3.1 Index of a fixed point

Each of the following systems has a fixed point at $(x^\star, y^\star) = (0, 0)$. For each system, either draw a (rough) phase plot (for example using StreamPlot[]) and obtain the index, or evaluate the index using an analytical integral formula.

You do not need to prepare and upload your phase plots in the parts a)-d) of this exercise, it is enough to upload the code you used to solve the problem (or StreamPlots if that was how you obtained the index). Moreover, you will not receive any feedback on the correctness of your answer for this exercise from OpenTA.

a) Give the index for the fixed point of $\dot{x} = y - x$, $\dot{y} = x^2$.

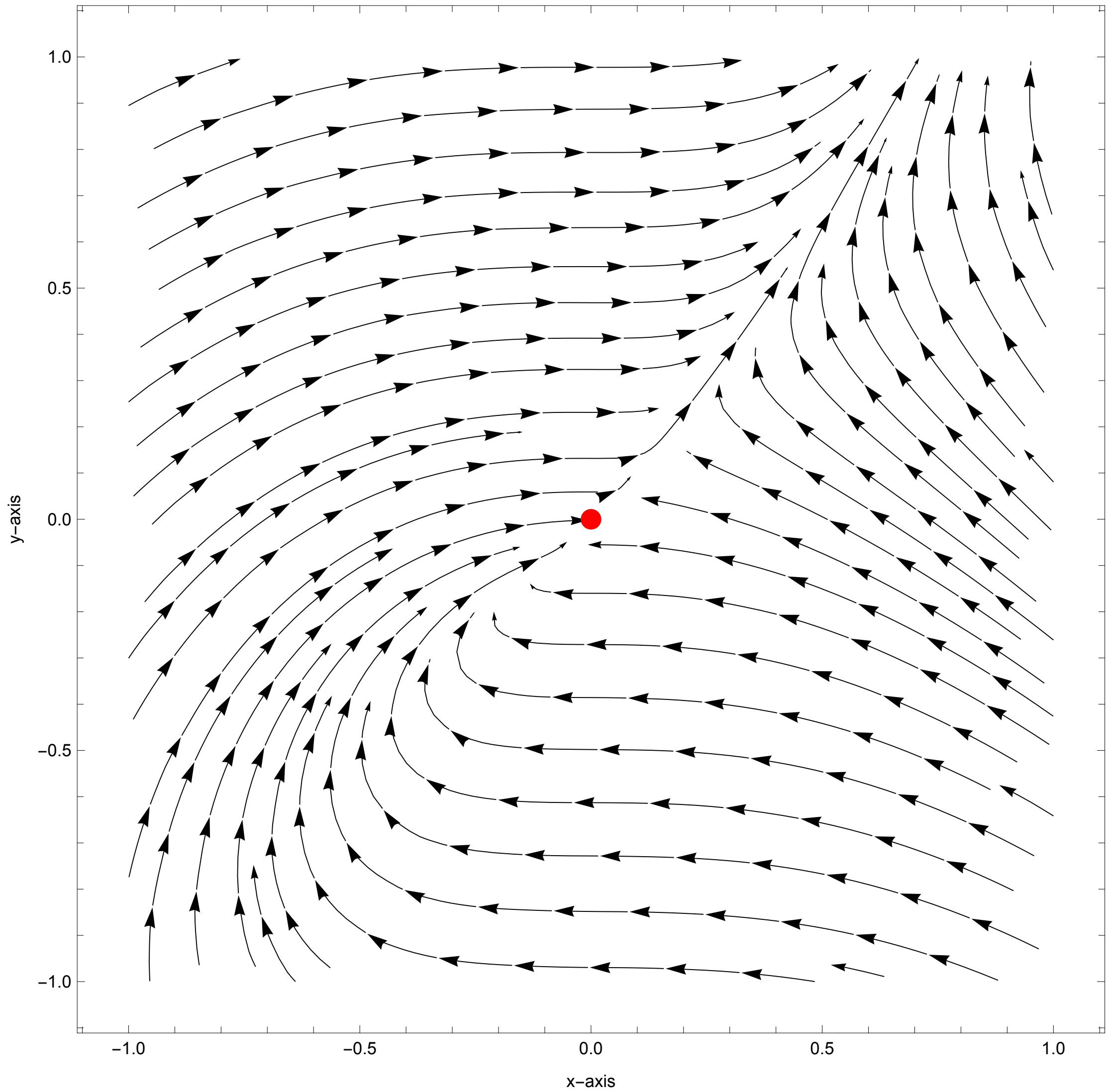
In[1169]:=

```
ClearAll["Global`*"]

equation1 = y-x;
equation2 = x^2;

StreamPlot[{equation1, equation2}, {x, -1, 1}, {y, -1, 1},
 PlotRange → All, ImageSize → Large, ColorFunction → (Black &),
 StreamStyle → Directive[Black], StreamColorFunction → None,
 Epilog → {Red, PointSize[0.02], Point[{0, 0}]},
 FrameLabel → {{"y-axis", None}, {"x-axis", None}}]
```

