4.1 Introduction to the Lorenz model

The three-dimensional Lorenz flow is given by

$$\dot{x} = \sigma\,(y - x)$$
$$\dot{y} = r\,x - y - x\,z$$
$$\dot{z} = x\,y - b\,z$$

The Lorenz system is named after the meteorologist Edward Norton Lorenz who studied it extensively. He found that the system (1) exhibits a fractal attractor for the parameter values $\sigma = 10$, $b = 8/3$ and $r = 28$. This attractor is nowadays called Lorenz attractor.

a) How many fixed points does the Lorenz system have, and how many of them are stable for the parameter values given above? Give your answer as the vector [number of fixed points, number of stable fixed points].

To find the fixed points start with checking the trivial solution, which happens to be fixed point 1 = (0,0,0). From the first equation one sees that $x = y$, using this in the second equation one finds $z = r - 1$ and applying this information to the third equation leads to $x = \pm\sqrt{b(r-1)}$. Thus fixed point 2 = $\left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$ and fixed point 3 = $\left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$. To check the stability of the fixed points, the Jacobian can be calculated and then evaluated at the fixed points.

The Jacobian is $[[-\sigma, \sigma, 0], [r - z, -1, -x], [y, x, -b]]$.

By applying $\sigma = 10$, $b = 8/3$ and $r = 28$ to the Jacobian and evaluating the eigenvalues one can insert the (x, y, z) of the fixed points and then analysing the stability.

<u>Fixed point 1</u>: The eigenvalues are $\lambda_1 = -22.828$, $\lambda_2 = -2.667$ and $\lambda_3 = 11.828$.

Fixed point 2: The eigenvalues are $\lambda_1 = -13.855$, $\lambda_2 = 0.094 - 10.195 * i$ and $\lambda_3 = 0.094 + 10.195 * i$.
Fixed point 3: The eigenvalues are $\lambda_1 = 8.973$ $\lambda_2 = -11.319 - 5.687 * i$ and $\lambda_3 = -11.319 + 5.687 * i$

In two dimensions it would suffice analyse the fixed points and their eigenvalues, which is done according to the following rules.

Complex Eigenvalues:
Positive real part: The fixed point is unstable and the system behaves like an unstable oscillator a.k.a unstable spiral.
Zero real part: The fixed point is stable and the system behaves like an undamped oscillator a.k.a center.
Negative real part: The fixed point is stable and the system behaves like a damped oscillator a.k.a stable spiral.
Real Eigenvalues:
Zero Eigenvalues: System is unstable (This is just a trivial case of the complex eigenvalue that has a zero part).
Positive distinct Eigenvalues: Unstable fixed point.
Negative distinct Eigenvalues: Stable fixed point.
Positive and negative Eigenvalues: Saddle node (unstable).

Even though it seems confusing at first that there are more than two eigenvalues it still holds that the fixed points must be unstable. This interpretation is called for since in each one of the evaluated set of eigenvalues there is always one that is positive, which lets the fixed point be unstable.

In[ ]:=
```
ClearAll["Global`*"]

(*Fixed points for the system*)
FixedPoint1 = {0,0,0};
FixedPoint2 = {Sqrt[b(r-1)],Sqrt[b(r-1)],r-1};
FixedPoint2 = {-Sqrt[b(r-1)],-Sqrt[b(r-1)],r-1};

f1 = s(y-x);
f2 = r*x-y-x*z;
f3 = x*y-b*z;

Jacobi = D[{f1,f2,f3},{{x,y,z}}];



(*Parameters*)
s = 10;
b = 8/3;
bC =N[(s(s+4)/(s-2))];
r = 28;
rHopf=N[s*((s+b+3)/(s-b-1))];

(*x=0;
y=0;
z=0;*)
x = Sqrt[b(r-1)];
y = Sqrt[b(r-1)];
z = r-1;
Eig = N[Eigenvalues[Jacobi]]
```

