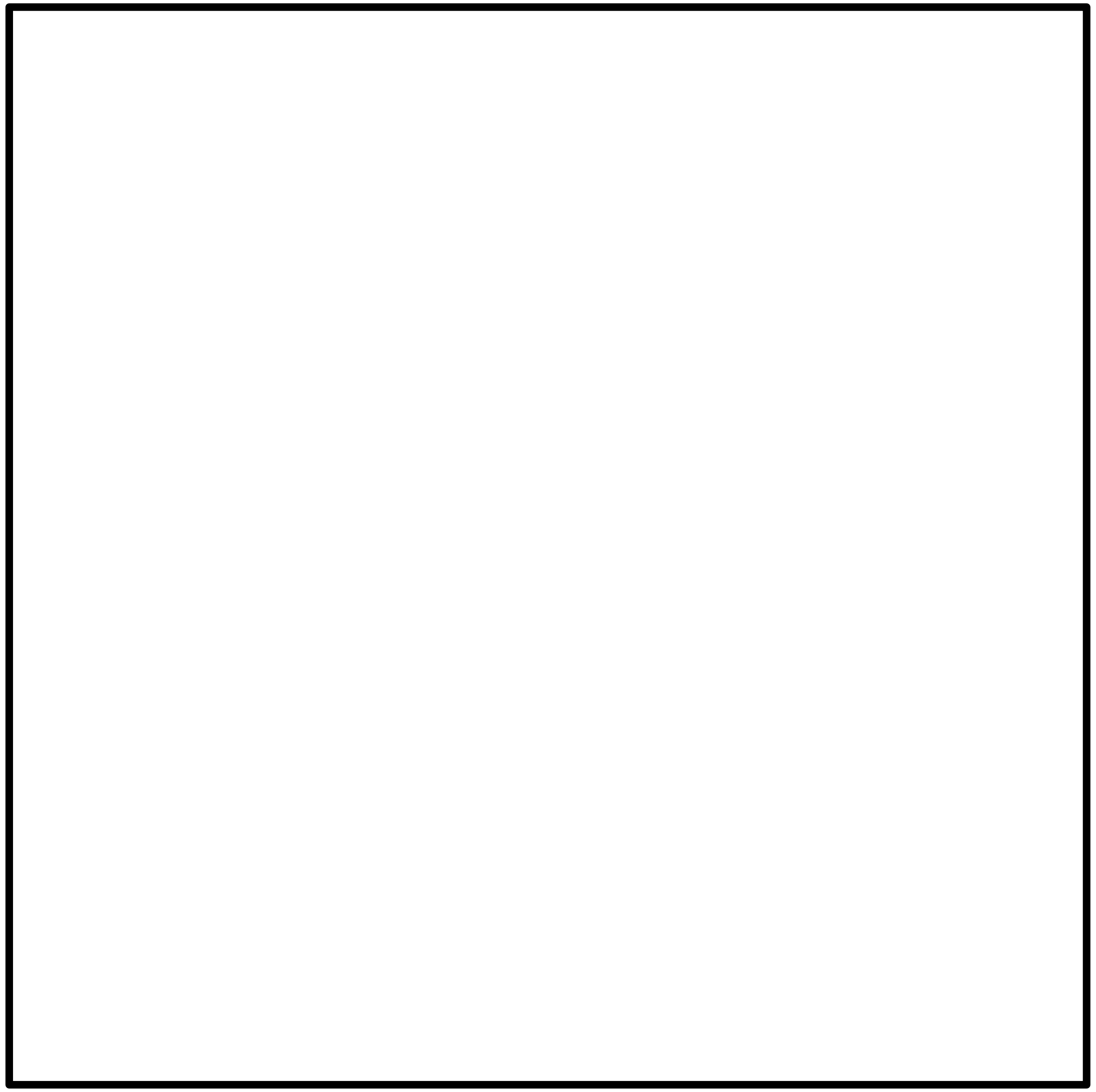
PATTERNS IN RECTANGULATIONS

Michaela A. Polley
Dartmouth College
joint work with Andrei Asinowski (University of Klagenfurt)

University of Klagenfurt Doctoral Seminar
Klagenfurt, Austria
December 18, 2025

What is a rectangulation?



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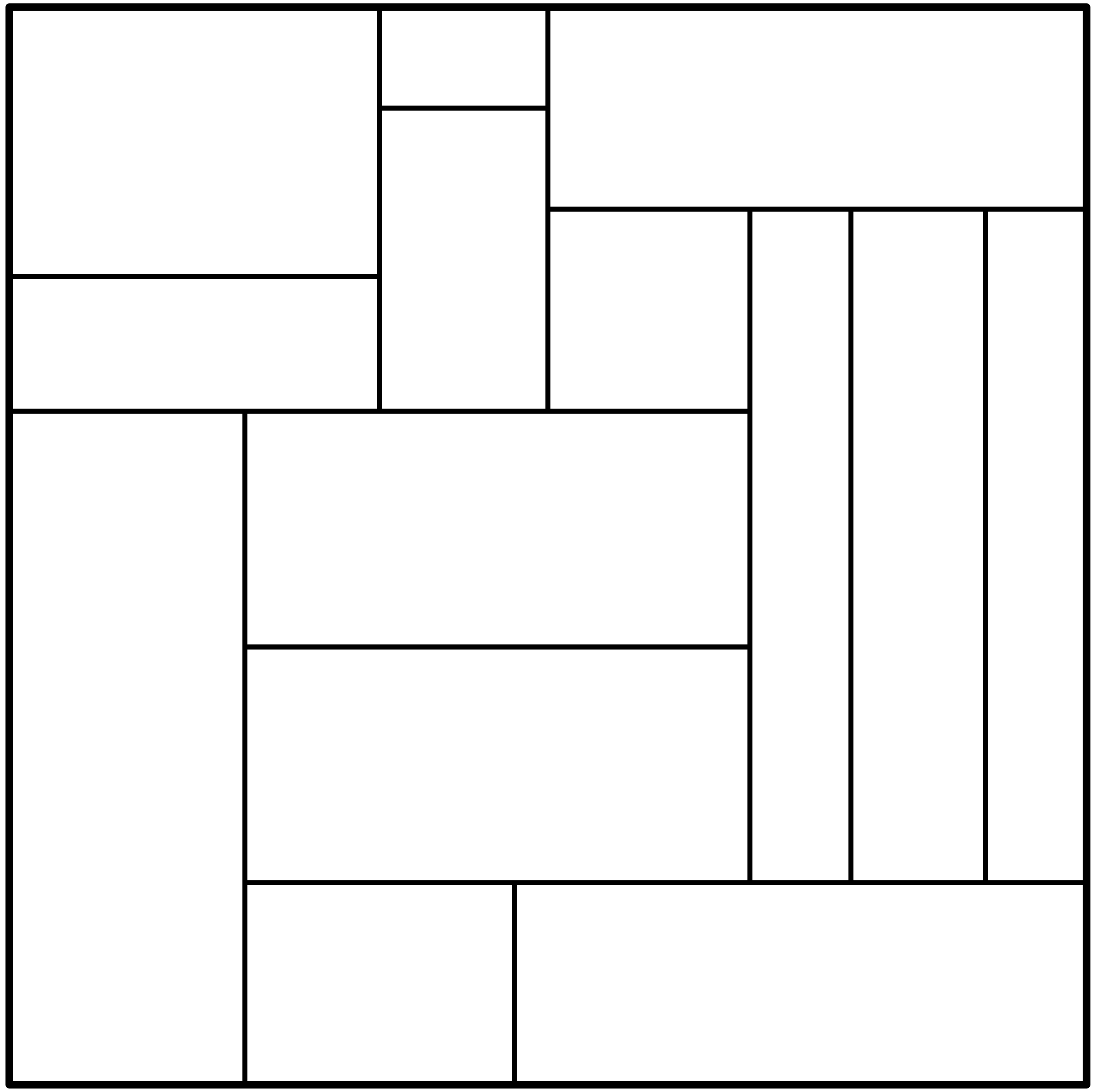

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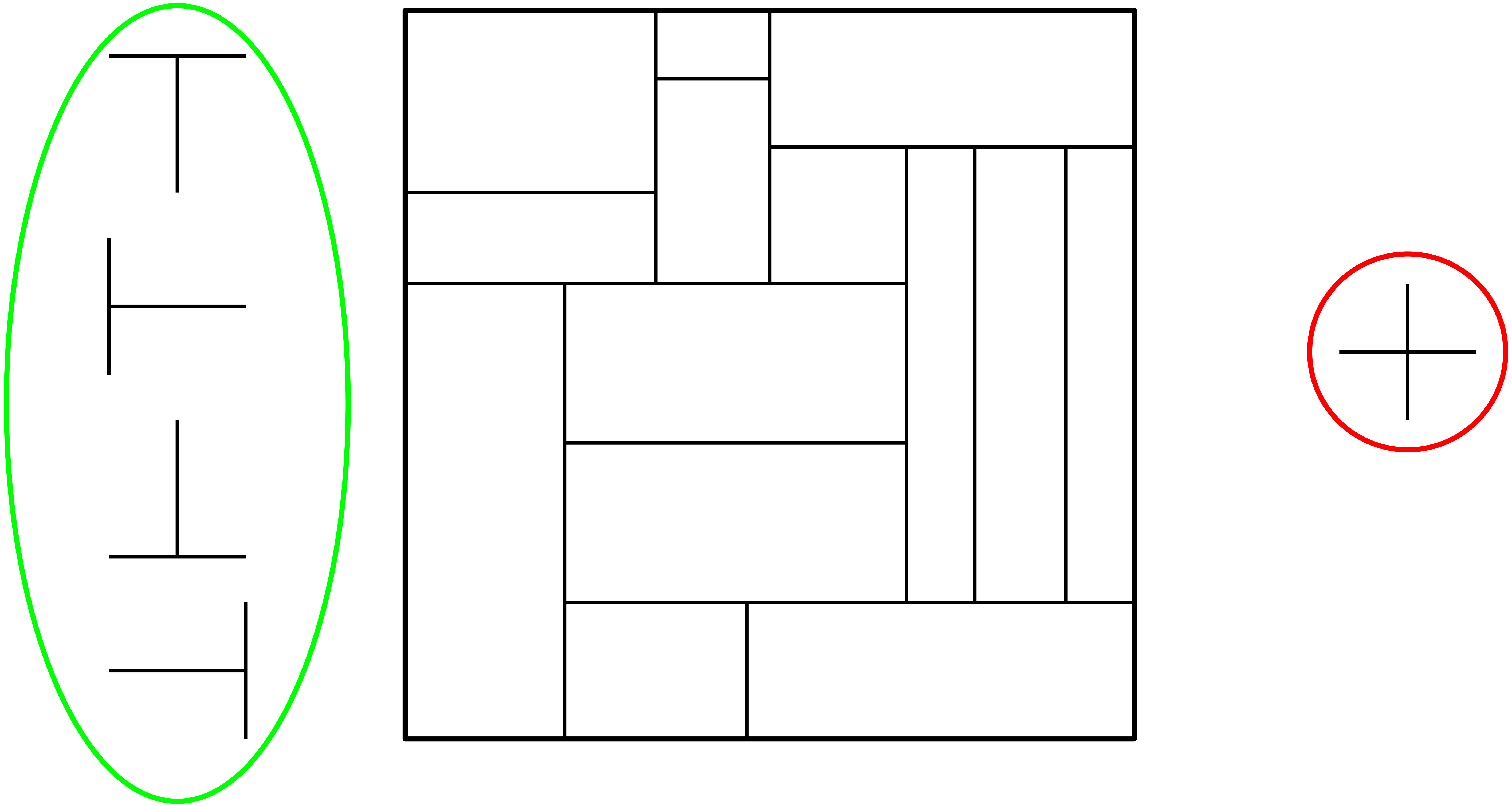

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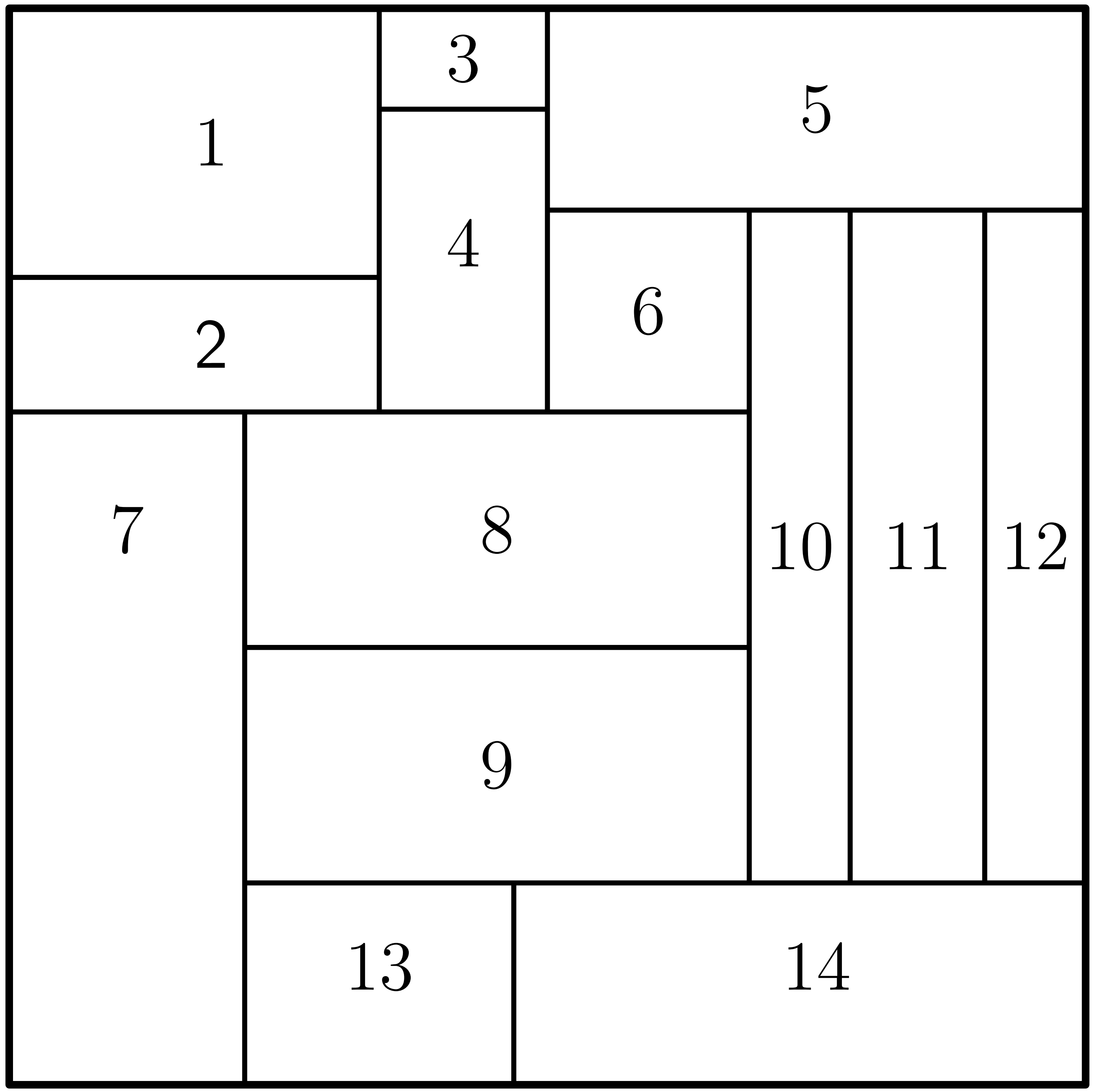

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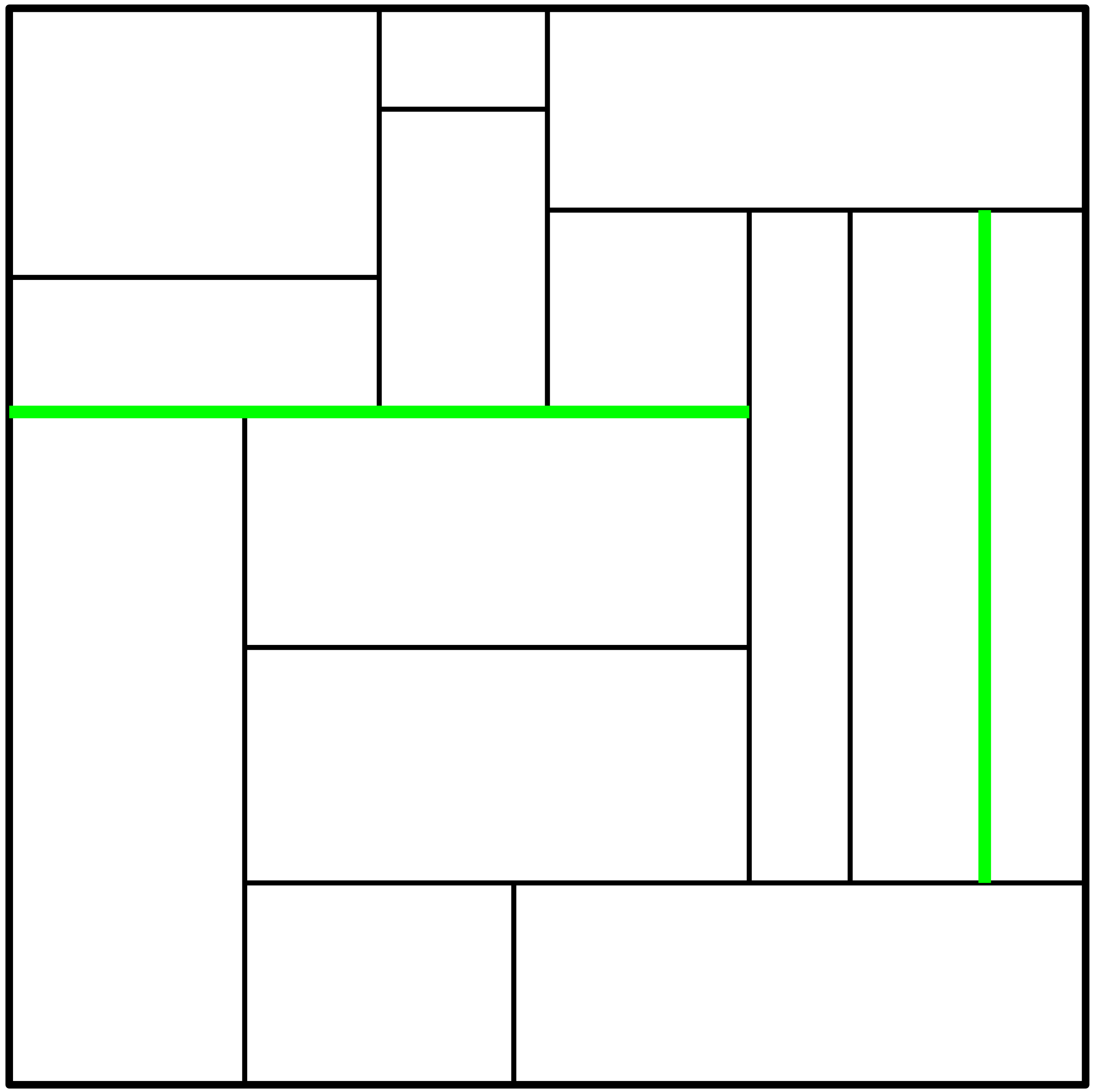

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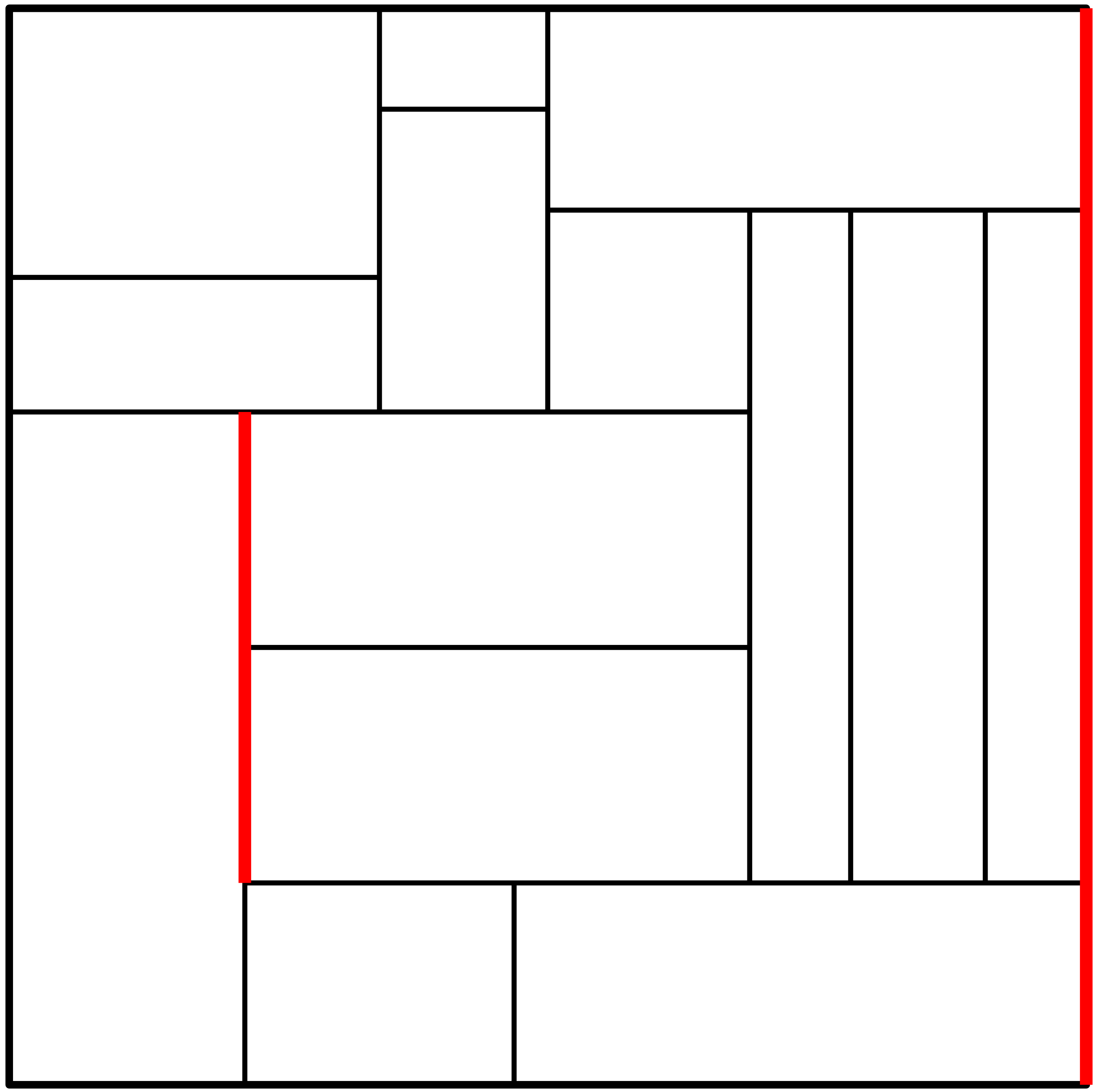

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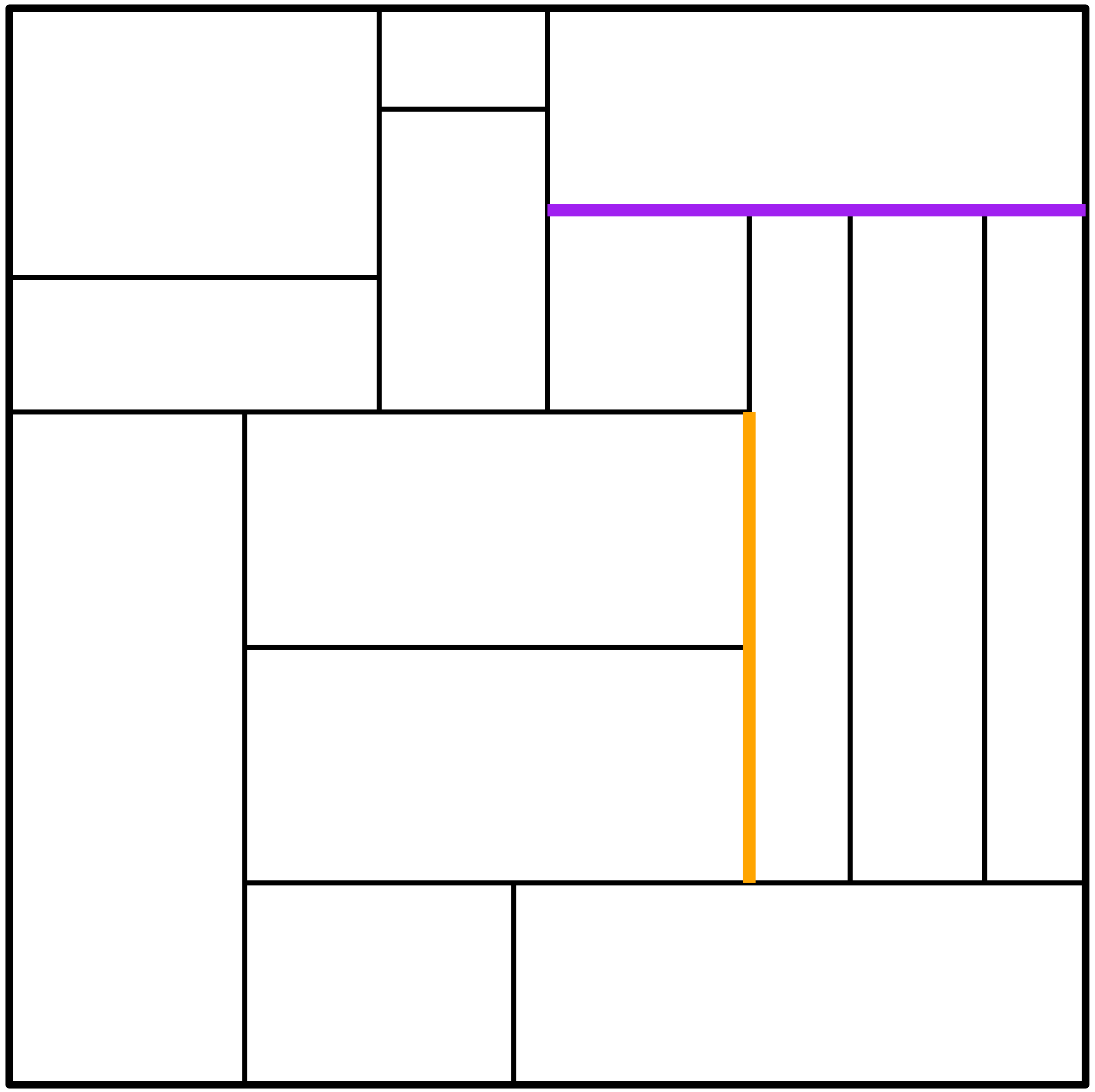

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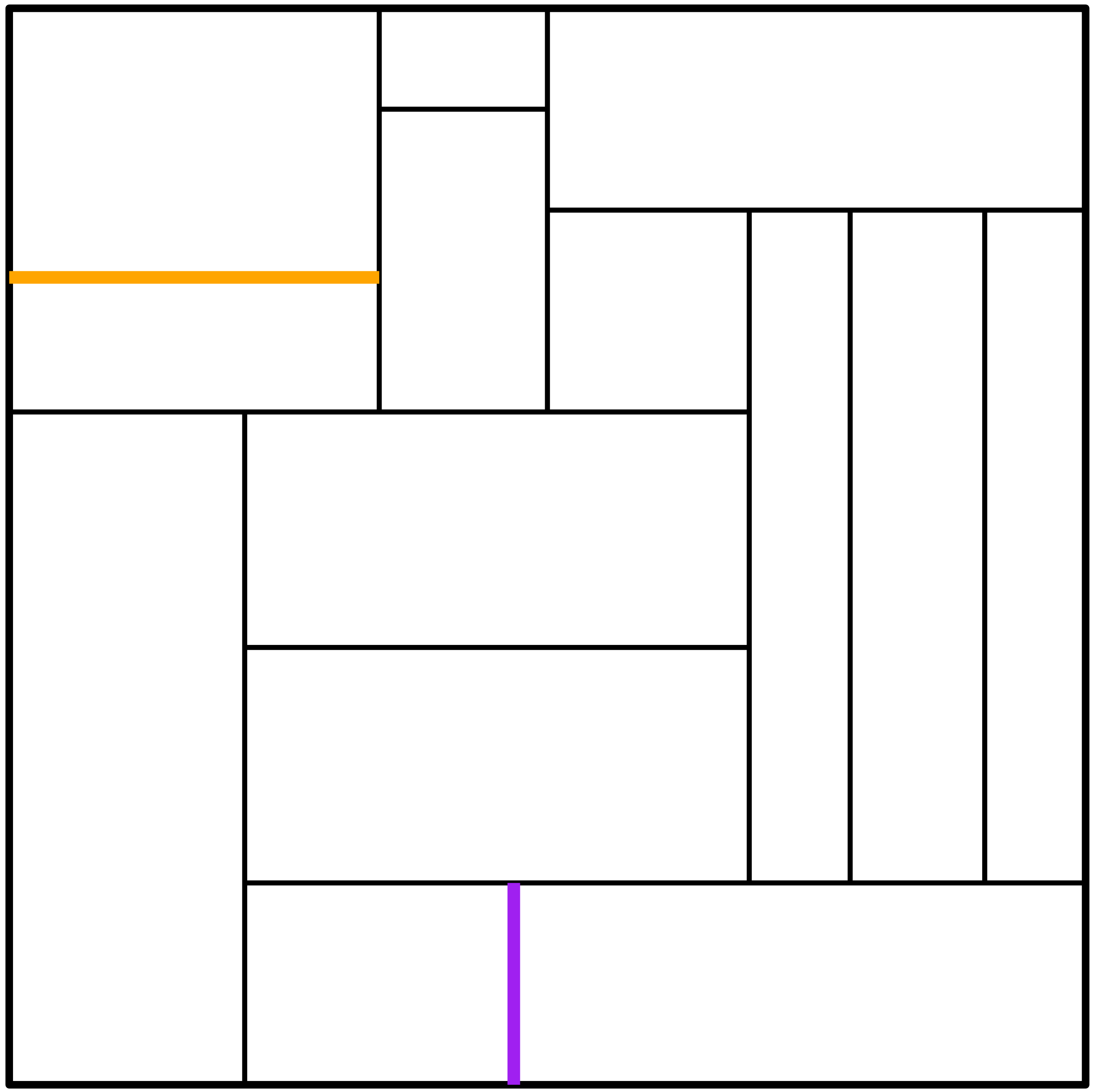

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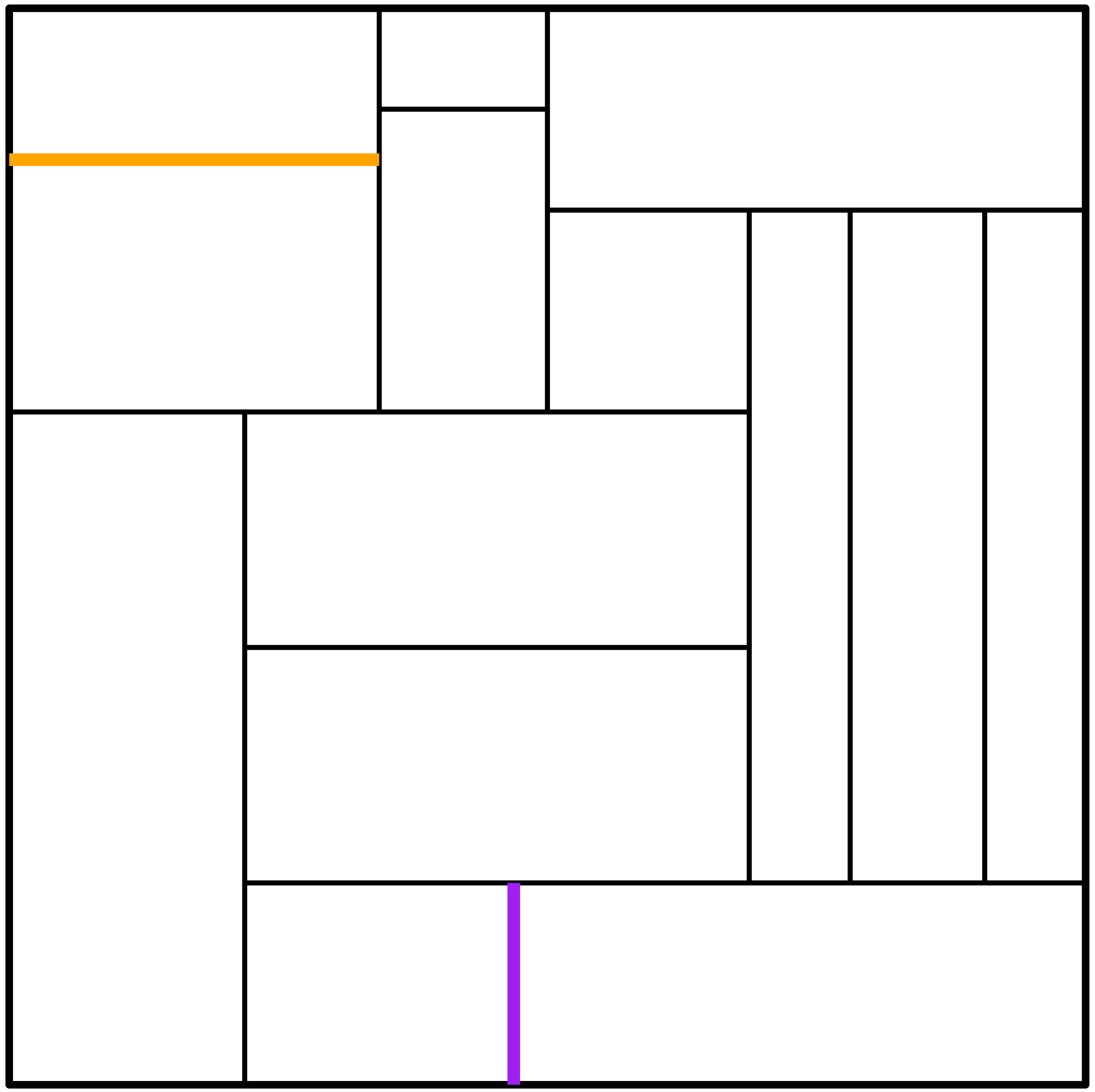

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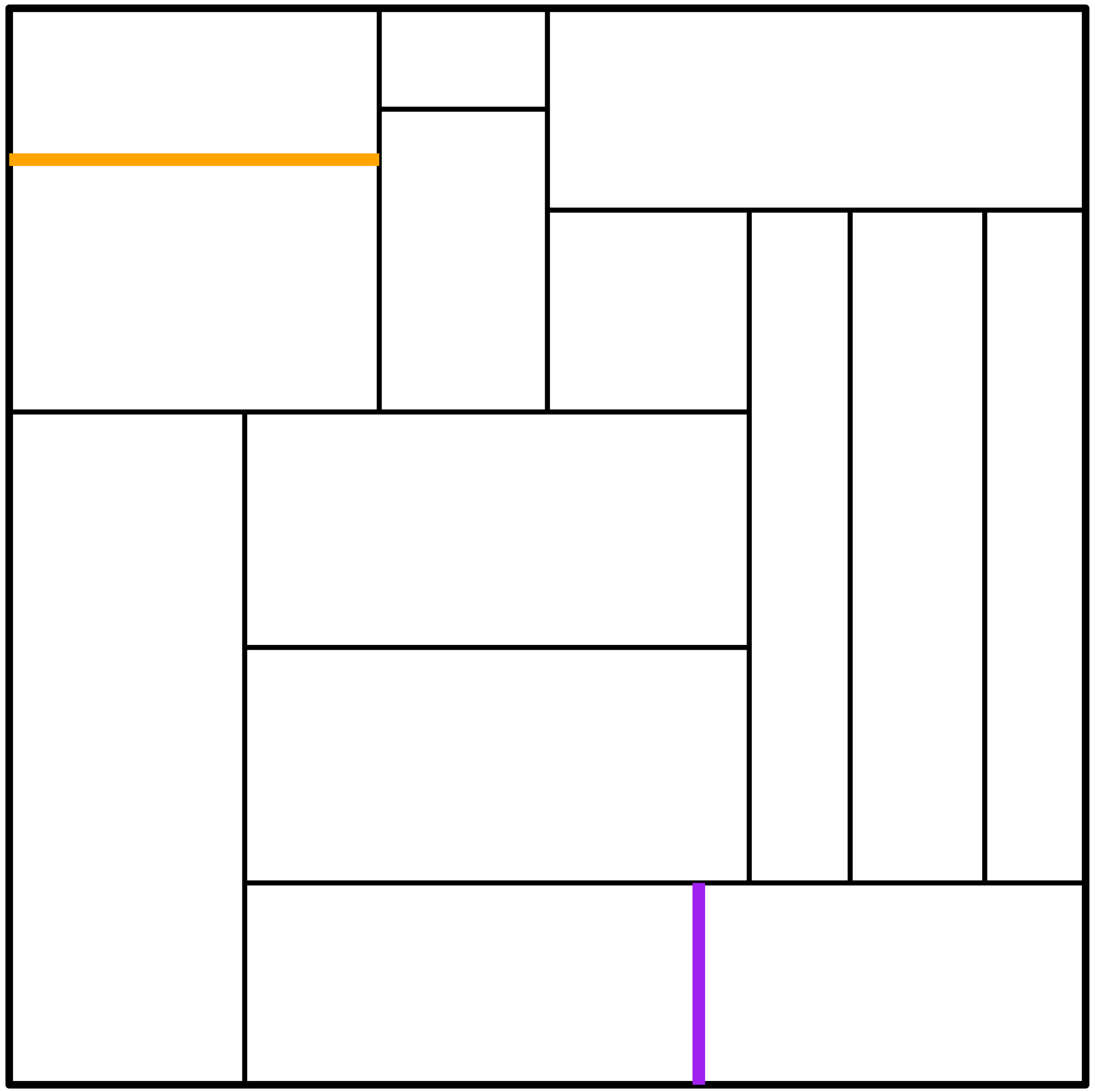

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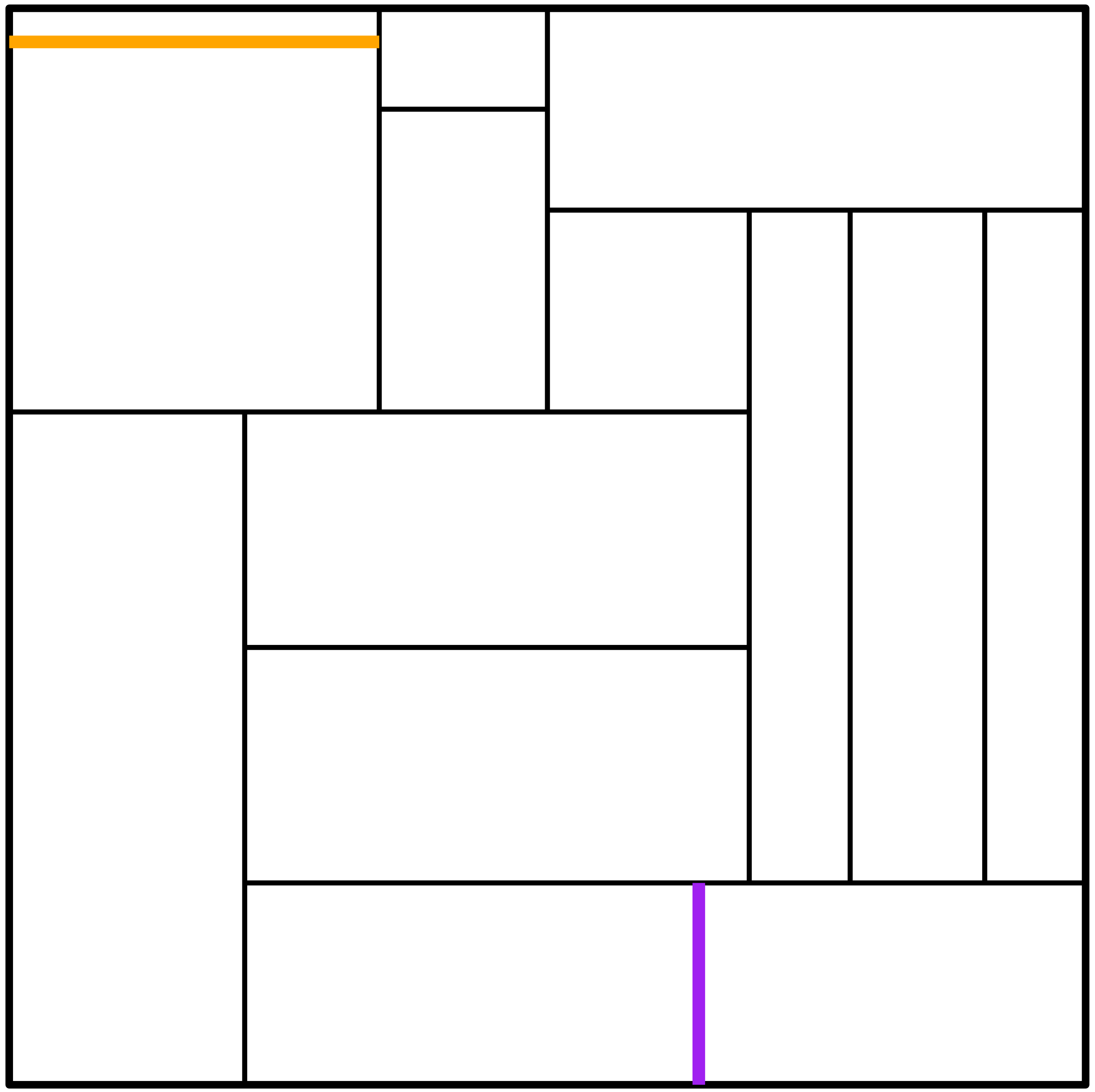

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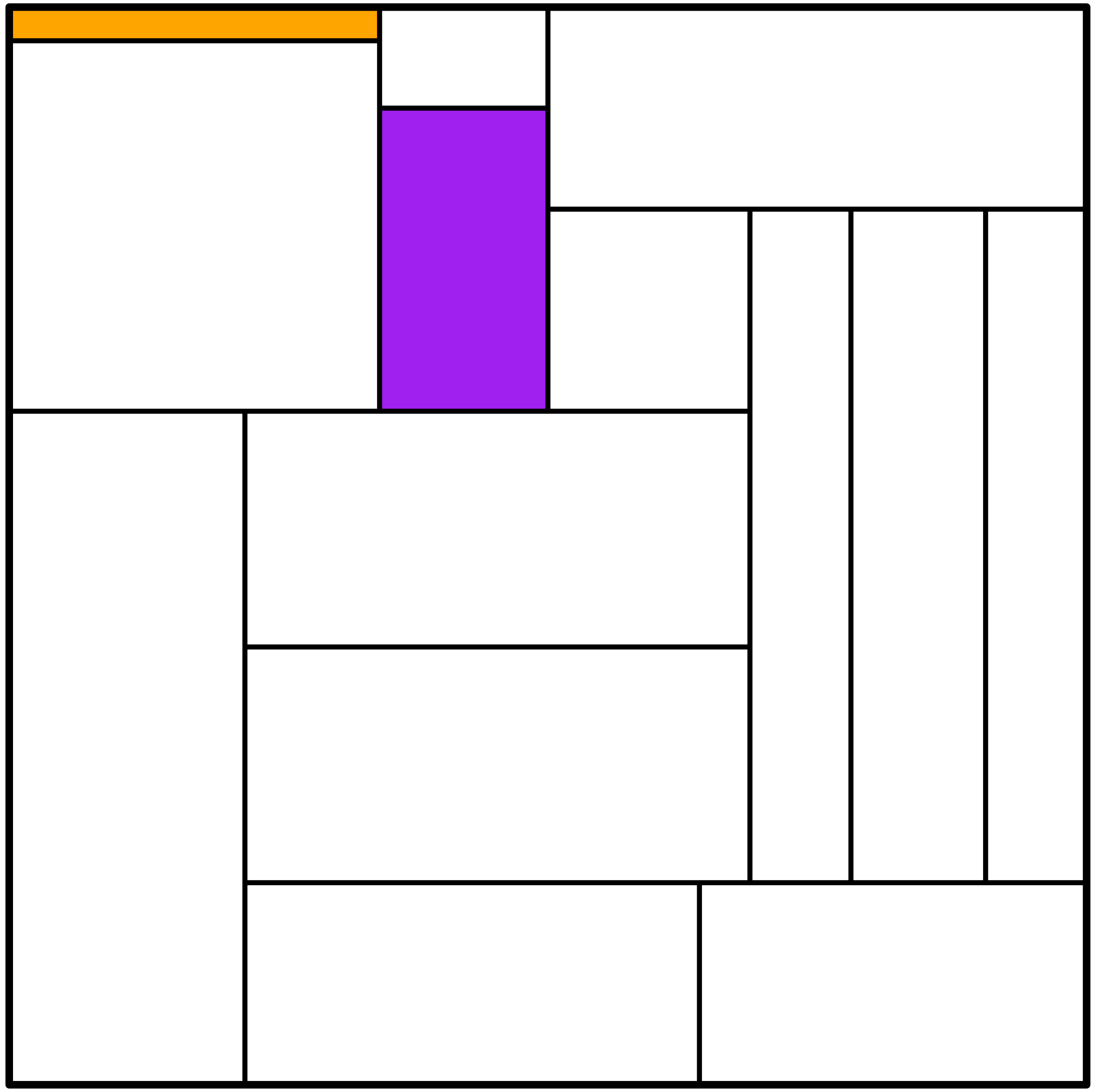

