3.2 Hopf bifurcation

See exercises 8.2.14-15 in Strogatz

For each of the following two-dimensional flows

$$\dot{x} = \mu x - 5y - x^3$$
$$\dot{y} = 5x + \mu y + 3y^3$$

and

$$\dot{x} = \mu x + y - x^2$$
$$\dot{y} = -x + \mu y + 2x^2$$

a Hopf bifurcation occurs at the origin for $\mu = 0$.

A system with these properties can, at the bifurcation, be brought into the form

$$\dot{x} = -\omega y + f(x, y)$$
$$\dot{y} = \omega x + g(x, y)$$

by a suitable change of coordinates. The functions f and g contain only higher-order (non-linear) terms that vanish at the origin.

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a) What is ω for the two systems (1) and (2), respectively? Write your answer as the vector [ω(1),ω(2)].

Rearrange the equations to:

$$\dot{x} = -5\,y - \mu\,x - x^3$$
$$\dot{y} = 5\,x + \mu\,y + 3\,y^3$$

and

$$\dot{x} = y + \mu\,x - x^2$$
$$\dot{y} = -x + \mu\,y + 2\,y^2$$

Now it's evident that $[\omega(1) = 5,\ \omega(2) = -1]$

b) Determine f and g for the systems (1) and (2). Write your solution as the matrix [[f(1),g(1)],[f(2),g(2)]].

From a) one can easily see that the solution is $[[f(1) = -x^3,\ g(1) = 3\,y^3],\ [f(2) = -x^2,\ g(2) = 2\,x^2]]$.

It can be shown that whether the bifurcation is subcritical or supercritical depends solely on the sign of the quantity a defined by

$$16\,a = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}$$
$$+\frac{1}{\omega}\left[f_{xy}\,(f_{xx} + f_{yy}) - g_{xy}\,(g_{xx} + g_{yy}) - f_{xx}\,g_{xx} + f_{yy}\,g_{yy}\right]$$

where the subscripts denote partial derivatives evaluated at the fixed point (0,0). According to this

criterion, the bifurcation is supercritical if $a < 0$ and subcritical if $a > 0$.

c) Determine a for the two systems (1) and (2). Write you solution as the vector [a(1),a(2)].

Using this result you should be able to determine what kind of bifurcations the systems (1) and (2) undergo at μ = 0.

$a > 0 \to$ subcritical

$a < 0 \to$ supercritical

In[32]:=
```
f1[x_,y_] := -x^3
g1[x_,y_] := 3y^3

firstTerm = D[f1[x,y],{x,3}]+D[f1[x,y],x,y,y]+D[g1[x,y],x,x,y]+D[g1[x,y],y,y,y];
secondTerm = 1/5*(D[f1[x,y],x,y]*(D[f1[x,y],x,x]+D[f1[x,y],y,y])-D[g1[x,y],x,y]*(D[g1[
a = (firstTerm+secondTerm)/16
```

Out[36]=

$$\frac{3}{4}$$

From this calculation one sees that $a = \frac{3}{4} > 0$ for the first system thus the bifurcation is a subcritical one.

In[60]:=
```
ClearAll["Global'*"]

f2[x_,y_] := -x^2
g2[x_,y_] := 2x^2

firstTerm = D[f2[x,y],{x,3}]+D[f2[x,y],x,y,y]+D[g2[x,y],x,x,y]+D[g2[x,y],y,y,y];
secondTerm = 1/(-1)*(D[f2[x,y],x,y]*(D[f2[x,y],x,x]+D[f2[x,y],y,y])-D[g2[x,y],x,y]*(D[
a = (firstTerm+secondTerm)/16
```

Out[65]=

$$-\frac{1}{2}$$

From this calculation one sees that $a = -\frac{1}{2} < 0$ for the first system thus the bifurcation is a supercritical one.

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d) Draw phase portraits of the global dynamics for positive and negative μ for each of the systems (1) and

(2). Make sure that these phase portraits verify the criteria you found in subtask c). Use a numerical solver, for example NDSolve[] in Mathematica (using StreamPlot[] will not give enough resolution for this task).

