3.2 Hopf bifurcation

See exercises 8.2.14-15 in Strogatz

For each of the following two-dimensional flows

$$\dot{x} = \mu x - 5y - x^3$$
$$\dot{y} = 5x + \mu y + 3y^3$$

and

$$\dot{x} = \mu x + y - x^2$$
$$\dot{y} = -x + \mu y + 2x^2$$

a Hopf bifurcation occurs at the origin for $\mu = 0$.

A system with these properties can, at the bifurcation, be brought into the form

$$\dot{x} = -\omega y + f(x, y)$$
$$\dot{y} = \omega x + g(x, y)$$

by a suitable change of coordinates. The functions f and g contain only higher-order (non-linear) terms that vanish at the origin.

a) What is ω for the two systems (1) and (2), respectively? Write your answer as the vector [ω(1),ω(2)].

Rearrange the equations to:

$$\dot{x} = -5\, y - \mu\, x - x^3$$
$$\dot{y} = 5\, x + \mu\, y + 3\, y^3$$

and

$$\dot{x} = y + \mu\, x - x^2$$
$$\dot{y} = -x + \mu\, y + 2\, y^2$$

Now it's evident that $[\omega\,(1) = 5,\ \omega\,(2) = -1]$

b) Determine f and g for the systems (1) and (2). Write your solution as the matrix [[f(1),g(1)],[f(2),g(2)]].

From a) one can easily see that the solution is $[[f\,(1) = -x^3,\ g\,(1) = 3\, y^3],\ [f\,(2) = -x^2,\ g\,(2) = 2\, x^2]]$.

It can be shown that whether the bifurcation is subcritical or supercritical depends solely on the sign of the quantity a defined by

$$16\, a \;=\; f_{x\,x\,x} + f_{x\,y\,y} + g_{x\,x\,y} + g_{y\,y\,y}$$
$$+\frac{1}{\omega}\,[f_{x\,y}\,(f_{x\,x} + f_{y\,y}) - g_{x\,y}\,(g_{x\,x} + g_{y\,y}) - f_{x\,x}\, g_{x\,x} + f_{y\,y}\, g_{y\,y}]$$

where the subscripts denote partial derivatives evaluated at the fixed point (0,0). According to this criterion, the bifurcation is supercritical if $a < 0$ and subcritical if $a > 0$.

c) Determine a for the two systems (1) and (2). Write you solution as the vector [a(1),a(2)].

Using this result you should be able to determine what kind of bifurcations the systems (1) and (2) undergo at $\mu = 0$.

$a > 0 \to$ subcritical

$a < 0 \to$ supercritical

In[1]:=
```
f1[x_,y_] := -x^3
g1[x_,y_] := 3y^3

firstTerm = D[f1[x,y],{x,3}]+D[f1[x,y],x,y,y]+D[g1[x,y],x,x,y]+D[g1[x,y],y,y,y];
secondTerm = 1/5*(D[f1[x,y],x,y]*(D[f1[x,y],x,x]+D[f1[x,y],y,y])-D[g1[x,y],x,y]*(D[g1[
a = (firstTerm+secondTerm)/16
```

Out[5]= $\frac{3}{4}$

From this calculation one sees that $a = \frac{3}{4} > 0$ for the first system thus the bifurcation is a subcritical one.

In[6]:=
```
ClearAll["Global'*"]

f2[x_,y_] := -x^2
g2[x_,y_] := 2x^2

firstTerm = D[f2[x,y],{x,3}]+D[f2[x,y],x,y,y]+D[g2[x,y],x,x,y]+D[g2[x,y],y,y,y];
secondTerm = 1/(-1)*(D[f2[x,y],x,y]*(D[f2[x,y],x,x]+D[f2[x,y],y,y])-D[g2[x,y],x,y]*(D[
a = (firstTerm+secondTerm)/16
```

Out[11]= $-\frac{1}{2}$

From this calculation one sees that $a = -\frac{1}{2} < 0$ for the first system thus the bifurcation is a supercritical one.

