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Abstract

This report is a summary of the work done under the guidance of Prof. Anuradha Misra for NIUS Camp 18.2, 2022.
It begins with a short introduction, followed by a thorough discussion of Relativistic Quantum Mechanics. Chapter 2 includes a description of scattering cross-sections, Feynman rules for QED and e^-e^+ and $e^-\mu^-$ cross-section calculations.
The next chapter is a brief interlude to the Quark Model emphasising on the geometric interpretation and construction of meson and baryon octets in the $Y - I_3$ space.
The penultimate chapter is dedicated to Quantum Field Theory. Canonical quantization of the Klein-Gordon Field, the Complex Scalar Field, the Dirac Field, and the theory of Interacting Fields are discussed; finally followed by the summary.

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1

Introduction

Physics, as practiced today, originated when the likes of Newton, Galileo and Kepler sought to find mathematical descriptions for various observed natural phenomena and effectively giving rise to what we now call the scientific method.
After several centuries of progress, this pursuit came to a head in the 20th century when with the advent of Quantum Mechanics and Relativity, physicists, including Feynman, Weinberg, Salam, Klein, among many others, were able to develop the best known (and the most thoroughly experimentally verified) description for the fundamental nature of reality that unifies the electromagnetic, weak and strong interactions: The Standard Model of Particle Physics.
The aim of NIUS Camp 18.2 camp was to develop an understanding of the theoretical formalism behind the development of the Standard Model. Quantum Field Theory, arrived at by combining Quantum Mechanics with Special Relativity, describes particles as excitations produced in vacuum by creation and annihilation operators corresponding to different sets of interacting particles.
The discussion here is mainly restricted to Quantum Electrodynamics, the field theory description of electromagnetic interactions and the canonical quantization procedure has been followed.
Future editions of the report will develop the concepts included here further to topics such as Compton Scattering, Renormalization, Ultraviolet Divergences and ultimately, Infrared Divergences.

2

Relativistic Quantum Mechanics

Quantum Mechanics, being inherently non-relativistic, breaks down at high energies and a more general treatment is required. This chapter details two simple yet powerful equations, namely the Klein-Gordon and Dirac Equations, which incorporate principles of the Special Theory of Relativity into Quantum Mechanics. Dirac equation is explored in much detail and its coupling with electromagnetic field is discussed. Further, the equation is found to be Lorentz Covariant and its free solutions are also discussed.

However, this discussion must be preceded by a review of Special Relativity and an introduction to the four-vector notation.

2.1. Relativistic Kinematics

Special Relativity treats space and time coordinates to be on equal footing in inertial frames of reference. This can be most conveniently represented mathematically using the four vector notation. Let

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3)$$

where c is the speed of light. In Cartesian system, this becomes

$$x^\mu = (ct, x, y, z)$$

In this formalism, Lorentz transformations can be written as:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.1.1}$$

For example, boost transformation along the x-direction is represented as

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and expanding $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ yields the expected expressions

$$t' = \gamma\left(t - \frac{\beta x}{c}\right); x' = \gamma(x - \beta t); y' = y; z' = z$$

The Invariant, defined as

$$I = x_\mu x^\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

is, as the name suggest, invariant under Lorentz transformation. It can also be written as

$$I = x_\mu x^\mu = g_{\mu\nu} x^\nu x^\mu$$

where

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is the metric tensor.

Energy and Momentum

Proper time (τ) is defined as the time observed by the particle in its rest frame. Hence, proper velocity is given by

$$\boldsymbol{\eta} = \frac{d\boldsymbol{x}}{d\tau}$$

as opposed to $\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}$, which is velocity in the lab frame. However, since $t = \gamma\tau$

$$\boldsymbol{\eta} = \gamma \boldsymbol{v}$$

It is convenient to work with proper time as it is an invariant. This convenience also extends to $\boldsymbol{\eta}$ as only it's numerator changes with change in frame of reference.
Proper velocity is a part of a four vector

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, v_x, v_y, v_z)$$

where $\eta_\mu \eta^\mu = c^2$.
Four-momentum can now be defined as

$$p^\mu = m\eta^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) \tag{2.1.2}$$

where E is the relativistic energy $E = \gamma mc^2$ and $\boldsymbol{p} = m\boldsymbol{\eta}$, and

$$p_\mu p^\mu = m^2 c^2 \tag{2.1.3}$$

2.2. The Klein-Gordon Equation

In non-relativistic Quantum Mechanics, Schrodinger's equation can be written by employing the correspondence principle and replacing the energy and momentum variables with their respective operators in the expression $E = \frac{p^2}{2m}$.
In relativistic regime

$$E^2 = |\boldsymbol{p}|^2 c^2 + m^2 c^4$$

Replacing Energy and momentum with corresponding operators and acting on ψ, we obtain a relativistic wave equation

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = (-\hbar^2 c^2 \nabla^2 + m^2 c^4)\psi \tag{2.2.1}$$

Defining the four-derivative

$$\partial_\mu \equiv \left(\frac{\partial}{\partial(ct)}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\Rightarrow \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box$$

Rewriting (2.2.1) gives the Klein-Gordon Equation:

$$\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0 \tag{2.2.2}$$

Three Problems with the Klein-Gordon Equation

1. **Negative Probability Density**

   The continuity equation, as derived from the K-G equation-

   $$\frac{\partial}{\partial(ct)}\left[\frac{\hbar}{2im}\left(\psi^*\frac{\partial\psi}{\partial(ct)}-\psi\frac{\partial\psi^*}{\partial(ct)}\right)\right]-\nabla\cdot\left[\frac{\hbar}{2im}(\psi^*\nabla\psi-\psi\nabla\psi^*)\right]=0$$

   leads to the probability density

   $$\rho=\frac{\hbar}{2im}\left(\psi^*\frac{\partial\psi}{\partial(ct)}-\psi\frac{\partial\psi^*}{\partial(ct)}\right)$$

   which is clearly not positive definite and cannot be interpreted as the probability density.

2. **Negative Energy Solutions**

   Free solutions of the K-G equation

   $$\psi(\boldsymbol{x},t)=e^{i(Et-\boldsymbol{p}\cdot\boldsymbol{x})\hbar} \tag{2.2.3}$$

   have $E=\pm\sqrt{p^2c^2+m^2c^4}$. Hence, the Klein-Gordon equation admits negative energy solutions that are not physically realizable.

3. **Spin**

   The Klein-Gordon equation doesn't account for the intrinsic spin angular momentum of elementary particles thus can only describe spin zero particles.

2.3. The Dirac Equation

The source of all the apparent shortcomings of the Klein-Gordon equation can be traced to the second order time derivative. Therefore, an equation first order in time and space is required which should also yield positive probability density.Such an equation is Dirac Equation:

$$i\hbar\frac{\partial\psi}{\partial t}=\left(\frac{\hbar c}{i}\alpha^k\partial_k+\beta mc^2\right)\psi=H\psi \tag{2.3.1}$$

where α^k and β must be $N\times N$ matrices and ψ is a N-dimensional column vector.
Demanding this equation to fulfil certain physical requirements helps restrict the form α^k, β can take. These requirements are

i) All physical particles must satisfy the relativistic Energy-momentum relation. Thus, all the components of ψ must satisfy the K-G equation.

   Squaring the Dirac equation on both sides and comparing with the K-G equation, we get

   $$-\hbar^2\frac{\partial^2\psi}{\partial t^2}+\hbar^2\sum_i\sum_j(\alpha^i\alpha^j+\alpha^j\alpha^i)\partial_i\partial_j\psi-\frac{\hbar}{i}mc^3\sum_i(\alpha^i\beta+\beta\alpha^i)\partial\psi-\beta^2m^2c^4\psi=-\hbar^2\frac{\partial^2\psi}{\partial t^2}+(\hbar^2c^2\nabla^2+m^2c^4)\psi$$

   which is satisfied if α^is and β satisfy

   $$\begin{aligned}\Rightarrow\{\alpha_i,\alpha_j\}&=2\delta_{ij}\mathbb{1}\\ \{\alpha_i,\beta\}&=0\\ \beta^2&=\mathbb{1}\end{aligned}$$

ii) Writing the continuity equation:

$$\frac{\partial \psi^\dagger \psi}{\partial t} = -c(\psi^\dagger \alpha^i \partial_i \psi + \partial_i \psi^\dagger \alpha^\dagger \psi) - \frac{imc^2}{\hbar}(\psi^\dagger \beta \psi - \psi^\dagger \beta^\dagger \psi)$$

and requiring that the RHS is the divergence of a probability current density $\Rightarrow \beta = \beta^\dagger$, $\alpha^i = (\alpha^i)^\dagger$.

