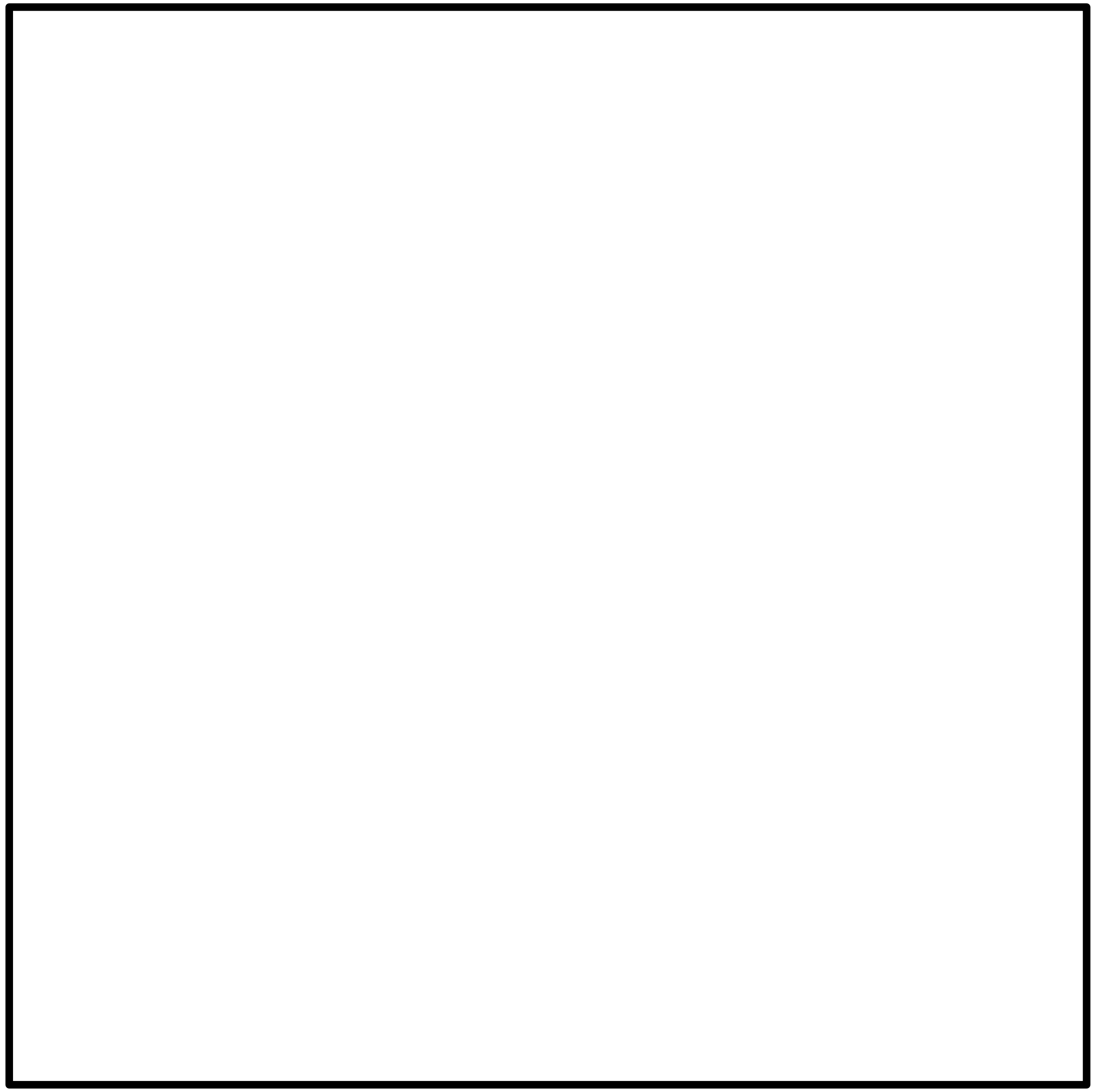
PATTERNS IN RECTANGULATIONS

Michaela A. Polley

Dartmouth College

Women in Mathematics in New England 2025
Northampton, MA, USA
September 20, 2025

What is a rectangulation?



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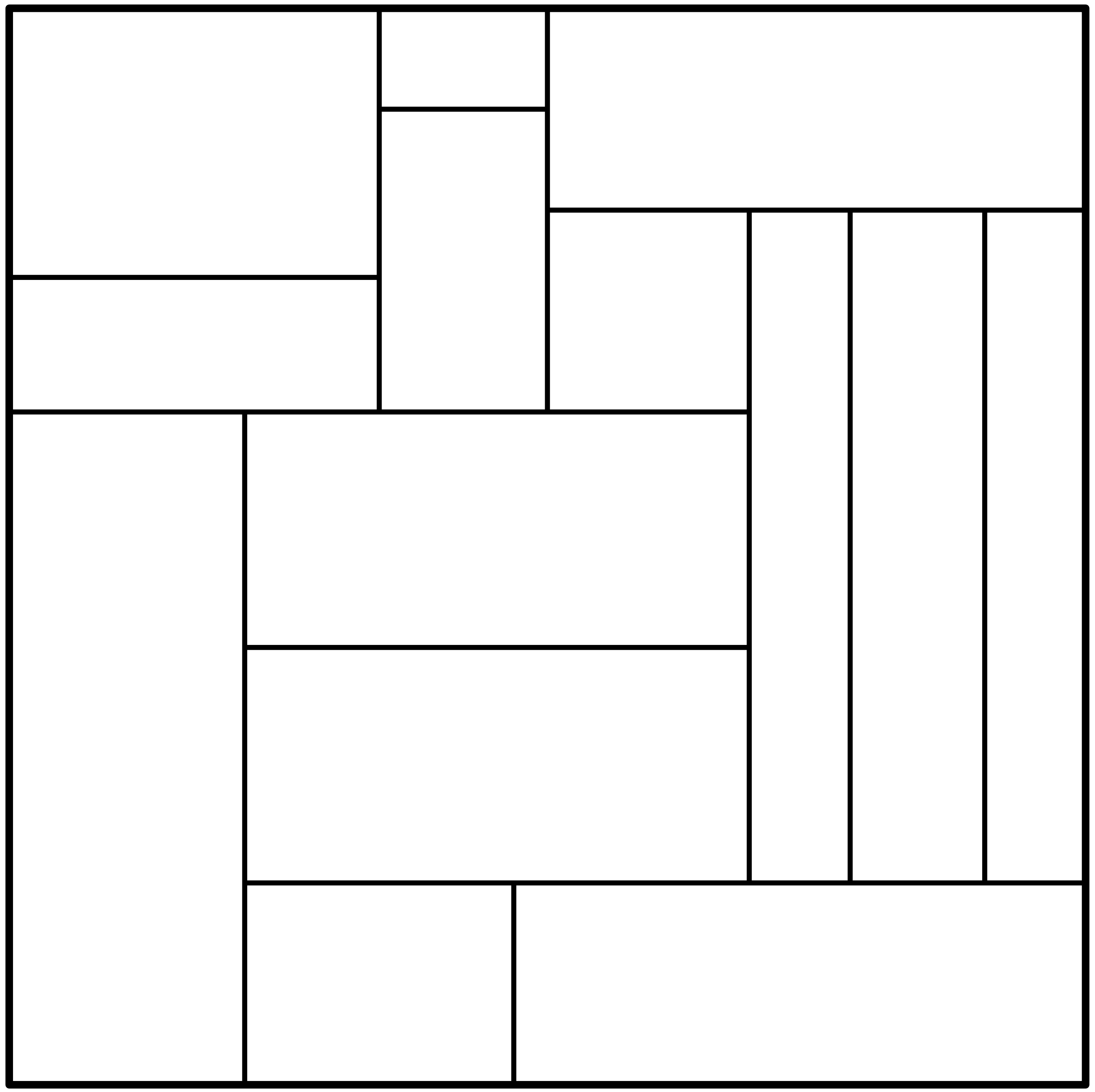

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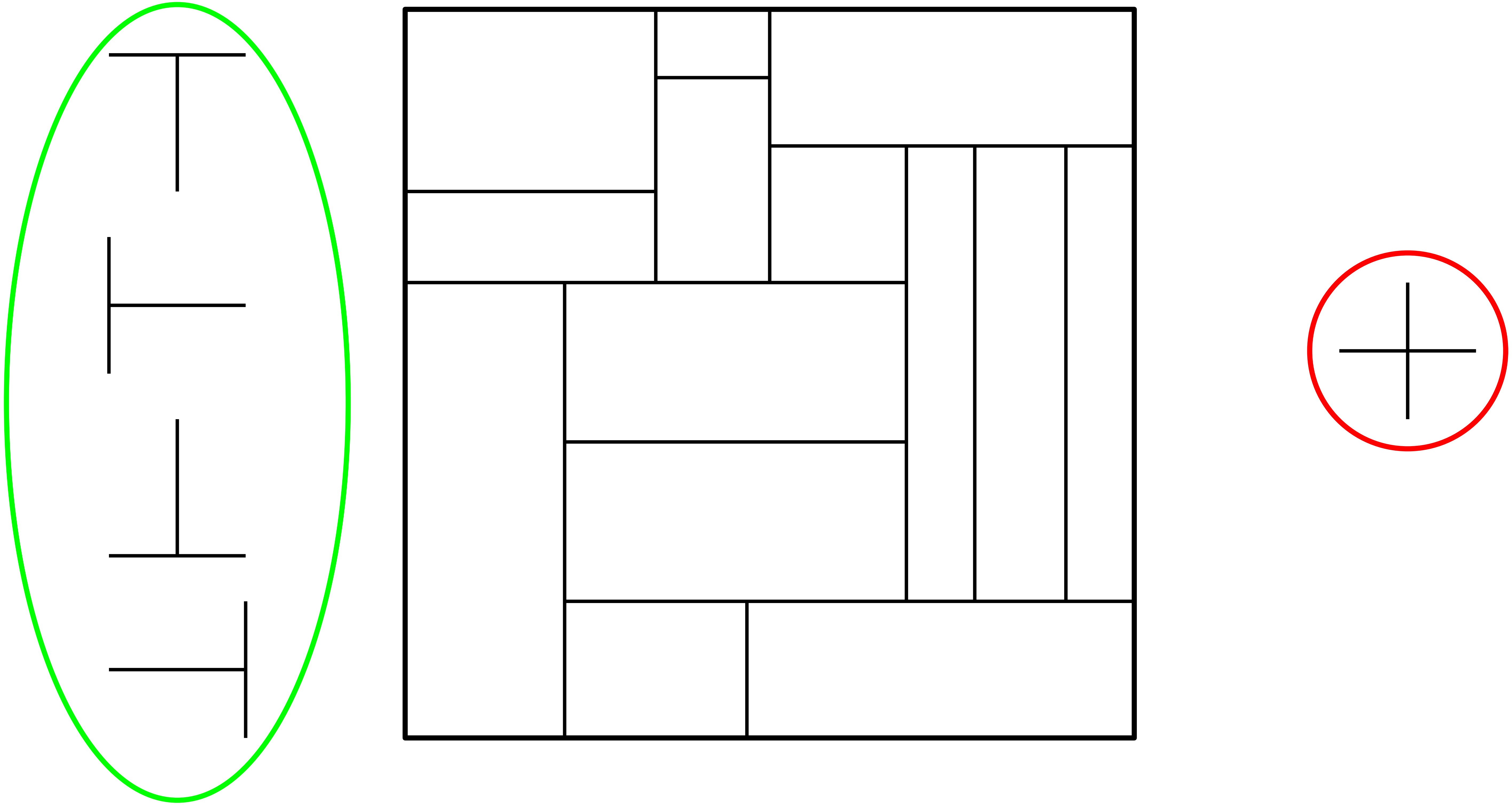