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d) Draw phase portraits of the global dynamics for positive and negative μ for each of the systems (1) and (2). Make sure that these phase portraits verify the criteria you found in subtask c). Use a numerical solver, for example NDSolve[] in Mathematica (using

StreamPlot[] will not give enough resolution for this task).

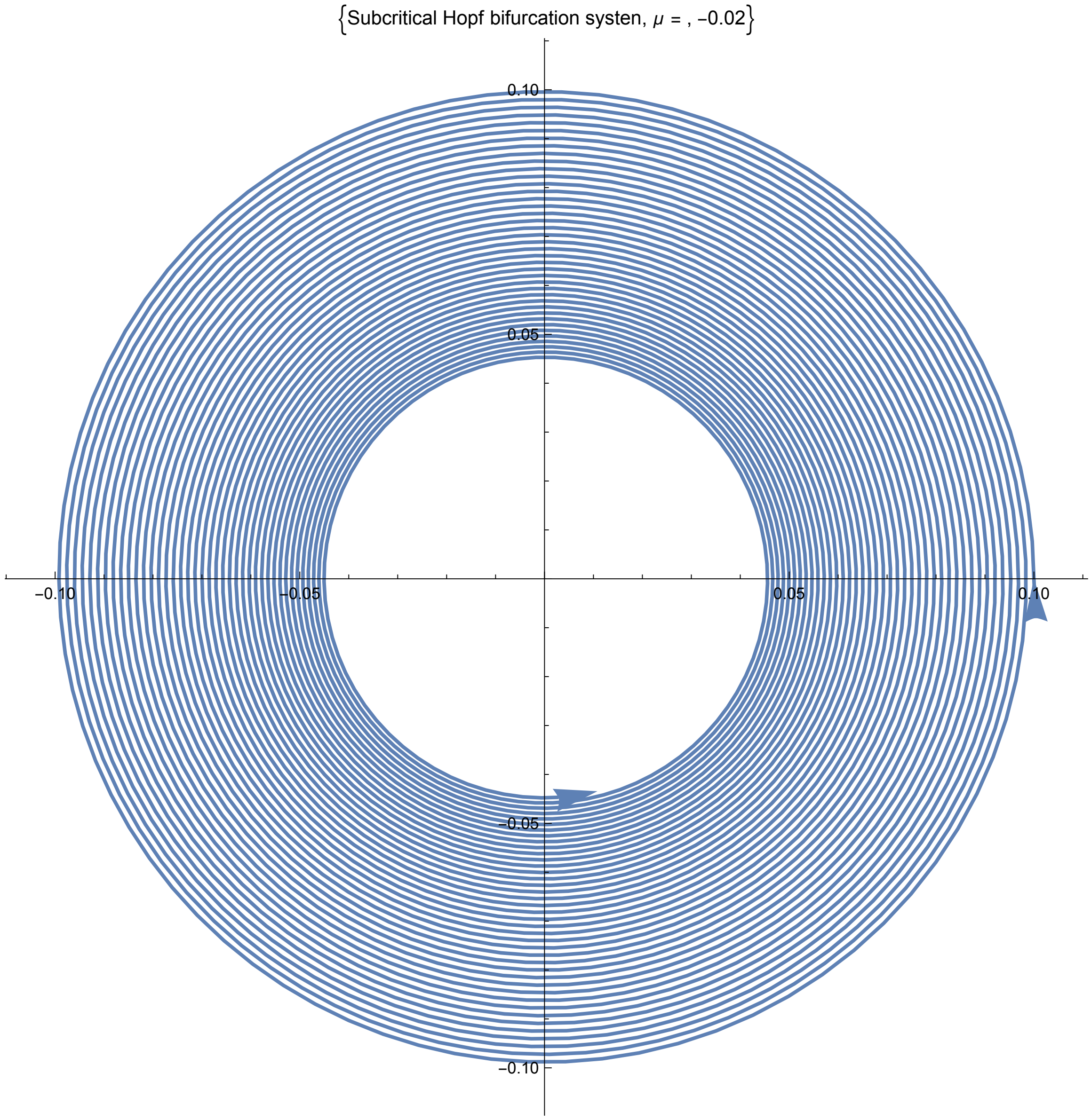
The first two plots are for the system which undergoes a subcritical Hopf bifurcation at $\mu = 0$. The first shows the case where $\mu < 0$, where we expect a stable fixed point at (0,0) and nothing else. This behaviour is shown to be true. The second plot should by the knowledge of subcritical Hopf bifurcation "explode" because $\mu > 0$, so the fixed point at (0,0) becomes unstable.