One can define $\rho = \psi^\dagger \psi$ and $j^k = c\psi^\dagger \alpha^k \psi$ such that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \equiv \partial_\mu j^\mu = 0$$

The probability current density $\rho = \psi^\dagger \psi = |\psi|^2$ is now a positive definite quantity.

From the anti-commutation relations, it can be concluded that:

1. $(\alpha^i)^2 = \beta^2 = \mathbb{1} \Rightarrow$ eigenvalues of α and β are ± 1
2. $\mathrm{Tr}(\alpha^i) = \mathrm{Tr}(\beta) = 0$

These properties combine to show that α^i and β must be even dimensional.
For N=2, the three Pauli matrices already satisfy all the properties except $\mathrm{Tr}(\sigma^i) = 0$ but it is impossible to find a set of 4 anti-commutating matrices that satisfy all the required conditions.
For N=4, there are several ways to achieve this.

The Standard/Dirac-Pauli Representation

The following set of 4×4 matrices $\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$

where $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ satisfy the required anti-commutation relations.

Defining the Dirac Matrices- $\gamma^0 = \beta$ and $\gamma^i = \beta \alpha^i$, we get the Covariant form of Dirac Equation,

$$[-i\hbar \gamma^\mu \partial_\mu + mc]\psi = 0 \tag{2.3.2}$$

Introducing the notation $\gamma^\mu \partial_\mu = \not{\partial}$, Dirac equation can be written compactly as

$$\left(-i\not{\partial} + \frac{mc}{\hbar}\right)\psi = 0 \tag{2.3.3}$$

where $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ and $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \text{-}\sigma^i & 0 \end{pmatrix}$ in the Dirac-Pauli representation.

Properties of Dirac Matrices

1. γ^0 is hermitian and $(\gamma^0)^2 = \mathbb{1}$
2. γ^k are anti-hermitian, i.e., $(\gamma^k)^\dagger = -\gamma^k$ and $(\gamma^0)^2 = -\mathbb{1}$
3. $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$, where $g^{\mu\nu}$ is the metric tensor $g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
4. We can define

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.3.4}$$

which in standard representation is

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\Rightarrow \gamma_5^2 = \mathbb{1}$$

5. $\{\gamma^\mu, \gamma_5\} = 0$

We can find a basis for the 16 dimensional space in which the Dirac matrices exist by forming products of γ^μs. The 16 independent matrices can be chosen as

$$\begin{aligned}
\Gamma^S &= \mathbb{1} \\
\Gamma^V_\mu &= \gamma_\mu \\
\Gamma^T_{\mu\nu} &= \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \\
\Gamma^A_\mu &= \gamma_5\gamma^\mu \\
\Gamma^P &= \gamma_5
\end{aligned} \tag{2.3.5}$$

having the properties

6. $(\Gamma^a)^2 = \pm\mathbb{1}$
7. For every Γ^a except Γ^b, $\exists$ a Γ^b such that $\{\Gamma^a, \Gamma^b\} = 0$
8. For $a \neq S$, $\mathrm{Tr}(\Gamma^a) = 0$
9. For every pair Γ^a, Γ^b $(a \neq b)$ $\exists$a Γ^c such that
$$\Gamma^a\Gamma^b = \beta\Gamma^c$$
where $\beta = \pm 1, \pm i$
10. The matrices $\{\Gamma^a\}$ are linearly independent.
11. If a 4×4 matrix X commutes with every Γ^a, then $X \propto \mathbb{1}$
12. *Pauli's Fundamental Theorem*
Given two sets of matrices γ and γ' both satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$, $\exists$a non-singular M such that $\gamma'^\mu = M\gamma^\mu M^{-1}$ and M is unique upto a constant factor.

Solutions for free particles at rest

The solutions to Dirac Equation are four component spinors. Expanding the equation in terms of these components-

$$i(\gamma^\mu\partial_\mu\psi)_\alpha + m\psi_\alpha = 0$$
$$i(\gamma^\mu)_{\alpha\beta}\partial_\mu\psi_\beta + m\delta_{\alpha\beta}\psi_\beta = 0$$

Explicitly,

$$\sum_{\beta=1}^{4}\left[\sum_{\mu=0}^{3} i(\gamma^\mu)_{\alpha\beta}\frac{\partial}{\partial x^\mu} - m\delta_{\alpha\beta}\right]\psi_\beta = 0, \tag{2.3.6}$$

Thus, the Dirac equation is a set of four coupled differential equations.
Each component ψ_β of ψ satisfies the Klein-Gordon equation, the four component eigensoltuions of Dirac equation have the form

$$\psi = u(\boldsymbol{p})e^{-ip\cdot x}$$

We revisit 2.3.1 to evaluate the energy eigenvectors,

$$Hu = (\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)u = Eu \tag{2.3.7}$$

For particles at rest, $\boldsymbol{p} = 0$,

$$\Rightarrow Hu = \beta u = \begin{pmatrix} m\mathbb{1} & 0 \\ 0 & -m\mathbb{1} \end{pmatrix} u$$

Combining the energy eigenvectors to their corresponding eigenvalues $E = m, m, -m, -m$ gives solutions for free particles at rest which are

$$u^{(\uparrow)}(m,\mathbf{0})\textbf{=}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}e^{-imt/\hbar} \qquad u^{(\downarrow)}(m,\mathbf{0})\textbf{=}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}e^{-imt/\hbar}$$

$$v^{(\uparrow)} = u^{(\uparrow)}(-m,\mathbf{0}) = \begin{bmatrix}0\\0\\-1\\0\end{bmatrix}e^{imt/\hbar} \qquad v^{(\downarrow)} = u^{(\downarrow)}(-m,\mathbf{0})\textbf{=}\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}e^{imt/\hbar}$$

Solutions for free particles in motion

For $\boldsymbol{p} \neq 0$,

$$Hu = \begin{pmatrix} m\mathbb{1} & \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} & -m\mathbb{1}\end{pmatrix}\begin{pmatrix}u_A\\u_B\end{pmatrix} = E\begin{pmatrix}u_A\\u_B\end{pmatrix}$$

where u has been divided into two component spinors u_A and u_B.
Thus,

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}u_B = (E-m)u_A$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}u_A = (E+m)u_B$$

Taking $u_A^{(s)} = \chi^{(s)}$, where

$$\chi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}, \chi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}$$

implies,

$$u_B^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)}$$

Therefore, the positive-energy (E>0) four-spinor solutions to Dirac equation are

$$u^{(s)} = N\begin{pmatrix}\chi^{(s)}\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)}\end{pmatrix} \tag{2.3.8}$$

where N is the normalization constant. Correspondingly, for E<0, let $u_B^{(s)} = \chi^{(s)}$

$$\Rightarrow u_A^{(s)} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E-m}\chi^{(s)} = -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E|+m}\chi^{(s)}$$

Hence, negative-energy solutions are

$$u^{(s+2)} = N\begin{pmatrix}\frac{-\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|E|+m}\chi^{(s)}\\ \chi^{(s)}\end{pmatrix} \tag{2.3.9}$$

We define

$$\bar{u} = u^\dagger\gamma^0 \tag{2.3.10}$$

and choose the normalization constant $N = \sqrt{\frac{E+m}{2m}}$ such that $\bar{u}u = 1$. Replacing this in the expression for u and expanding $(\sigma\cdot k)$, u and $\bar{u}$ become

$$u(\boldsymbol{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}E+m\\0\\p^3\\p^1+ip^2\end{pmatrix}$$

$$\bar{u}(\boldsymbol{p}) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix}E+m & 0 & p^3 & p^1-ip^2\end{pmatrix}$$