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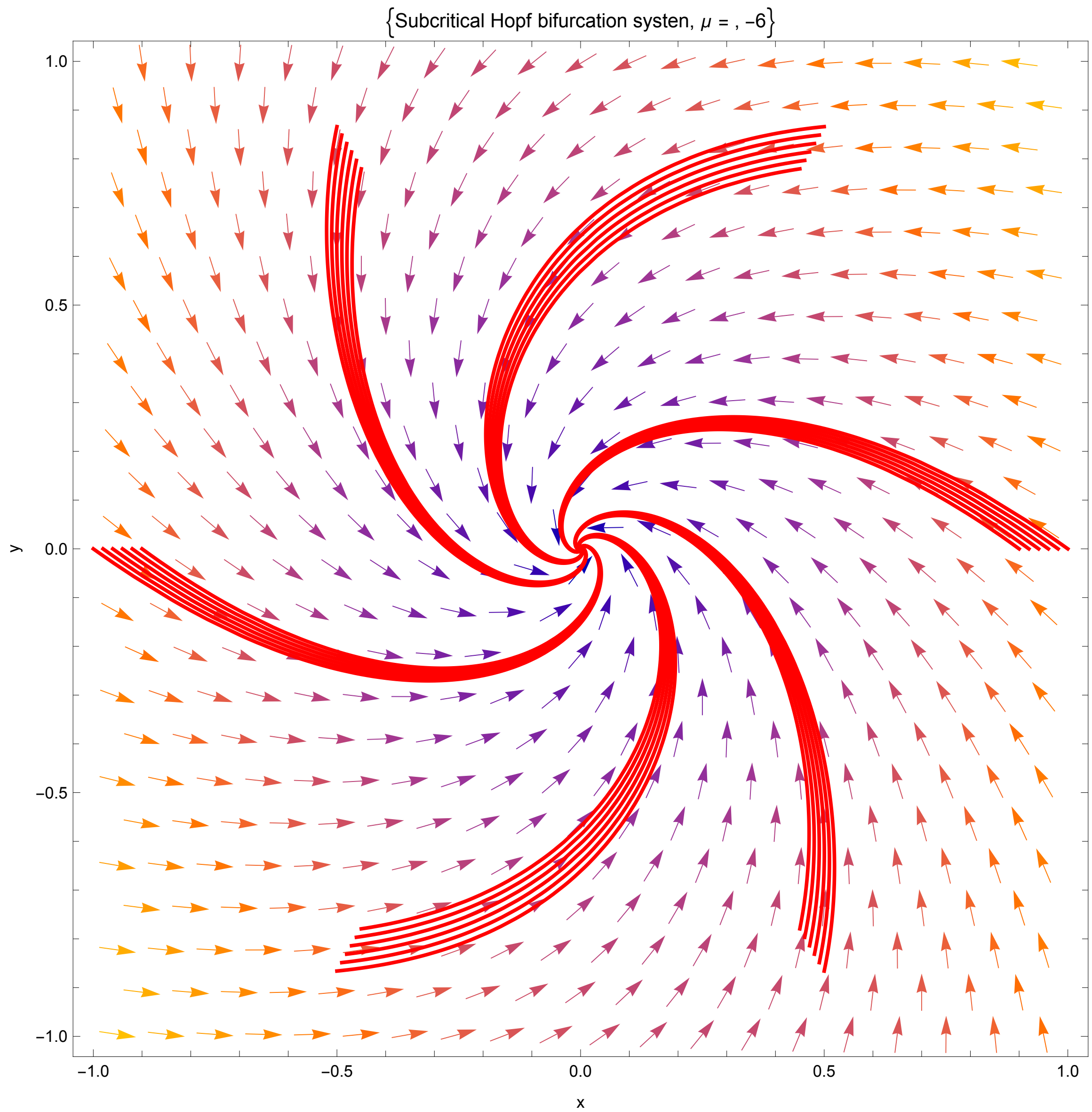
In[1118]:=

```
ClearAll["Global`*"]

u = -6;

Equation1 = x'[t] == -5*y[t] + u*x[t] - x[t]^3;
Equation2 = y'[t] == 5*x[t] + u*y[t] + 3*y[t]^3;

System1 = {Equation1, Equation2};

FixPoint1 = {0, 0};

t0 = 0;
tMax = 5;

(* Create 10 circles with different radii on which the initialPoints lie *)
nrOfInitialPoints = 6;
radiusMin = 0.9;
radiusMax = 1.0;

initialPoints = Table[
   {x[0] == FixPoint1[[1]] + radius*Cos[2 Pi k/nrOfInitialPoints],
    y[0] == FixPoint1[[2]] + radius*Sin[2 Pi k/nrOfInitialPoints]},
   {radius, Range[radiusMin, radiusMax, (radiusMax - radiusMin)/(nrOfInitialPoints - 1
   {k, nrOfInitialPoints}
];

(* Flatten the list to get a single list of initial points *)
initialPoints = Flatten[initialPoints, 1];

(* Use NDSolve for each initial point and store the solutions in a list *)
solutions = Table[NDSolve[{System1, initialPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Len

(* Plot the trajectories for each solution with arrows indicating the direction *)
TrajectoryPlotArrows = Table[
   ParametricPlot[Evaluate[{x[t], y[t]} /. solution], {t, t0, tMax}, PlotStyle → Red],
   {solution, solutions}
];

(* Add arrows indicating the direction using VectorPlot *)
VectorPlotArrows = VectorPlot[{-5*y+u*x -x^3,5*x+u*y+3*y^3}, {x, -1, 1}, {y, -1, 1},
   VectorPoints → 20, VectorScale → 0.03, VectorStyle → Directive[Black]];

(* Show the combined plot *)
Show[VectorPlotArrows, TrajectoryPlotArrows, FrameLabel → {"x", "y"},
  PlotLabel → {"Subcritical Hopf bifurcation systen, μ = ", u}, ImageSize → 600]
```