The third plot shows the supercritical Hopf bifurcation which has a stable fixed point at (0,0), the plot indicates this to be true. The fourth plot shows the that for $\mu > 0$, the trajectories do not explode but get close to a limit cycle which is very much expected for a supercritical Hopf bifurcation.

In[1628]:=

```
ClearAll["Global`*"]

u = -0.02;

Equation1 = x'[t] == -5*y[t] + u*x[t] - x[t]^3;
Equation2 = y'[t] == 5*x[t] + u*y[t] + 3*y[t]^3;

System1 = {Equation1, Equation2};

FixPoint1 = {0, 0};

t0 = 0;
tMax = 50;

(* Create 10 circles with different radii on which the initialPoints lie *)
nrOfInitialPoints = 1;
radiusMin = 0.1;
radiusMax = 0.11;

initialPoints = Table[
   {x[0] == FixPoint1[[1]] + radius*Cos[2 Pi k/nrOfInitialPoints],
    y[0] == FixPoint1[[2]] + radius*Sin[2 Pi k/nrOfInitialPoints]},
   {radius, Range[radiusMin, radiusMax, (radiusMax - radiusMin)/(nrOfInitialPoints - 1
   {k, nrOfInitialPoints}
];

(* Flatten the list to get a single list of initial points *)
initialPoints = Flatten[initialPoints, 1];

(* Use NDSolve for each initial point and store the solutions in a list *)
solutions = Table[NDSolve[{System1, initialPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Len

(* Plot the trajectories for each solution with arrows indicating the direction *)
TrajectoryPlotArrows = Table[
    ParametricPlot[Evaluate[{x[t], y[t]} /. solution], {t, t0, tMax},
    PlotStyle → style] /. Line[x_] :→ {Arrowheads[{0.04, 0.04}], Arrow[x]},{solution, s


(* Show the combined plot *)
Show[TrajectoryPlotArrows, FrameLabel → {"x", "y"},
  PlotLabel → {"Subcritical Hopf bifurcation systen, μ = ", u}, ImageSize → 600]
```