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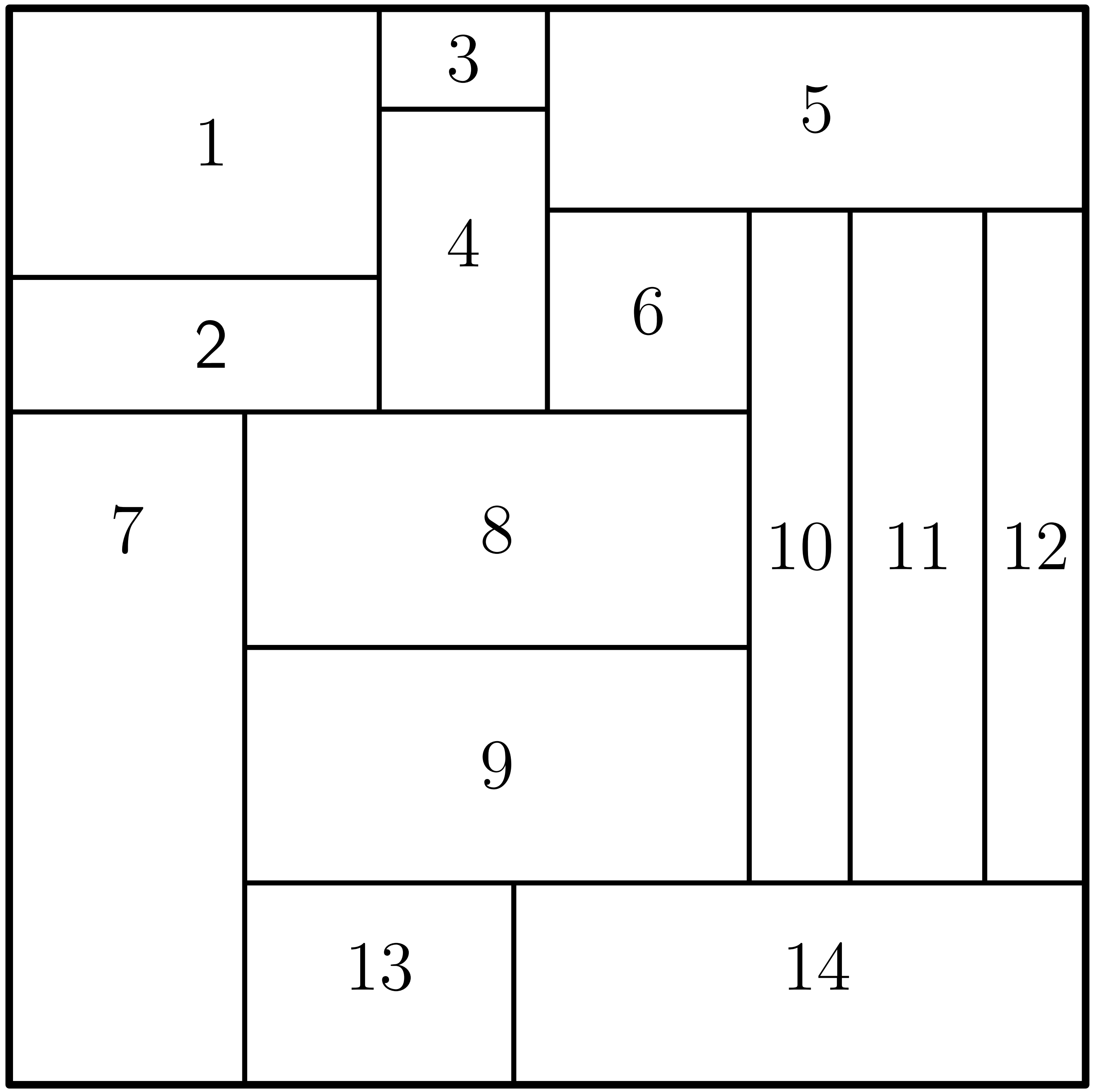

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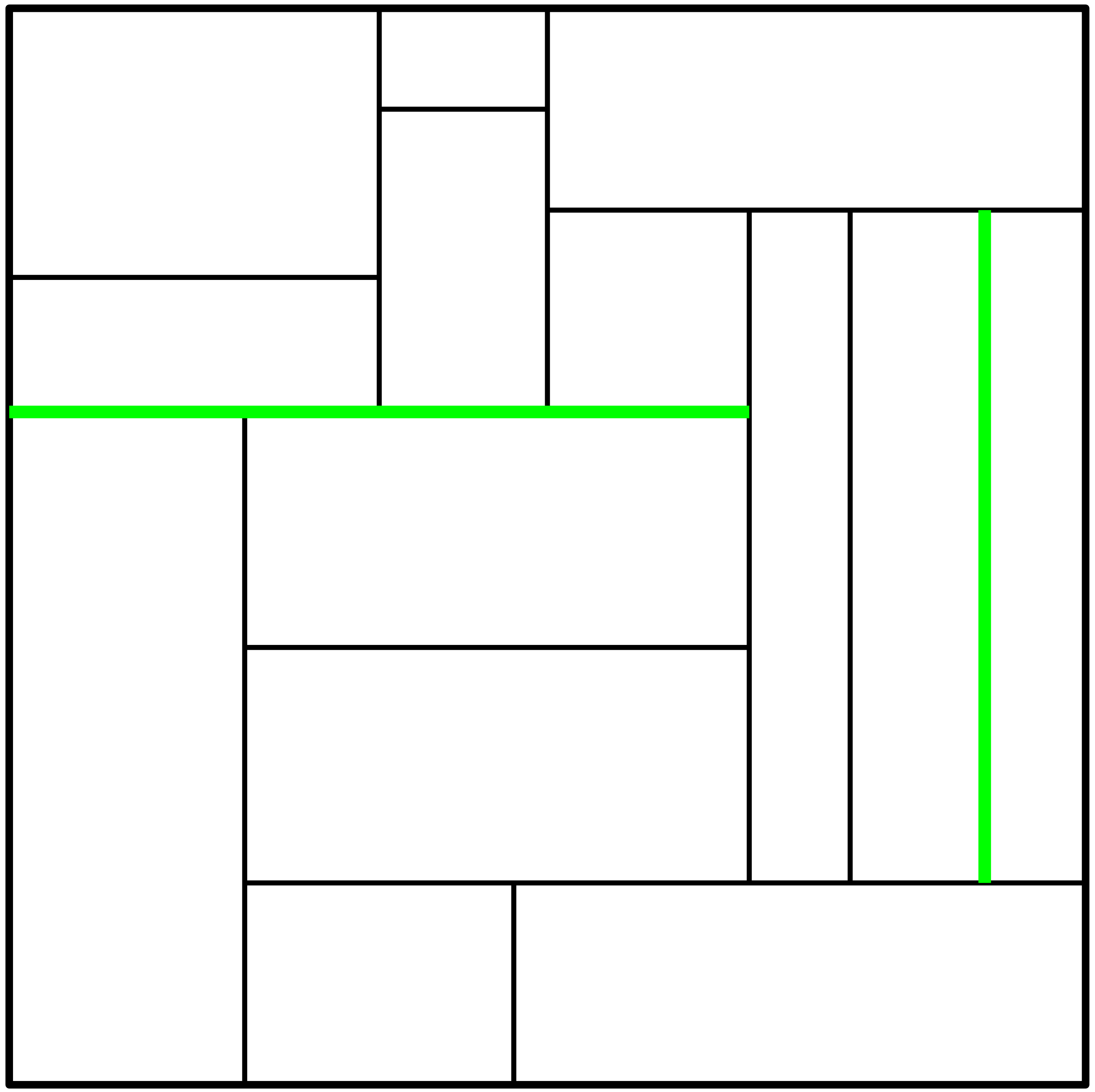

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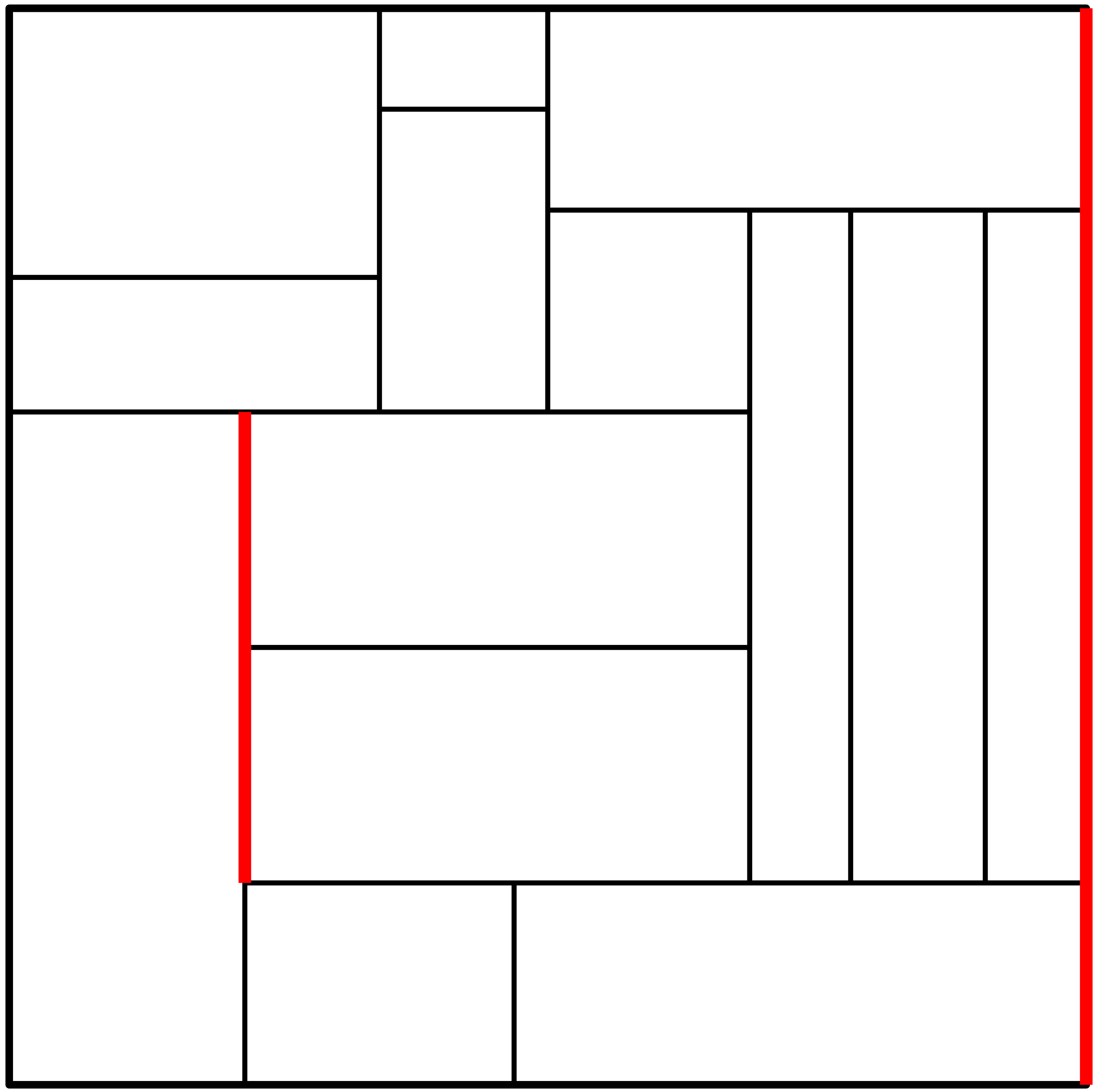

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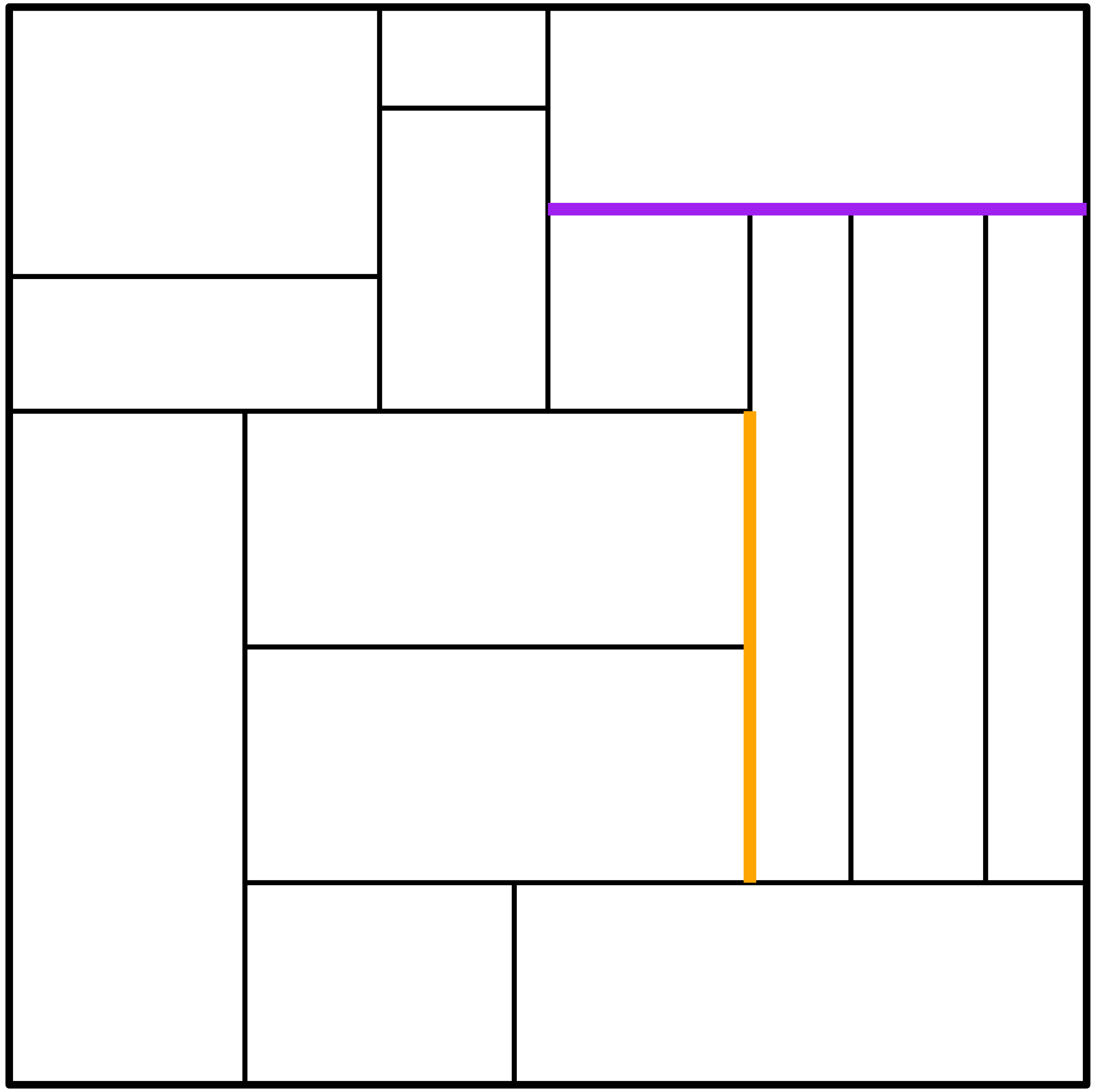