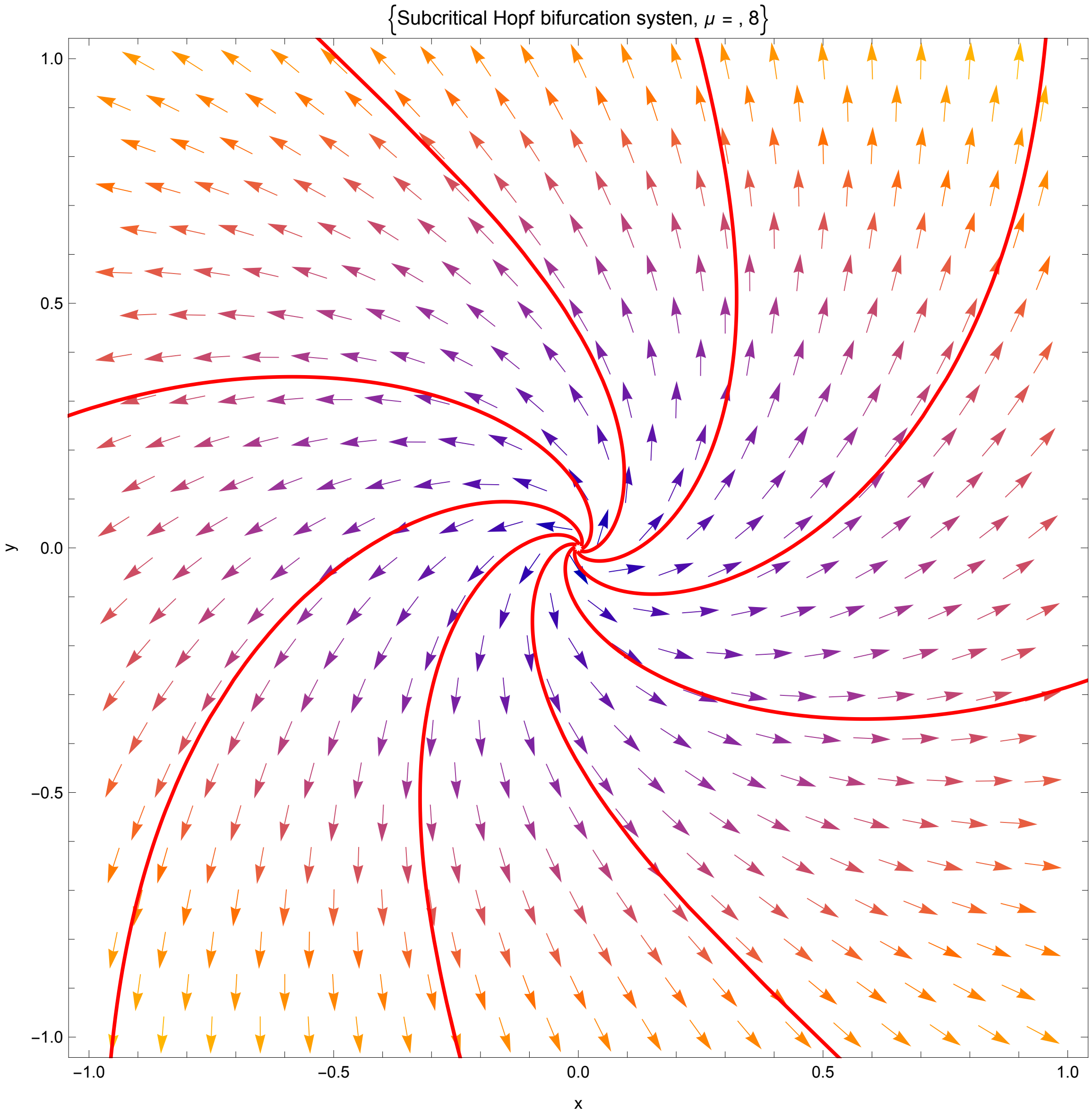
Out[1134]=


{Subcritical Hopf bifurcation systen, μ = , −6}
1.0
0.5
0.0
−0.5
−1.0
y
−1.0
−0.5
0.0
0.5
1.0
x

In[1084]:=

```
ClearAll["Global`*"]

u = 8;

Equation1 = x'[t] == -5*y[t] + u*x[t] - x[t]^3;
Equation2 = y'[t] == 5*x[t] + u*y[t] + 3*y[t]^3;

System1 = {Equation1, Equation2};

FixPoint1 = {0, 0};

t0 = 0;
tMax = 5;

(* Create 10 circles with different radii on which the initialPoints lie *)
nrOfInitialPoints = 8;
radiusMin = 0.01;
radiusMax = 0.01;

initialPoints = Table[
   {x[0] == FixPoint1[[1]] + radius*Cos[2 Pi k/nrOfInitialPoints],
    y[0] == FixPoint1[[2]] + radius*Sin[2 Pi k/nrOfInitialPoints]},
   {radius, Range[radiusMin, radiusMax, (radiusMax - radiusMin)/(nrOfInitialPoints - 1
   {k, nrOfInitialPoints}
];

(* Flatten the list to get a single list of initial points *)
initialPoints = Flatten[initialPoints, 1];

(* Use NDSolve for each initial point and store the solutions in a list *)
solutions = Table[NDSolve[{System1, initialPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Len

(* Plot the trajectories for each solution with arrows indicating the direction *)
TrajectoryPlotArrows = Table[
   ParametricPlot[Evaluate[{x[t], y[t]} /. solution], {t, t0, tMax}, PlotStyle → Red],
   {solution, solutions}
];

(* Add arrows indicating the direction using VectorPlot *)
VectorPlotArrows = VectorPlot[{-5*y+u*x -x^3,5*x+u*y+3*y^3}, {x, -1, 1}, {y, -1, 1},
   VectorPoints → 20, VectorScale → 0.03, VectorStyle → Directive[Black]];

Show[VectorPlotArrows, TrajectoryPlotArrows, FrameLabel → {"x", "y"},
  PlotLabel → {"Subcritical Hopf bifurcation systen, μ = ", u}, ImageSize → 600]
```