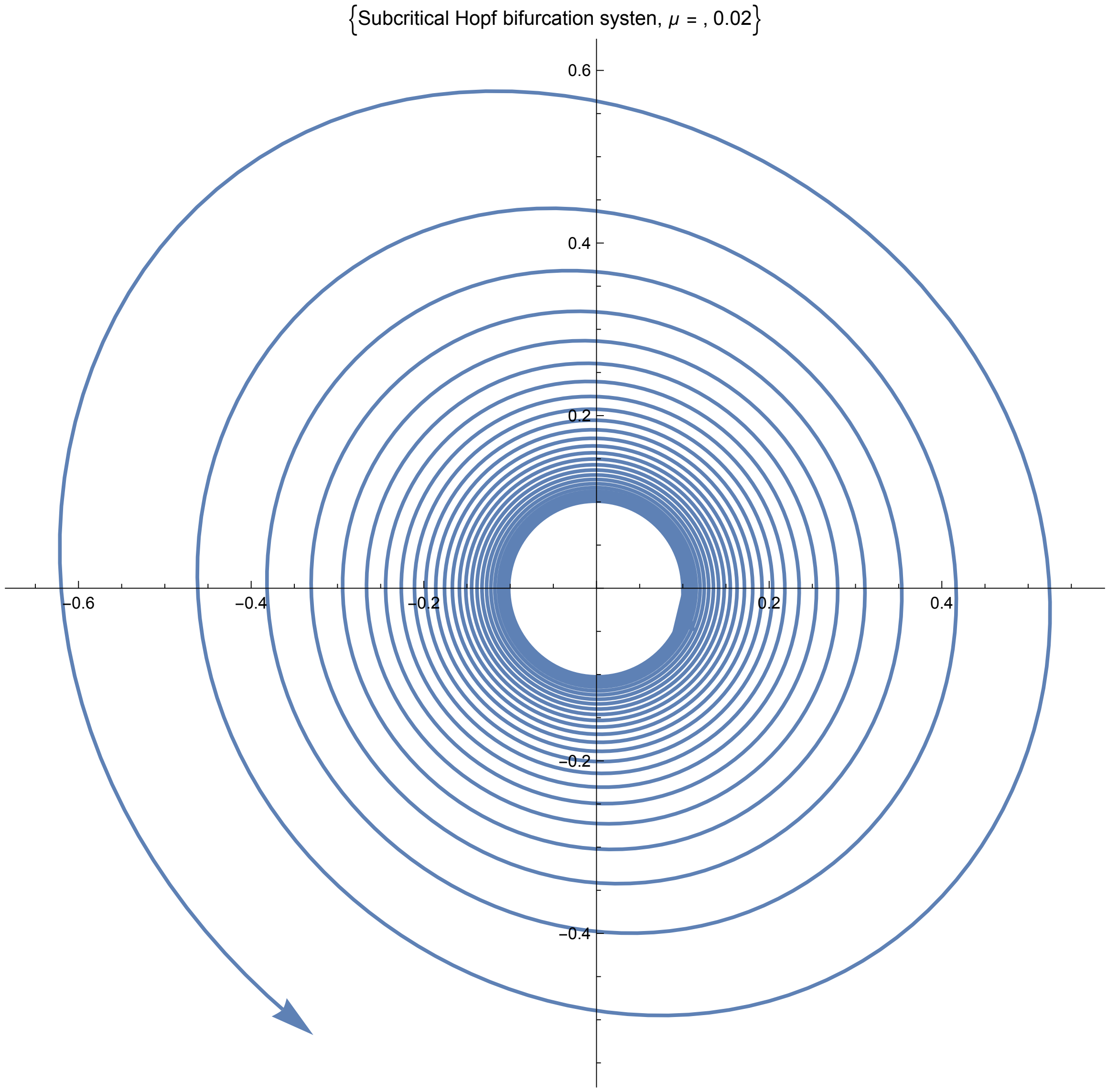
Out[1643]=


{Subcritical Hopf bifurcation systen, μ = , −0.02}
0.10
0.05
−0.05
−0.10
−0.10
−0.05
0.05
0.10

In[1644]:=

```
ClearAll["Global`*"]
ParametricPlot[Evaluate[{x[t], y[t]} /. sol], {t, t0, tMax},
    PlotStyle → style] /. Line[x_] :→ {Arrowheads[{0., 0.04, 0.04, 0.05, 0.}], Arrow[x]

u = 0.02;

Equation1 = x'[t] == -5*y[t] + u*x[t] - x[t]^3;
Equation2 = y'[t] == 5*x[t] + u*y[t] + 3*y[t]^3;

System1 = {Equation1, Equation2};

FixPoint1 = {0, 0};

t0 = 0;
tMax = 31;

(* Create 10 circles with different radii on which the initialPoints lie *)
nrOfInitialPoints = 1;
radiusMin = 0.1;
radiusMax = 0.11;

initialPoints = Table[
   {x[0] == FixPoint1[[1]] + radius*Cos[2 Pi k/nrOfInitialPoints],
    y[0] == FixPoint1[[2]] + radius*Sin[2 Pi k/nrOfInitialPoints]},
   {radius, Range[radiusMin, radiusMax, (radiusMax - radiusMin)/(nrOfInitialPoints - 1
   {k, nrOfInitialPoints}
];

(* Flatten the list to get a single list of initial points *)
initialPoints = Flatten[initialPoints, 1];

(* Use NDSolve for each initial point and store the solutions in a list *)
solutions = Table[NDSolve[{System1, initialPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Len

(* Plot the trajectories for each solution with arrows indicating the direction *)
TrajectoryPlotArrows = Table[
    ParametricPlot[Evaluate[{x[t], y[t]} /. solution], {t, t0, tMax},
    PlotStyle → style] /. Line[x_] :→ {Arrowheads[{0.04, 0.04}], Arrow[x]},{solution, s

(* Show the combined plot *)
Show[TrajectoryPlotArrows, FrameLabel → {"x", "y"},
  PlotLabel → {"Subcritical Hopf bifurcation systen, μ = ", u}, ImageSize → 600]
```