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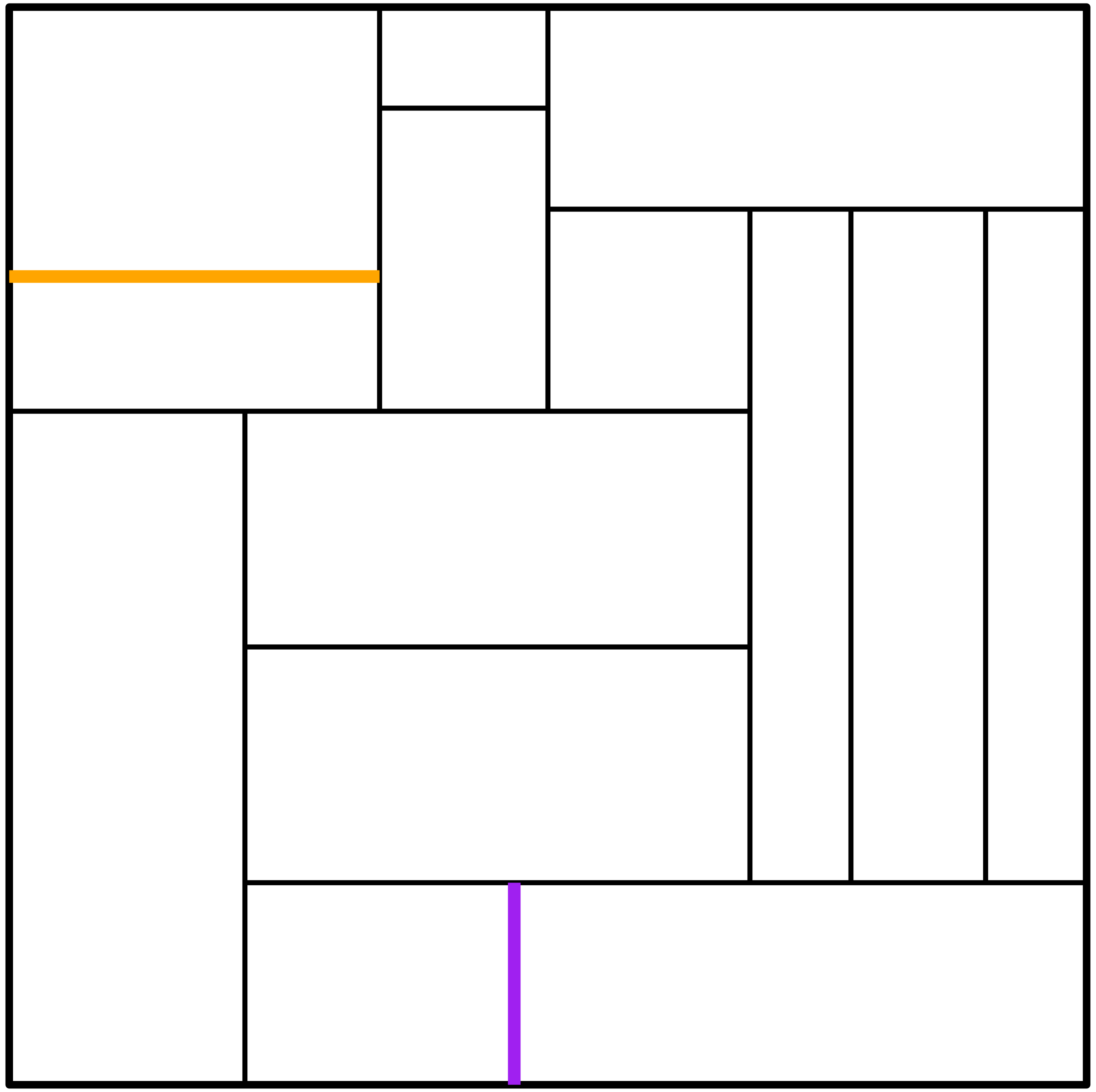

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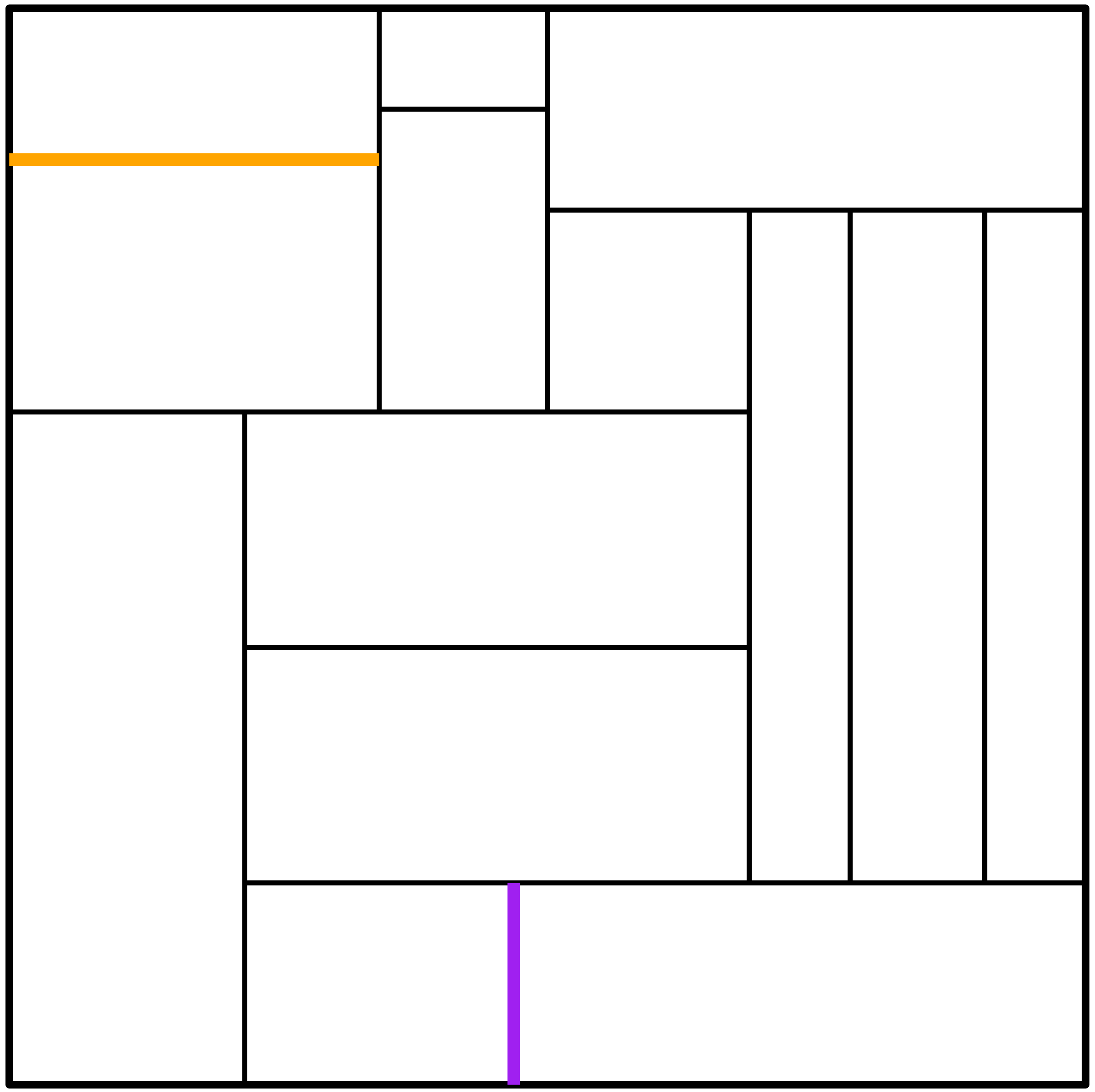

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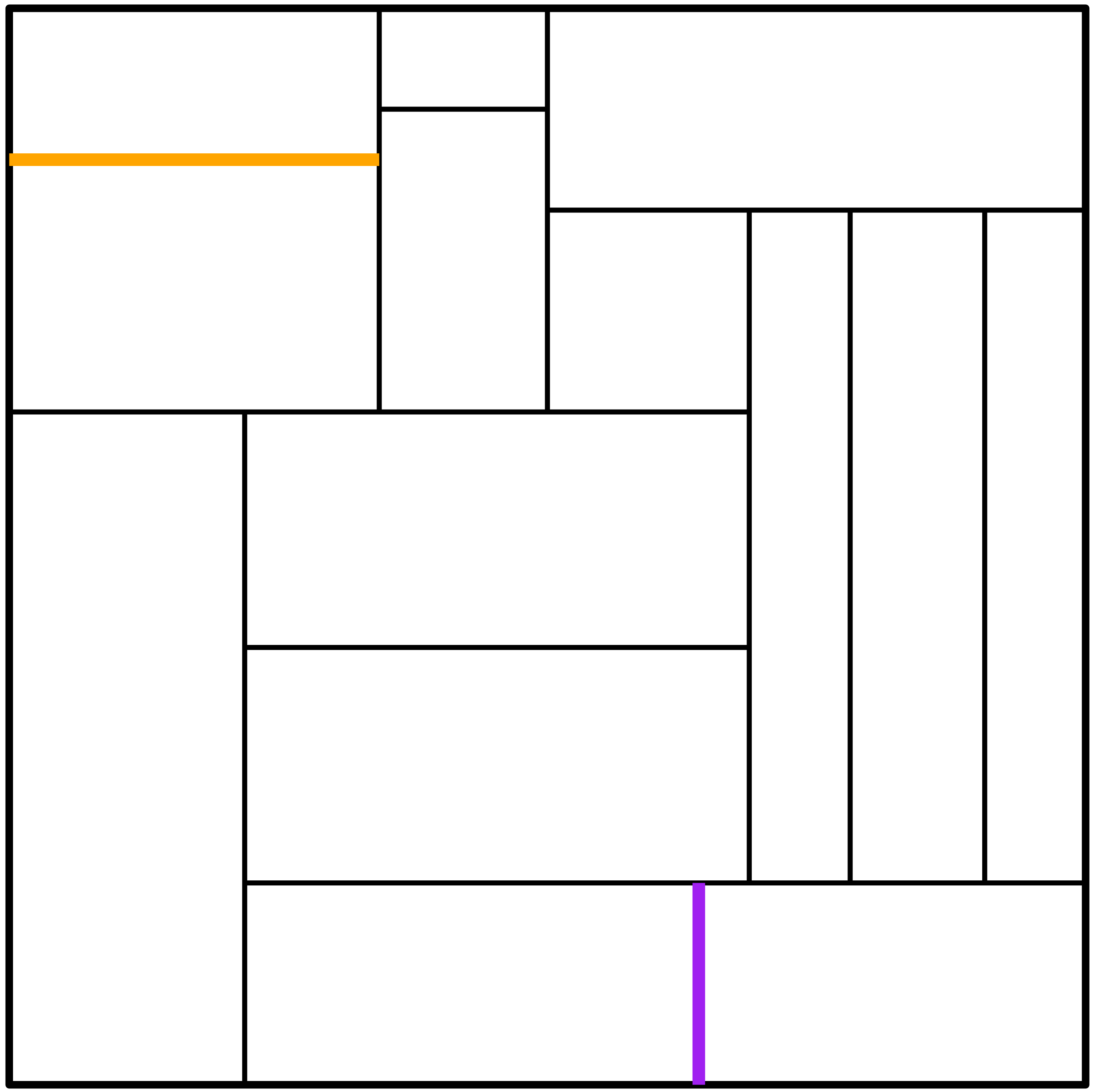