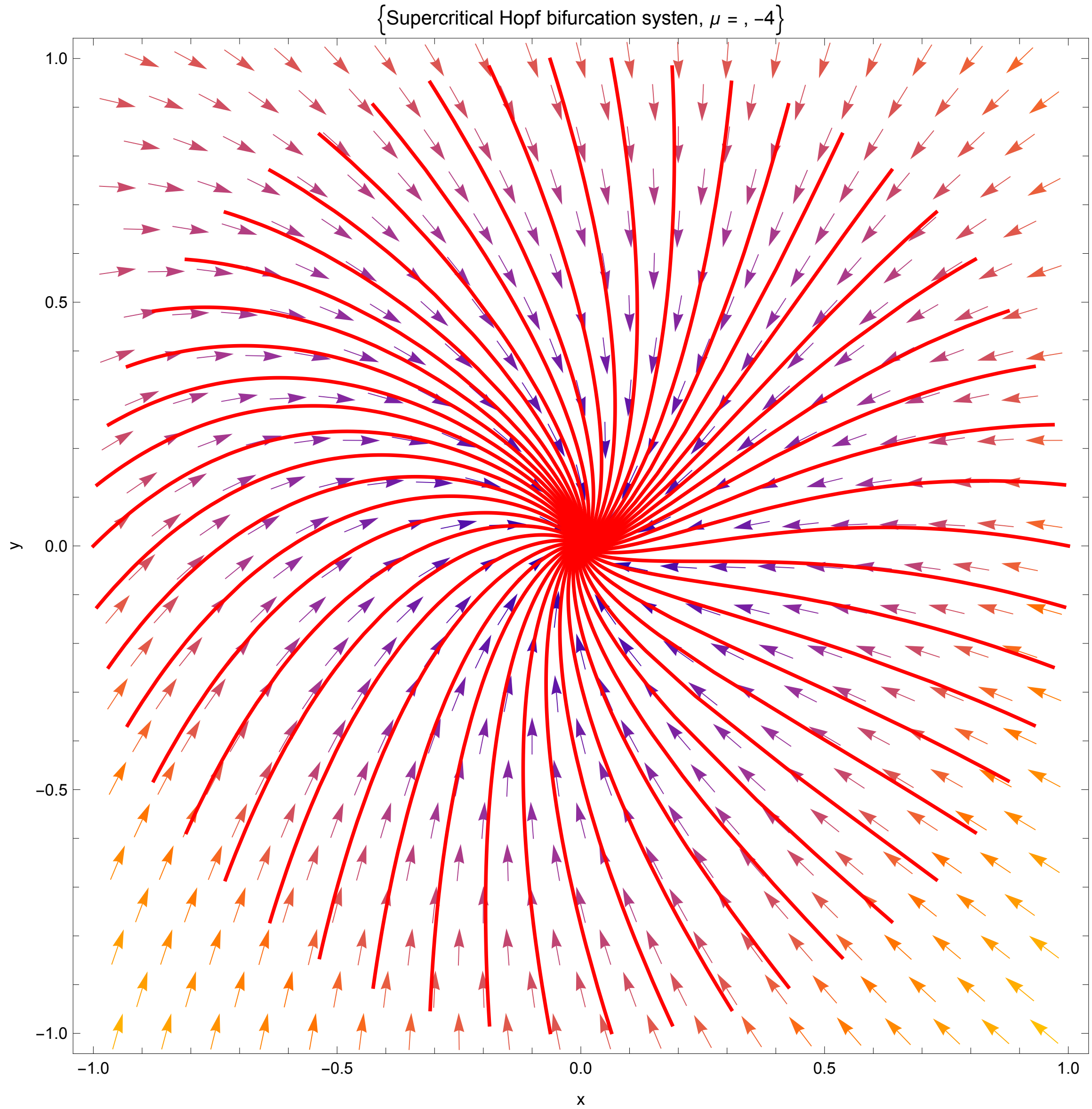
Out[1100]=


{Subcritical Hopf bifurcation systen, μ = , 8}
1.0
0.5
0.0
−0.5
−1.0
y
−1.0
−0.5
0.0
0.5
1.0
x