Out[1172]=



b) Give the index for the fixed point of the Cartesian system $\dot{x}$ and $\dot{y}$ corresponding to $\dot{r} = h(r)$, $\dot{\theta} = 0$, where r and θ are polar coordinates. Let $h(r)$ be a smooth function with $h(r) \sim a\,r + O(r^2)$ for small values of r and $a \neq 0$.

To solve this use the fact that in $h(r) \sim a\,r + O(r^2)$ the terms of $O(r^2)$ can be ignored. Thus we receive the following equation system.

$$\dot{r} = a\,r$$

$$\dot{\theta} = 0$$

Since we want to transform from polar to Cartesian coordinates we can use the $x = r\cos(\theta)$ and

$y = r\sin(\theta)$. Since both $r(t)$ and $\theta(t)$it is possible to derivate with respect to t. One then receives:

$$\dot{x} = \dot{r}\cos(\theta) - r\sin(\theta)\,\dot{\theta}$$
$$\dot{y} = \dot{r}\sin(\theta) + r\cos(\theta)\,\dot{\theta}$$

We know from above that $\dot{\theta} = 0$, thus the equations become

$$\dot{x} = \dot{r}\cos(\theta)$$
$$\dot{y} = \dot{r}\sin(\theta)$$

Now for small a and r, $\dot{r} = h(r) \sim a\,r$ thus substitute to get

$$\dot{x} = a\,r\cos(\theta)$$
$$\dot{y} = a\,r\sin(\theta)$$

And in the last step substitute r according to x = $r\cos(\theta)$ and $y = r\sin(\theta)$ to get the transformed system:

$$\dot{x} = a\,x$$
$$\dot{y} = a\,y$$

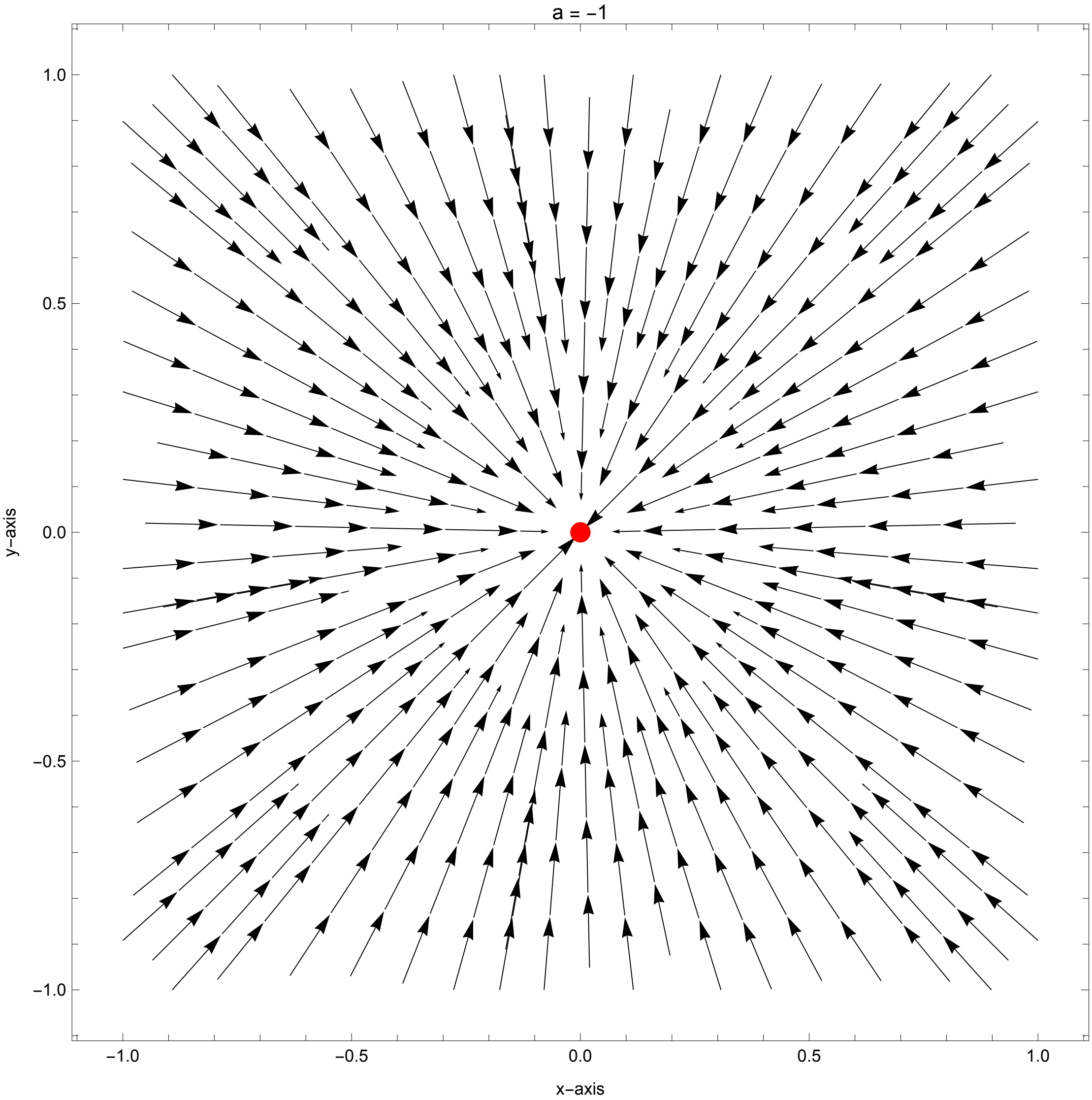
While plotting one has to remember that there are two cases to check, $a < 0$ and $a > 0$. In both cases the index is equal to one. When changing the sign of a has the equivalent effect as changing time and thus the flow changes direction.