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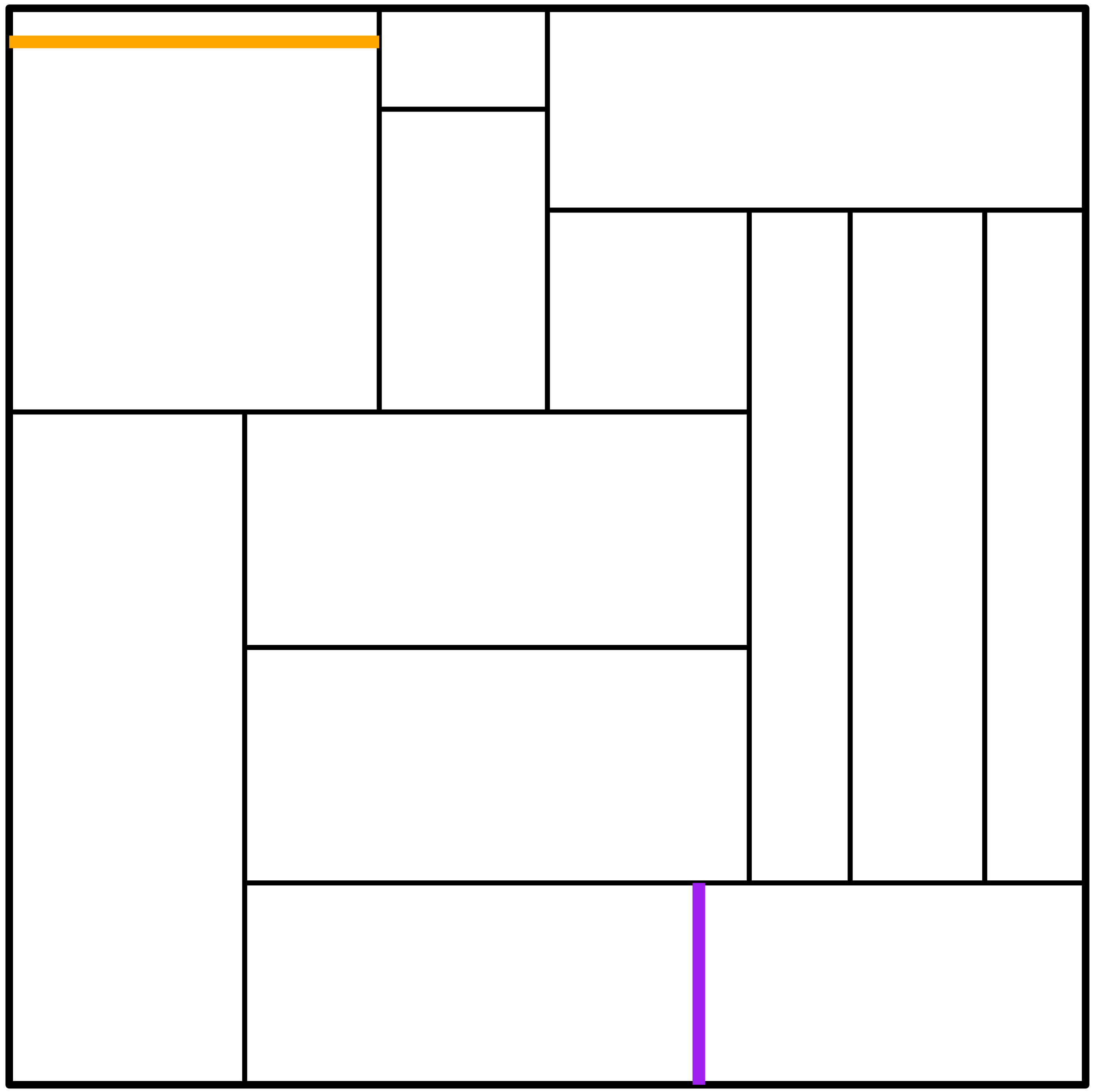

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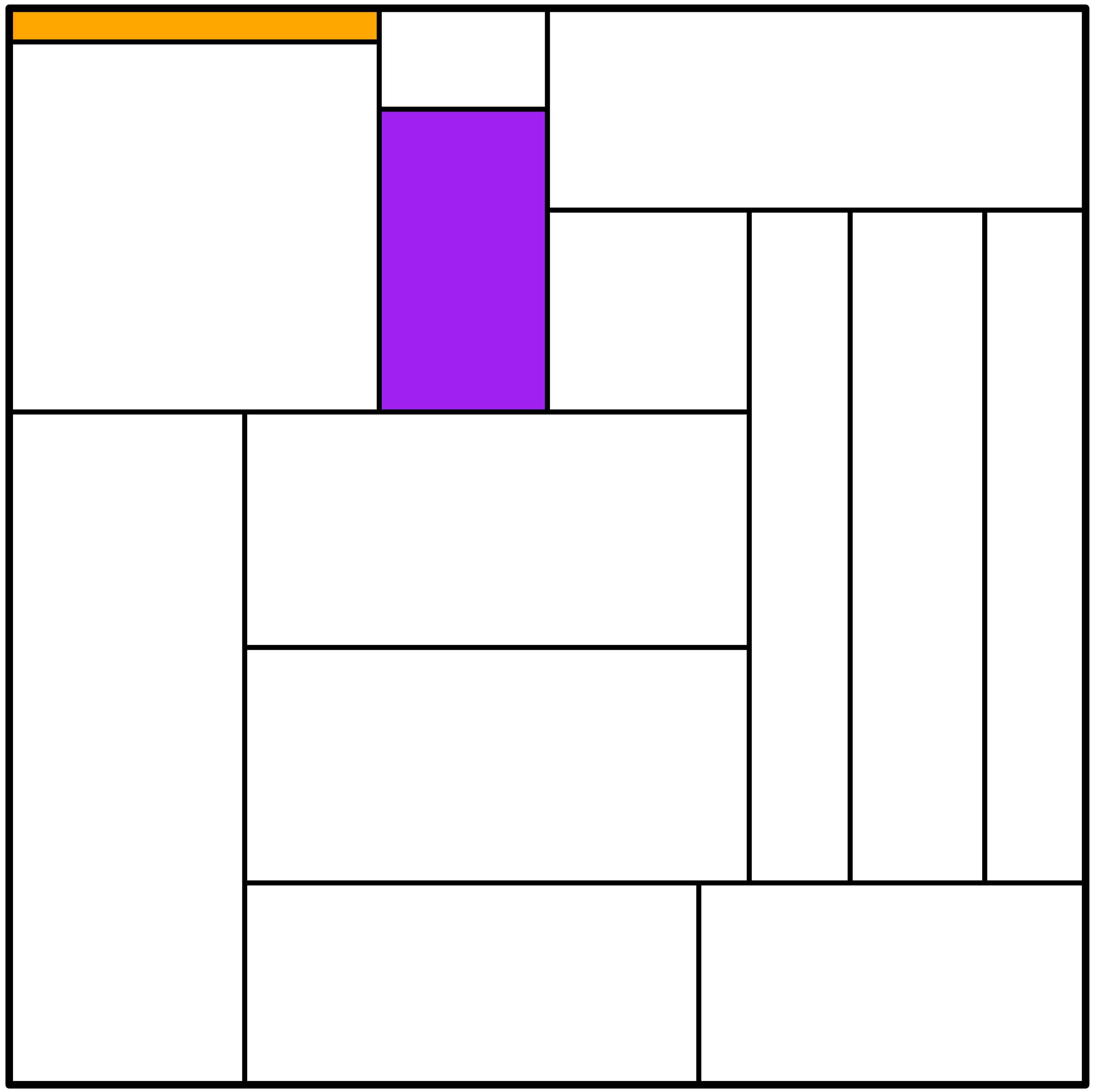

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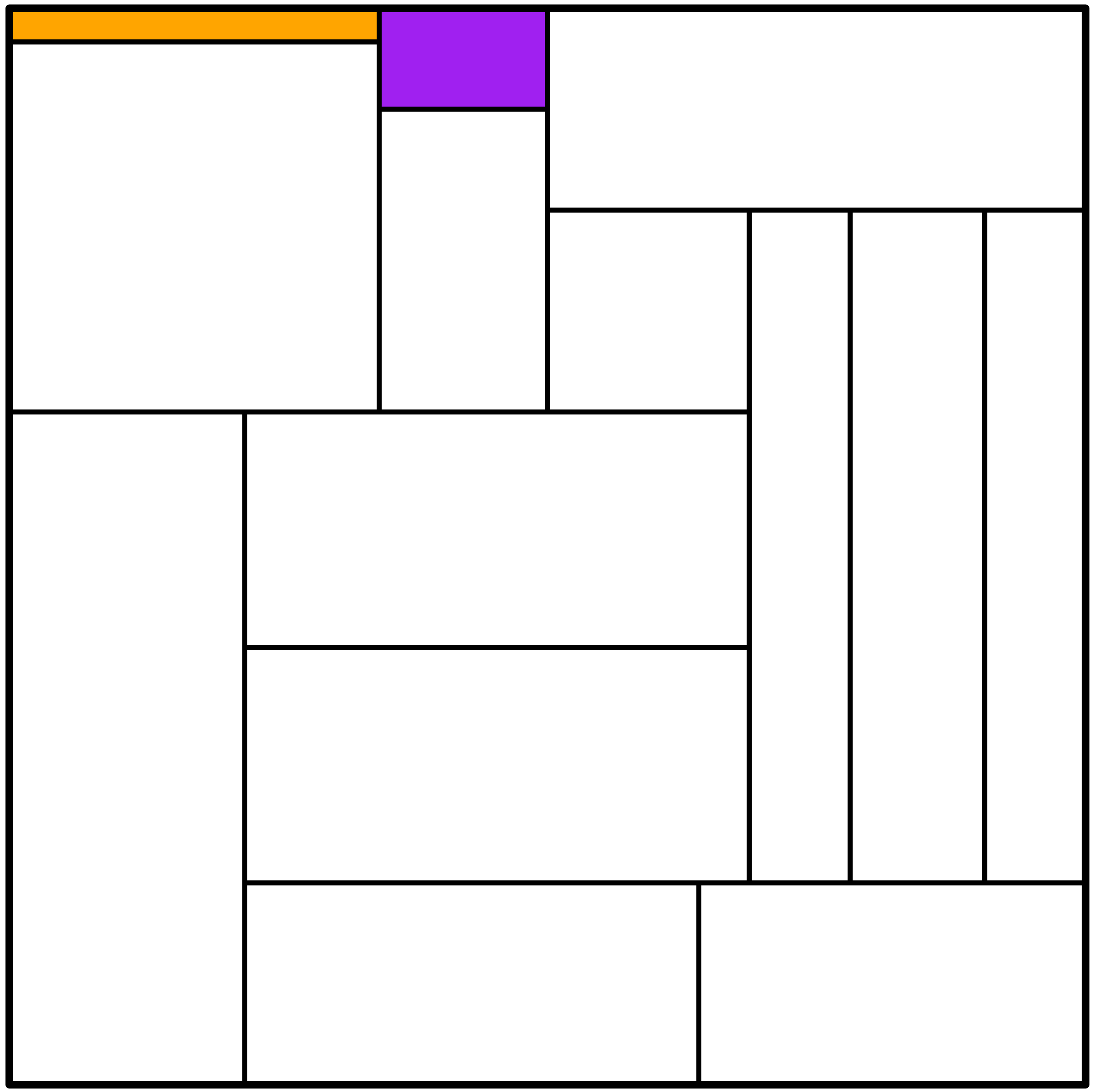

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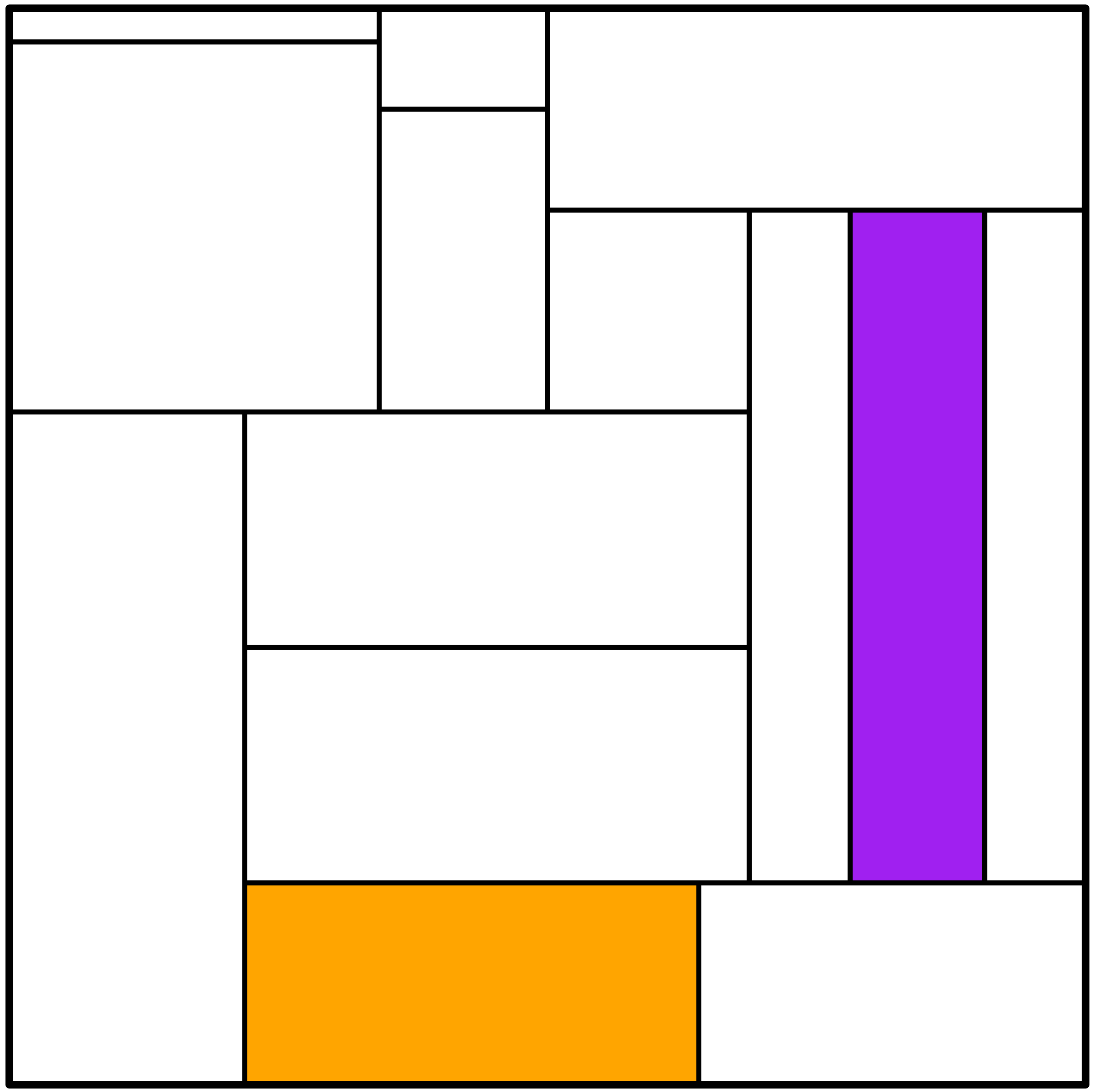

