✓

$$\Rightarrow K = [K_1 \quad K_2] = [5 \quad 7]$$

$\text{eig}(\bar{A}) = \{-1, -1\}$

$$\Rightarrow u = -Kx = -[5 \quad 7]x$$

S2:

$$\bar{A} = \begin{bmatrix} -1 & -2 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$ where $K = [K_1 \; K_2]$

$$= \begin{bmatrix} 0.5K_1 - 1 & 0.5K_2 - 2 \\ 6 - K_1 & 7 - K_2 \end{bmatrix}$$

$$\begin{matrix} -0.5K_1 & -0.5K_2 \\ K_1 & K_2 \end{matrix}$$

rank([B AB]) = 2 => controllable and stabilizable

$$\chi_A(\lambda) = \det(\lambda I - \bar{A}) = \det \begin{bmatrix} \lambda - 0.5K_1 + 1 & 2 - 0.5K_2 \\ K_1 - 6 & \lambda - 7 + K_2 \end{bmatrix}$$

$$= \lambda^2 + \lambda(-7 + K_2) + \lambda(-0.5K_1 + 1) + 3.5K_1 - \cancel{0.5K_1 K_2} + K_2 - 7$$
$$- (2K_1 - \cancel{0.5K_1 K_2} - 12 + 3K_2)$$

$$= \lambda^2 + (-0.5K_1 + K_2 - 6)\lambda + (1.5K_1 - 2K_2 + 5) = 0$$

for $\text{eig}(\bar{A}) = \{-1, -1\}$

$\chi_A(\lambda) = \lambda^2 + 2\lambda + 1$ =>
$$-0.5K_1 + K_2 - 6 = 2 \quad ①$$
$$1.5K_1 - 2K_2 + 5 = 1 \quad ②$$

using MATLAB to solve ① + ②

=> $K_1 = 24, \; K_2 = 20$

Check ✓

$\text{eig}(\bar{A}) = \{-1, -1\}$

=> $K = [K_1 \; K_2] = [24 \;\; 20]$

=> $u = -Kx = -[24 \;\; 20]x$

S3: $\bar{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{13} \end{bmatrix}$

$\text{rank}([B \ \ AB \ \ A^2B]) = 3 \Rightarrow$ Controllable and stabilizable

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{11}+K_{21} & K_{12}+K_{22} & K_{13}+K_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 1-K_{11} & 2-K_{11} & 3-K_{13} \\ 4-K_{21} & 5-K_{22} & 6-K_{23} \\ 7-K_{11}-K_{21} & 8-K_{12}-K_{22} & 9-K_{13}-K_{23} \end{bmatrix}$$

We want $\bar{A}$ to be upper triangular so that the diagonals are the eigenvalues. So, first choose $4-K_{21}=0$

$\Rightarrow \underline{K_{21} = 4}$ ①

Now, $7-K_{11}-K_{21}=0 \Rightarrow \underline{K_{11}=3}$ ②

Next, $8-K_{12}-K_{22}=0$ and $5-K_{22}<0$

choose $\underline{K_{22}=6} \Rightarrow \underline{K_{12}=2}$

Lastly, $9-K_{13}-K_{23}<0$ choose $\underline{K_{13}=0,\ K_{23}=10}$

$\Rightarrow \bar{A} = \begin{bmatrix} -2 & 0 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{bmatrix}, \Rightarrow \text{eig}(\bar{A}) = \{-2, -1, -1\}$

check ✓

$$\Rightarrow u = -Kx = -\begin{bmatrix} 3 & 2 & 0 \\ 4 & 6 & 10 \end{bmatrix} x$$

Orbit-plane motion (30 points)

The equations for the orbit-plane motion of a satellite in orbit about a planet with an ideal inverse-square gravity field are:

$$\ddot{r} - \dot{\theta}^2 r = -\frac{\mu}{r^2} + a_r$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_i$$

where a_r and a_i are the radial and in-track components of any acceleration terms due to thrust, drag, gravitational anomalies, and the like. These will be treated as components of the input vector $u = \begin{bmatrix} a_r & a_i \end{bmatrix}^T$. Use $\mu = 4.302 \times 10^{-3}$ and $R = 200$ (Note, this is not a realistic number, but it does allow you to simulate in a small amount of time).