In[1173]:=

```
ClearAll["Global`*"]

equation1 = a*x;
equation2 = a*y;
a = -1;
StreamPlot[{equation1, equation2}, {x, -1, 1}, {y, -1, 1},
 PlotRange → All, ImageSize → Large, ColorFunction → (Black &),
 StreamStyle → Directive[Black], StreamColorFunction → None,
 Epilog → {Red, PointSize[0.02], Point[{0, 0}]},
 FrameLabel → {{"y-axis", None}, {"x-axis", None}},
 PlotLabel → "a = -1"]
```

Out[1177]=

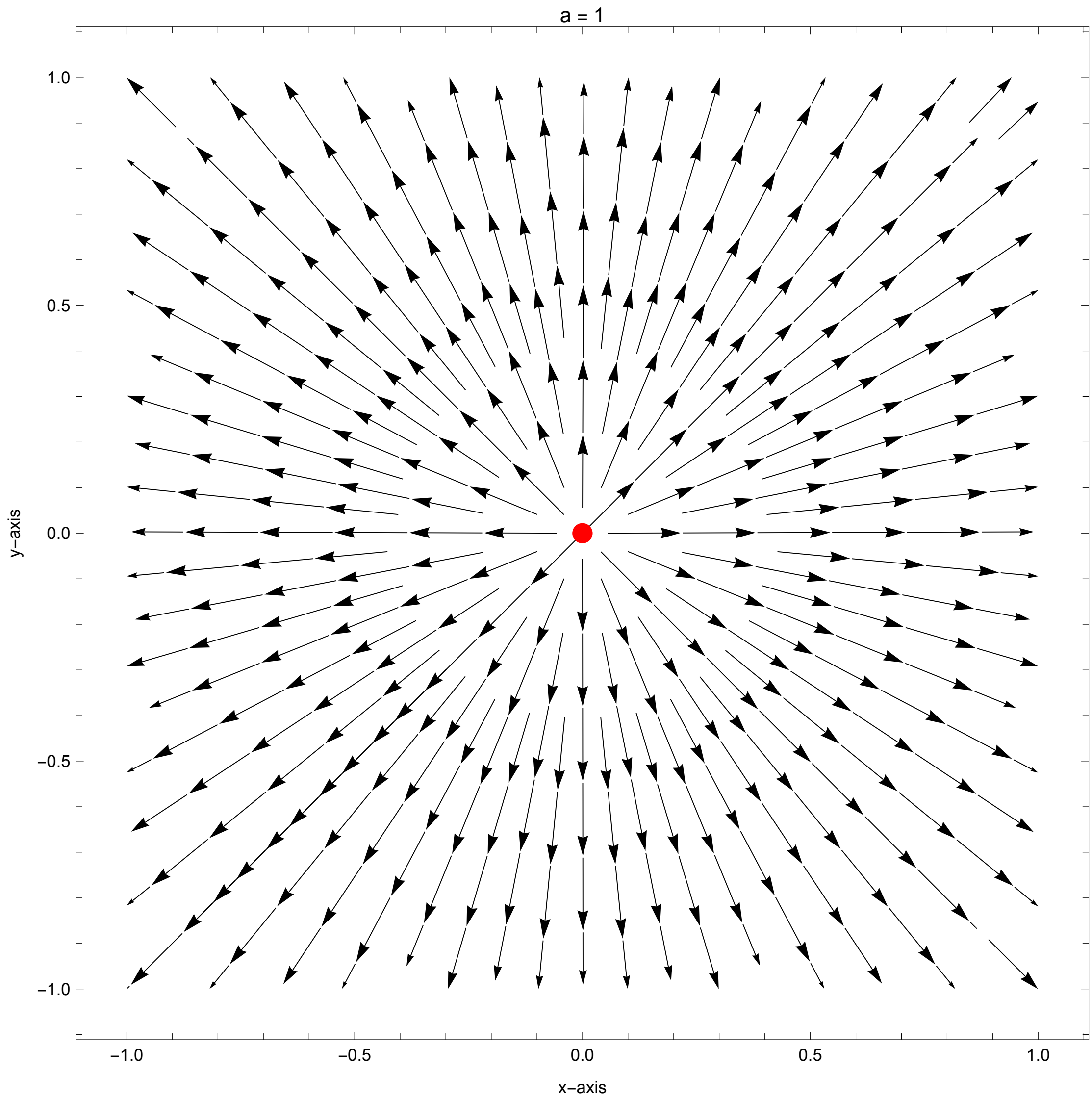


In[1178]:=

```
ClearAll["Global`*"]

equation1 = a*x;
equation2 = a*y;
a = 1;
StreamPlot[{equation1, equation2}, {x, -1, 1}, {y, -1, 1},
 PlotRange → All, ImageSize → Large, ColorFunction → (Black &),
 StreamStyle → Directive[Black], StreamColorFunction → None,