In[1135]:=

```
ClearAll["Global`*"]

u = -4;

Equation1 = x'[t] == y[t] + u*x[t] - x[t]^2;
Equation2 = y'[t] == -x[t] + u*y[t] + 2*x[t]^2;

System1 = {Equation1, Equation2};

FixPoint1 = {0, 0};

t0 = 0;
tMax = 5;

(* Create 10 circles with different radii on which the initialPoints lie *)
nrOfInitialPoints = 50;
radiusMin = 1.0;
radiusMax = 1.0;

initialPoints = Table[
   {x[0] == FixPoint1[[1]] + radius*Cos[2 Pi k/nrOfInitialPoints],
    y[0] == FixPoint1[[2]] + radius*Sin[2 Pi k/nrOfInitialPoints]},
   {radius, Range[radiusMin, radiusMax, (radiusMax - radiusMin)/(nrOfInitialPoints - 1
   {k, nrOfInitialPoints}
];

(* Flatten the list to get a single list of initial points *)
initialPoints = Flatten[initialPoints, 1];

(* Use NDSolve for each initial point and store the solutions in a list *)
solutions = Table[NDSolve[{System1, initialPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Len

(* Plot the trajectories for each solution with arrows indicating the direction *)
TrajectoryPlotArrows = Table[
   ParametricPlot[Evaluate[{x[t], y[t]} /. solution], {t, t0, tMax}, PlotStyle → Red],
   {solution, solutions}
];

(* Add arrows indicating the direction using VectorPlot *)
VectorPlotArrows = VectorPlot[{y+u*x -x^2,-x+u*y+2*x^2}, {x, -1, 1}, {y, -1, 1},
   VectorPoints → 20, VectorScale → 0.03, VectorStyle → Directive[Black]];

(* Show the combined plot *)
Show[VectorPlotArrows, TrajectoryPlotArrows, FrameLabel → {"x", "y"},
  PlotLabel → {"Supercritical Hopf bifurcation systen, μ = ", u}, ImageSize → 600]
```