1. Determine the equations that describe small perturbations about a circular orbit of radius R
   - Let the state vector be defined as $x = \begin{bmatrix} r & \theta & \dot{r} & \dot{\theta} \end{bmatrix}^T$
   - Assume that the nominal inputs are zero
   - Hint: You should find that the resulting system is LTI even though you linearize about a trajectory
2. What can be said about stability using the linearized system?
3. Is the linearized system completely controllable, stabilizable, or not stabilizable (Justify your answer)
4. Design a feedback control law which will render the linearized system LAS or state that it is not possible. If it is possible, explicitly state the control law.
5. Simulate the system to show convergence of the state to the desired trajectory. Turn in plots of the states and controls over time. (Files are in the Homework Week 7 folder)
   - SatelliteControlDesign.m: Is a shell for symbolically creating the control design
   - SatelliteSimulation.m: Simulation for the satellite.
     - Change lines 22 and 23 to incorporate the feedback matrix from SatelliteControlDesign.m
     - Update the "getDesiredState(...)" function to input the desired state
     - Update the "input(...)" function to calculate the control input

1. $\ddot{r} - \dot{\theta}^2 r = -\frac{\mu}{r^2} + a_r \Rightarrow \ddot{r} = \dot{\theta}^2 r - \frac{\mu}{r^2} + a_r$

$r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_i \Rightarrow \ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{a_i}{r}$

$$x = \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix}, \quad u = \begin{bmatrix} a_r \\ a_i \end{bmatrix} \quad f(x,u) = \begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_4^2 x_1 - \frac{\mu}{x_1^2} + u_1 \\ -\frac{2x_3x_4}{x_1} + \frac{u_2}{x_1} \end{bmatrix} = \dot{x}$$

Finding solution

$$x_{sol} = \begin{bmatrix} R \\ \omega t \\ 0 \\ \omega \end{bmatrix}, \quad u_{sol} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \dot{x}_{sol} = \begin{bmatrix} 0 \\ \omega \\ 0 \\ 0 \end{bmatrix}$$

$$f(x_{sol}, u_{sol}) = \begin{bmatrix} 0 \\ \omega \\ \omega^2 R - \frac{\mu}{R^2} \\ 0 \end{bmatrix}$$

if

$$\omega = \sqrt{\frac{\mu}{R^3}}$$

$$f(x_{sol}, u_{sol}) = \begin{bmatrix} 0 \\ \omega \\ 0 \\ 0 \end{bmatrix} = \dot{x}_{sol} \quad \checkmark$$

$$\Rightarrow x_{sol} = \begin{bmatrix} R \\ \sqrt{\frac{\mu}{R^3}}\, t \\ 0 \\ \sqrt{\frac{\mu}{R^3}} \end{bmatrix}, \quad u_{sol} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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from SatelliteLinearDesign.m

$$\hookrightarrow \left.\frac{\partial f}{\partial x}\right|_{x_{sol}, u_{sol}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2\mu}{R^3} + \omega^2 & 0 & 0 & 2R\omega \\ 0 & 0 & -\frac{2\omega}{R} & 0 \end{bmatrix}$$

where $\omega = \sqrt{\frac{\mu}{R^3}}$

$$\left.\frac{\partial f}{\partial u}\right|_{x_{sol}, u_{sol}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{R} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1.6e^{-9} & 0 & 0 & 0.0093 \\ 0 & 0 & -2.3e^{-7} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0.005 \end{bmatrix}$$

$\Rightarrow \delta\dot{x} = A\delta x + B\delta u$ where $x = x_{sol} + \delta x$

$u = u_{sol} + \delta u = \delta u$

2. Eigenvalues of $A = \{0, 0, \pm 0.2319j\}$

$\Rightarrow$ No conclusions drawn about stability using linearized system

3. $\text{rank}(\Gamma) = 4 = n$, where $\Gamma = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$, $n=4$

$\Rightarrow$ Linearized system is completely controllable and stabilizable

4.

