

# Reinforcement Learning

## An Introductory Note

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## 6 Model-Free Prediction

To begin, we firstly give definition of the term *episode* then give the prediction algorithm based MC and TD methods.

**Episodes:** An episode  $\tau$  is a sequence of states and actions in the environment,

$$\tau = (s_0, a_0, r_1, \dots, s_{T-1}, a_{T-1}, r_T).$$

We call an episode is complete if it ends with the terminal state.

In model free method, we only utilize the episode itself without further exploitation.

### 6.1 Monte-Carlo Policy Evaluation

Monte-Carlo (MC) policy evaluation methods learn directly from the *complete* episodes, thus it needs no knowledge of MDP transitions or rewards. The limitation of MC policy method methods is that it requires MDPs are episodic.

**Monte-Carlo Policy Evaluation:** Given some complete episodes under the policy  $\pi$ , we can approximate the value of  $s$  by the average returns observed after visiting to  $s$ .

Depending on when average returns for state  $s$  in an episode, there are two different implementations.

**First-Visit Monte-Carlo Policy Evaluation:** Only at the first time-step  $t$  that state  $s$  is visited in an episode, we do the following procedure:

- Increment counter  $N(s) \leftarrow N(s) + 1$ ;
- Increment the total return  $return(s) \leftarrow return(s) + G_t$ ;
- Update the value by the mean return  $v(s) \leftarrow return(s)/N(s)$ .

**Every-Visit Monte-Carlo Policy Evaluation:** Every time-step  $t$  that state  $s$  is visited in an episode, we do the following procedure:

- Increment counter  $N(s) \leftarrow N(s) + 1$ ;
- Increment the total return  $return(s) \leftarrow return(s) + G_t$ ;
- Update the value by the mean return  $v(s) \leftarrow return(s)/N(s)$ .

**Algorithm 10** First-Visit Monte-Carlo Policy Evaluation

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```

1: initialize  $v(s) \in \mathbb{R}$  arbitrarily for all  $s \in \mathcal{S}$ ;
2: initialize  $return(s) \leftarrow$  an empty list for all  $s \in \mathcal{S}$ ;
3: input the policy  $\pi$  to be evaluated;
4: for true do:
    # variants for this alg. can be start with  $s_0 \in \mathcal{S}, a_0 \in \mathcal{A}(s_0)$  randomly,  $\varepsilon$ -greedy, etc.
5:   Generate a complete episode  $\tau = (s_0, a_0, r_1, \dots, s_{T-1}, a_{T-1}, r_T)$ ;
6:    $G \leftarrow 0$ ;
7:   for  $t = T - 1, T - 2, \dots, 0$  do:
8:      $G \leftarrow \gamma G + r_{t+1}$ ;
9:     if  $s_t$  appears in  $(s_0, s_1, \dots, s_{t-1})$  then:
10:      Append  $G$  to  $return(s_t)$ ;
11:       $v(s_t) \leftarrow \text{average}(return(s_t))$ ;
12:     end if
13:   end for
14: end for

```

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**Algorithm 11** Every-Visit Monte-Carlo Policy Evaluation

---

