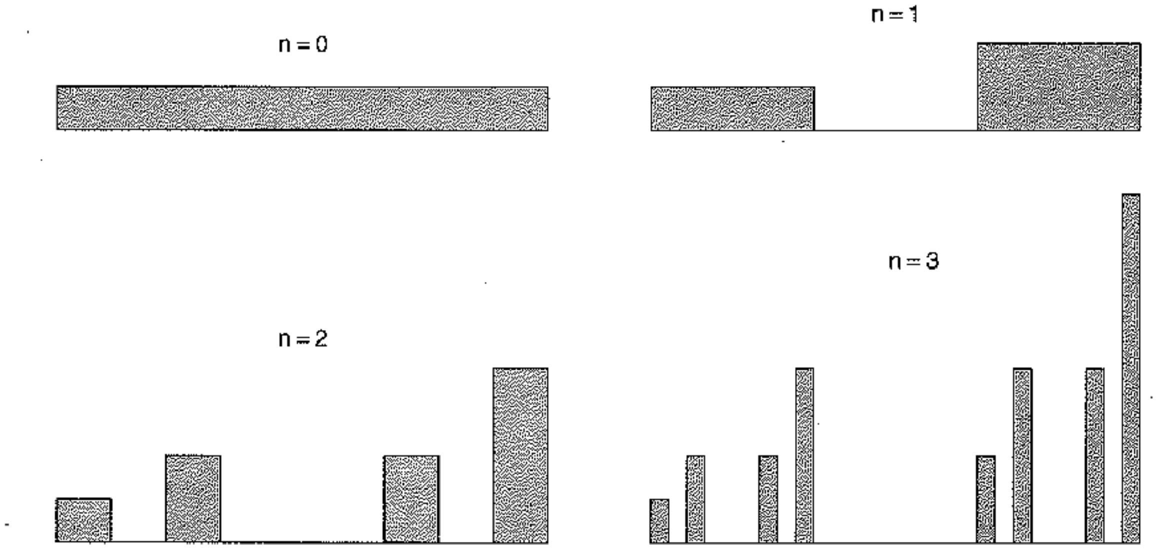
5.2 Renyi dimension of weighted Cantor set



In this exercise, your are going to calculate and analyze the Renyi dimension spectrum of the a weighted Cantor set. The symmetric Cantor set was discussed in the lecture and you know that the box counting dimension of this set is $D_0 = \dfrac{\ln 2}{\ln 3} \approx 0.63093$. Recall that the Cantor set was generated by removing the middle third from the unit interval, then the remaining two subintervals had their middle thirds removed and so on.

Now extend this construction by allocating a probability (or “mass”) to each subinterval at each division. Allocate

$\frac{2}{3}$ of the existing probability in an interval being divided to the right-hand side of the subinterval, and $\frac{1}{3}$ to the left as is shown in the figure above.

$$D_q = \frac{1}{1-q} \lim_{\epsilon \to \infty} \frac{\ln(I_q(\epsilon))}{\ln(1/\epsilon)}$$

$q = 0 \to$ box counting

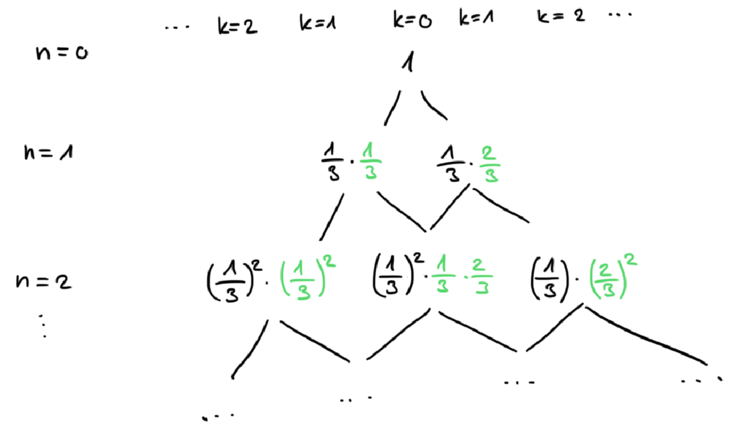
$q = 1 \to$ information dimension

$q = 2 \to$ correlation dimension

---

a) Calculate analytically the Renyi dimension spectrum D_q of the weighted Cantor set. Make sure that for $q = 0$, you recover the box counting dimension of the Cantor set. Give your result as a function of q.