$\bar{A} = A - BK$ where $K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix}$

Note: can use place() function

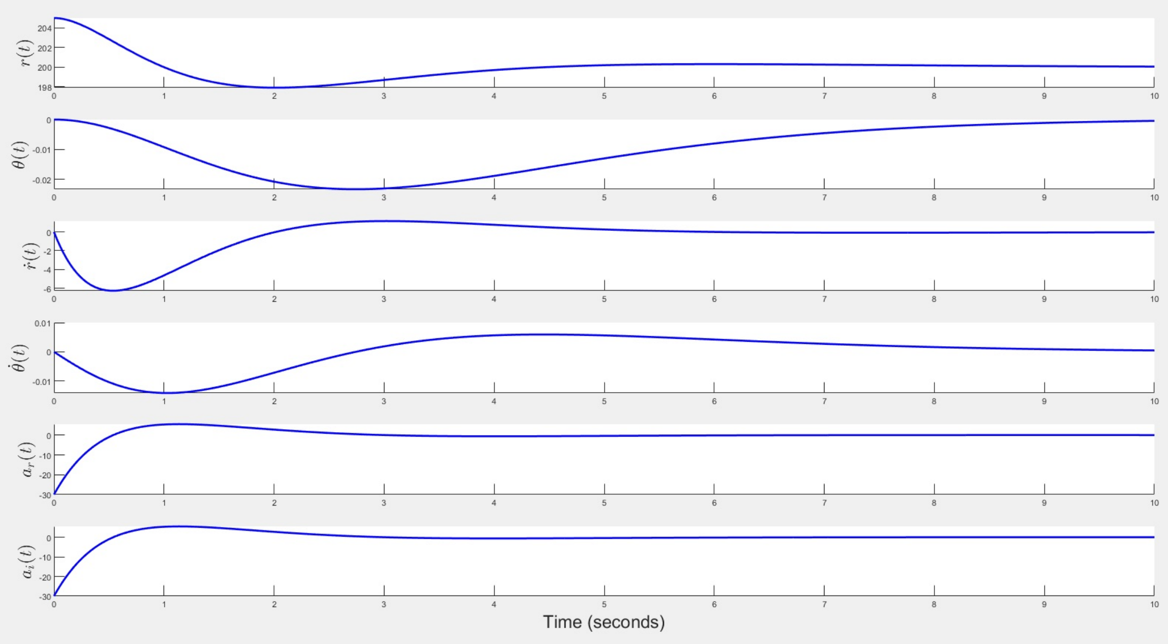
using MATLAB charpoly($\bar{A}$) and solve()
with a desired polynomial created by poly([-1 -1 -1 -1])

$$\Rightarrow K = \begin{bmatrix} 5.9998 & -200 & 4 & -799.9814 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where $\delta u = -K(dx)$ $\quad u - u_{sol} = -K(x - x_{sol})$

Results in closed loop eigenvalues of $\{-1, -1, -1, -1\}$ ✓

5.



Controllability proof (20 points)

- Prove that the controllable subspace is equal to the image of the Controllability Grammian (i.e. $\mathcal{C}(t_0, t_1) = Im\Big(W_C(t_0, t_1)\Big)$)
  - *Hint: see Lecture S03_L01*

$$(i)\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau)$$
$$(ii)\frac{d}{d\tau}\Phi(t,\tau) = \Phi(t,\tau)A(\tau)$$
$$(iii)\Phi(t,t) = I$$
$$(iv)\Phi(t_1,t_0)^{-1} = \Phi(t_0,t_1)$$
$$(v)\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0)$$

