Please enter your name and uID below.

Name:

uID:

Collaborators, if any, and how you collaborated:

Submission notes

- Due at 11:59 pm (midnight) on Thursday, Dec 1st.
- Solutions must be typeset using one of the template files. For each problem, your answer must fit in the space provided (e.g. not spill onto the next page) *without* space-saving tricks like font/margin/line spacing changes.
- Upload a PDF version of your completed problem set to Gradescope.
- Teaching staff reserve the right to request original source/tex files during the grading process, so please retain these until an assignment has been returned.
- Please remember that for this problem set, you are allowed to collaborate in detail with your peers, as long as you cite them. However, you must write up your own solution, alone, from memory. If you do collaborate with other students in this way, you must identify the students and describe the nature of the collaboration. You are not allowed to create a group solution, and all work that you hand in must be written in your own words. Do not base your solution on any other written solution, regardless of the source.

1. (PARTITION in P: 40pts)

   The PARTITION problem[1] is: Given a set S of n integers, can it be split into two subsets with equal sums? That is, are there subsets A and B such that $A \cup B = S$, $A \cap B = \emptyset$, and $\sum_{a \in A} a = \sum_{b \in B} b$?

   For example, for $S = \{1, 2, 3\}$, then the answer is YES, because $A = \{1, 2\}$ and $B = \{3\}$ give a partition. However, if $S = \{1, 2, 3, 100\}$, the answer is NO.

   (a) Describe and analyze an algorithm to solve PARTITION in time $O(nM)$, where n is the size of the input set and M is the sum of the absolute values of its elements. *Hint: Use an algorithm design strategy we've learned earlier this class.*

   (b) Why doesn't this algorithm imply that $P = NP$?

[1] This is also defined on p405 of the textbook.

2. (6PARTITION$^+$: 60pts)

The 6PARTITION$^+$problem is very similar to the 3PARTITION problem defined on p405 of the textbook, except for the subsets must have size 6, and the $^+$ indicates that duplicate elements are allowed.[2]

6PARTITION$^+$ problem is defined as: given a multiset[3] S of $6n$ integers, can it be partitioned into n 6-element multisets, such that each subset[4] has exactly the same sum?

a. Prove that 6PARTITION$^+$ is in NP by describing a certificate of a YES-instance, describing an algorithm for verifying a certificate, and showing that the verification algorithm takes polynomial time (these descriptions can be brief).

b. We would like to prove that 6PARTITION$^+$ is NP-hard by reducing 3PARTITION to it. Consider the following reduction idea: given some arbitrary set S of $3n$ integers, construct a new S' by making 2 copies of every element of S. That is:

$$\text{if } S = \{0, 1, 2, 3, 4, 6\}, \text{ then } S' = \{0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 6, 6\}$$

We would like to say that $S \in$ 3PARTITION if and only if $S' \in$ 6PARTITION$^+$, but this reduction idea *will not work*, and is not a valid way to prove that 6PARTITION$^+$ is NP-hard. Explain, in detail, why this reduction will not work.

c. Consider instead the following reduction idea. Given some arbitrary set S of $3n$ integers, define a very large number $T = 100 * \max_{s \in S} |s|$, and construct a new S' that has all elements of S, plus $3n$ copies of T. That is:

$$\text{if } S = \{0, 1, 2, 3, 4, 6\}, \text{ then } S' = \{0, 1, 2, 3, 4, 6, T, T, T, T, T, T\}$$

This reduction idea *does* work. Show that 6PARTITION$^+$ is NP-hard by proving that $S \in$ 3PARTITION if and only if $S' \in$ 6PARTITION$^+$.

[2] This is less similar to PARTITION, which asks about dividing some set into two subsets that can have any size. In 6PARTITION$^+$, every subset must have size exactly 6.

[3] *Multiset* is a set where duplicate elements are allowed.

[4] Technically, sub*multi*set