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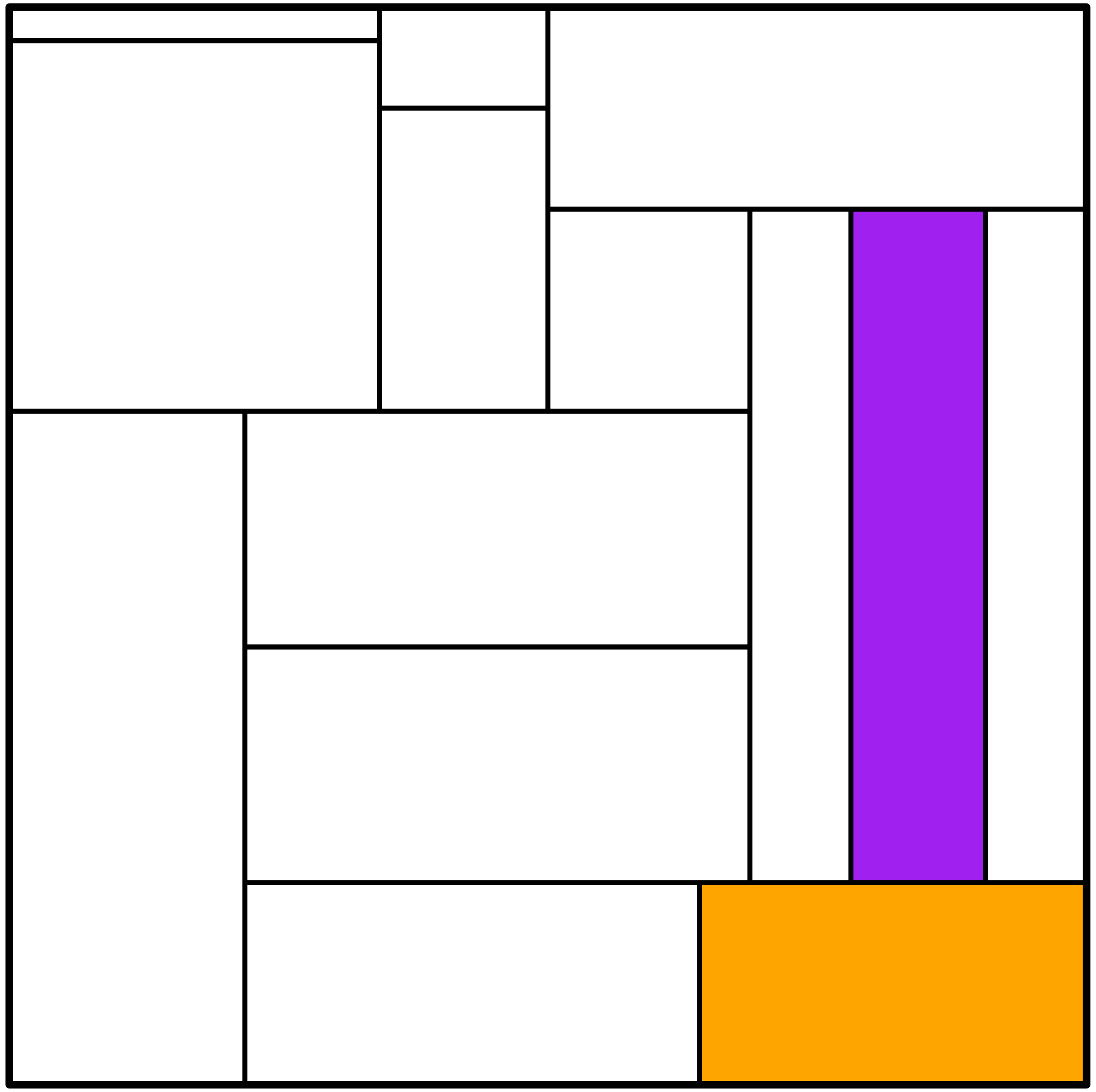
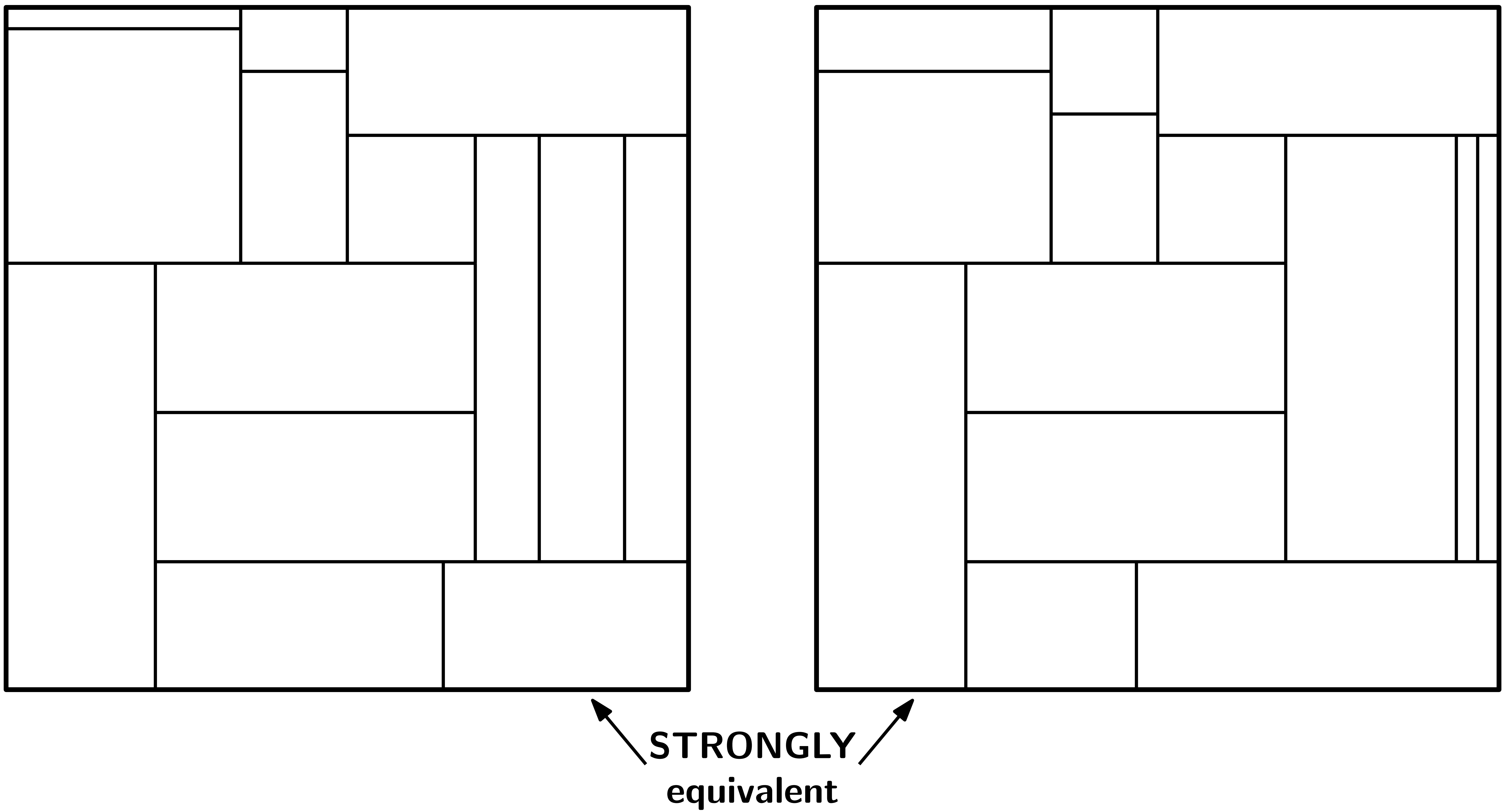


Weak and Strong Equivalence



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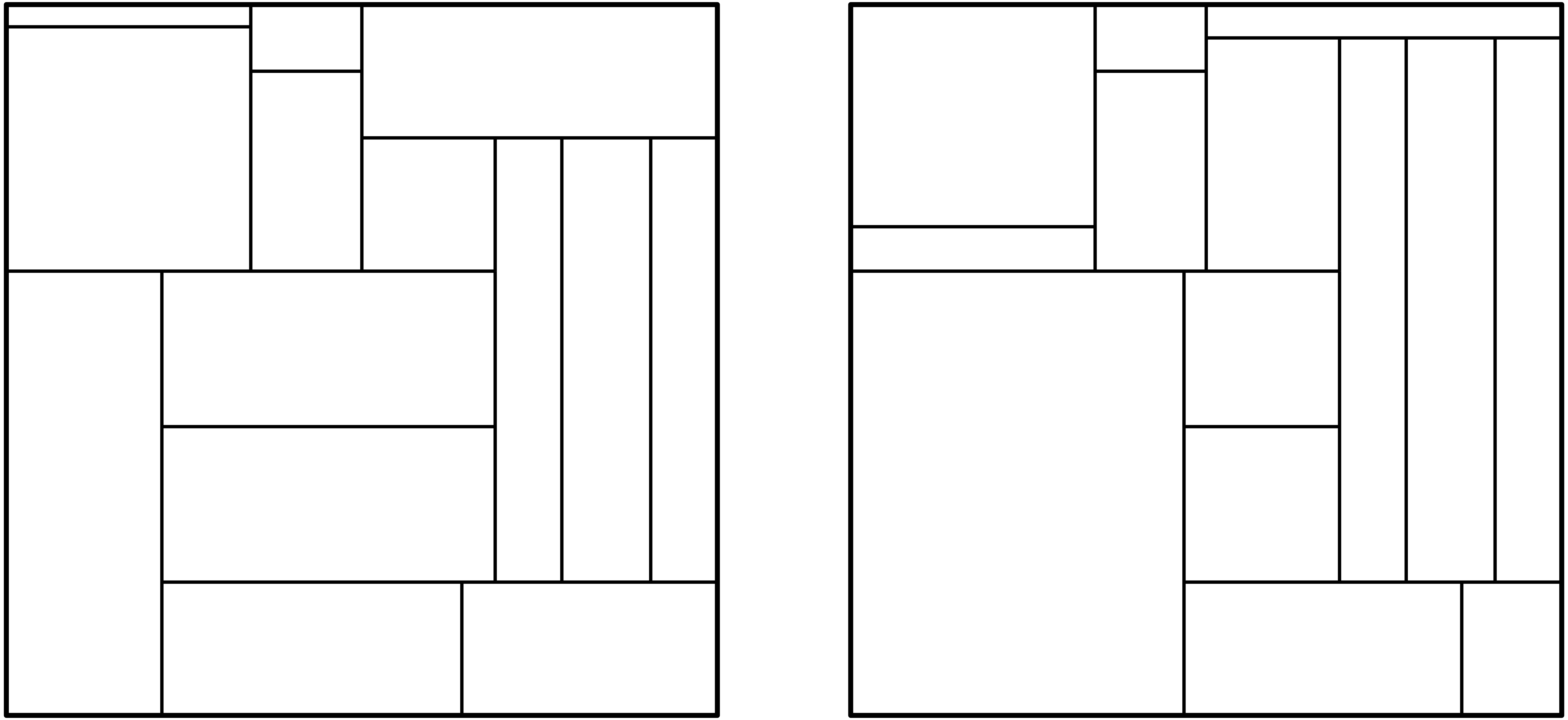
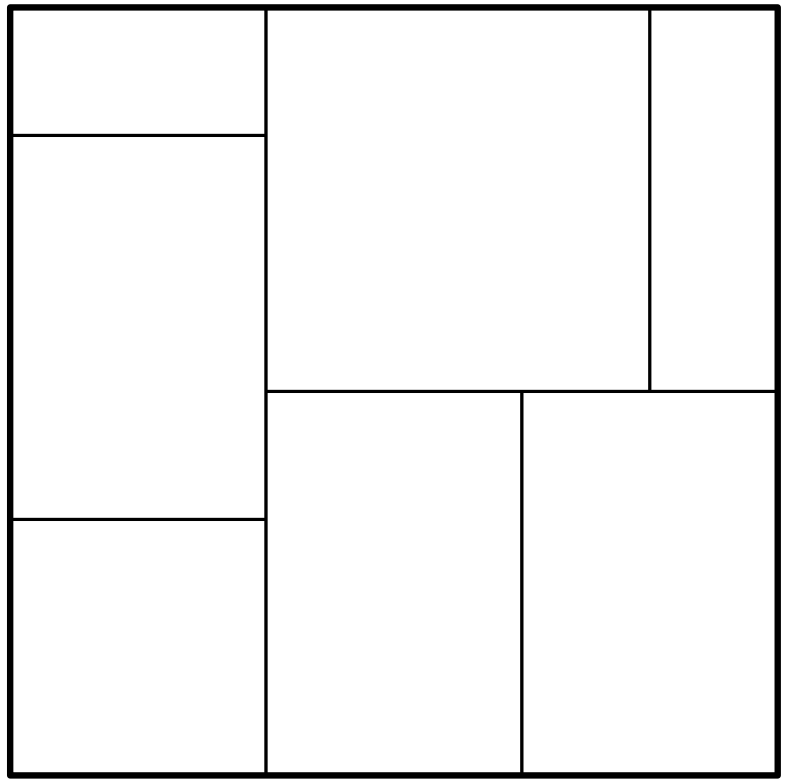
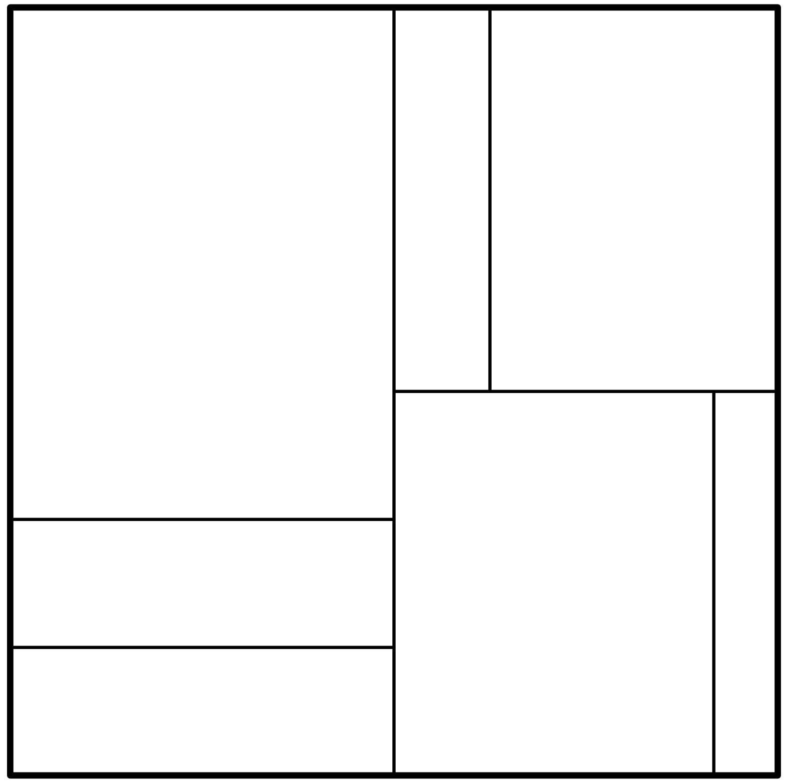