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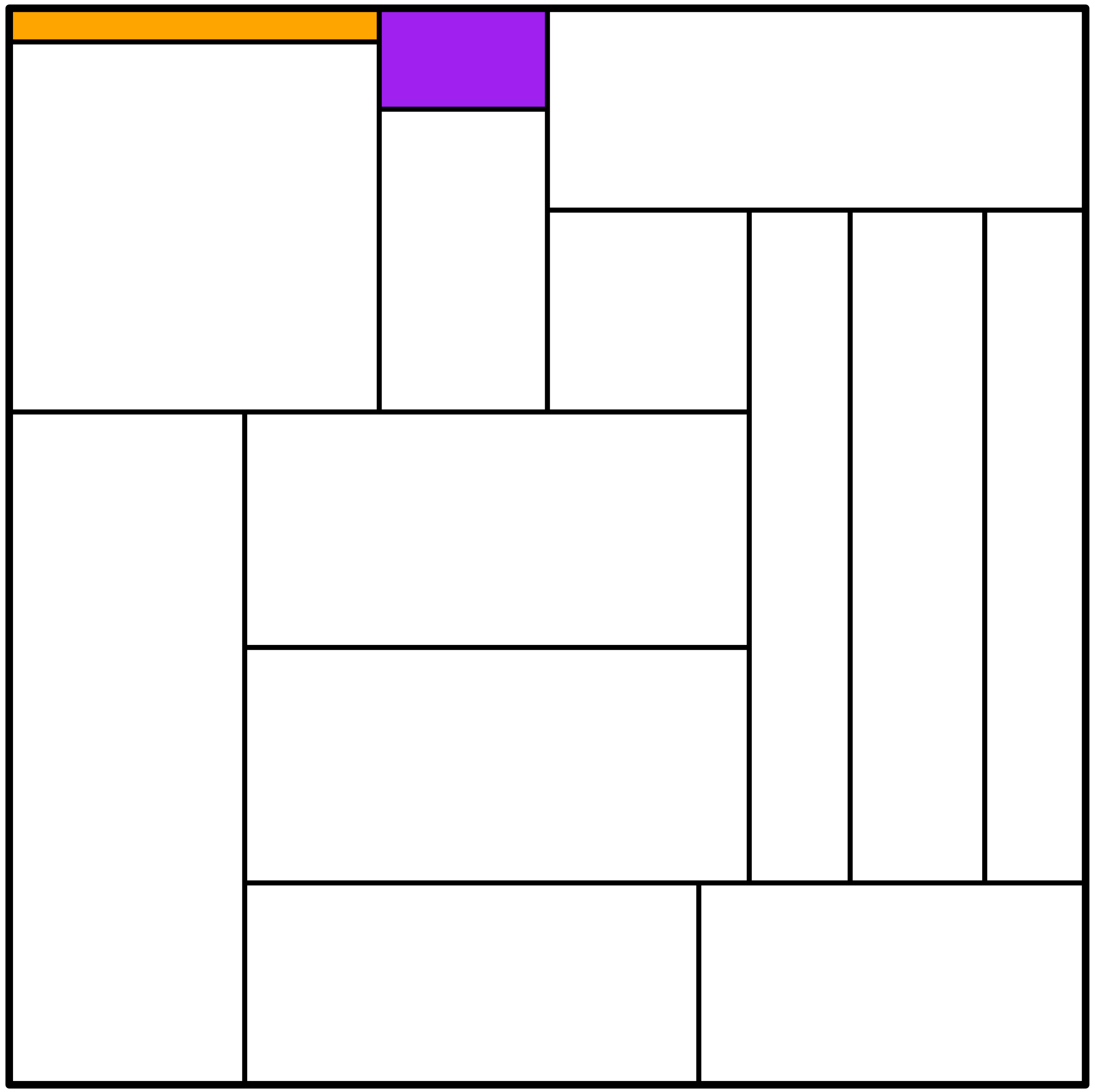

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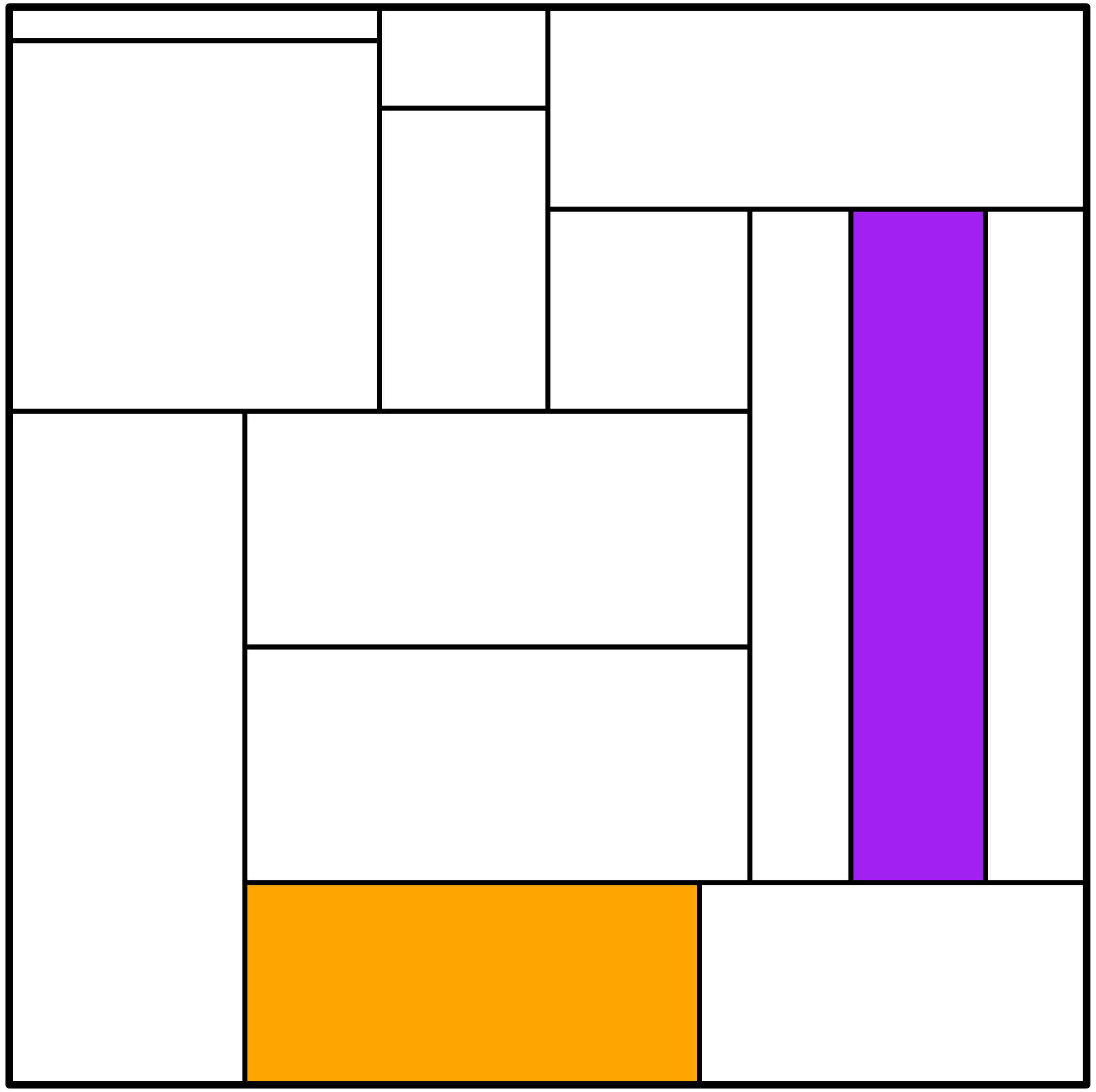

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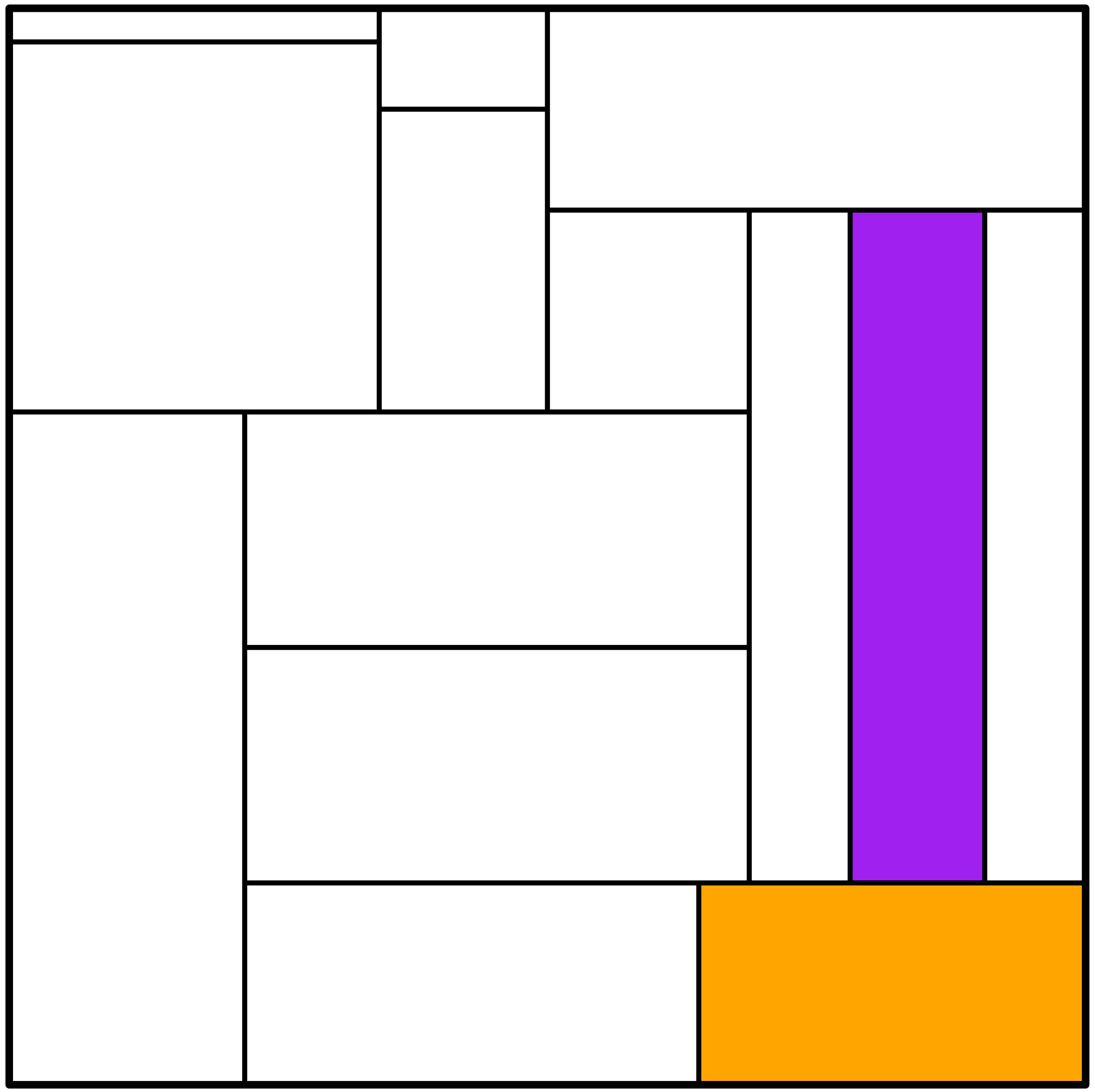
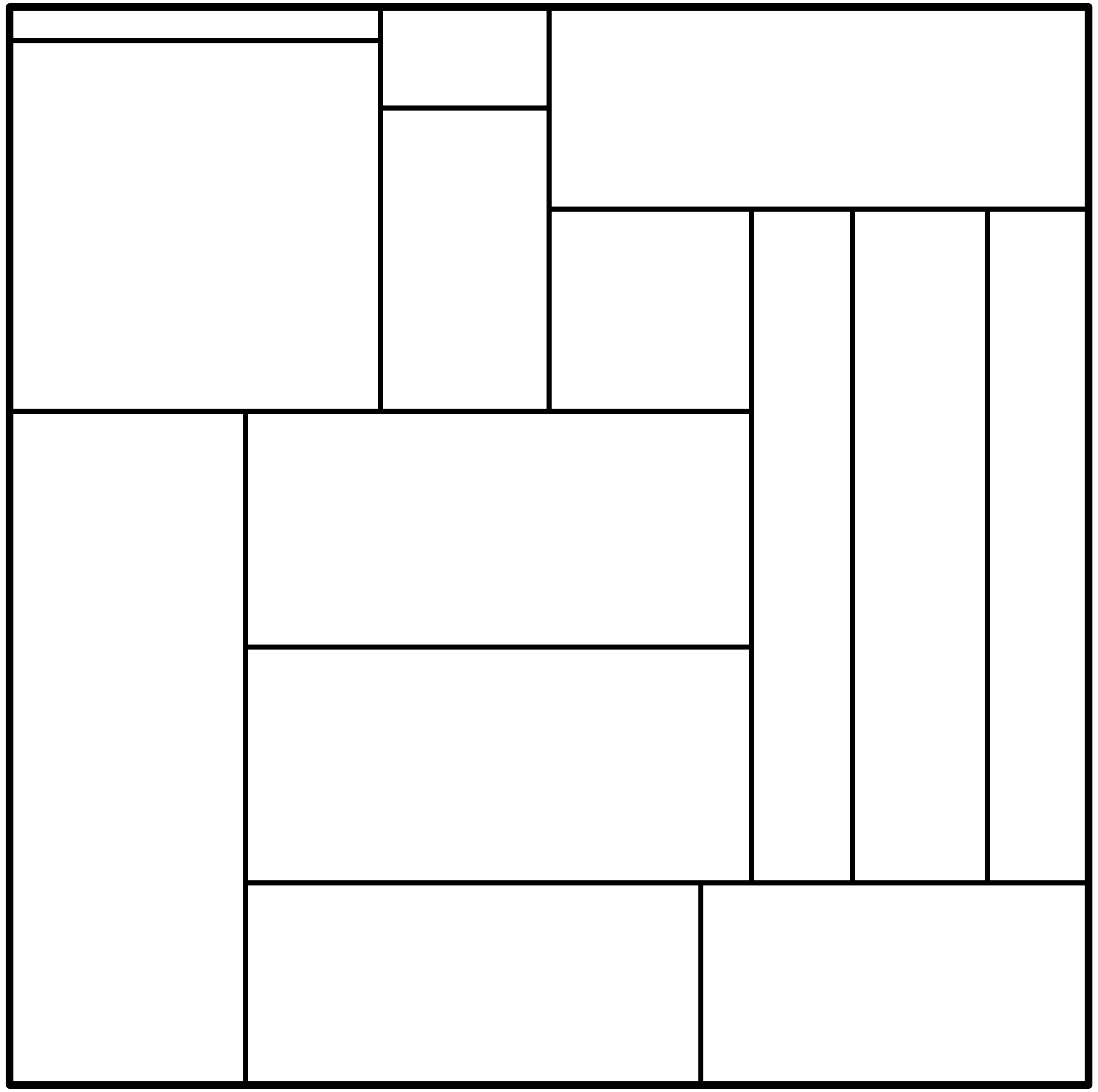
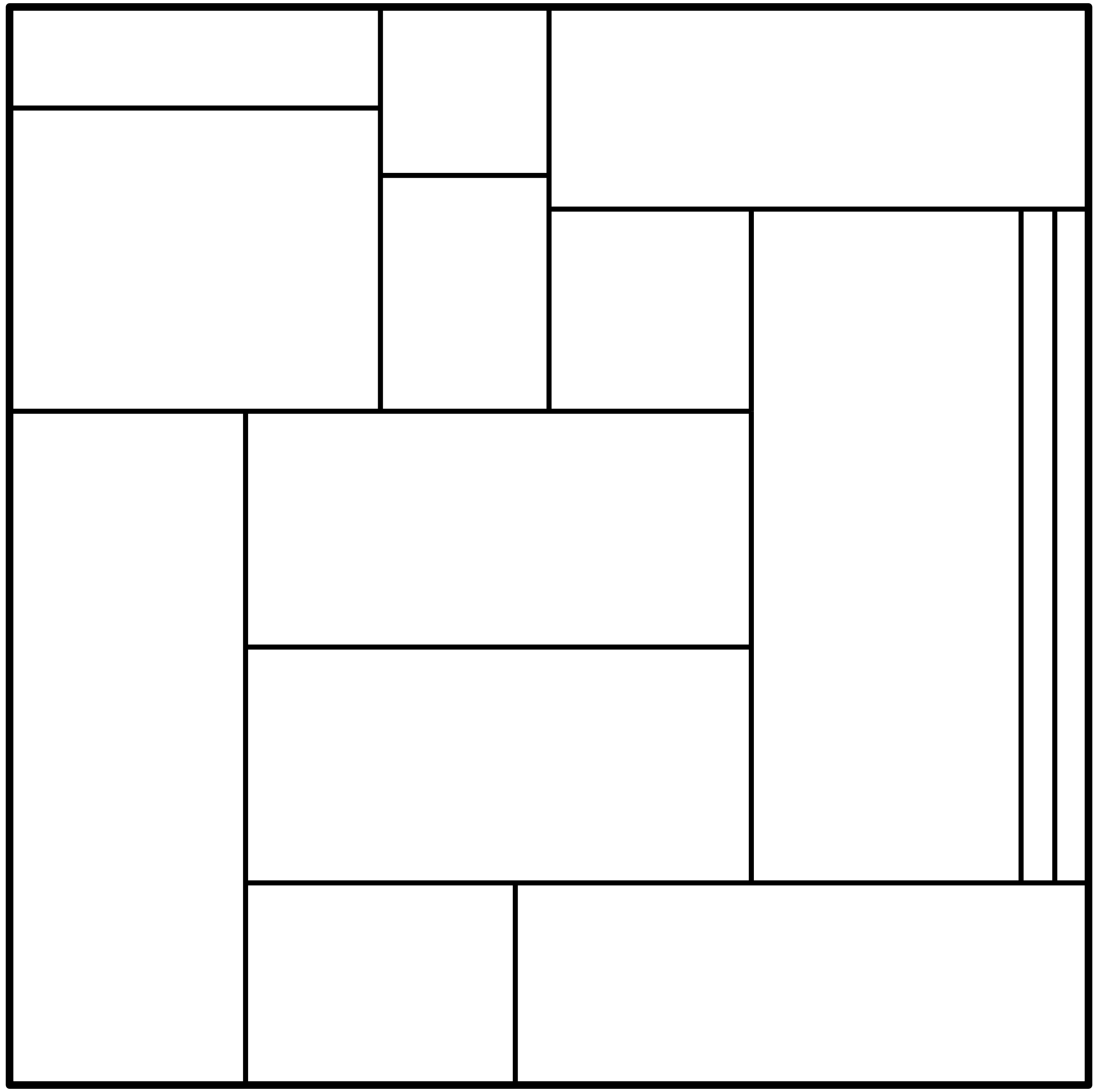


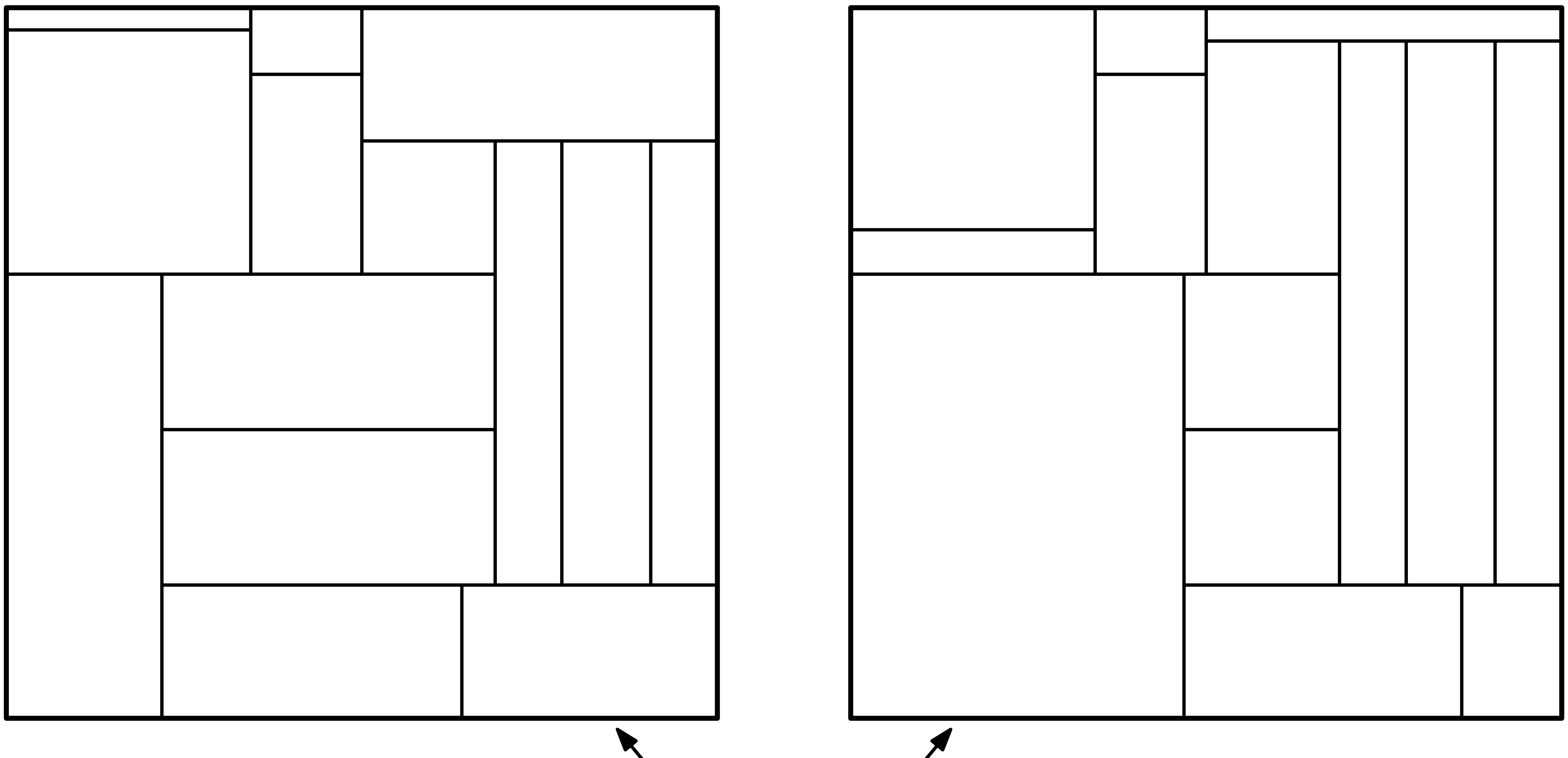
Weak and Strong Equivalence





STRONGLY
equivalent

Weak and Strong Equivalence



WEAKLY
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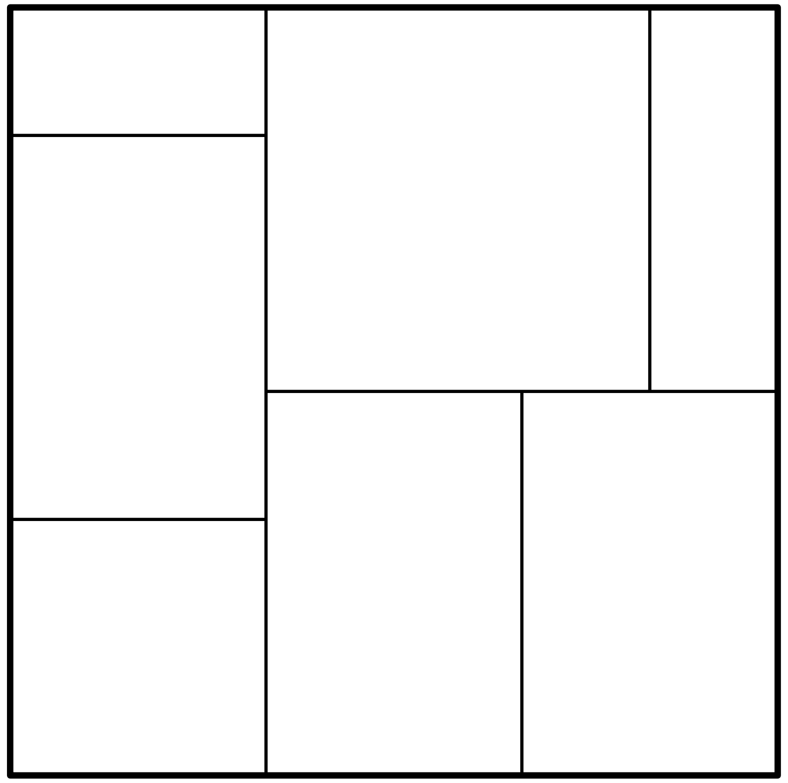
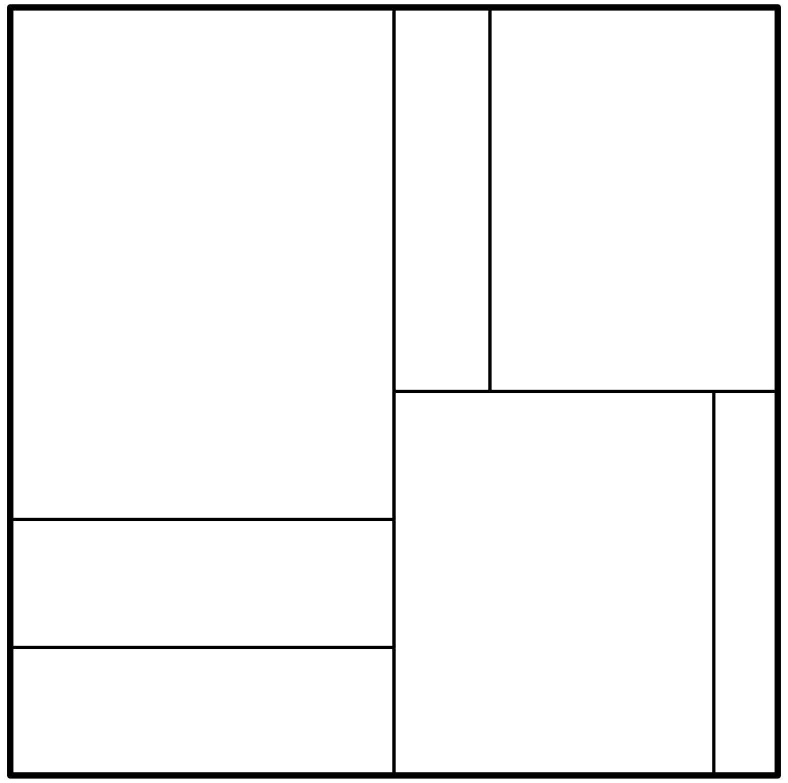
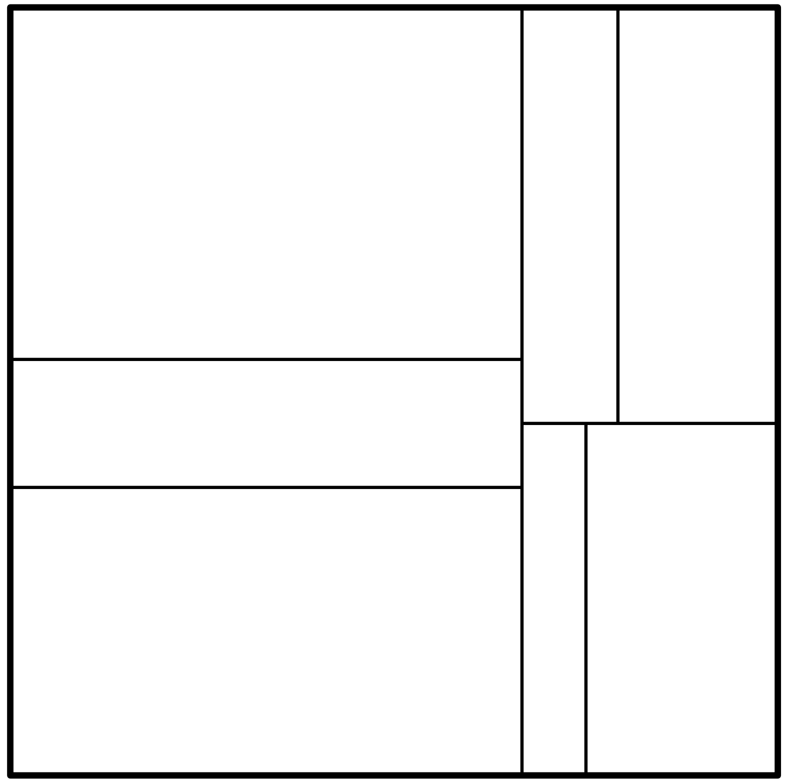


$\mathcal{A}$



$\mathcal{B}$



$\mathcal{C}$

Pattern Avoidance: $R(\top, \dashv, \vdash)$

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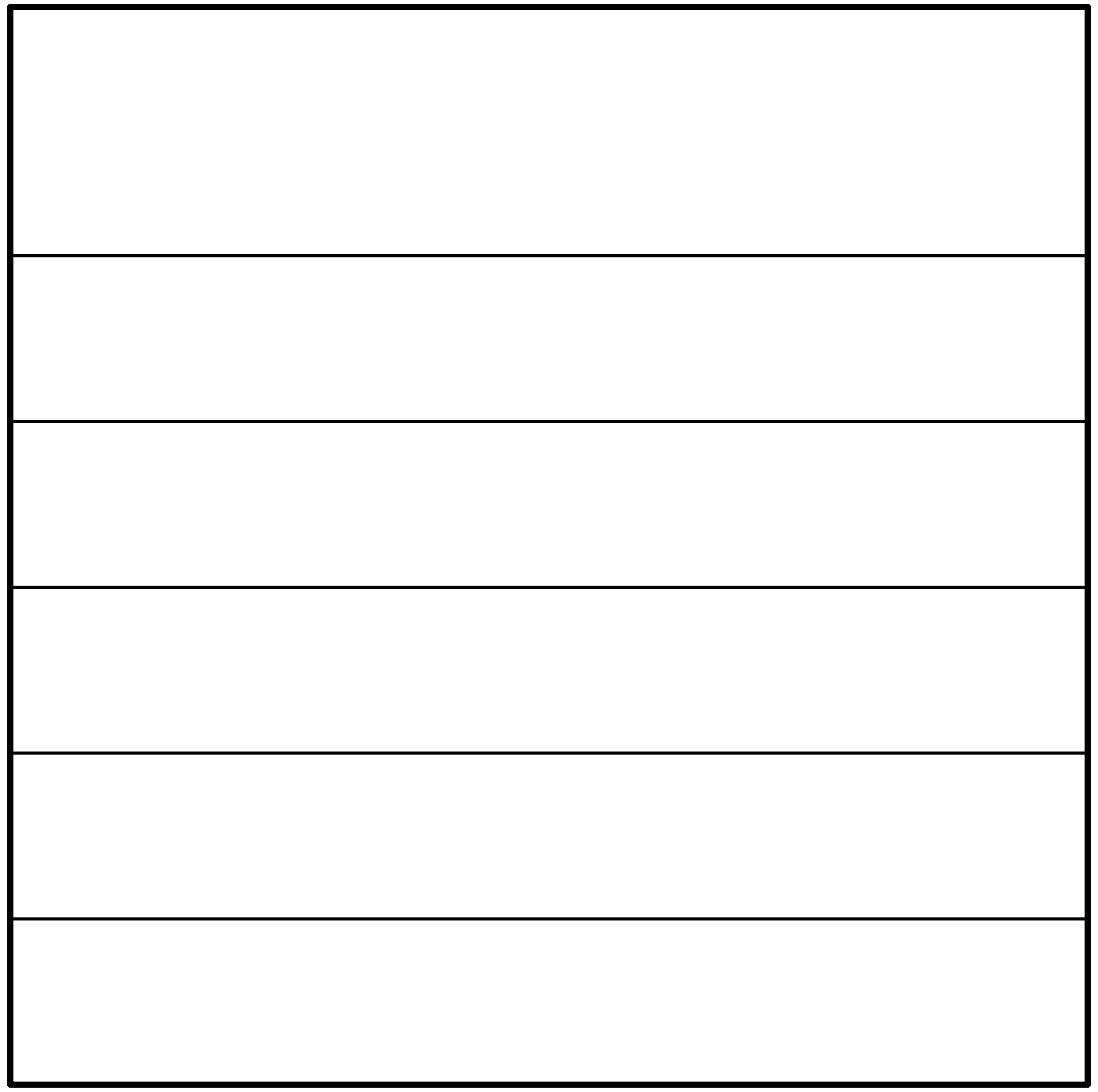
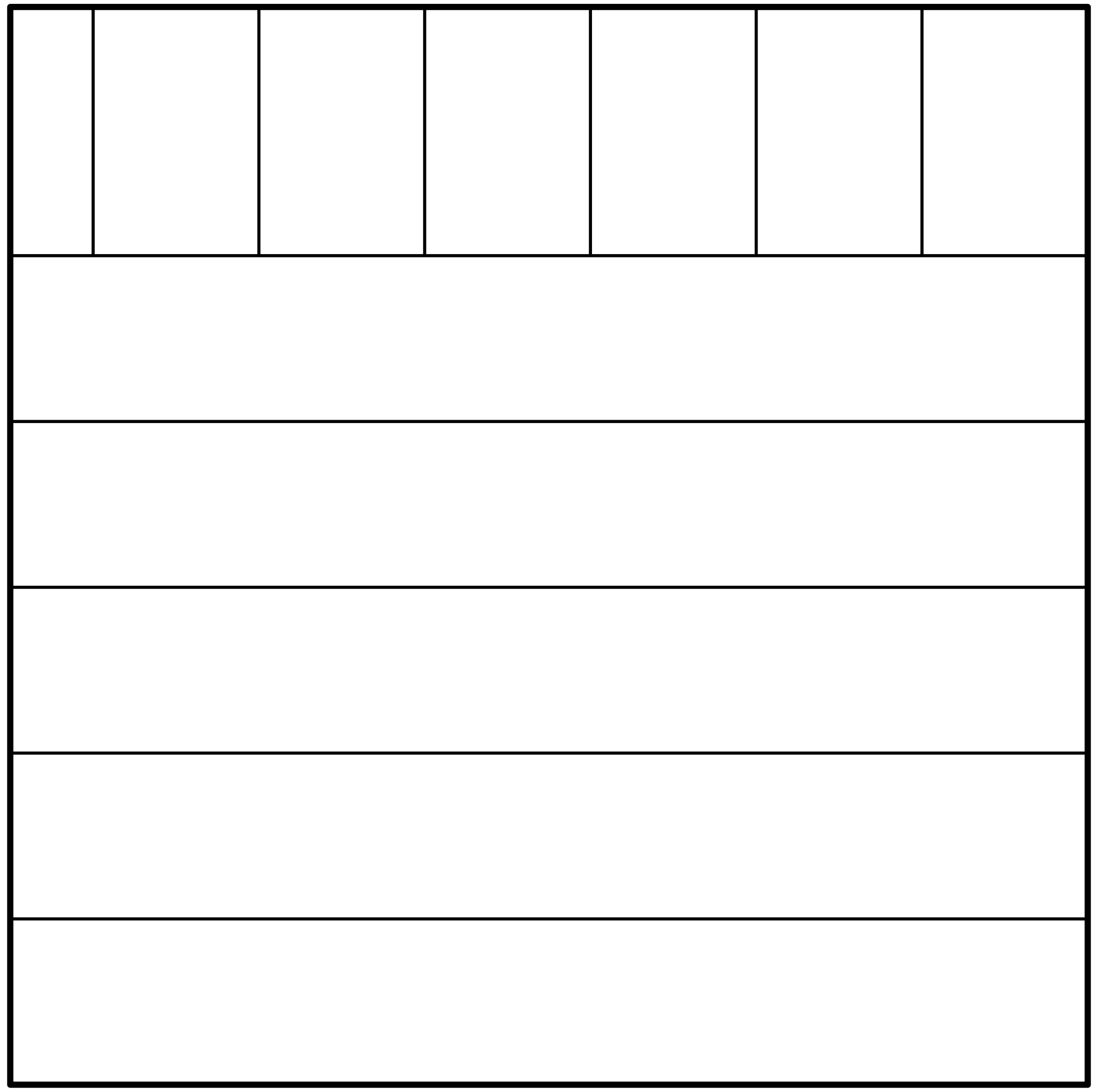


Pattern Avoidance: $R(\top, \dashv, \vdash)$



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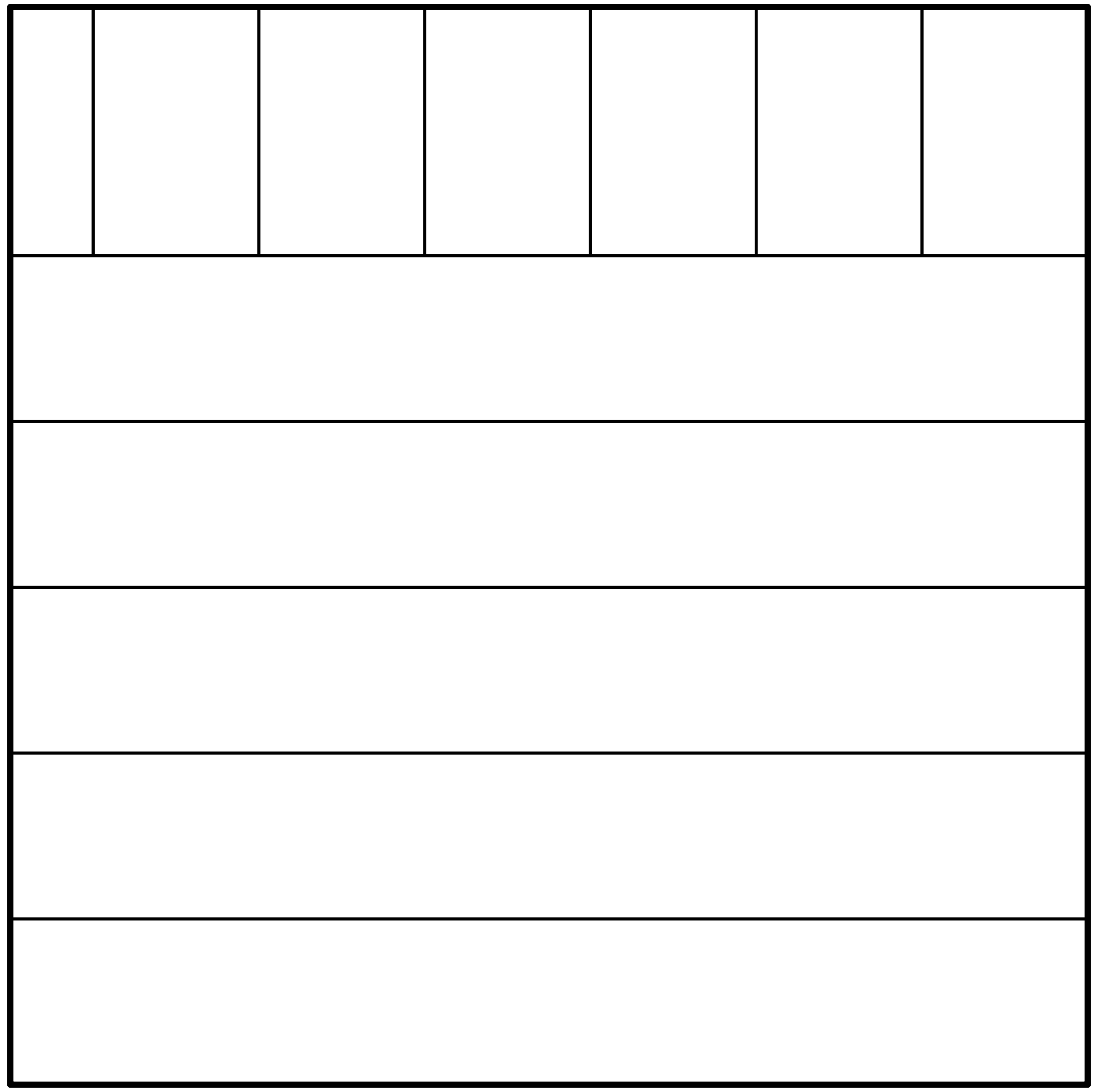

Proposition

The number of rectangulations of size n that avoid $\top$, $\dashv$, and $\vdash$, denoted $R_n(\top, \dashv, \vdash)$ is n.