 Epilog → {Red, PointSize[0.02], Point[{0, 0}]},
 FrameLabel → {{"y-axis", None}, {"x-axis", None}},
 PlotLabel → "a = 1"]
```

Out[1182]=



c) Give the index for the fixed point of $\dot{x} = y^3$, $\dot{y} = x$.

As we can see from the plot this is a saddle point and the index is minus one.

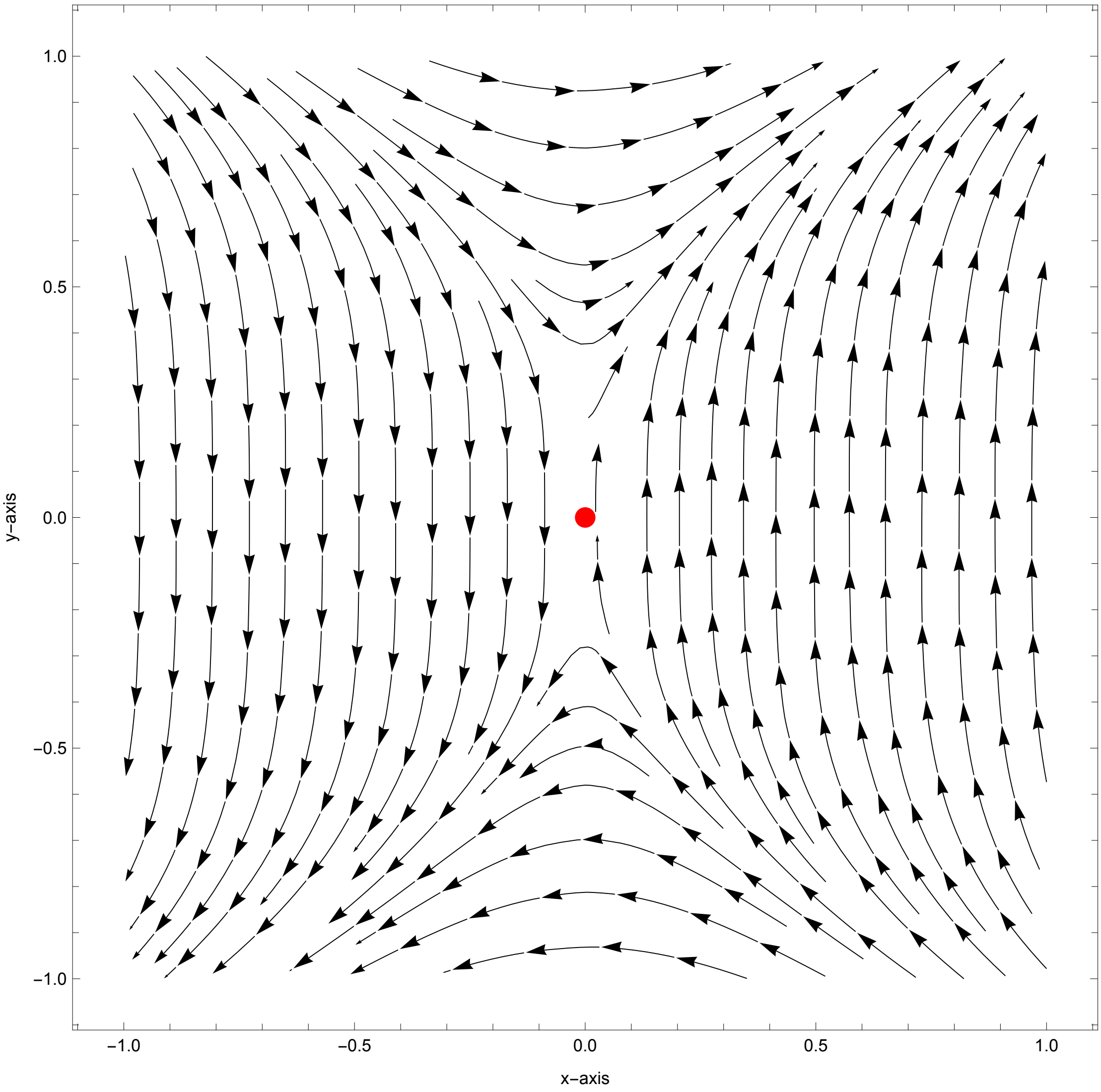
In[1183]:=

```
ClearAll["Global`*"]

equation1 = y^3;
equation2 = x;

StreamPlot[{equation1, equation2}, {x, -1, 1}, {y, -1, 1},
 PlotRange → All, ImageSize → Large, ColorFunction → (Black &),
 StreamStyle → Directive[Black], StreamColorFunction → None,
 Epilog → {Red, PointSize[0.02], Point[{0, 0}]},
 FrameLabel → {{"y-axis", None}, {"x-axis", None}}]
```

