FIM 740

Problem 1

d) $\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t-T)}{K}\right)\left(\frac{N(t)}{A} - 1\right)$

$r = \left[\frac{1}{s}\right]$ $\quad u = \frac{N}{N_0}$ 

$K = [\#\text{ of } N]$ $\quad \tau = \frac{t}{t_0}$

$A = [\#\text{ of } N]$ $\quad D = \frac{T}{t_0}$

$t = [s]$

$$\frac{d(N_0 u)}{d(\tau t_0)} = \frac{\cancel{N_0}}{t_0}\frac{du}{d\tau} = r\cancel{N_0}u\left(1 - \frac{N_0 u(t_0\tau - Dt_0)}{K}\right)\left(\frac{N_0 u(t_0\tau)}{A} - 1\right)$$

$$\frac{du}{d\tau} = r\cdot t_0\, u\left(1 - \frac{N_0 u(t_0\tau - Dt_0)}{K}\right)\left(\frac{N_0 u(t_0\tau)}{A} - 1\right)$$

$t_0 = \frac{1}{r}$ , $N_0 = K$

$$\frac{du}{d\tau} = u(1 - u(\tau - D))\left(\frac{K u(\tau)}{A} - 1\right)$$

$\frac{A}{K} = B$ $\qquad f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$$\frac{du}{d\tau} = u(1 - u(\tau - D))\left(\frac{u(\tau)}{B} - 1\right)$$

$u = u^* + \eta$

$$\underbrace{\frac{du^*}{d\tau}}_{0} + \frac{d\eta}{d\tau} = (u^* + \eta)(1 - u^* - \eta(\tau - D))\left(\frac{u^* + \eta(\tau)}{B} - 1\right)$$

$$\frac{d\eta}{d\tau} = \left[\mu^*(1-\mu^*) - \mu^*\eta(\tau-D) + \eta(1-\mu^*) - \underbrace{\eta(\tau)\eta(\tau-D)}_{0}\right]\left(\frac{u^* + \eta(\tau)}{B} - 1\right)$$

$$\frac{d\eta}{d\tau} = \mu^*(1-\mu^*)\cdot\frac{\mu^* + \eta}{B} - \frac{\mu^*\eta(\tau-D)}{B} - \underbrace{\frac{\mu^*\eta(\tau-D)\eta(\tau)}{B}}_{=0} + \frac{\eta(1-\mu^*)\mu^*}{B} + \underbrace{\frac{\eta(1-\mu)\eta(\tau)}{B}}_{=0} - [\quad]$$

$$\frac{d\eta}{d\tau} = u^*(1-u^*)\frac{u^*+\eta(\tau)}{B} - \frac{u^{*2}\,\eta(\tau-D)}{B} + \frac{\eta(\tau)(1-\mu^*)\mu^*}{B}$$

$$- u^*(1-u^*) + u^*\cdot\eta(\tau-D) - \eta(\tau)(1-u^*)$$

$$= (u^*-u^{*2})\frac{u^*+\eta(\tau)}{B} - \frac{u^{*2}\cdot\eta(\tau-D)}{B} + \frac{\eta(\tau)(u^*-u^{*2})}{B} - u^* + u^{*2} + u^*\eta(\tau-D) - \eta(\tau) + \eta(\tau)u^*$$

$$= \frac{u^*(u^*-u^{*2})}{B} + \frac{\eta(\tau)(u^*-u^{*2})}{B} - \frac{u^{*2}\cdot\eta(\tau-D)}{B} + \frac{\eta(\tau)(u^*-u^{*2})}{B} - u^* + u^{*2} + u^*\eta(\tau-D) - \eta(\tau) + \eta(\tau)\,u^*$$

$$= \underbrace{\frac{2\cdot\eta(\tau)(u^*-u^{*2})}{B}}_{=0} + \frac{u^{*2}(1-u^*-\eta(\tau-D))}{B} \underbrace{- u^* + u^{*2}}_{=0} + u^*\eta(\tau-D) \underbrace{- \eta(\tau) + \eta(\tau)\cdot u^*}_{=0}$$