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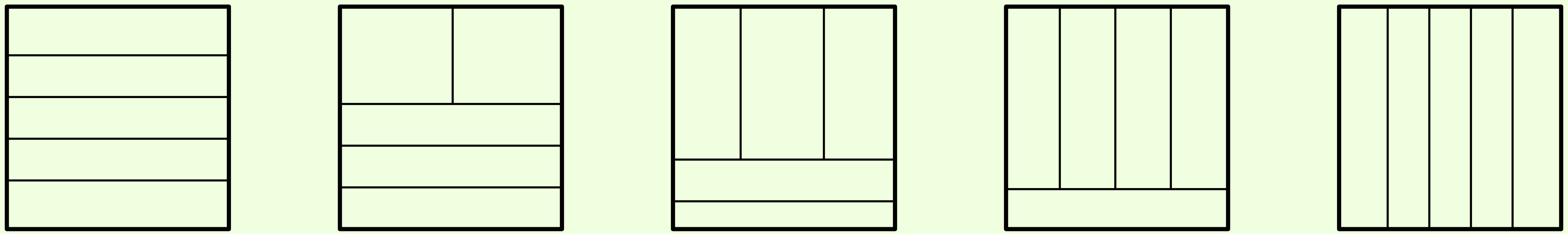
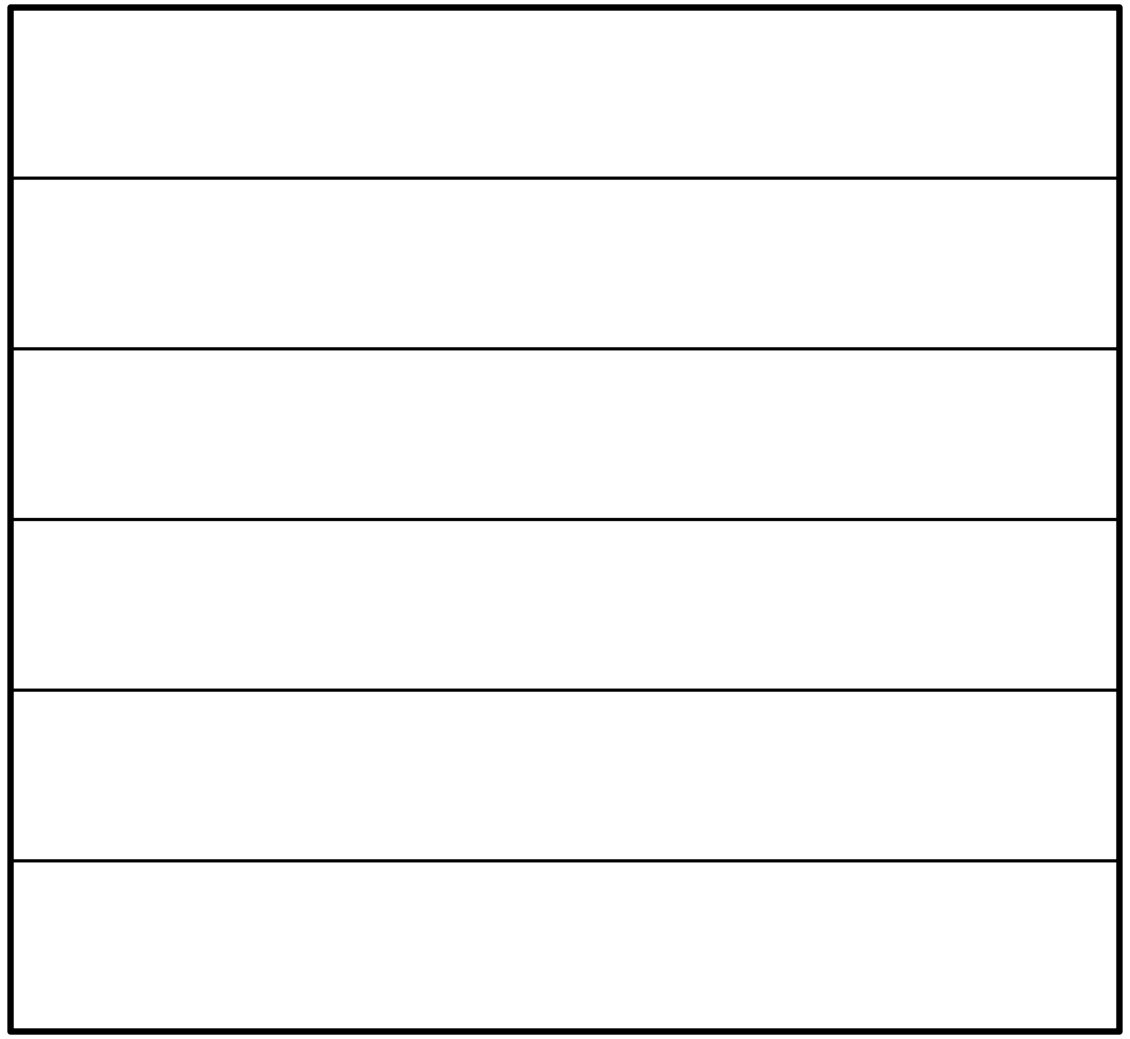
Example

For $n = 5$, there are five rectangulations that avoid $\top$, $\dashv$, and $\vdash$:



Pattern Avoidance: $R(\dashv, \vdash)$



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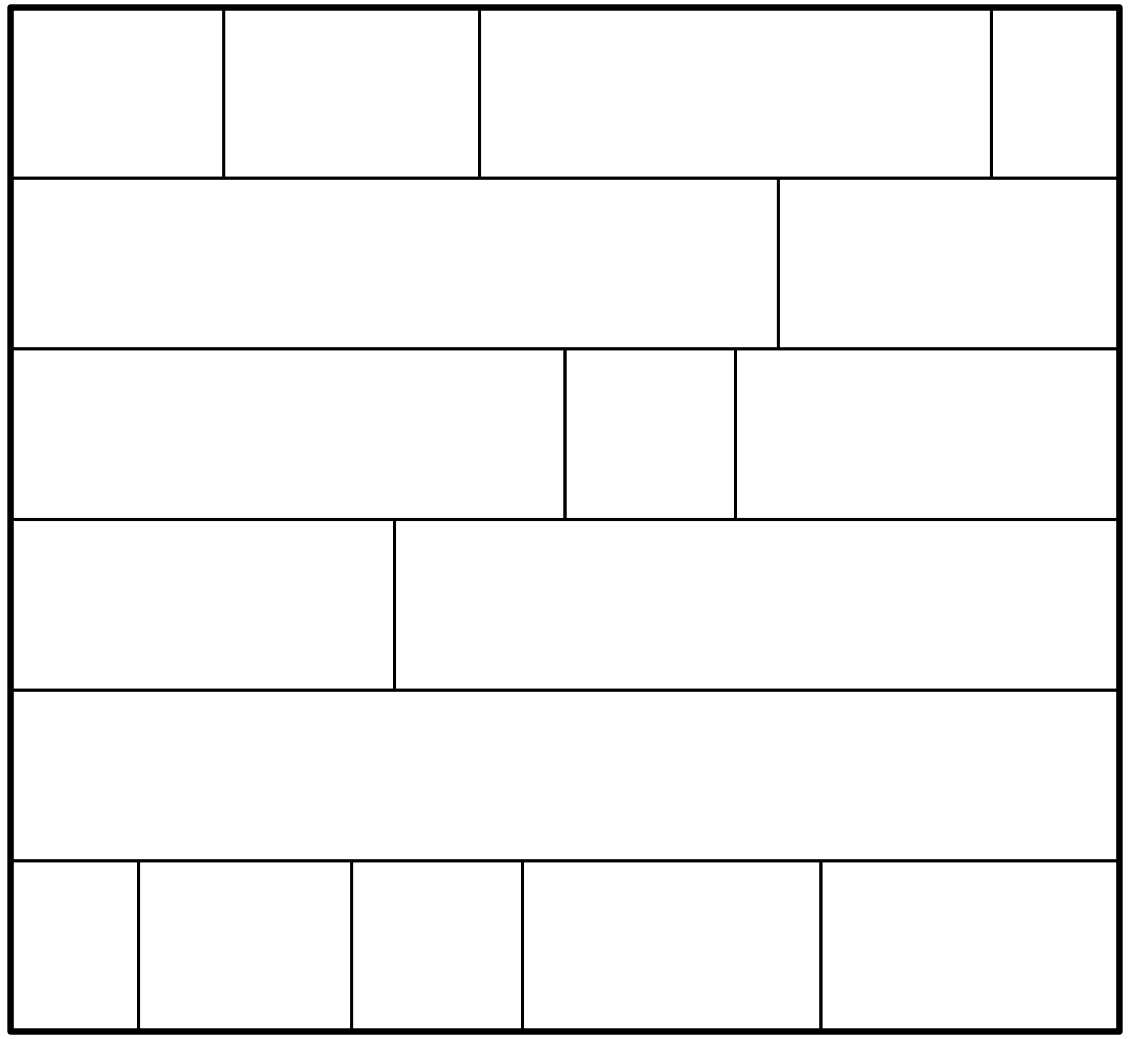

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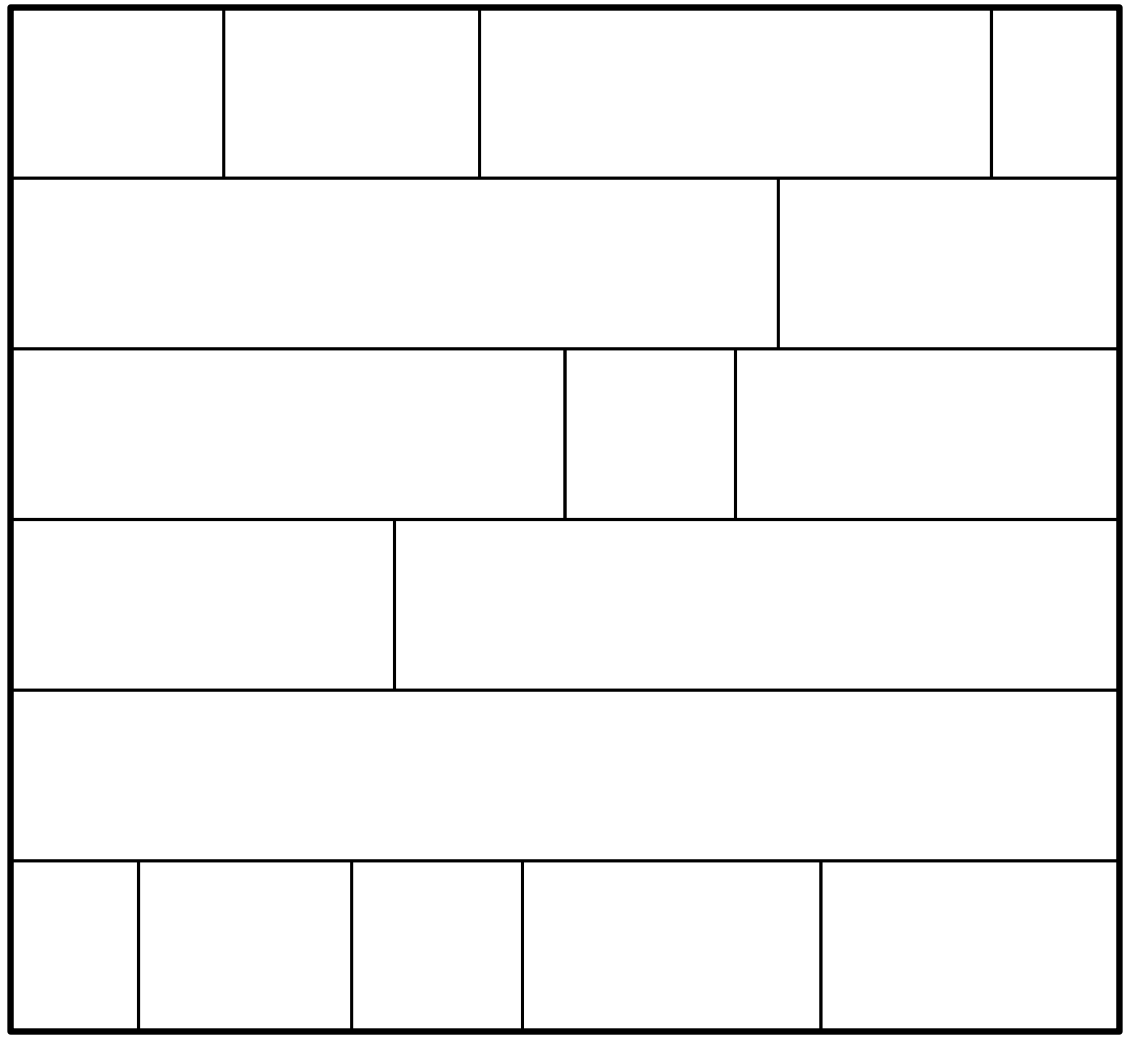

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The number of weak rectangulations of size n that avoid $\dashv$ and $\vdash$, denoted $R_n^w(\dashv, \vdash)$ is 2^{n-1}.

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Definition

A *composition* of n is an ordered list of positive integers $(a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \ldots + a_k = n$.

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There are 2^{n-1} compositions of n:

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Pattern Avoidance: $R(\dashv, \vdash)$

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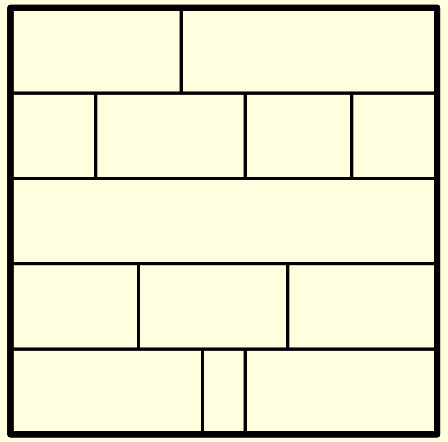
Proof

Establish a bijection between $R_n^w(\dashv, \vdash)$ and compositions of n.



$\longleftrightarrow$ $2+4+1+3+3$

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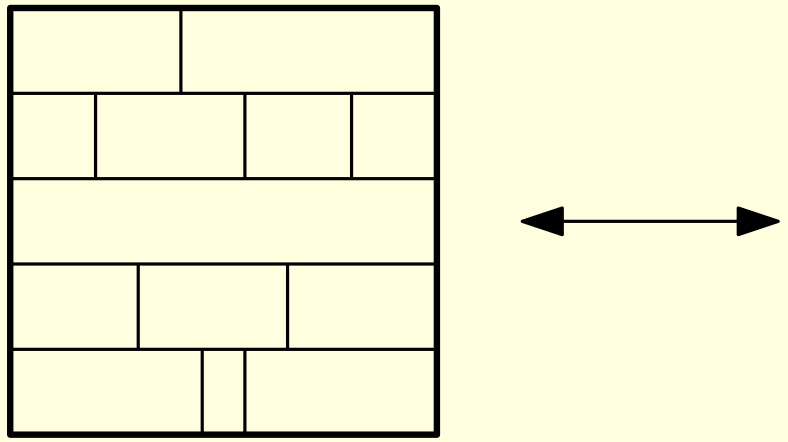

$2+4+1+3+3$

Question: What about strong rectangulations?

Pattern Avoidance: $R(\dashv, \vdash)$

By complete enumeration (using a computer), we find that the number of *strong* rectangulations of size n which avoid $\vdash$ and $\dashv$ for $n = 1, \ldots, 8$ are $1, 2, 4, 9, 22, 57, 154, 430$.

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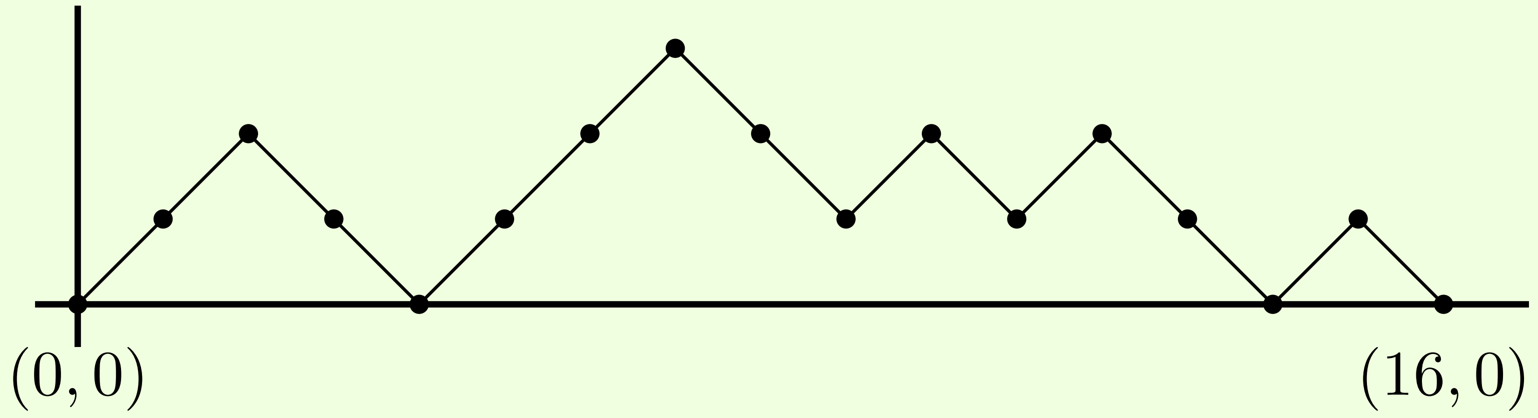
A *Dyck path* of semi-length n is a lattice path from $(0, 0)$ to $(2n, 0)$ consisting of $(1, 1)$ (U) and $(1, -1)$ (D) steps which never goes below the x-axis.



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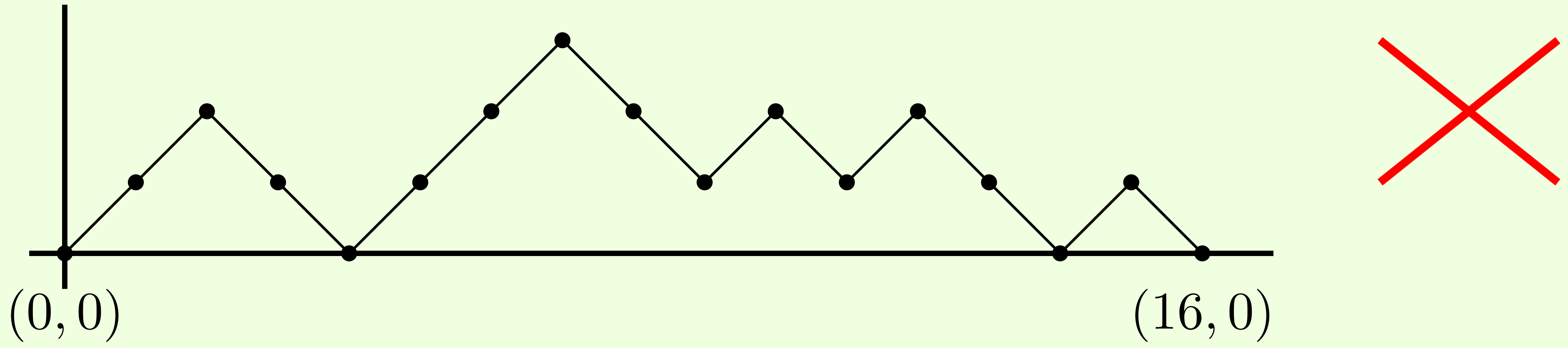

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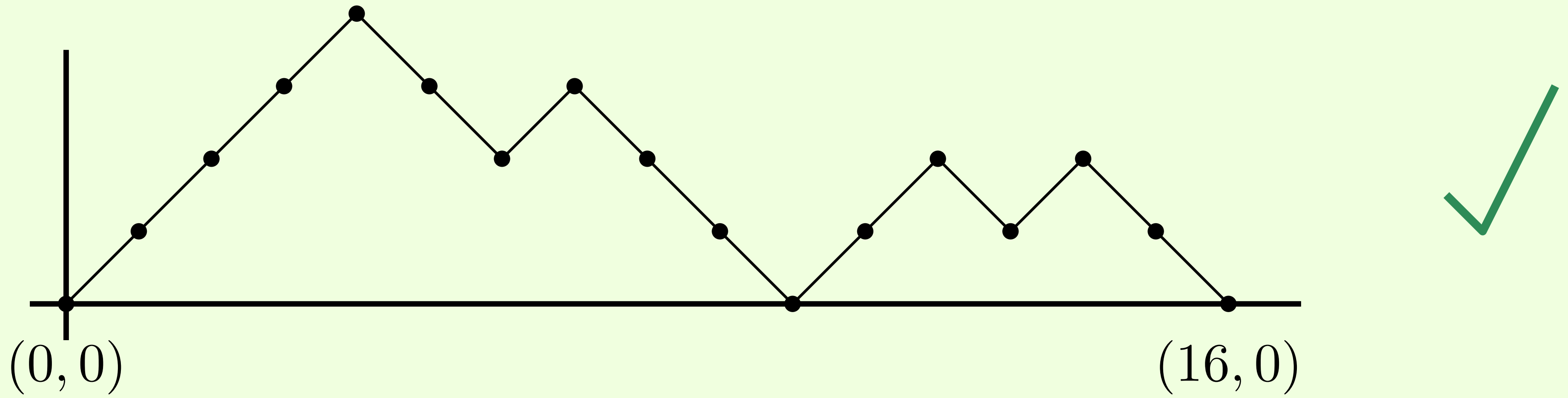

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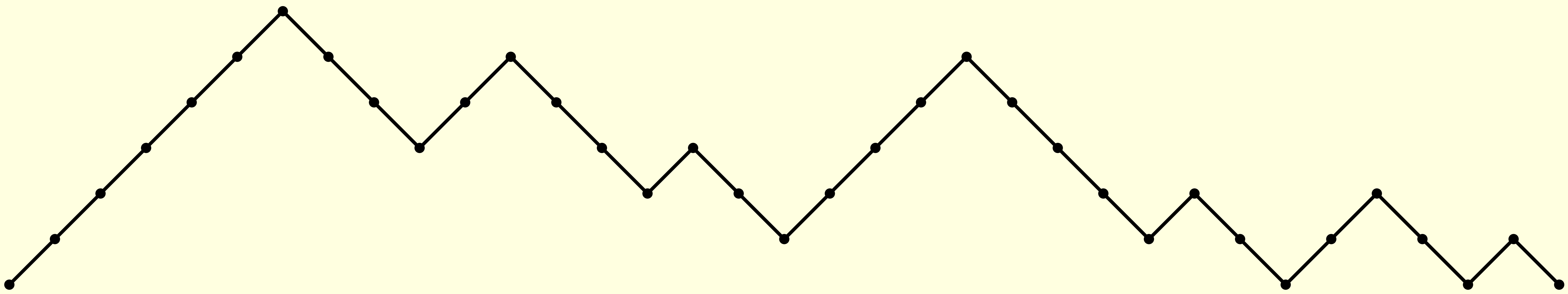

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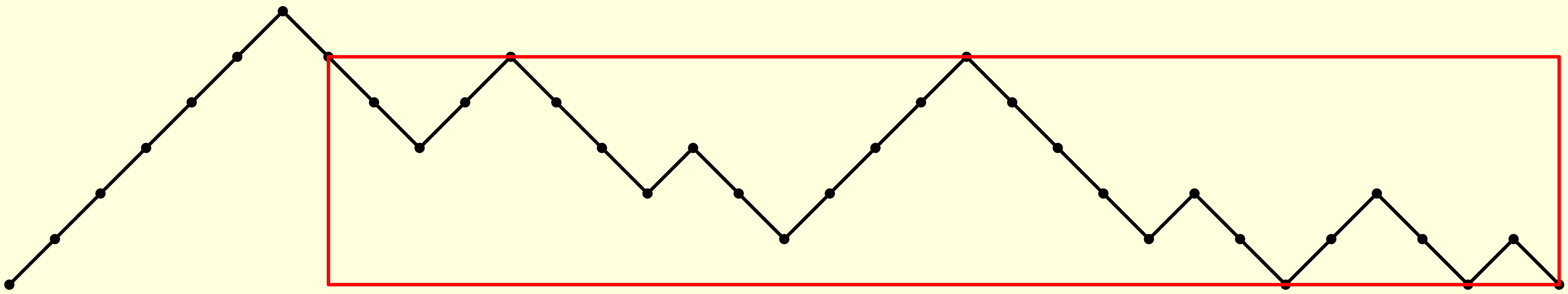

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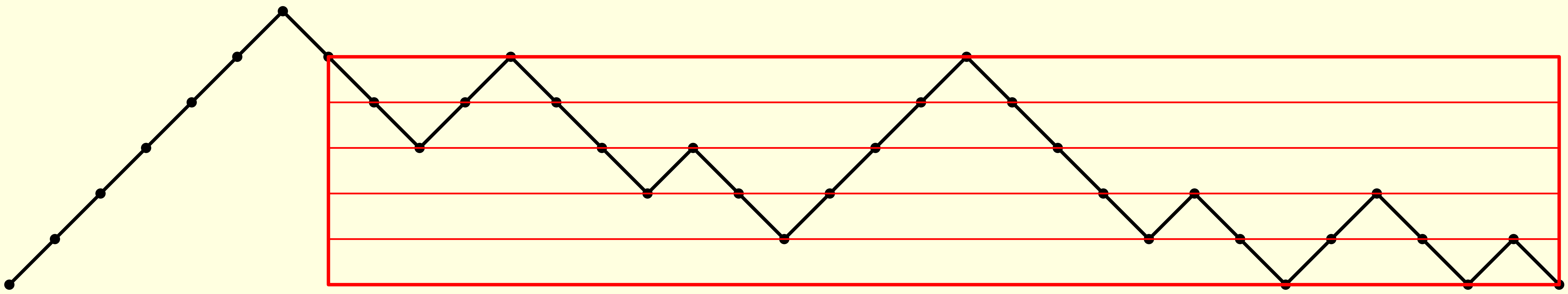

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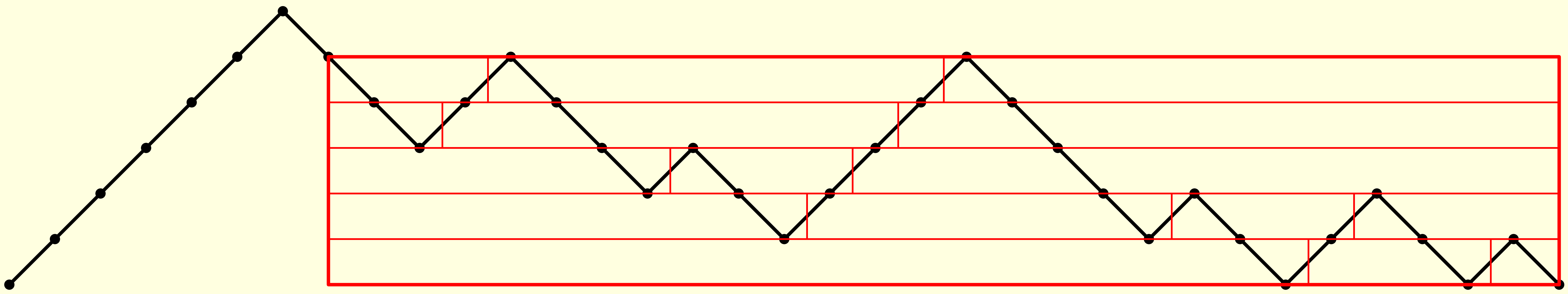

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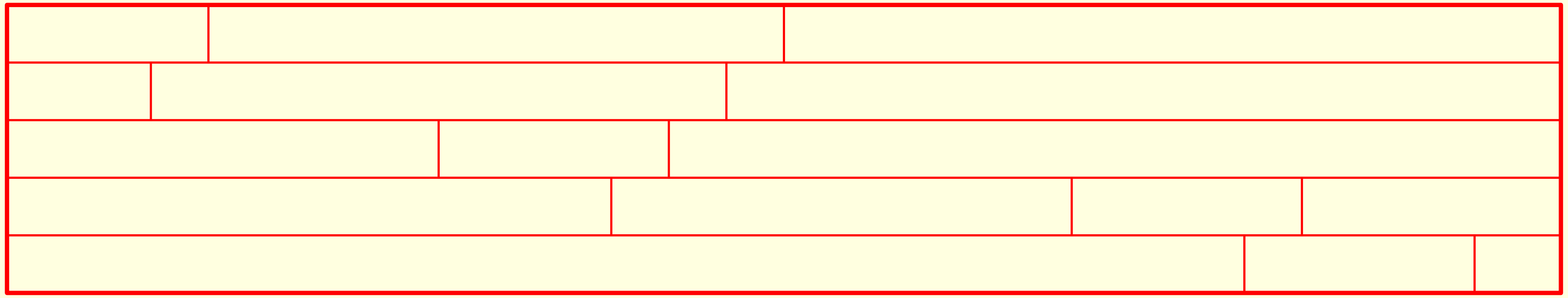

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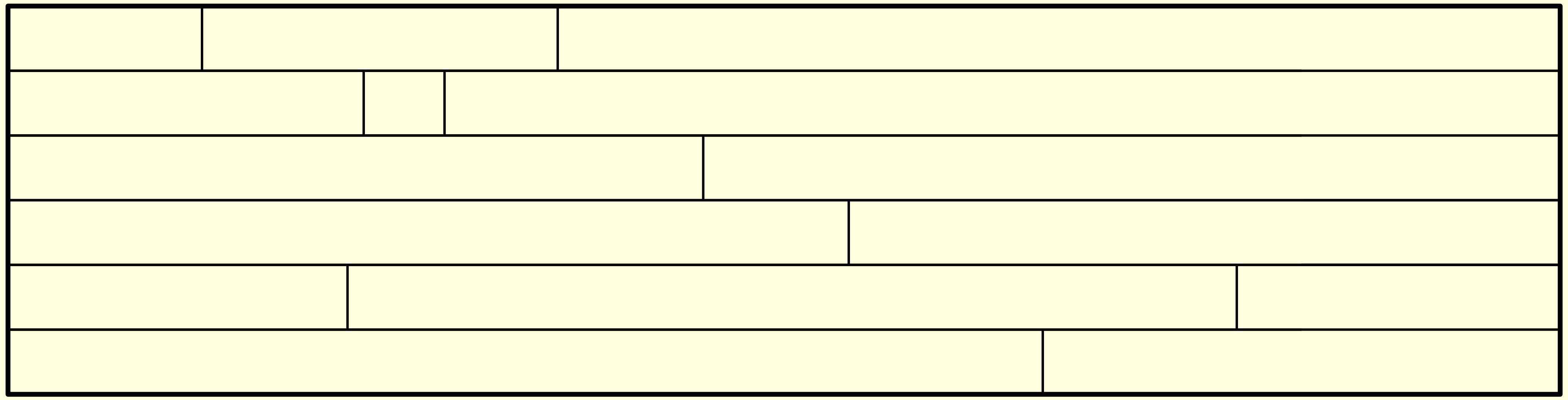

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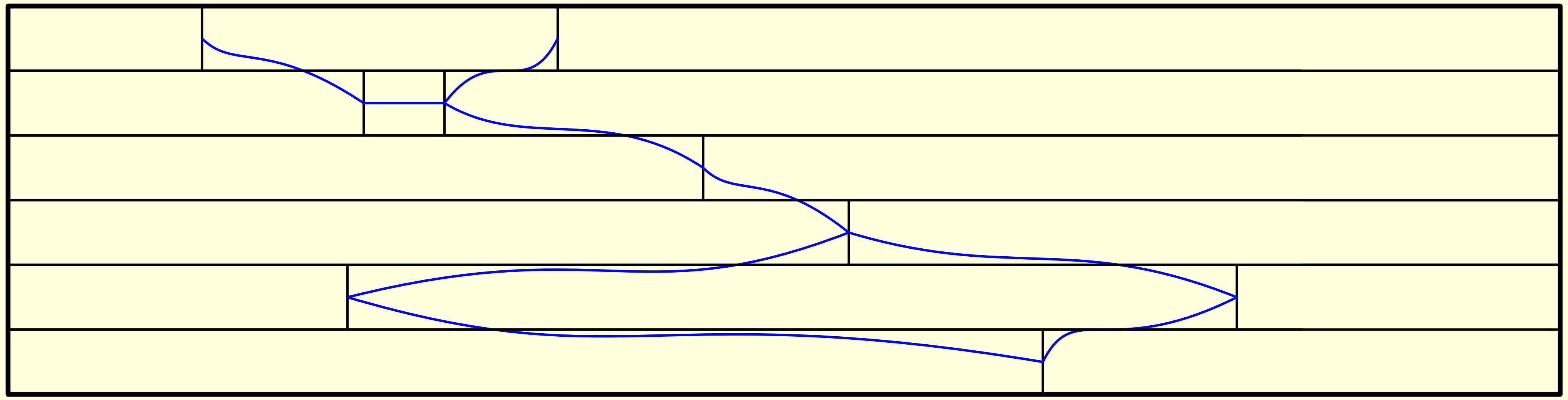

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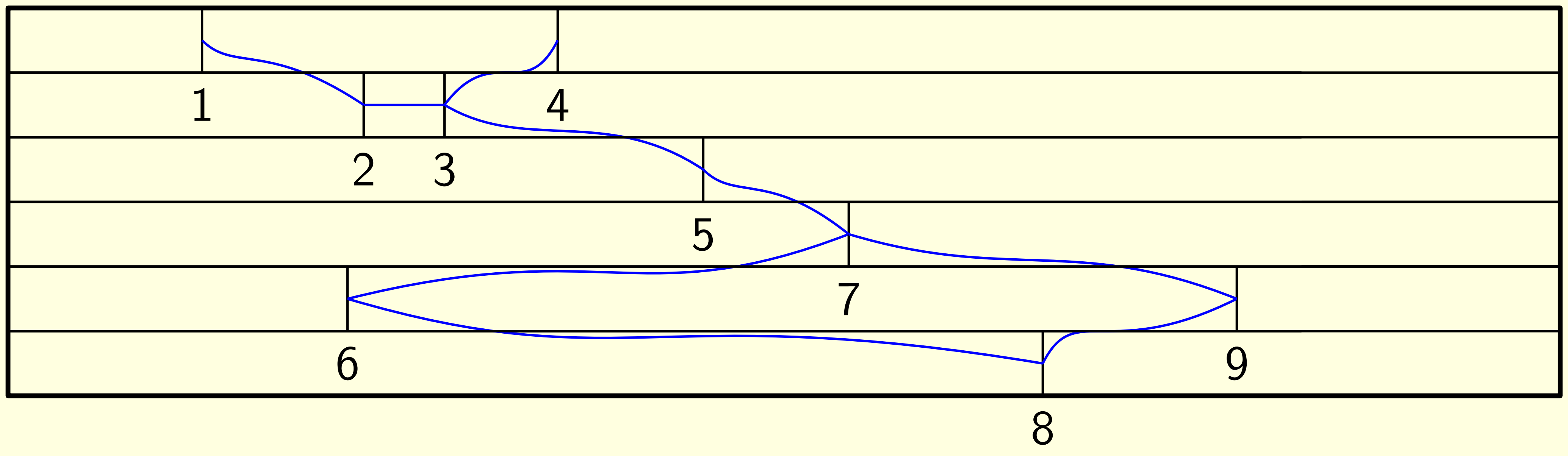

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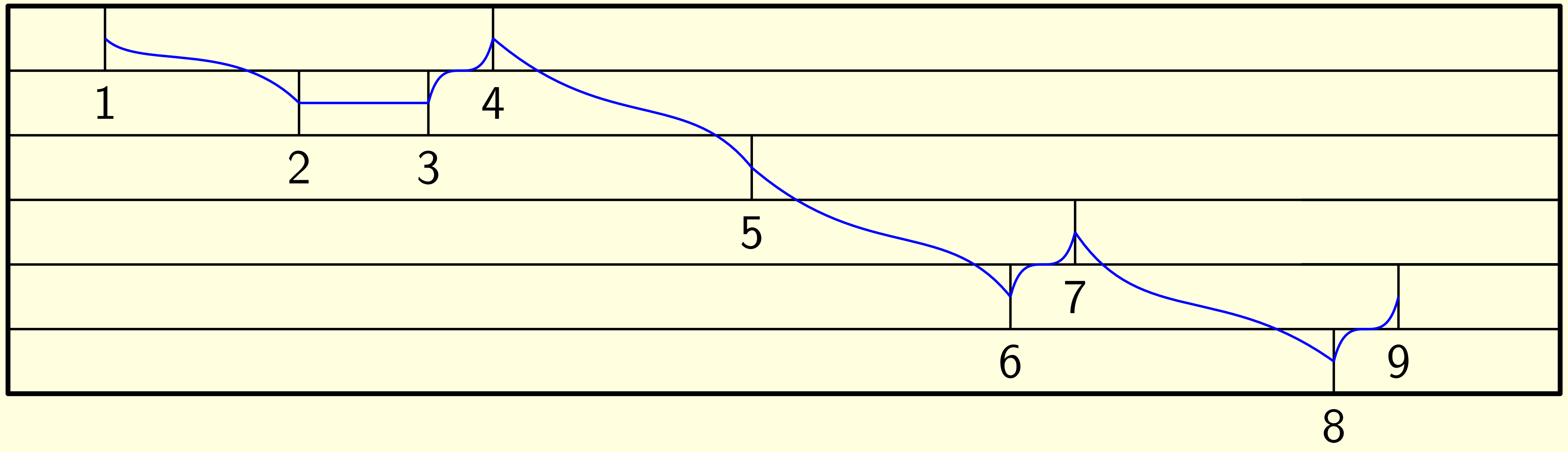

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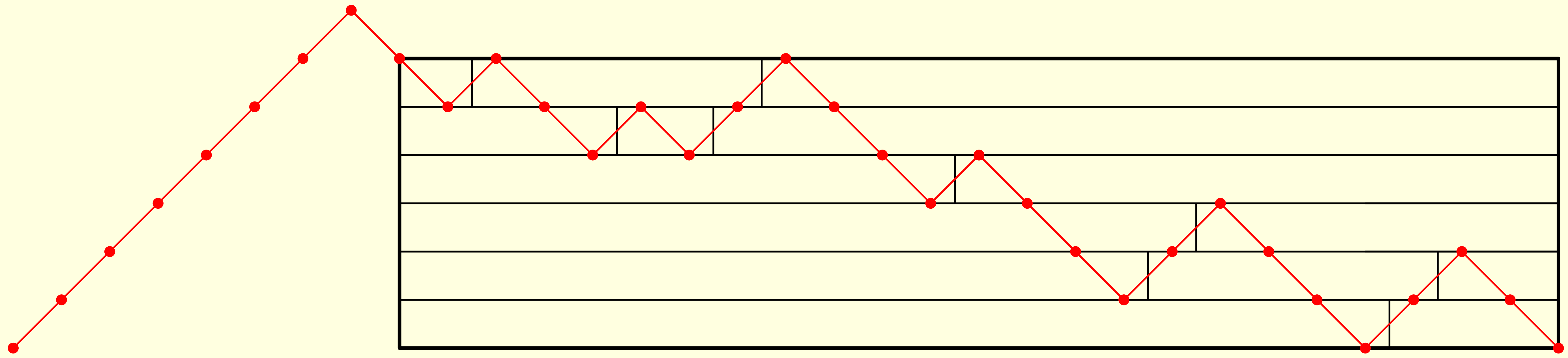

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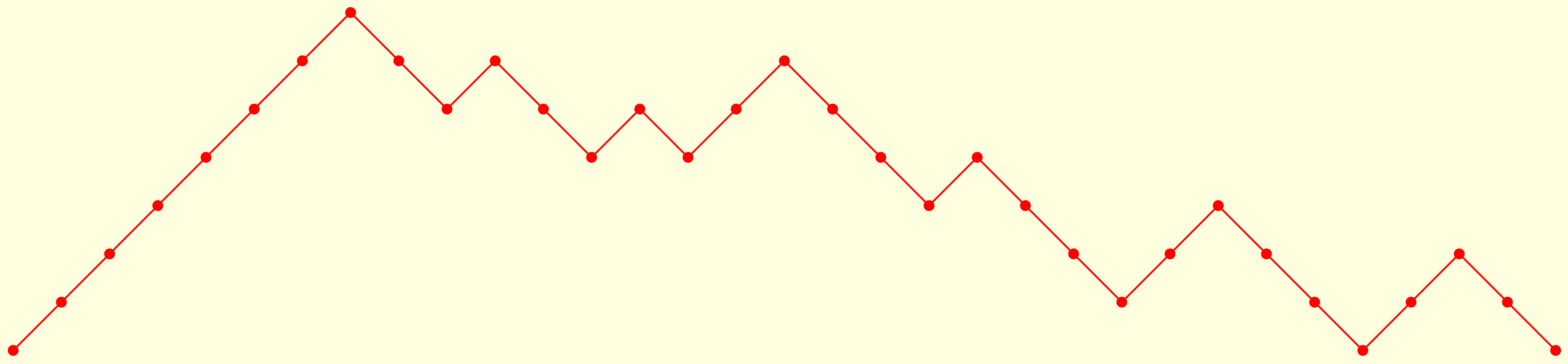

Pattern Avoidance: $R(\top)$

Theorem (Williams)

The number of weak rectangulations of size n that avoid $\top$, denoted $R_n^w(\top)$ is enumerated by the Catalan numbers.

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Definition

A *inversion sequence* of length n is a list of n numbers $s = (s_1, \ldots, s_n)$ such that for each $1 \leq i \leq n$, $0 \leq s_i < i$.

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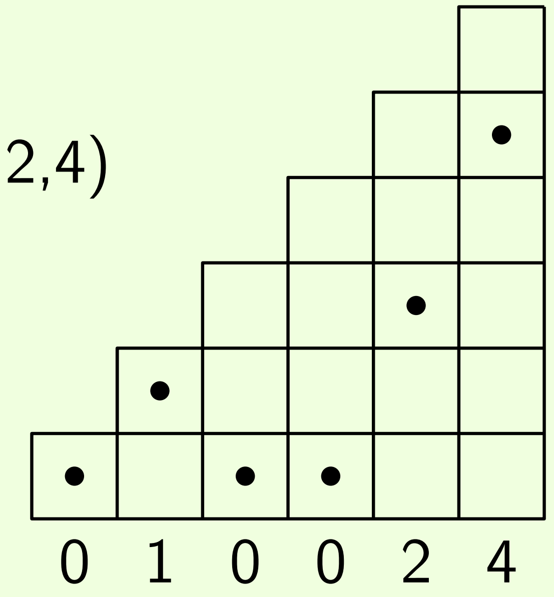
A *inversion sequence* of length n is a list of n numbers $s = (s_1, \ldots, s_n)$ such that for each $1 \leq i \leq n$, $0 \leq s_i < i$.

(0,1,0,0,2,4)



Pattern Avoidance: $R(\top)$

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We say s contains a pattern t if there is a subsequence of s which is order isomophic to t.

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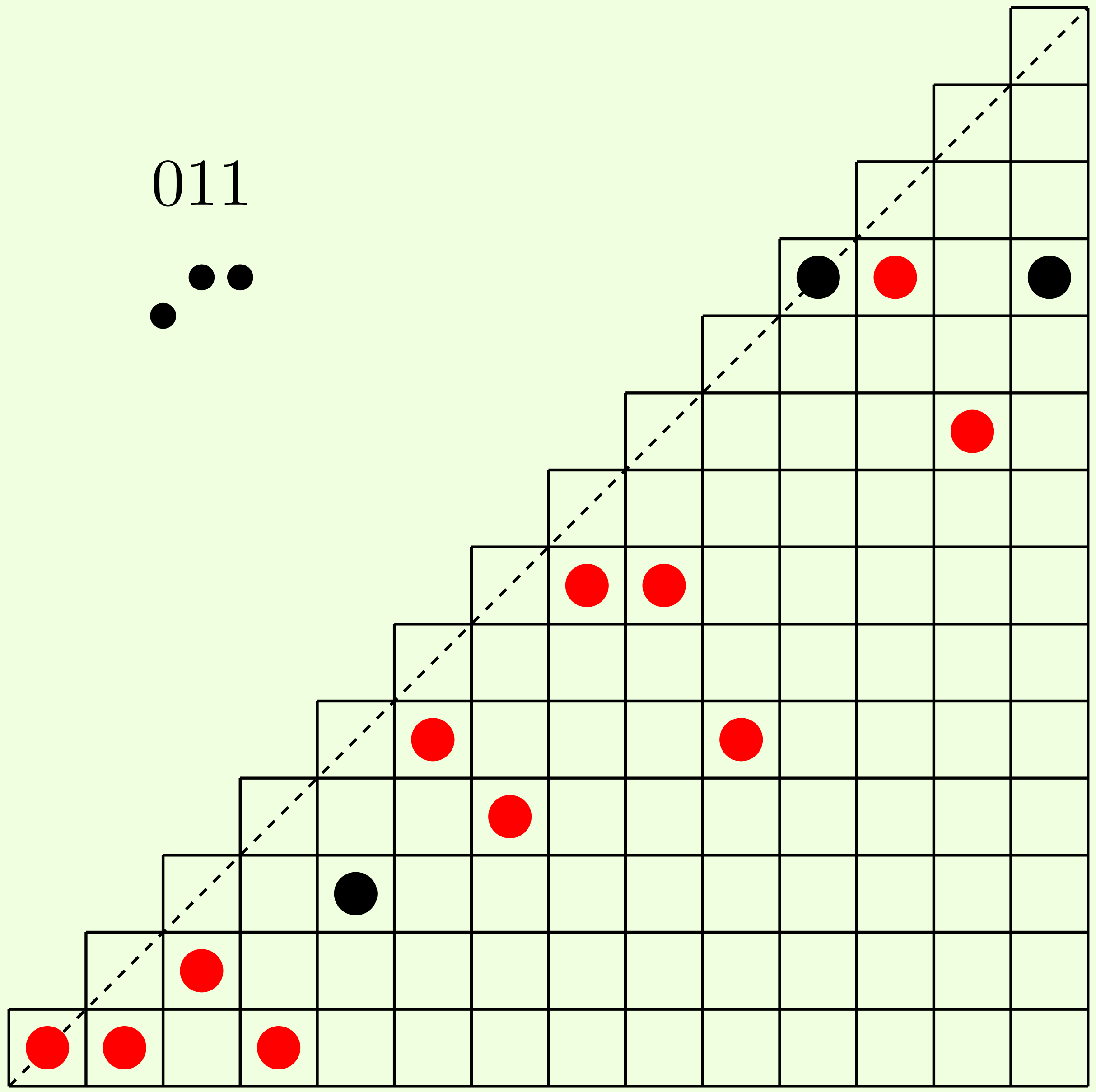

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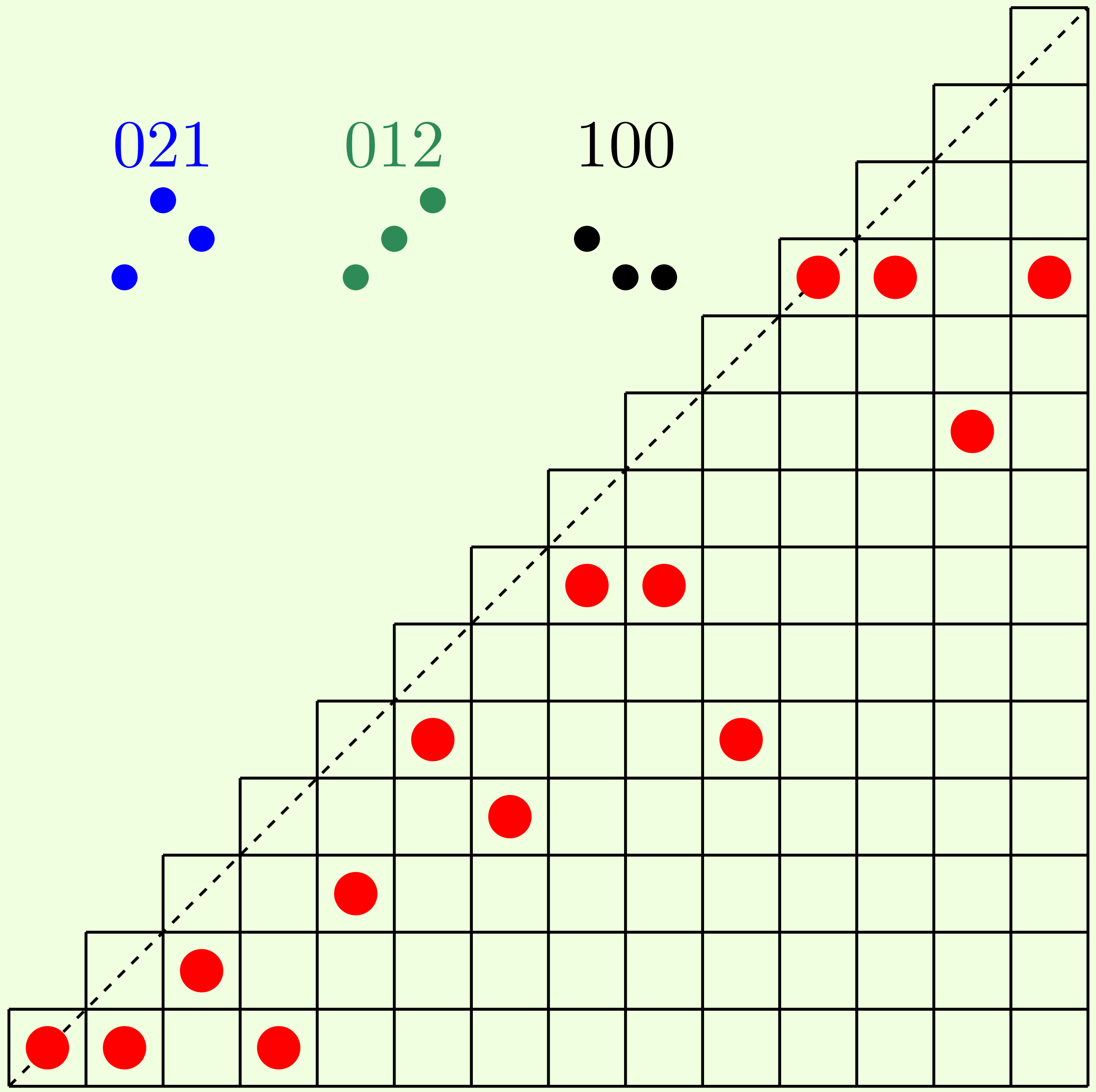

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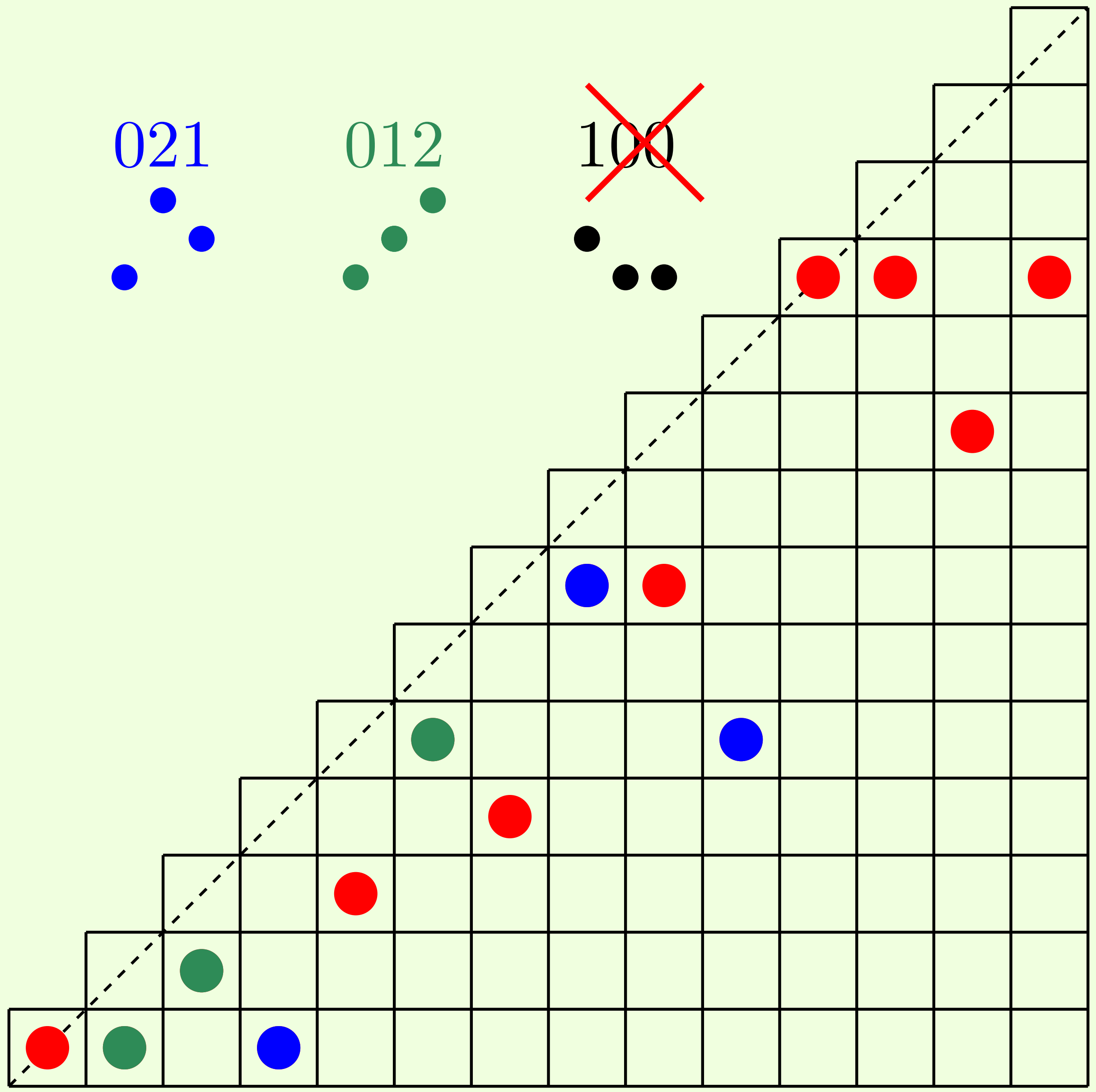

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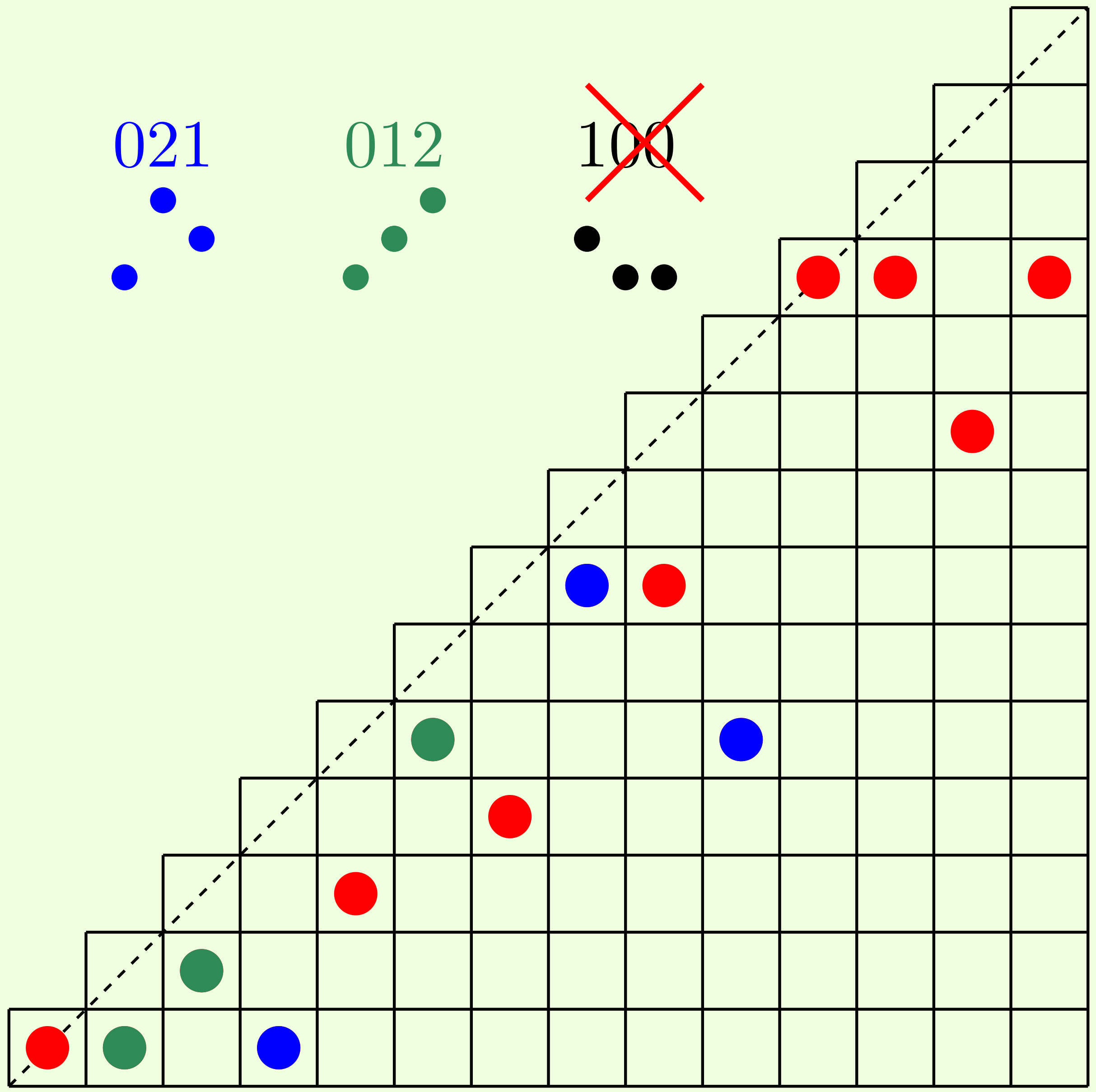

If s does not contain t, then we say that s avoids t. Denote by $I_n(L)$ the set of inversion sequences of length n which avoid all of the patterns in L.