Out[1186]=



d) Give the index for the fixed point of $\dot{x} = (x^2 + y^2)^{|n|/2} \cos[n \arctan(y/x)]$, $\dot{y} = (x^2 + y^2)^{|n|/2} \sin[n \arctan(y/x)]$, where n is a non-zero

integer number and arctan(y/x) is evaluated on the suitable branch.

Given the two n dependent coordinates

$$\dot{x} = \left(x^2 + y^2\right)^{|n|/2} \cos\left(n \arctan\left(y/x\right)\right)$$

$$\dot{y} = \left(x^2 + y^2\right)^{|n|/2} \sin\left(n \arctan\left(y/x\right)\right)$$

originally three cases ought to be considered $n < 0$, $n = 0$ and $n > 0$. The case $n = 0$, is not necessary for this task, but still somewhat interesting.

While using interactive plots it is easy to check that the index is n.

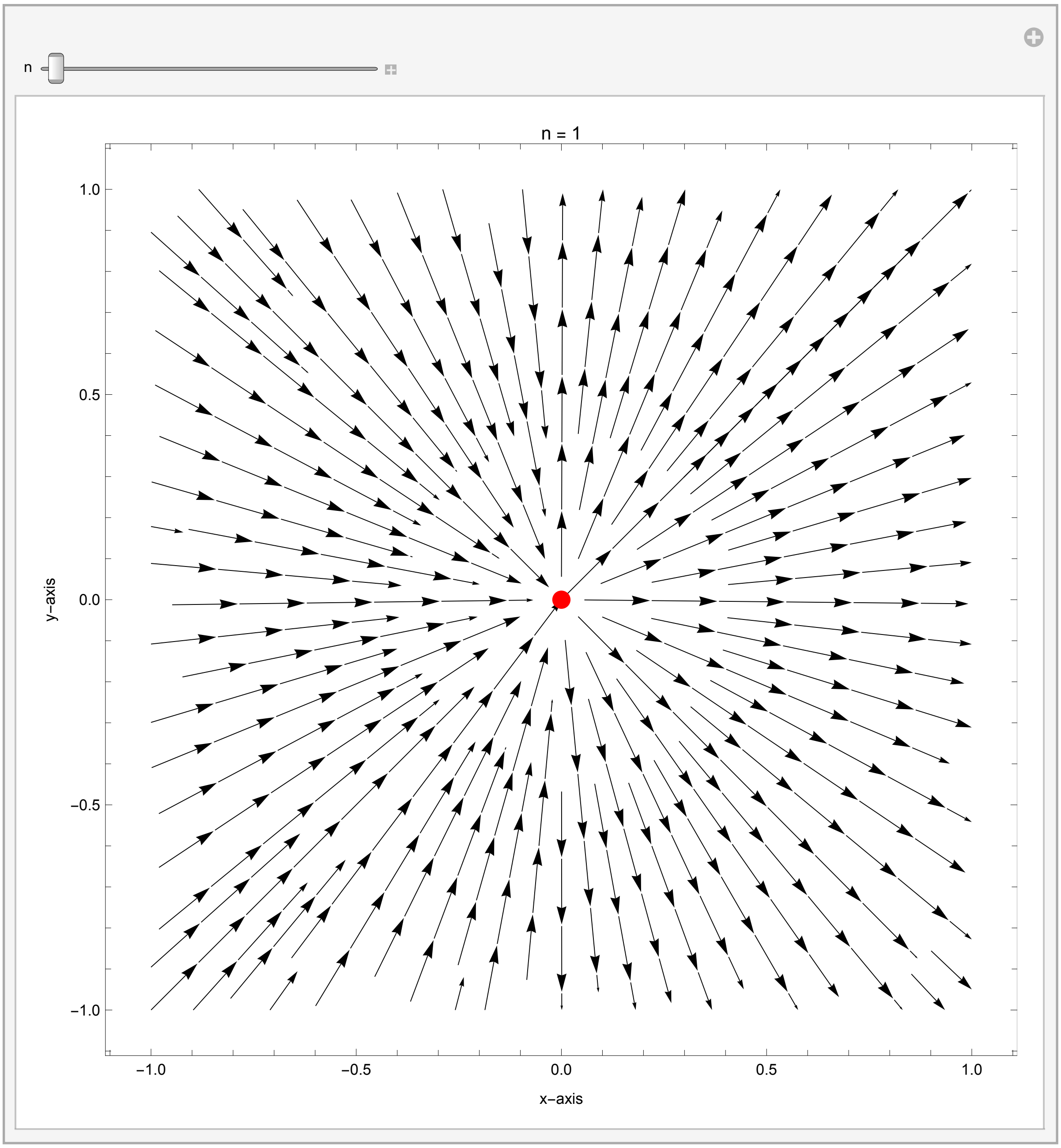
In[1187]:=

```
ClearAll["Global`*"]

equation1[n_, x_, y_] := (x^2 + y^2)^(Abs[n]/2) * Cos[n * ArcTan[y/x]];
equation2[n_, x_, y_] := (x^2 + y^2)^(Abs[n]/2) * Sin[n * ArcTan[y/x]];

Manipulate[
  StreamPlot[{equation1[n, x, y], equation2[n, x, y]}, {x, -1, 1}, {y, -1, 1},
    PlotRange → All, ImageSize → Large, ColorFunction → (Black &),
    StreamStyle → Directive[Black], StreamColorFunction → None,
    Epilog → {Red, PointSize[0.02], Point[{0, 0}]},
    FrameLabel → {{"y-axis", None}, {"x-axis", None}},
    PlotLabel → Row[{"n = ", n}]],
  {n, 1, 10, 1}
]
```