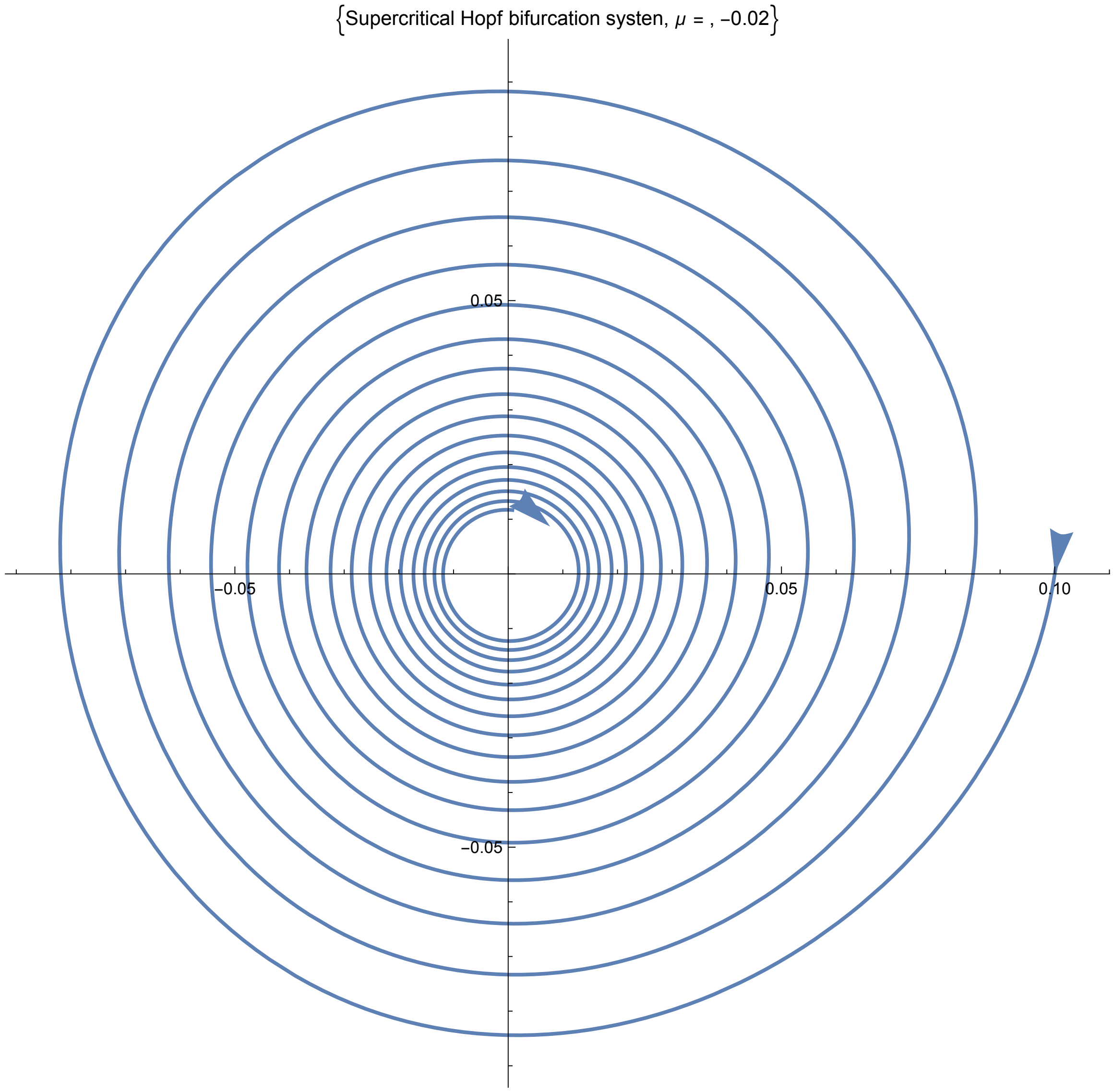
Out[1660]=


{Subcritical Hopf bifurcation systen, μ = , 0.02}
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.4
−0.2
0.2
0.4