Out[ ]=
```
{-13.8546, 0.0939556 + 10.1945 i, 0.0939556 - 10.1945 i}
```

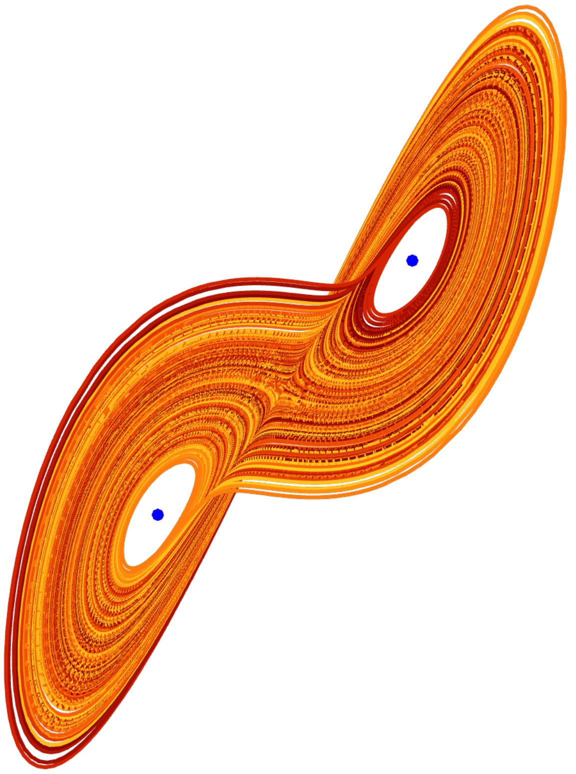
b) Solve the equations (1) numerically using the parameters stated above for some initial condition close to the origin. Plot an approximation of Lorenz attractor obtained by discarding the initial part of the solution.

Upload your figure as .pdf or .png.

In[ ]:=
```
ClearAll["Global`*"]

(* Lorenz variables *)
s = 16;
r = 330;   (*At a little smaller than r= 24.058 there plot changes completely, calcula
b = 5;

(* Differential equation system *)
Equation1 = x'[t] == s*(y[t] - x[t]);
Equation2 = y'[t] == r*x[t] - y[t] - x[t]*z[t];
Equation3 = z'[t] == x[t]*y[t] - b*z[t];
System = {Equation1, Equation2, Equation3};

(* Fixed points of the differential equation system *)
FixedPoint1 = {0, 0, 0};
FixedPoint2 = {Sqrt[b (r - 1)], Sqrt[b (r - 1)], r - 1};
FixedPoint3 = {-Sqrt[b (r - 1)], -Sqrt[b (r - 1)], r - 1};

(* Parameters for the plot *)
eta = 0.0001;
t0 = 0;
tMax = 1000;
tZeroPlot = 20;

(* Starting point of the trajectory (distance from fixed points handled via eta) *)
StartingPoint = {x[0] == FixedPoint1⟦1⟧ + eta, y[0] == FixedPoint1⟦2⟧ - eta, z[0] == Fix
(*StartingPoint = {x[0] == + eta, y[0] == + eta, z[0] == - eta};*)

Solution = NDSolve[{System, StartingPoint}, {x, y, z}, {t, t0, tMax},MaxSteps → ∞];

(* Plotting *)
plot = ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. Solution], {t, tZeroPlot, 400},
PlotPoints → 1000,
PlotLabel → Style["(x,z)-plot", FontSize → 18],
PlotStyle → Directive[Thick, RGBColor[.8, 0, 0]],
ColorFunction → (ColorData["SolarColors", #4] &),
PlotRange → All, RotationAction → "Clip", Boxed → False, ViewPoint → {0,Infinity,0},

(*For exercise b*)
fixedPoints = Graphics3D[{Blue, PointSize[0.02], Point[{FixedPoint2, FixedPoint3}]}];
Show[plot, fixedPoints]


(*Toying around*)
(*fixedPoints = Graphics3D[{Blue, PointSize[0.02], Point[{FixedPoint1,FixedPoint2,Fixe
fixedPoints1 = Graphics3D[{Blue, PointSize[0.02], Point[{0,0,0}]}];
Show[plot,fixedPoints]*)
```