Out[1190]=

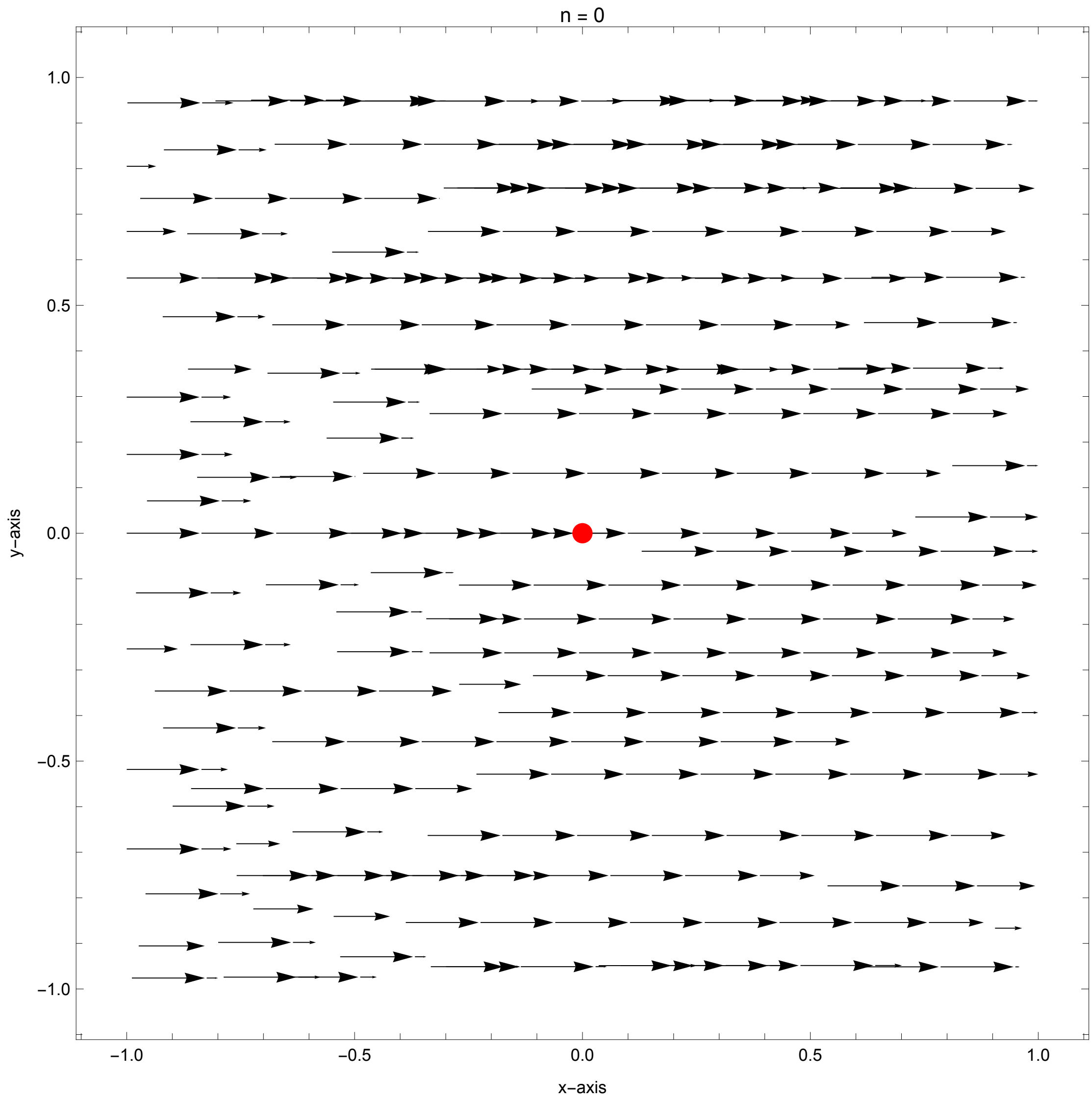


In[1191]:=

```
ClearAll["Global`*"]

equation1 = (x^2+y^2)^(Abs[n]/2)*Cos[n*ArcTan[y/x]];
equation2 = (x^2+y^2)^(Abs[n]/2)*Sin[n*ArcTan[y/x]];
n = 0;
StreamPlot[{equation1, equation2}, {x, -1, 1}, {y, -1, 1},
 PlotRange → All, ImageSize → Large, ColorFunction → (Black &),
 StreamStyle → Directive[Black], StreamColorFunction → None,
 Epilog → {Red, PointSize[0.02], Point[{0, 0}]},
 FrameLabel → {{"y-axis", None}, {"x-axis", None}},
 PlotLabel → "n = 0"]
```