Lets start by recognising that we are not looking at the classical cantor set, but at a weighted cantor set. The conditions explained in the task can be visualised as follows:



the green factors are the weights. this clearly looks like a pascals triangle. From this we can derive the following expression for the probabilities (=mass) one is looking for:

$$\sum_{k=0}^{n} \binom{n}{k} \left(p^k \cdot (1-p)^{n-k} \right).$$

With the definition $D_q = \frac{1}{1-q} \lim_{\varepsilon \to 0} \frac{\ln(I_q(\varepsilon))}{\ln\left(\frac{1}{\varepsilon}\right)}$

where $I_q(\varepsilon) = \sum_{j=0}^{N_{box}} P_j^q(\varepsilon)$ and $P_j^q(\varepsilon) = \frac{N_i}{N_{box}}$

it is possible to solve the problem. In our case I is

$$I_q(\varepsilon) = \sum_{k=0}^{n} \binom{n}{k} \left(p^k \cdot (1-p)^{n-k} \right)^q$$

$$= \{\text{with binomial theorem}\}$$

$$= \left(p^q + (1-p)^q \right)^n$$

The precision ε for this weighted Cantor set is $\frac{1}{3}$.

$$\Rightarrow D_q = \frac{1}{1-q} \frac{\ln\left((p^q + (1-p)^q)^n \right)}{\ln(3^n)}$$

$$= \frac{1}{1-q} \frac{\ln(p^q + (1-q)^q)}{\ln(3)}$$

b) Using the expression derived in (a), make a plot of D_q as a function of q for $q \in [-20, 20]$.