WEAKLY
equivalent

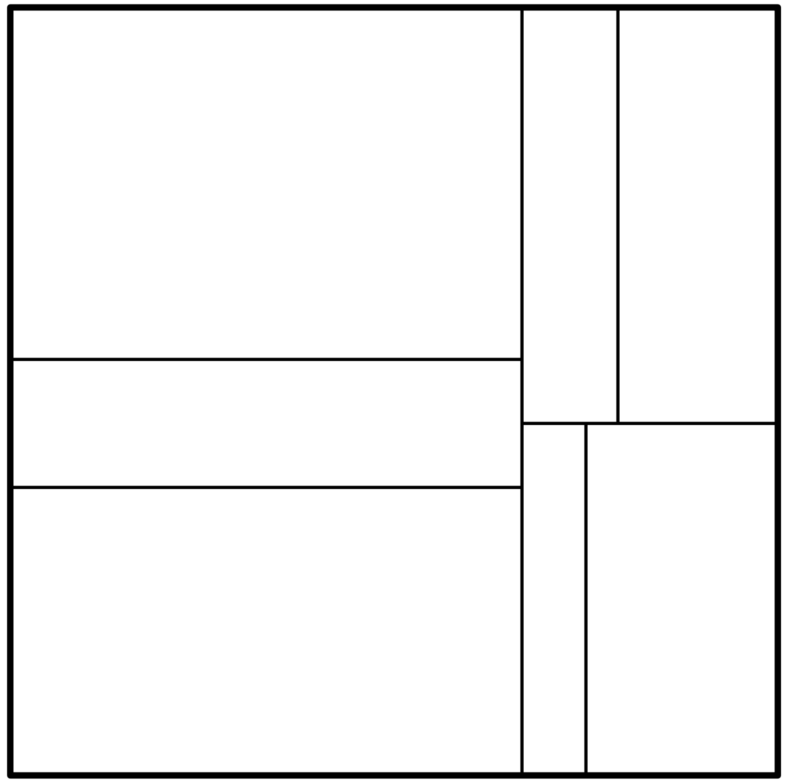
Weak and Strong Equivalence



$\mathcal{A}$



$\mathcal{B}$



$\mathcal{C}$

What is combinatorics?

Combinatorics is the area of math that studies the properties of discrete structures (such as rectangulations) and tries to answer questions such as:

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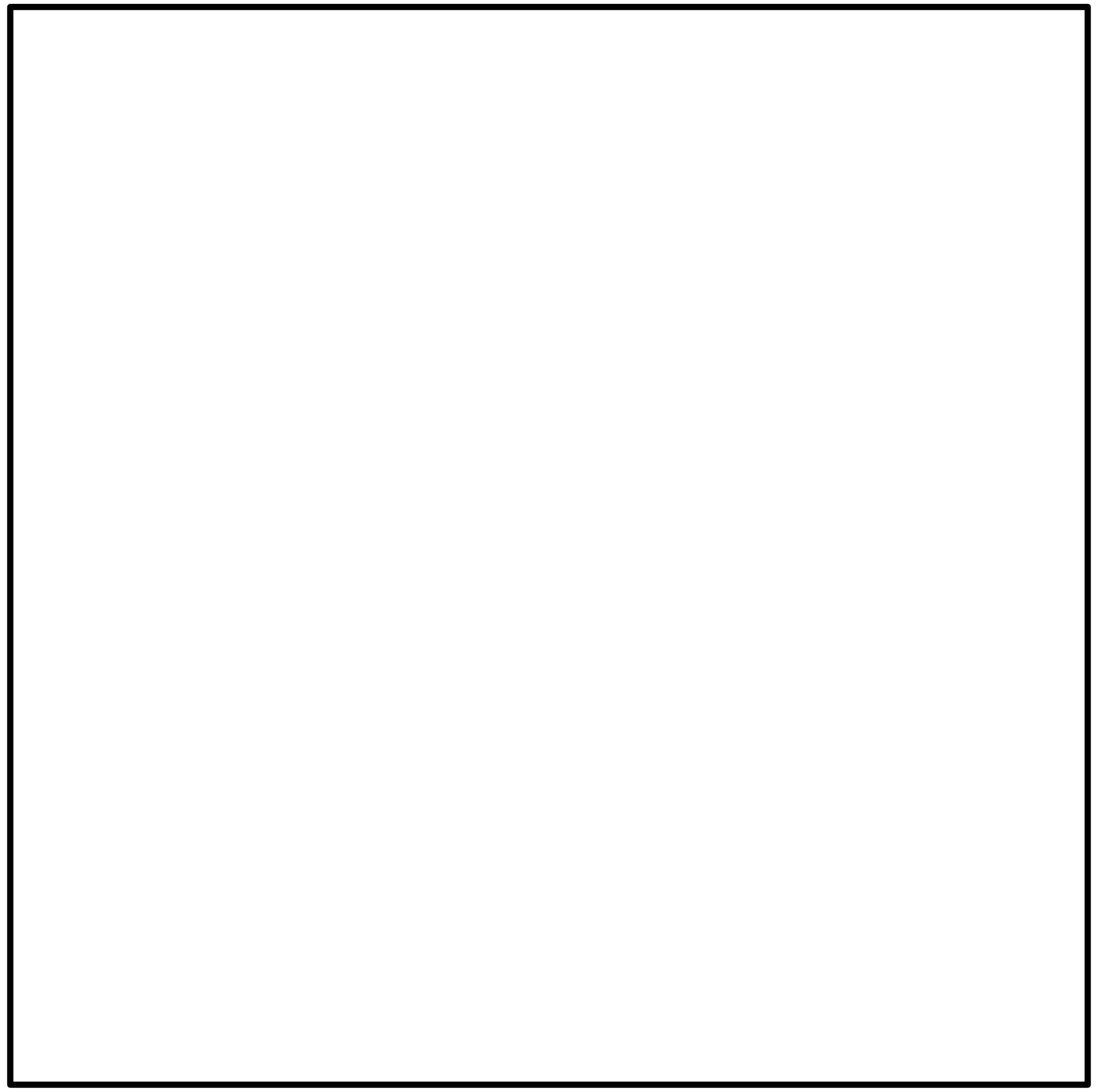
Pattern Avoidance: $R(\top, \dashv, \vdash)$



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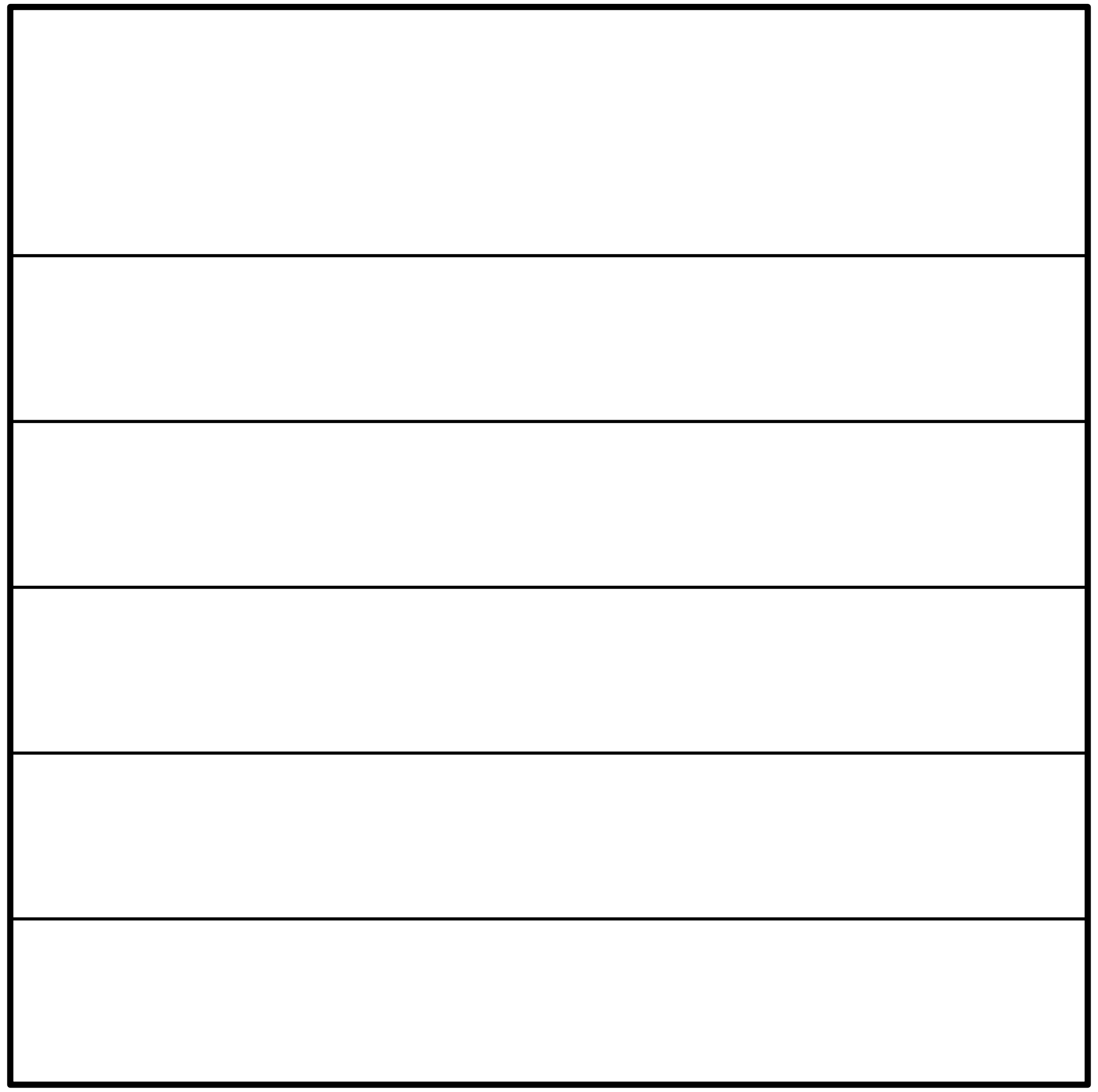