$\boxed{x(t)}$

First, to prove if $x_0 \in Im(W_c(t_0, t_1)) \Rightarrow x_0 \in \mathcal{C}(t_0, t_1)$

We know if $x_0 \in Im(W_c(t_0, t_1))$, $\exists\, \eta_0$ s.t. $x_0 = W_c(t_0, t_1)\eta_0$

<u>State evolution from x_0</u>

$$x(t_1) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)\,d\tau$$

where $u(t) = -B^T(t)\Phi^T(t_0, t)\eta_0$

$$x(t_1) = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)(-B^T(\tau))\Phi^T(t_0, \tau)\eta_0\,d\tau$$

left multiply by $\Phi(t_0, t_1)$

$$\Phi(t_0, t_1)\,x(t_1) = \Phi(t_0, t_1)\Phi(t_1, t_0)x_0 - \Phi(t_0, t_1)\int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)\eta_0\,d\tau$$

$$\Phi(t_0, t_1)\,x(t_1) = x_0 - \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)\,d\tau\,\eta_0$$

$$\Phi(t_0, t_1)\,x(t_1) = x_0 - W_c(t_0, t_1)\eta_0$$

$$\Phi(t_0, t_1)\,x(t_1) = x_0 - x_0 = 0$$

$\Rightarrow x(t_1) = 0$, since the state from $t_0 \to t_1$ evolves to 0 from $x_0 \in Im(W_c(t_0, t_1))$,

$x_0 \in \mathcal{C}[t_0, t_1]$

So if $x_0 \in Im(W_c(t_0, t_1)) \Rightarrow x_0 \in \mathcal{C}[t_0, t_1]$

Second, to prove if

$$x_0 \in \mathcal{C}[t_0, t_1] \Rightarrow x_0 \in Im(W_c(t_0, t_1))$$

Since $x_0 \in \mathcal{C}[t_0, t_1] \Rightarrow \exists\, u(\cdot) \in \mathcal{U}$ s.t.

$$0 = \Phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

$$\Rightarrow x_0 = -\Phi^{-1}(t_1, t_0)\int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

$$= -\int_{t_0}^{t_1} \Phi(t_0, t_1)\Phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

$$= -\int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau \quad (a)$$

Now, let $\eta_1 \in N(W_c[t_0, t_1])$, $\Rightarrow W_c[t_0, t_1]\eta_1 = 0$

and $\eta_1^T W_c[t_0, t_1]\eta_1 = 0$

$$\eta_1^T W_c[t_0, t_1]\eta_1 = \int_{t_0}^{t_1} \eta_1^T \Phi(t_0, \tau)B(\tau)\underbrace{B^T(\tau)\Phi^T(t_0, \tau)\eta_1}_{\text{let } v(\tau) =} d\tau$$

$$= \int_{t_0}^{t_1} v^T(\tau)v(\tau)d\tau = \int_{t_0}^{t_1} \|v(\tau)\|^2 d\tau = 0$$

$$\Rightarrow v(\tau) = 0$$

(b)

$$\Rightarrow \underline{B^T(\tau)\Phi^T(t_0, \tau)\eta_1 = 0}$$

Next, if $x_0^T \eta_1 = 0$

$$\Rightarrow x_0 \in (N(W_c[t_0, t_1]))^{\perp}$$

using (a)

$$x_0^T \eta_1 = -\int_{t_0}^{t_1} u^T[\tau] B^T[\tau] \Phi^T(t_0, \tau) \eta_1 \, d\tau = 0$$

$$\Rightarrow x_0 \in (N(W_c[t_0, t_1]))^{\perp}$$

$$\Rightarrow x_0 \in \operatorname{Im}(W_c[t_0, t_1])$$

So, if $x_0 \in \mathcal{C}[t_0, t_1]$, then $x_0 \in \operatorname{Im}(W_c(t_0, t_1))$

Therefore, since

① $x_0 \in \operatorname{Im}(W_c(t_0, t_1)) \Rightarrow x_0 \in \mathcal{C}(t_0, t_1)$

and

② $x_0 \in \mathcal{C}(t_0, t_1) \Rightarrow x_0 \in \operatorname{Im}(W_c(t_0, t_1))$

$$\mathcal{C}(t_0, t_1) = \operatorname{Im}(W_c(t_0, t_1))$$

$\square$

Exact control design (20 points):

a) Design an open-loop controller that will take the system from x_1 to x_2 in exactly 5 seconds.