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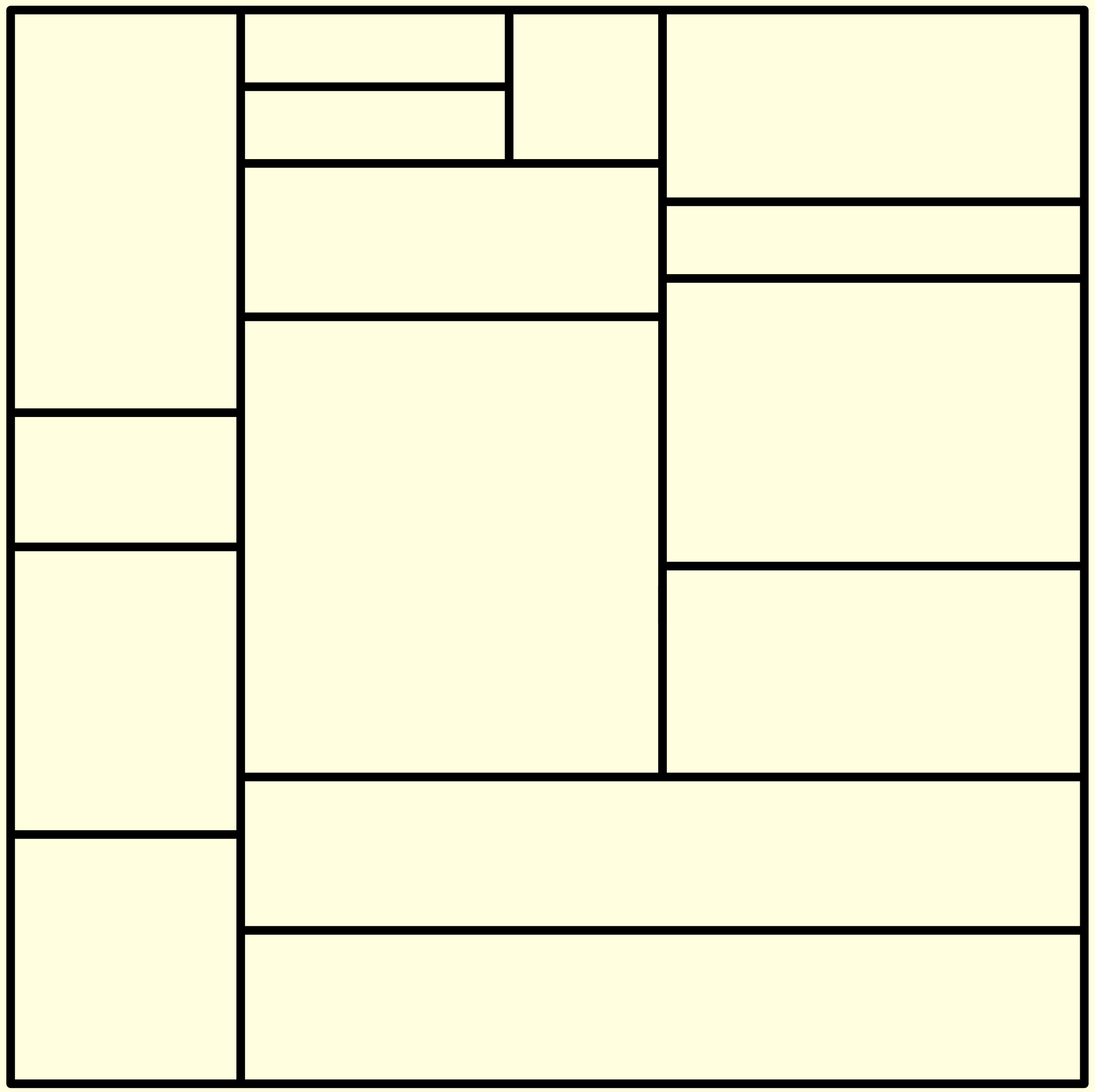
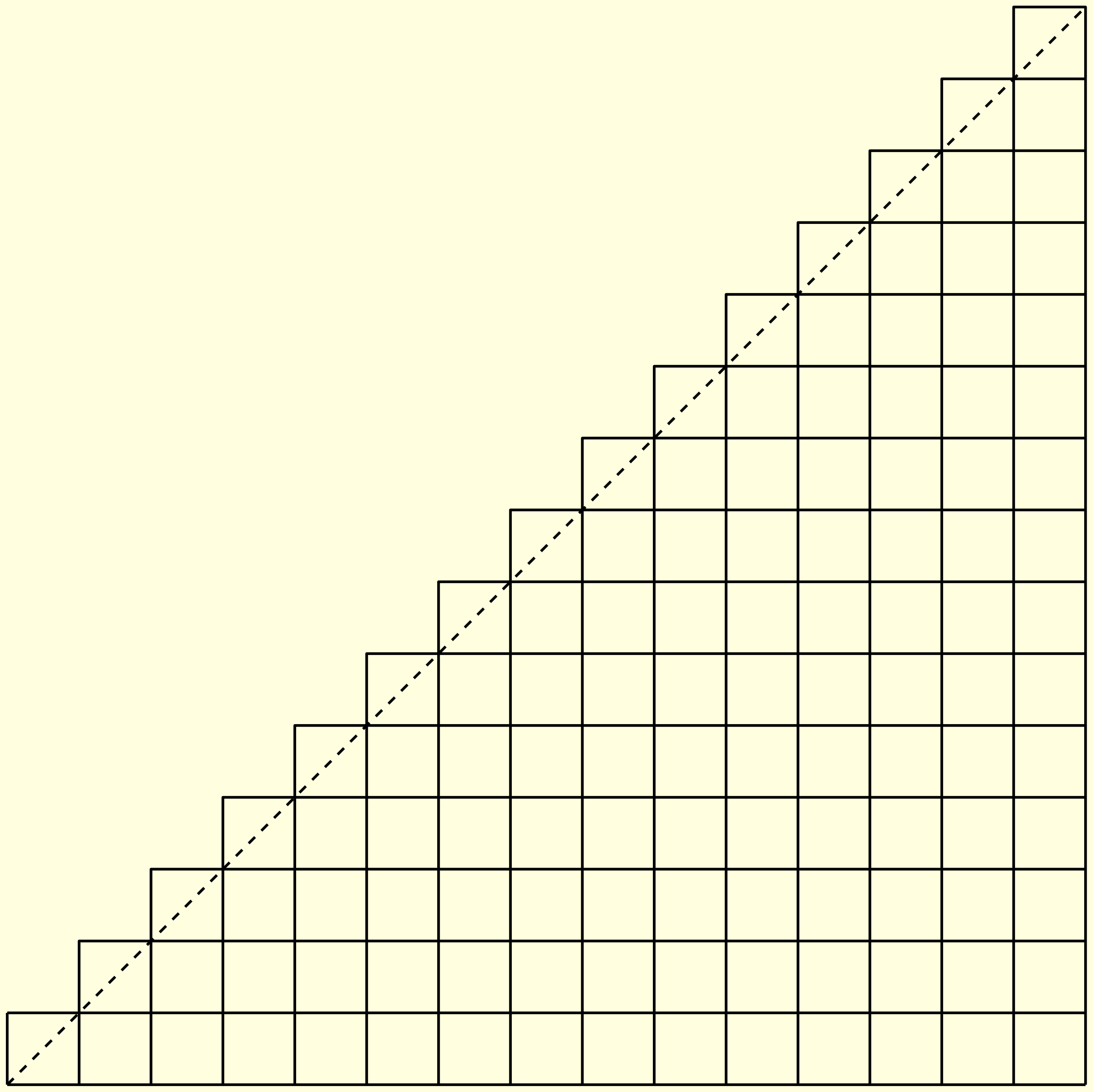
Idea: Construct a bijection to Dyck paths via inversion sequences





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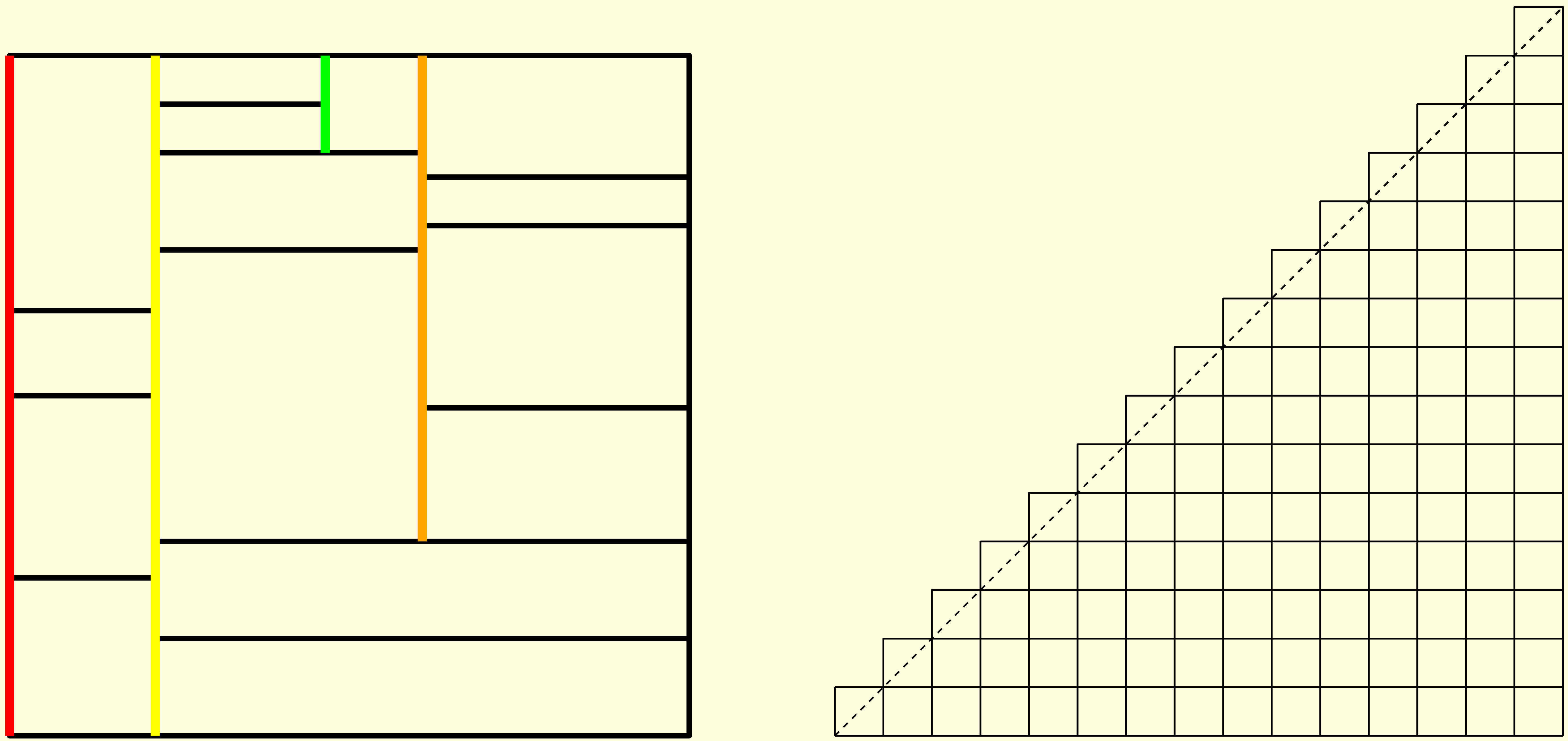

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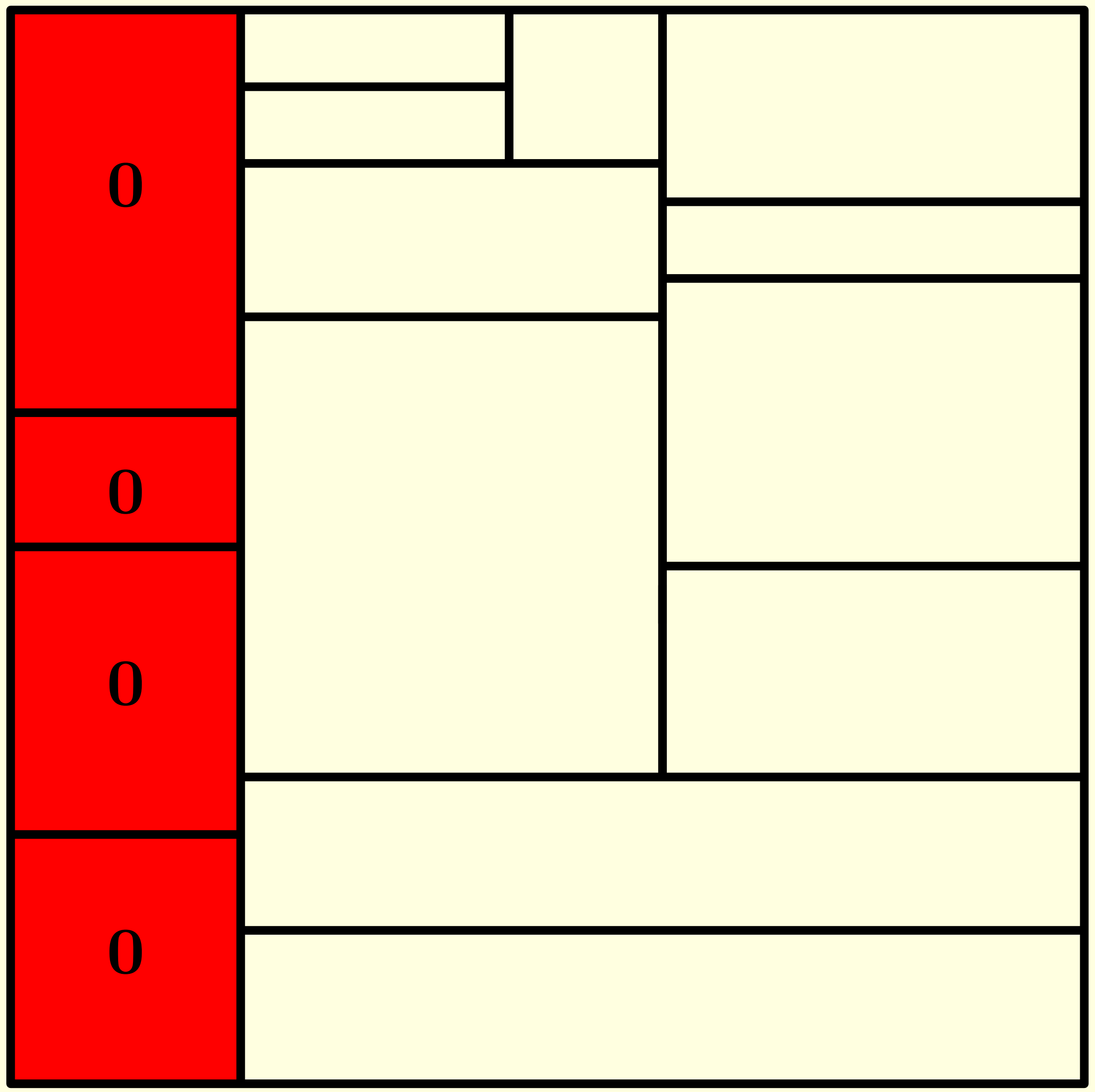

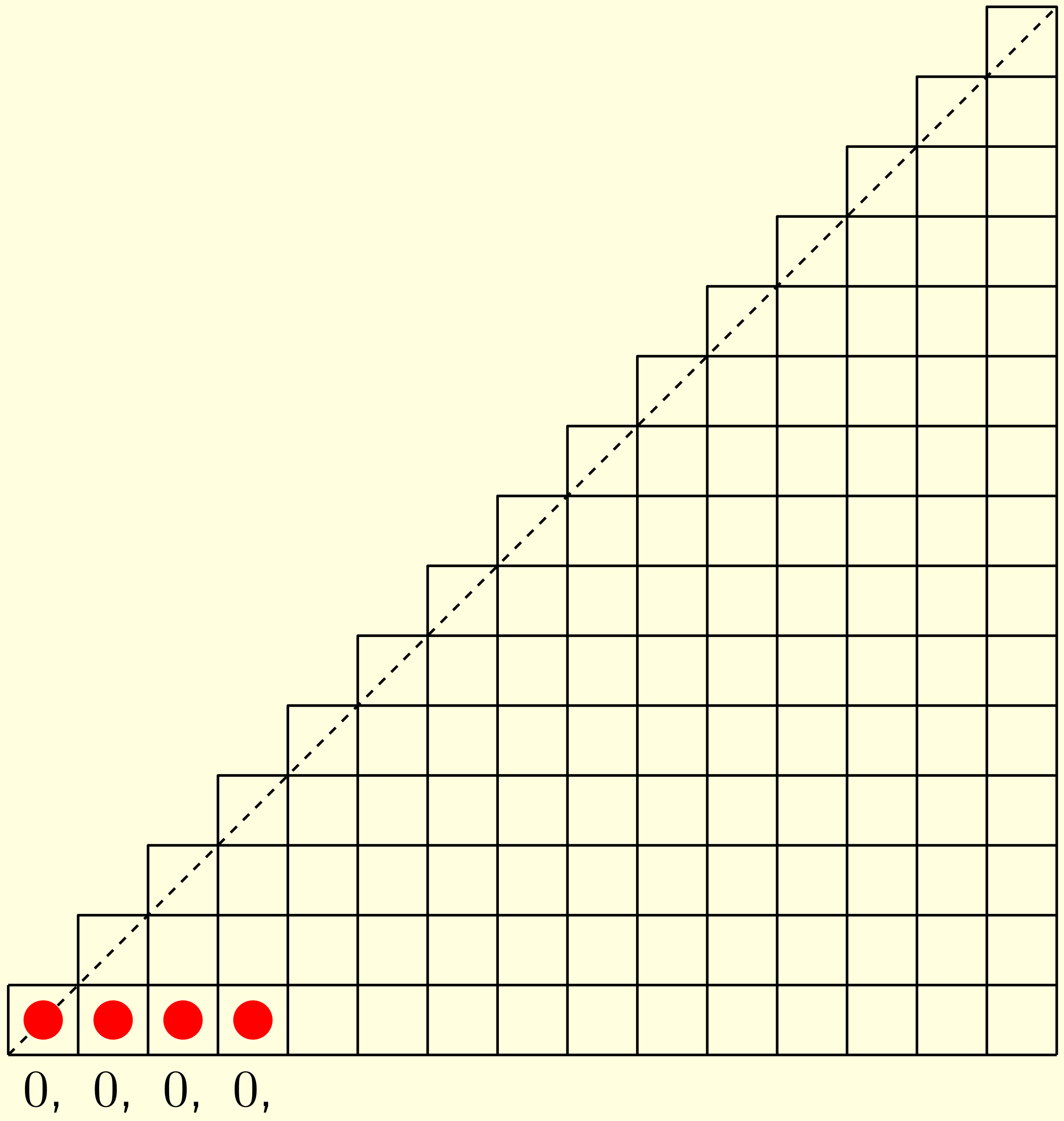

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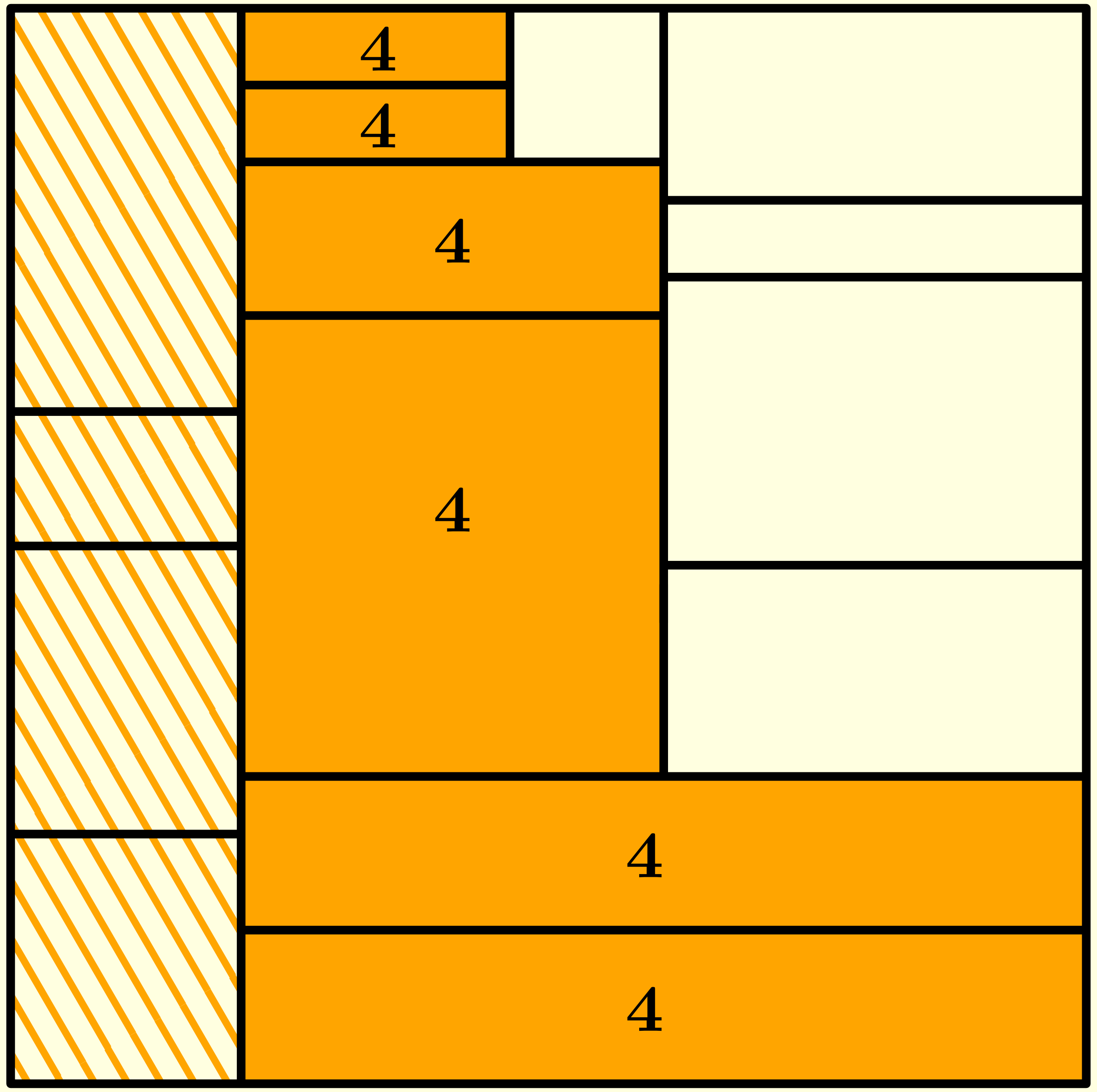

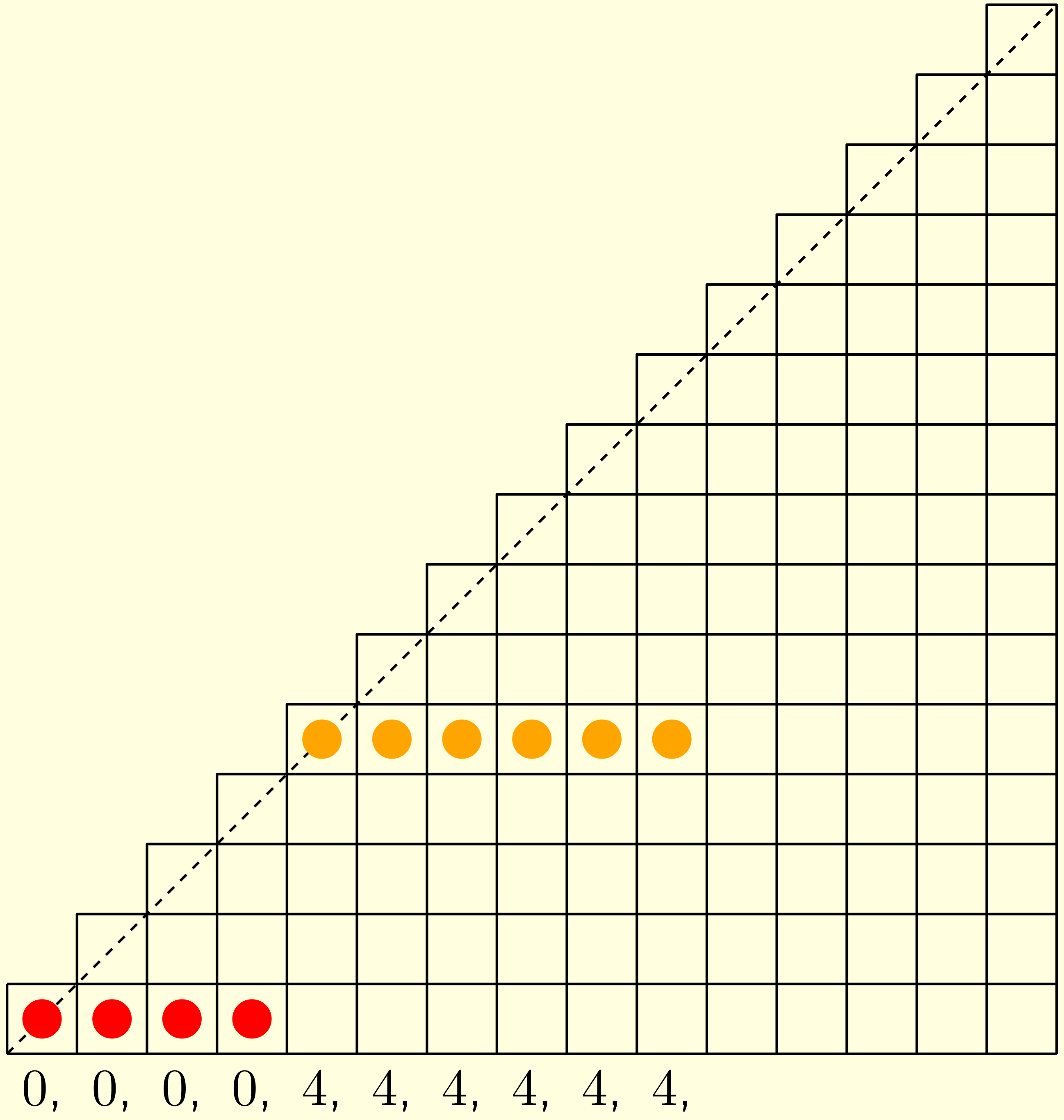

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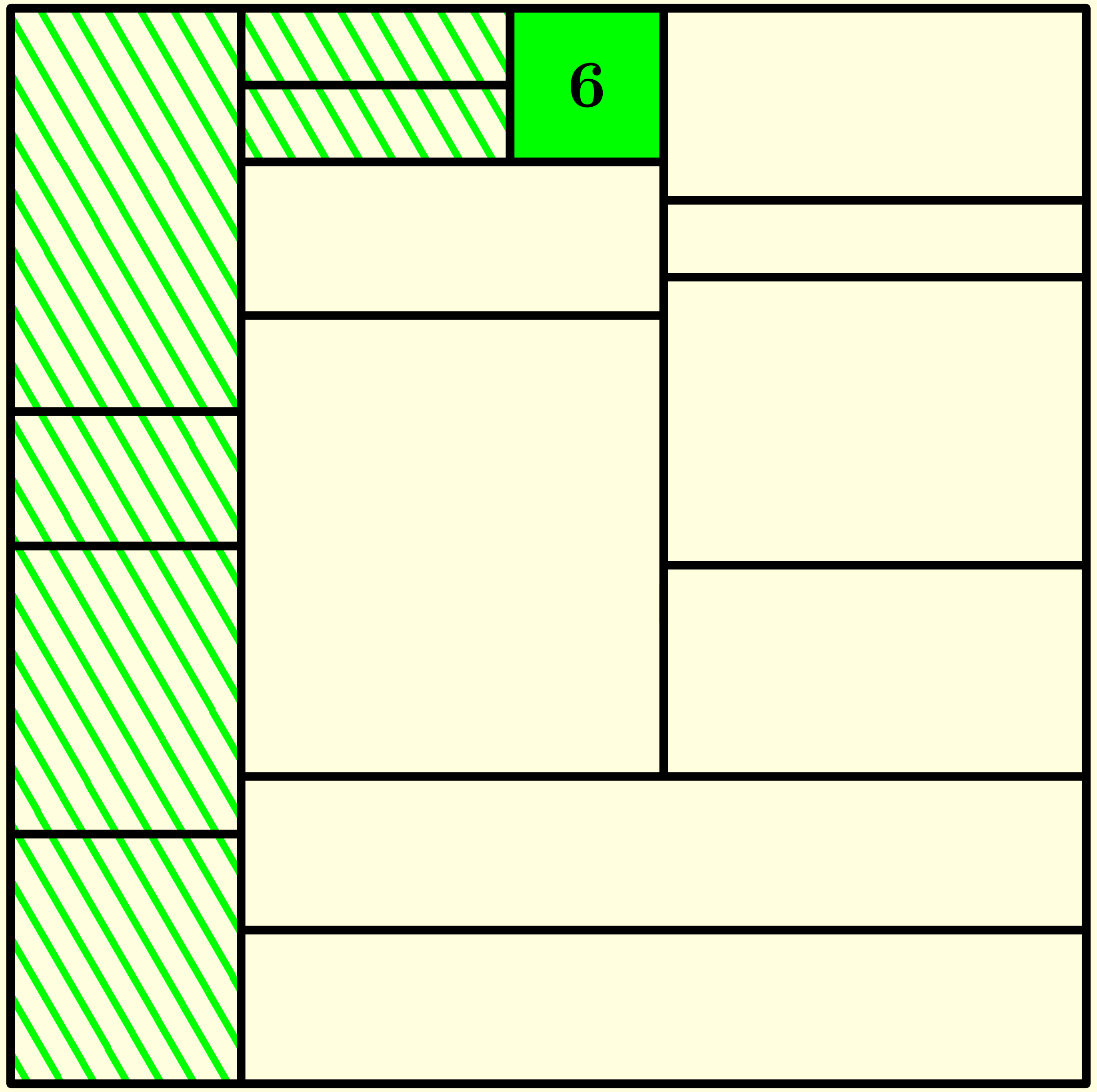

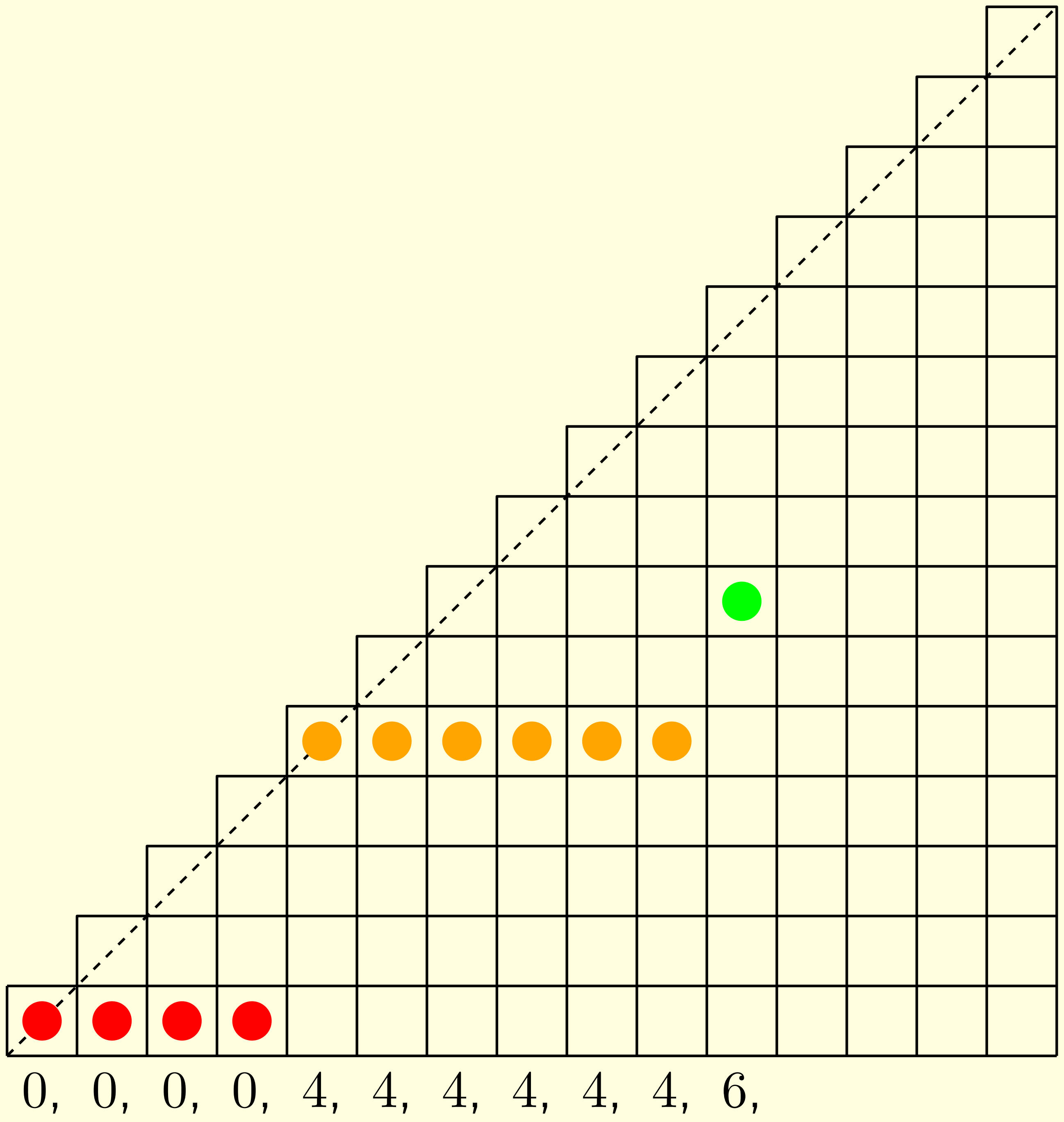

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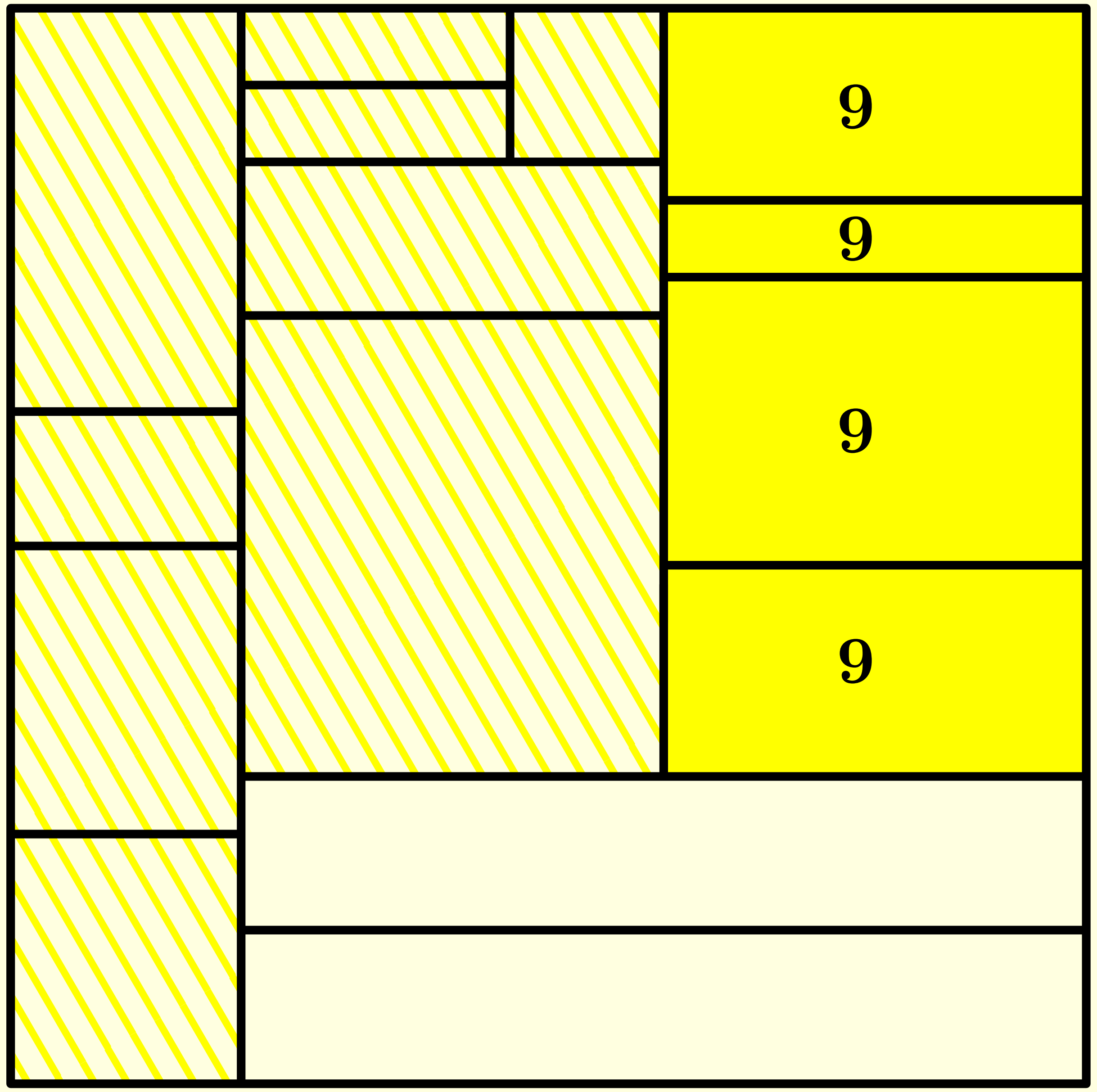