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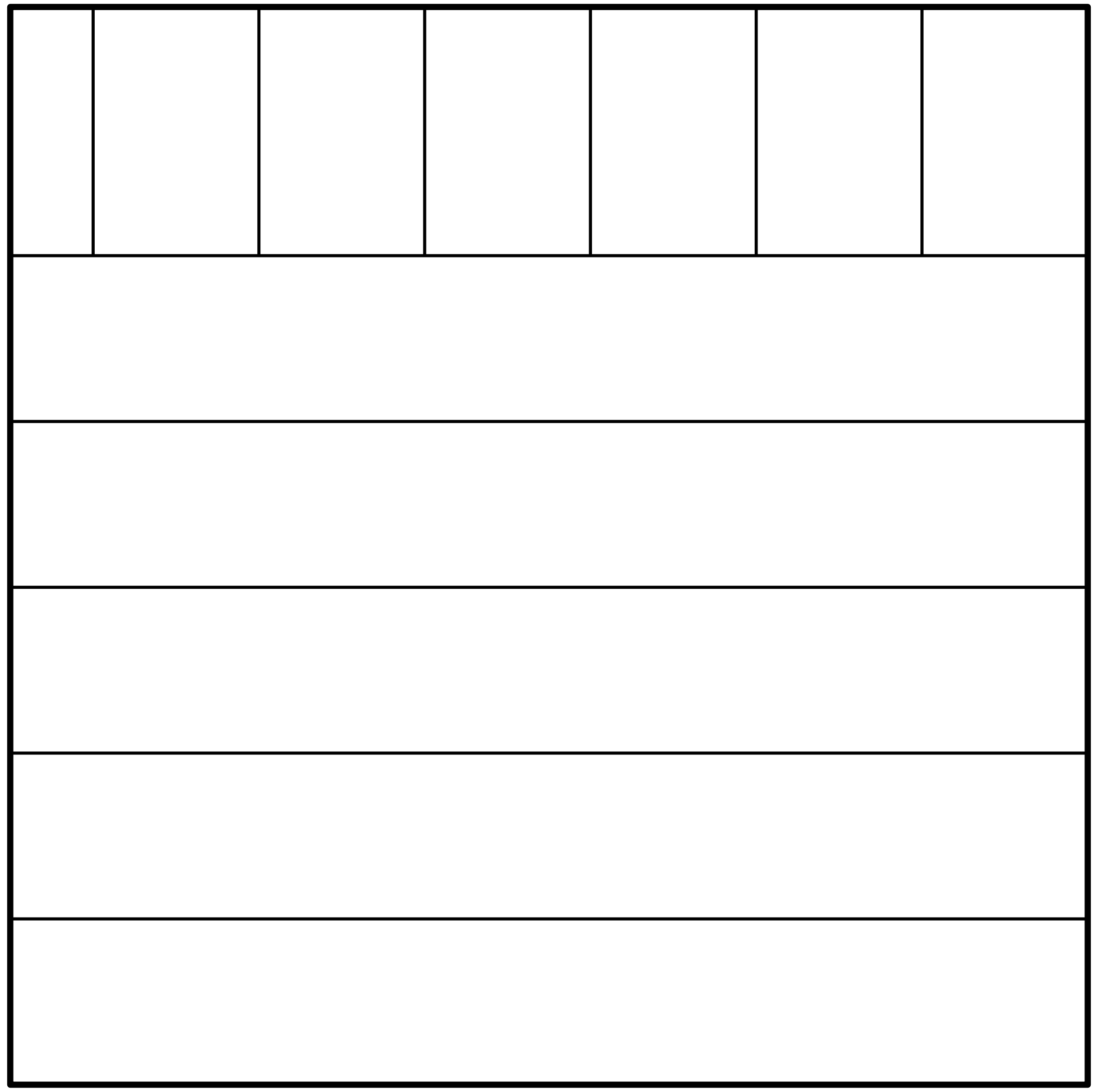

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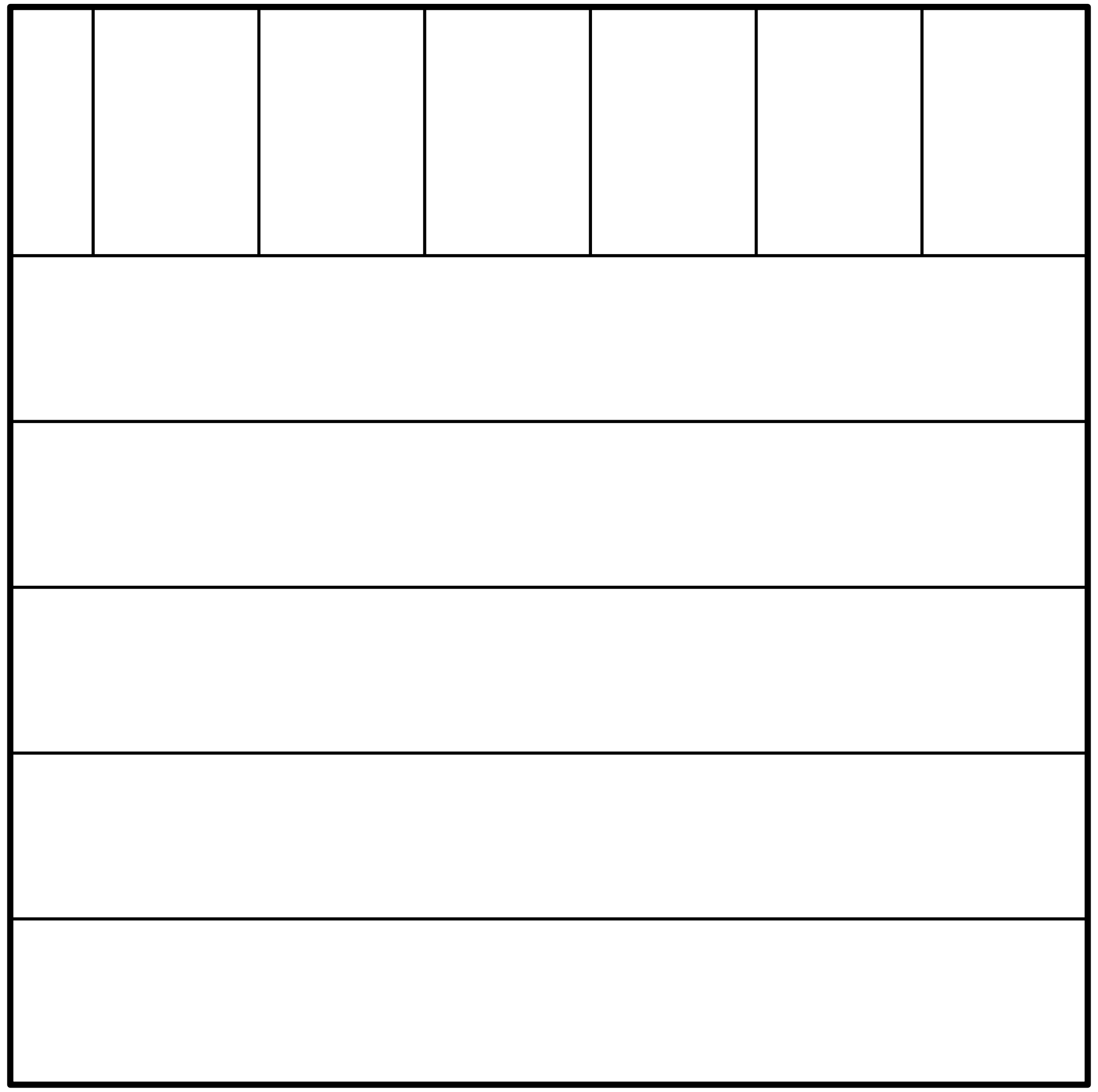

Proposition

The number of rectangulations of size n that avoid $\top$, $\dashv$, and $\vdash$, denoted $R_n(\top, \dashv, \vdash)$ is n.

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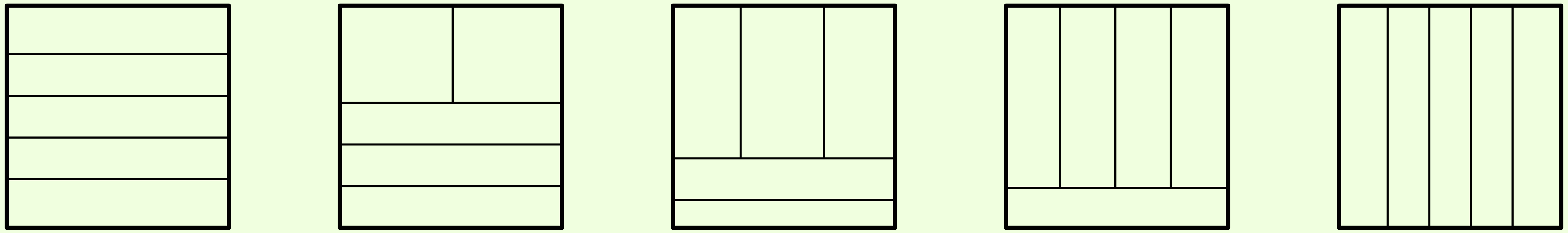
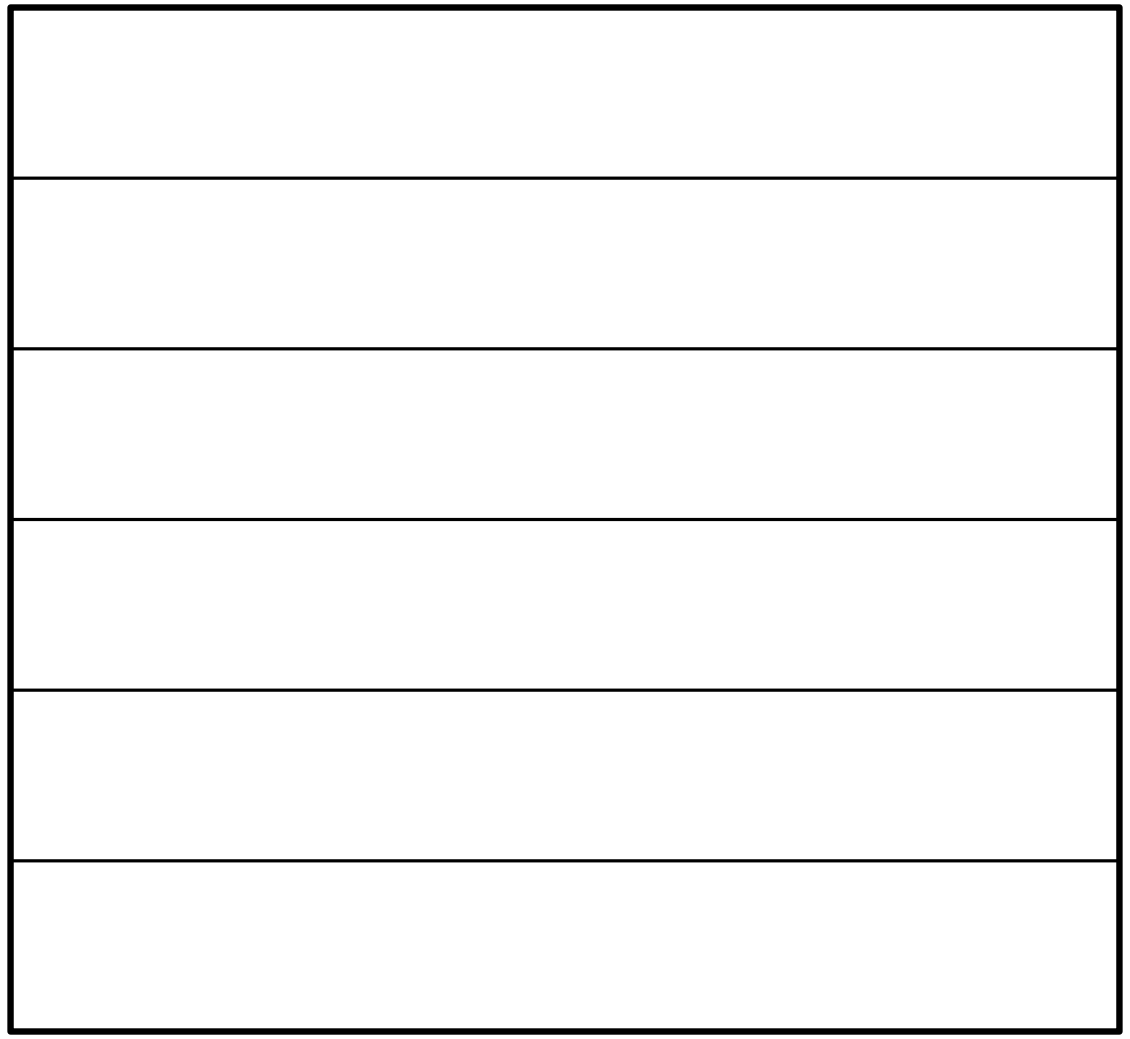
Example

For $n = 5$, there are five rectangulations that avoid $\top$, $\dashv$, and $\vdash$:



Pattern Avoidance: $R(\dashv, \vdash)$



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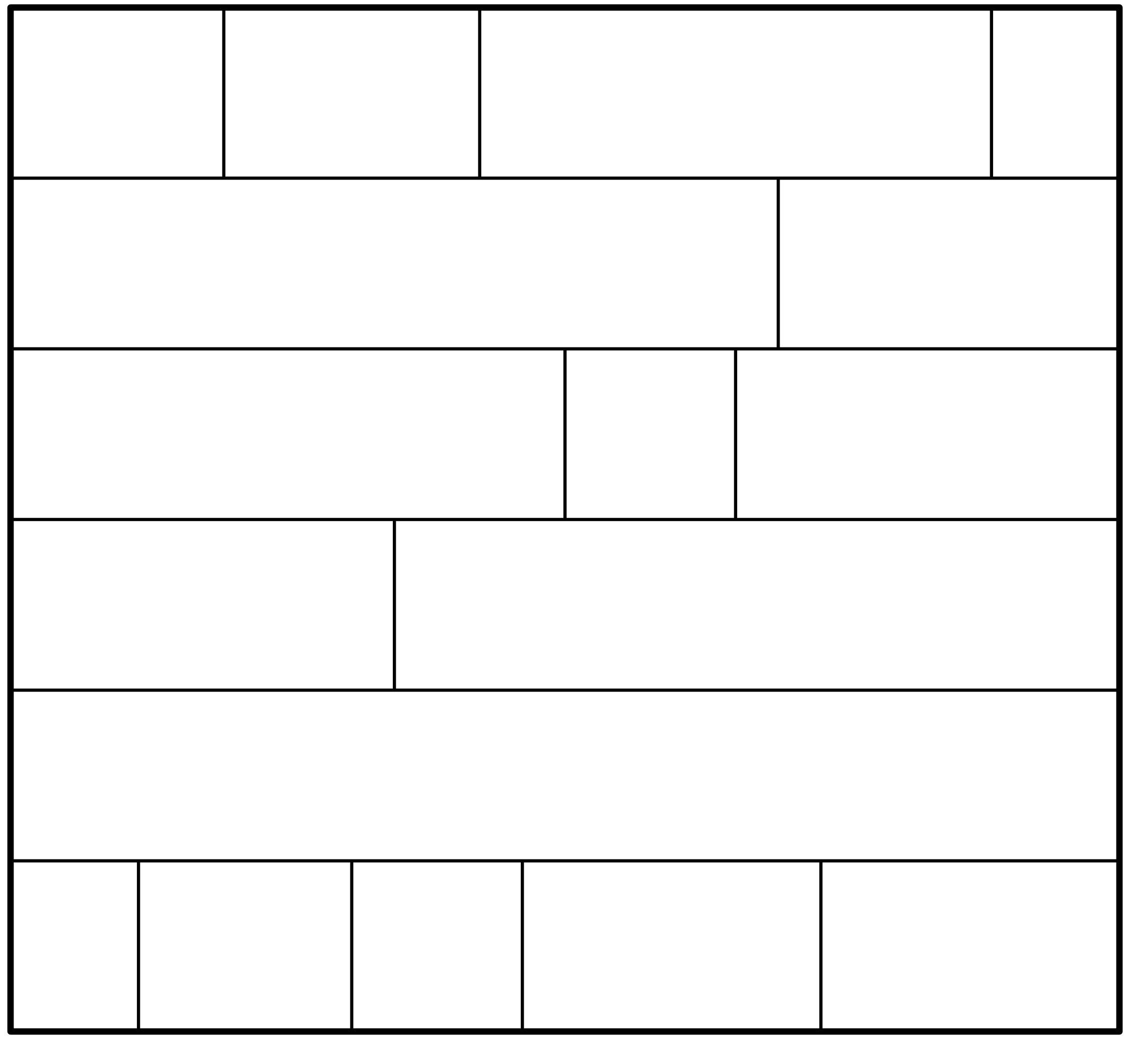

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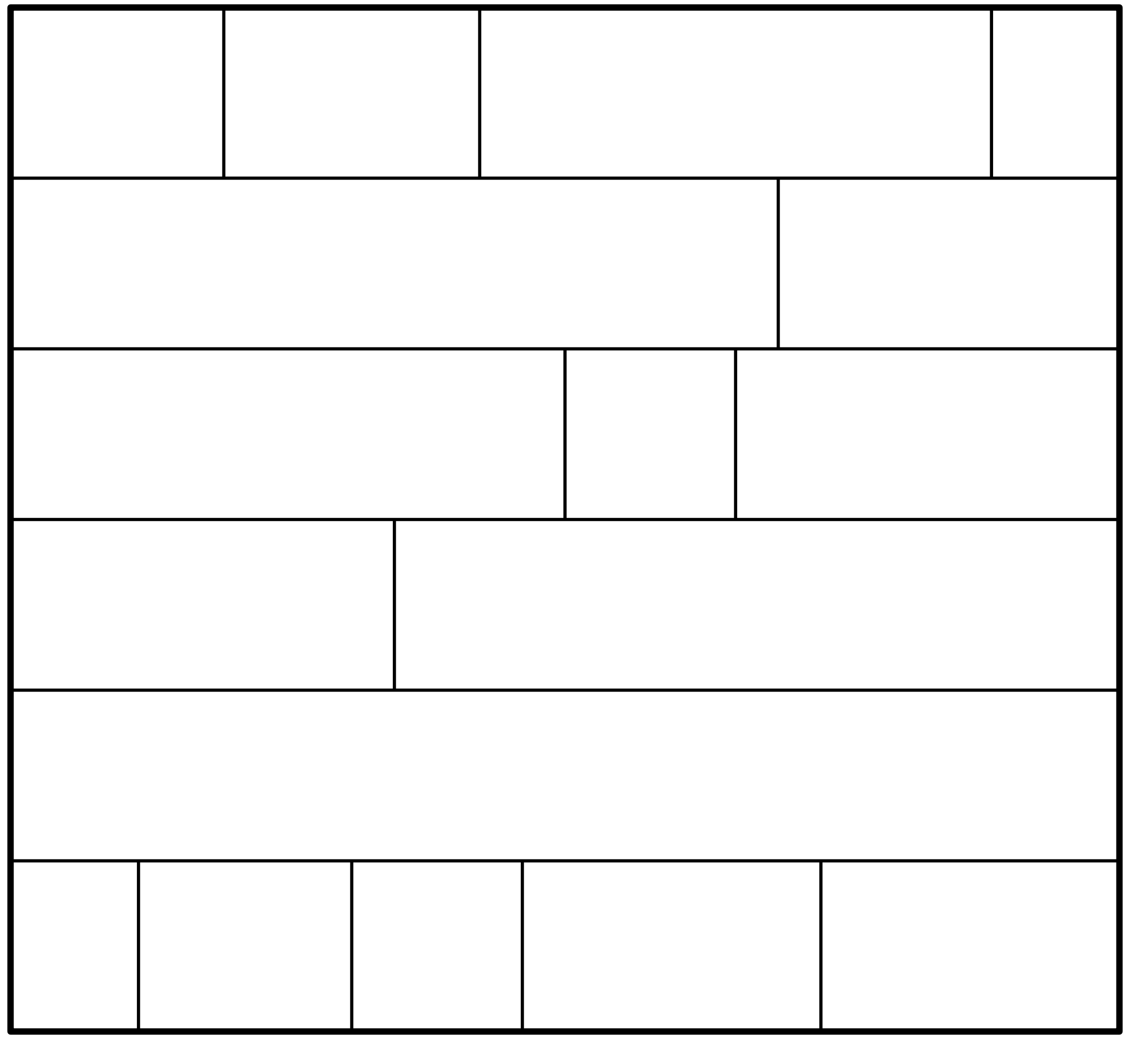

Proposition

The number of weak rectangulations of size n that avoid $\dashv$ and $\vdash$, denoted $R_n^w(\dashv, \vdash)$ is 2^{n-1}.

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Definition

A *composition* of n is an ordered list of positive integers $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \ldots + a_k = n$.

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$$3 \qquad\qquad 2+1 \qquad\qquad 1+2 \qquad\qquad 1+1+1$$

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There are 2^{n-1} compositions of n:

$$\bullet\ \bullet\ \bullet \qquad \bullet\ \bullet|\bullet \qquad \bullet|\bullet\ \bullet \qquad \bullet|\bullet|\bullet$$

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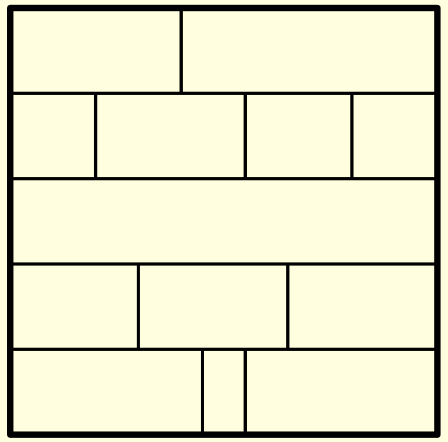
Proof

Establish a bijection between $R_n^w(\dashv, \vdash)$ and compositions of n.



$\longleftrightarrow$ $2+4+1+3+3$

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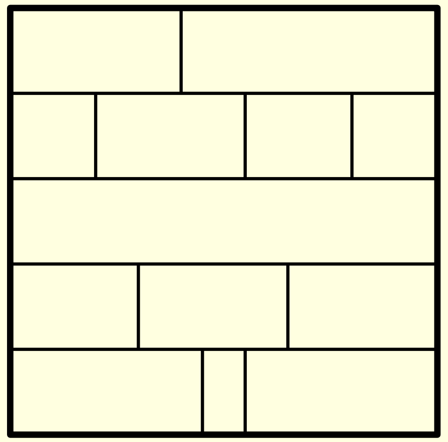

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Question: What about strong rectangulations?

Pattern Avoidance: $R(\dashv, \vdash)$

By complete enumeration (using a computer), we find that the number of *strong* rectangulations of size n which avoid $\vdash$ and $\dashv$ for $n = 1, \ldots, 8$ are $1, 2, 4, 9, 22, 57, 154, 430$.

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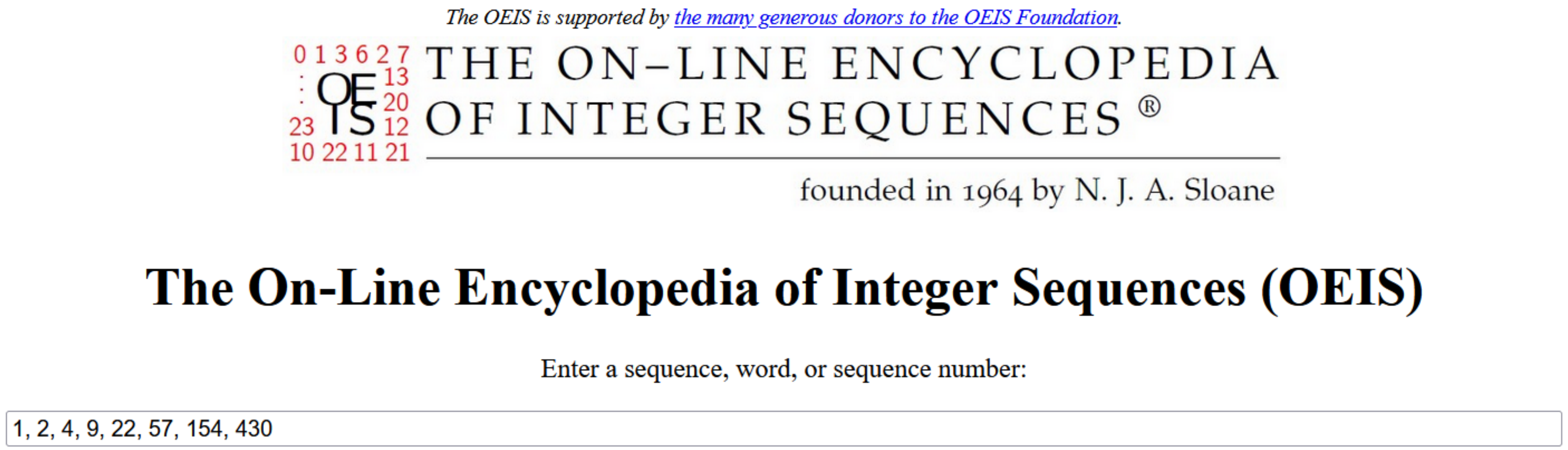

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Pattern Avoidance: $R(\dashv, \vdash)$

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1, 2, 4, 9, 22, 57, 154, 430 | Search | Hints

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Sort: relevance | references | number | modified | created Format: long | short | data

A287709 Number of Dyck paths of semilength n such that every peak at level y > 1 is preceded by (at least) one peak at level y-1. +30 2

1, 1, **1, 2, 4, 9, 22, 57, 154, 430**, 1234, 3625, 10865, 33136, 102598, 321913, 1021963, 3278543, 10617413, 34678693, 114151769, 378436049, 1262822229, 4239469076, 14312153289, 48567846377, 165610404277, 567259571451, 1951218773118, 6738242931451, 23356148951482

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COMMENTS Also the number of Dyck paths of semilength (n-1) whose maximum height is attained by the initial ascent. (That is, Dyck paths with prefix U^kD, k>=1, and maximum height k.) For a(3)=2: UDUD, UUDD. For a(4)=3: UDUDUD, UUDUDD, UUDDUD, UUUDDD. (Andrei Asinowski and Vít Jelínek) - Andrei Asinowski, Jun 21 2021

From Andrei Asinowski, Sep 01 2025: (Start)

Also the number of strong rectangulations of size (n-1) that avoid the patterns "top" and "bottom" (that is, the T-shape and the upside-down T-shape). (Andrei Asinowski and Michaela Polley, Thm. 13).

Also the number of (010,101,120,201)-avoiding inversion sequences e of length (n-1) in which all left-to-right maxima e_j satisfy e_j=j-1. Also the number of (010,110,120,210)-avoiding inversion sequences of length (n-1) that satisfy this condition. Also the number of (010,100,120,210)-avoiding inversion sequences of length (n-1) that satisfy this condition. (Andrei Asinowski and Michaela Polley, Prop. 14).

Also the number of (011,201)-avoiding inversion sequences e of length (n-1) in which the set of values is precisely {0,1,2,...,M}, where M is the maximum value of e. (Andrei Asinowski and Michaela Polley, Prop. 15). (End)

REFERENCES Andrei Asinowski and Vít Jelínek. Two types of Dyck paths (unpublished manuscript).

LINKS Alois P. Heinz, Table of n, a(n) for n = 0..1000

Andrei Asinowski and Michaela A. Polley, Patterns in rectangulations. Part I: T-like patterns, inversion sequence classes I(010, 101, 120, 201) and I(011, 201), and rushed Dyck paths, arXiv:2501.11781 [math.CO], 2025. See pp. 1, 7, 25, 27.

Axel Bacher, Progressive and rushed Dyck paths, arXiv:2403.08120 [math.CO], 2024

https://oeis.org/A287709