Dirac spinors are normalized

$$\bar{u}u = u^\dagger\gamma^0 u = \frac{1}{2m(E+m)}\begin{pmatrix} E+m \\ 0 \\ p^3 \\ p^1+ip^2 \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} E+m & 0 & p^3 & p^1-ip^2 \end{pmatrix}$$

$$\Rightarrow \bar{u}u = \left[\frac{E+m}{2m} - \frac{\boldsymbol{p}^2}{2m(E+m)}\right] = \left[\frac{(E^2-\boldsymbol{p}^2)+m^2+2Em}{2m(E+m)}\right] = 1$$

The spinors $u^{(s+2)}$ (to be interpreted later as *antiparticles*) are denoted by $v^{(s)}$

Σ_k and the Helicity Operator

The solutions to Dirac equation for a free particle at rest $\{u^{(\uparrow)}, u^{(\downarrow)}, v^{(\uparrow)}, v^{(\downarrow)}\}$ are eigenfunctions of the operator Σ_3 with eigenvalues +1,-1,+1,-1 respectively. However, when $\boldsymbol{p} \neq 0$, this is not the case.
We can choose the z-axis direction such that $p_1 = p_2 = 0$ which would again make $u^{(s)}$ with s=1,2,3,4 eigenstates of Σ_3. In general, we can always choose free states such that they are eigenfunctions of the operator

$$\Sigma\cdot\hat{\boldsymbol{p}} = \Sigma\cdot\frac{\boldsymbol{p}}{|\boldsymbol{p}|}$$

called the Helicity Operator.
Eigenstates of helicity eigenvalue $+1$ are referred to as *right-handed states* while those with eigenvalue -1 are called *left-handed states*

2.4. Coupling to Electromagnetic Field

Non-relativistic Case

The Hamiltonian for a particle in static electric field is given by

$$H = \frac{p^2}{2m} + e\Phi$$

Using the Principle of Minimal Coupling:

$$\boldsymbol{p} \to \boldsymbol{p} - \frac{e\boldsymbol{A}}{c}$$

and replacing p by the corresponding operator in the position space representation we obtain the Hamiltonian for the particle in the presence of an electromagnetic field

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \frac{e\boldsymbol{A}}{c}\right) + e\Phi = -\frac{\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{2mc}(\nabla\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\nabla) + \frac{e^2}{2mc^2}A^2 + e\Phi \tag{2.4.1}$$

For a constant magnetic field,

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B}\times\boldsymbol{r})$$

$$\Rightarrow (p\cdot\boldsymbol{A} + \boldsymbol{A}\cdot p) = \frac{1}{2}(\boldsymbol{B}\cdot L);\, A^2 = \frac{1}{4}(B^2 r_\perp^2)$$

Hence, the Hamiltonian reduces to

$$H = H_0 - \frac{e}{2mc}\boldsymbol{B}\cdot\boldsymbol{L} + \frac{e^2}{8mc^2}B^2 r_\perp^2 \tag{2.4.2}$$

The term $-\frac{e}{2mc}\boldsymbol{B}\cdot\boldsymbol{L}$ is the energy of a magnetic dipole in a field $\boldsymbol{B}$ with magnetic moment

$$\boldsymbol{\mu} = \frac{e\boldsymbol{L}}{2mc}$$

Neglecting the $\mathcal{O}(e^2)$ term,

$$H = H_0 - \boldsymbol{\mu} \cdot \boldsymbol{B} \tag{2.4.3}$$

The eigenstates of H_0 (for Coulomb Potential), $|n, l, m\rangle$ have a $(2l+1)$ fold degeneracy. Application of external magnetic field splits each of these energy levels into two equidistant levels. This can be explained by assigning an intrinsic 'spin' angular momentum to the electron, which arises out of a necessity to explain experimental results and contributes to the magnetic moment

$$\boldsymbol{\mu_S} = g_S \frac{e}{2mc} \boldsymbol{S}$$

For spin $\frac{1}{2}$ particles, $g_S = 2$, a fact that emerges from coupling the Electromagnetic field to Dirac Equation. The wave function is now modified to

$$\psi = \psi(\boldsymbol{x}, s_z, t; s_z) = \pm\frac{\hbar}{2}$$

This can now be represented as a two component spinor

$$\psi = \begin{pmatrix} \psi_1(\boldsymbol{r}, t) \\ \psi_2(\boldsymbol{r}, t) \end{pmatrix} \tag{2.4.4}$$

where $\psi_1(\boldsymbol{r}, t) = \psi(\boldsymbol{r}, +\frac{\hbar}{2}, t)$ and $\psi_2(\boldsymbol{r}, t) = \psi(\boldsymbol{r}, -\frac{\hbar}{2}, t)$.

$$\Rightarrow \psi = \psi_1(\boldsymbol{r}, t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_2(\boldsymbol{r}, t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \psi_1(\boldsymbol{r}, t)\chi_+ + \psi_2(\boldsymbol{r}, t)\chi_- \tag{2.4.5}$$

where $\chi_\pm$ are eigenstates of σ_z with eigenvalues ± 1.
Hence, the Pauli-Schrodinger Equation,

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)^2 + e\Phi + \mu_B \boldsymbol{\sigma} \cdot \boldsymbol{B}\right]\psi \tag{2.4.6}$$

describes the electron under electromagnetic potential, takes into account the intrinsic spin, and has solutions represented by two component Pauli spinors $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

Relativistic Case

Applying the principle of minimal coupling

$$p_\mu \to p_\mu - \frac{eA_\mu}{c}$$

where $A^\mu = (\Phi, \boldsymbol{A})$ is the four-potential, to the Dirac Equation, we obtain the wave equation in presence of electromagnetic field in covariant form

$$[-\gamma^\mu(i\hbar\partial_\mu - (e/c)A_\mu) + mc]\psi = 0 \tag{2.4.7}$$

which can also be written in the Hamiltonian form as

$$i\hbar\frac{\partial\psi}{\partial t} - e\Phi\psi = \left[c\boldsymbol{\alpha} \cdot \left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right) + \beta mc^2\right]\psi \tag{2.4.8}$$

Let $\boldsymbol{\Pi} = \boldsymbol{p} - \frac{e\boldsymbol{A}}{c}$. Then

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\boldsymbol{\alpha} \cdot \boldsymbol{\Pi} + e\Phi + \beta mc^2\right]\psi \tag{2.4.9}$$

Replacing the four component Dirac spinor by $\psi = \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix}$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} = c\begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\Pi} \\ \boldsymbol{\sigma} \cdot \boldsymbol{\Pi} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} + e\Phi\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} + mc^2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} \tag{2.4.10}$$

In non-relativistic limit, $mc^2 \gg e|\Phi|, |\boldsymbol{p}|c$, and we can write

$$\Rightarrow \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} = e^{-imc^2t/\hbar} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{2.4.11}$$

where ϕ and χ are slowly varying functions of time and contribute negligibly to the time derivative.
Substituting this in the coupled Dirac Equation and simplifying

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \phi \\ \chi \end{pmatrix} + mc^2\begin{pmatrix} \phi \\ \chi \end{pmatrix} = c\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\chi \\ \boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\phi \end{pmatrix} + e\begin{pmatrix} \phi \\ \chi \end{pmatrix} + mc^2\begin{pmatrix} \phi \\ -\chi \end{pmatrix} \tag{2.4.12}$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = c\begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\chi \\ \boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\phi \end{pmatrix} + e\begin{pmatrix} \phi \\ \chi \end{pmatrix} - 2mc^2\begin{pmatrix} 0 \\ \chi \end{pmatrix} \tag{2.4.13}$$

Again, in NR limit, $e\Phi\chi$, $i\hbar\frac{\partial\chi}{\partial t} \ll mc^2\chi \Rightarrow 2mc^2\chi \simeq c\,\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\phi$

$$\Rightarrow \chi \simeq \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}}{2mc}\phi \tag{2.4.14}$$

Recall, $\boldsymbol{\sigma}\cdot\boldsymbol{\Pi} \sim v \Rightarrow \chi \simeq \mathcal{O}\left(\frac{v}{c}\right)\phi$. Since in NR limit, $\left(\frac{v}{c}\right) \ll 1$, we have $\chi \ll \phi$.
This leaves only

$$i\hbar\frac{\partial\phi}{\partial t} = \frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi})}{2m}\phi + e\Phi\phi \tag{2.4.15}$$

Replacing $(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}) = \boldsymbol{\Pi}\cdot\boldsymbol{\Pi} - \frac{e\hbar}{c}\boldsymbol{\sigma}\cdot\boldsymbol{B}$ and $\boldsymbol{\Pi} = \left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)$, we finally obtain

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{1}{2m}\left(\boldsymbol{p} - \frac{e\boldsymbol{A}}{c}\right)^2 - \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B} + e\Phi\right]\psi \tag{2.4.16}$$

Hence, the non-relativistic limit of Dirac Equation returns the Pauli-Schrodinger Equation.