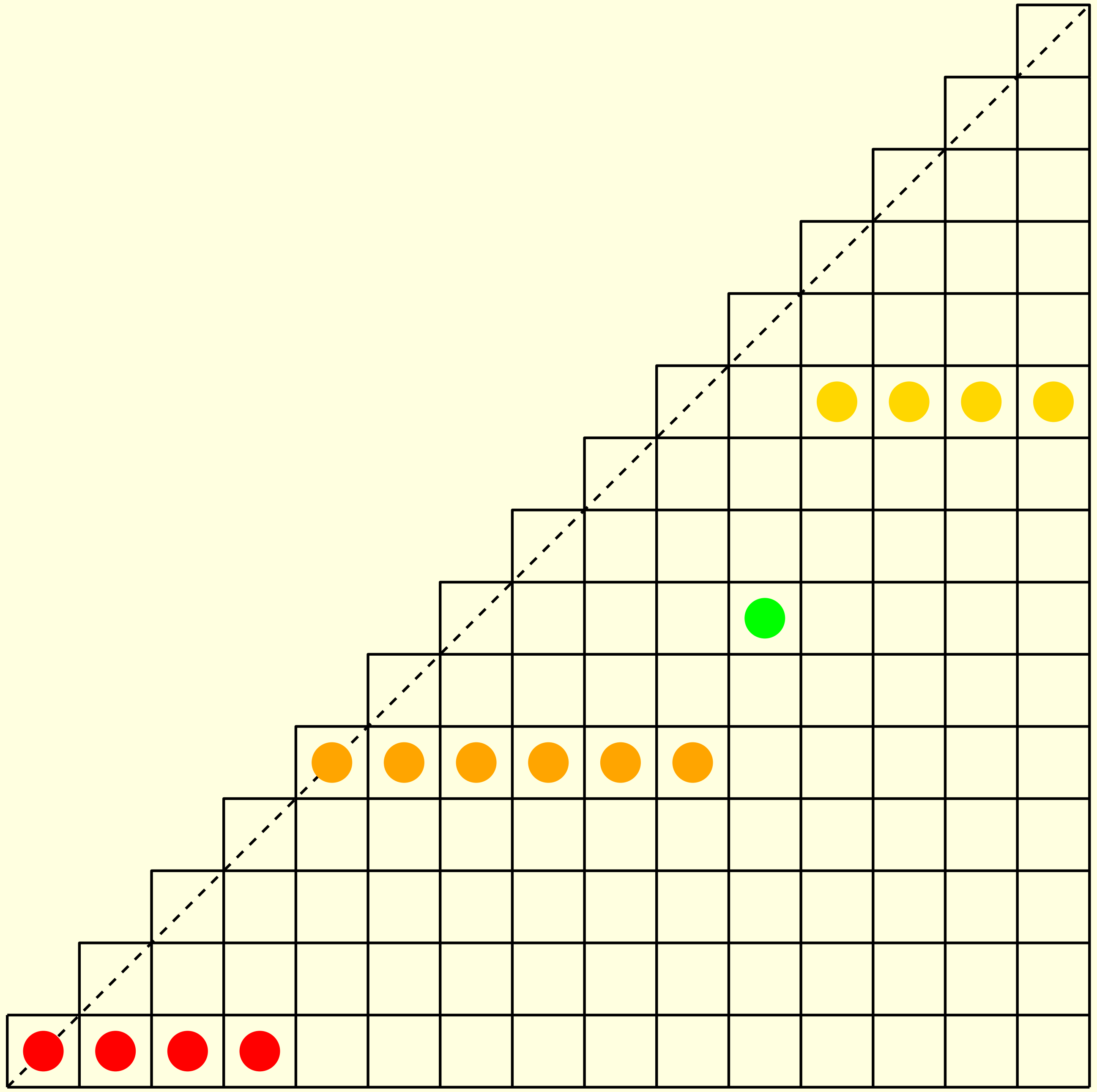

$0,\ 0,\ 0,\ 0,\ 4,\ 4,\ 4,\ 4,\ 4,\ 4,\ 6,\ 9,\ 9,\ 9,\ 9$

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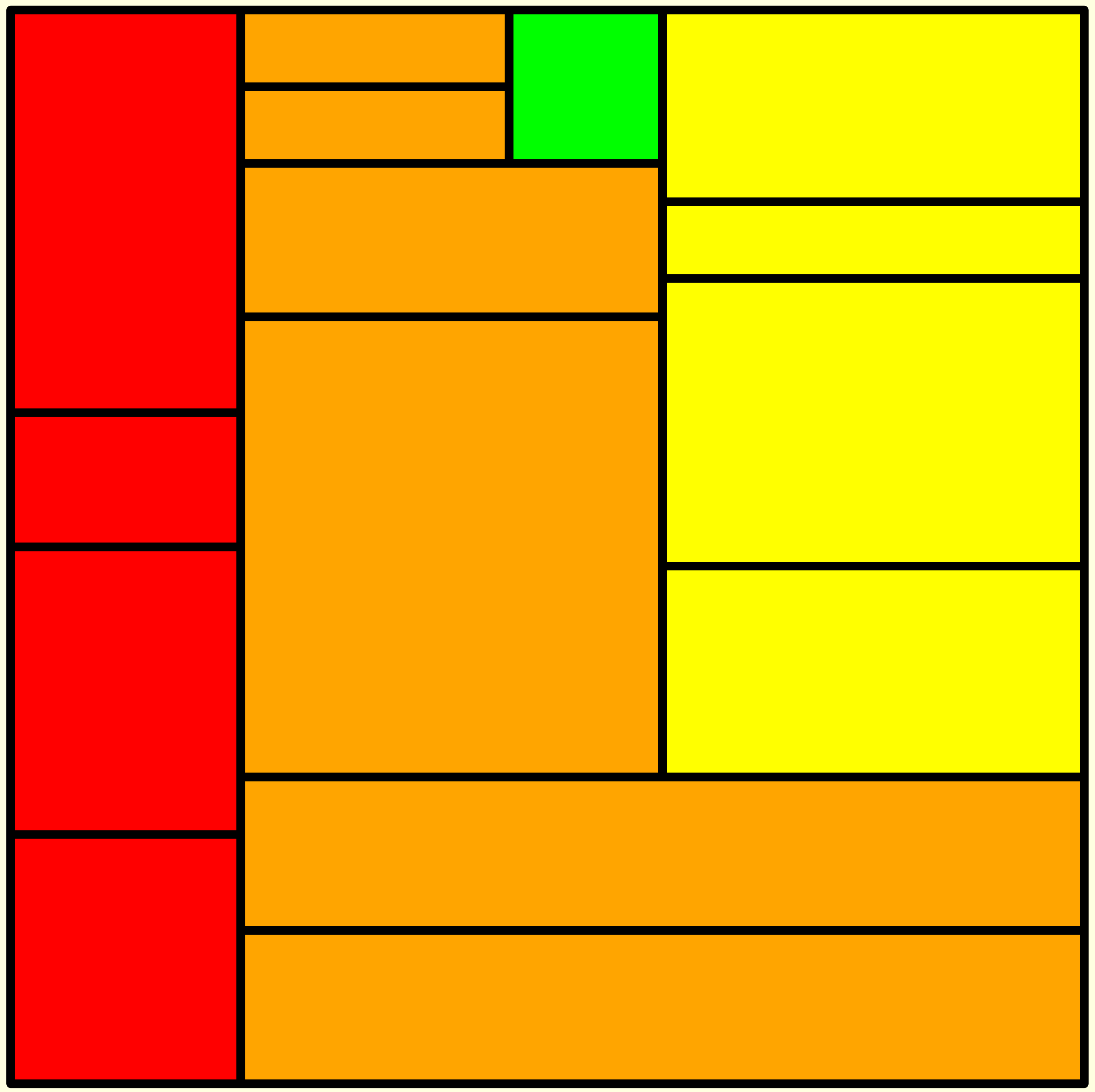

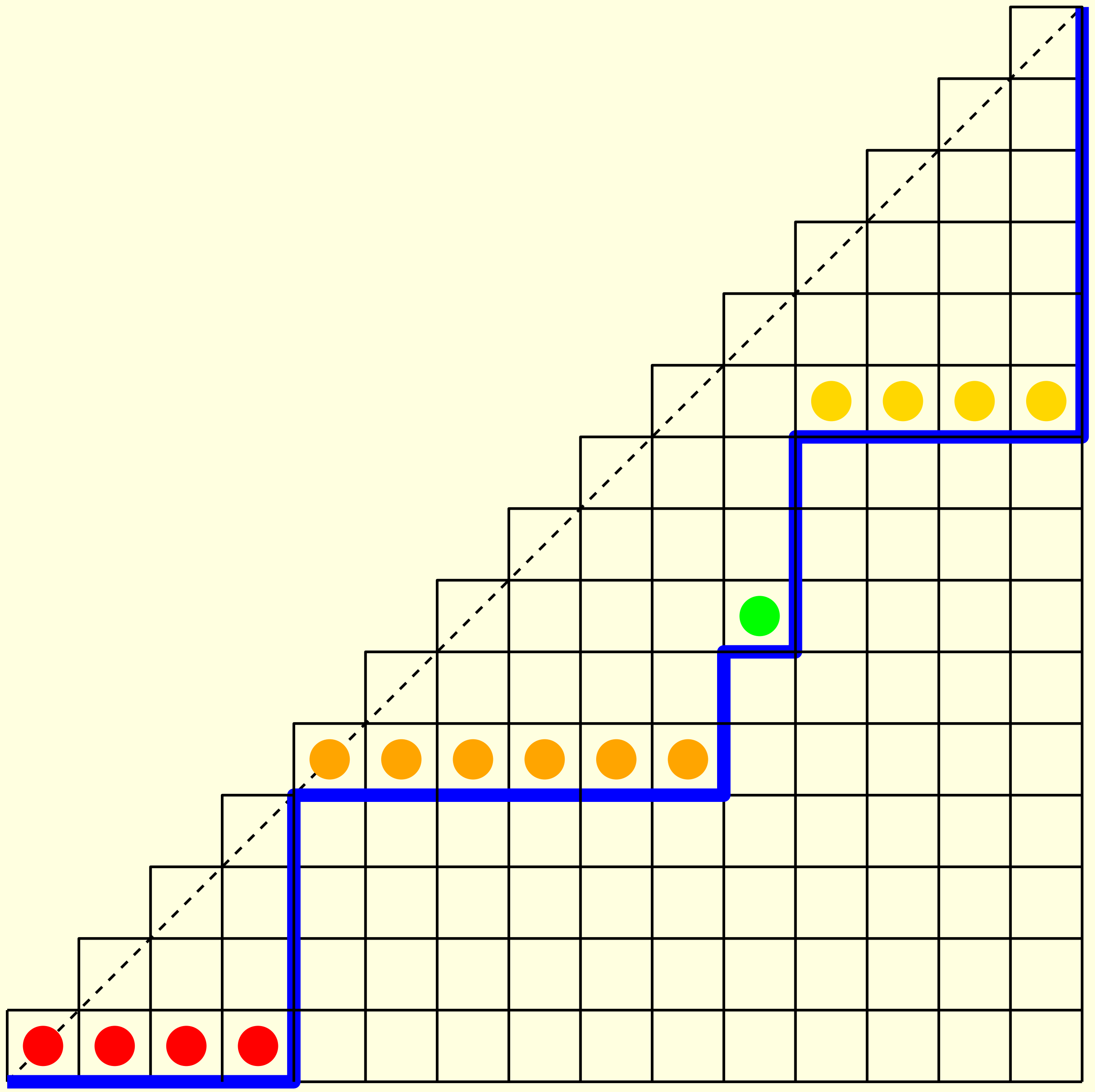

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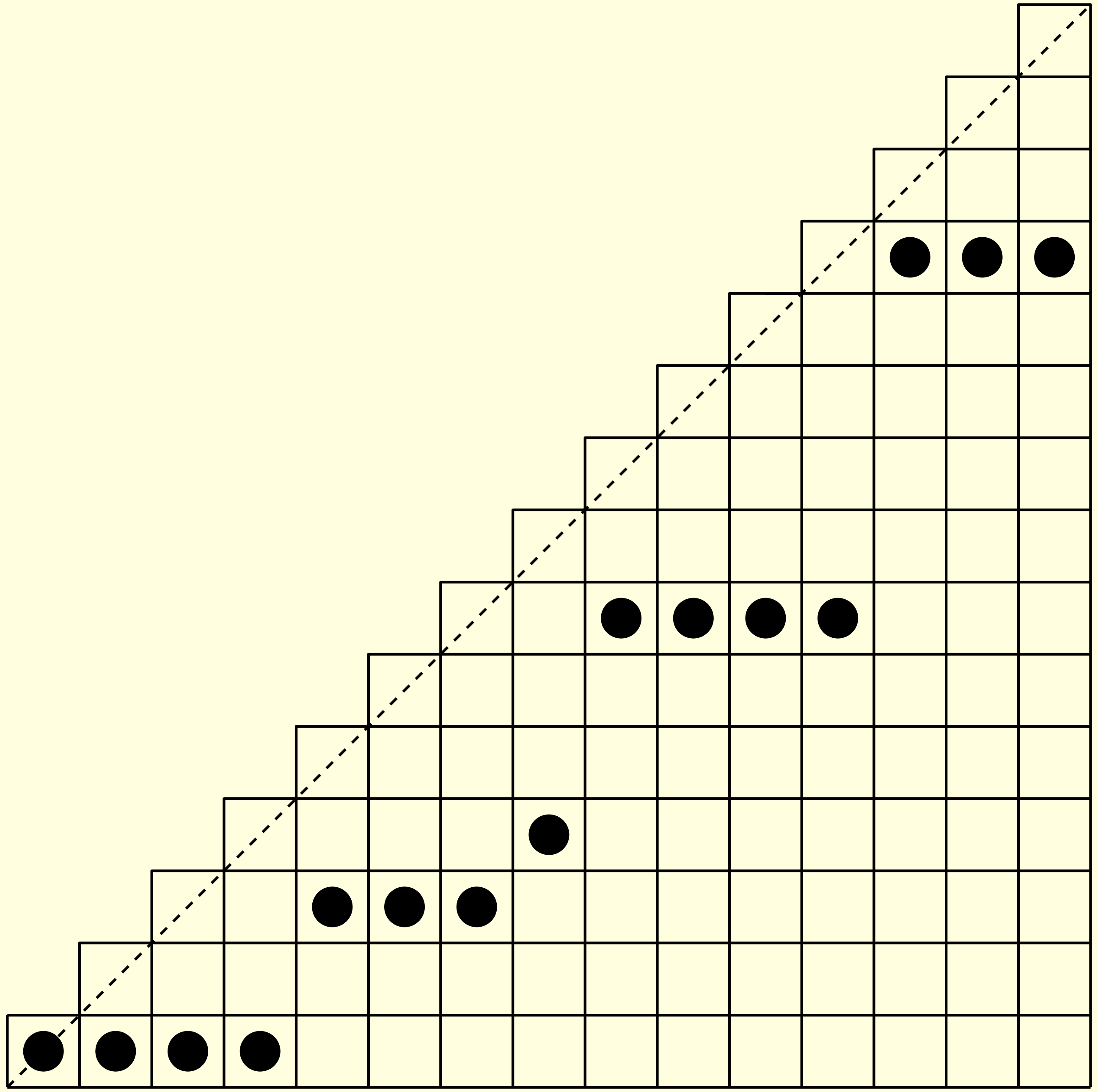

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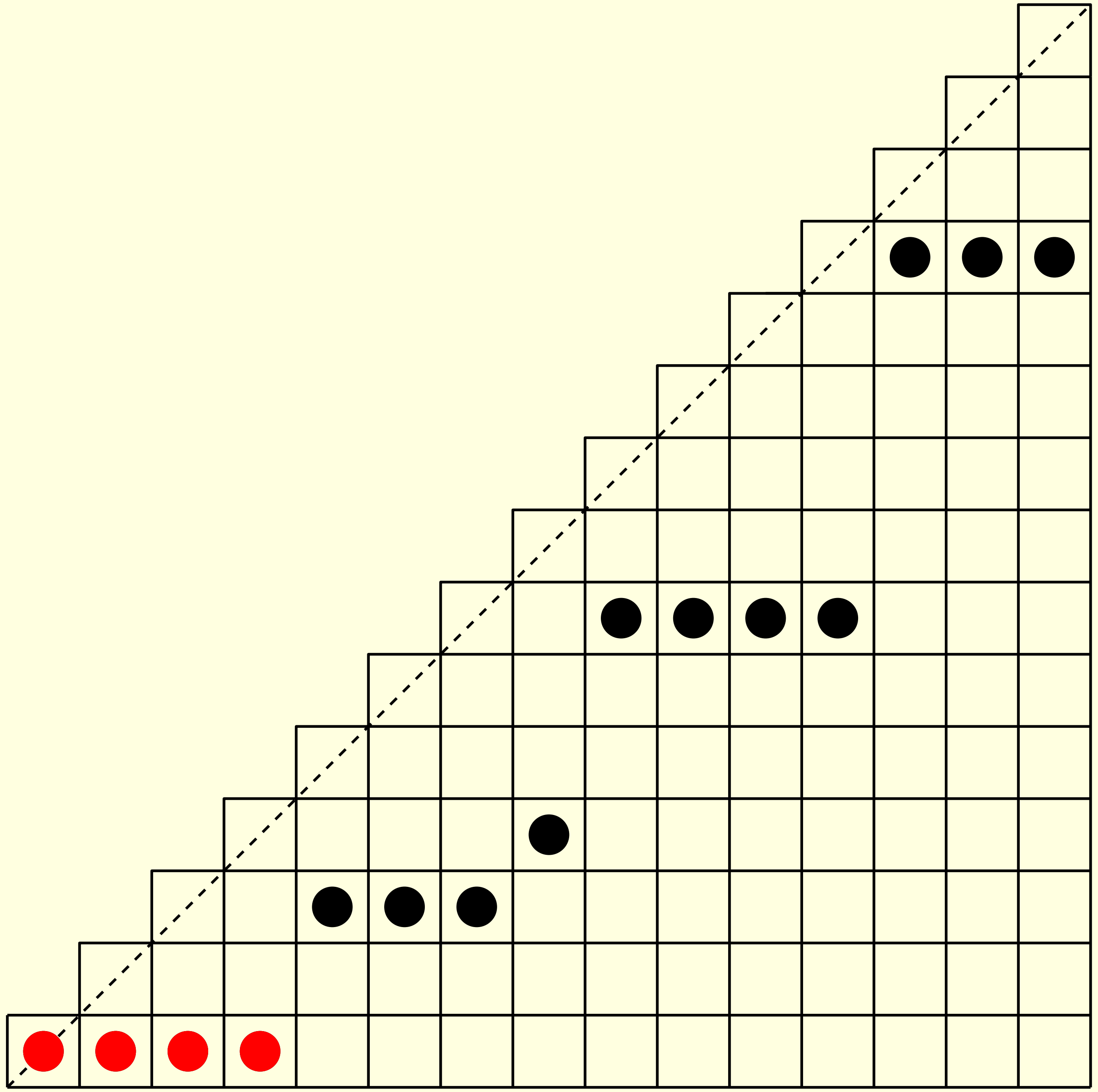

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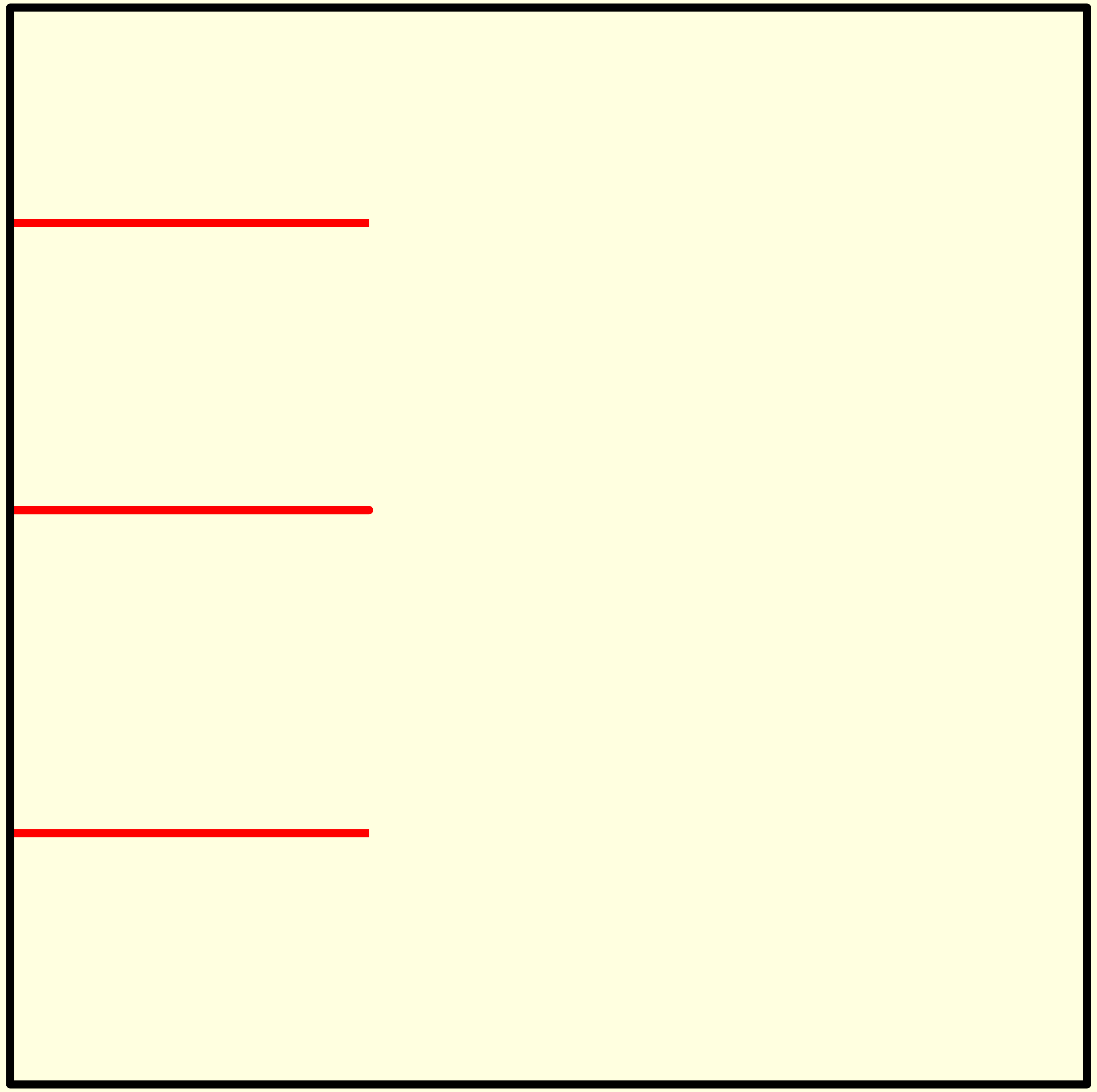

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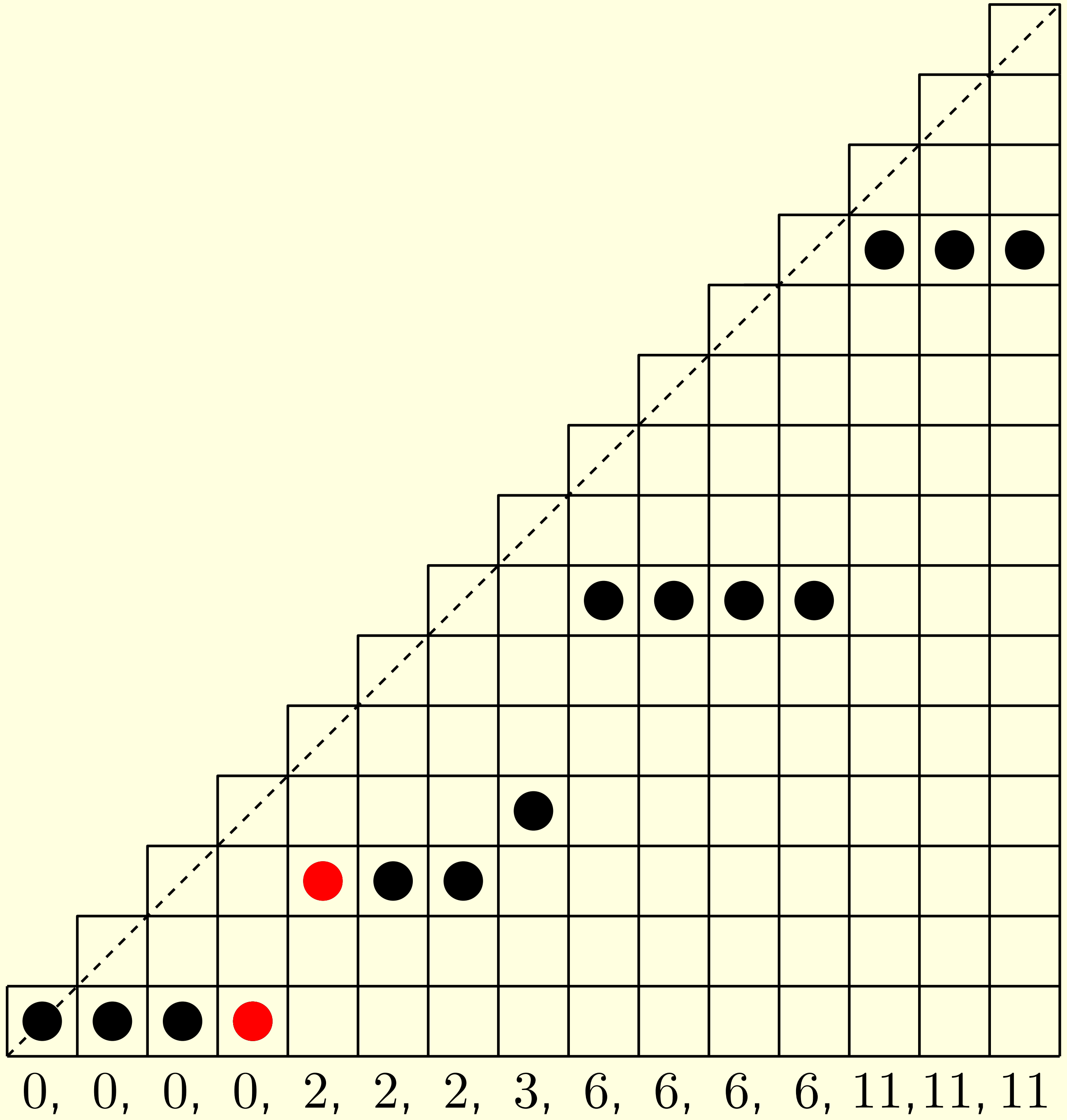

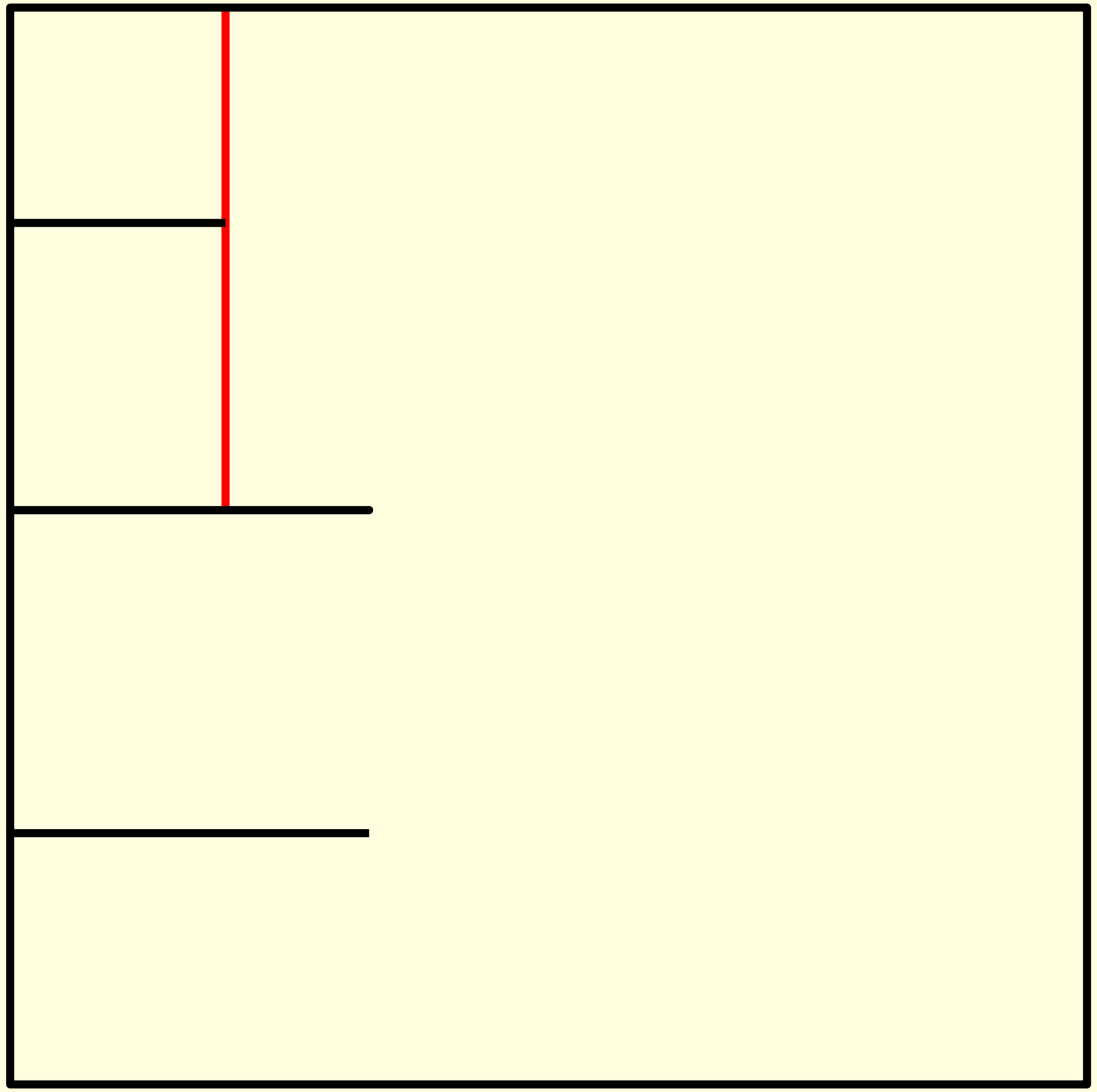

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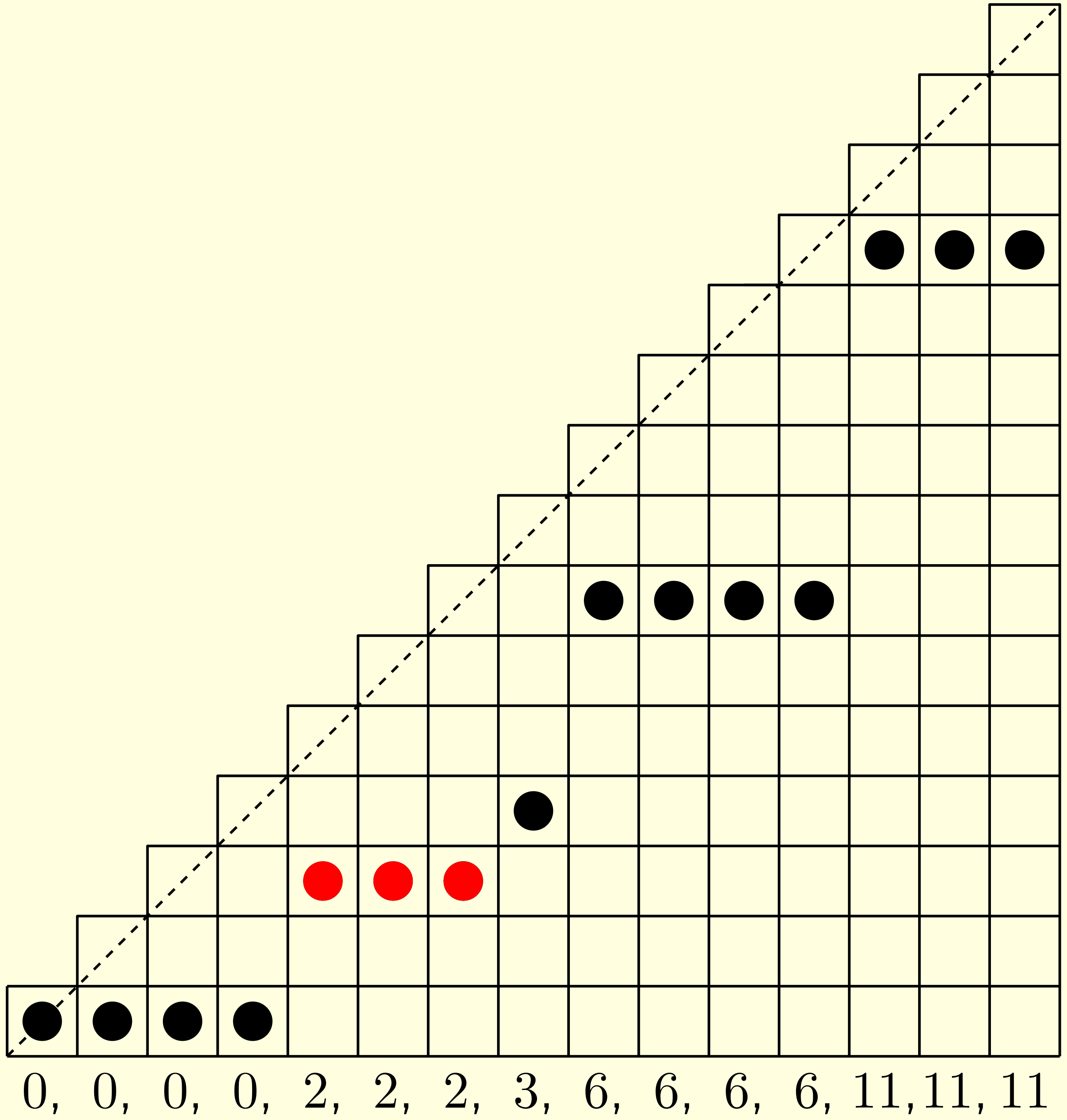

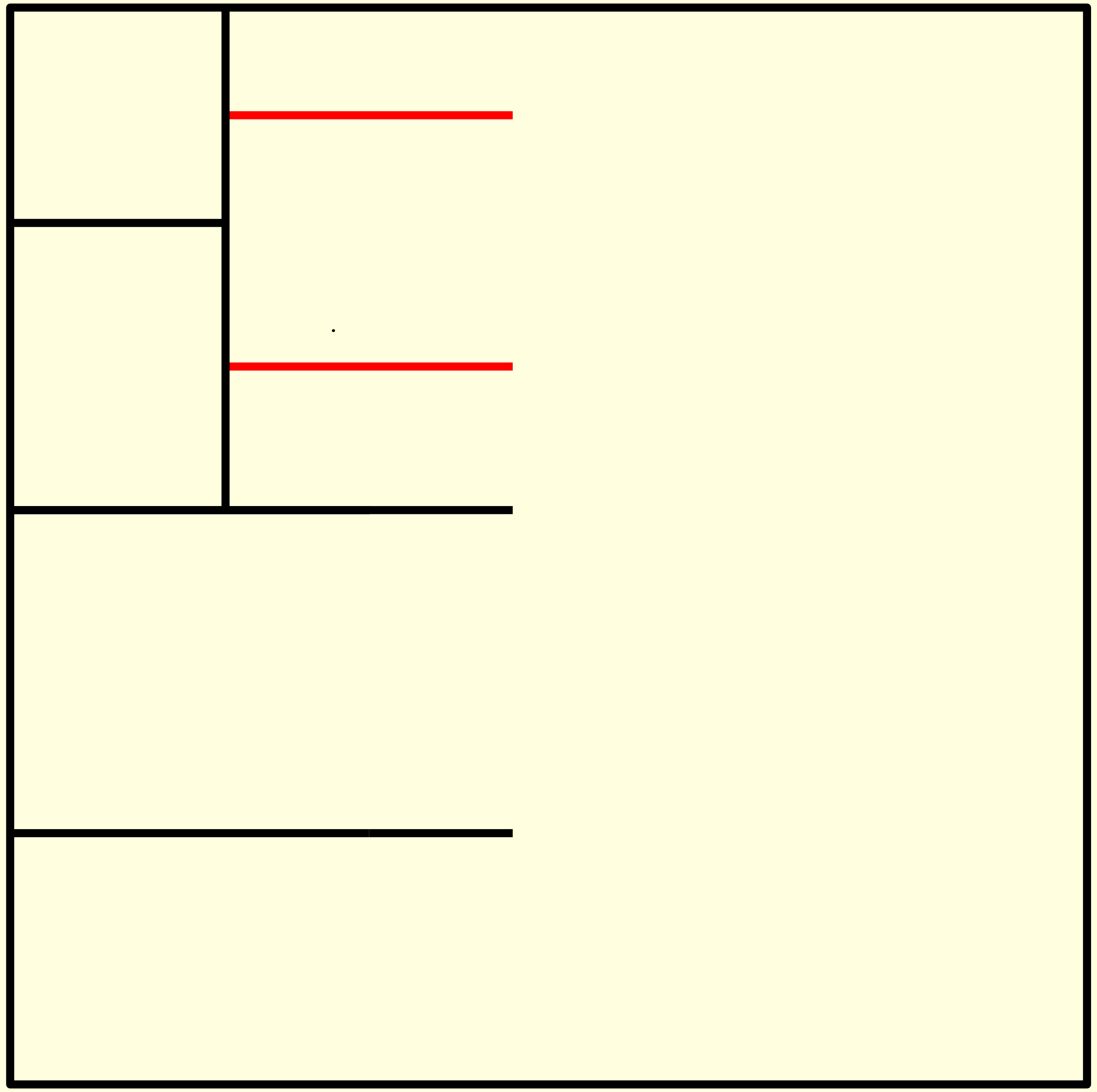

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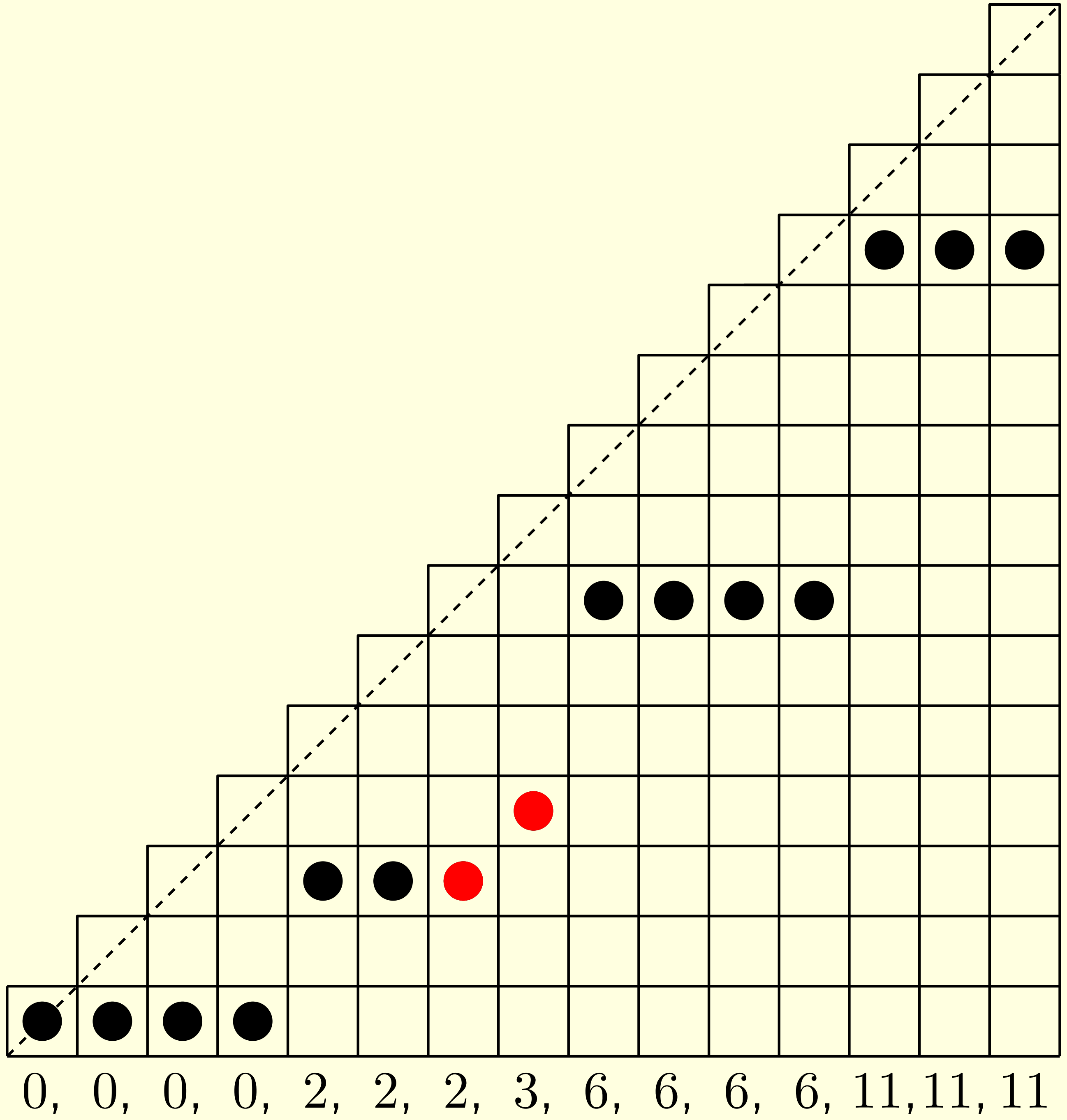

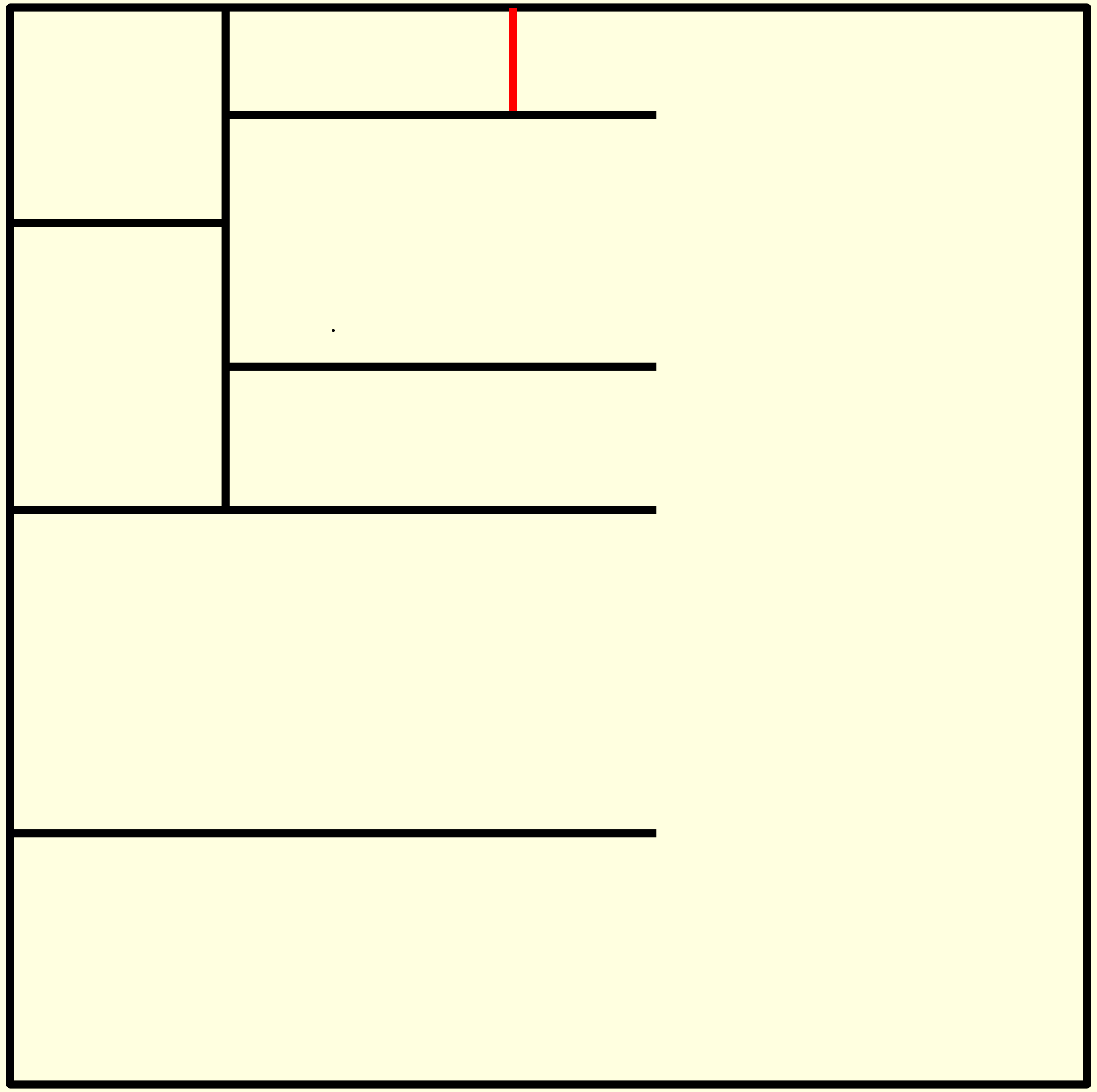

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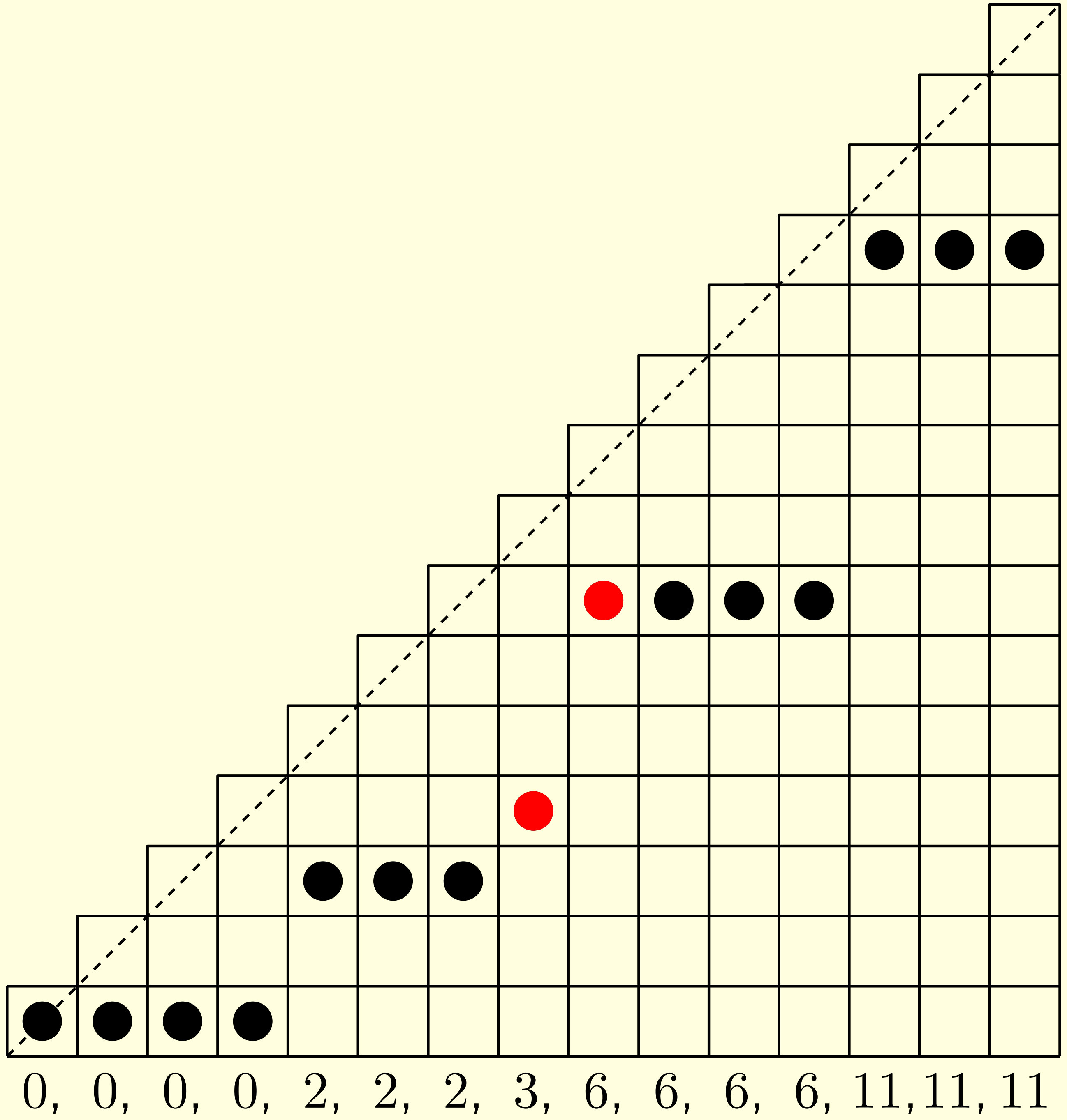

