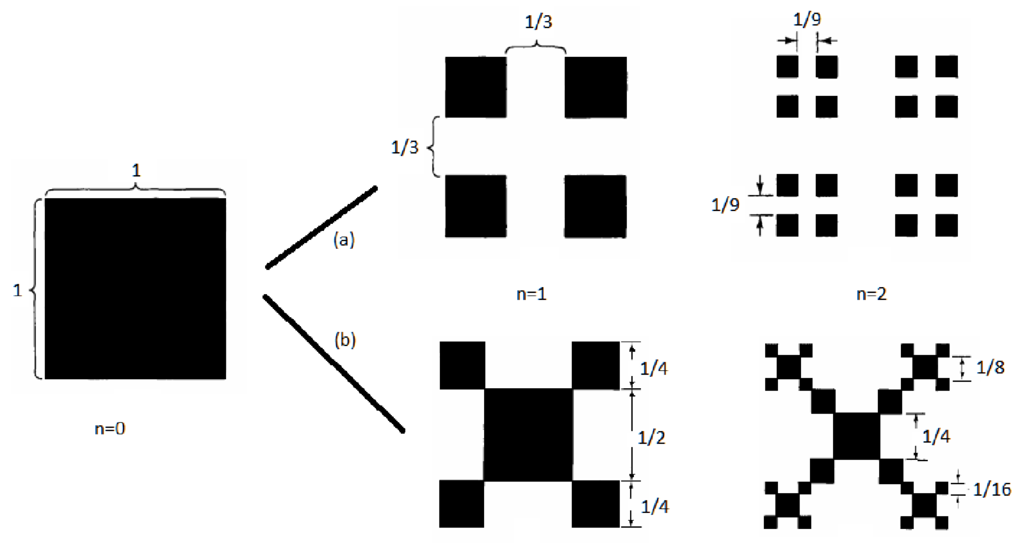
5.1 Box-counting dimension



The figure above shows two ways, (a) and (b), to evolve the unit square into two fractals sets (defined by iterating to generation n=∞). In each generation in (a), every square is replaced by four squares, with side lengths 1/3 of the original square. In each generation in (b), every square is replaced by five smaller squares, four with side lengths 1/4 and one with 1/2 of the original square.

(a) Analytically find an expression for the box-counting dimension of the fractal obtained by evolving the unit square according to way (a) in the figure above.

Fractals have a dimension between integers and fill the space there.
Point has dim = 0.
Line has dim = 1.
Area has dim = 2.

$N_{\text{boxes}} \sim A\,\epsilon^{-D_0}$ where D_0 is the dimension and $\epsilon :=$ size of the small boxes.

$n = 0$, 1 boxes, $\epsilon = 1$

$n = 1$, 4 boxes, $\epsilon = 1/3$

$n = 2$, 16 boxes, $\epsilon = 1/9$

...

$n = n$, 4^n boxes, $(1/3)^n$

$$\ln N_{\text{box}} = D_0 \ln(1/\epsilon)$$
$$D_0 = \frac{\ln(N_{\text{box}})}{\ln(1/\epsilon)} = \frac{\ln(4^n)}{\ln \frac{1}{(1/3)^n}}$$

(b) Analytically find an expression for the box-counting dimension of the fractal obtained by evolving the unit square according to way (b) in the figure above. Hint: The result is not $\frac{3}{2}$!

$N_{\text{box}} = 4\,N_a + 1\,N_b$ there is self similarity
$N_{\text{box}}^{(\epsilon)} = N_a\,(\epsilon/\epsilon_a) + N_b\,(\epsilon/\epsilon_b)$

$\epsilon_a = 1/4$, $\epsilon_b = 1/2$, then use this for $A\,\epsilon^{-D_0} = 4\,A\left(\frac{\epsilon}{\epsilon_a}\right)^{-D_0} + 1\,A\left(\frac{\epsilon}{\epsilon_b}\right)^{-D_0}$

$1 = 4\left(\frac{1}{\epsilon_a}\right)^{-D_0} + 1\left(\frac{1}{\epsilon_b}\right)^{-D_0}$ With $\frac{1}{2^{D_0}} = x$

$1 = 4\left(\frac{1}{2}\right)^{2\,D_0} + 1\left(\frac{1}{2}\right)^{D_0}$

$1 = 4\,x^2 + x$

In[236]:=

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sol1 = Solve[1 == 4 * (1/2)^(2 D) + (1/2)^D, D, Reals] // TraditionalForm // Simplify
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Out[236]//TraditionalForm=

$$\left\{\left\{D \to \frac{\log\left(1+\sqrt{17}\right)}{\log(2)} - 1\right\}\right\}$$