- Showing $g_S = 2$

  On expanding,

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{1}{2m}\left(\boldsymbol{p}^2 - \frac{e}{c}(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}) + \frac{e^2}{c^2}A^2\right) - \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B} + e\Phi\right]\phi$$

  Here, $\frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B} = \frac{e}{mc}\boldsymbol{S}\cdot\boldsymbol{B}$ and $(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}) = \boldsymbol{L}\cdot\boldsymbol{B}$

$$\Rightarrow i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{1}{2m}\boldsymbol{p}^2 - \frac{e}{2mc}(\boldsymbol{L} + 2\boldsymbol{S})\cdot\boldsymbol{B} + \frac{e^2}{2mc^2}A^2 + e\Phi\right]\phi$$

  $\Rightarrow$ Net magnetic moment, $\boldsymbol{\mu} = \boldsymbol{\mu}_{orb} + \boldsymbol{\mu}_{spin} = \frac{e}{2mc}(\boldsymbol{L} + 2\boldsymbol{S})$ and since $\boldsymbol{\mu}_{spin} = g_S\frac{e}{2mc}\boldsymbol{S}$, $g_S = 2$.

2.5. Lorentz Covariance of Dirac Equation

For the theory to be consistent with Special Relativity, it must be Lorentz covariant (an explicit illustration of Einstein's first postulate that the laws of Physics are the same in all inertial reference frames). Hence, if in frame $\mathcal{I}$, Dirac Equation reads (in natural units),

$$[-i\gamma^\mu\partial_\mu + m]\psi(x) = 0$$

then on Lorentz transforming to frame $\mathcal{I}'$

$$[-i\gamma^\mu\partial'_\mu + m]\psi'(x') = 0$$

where $\partial_\mu = \Lambda^\nu{}_\mu\partial'_\nu$ and $\psi'(x') = S(\Lambda)\psi(x)$. Replacing these in the equations and simplifying yields the condition for Dirac Equation to be covariant under Lorentz transformation is that $S(\Lambda)$ satisfies

$$S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu\gamma^\mu \tag{2.5.1}$$

Determination of $S(\Lambda)$

Consider an infinitesimal transformation

$$\Lambda^{\nu}{}_{\mu} = g^{\nu}_{\mu} + \Delta\omega^{\nu}{}_{\mu}$$

Since, $\Lambda^{\lambda}{}_{\mu} g^{\mu\nu} \Lambda^{\rho}{}_{\nu} = g^{\lambda}_{\rho}$ we have $\Delta\omega^{\mu\nu} = -\Delta\omega^{\nu\mu}$.
Under this infinitesimal transformation,

$$S(\Lambda) = \mathbb{1} + \tau \tag{2.5.2}$$

where $\tau \ll \mathbb{1}$. Applying the condition for Lorentz covariance

$$S^{-1}(\Lambda)\gamma^{\nu} S(\Lambda) = \Lambda^{\nu}{}_{\mu}\gamma^{\mu}$$

constrains τ such that

i) $Tr(\tau) = 0$ and,

ii) $[\gamma, \tau] = \Delta\omega^{\mu}_{\nu}\gamma^{\nu}$

Defining

$$\tau = \frac{1}{8}\Delta\omega^{\mu\nu}(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}) = \frac{-i}{4}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu}, \gamma_{\mu}]$, τ satisfies the two conditions above. Moreover, since $\Delta\omega^{\mu\nu}$ in anti-symmetric for $\mu \neq \nu$ and γ_{μ} and γ_{ν} anti-commute,

$$\tau = \frac{1}{4}\Delta\omega^{\mu\nu}\gamma_{\mu}\gamma_{\nu} = \frac{-i}{2}\Delta\omega^{\mu\nu}\sigma_{\mu\nu} \tag{2.5.3}$$

$$\Rightarrow S(\Lambda) = \mathbb{1} + \tau = \mathbb{1} + \frac{1}{4}\Delta\omega^{\mu\nu}\gamma_{\mu}\gamma_{\nu} = \mathbb{1} - \frac{i}{2}\Delta\omega^{\mu\nu}\sigma_{\mu\nu} \tag{2.5.4}$$

Finite transformations can be obtained by repeated application of this infinitesimal transformation

$$S(\Lambda) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{2}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}\right)^{N}$$

$$S(\Lambda) = e^{-\frac{i}{2}\Delta\omega^{\mu\nu}\sigma_{\mu\nu}} \tag{2.5.5}$$

Bilinear Forms and their transformation properties

Transformation of bilinear forms, such as $\bar{\psi}\gamma^{\mu}\psi$, can be obtained by employing the following properties of $S(\Lambda)$

i. $S^{\dagger}\gamma^0 = b\gamma^0 S^{-1}$

ii. $\det(S^{\dagger}\gamma^0 S\gamma^0) = 1 = \det(b\mathbb{1}) = b^4$
$\Rightarrow b^2 = \pm 1 \Rightarrow b = \pm 1$

iii. $b = +1$ for orthochronous Lorentz transformations $[\Lambda^0_0 \geq 1]$
$b = -1$ for non-orthochronous Lorentz transformations $[\Lambda^0_0 \leq -1]$

Hence, under orthochronous Lorentz transformations we have,

1. $\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x)$
2. $\bar{\psi}'(x')\gamma^{\mu}\psi'(x') = \Lambda^{\mu}{}_{\nu}\bar{\psi}(x)\gamma^{\nu}\psi(x)$
3. $\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\tau}\bar{\psi}(x)\sigma^{\rho\tau}\psi(x)$

3

Quantum Electrodynamics: Feynman Rules

One of the most effective and widely used experimental techniques to investigate the atomic and subatomic structure of matter is simply the following: smash two particles against each other and observe the outcome of their interaction. Formally, this process is called Scattering.

One of the important observables that can be calculated theoretically and verified experimentally is the scattering cross-section. It has the following structure

$$d\sigma \sim |\mathcal{M}|^2 \left(\frac{dQ}{F}\right) \tag{3.0.1}$$

where $|\mathcal{M}|$ is the dynamical part and can be calculated using Feynman Rules and (dQ/F) is the kinematic part and depends only on the initial and final conditions of the scattering process.

Physically, it can be interpreted as the effective area responsible for the scattering of the particles.

3.1. Scattering Cross-section

More explicitly, the cross-section has the form

$$\sigma = W_{fi}\frac{\text{(Phase-Space Factor)}}{\text{(Initial Flux)}} \tag{3.1.1}$$

The terms in this expression have the following interpretation:

- *Transition rate per unit volume* (W_{fi})

  As the name suggests, W_{fi} is defined as

  $$W_{fi} = \frac{|T_{fi}|^2}{VT}$$

  where VT is the spacetime volume. Replacing $T_{fi} = -i(2\pi)^4\delta^4(p_A + p_B - p_C - p_D)\frac{\mathcal{M}}{\sqrt{V}}$ and rewriting one of the delta functions as $\delta^4(p_A + p_B - p_C - p_D) = \int \frac{1}{(2\pi)^3} d^4x e^{i(p_A+p_B-p_C-p_D)\cdot x}$, we get

  $$W_{fi} = \frac{(2\pi)^4\delta^4(p_A + p_B - p_C - p_D)}{V^4}|\mathcal{M}|^2 \tag{3.1.2}$$

  where it is assumed the number of particles in a given volume is invariant under Lorentz transformations.