In[1515]:=

```
ClearAll["Global`*"]

u = -0.02;

Equation1 = x'[t] == y[t] + u*x[t] - x[t]^2;
Equation2 = y'[t] == -x[t] + u*y[t] + 2*x[t]^2;

System1 = {Equation1, Equation2};

FixPoint1 = {0, 0};

t0 = 0;
tMax = 100;

(* Create 10 circles with different radii on which the initialPoints lie *)
nrOfInitialPoints = 1;
radiusMin = 0.1;
radiusMax = 0.11;

initialPoints = Table[
   {x[0] == FixPoint1[[1]] + radius*Cos[2 Pi k/nrOfInitialPoints],
    y[0] == FixPoint1[[2]] + radius*Sin[2 Pi k/nrOfInitialPoints]},
   {radius, Range[radiusMin, radiusMax, (radiusMax - radiusMin)/(nrOfInitialPoints - 1
   {k, nrOfInitialPoints}
];

(* Flatten the list to get a single list of initial points *)
initialPoints = Flatten[initialPoints, 1];

(* Use NDSolve for each initial point and store the solutions in a list *)
solutions = Table[NDSolve[{System1, initialPoints[[i]]}, {x, y}, {t, t0, tMax}], {i, Len

(* Plot the trajectories for each solution with arrows indicating the direction *)
TrajectoryPlotArrows = Table[
    ParametricPlot[Evaluate[{x[t], y[t]} /. solution], {t, t0, tMax},
    PlotStyle → style] /. Line[x_] :→ {Arrowheads[{0.04, 0.04}], Arrow[x]},{solution, s

(* Show the combined plot *)
Show[TrajectoryPlotArrows, FrameLabel → {"x", "y"},
  PlotLabel → {"Supercritical Hopf bifurcation systen, μ = ", u}, ImageSize → 600]
```