Out[1195]=



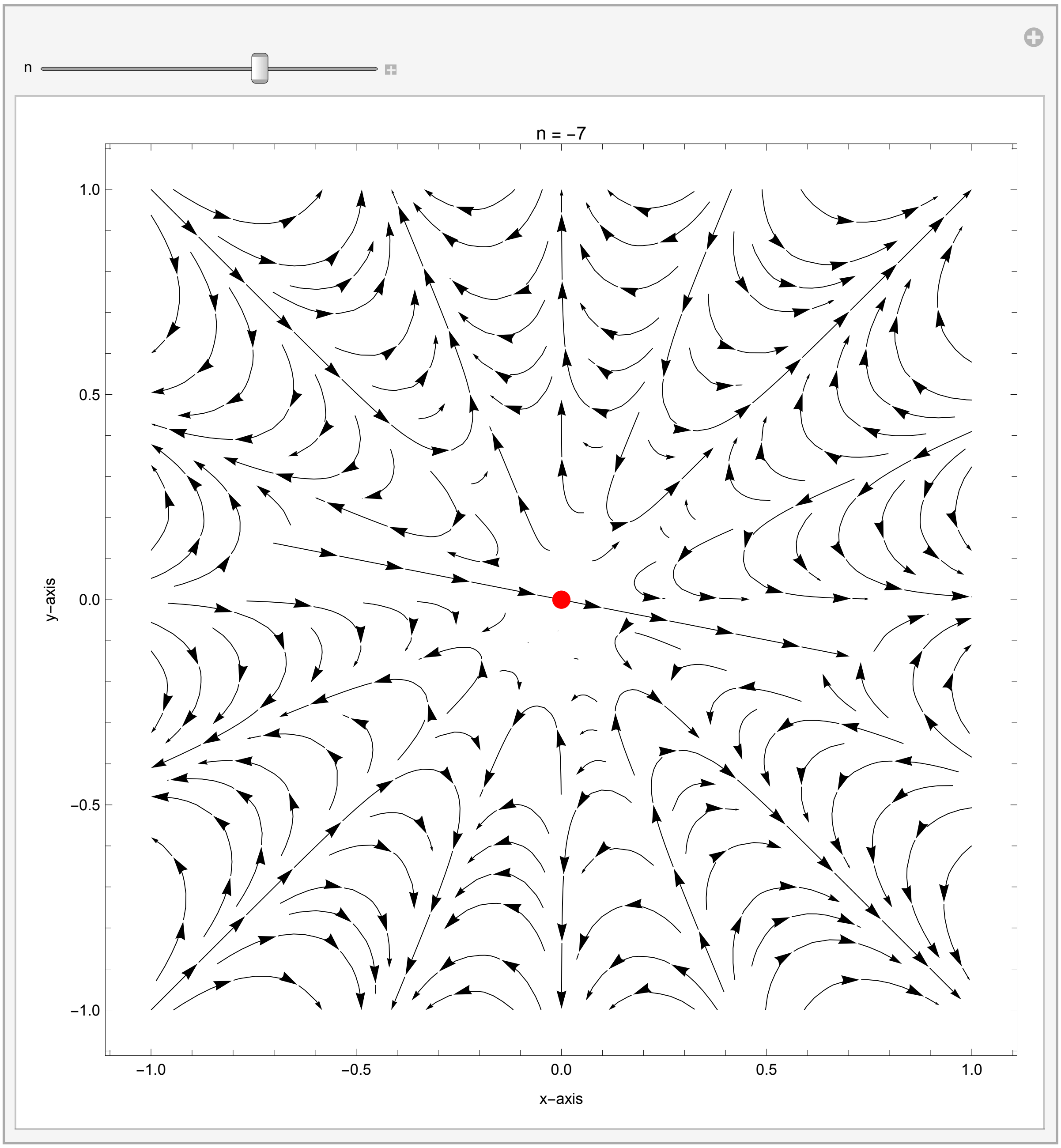
In[1200]:=

```
ClearAll["Global`*"]

equation1[n_, x_, y_] := (x^2 + y^2)^(Abs[n]/2) * Cos[n * ArcTan[y/x]];
equation2[n_, x_, y_] := (x^2 + y^2)^(Abs[n]/2) * Sin[n * ArcTan[y/x]];

Manipulate[
  StreamPlot[{equation1[n, x, y], equation2[n, x, y]}, {x, -1, 1}, {y, -1, 1},
    PlotRange → All, ImageSize → Large, ColorFunction → (Black &),
    StreamStyle → Directive[Black], StreamColorFunction → None,
    Epilog → {Red, PointSize[0.02], Point[{0, 0}]},
    FrameLabel → {{"y-axis", None}, {"x-axis", None}},
    PlotLabel → Row[{"n = ", n}]],
  {n, -1, -10, 1}
]
```

Out[1203]=



$$I = \frac{\Delta\Phi}{2\pi}$$

If we have two equations

$\dot{x} = f(x, y)$

$\dot{y} = g(x, y)$

$$\Phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) = \tan^{-1}\left(\frac{g}{f}\right)$$

$$I = \frac{1}{2\pi}\int d\Phi$$

$$d\Phi = \partial_x \Phi + \partial_y \Phi$$

$$= \frac{f^2}{f^2+g^2}\left[\left(\frac{f\,\partial_x g - g\,\partial_x f}{f^2}\right) + \left(\frac{f\,\partial_y g + g\,\partial_y f}{f^2}\right)\right]$$
$$= \frac{1}{f^2+g^2}\left[f\,(\partial_x + \partial_y)\,g - g\,(\partial_x + \partial_y)\,f\right]$$
$$I = \frac{1}{2\pi}\int d\Phi = \frac{1}{2\pi}\left[\int_{x_1}^{x_2} d\Phi\, d\,x + \int_{y_1}^{y_2} d\Phi\, d\,y + \int_{y_2}^{x_1} d\Phi\, d\,x + \int_{y_2}^{y_1} d\Phi\, d\,y\right]$$

Sometimes its much easier to transform to polar coordinates.

$x = r\cos\theta,\ y = r\sin\theta,\ r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$