- *Phase-Space Factor*

  This term effectively represents the number of final states. In momentum space representation,

$$\text{\# of states per unit particle} = \frac{V d^3 p}{(2\pi)^3 2E} \tag{3.1.3}$$

  Similarly, for a process with two final states 3 and 4.

$$\text{Phase-Space Factor}(A + B \to C + D) = \frac{V d^3 p_C}{(2\pi)^3 2E_C} \cdot \frac{V d^3 p_D}{(2\pi)^3 2E_D} \tag{3.1.4}$$

- *Initial Flux*

  In the lab frame, consider a particle A approaching a target B at a velocity $\boldsymbol{v_A}$. We choose a normalization such that

$$\text{\# of target particles per unit volume} = \frac{2E_B}{V}$$

$$\text{\# of beam particles per unit area per unit time} = |\boldsymbol{v_A}|\frac{2E_A}{V}$$

  Therefore,

$$\text{Initial Flux} = |\boldsymbol{v_A}|\frac{2E_A}{V} \cdot \frac{2E_B}{V} \tag{3.1.5}$$

  Note: A more general, Lorentz invatiant expression for the Flux factor is

$$F = 4\sqrt{(p_A \cdot p_B)^2 - m_A m_B} \tag{3.1.6}$$

Replacing these expressions in 3.1.1 and simplifying yields

$$d\sigma = \frac{1}{|\boldsymbol{v_1}|(2E_A)(2E_B)}|\mathcal{M}|^2 (2\pi)^4 \delta^4(p_A + p_B - p_C - p_D)\frac{V d^3 p_C}{(2\pi)^3 2E_C} \cdot \frac{V d^3 p_D}{(2\pi)^3 2E_D} \tag{3.1.7}$$

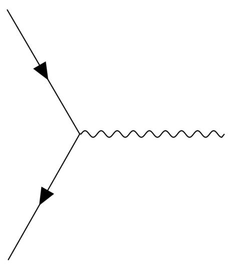
3.2. Feynman Rules for Quantum Electrodynamics

1. Propagators:

$$\text{Photon Propagator} \equiv \sim\!\sim\!\sim\!\sim\!\sim \equiv \frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \tag{3.2.1}$$

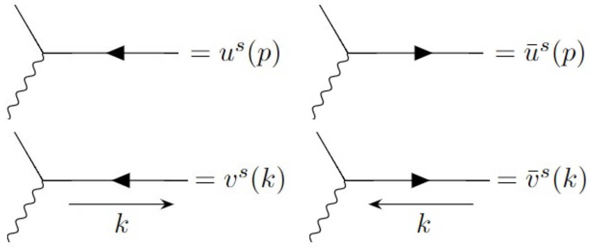
$$\text{Fermion Propogator} \equiv \psi(x)\psi(y) \equiv \longrightarrow \equiv \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \tag{3.2.2}$$

2. Vertex:



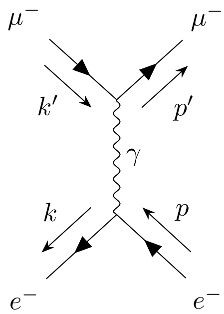
$$\equiv -ie\gamma^\mu \tag{3.2.3}$$

3. External leg contractions:



4. Impose momentum conservation at each vertex.
5. Integrate over each undetermined loop momentum.
6. Figure out the overall sign of the diagram

3.3. $e^-e^+ \to \mu^-\mu^+$ scattering using Feynman rules



Using the Feynman Rules, we can write

$$i\mathcal{M} = [ie\bar{v}(p)\gamma^\mu u(k)] \left(\frac{-ig_{\mu\nu}}{q^2}\right) [ie\bar{u}(p')\gamma^\mu v(k')] \tag{3.3.1}$$

$$\begin{aligned} \Rightarrow |\mathcal{M}|^2 &= \frac{e^4}{q^4} \left[(\bar{v}(p)\gamma^\mu u(k))(\bar{u}(p')\gamma_\mu v(k'))\right] \left[(\bar{u}(k)\gamma^\nu v(p))(\bar{v}(k')\gamma_\nu u(p'))\right] \\ &= \frac{e^4}{q^4} \mathcal{L}^{\mu\nu}_{e^-} \mathcal{L}^{\text{muon}}_{\mu\nu} \end{aligned} \tag{3.3.2}$$

where

$$\mathcal{L}^{\mu\nu}_{e^-} = \frac{1}{2} \sum_{e^- \text{ spin}} (\bar{v}(p)\gamma^\mu u(k))(\bar{u}(k)\gamma^\nu v(p))$$

and

$$\mathcal{L}^{\text{muon}}_{\mu\nu} = \frac{1}{2} \sum_{\mu^- \text{ spin}} (\bar{u}(p')\gamma_\mu v(k'))(\bar{v}(k')\gamma_\nu u(p'))$$

Simplifying,

$$\mathcal{L}^{\mu\nu}_{e^-} = \frac{1}{2} \sum_{s'} \left(\bar{v}_\alpha(p,s')(\gamma^\mu)_{\alpha\beta} \left(\sum_s u_\beta(k,s)\bar{u}_\gamma(k,s) \right) (\gamma^\nu)_{\gamma\delta} v_\delta(p,s') \right)$$

Replacing,

$$\sum_s u\bar{u} = (\not{k} + m)$$

$$\sum_{s'} v\bar{v} = (\not{p} - m)$$

and summing over the indices

$$\mathcal{L}_{e^-}^{\mu\nu} = \frac{1}{2}\mathrm{Tr}[(-\not{p} + m)\gamma^\mu(\not{k} + m)\gamma^\nu] \tag{3.3.3}$$

Similarly,

$$\mathcal{L}^{\mathrm{muon}}_{\mu\nu} = \frac{1}{2}\mathrm{Tr}[(\not{p}' + m)\gamma_\mu(-\not{k}' + m)\gamma_\nu] \tag{3.3.4}$$

Using Trace Theorems (Appendix A),

$$\mathcal{L}_{e^-}^{\mu\nu} = 2[-p^\mu k^\nu - p^\nu k^\mu + (p\cdot k + m^2)g^{\mu\nu}] \tag{3.3.5}$$

$$\mathcal{L}^{\mathrm{muon}}_{\mu\nu} = 2[-p'_\mu k'_\nu - p'_\nu k'_\mu + (p'\cdot k' + M^2)g_{\mu\nu}] \tag{3.3.6}$$

where m is the mass of the electron/positron and M is the mass of the muon/anti-muon. Multiplying, taking all the four products and simplifying:

$$|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p\cdot p')(k\cdot k') + (k\cdot p')(p\cdot k') - m^2(k'\cdot p') - M^2(p\cdot k) - 2M^2m^2\right]$$

In high-energy limit (masses are negligible as compared to the momentum):

$$|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p\cdot p')(k\cdot k') + (k\cdot p')(p\cdot k')\right] \tag{3.3.7}$$

where $q = (p+k)^2$. In this limit, the Mandelstam variables are

$$\begin{aligned} s &= (p+k)^2 = (p'+k')^2 \simeq 2(p\cdot k) \simeq 2(p'\cdot k') \\ t &= (k-k')^2 = (p-p')^2 \simeq -2(k\cdot k') \simeq -2(p\cdot p') \\ u &= (k-p')^2 = (p-k')^2 \simeq -2(p'\cdot k) \simeq -2(p\cdot k') \end{aligned} \tag{3.3.8}$$