$u = \frac{N}{N_0}$ dedimensional: $N_0 = K$ , $N^* = K \rightarrow u^* = \frac{K}{K} = 1$

$$\frac{d\eta}{d\tau} = \frac{-\eta(\tau-D)}{B} + \eta(\tau-D)$$

$$\frac{d\eta}{d\tau} = \eta(\tau-D)\left(1-\frac{1}{B}\right)$$

Ansatz: $\eta(\tau) = \eta_0\cdot e^{\lambda\tau}$

$$\lambda\eta_0 e^{\lambda\tau} = \eta_0\cdot e^{\lambda(\tau-D)} - \frac{\eta_0}{B}\cdot e^{\lambda(\tau-D)} \quad \Big| \eta_0$$

$$\lambda e^{\lambda\tau} = \frac{e^{\lambda\tau}}{e^{\lambda D}}\left(1-\frac{1}{B}\right) \quad \Big| e^{\lambda\tau}$$

$$\lambda = e^{-\lambda D}\left(1-\frac{1}{B}\right) \qquad D = \frac{T_H}{t_0}$$

$$\boxed{\lambda = e^{-\lambda D} - \frac{e^{-\lambda D}}{B}}$$

$$\lambda' + i\lambda'' = e^{-\lambda' D}\left(\cos(\lambda'' D) - i\sin(\lambda'' D)\right) - \frac{e^{-\lambda' D}}{B}\left(\cos(\lambda'' D) - i\sin(\lambda'' D)\right)$$

$$\mathrm{Re}[\lambda] = \lambda' = e^{-\lambda' D}\cdot\cos(\lambda'' D)\cdot\left(1-\frac{1}{B}\right) \overset{!}{=} 0$$

$$\mathrm{Im}[\lambda] = \lambda'' = -e^{\lambda' D}\cdot\sin(\lambda'' D)\cdot\left(1-\frac{1}{B}\right)$$

If $Re[\lambda] = \lambda' \stackrel{!}{=} 0$ then the Hopf bifurcation happens.

$$\frac{1}{\frac{20}{100}} = \frac{100}{20} = 5$$

$$\lambda' = \underbrace{e^{-\lambda' D}}_{=1} \cdot \cos(\lambda'' \cdot D) \cdot \underbrace{\left(1 - \frac{1}{B}\right)}_{=-4<0 \text{ with } B = \frac{A}{k} = \frac{20}{100} \Rightarrow \left(1-\frac{1}{B}\right) = -4} = 0$$

$$\Rightarrow -4\cos(\lambda'' \cdot D) = 0$$

$$\Rightarrow \boxed{\lambda'' D = \frac{\pi}{2}}$$

$$\left.\begin{aligned} \lambda' &= e^{-\lambda' D} \cdot \cos(\lambda'' \cdot D) \cdot \left(1 - \frac{1}{B}\right) \\ \lambda'' &= -e^{-\lambda' D} \cdot \sin(\lambda'' D) \cdot \left(1 - \frac{1}{B}\right) \end{aligned}\right\} \text{ insert } \lambda' = 0$$

$$0 = \cos(\lambda'' D) \cdot \left(1 - \frac{1}{B}\right) = -4\cos(\lambda'' D)$$

$$\lambda'' = -\sin(\lambda'' D)\left(1 - \frac{1}{B}\right) = 4\sin(\lambda'' D)$$

$$\lambda'' = 4\sin\left(\frac{\pi}{2}\right) \rightarrow \boxed{\lambda'' = \pm 4}$$ , ± because

$$D = \frac{\pi}{2 \cdot \lambda'} = \pm\frac{\pi}{8} \Rightarrow D = \frac{T}{t_0} = \left\{\frac{1}{r} = t_0\right\} = T \cdot r = \frac{\pi}{8}$$

$$T = \frac{\pi}{8 \cdot r}$$

$$\boxed{\begin{aligned} T &= 3.927 \\ &\approx 3.9 \end{aligned}}$$

$$\frac{d\eta}{d\tau} = \left[\underbrace{1 \cdot (1-1)}_{=0} - 1 \cdot \eta(\tau - D) + \underbrace{\eta(\tau)(1-1)}_{=0} - \underbrace{\eta(\tau) \cdot \eta(\tau - D)}_{=0}\right]\left(\frac{1+\eta(\tau)}{B} - 1\right)$$

$$= -\eta(\tau - D)\left(\frac{1+\eta(\tau)}{B} - 1\right) = \frac{-\eta(\tau-D)}{B} + \underbrace{\frac{-\eta(\tau)\cdot\eta(\tau-D)}{B}}_{=0} + \eta(\tau - D)$$

$$\frac{d\eta}{d\tau} = \eta(\tau - D)\left(1 - \frac{1}{B}\right)$$