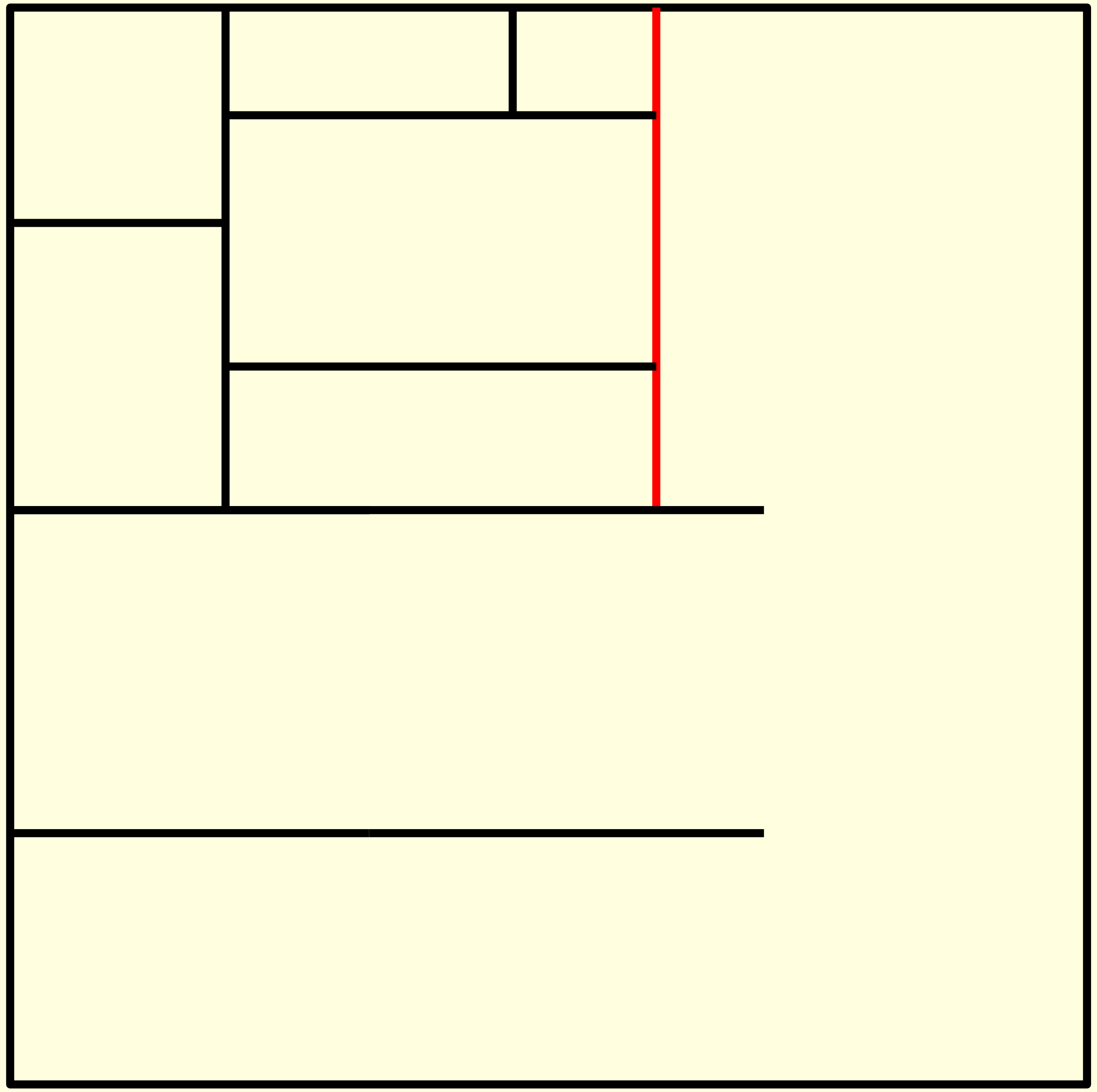

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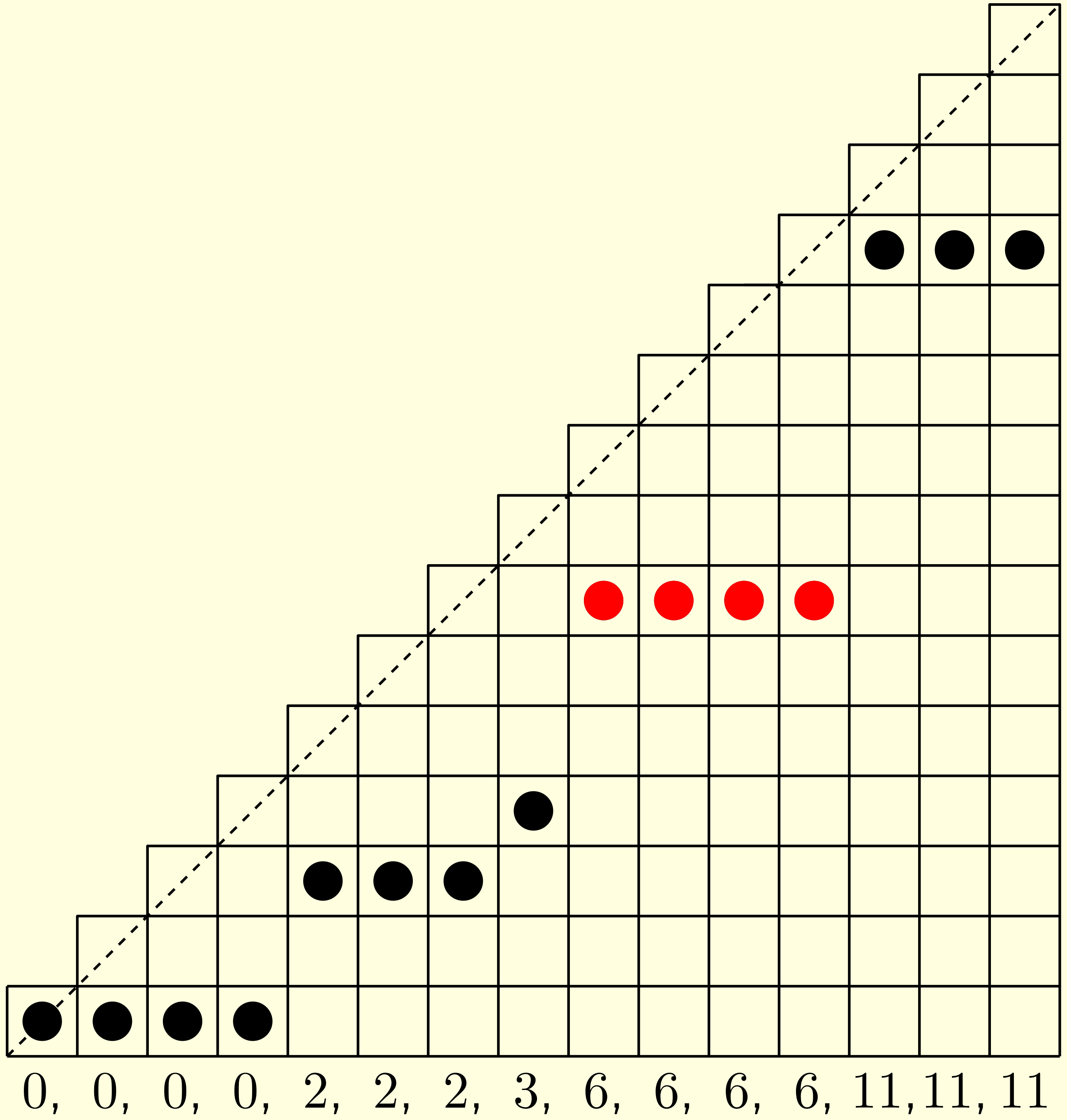

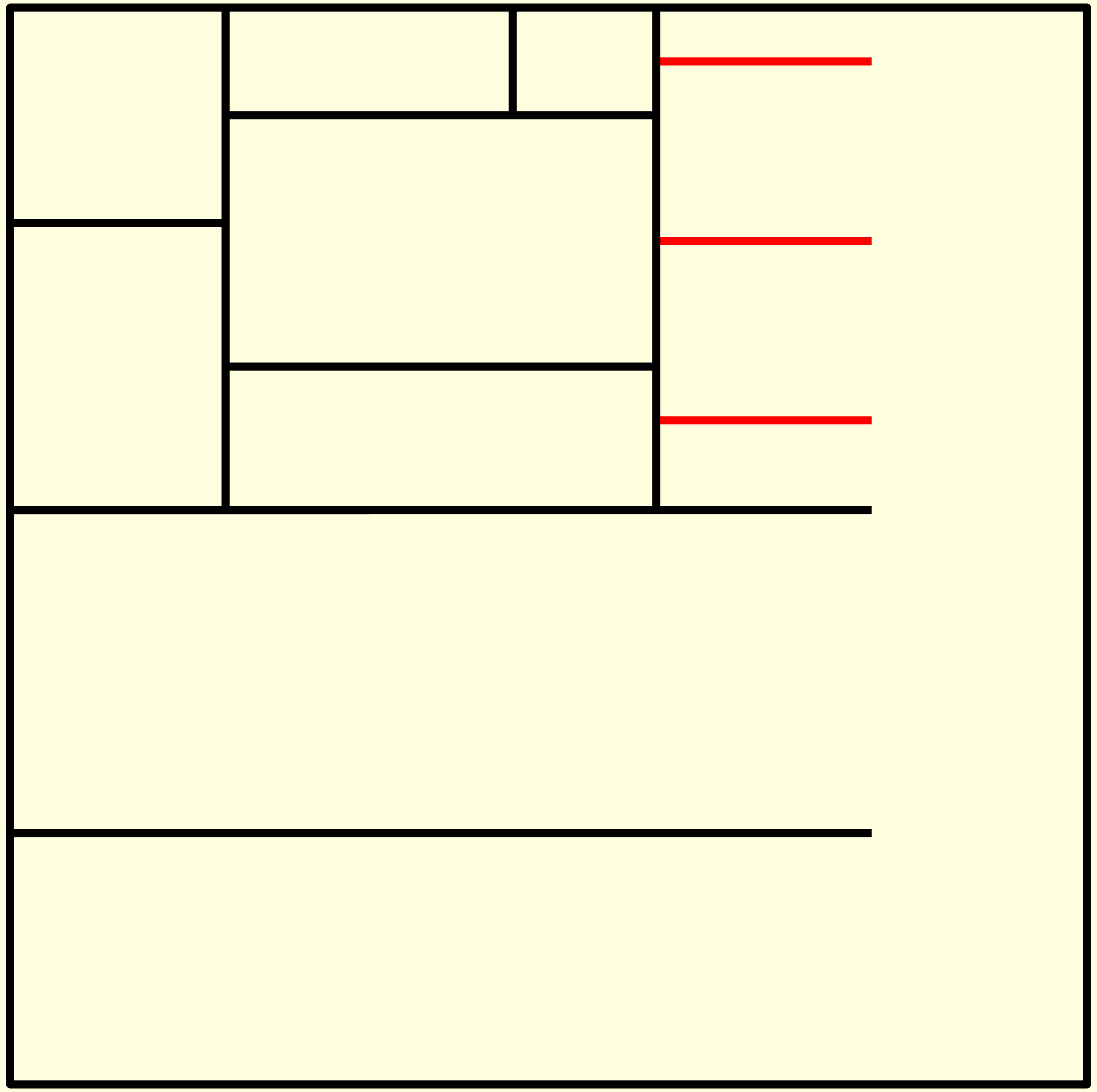

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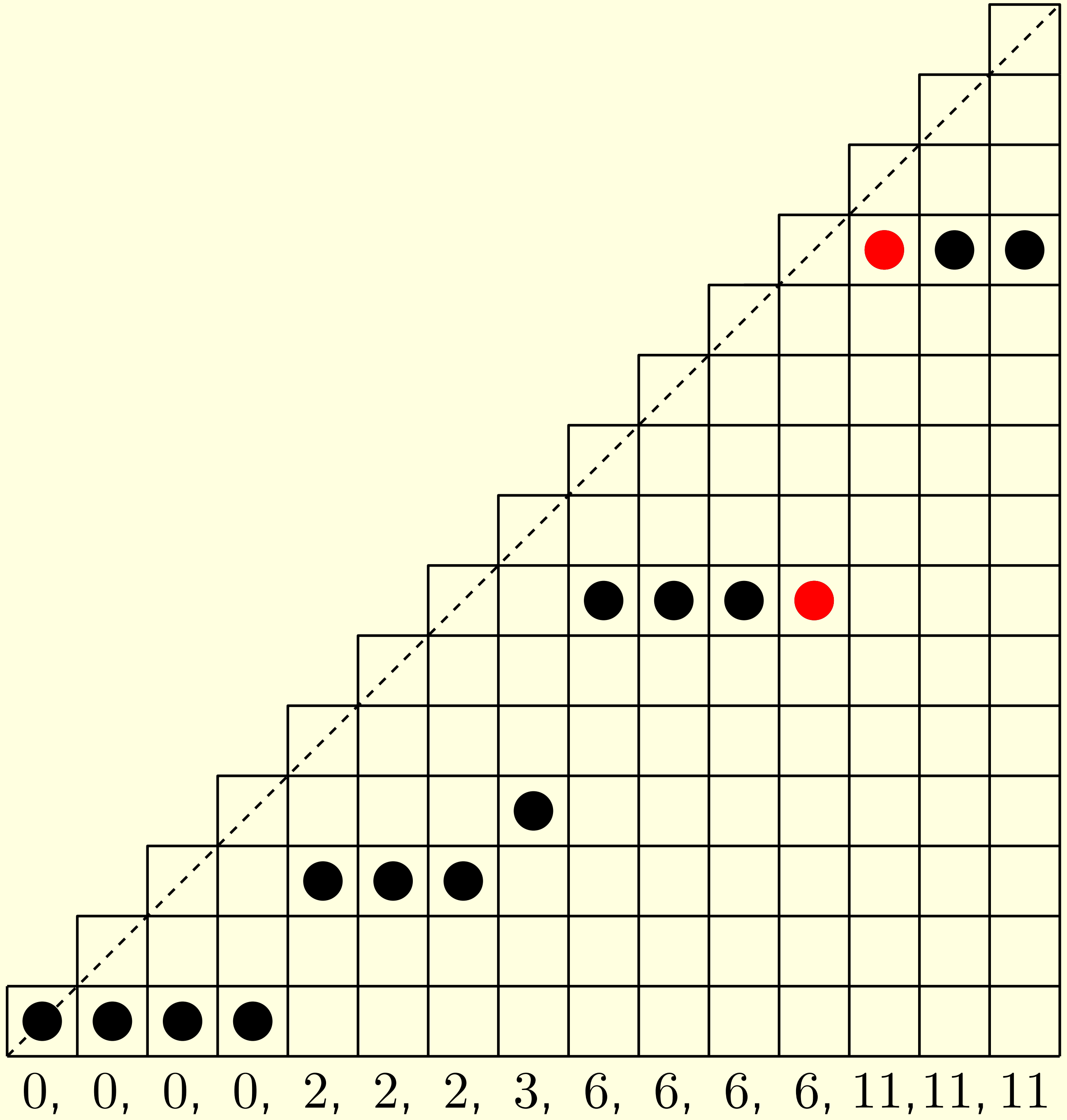

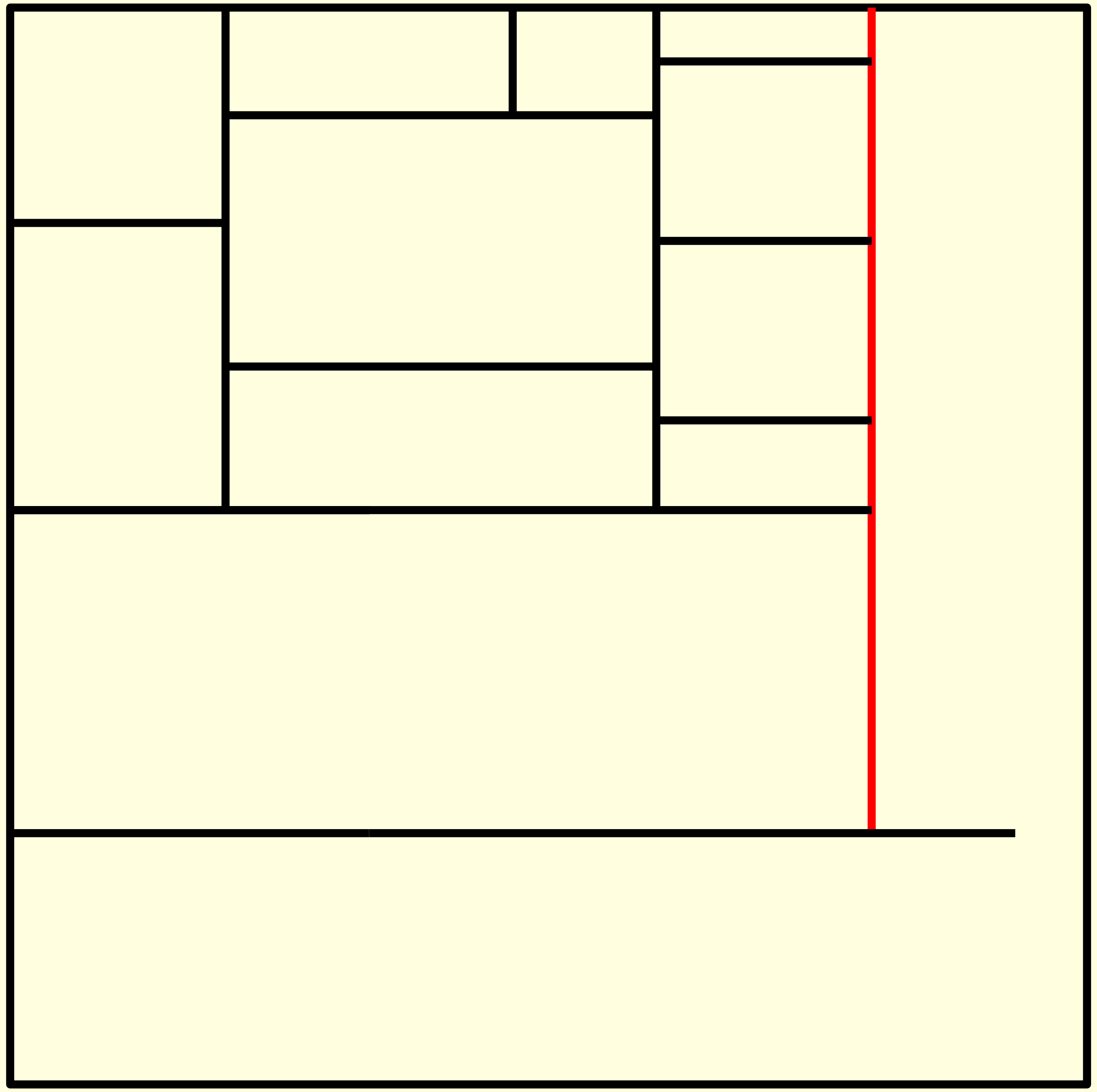

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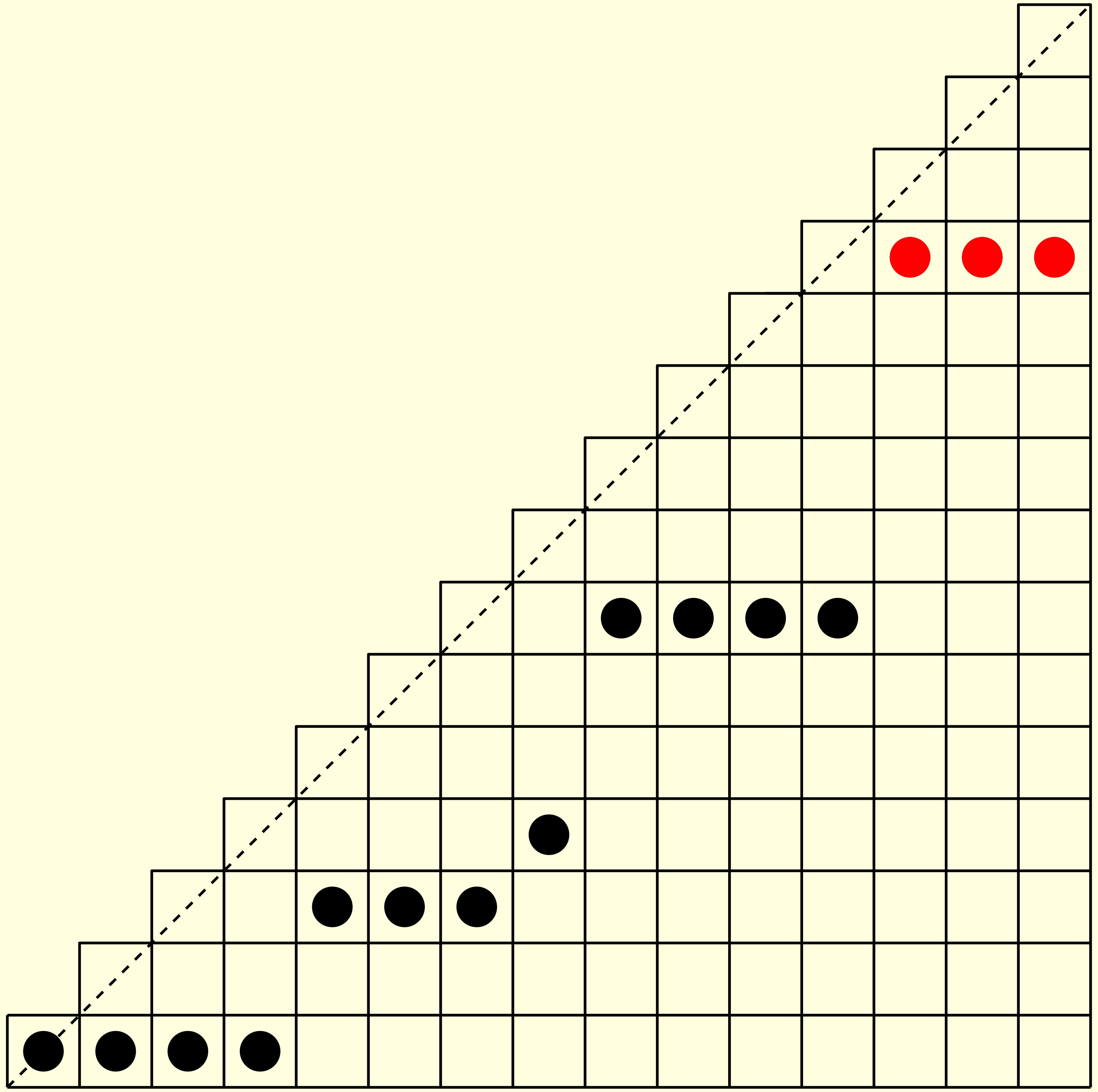

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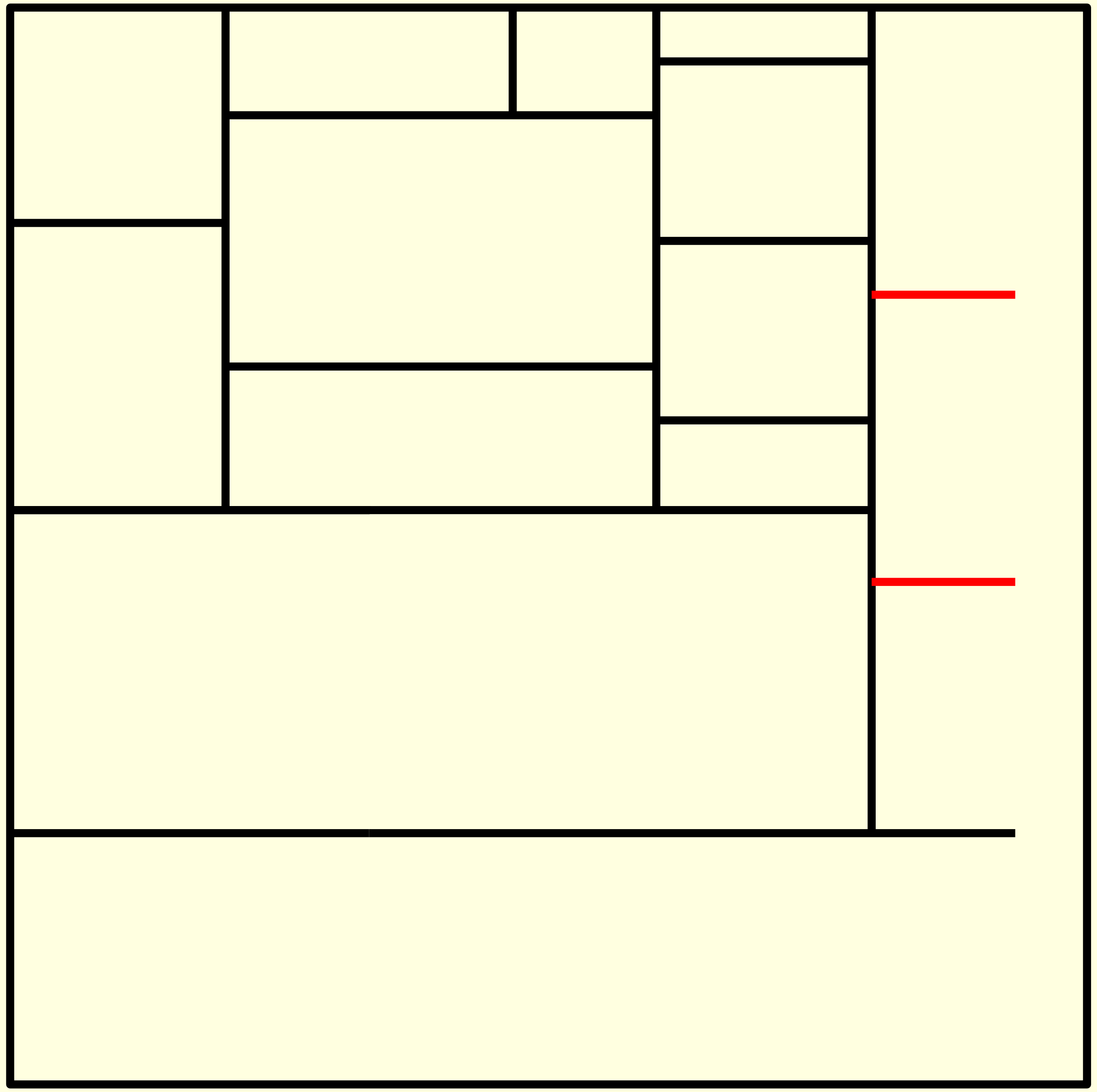

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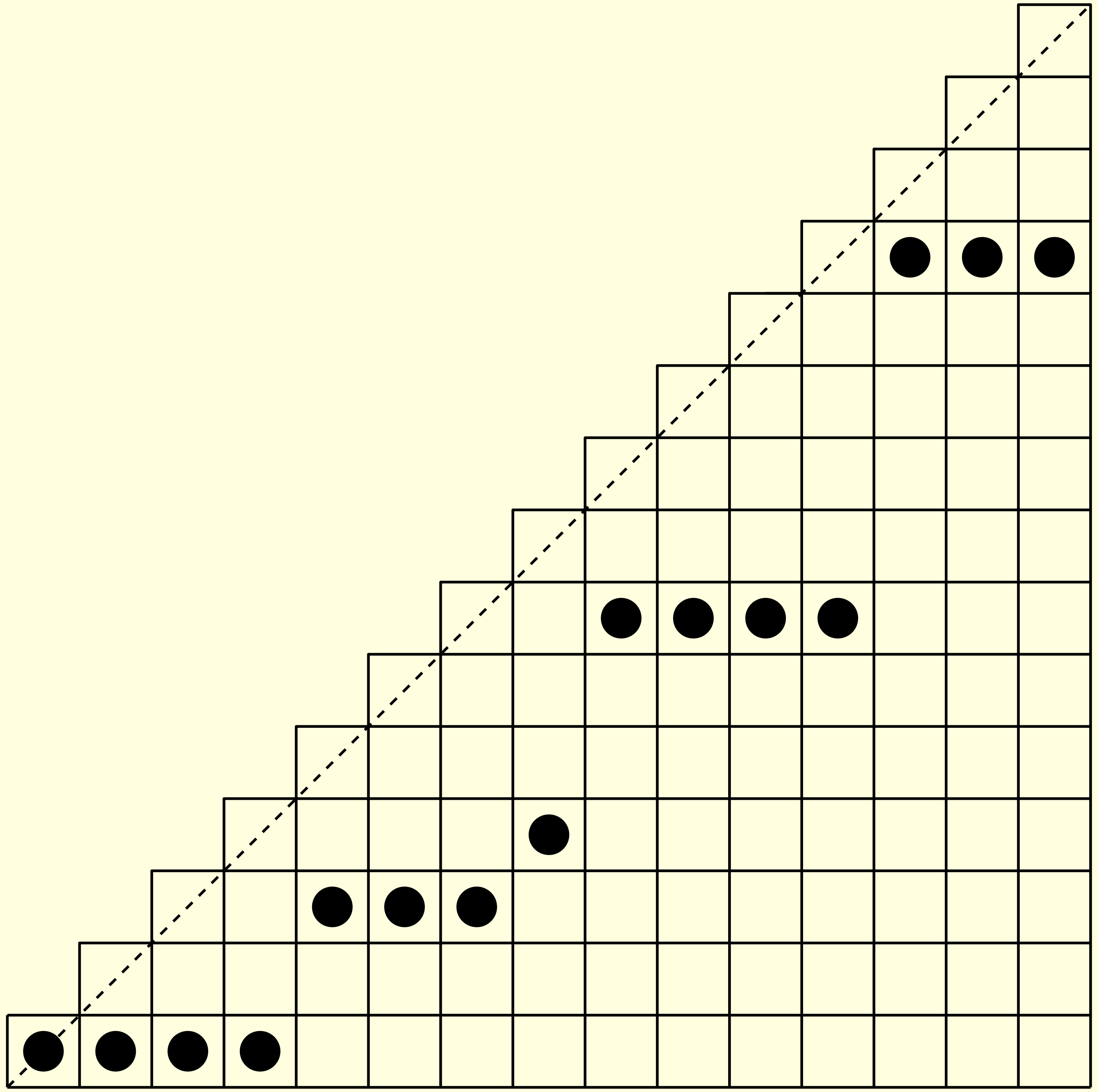

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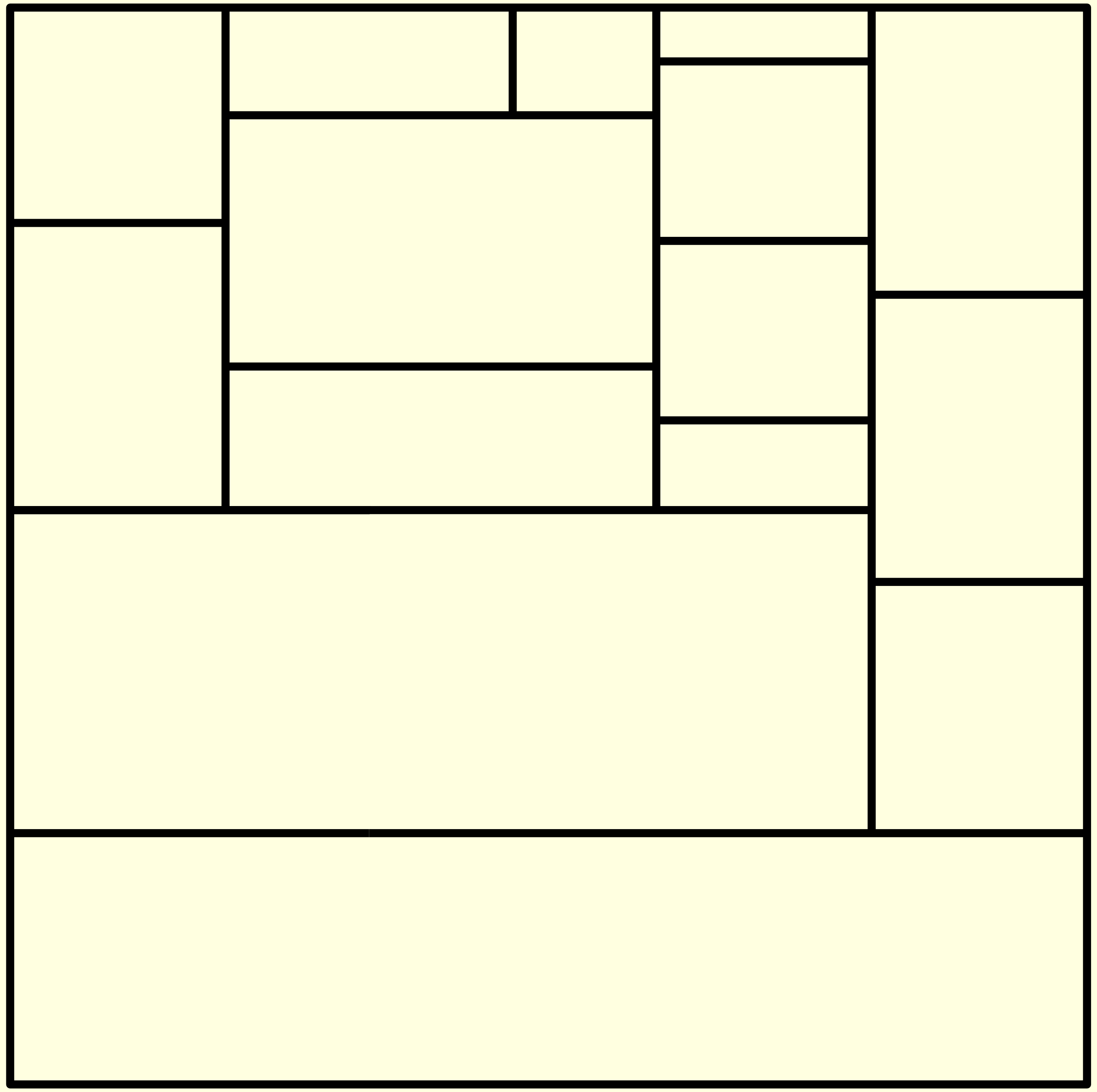

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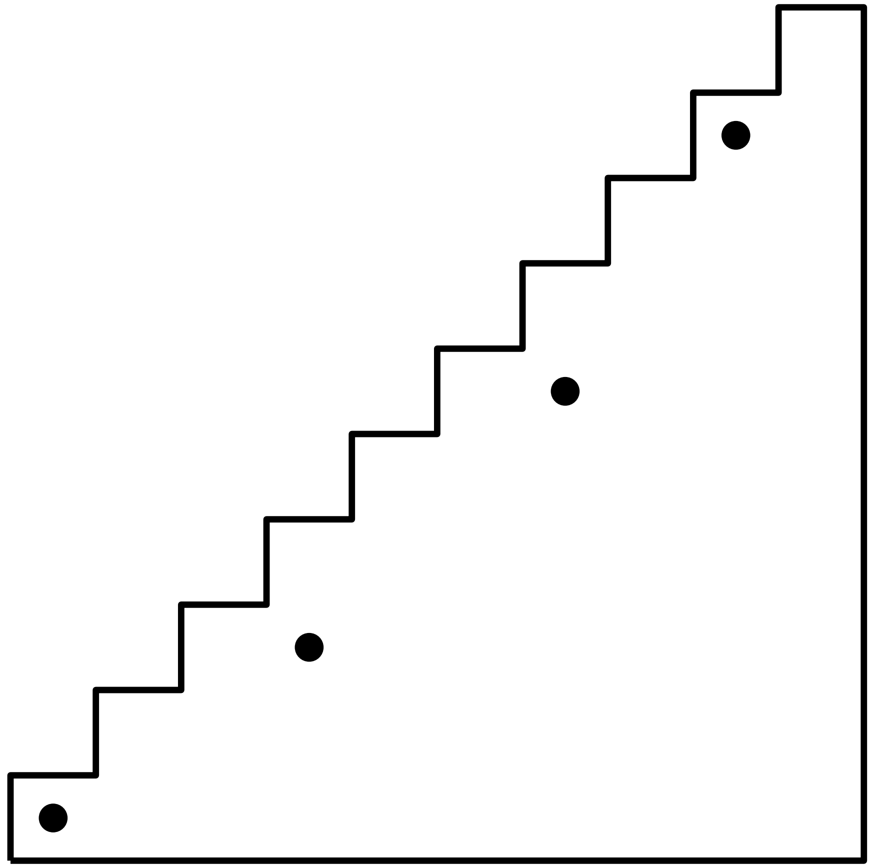
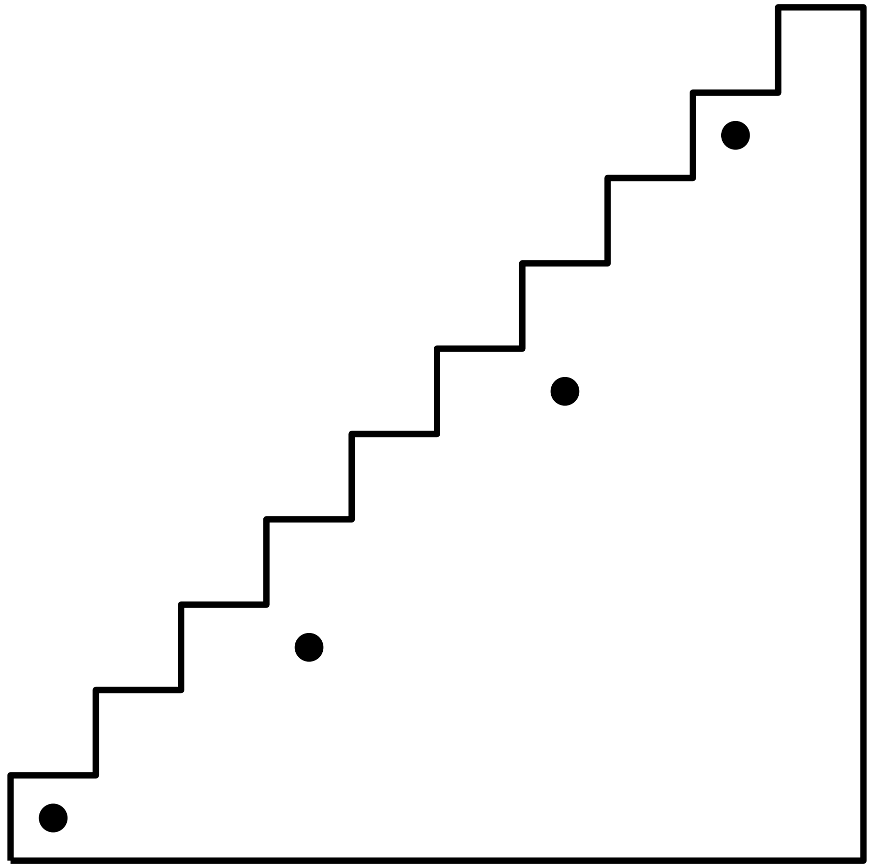


$I_n(010, 110, 120, 210)$



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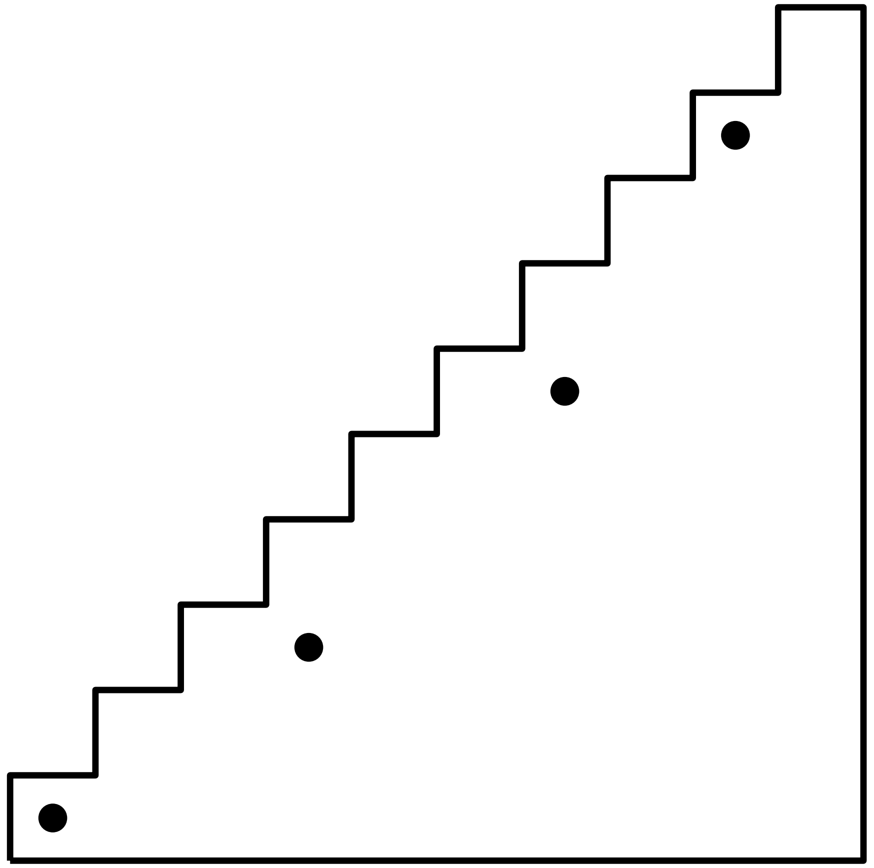

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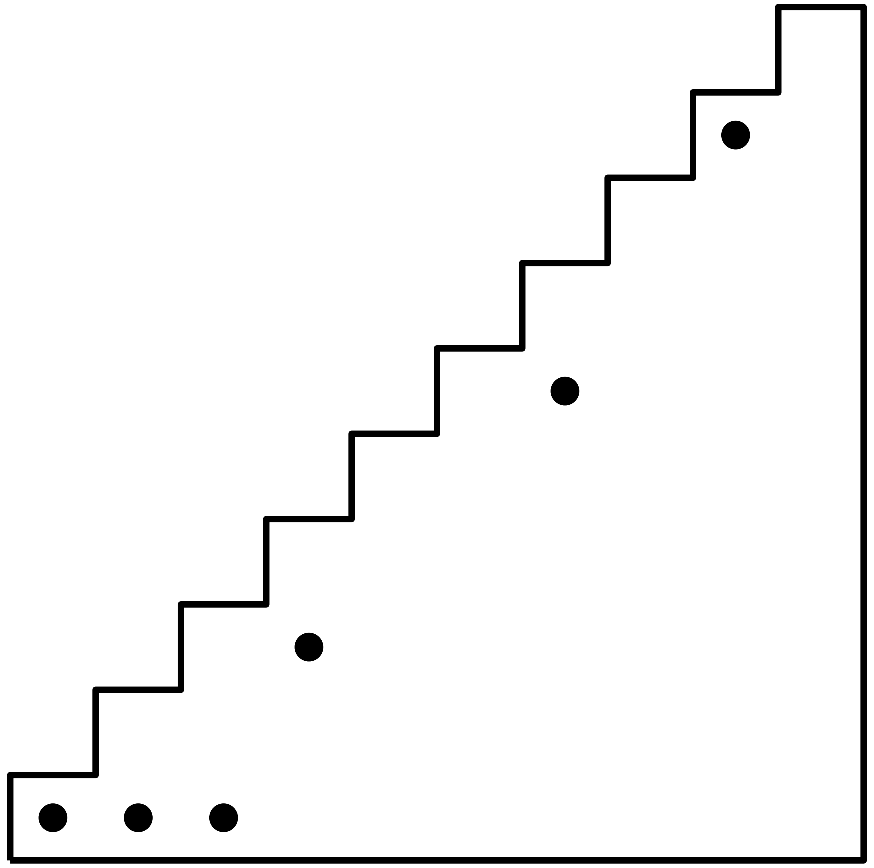