Replacing these into 3.3.7

$$|\mathcal{M}|^2 = 2e^4\left(\frac{t^2+u^2}{s^2}\right) \tag{3.3.9}$$

3.4. $e^-\mu^- \to e^-\mu^-$ in lab frame

Consider the scattering process $e^-\mu^- \to e^-\mu^-$ as shown in Fig. 3.1

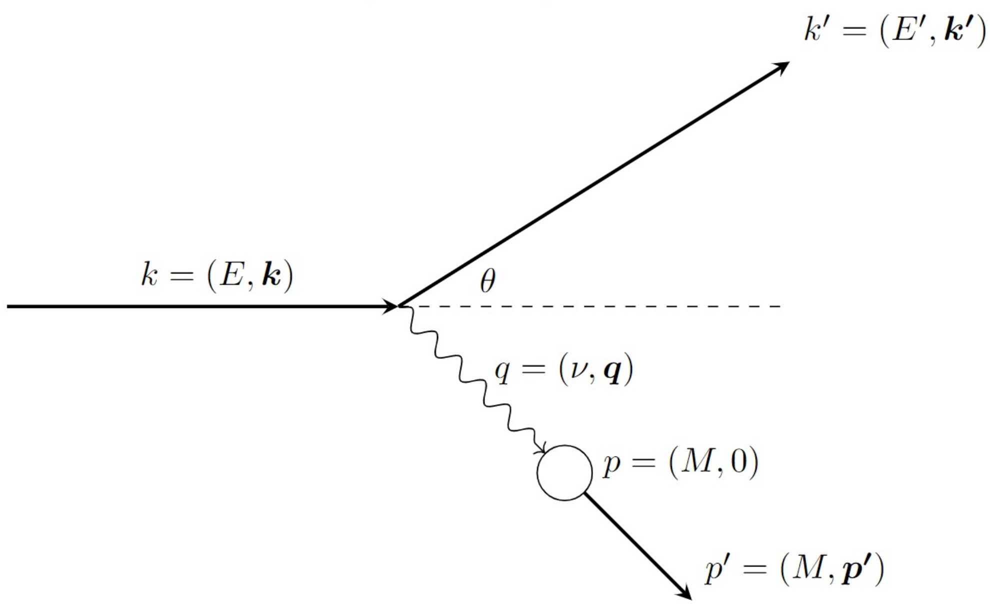


Figure 3.1: $e^-\mu^- \to e^-\mu^-$ in lab frame with the muon initially at rest

For electron-muon scattering, we have the amplitude

$$|\bar{\mathcal{M}}|^2 = \frac{8e^4}{q^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(p'\cdot k) - m^2(k'\cdot p') - M^2(p\cdot k) - 2M^2m^2\right] \tag{3.4.1}$$

In the limit that $m \ll M$, this reduces to

$$|\bar{\mathcal{M}}|^2 = \frac{8e^4}{q^4}\left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(p'\cdot k) - M^2(p\cdot k)\right] \tag{3.4.2}$$

where $q = k - k' \Rightarrow q^2 \simeq -2k\cdot k'$ because in this limit, $k^2 = k'^2 \simeq 0$. Replacing $p' = k + p - k'$,

$$|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[-\frac{q^2}{2}(k'\cdot p - k'\cdot p) + 2(k'\cdot p)(p\cdot k) + \frac{1}{2}M^2q^2\right] \tag{3.4.3}$$

Since, $p = (M, \mathbf{0})$ and $k = (E, k)$, $k' = (E', k')$,

$$\begin{aligned}|\mathcal{M}|^2 &= \frac{8e^4}{q^4}\left[-\frac{q^2}{2}(E - E') + 2(k'\cdot p)(p\cdot k) + \frac{1}{2}M^2q^2\right] \\ &= \frac{8e^4}{q^4}(2M^2)EE'\left[1 + \frac{q^2}{4EE'} - \frac{q^2}{2M^2}\left(\frac{M(E-E')}{2EE'}\right)\right]\end{aligned} \tag{3.4.4}$$

From Fig. 3.1, we can see that $q^2 \simeq -2k' = -2EE'(1 - cos\theta) = -4EE'\sin^2\frac{\theta}{2}$ and $q^2 = -2p\cdot q = -2\nu M \Rightarrow \nu \equiv (E - E') = -\frac{q^2}{2M}$. Thus, 3.4.4 reduces to

$$|\mathcal{M}|^2 = \frac{8e^4}{q^4}(2M^2)EE'\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \tag{3.4.5}$$

Replacing this in the general expression for the cross-section,

$$\begin{aligned}d\sigma &= \frac{1}{4ME}\frac{|\mathcal{M}|^2}{4\pi^2}\frac{d^3k'}{2E'}\frac{d^3p'}{2p'_0}\delta^{(4)}(p - p' + (k - k')) \\ &= \frac{1}{4ME}\frac{|\mathcal{M}|^2}{4\pi^2}\frac{E'}{2}dE'd\Omega\frac{d^3p'}{2p'_0}\delta^{(4)}(p - p' + q)\end{aligned} \tag{3.4.6}$$

To bring this to a simpler form, we use the identity

$$\delta(f(x)) = \sum_{k=1}^{n}\frac{1}{|f'(x_k)|}\delta(x - x_k)$$

where x_k roots of $f(x)$. Then, for a function

$$f(p_0) = p_0^2 - E_p^2$$

$$\Rightarrow \delta(p^2 - M^2) = \frac{1}{2E_p}\delta(p_0 - E_p) + \frac{1}{-2E_p}\delta(p_0 + E_p)$$

Multiplying both sides by the Heaviside Step function $\Theta(p_0)$ (to ensure only the part with $p_0 > 0$ contributes) and integrating over p_0 we get

$$\int dp_0\delta(p^2 - M^2)\Theta(p_0) = \frac{1}{2p_0} \tag{3.4.7}$$

We have,

$$\begin{aligned}\int\frac{d^3p'}{2p'_0}\delta^{(4)}(p + q - p') &= \int d^3p'\delta^{(4)}(p + q - p')\left(\frac{1}{2p'_0}\right) \\ &= \int d^3p'dp'_0\delta^{(4)}(p + q - p')\delta(p^2 - M^2)\Theta(p_0) \\ &= \int d^4p'\delta^{(4)}(p + q - p')\delta(p'^2 - M^2)\Theta(p'_0) \\ &= \int d^4p'\delta((p + q)^2 - M^2)\Theta(p'_0) \\ &= \delta(p^2 + 2p\cdot q + q^2 - M^2) = \delta(-2M(E - E') - 4EE'\sin^2\frac{\theta}{2})\end{aligned}$$

Defining $A = 1 + (2E/M)\sin^2\frac{\theta}{2}$,

$$\int \frac{d^3p'}{2p'_0}\delta^{(4)}(p+q-p') = \frac{1}{2MA}\delta\left(E' - \frac{E}{A}\right) \tag{3.4.8}$$

Substituting this back to 3.4.6 gives

$$\frac{d\sigma}{d\Omega dE'} = \frac{(2\alpha E')^2}{q^4}\left\{\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right\}\delta\left(E' - \frac{E}{A}\right) \tag{3.4.9}$$

Integrating over E'

$$\left.\frac{d\sigma}{d\Omega}\right|_{lab} = \left[\frac{\alpha^2}{4E^2\sin^2\frac{\theta}{2}}\right]\frac{E'}{E}\left\{\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right\} \tag{3.4.10}$$

4

The Quark Model

In this chapter, we will briefly discuss the group SU(3) and its relation to the Quark Model. The mathematical background for this chapter: Group theory and Lie algebra can be found in Appendix B

4.1. SU(3)

In general, SU(n) has $n^2 - 1$ generators ⇒ SU(3) has 8 generators. A representation of this group in terms of the Gell-Mann Matrices is

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

These satisfy the lie algebra

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2} \tag{4.1.1}$$

where $f_{123} = f_{312} = f_{231} = 1$, $f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2}$, $f_{156} = f_{367} = \frac{-1}{2}$, and $f_{458} = f_{678} = \frac{\sqrt{3}}{2}$. Let $\frac{\lambda_a}{2} = F_a$. Then the Lie Algebra for $SU(3)$ is

$$[F_a, F_b] = i f_{abc} F_c \tag{4.1.2}$$

Defining raising and lowering operators

$$T_\pm = F_1 \pm F_2,\ V_\pm = F_4 \pm F_5,\ U_\pm = F_6 \pm F_7$$

and also,

$$T_3 = F_3,\ Y = \frac{2}{\sqrt{3}} F_8$$

One can express the lie algebra in terms of these operators

$$[T_3, T_\pm] = T\pm, \quad [Y, T_\pm] = 0$$
$$[T_3, U_\pm] = \mp\frac{1}{2}U_\pm, \quad [Y, U_\pm] = \pm U_\pm$$
$$[T_3, V_\pm] = \pm\frac{1}{2}V_\pm, \quad [Y, V_\pm] = \pm V_\pm$$
$$[T_+, T_-] = 2T_3,$$
$$[U_+, U_- =]\frac{3}{2}Y - T_3 = 2U_3, \quad [V_+, V_- =]\frac{3}{2}Y + T_3 = 2V_3$$