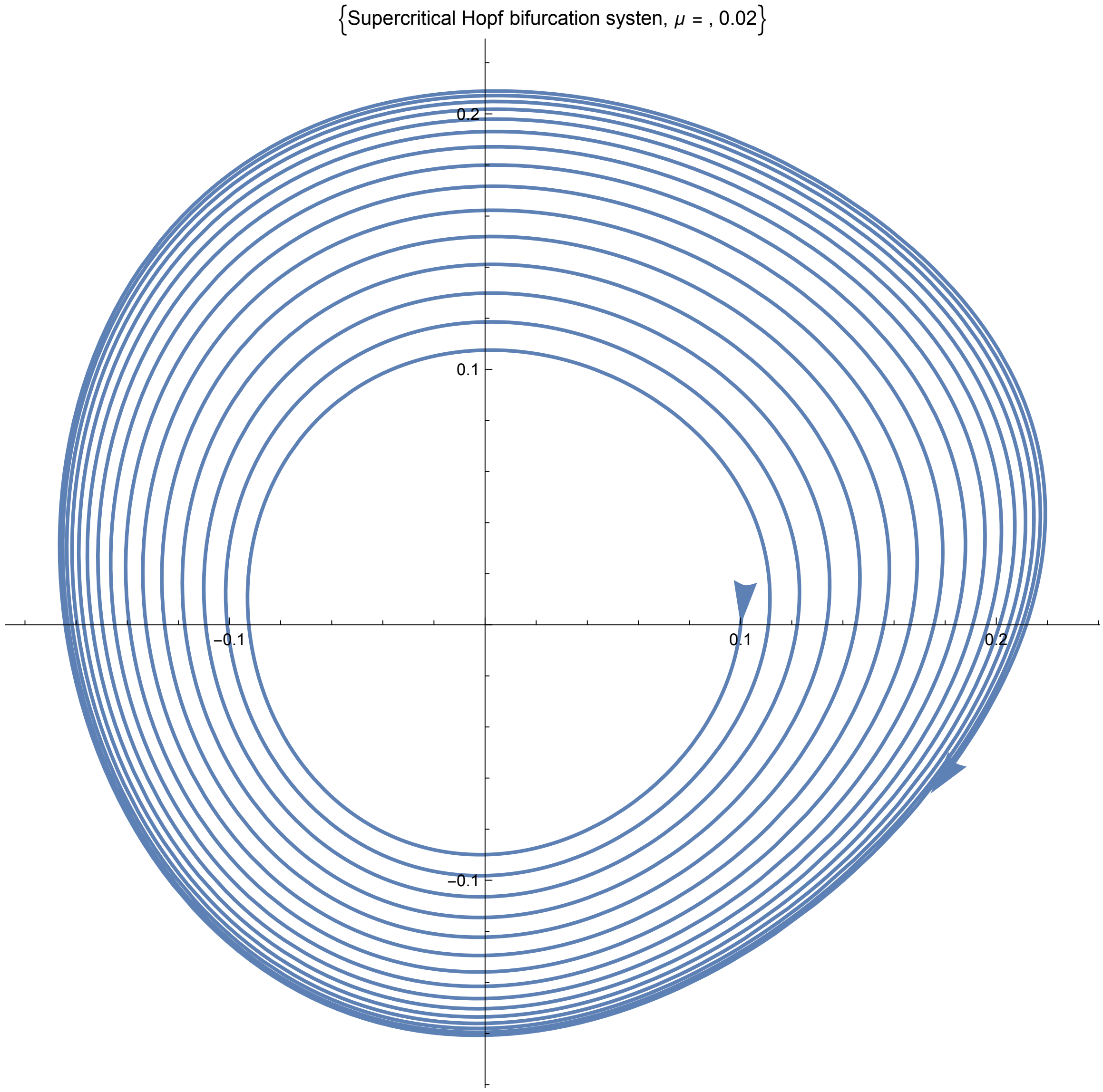
Out[1530]=


{Supercritical Hopf bifurcation systen, μ = , -0.02}
0.05
-0.05
-0.05
0.05
0.10

In[1499]:=

```
ClearAll["Global`*"]

u = 0.02;

Equation1 = x'[t] == y[t] + u*x[t] - x[t]^2;
Equation2 = y'[t] == -x[t] + u*y[t] + 2*x[t]^2;

System1 = {Equation1, Equation2};

FixPoint1 = {0, 0};

t0 = 0;
tMax = 100;

(* Create 10 circles with different radii on which the initialPoints lie *)
nrOfInitialPoints = 1;
radiusMin = 0.1;
radiusMax = 0.11;

initialPoints = Table[
   {x[0] == FixPoint1〚1〛 + radius*Cos[2 Pi k/nrOfInitialPoints],
    y[0] == FixPoint1〚2〛 + radius*Sin[2 Pi k/nrOfInitialPoints]},
   {radius, Range[radiusMin, radiusMax, (radiusMax - radiusMin)/(nrOfInitialPoints - 1
   {k, nrOfInitialPoints}
];

(* Flatten the list to get a single list of initial points *)
initialPoints = Flatten[initialPoints, 1];

(* Use NDSolve for each initial point and store the solutions in a list *)
solutions = Table[NDSolve[{System1, initialPoints〚i〛}, {x, y}, {t, t0, tMax}], {i, Len

(* Plot the trajectories for each solution with arrows indicating the direction *)
TrajectoryPlotArrows = Table[
    ParametricPlot[Evaluate[{x[t], y[t]} /. solution], {t, t0, tMax},
    PlotStyle → style] /. Line[x_] :→ {Arrowheads[{0.04, 0.04}], Arrow[x]},{solution, s

(* Show the combined plot *)
Show[TrajectoryPlotArrows, FrameLabel → {"x", "y"},
  PlotLabel → {"Supercritical Hopf bifurcation systen, μ = ", u}, ImageSize → 600]
```

Out[1514]=


{Supercritical Hopf bifurcation systen, μ = , 0.02}
0.2
0.1
−0.1
0.1
0.2
−0.1