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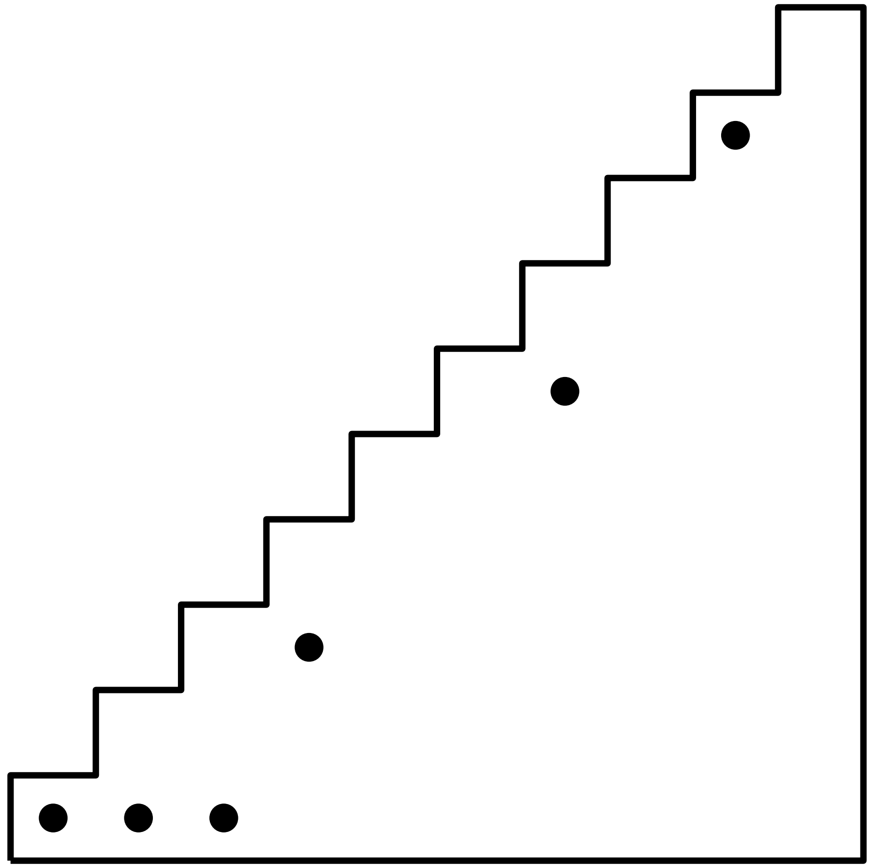

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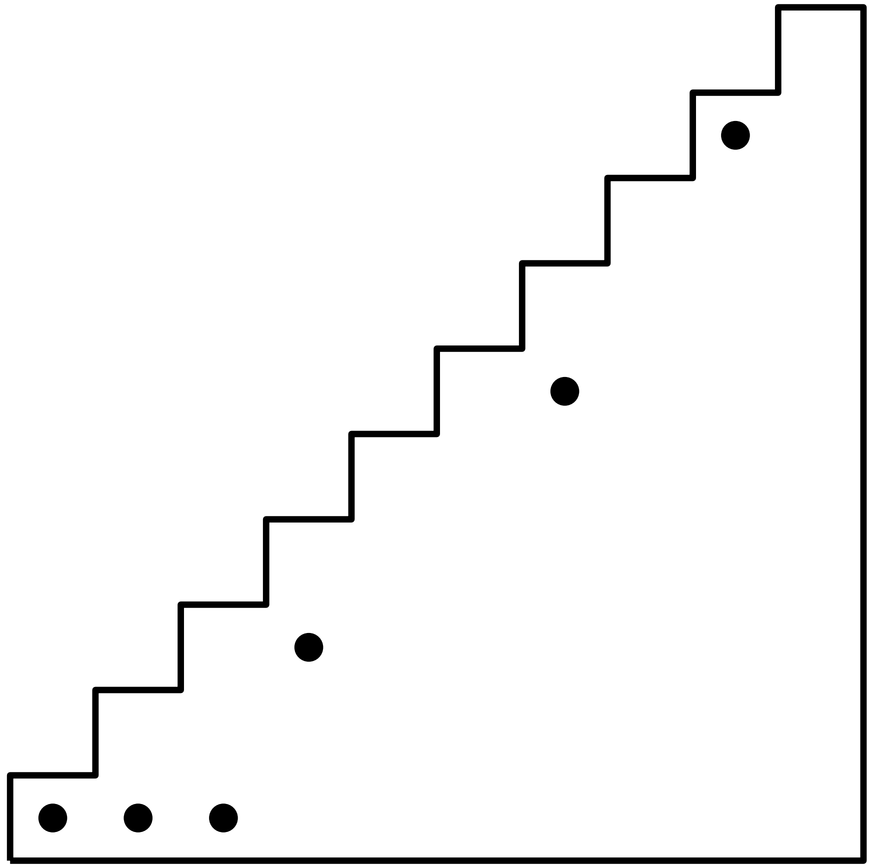

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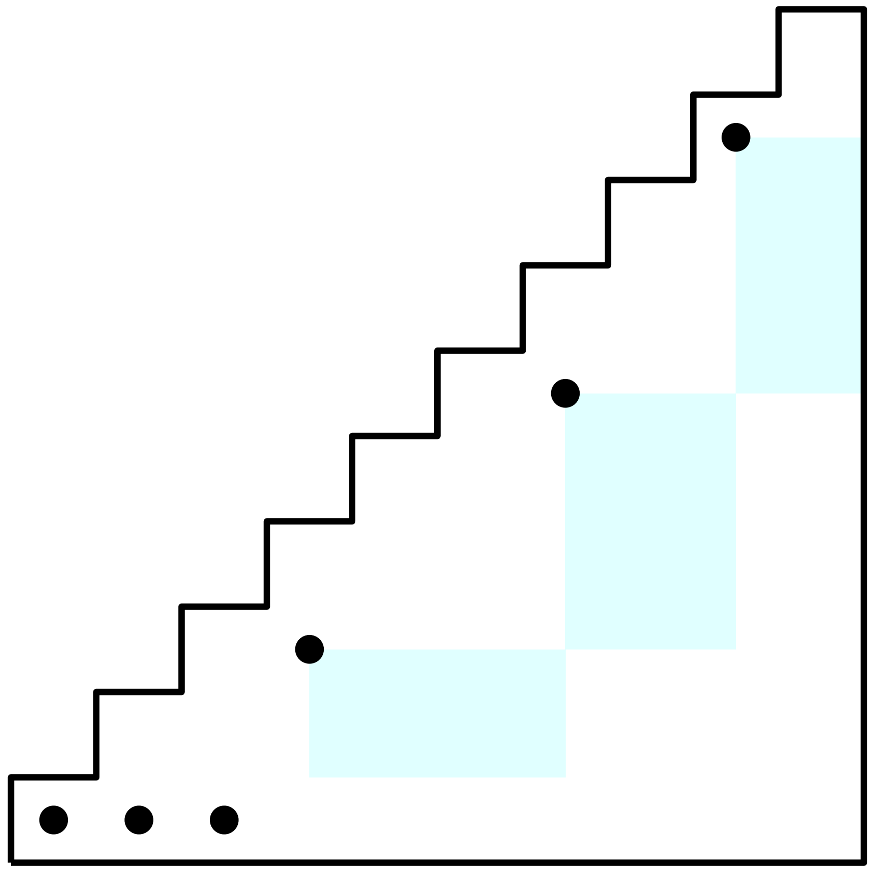

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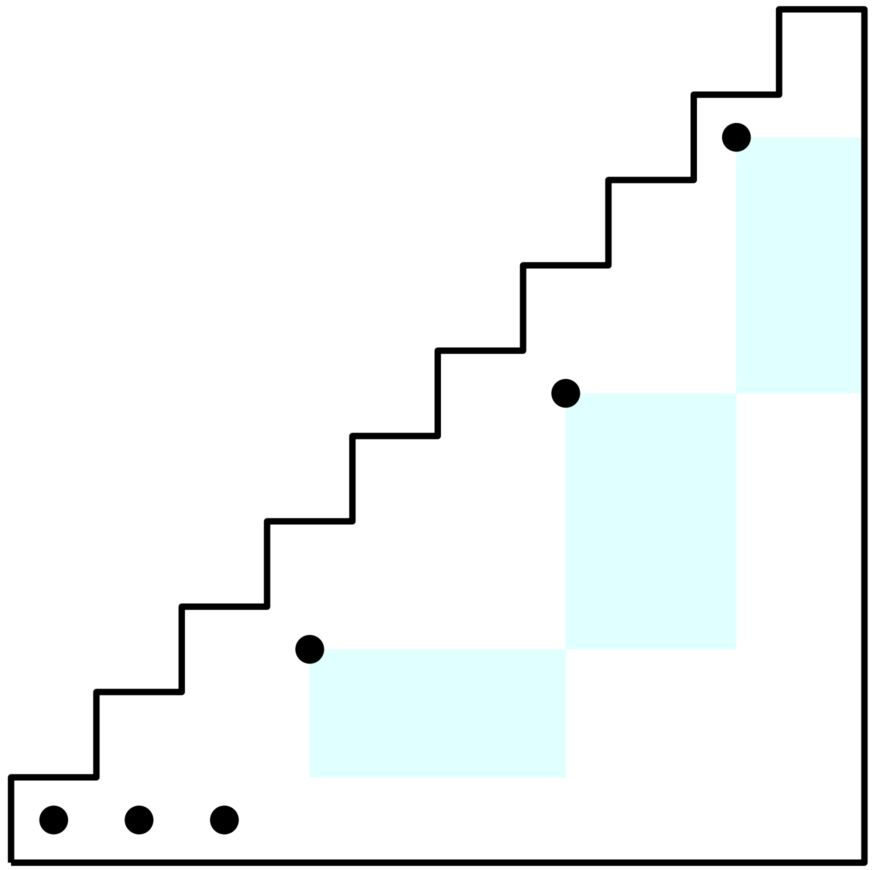

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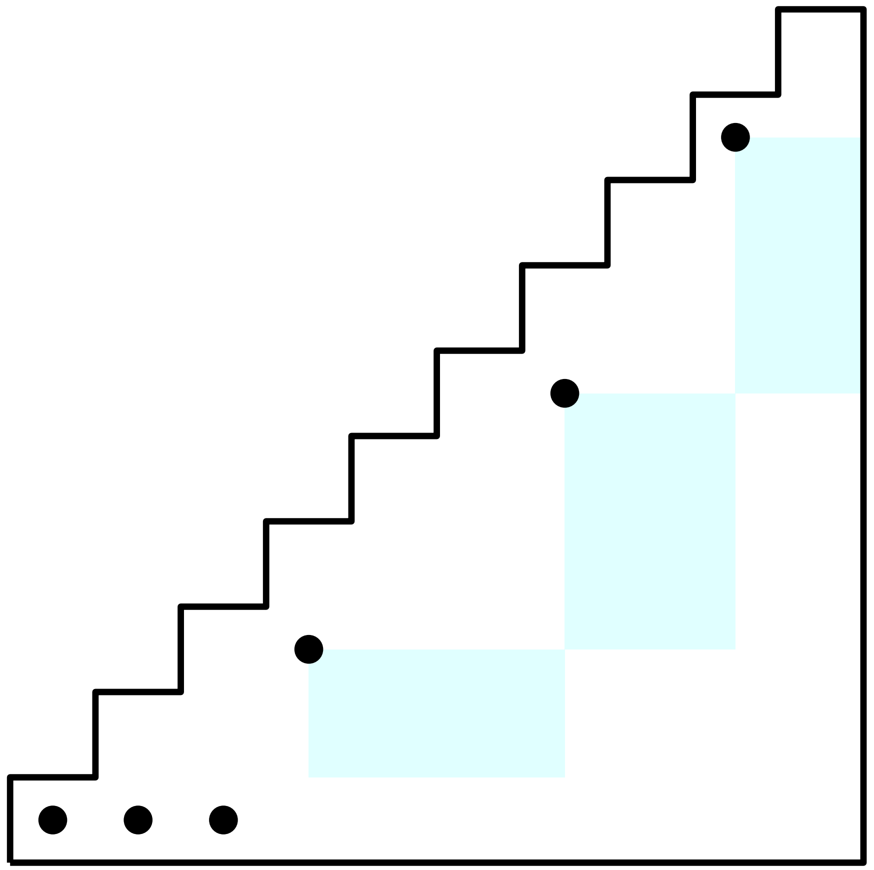

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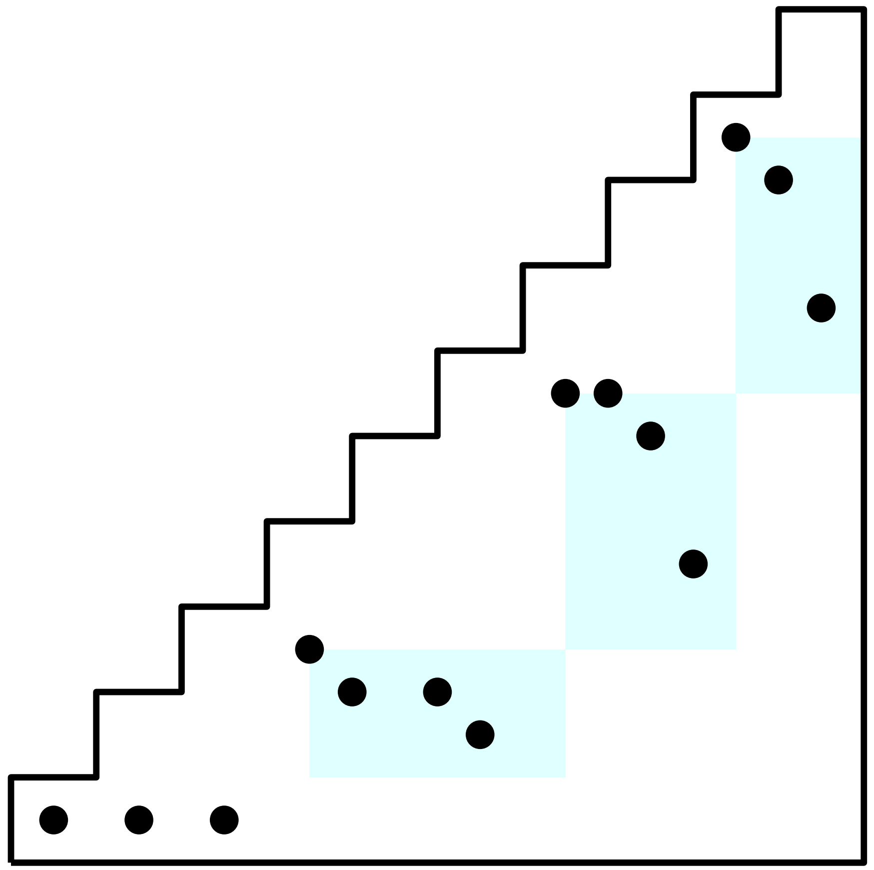

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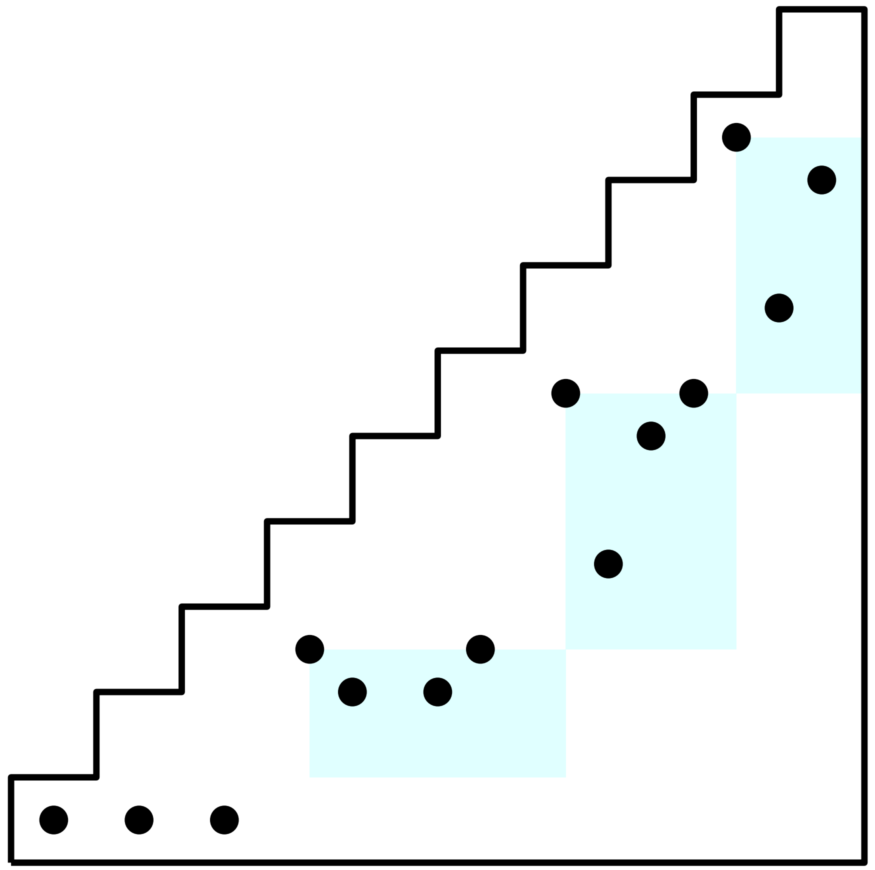

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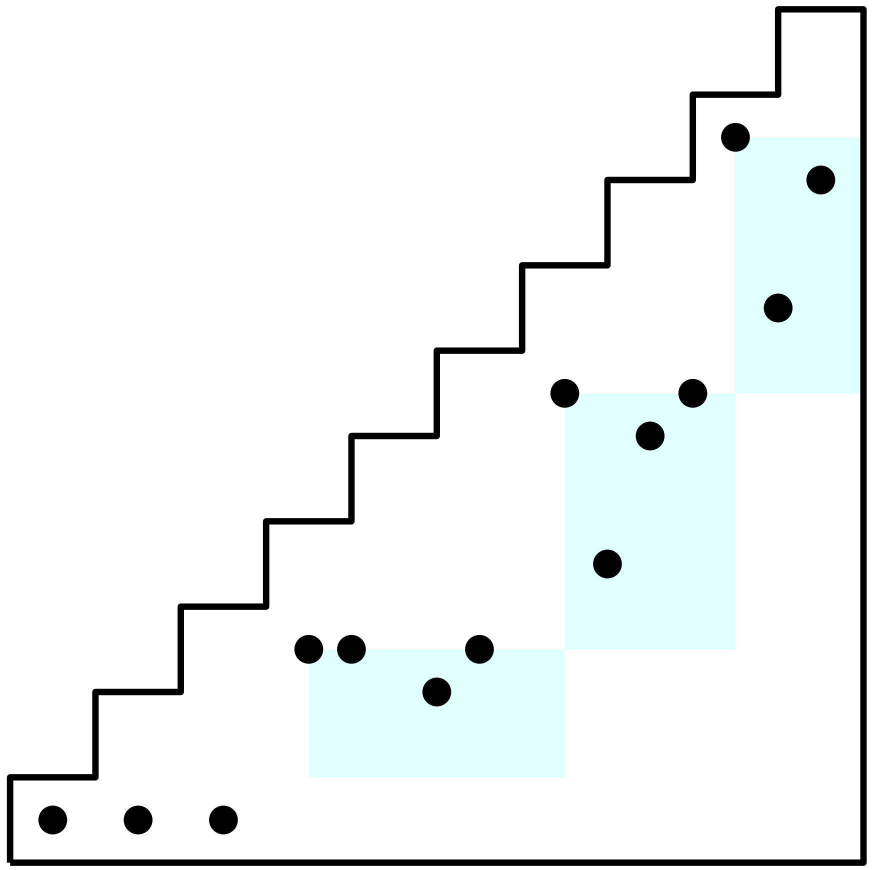

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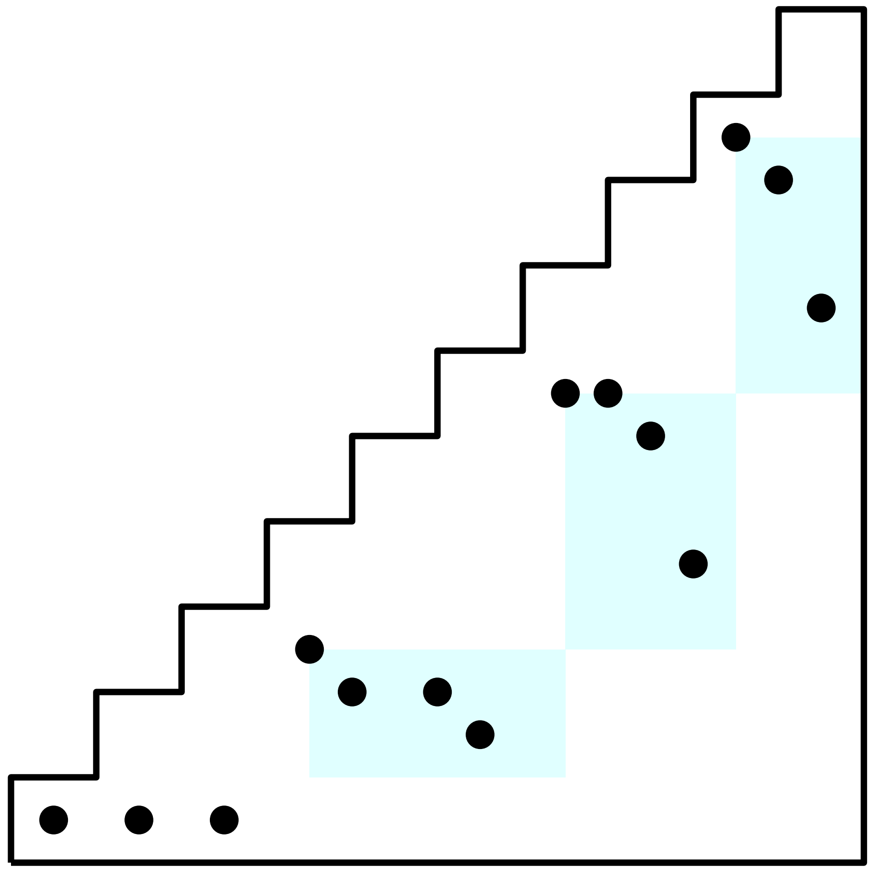

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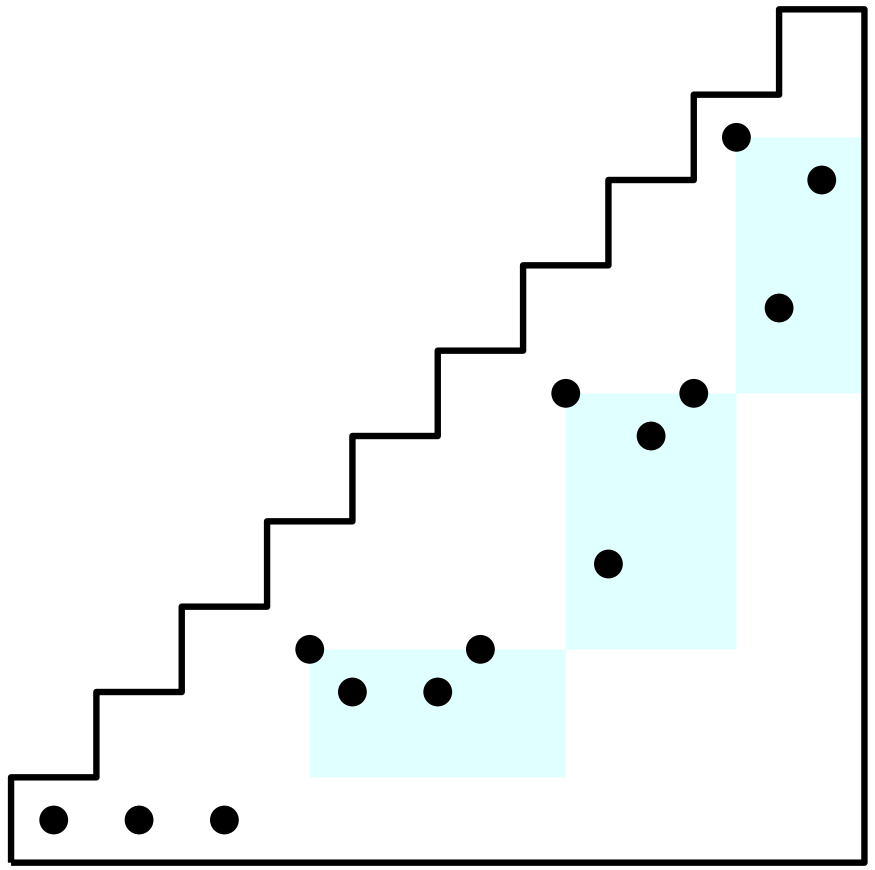

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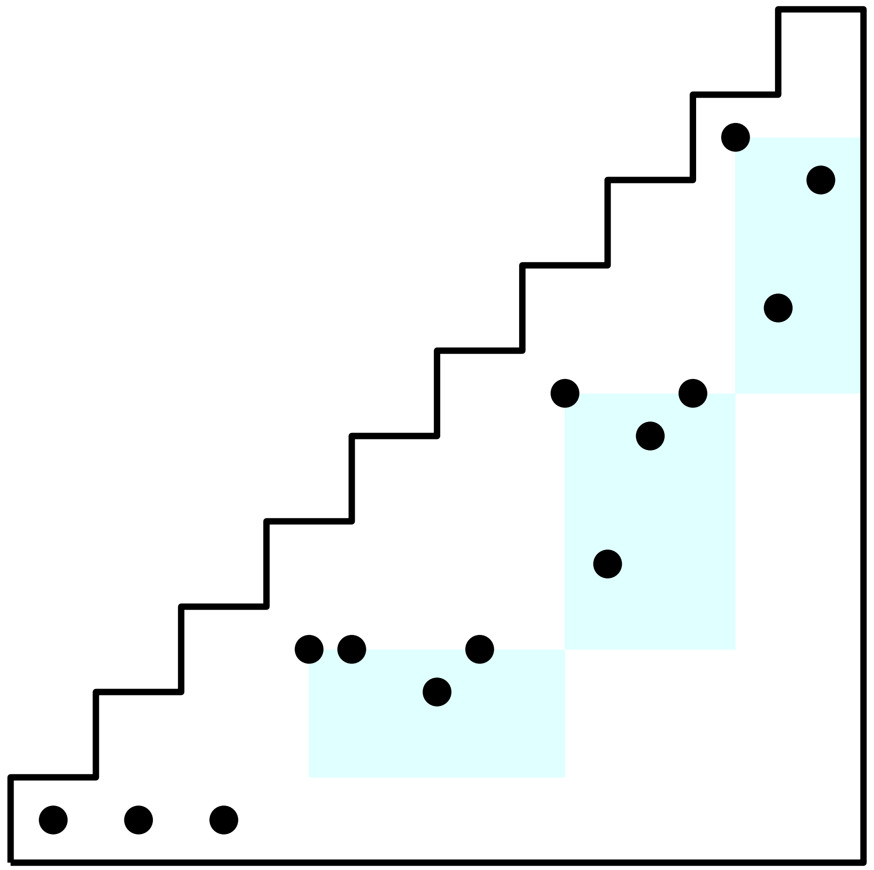

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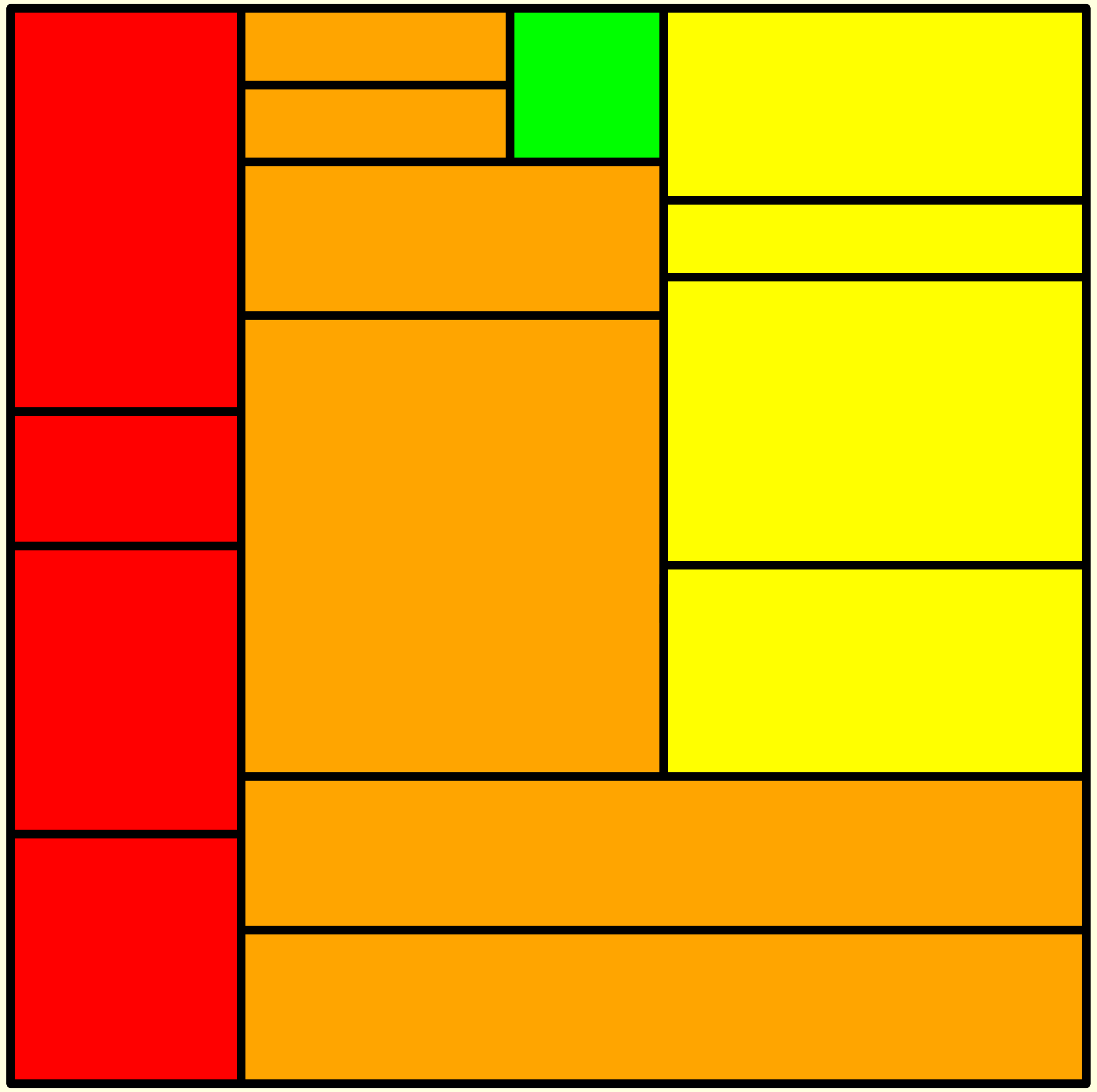

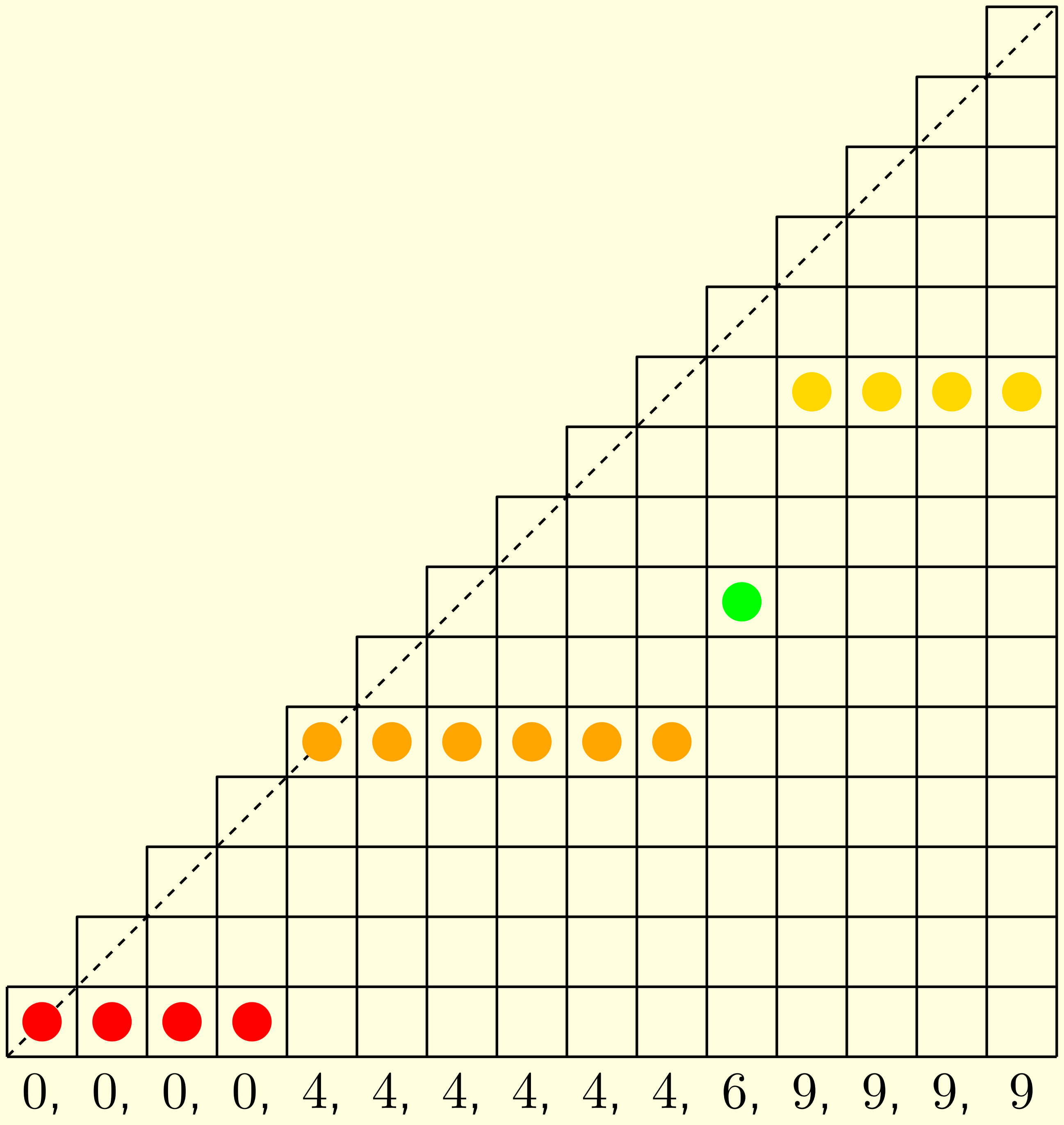

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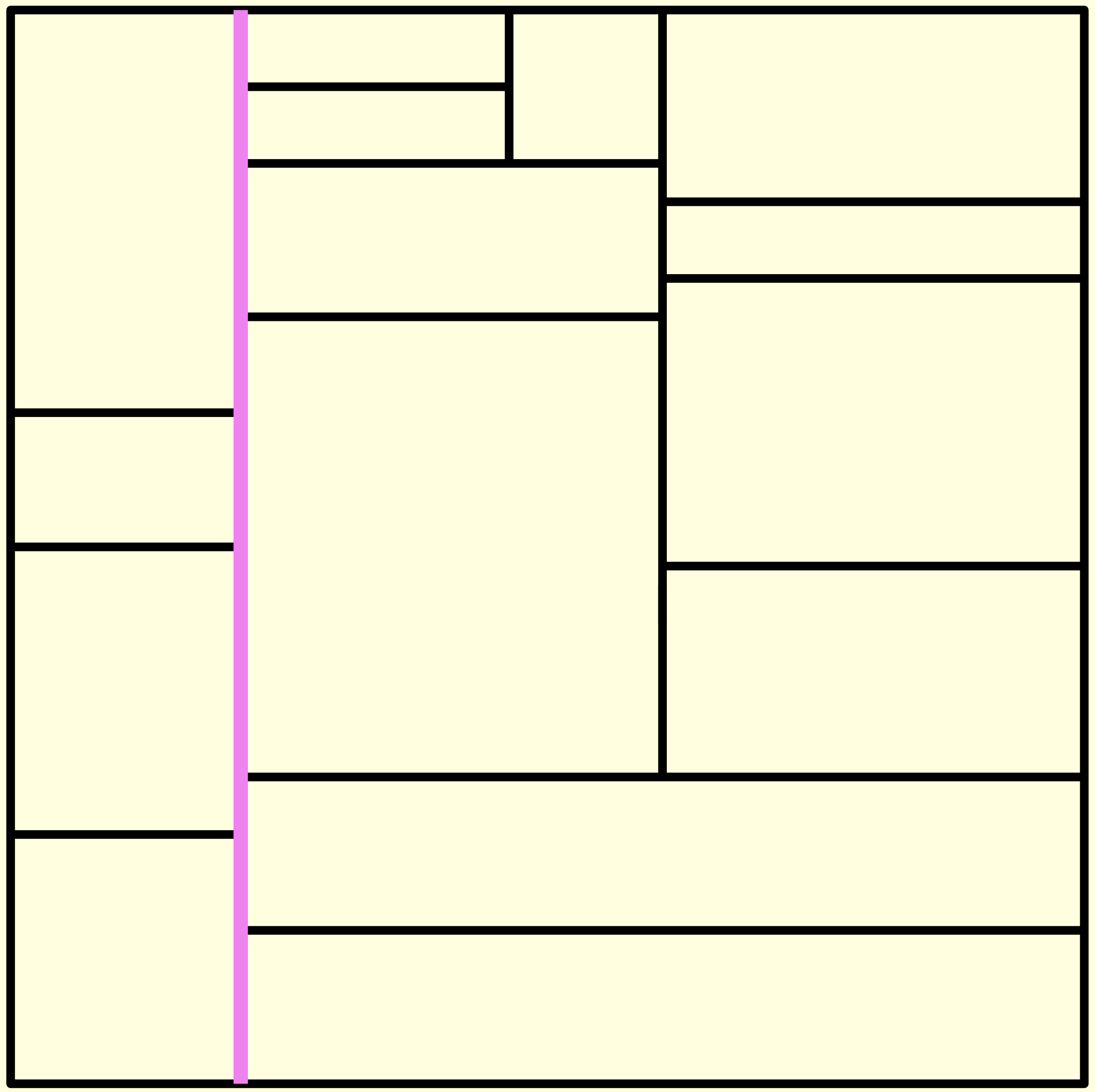

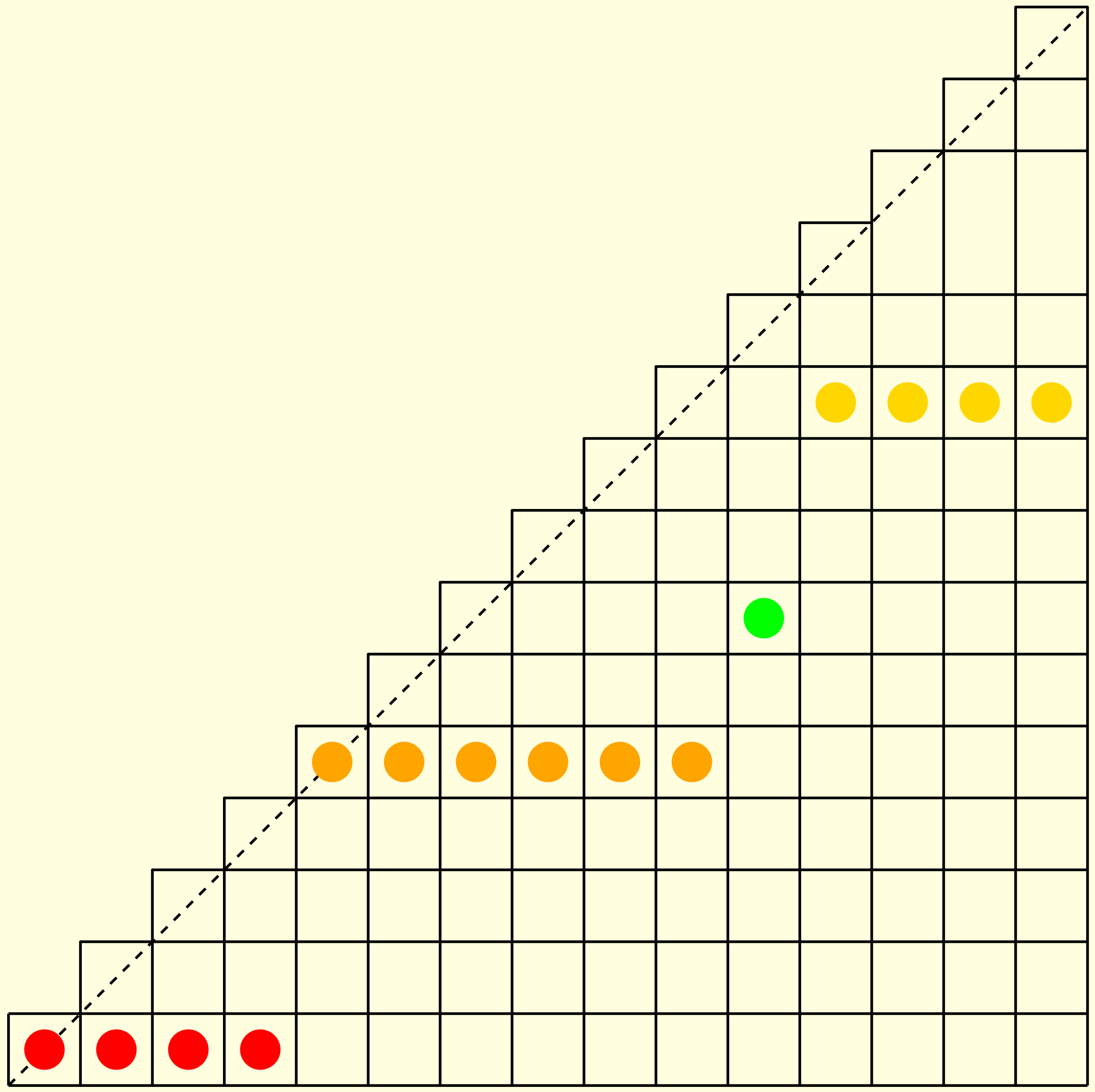

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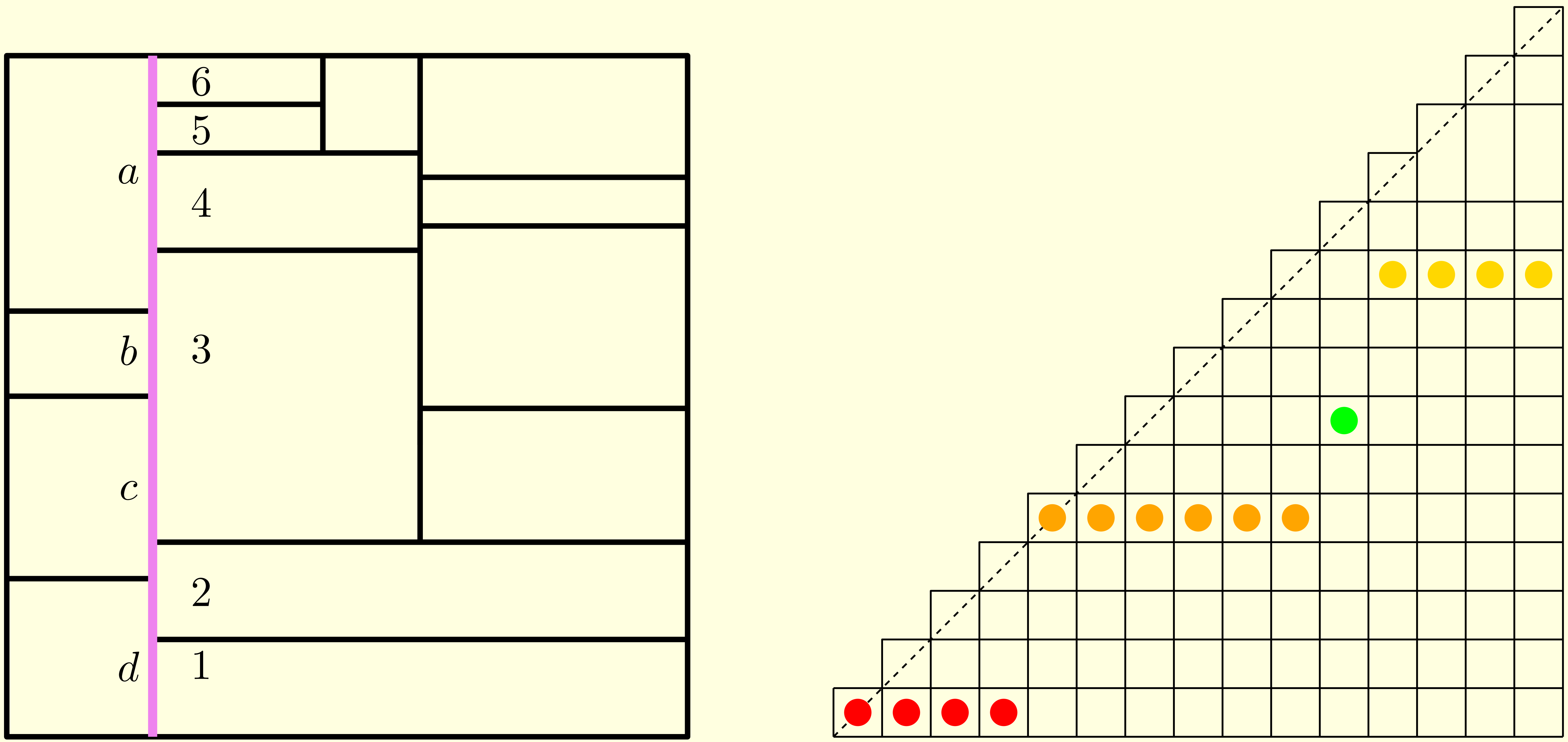