The Fundamental Representation

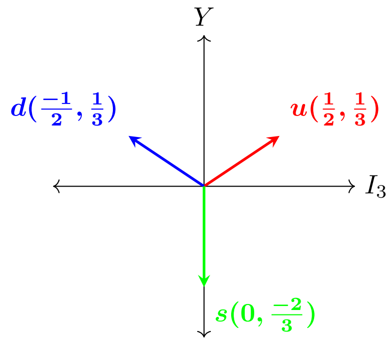
The states of SU(3) can be labeled by the eigenvalues of the diagonal generators T_3 and Y: (t_3, y) and the action of the raising and lowering operators on the states is as follows-

$$T_{\pm}(t_3, y) = (t_3 \pm 1, y)$$

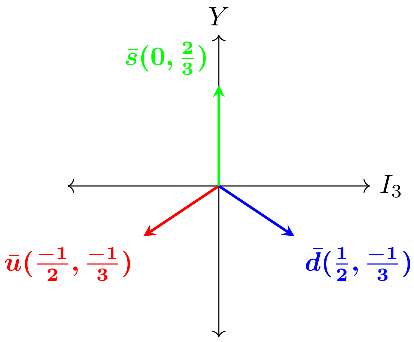
$$U_{\pm}(t_3, y) = (t_3 \mp \frac{1}{2}, y \pm 1)$$

$$V_{\pm}(t_3, y) = (t_3 \pm \frac{1}{2}, y \pm 1)$$

For the Quark Model, $T_3 \rightarrow I_3$, the isospin, and (u, d, s) form the fundamental representation. Then, in the (i_3, y) space

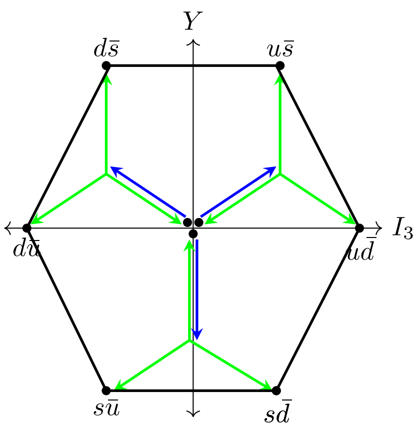


Similarly, the conjugate representation, composed of the anti-quarks $(\bar{u}, \bar{d}, \bar{s})$:



4.2. Meson and Baryon Octets

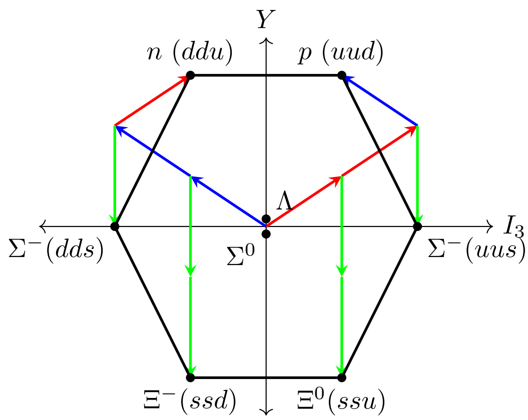
Using these representations, we can easily construct the meson octet as follows (using blue arrows for (u, d, s) and green ones for $(\bar{u}, \bar{d}, \bar{s})$)



where at the center we have

$$\eta' = \frac{u\bar{u} + d\bar{d} + s\bar{s}}{\sqrt{3}}$$

For the spin 1/2 Baryon Octet, since all particles involved are quarks, we use red color to represent u, blue to represent d and green to represent s, just as it is shown in the fundamental representation above.



5

Quantum Field Theory

5.1. The Klein-Gordon Field

The Lagrangian for classical Klein-Gordon field is given by

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{5.1.1}$$

and the conjugate momentum density to $\psi(x)$ is $\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(x)$. Then the Hamiltonian is

$$H = \int d^3x\mathcal{H} = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi) + \frac{1}{2}m^2\phi^2\right] \tag{5.1.2}$$

Canonical quantization, or second quantization involves promoting the the dynamical variables, in this case the real Klein-Gordon field to operators and imposing suitable commutation relations.
Recall, for a discrete system,

$$[q_i, p_j] = i\delta_{ij};$$

$$[q_i, q_j] = [p_i, p_j] = 0$$

By analogy, $\phi(x)$ and $\pi(y)$ obey

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \tag{5.1.3}$$

$$[\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{y})] = 0 \tag{5.1.4}$$

In Fourier space, the classical Klein-Gordon field can be expanded as

$$\phi(\boldsymbol{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\phi(\boldsymbol{p}, t) \tag{5.1.5}$$

where $\phi^*(\boldsymbol{p}) = \phi(-\boldsymbol{p})$ so that $\phi(\boldsymbol{x})$ is real.
Replacing this in the Klein-Gordon equation gives

$$\left[\frac{\partial^2}{\partial t^2} + (|\boldsymbol{p}|^2 + m^2)\right]\phi(\boldsymbol{p}, t) = 0 \tag{5.1.6}$$

which is the equation of a simple harmonic oscillator. By analogy with the quantum mechanical simple harmonic oscillator, we can say

$$\phi(\boldsymbol{p}) = \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right)$$

and

$$\pi(\boldsymbol{p}) = -i\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right)$$

where

$$\omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2} \tag{5.1.7}$$

is the frequency of the oscillator corresponding to momentum $\boldsymbol{p}$. Therefore,

$$\phi(\boldsymbol{x}, t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}) \tag{5.1.8}$$

$$\pi(\boldsymbol{x}, t) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}) \tag{5.1.9}$$

The corresponding commutation relation between the creation and annihilation operators, $a^\dagger$ and a respectively, is

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p'}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p'}) \tag{5.1.10}$$

Further, we can rewrite the Hamiltonian in terms of the ladder operators

$$H = \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\boldsymbol{p}+\boldsymbol{p'})\cdot\boldsymbol{x}}\left[-\frac{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p'}}}}{4}(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}})(a_{\boldsymbol{p'}} - a^\dagger_{-\boldsymbol{p'}}) + \frac{(-\boldsymbol{p}\cdot\boldsymbol{p'}) + m^2}{4\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p'}}}}(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}})(a_{\boldsymbol{p'}} + a^\dagger_{-\boldsymbol{p'}})\right]$$

$$\Rightarrow H = \int \frac{d^3p}{(2\pi)^3}\frac{\omega_{\boldsymbol{p}}}{2}\left(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}\right) = \int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}\left(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + \frac{1}{2}[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}]\right) \tag{5.1.11}$$

Note: Recognize that the second term in the Hamiltonian is $\delta(0)$ and is thus infinite. This arises because we have summed over the ground state energies $\omega_{\boldsymbol{p}}/2$ corresponding to all the momenta $\boldsymbol{p}$. We ignore this infinite constant on the grounds that only the change in energy can be measured experimentally and this absolute energy would not be measured.

The Klein-Gordon Propagator

Writing $\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle$ as a four dimensional integral, assuming $x^0 > y^0$

$$\begin{aligned}\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}) \\ &= \int \frac{d^3p}{(2\pi)^3}\int \frac{dp^0}{2\pi}\frac{i}{p^2 - m^2}e^{-ip\cdot(x-y)} \text{ for } x^0 > y^0\end{aligned} \tag{5.1.12}$$

Defining

$$D_R(x - y) \equiv \theta(x^0 - y^0)\,\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle \tag{5.1.13}$$

the Green's function of the Klein-Gordon equation. Using Feynman prescription for contour integration, we obtain

$$D_R(x - y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip\cdot(x-y)} \tag{5.1.14}$$

In general,

$$\begin{aligned}D_F(x - y) &= \begin{cases} D(x - y) \text{ for } x^0 > y^0 \\ D(y - x) \text{ for } x^0 < y^0 \end{cases} \\ &= \theta(x^0 - y^0)\,\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle + \theta(y^0 - x^0)\,\langle 0|\,[\phi(y), \phi(x)]\,|0\rangle \\ &\equiv \langle 0|\,T\phi(x), \phi(y)\,|0\rangle\end{aligned} \tag{5.1.15}$$

where T is the "time-ordering" symbol.
$D_F(x - y)$ is called the Feynman Propagator for the Klein-Gordon Field and is the propagation amplitude of the particle.