- Explicitly state the open-loop controller using the system matrices and x_1, x_2.
- Hint: you can create the open-loop control without any knowledge of (A,B) except to know that the system is completely controllable.

b) Consider an 8 state, completely controllable linearized model of a helicopter ($\dot{x} = Ax + Bu$) with the specific points $x_1 = 0$, $x_2 = [0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5]^T$. Modify Linear_Helicopter_exact_control_design.m to simulate the state from 0 to 5 seconds under the following conditions (with $\hat{x}(0) = x_1$)

- $x(0) = x_1$
- $x(0) = rand(8, 1)$ (matlab syntax)

Turn in the produced plots and answer the question, "Why would you advise to not use the open-loop control law that you specified?"

Hints:

- To produce such a control law, look at the proofs for controllability and reachability

a) Assume $x_2 - \Phi(t_1, t_0) x_1 \in Im(W_R(t_0, t_1))$ since completely controllable

$\Rightarrow \exists\, \eta$ s.t. $W_R(t_0, t_1)\eta = x_2 - \Phi(t_1, t_0) x_1$

$\Rightarrow \eta = W_R(t_0, t_1)^{-1}(x_2 - \Phi(t_1, t_0) x_1)$

So, the control input to get from x_1 to x_2 is

$$u(t) = B^T(t)\,\Phi^T(t_1, t)\,\eta$$

$$u(t) = B^T(t)\,\Phi^T(t_1, t)\,W_R(t_0, t_1)^{-1}(x_2 - \Phi(t_1, t_0) x_1)$$

for $t_0 = 0$, $t_1 = 5$ or any t_0, $t_1 = t_0 + 5 \in \mathbb{R}$

Assume LTI

so $e^{A(t_1 - t_0)} = \Phi$

$$u(t) = B^T(t)\,\Phi^T(t_1, t)\left(\int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) B^T(\tau) \Phi^T(t_1, \tau)\, d\tau\right)^{-1}\left[x_2 - \Phi(t_1, t_0) x_1\right]$$

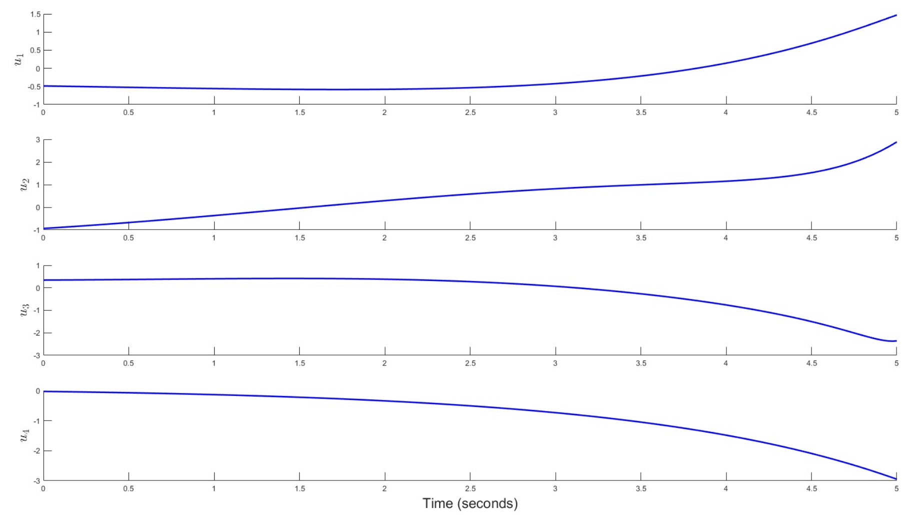
$$u(t) = B^T(t)\, e^{A^T(t_1 - t)}\left[\int_{t_0}^{t_1} e^{A(t_1 - \tau)} B(\tau) B^T(\tau)\, e^{A^T(t_1 - \tau)}\, d\tau\right]^{-1}\left[x_2 - e^{A(t_1 - t_0)} x_1\right]$$