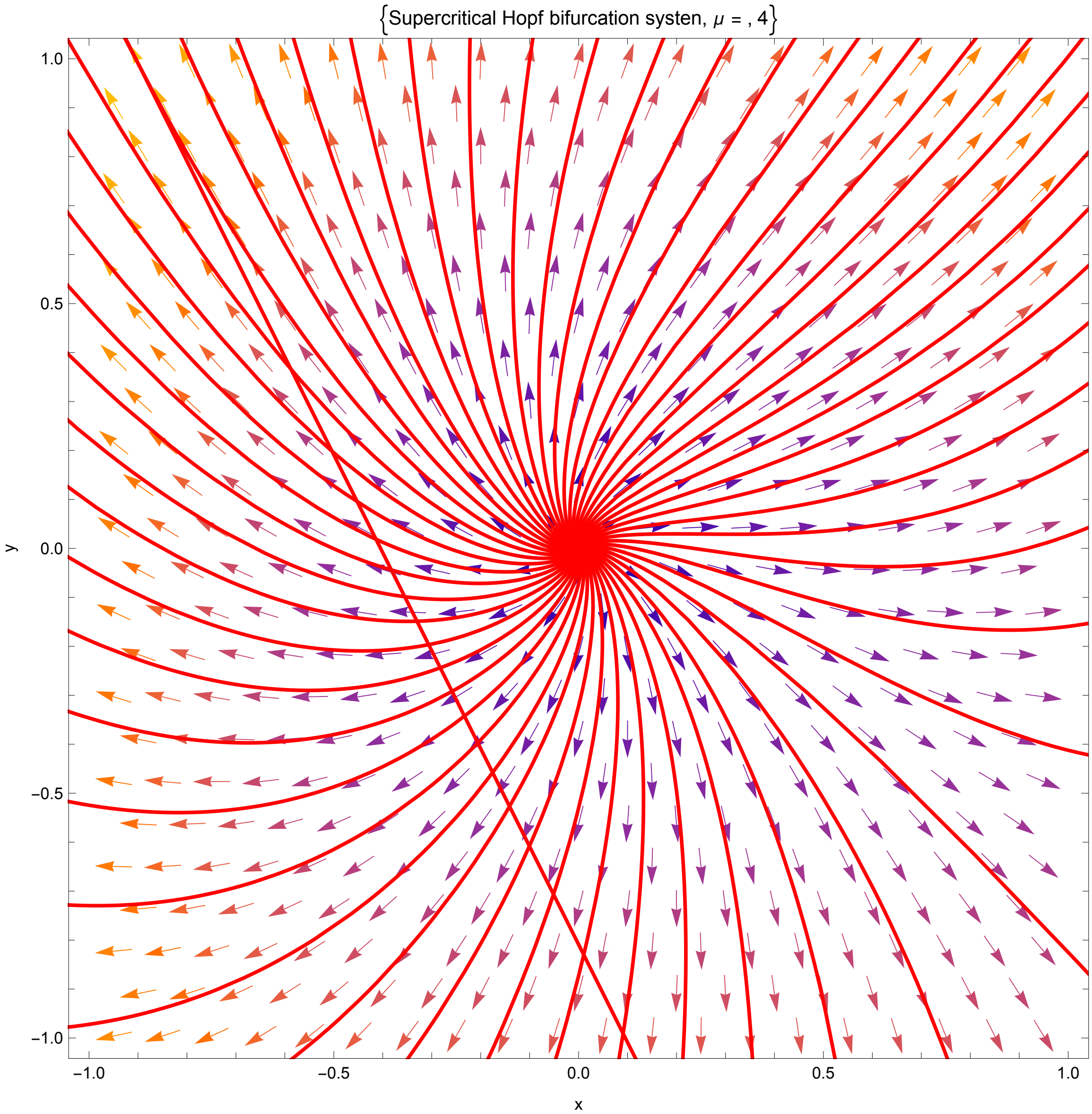
Out[1151]=


{Supercritical Hopf bifurcation systen, μ = , −4}
1.0
0.5
0.0
−0.5
−1.0
y
−1.0
−0.5
0.0
0.5
1.0
x

In[1152]:=

```
ClearAll["Global`*"]

u = 4;

Equation1 = x'[t] == y[t] + u*x[t] - x[t]^2;
Equation2 = y'[t] == -x[t] + u*y[t] + 2*x[t]^2;

System1 = {Equation1, Equation2};

FixPoint1 = {0, 0};

t0 = 0;
tMax = 5;

(* Create 10 circles with different radii on which the initialPoints lie *)
nrOfInitialPoints = 50;
radiusMin = 0.001;
radiusMax = 0.001;

initialPoints = Table[
   {x[0] == FixPoint1[[1]] + radius*Cos[2 Pi k/nrOfInitialPoints],
    y[0] == FixPoint1[[2]] + radius*Sin[2 Pi k/nrOfInitialPoints]},
   {radius, Range[radiusMin, radiusMax, (radiusMax - radiusMin)/(nrOfInitialPoints - 1
   {k, nrOfInitialPoints}
];

(* Flatten the list to get a single list of initial points *)
initialPoints = Flatten[initialPoints, 1];

(* Use NDSolve for each initial point and store the solutions in a list *)
solutions = Table[NDSolve[{System1, initialPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Len

(* Plot the trajectories for each solution with arrows indicating the direction *)
TrajectoryPlotArrows = Table[
   ParametricPlot[Evaluate[{x[t], y[t]} /. solution], {t, t0, tMax}, PlotStyle → Red],
   {solution, solutions}
];

(* Add arrows indicating the direction using VectorPlot *)
VectorPlotArrows = VectorPlot[{y+u*x -x^2,-x+u*y+2*x^2}, {x, -1, 1}, {y, -1, 1},
   VectorPoints → 20, VectorScale → 0.03, VectorStyle → Directive[Black]];

Show[VectorPlotArrows, TrajectoryPlotArrows, FrameLabel → {"x", "y"},
  PlotLabel → {"Supercritical Hopf bifurcation systen, μ = ", u}, ImageSize → 600]
```

Out[1168]=


{Supercritical Hopf bifurcation systen, μ = , 4}
1.0
0.5
0.0
−0.5
−1.0
y
−1.0
−0.5
0.0
0.5
1.0
x