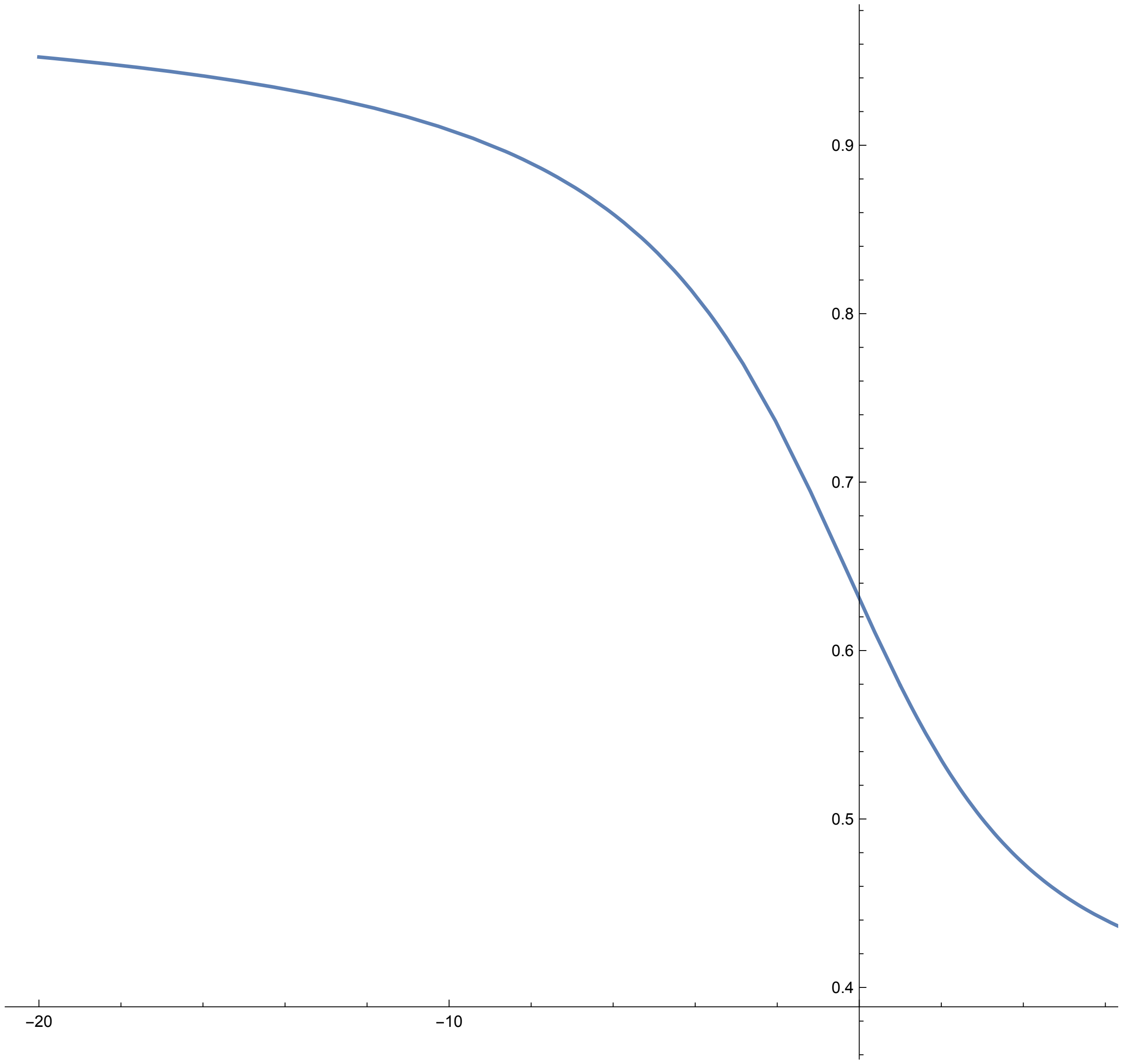
In[63]:=

```
funcD[q_]:= 1/(1-q) * Log[p^q+(1-p)^q]/Log[3]

p=1/3;

Plot[funcD[q],{q,-20,20}]
```

Out[65]=



c) Using the expression derived in (a), compute explicitly D_1 (information dimension) and D_2 (correlation dimension) of the weighted Cantor set. Give your result as the vector $[D_1, D_2]$

Since we take the limes we can use l'Hospital to calculate the "problematic" D_1.

In[190]:=

```
Clear[q]
nominator[q_] := Log[p^q+(1-p)^q];
denominator[q_] := (1-q)*Log[3];
derivNominator = D[nominator[q],q];
derivDenominator = D[denominator[q],q];

q = 1;
D1 = derivNominator/derivDenominator//Simplify;
q = 2;
D2 = 1/(1-q)*Log[p^q+(1-p)^q]/Log[3];
Result = {D1,D2}//TraditionalForm
```

Out[199]//TraditionalForm=

$$\left\{\frac{\log\left(\frac{27}{4}\right)}{\log(27)}, \frac{\log\left(\frac{9}{5}\right)}{\log(3)}\right\}$$

---

d) Using the expression derived in (a), compute explicitly $D_{-\infty} = \lim\limits_{q\to-\infty} D_q$ and $D_\infty = \lim\limits_{q\to-\infty} D_q$ of the weighted Cantor set. Give your result as the vector $[D_{-\infty}, D_\infty]$.

In[216]:=

```
Clear[q]
funcD[q_] := 1/(1-q)*Log[p^q+(1-p)^q]/Log[3]

L1 = Limit[funcD[q],q→-Infinity]//TraditionalForm;
L2 = Limit[funcD[q],q→Infinity]//TraditionalForm;

ResultD = {L1,L2}//TraditionalForm
```

Out[220]//TraditionalForm=

$$\left\{1, \frac{\log\left(\frac{3}{2}\right)}{\log(3)}\right\}$$