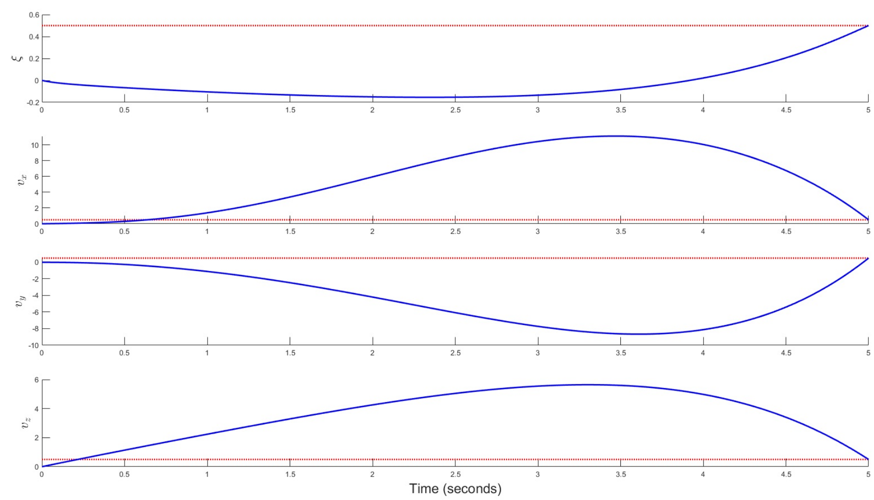
b) Using the above $u(t)$ as a control law

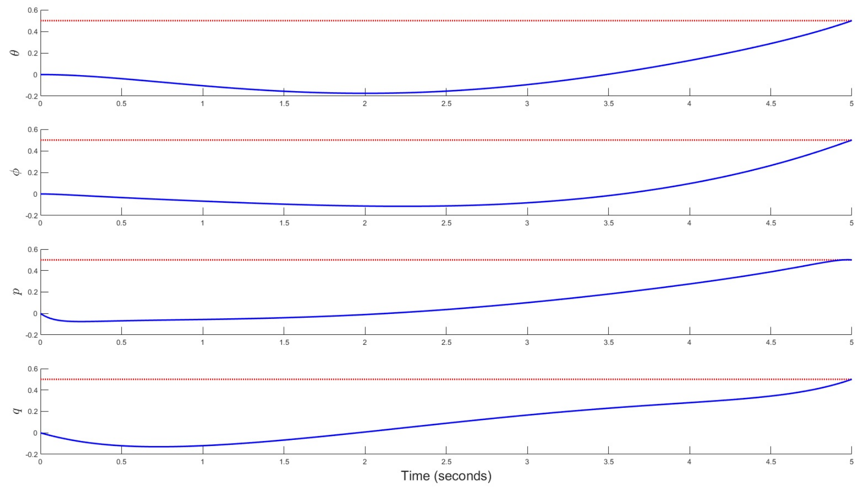
with $\Phi(t_1, t_0) = e^{A(t_1 - t_0)}$,

See plots below and MATLAB

When x(0) = x1: Notice that the 8 different states converge to the desired state after 5 seconds exactly. Also, note the input to compare with the next set of plots







When x(0) = rand(8,1): Notice that the input is identical to the previous set of plots and that the 8 different states do not converge to the desired state after 5 seconds.

I would advise against using this open-loop control law I specified since the final point will only reached the desired state if the initial state is perfectly at the right state and that there are no disurbances or errors in the plant. Everything must be 100% ideal to achieve the desired state with the open-loop input control