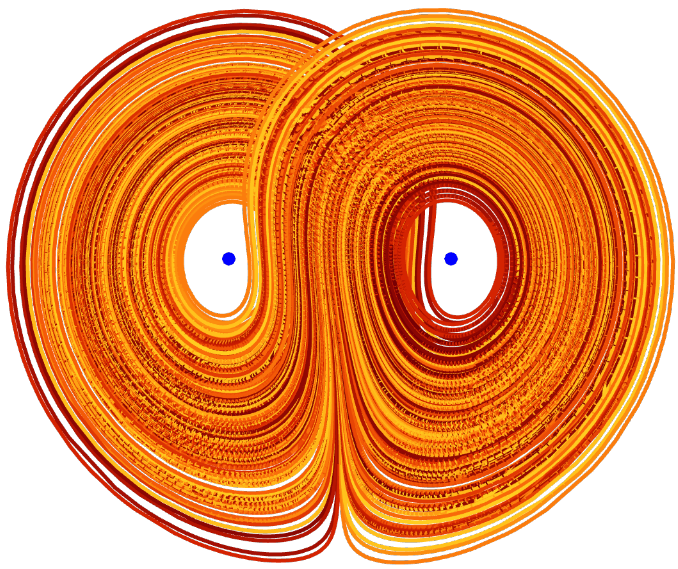
Out[ ]=

(x,y)–plot



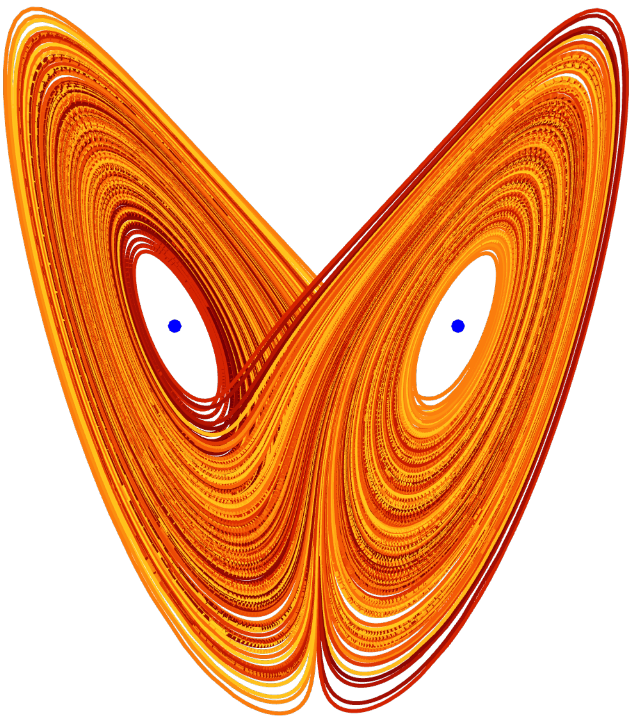
Out[ ]=

(y,z)–plot



Out[ ]=

(x,z)−plot



c) Compute the stability matrix $J_{ij} = \partial F_i / \partial x_j$ of the flow (1). Give your result as the matrix $[(J_{11}, J_{12}, J_{13}], [J_{21}, J_{22}, J_{23}], [J_{31}, J_{32}, J_{33}]]$.

In[ ]:=

```
ClearAll["Global`*"]

f1 = s(y-x);
f2 = r*x-y-x*z;
f3 = x*y-b*z;

Jacobi = D[{f1,f2,f3},{{x,y,z}}];

MatrixForm[Jacobi]

eigenvalues=Eigenvalues[Jacobi];
sum = eigenvalues[[1]]+eigenvalues[[2]]+eigenvalues[[3]]//Simplify
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -s & s & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

Out[ ]=

$-1 - b - s$

d) Confirm that the trace of the stability matrix, traceJ, is independent of the coordinates (x, y, z). From what you have learned in the lectures and read in the course book, you should now be able to compute the sum of Lyapunov exponents for the Lorenz system (and thus the Lorenz attractor). Give your result for $\lambda_1 + \lambda_2 + \lambda_3$ for general parameter values.

The trace of the stability matrix (a.k.a. Jacobian) is trace(J) $= -\sigma - 1 - b$ which shows that it is independent of the values (x, y, z) since they are parameters that do not depend on either of the coordinates. Then thank Stephen Wolfram that he invented Mathematica to easily calculate the sum of the three eigenvalues of the stability matrix, $\lambda_1 + \lambda_2 + \lambda_3 = -1 - b - \sigma$ (result from calculation in c).