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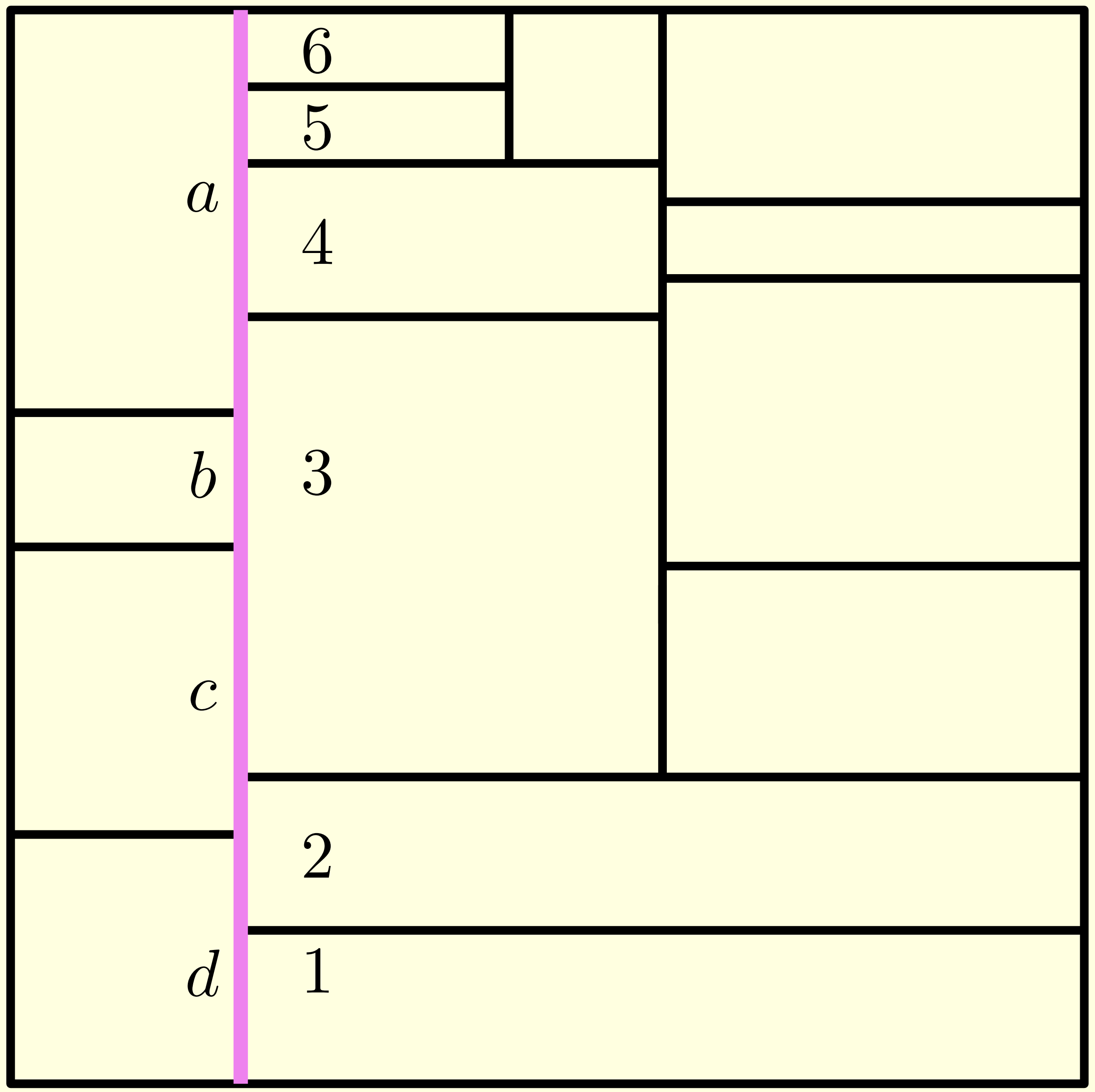

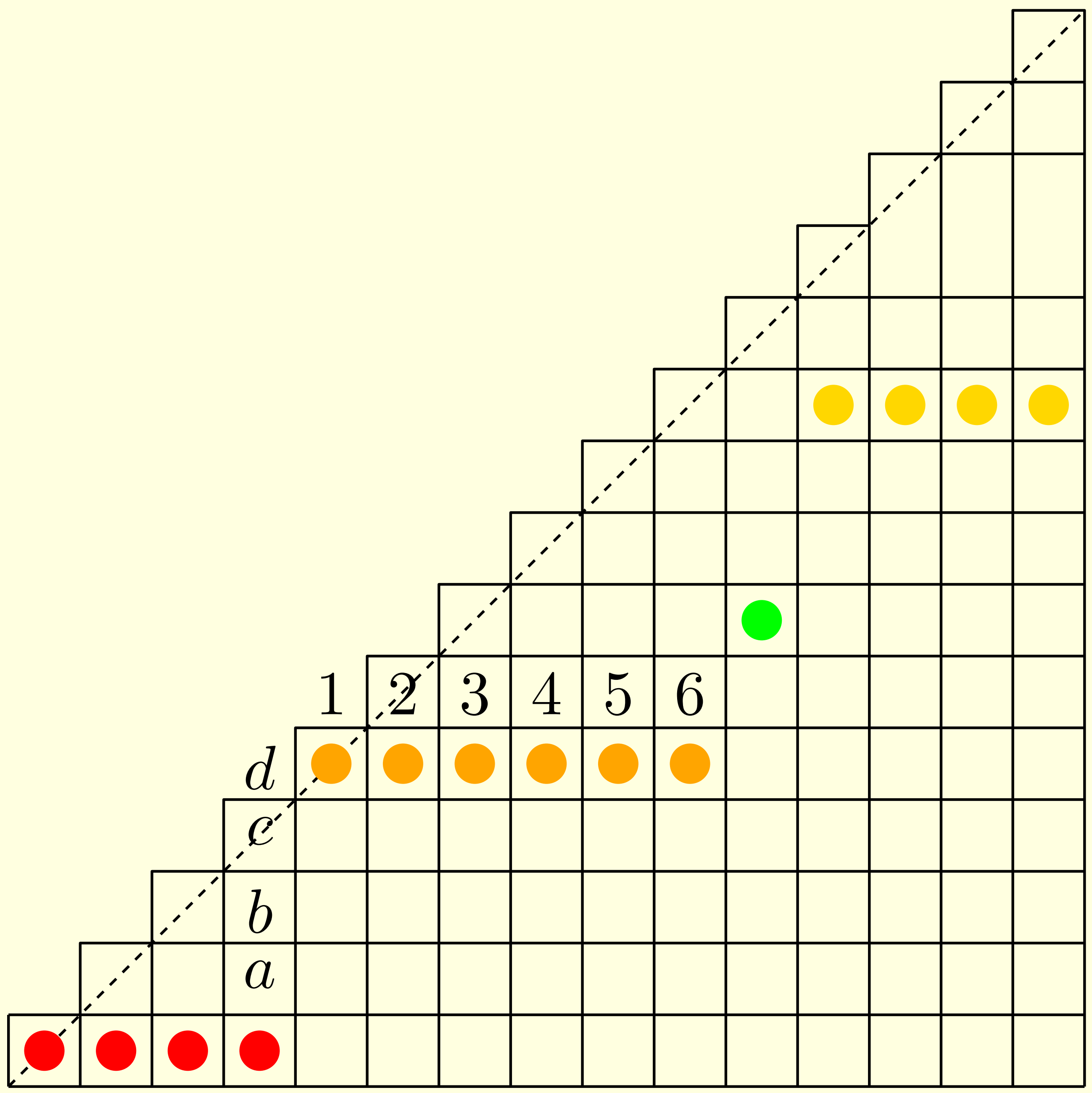

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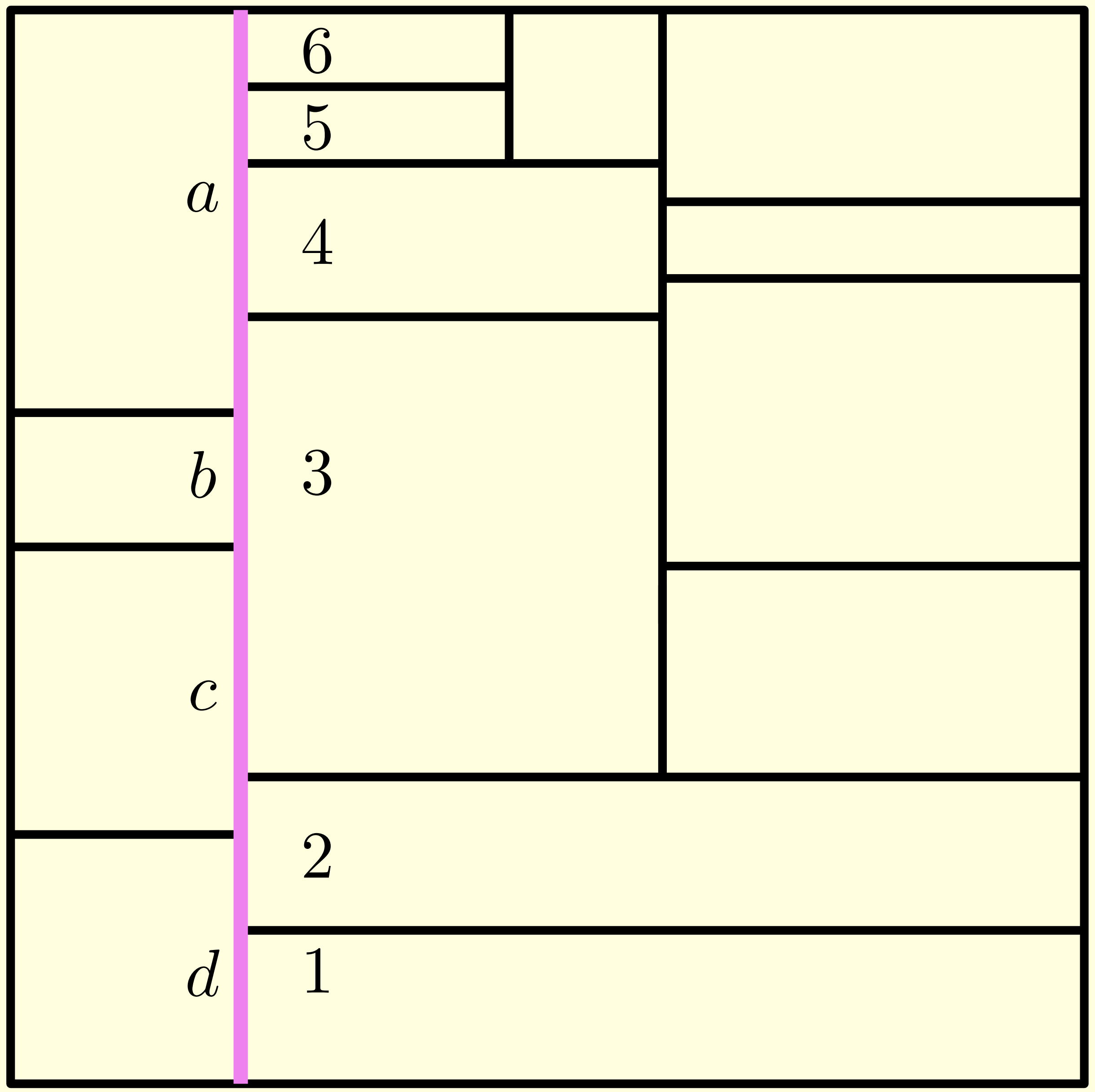

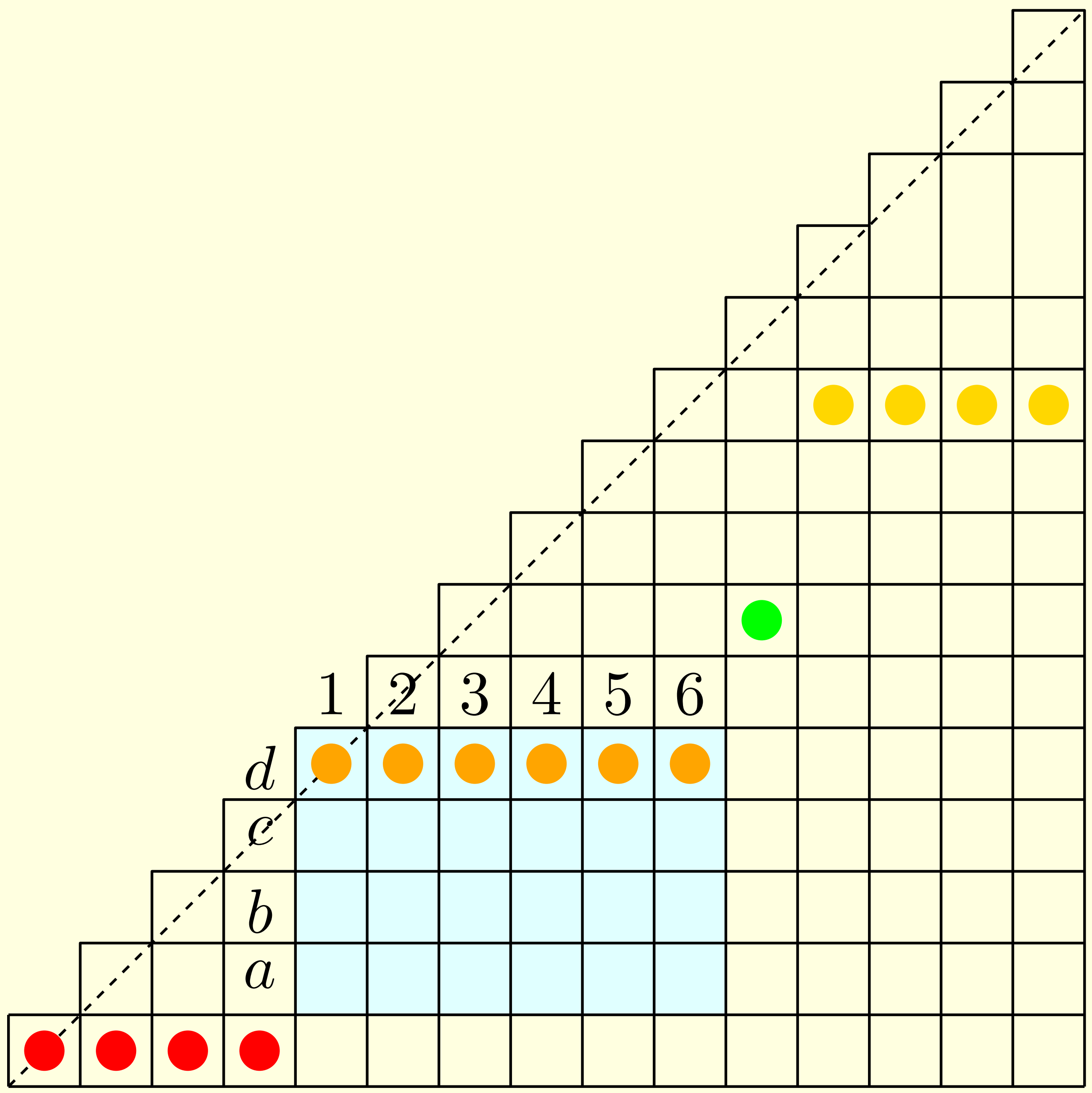

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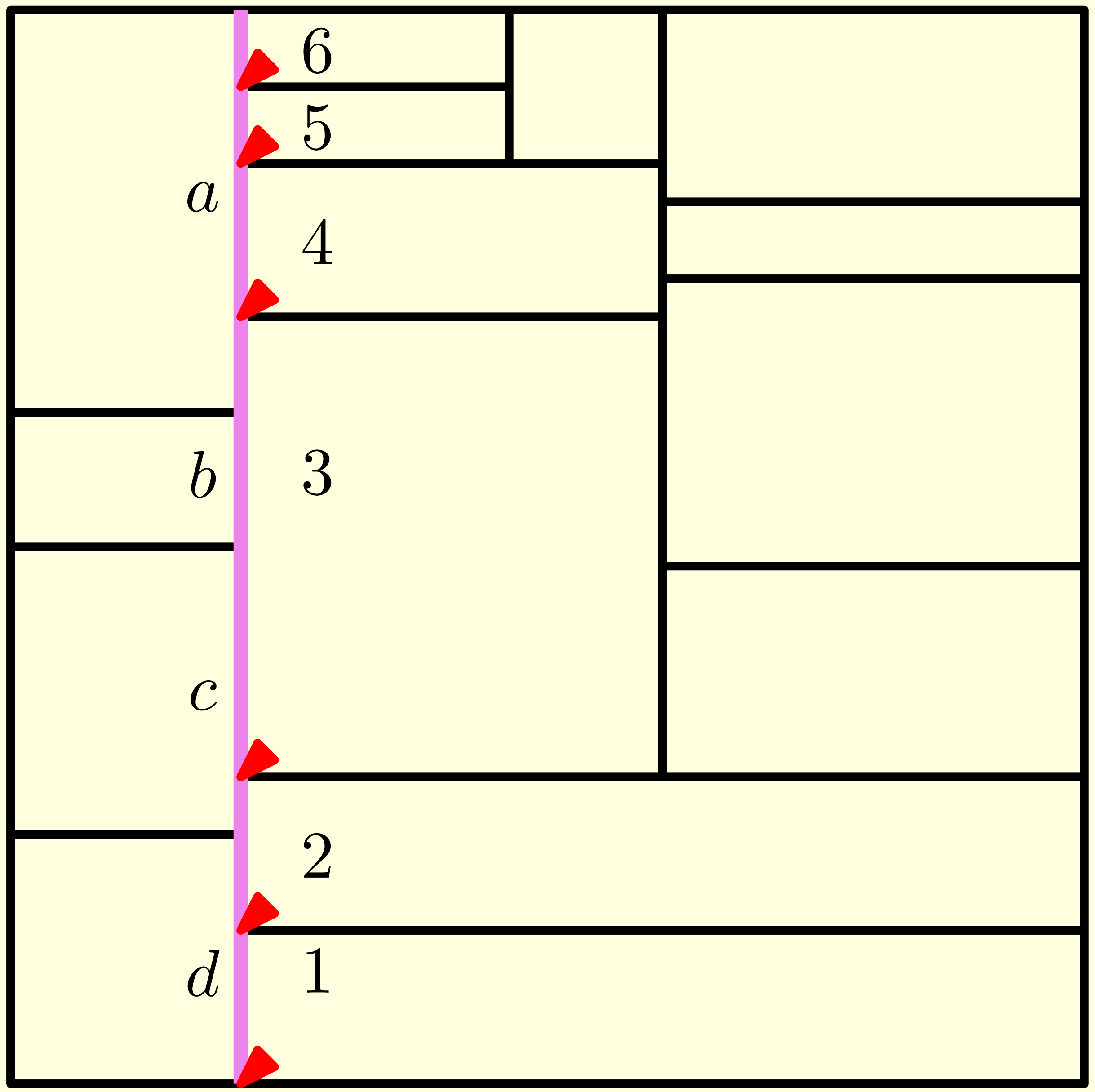

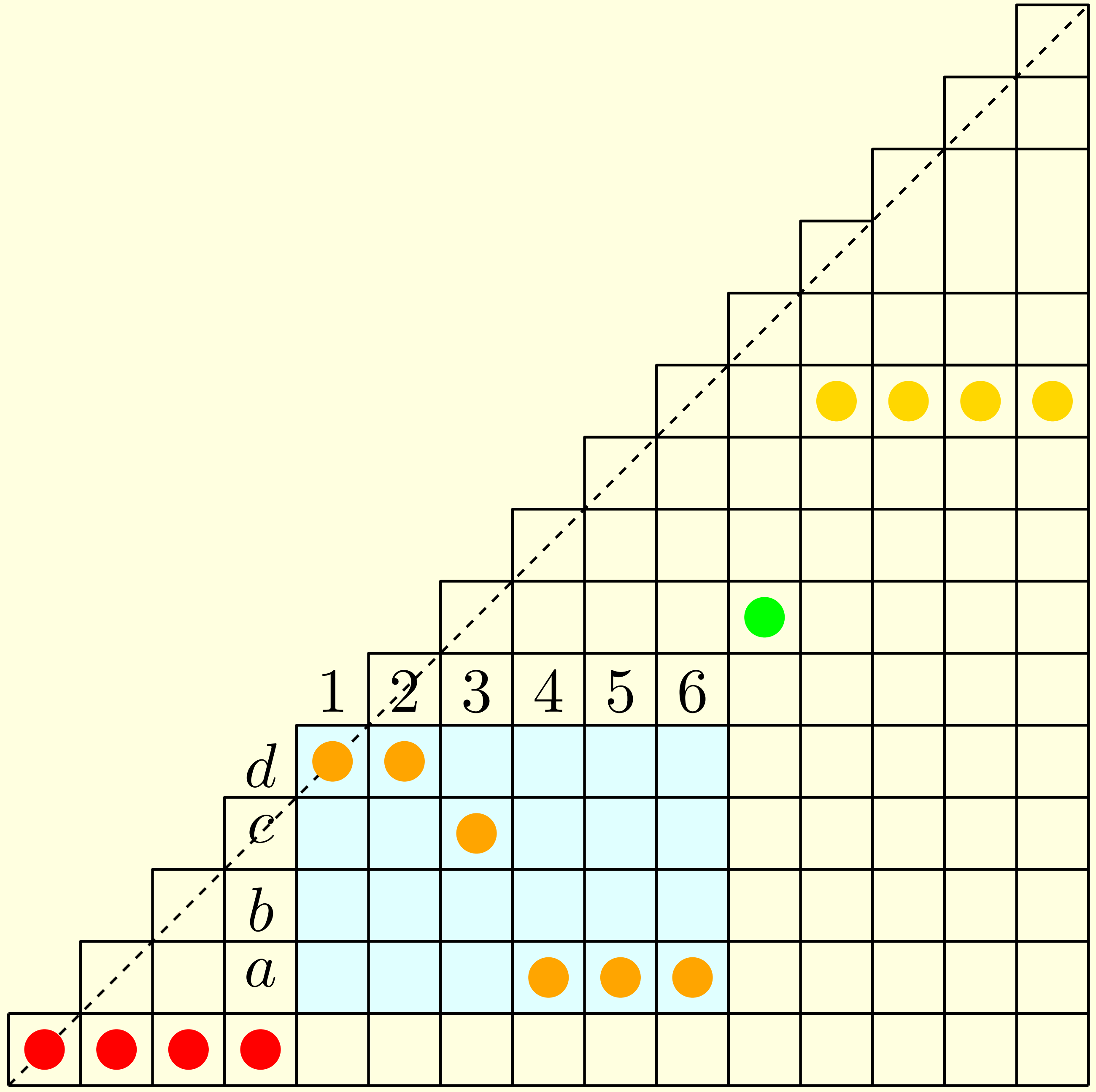

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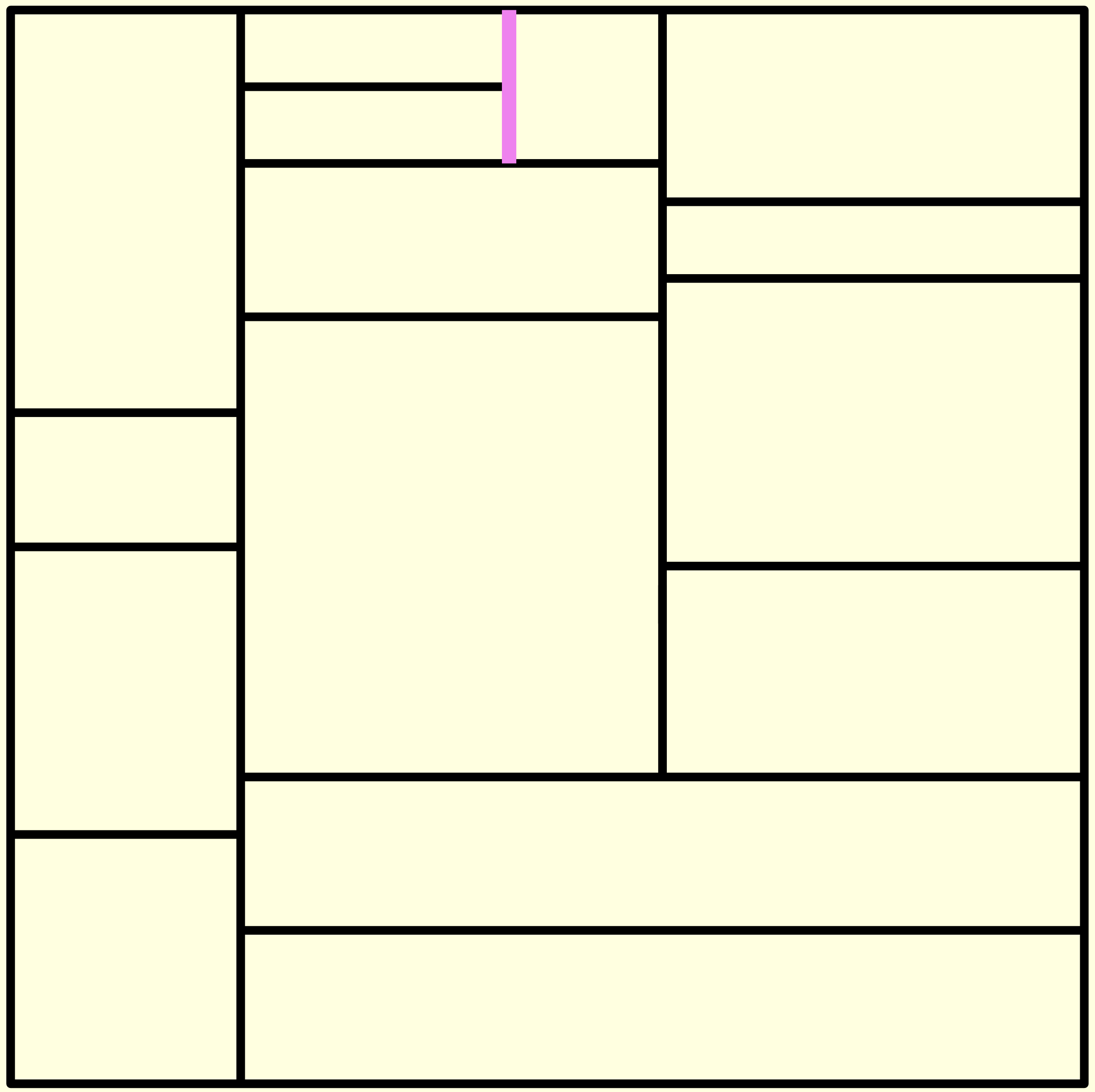

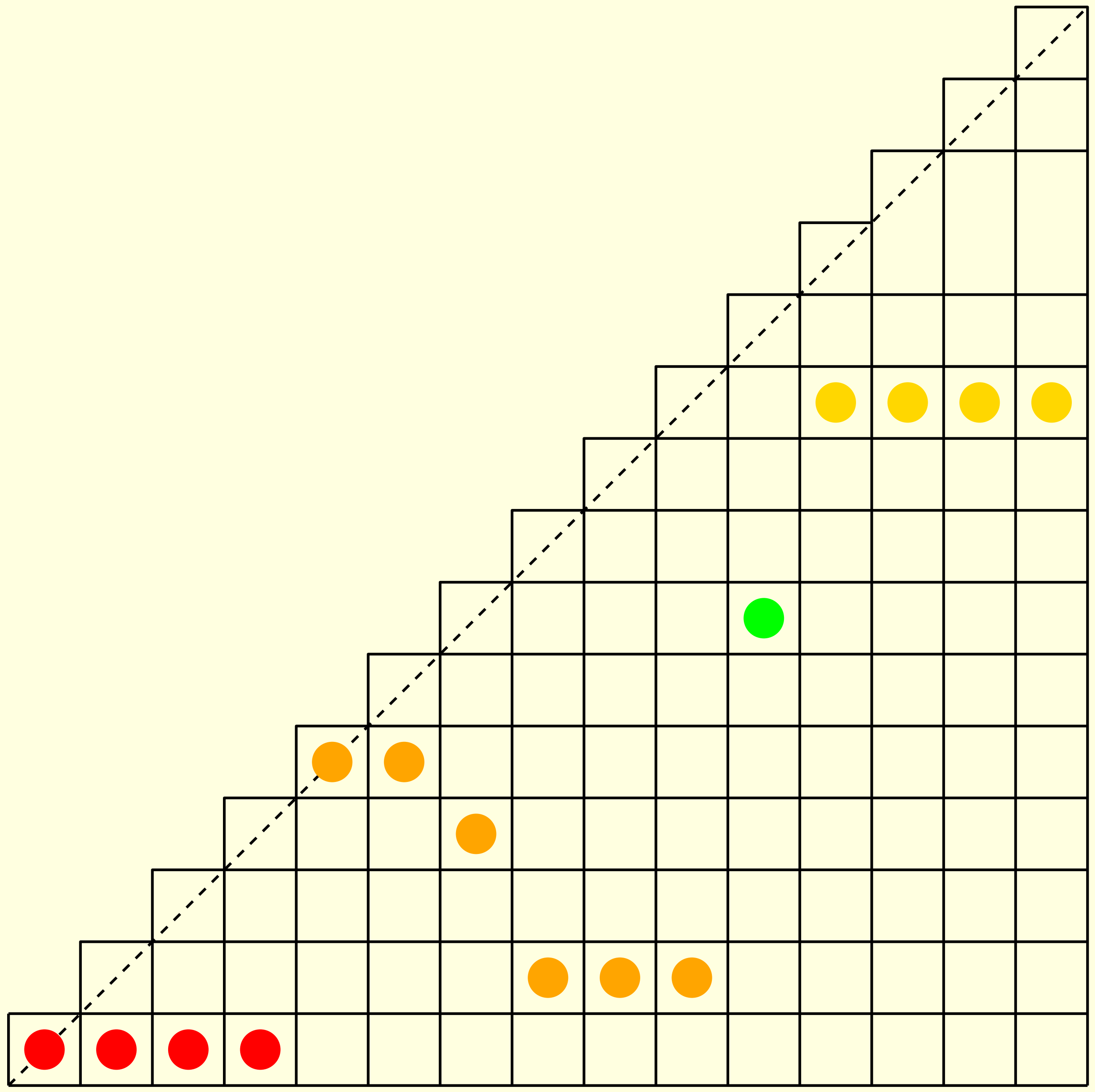

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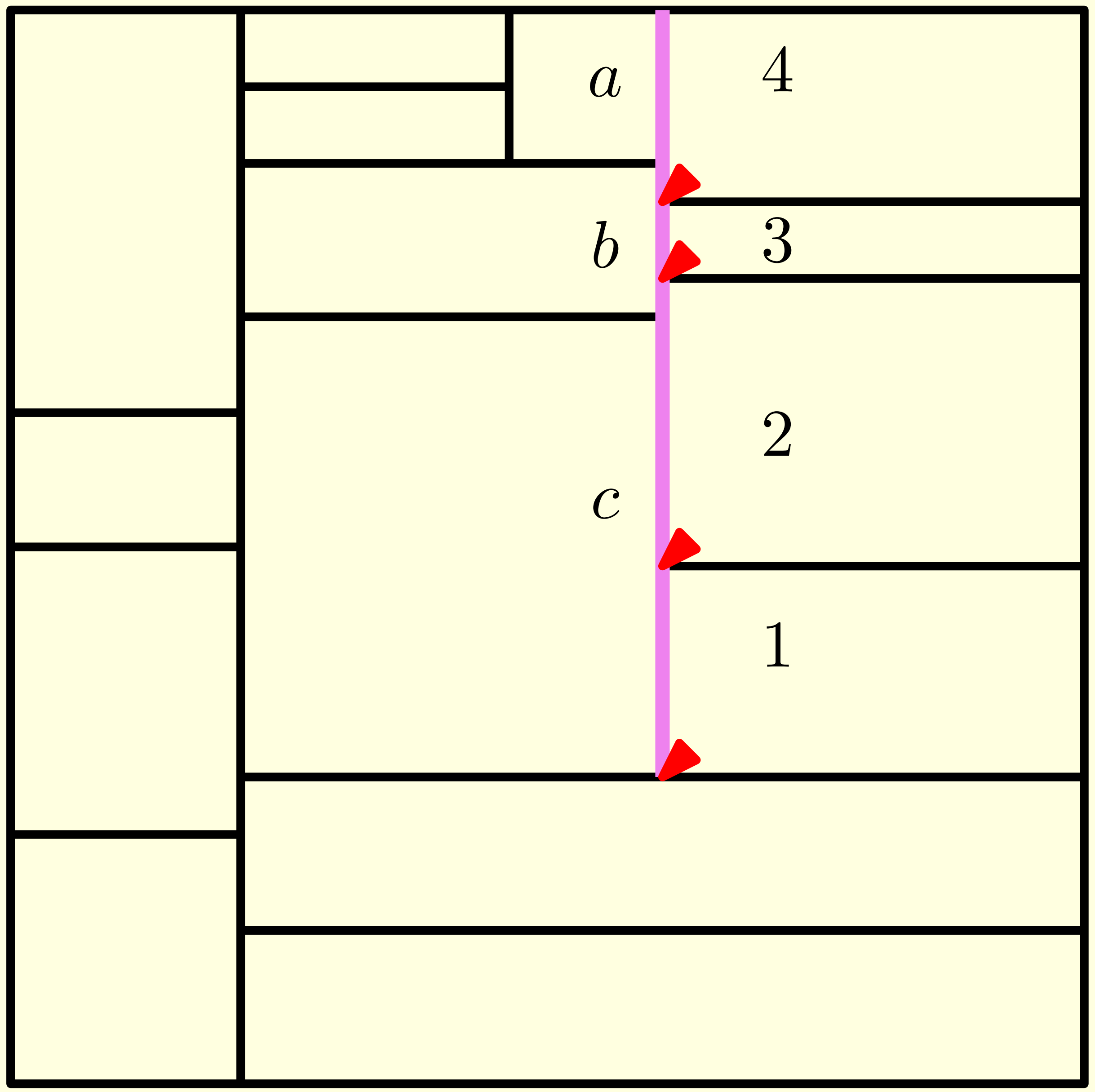

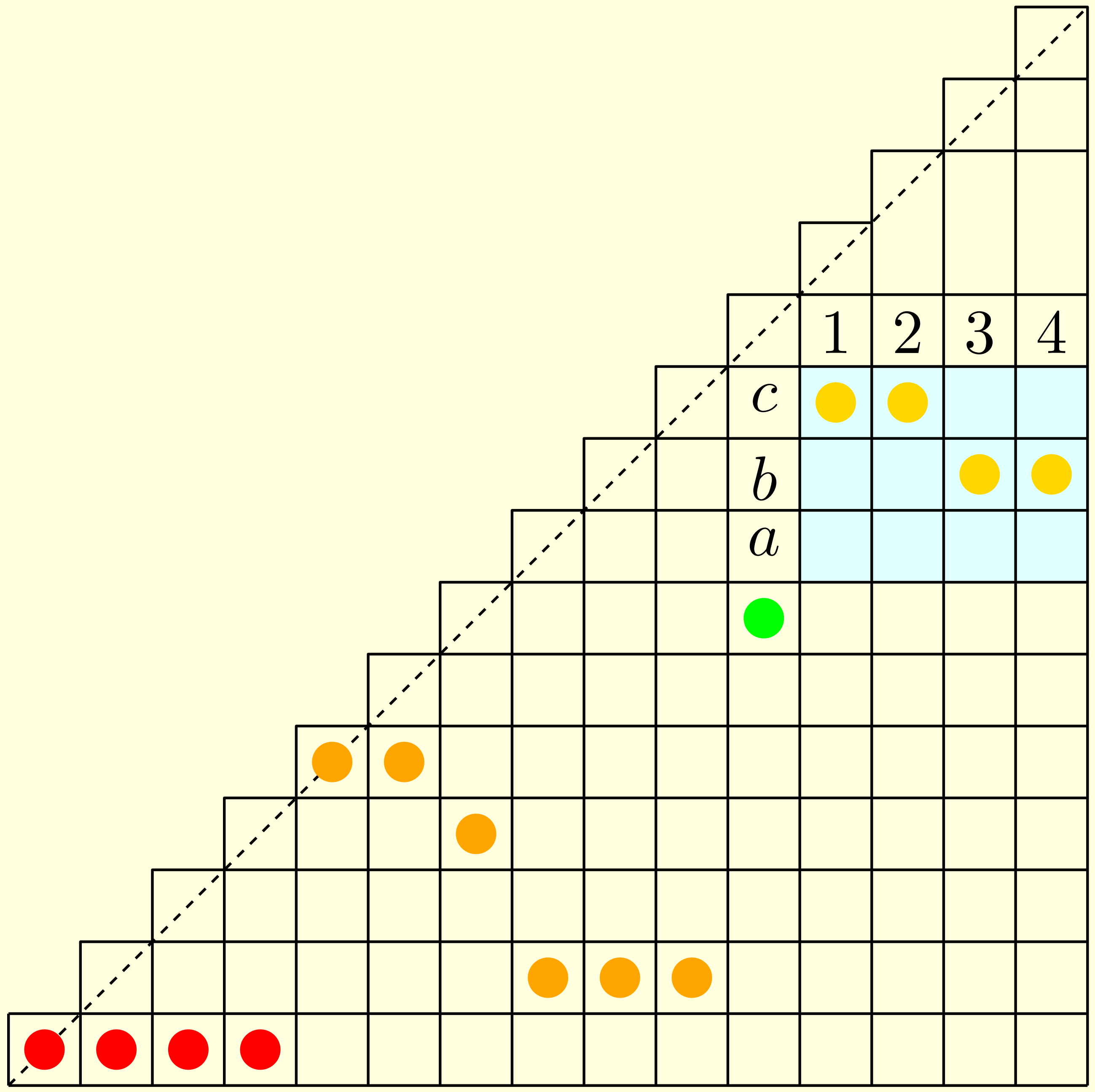

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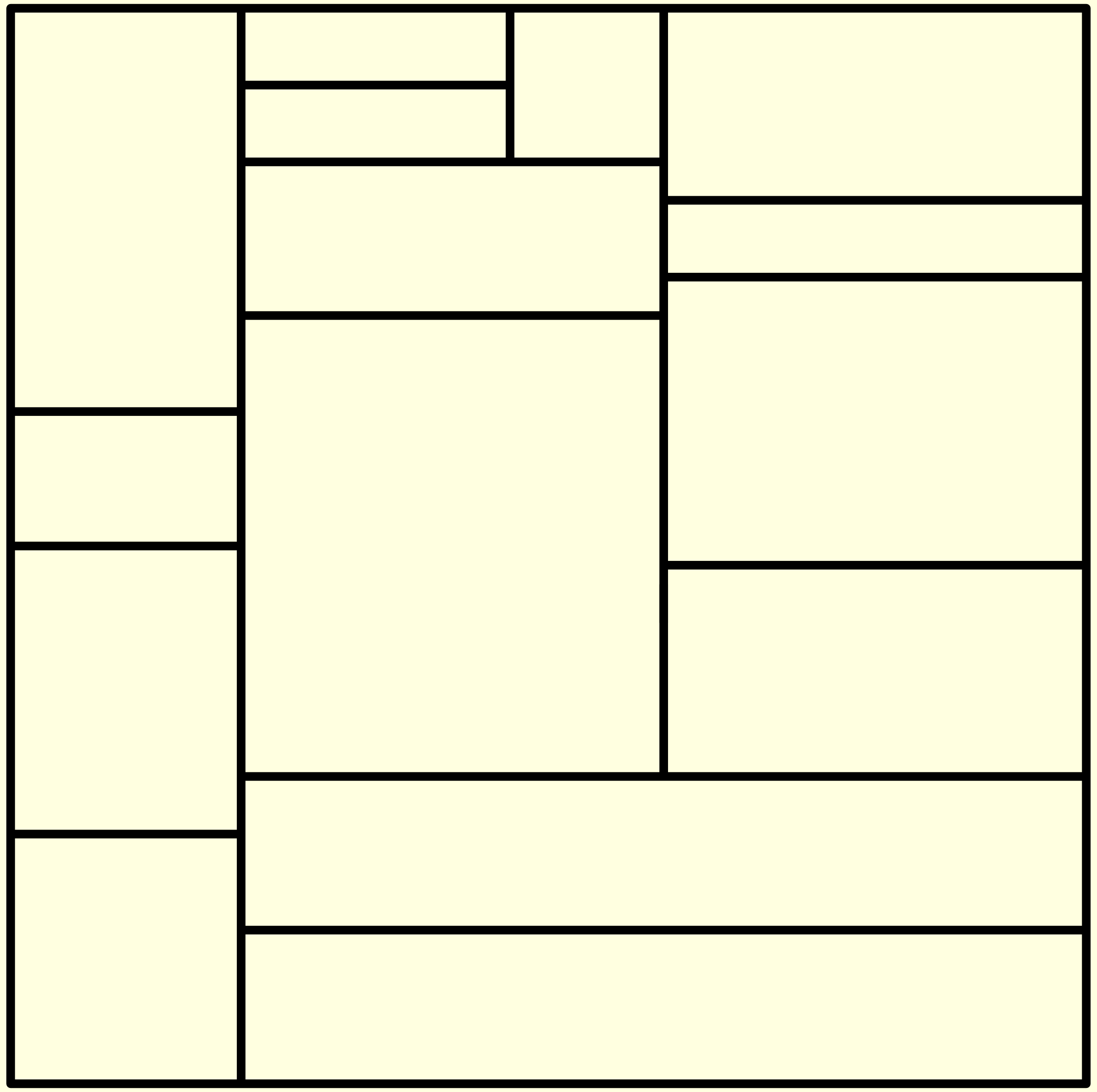
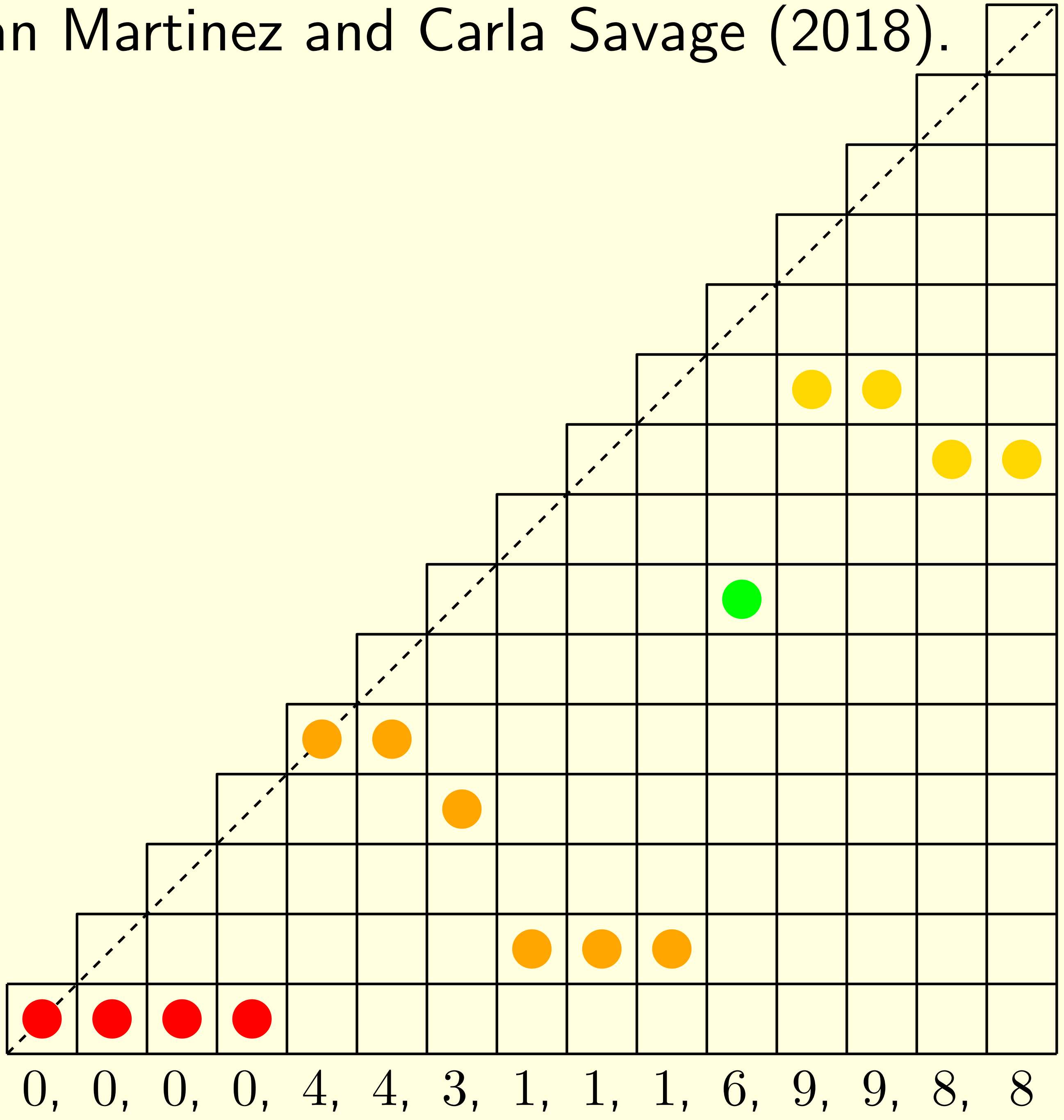
First geometric interpretation of sequence, sequence previously appeared in paper examining pattern avoidance in inversion sequences from Megan Martinez and Carla Savage (2018).





Summary

	Weak Equivalence	Strong Equivalence
$\top$	$\lvert R_n^w(\top)\rvert = C_n$	OEIS A279555
$\top, \bot$	$\lvert R_n^w(\top,\bot)\rvert = 2^{n-1}$	OEIS A287709
$\top, \vdash$	$\lvert R_n(\top,\vdash)\rvert = 2^{n-1}$	
$\top, \bot, \vdash$	$\lvert R_n(\top,\bot,\vdash)\rvert = n$	
$\top, \bot, \vdash, \dashv$	$\lvert R_n(\top,\bot,\vdash,\dashv)\rvert = 2$	

Summary

<table>
<tr><th></th><th>Weak Equivalence</th><th>Strong Equivalence</th></tr>
<tr><td>$\top$</td><td>$|R_n^w(\top)| = C_n$</td><td>OEIS A279555</td></tr>
<tr><td>$\top, \perp$</td><td>$|R_n^w(\top, \perp)| = 2^{n-1}$</td><td>OEIS A287709</td></tr>
<tr><td>$\top, \vdash$</td><td colspan="2">$|R_n(\top, \vdash)| = 2^{n-1}$</td></tr>
<tr><td>$\top, \perp, \vdash$</td><td colspan="2">$|R_n(\top, \perp, \vdash)| = n$</td></tr>
<tr><td>$\top, \perp, \vdash, \dashv$</td><td colspan="2">$|R_n(\top, \perp, \vdash, \dashv)| = 2$</td></tr>
</table>

THANK YOU!