```

1: input the policy  $\pi$  to be evaluated;
2: initialize  $v(s) \in \mathbb{R}$  arbitrarily for all  $s \in \mathcal{S}$ ;
3: initialize  $return(s) \leftarrow$  an empty list for all  $s \in \mathcal{S}$ ;
4: for true do:
5:   Generate a complete episode  $\tau = (s_0, a_0, r_1, \dots, s_{T-1}, a_{T-1}, r_T)$ ;
6:   for  $t = T - 1, T - 2, \dots, 0$  do:
7:      $G \leftarrow \gamma G + r_{t+1}$ ;
8:     Append  $G$  to  $return(s_t)$ ;
9:      $v(s_t) \leftarrow \text{average}(return(s_t))$ ;
10:  end for
11: end for

```

---

By law of large numbers, both of two methods can achieve  $v(s) \rightarrow v^\pi(s)$  as  $N(s) \rightarrow \infty$ . In practice, we can use a trick *incremental mean* to simplify the calculation.

Differences between DP and MC for policy evaluation:

- DP computes  $v_k$  by bootstrapping the rest of the expected return calculated with  $v_{k-1}$ ;
- DP iterates on Bellman expectation backup:

$$v_k(s) \leftarrow \sum_{a \in \mathcal{A}} \pi(a|s) \left( R_s^a + \gamma \sum_{s' \in \mathcal{S}} P(s'|s, a) v_{k-1}(s') \right).$$

- MC updates the empirical mean return with a sampled episode:

$$v(s_t) \leftarrow v(s_t) + \alpha(G_{k,t} - v(s_t)).$$

Advantages of MC over DP:

- MC can work when the environment is unknown;
- Working with sampled episodes has a huge advantage. Even with the complete knowledge of the environment's dynamics, the complexity still could be a challenge;
- Cost of estimating a single state's value is independent of the total number of states. So one can sample episodes starting from the states of interest then average returns.

## 6.2 Temporal-Difference Learning

Unlike MC methods, temporal-difference (TD) does not require the episodes are complete. TD methods can learn from incomplete episodes by bootstrapping.

**Temporal-Difference Learning:** *Given some incomplete episodes under the policy  $\pi$ , we update value  $v(s_t)$  toward estimated return  $r_{t+1} + \gamma v(s_{t+1})$ :*

$$v(s_t) \leftarrow v(s_t) + \alpha(r_{t+1} + \gamma v(s_{t+1}) - v(s_t)),$$

where  $r_{t+1} + \gamma v(s_{t+1})$  is called the TD target,  $\alpha$  is the step-size, and we call

$$\delta_t = r_{t+1} + \gamma v(s_{t+1}) - v(s_t)$$

the TD error.

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### Algorithm 12 TD(0) Evaluation

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```

1: initialize  $v(s) \in \mathbb{R}$  arbitrarily for all  $s \in \mathcal{S}$  except that  $v(\text{terminal}) = 0$ ;
2: input the policy  $\pi$  to be evaluated; the step size  $\alpha \in (0, 1]$ ;
3: for true do:
4:   Generate initial state  $s$ ;
5:   while  $s$  is not terminal do:
6:      $a \leftarrow \pi(s)$ ;
7:      $r, s' \leftarrow \text{environment}(s, a)$ ;
8:      $v(s) \leftarrow v(s) + \alpha(r + \gamma v(s') - v(s))$ ;
9:      $s \leftarrow s'$ ;
10:  end while
11: end for

```

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Differences between MC and TD for policy evaluation:

- TD can learn online after every step;
- MC must wait until end of episode before return is known;
- TD can learn from incomplete sequences;
- MC can only learn from complete sequences;
- TD works in continuing (non-terminating) environments;
- MC only works in episodic (terminating) environments;
- TD exploits Markov property, and it is efficient in Markov environments;
- MC does not exploit Markov property, thus it is relatively effective in non-Markov environments.

*Bootstrapping*: involves old values, it is something like in-place update.

*Sampling*: samples to get an expectation.

A summary of *bootstrapping* and *sampling* for DP, MC, and TD is shown in Table 2.

Algorithm	Bootstraps	Sampling
Dynamic Programming	✓	
Monte-Carlo		✓
Temporal-Difference	✓	✓

Table 2: *Bootstrapping* and *sampling* for DP, MC, and TD.

**n-step TD methods:** *Unlike the simplest TD method, in n-step TD methods, the updating rule of value  $v(s_t)$  is*

$$v(s_t) \leftarrow v(s_t) + \alpha(G_t^{(n)} - v(s_t)),$$

where  $G_t^{(n)}$  is the  $n$ -step return

$$G_t^{(n)} = r_{t+1} + \gamma r_{t+2} + \dots + \gamma^n v(s_{t+n}).$$

Notice that with additional definition, we can generalize TD to MC when  $n \rightarrow \infty$ .

**TD( $\lambda$ ) methods:** *To make use of the information from all time-steps, we can use weight  $(1 - \lambda)\lambda^{n-1}$  to average  $n$ -step returns over different  $n$  as*

$$G_t^\lambda = (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)},$$

thus the updating rule for the value becomes

$$v(s_t) \leftarrow v(s_t) + \alpha(G_t^\lambda - v(s_t)).$$