5.2. Complex Scalar Field Theory

Consider a complex valued scalar field obeying the Klein-Gordon Equation and having action

$$S = \int \mathrm{d}^4 x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi\phi^*) \tag{5.2.1}$$

where $\phi(x)$ and $\phi^*(x)$ are the dynamical variables. In general,

$$\begin{aligned} S &= \int \mathrm{d}^4 x \mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*) \\ \Rightarrow \mathcal{L} &= \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi\phi^* \end{aligned} \tag{5.2.2}$$

By definition,

$$\begin{aligned} \pi_1(x) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ &= \frac{\partial}{\partial \dot{\phi}} (\dot{\phi}\dot{\phi}^* - (\nabla\phi)\cdot(\nabla\phi^*) - m^2\phi\phi^*) \\ \Rightarrow \pi(x) &= \dot{\phi}^*(x) \end{aligned} \tag{5.2.3}$$

Similarly,

$$\pi_2(x) = \dot{\phi}(x) \tag{5.2.4}$$

Hence, we can say $\pi_1 = \pi_2^* = \pi(x)$ and

$$\pi(x) = \dot{\phi}^*(x) \text{ and } \pi^*(x) = \dot{\phi}(x) \tag{5.2.5}$$

Commutation Relations

$$\begin{aligned} [\phi(x), \pi(y)] &= i\delta(x-y) \\ [\phi^*(x), \pi^*(y)] &= -i\delta(x-y) \\ [\phi(x), \pi^*(y)] &= [\phi^*(x), \pi(y)] = 0 \end{aligned}$$

Hamiltonian

$$\begin{aligned} \mathcal{H} &= \sum \pi(x)\phi(x) - \mathcal{L} \\ &= \pi\pi^* + \pi^*\pi - \dot{\phi}\dot{\phi}^* + (\nabla\phi)\cdot(\nabla\phi^*) + m^2\phi\phi^* \\ &= \dot{\phi}\dot{\phi}^* + \dot{\phi}\dot{\phi}^* - \dot{\phi}\dot{\phi}^* + (\nabla\phi)\cdot(\nabla\phi^*) + m^2\phi\phi^* \\ \Rightarrow \mathcal{H} &= \pi\pi^* + (\nabla\phi)\cdot(\nabla\phi^*) + m^2\phi\phi^* \end{aligned} \tag{5.2.6}$$

Heisenberg Equations of Motion

In general

$$i\frac{\partial \mathcal{O}}{\partial t} = [\mathcal{O}, H]$$

Thus,

$$\begin{aligned} i\frac{\partial \phi(x')}{\partial t} &= \left[\phi(x'), \int \mathrm{d}^3 x \left(\pi\pi^* + (\nabla\phi)\cdot(\nabla\phi^*) + m^2\phi\phi^*\right)\right] \\ &= \int \mathrm{d}^3 x (\phi'\pi\pi^* - \pi\pi^*\phi' + \phi'(\nabla\phi)\cdot(\nabla\phi^*) - (\nabla\phi)\cdot(\nabla\phi^*)\phi' + m^2\phi'\phi\phi^* - m^2\phi\phi^*\phi') \end{aligned} \tag{5.2.7}$$

Using the commutation relations,

$$\begin{aligned} i\frac{\partial \phi(x')}{\partial t} &= \int \mathrm{d}^3 x (i\delta(x'-x)\pi^* + \pi\phi'\pi^* - \pi\phi'\pi^*) \\ &= i\int \mathrm{d}^3 x \delta(x'-x)\pi^* \\ &= i\pi'^* \\ &= i\frac{\partial \phi(x')}{\partial t} \end{aligned} \tag{5.2.8}$$

which is a trivial result. However, for the conjugate momentum:

$$\begin{aligned} i\frac{\partial \pi(x')}{\partial t} &= \left[\phi(x'), \int \mathrm{d}^3x \left(\pi\pi^* + (\nabla\phi)\cdot(\nabla\phi^*) + m^2\phi\phi^*\right)\right] \\ &= \int \mathrm{d}^3x(\pi'\pi\pi^* - \pi\pi^*\pi' + \pi'(\nabla\phi)\cdot(\nabla\phi^*) - (\nabla\phi)\cdot(\nabla\phi^*)\pi' + m^2\pi'\phi\phi^* - m^2\phi\phi^*\pi') \end{aligned} \tag{5.2.9}$$

Using the commutation relations

$$\begin{aligned} i\frac{\partial \pi(x')}{\partial t} &= \int \mathrm{d}^3x(\pi'(\nabla\phi)\cdot(\nabla\phi^*) - (\nabla\phi)\cdot(\nabla\phi^*)\pi' + m^2\pi'\phi\phi^* - m^2\phi\phi^*\pi') \\ &= \int \mathrm{d}^3x(-i(\nabla\delta(x'-x))\nabla\phi^* - im^2\delta(x'-x)\phi^*) \end{aligned} \tag{5.2.10}$$

Integrating the first term by parts evaluating the integral yields

$$\begin{aligned} i\frac{\partial \pi}{\partial t} &= i(\nabla^2\phi^* - m^2\phi^*) = i\frac{\partial^2\phi^*}{\partial t^2} \\ &\Rightarrow (\partial_\mu\partial^\mu + m^2)\phi^* = 0 \end{aligned} \tag{5.2.11}$$

i.e. ϕ^* satisfies the Klein-Gordon Equation. The same can be said for ϕ

$$(\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{5.2.12}$$

The Creation and Annihilation Operators

For the real Klein-Gordon Field,

$$\begin{aligned} \phi(\boldsymbol{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}})e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \\ \pi(\boldsymbol{x}) &= \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}})e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \end{aligned} \tag{5.2.13}$$

However, since ϕ and ϕ^* are two different dynamical variables in the Complex Scalar Field Theory, by analogy, we have

$$\pi^*(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}(a_{\boldsymbol{p}} - b^\dagger_{-\boldsymbol{p}})e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{5.2.14}$$

$$\phi^*(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(b_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}})e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{5.2.15}$$

and the creation and annihilation operators have the commutators

$$\begin{aligned} [a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p'}}] &= (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p'}) \\ [b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{p'}}] &= (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p'}) \end{aligned} \tag{5.2.16}$$

All other combinations commute.
Expressing the Hamiltonian in terms of creation and annihilation operators

$$\begin{aligned} H = \int d^3x \int \frac{d^3pd^3p'}{(2\pi)^6} & \frac{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p'}}}}{2}(-1)(b_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}})(a_{\boldsymbol{p'}} - b^\dagger_{-\boldsymbol{p'}})e^{i(\boldsymbol{p}+\boldsymbol{p'})\cdot\boldsymbol{x}} \\ &+ \frac{(-\boldsymbol{p}\cdot\boldsymbol{p'}) + m^2}{2\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p'}}}}(a_{\boldsymbol{p}} + b^\dagger_{-\boldsymbol{p}})(b_{\boldsymbol{p'}} + a^\dagger_{-\boldsymbol{p'}})e^{i(\boldsymbol{p}+\boldsymbol{p'})\cdot\boldsymbol{x}} \end{aligned}$$

Integrating over $\boldsymbol{x}$ and $\boldsymbol{p'}$ gives

$$H = \int \frac{d^3p}{(2\pi)^3}\frac{E_{\boldsymbol{p}}}{2}\left[(-1)(b_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}})(a_{-\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}}) + (a_{\boldsymbol{p}} + b^\dagger_{-\boldsymbol{p}})(b_{-\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}})\right]$$

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{E_{\boldsymbol{p}}}{2} \left(-b_{\boldsymbol{p}} a_{-\boldsymbol{p}} + b_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}} a_{-\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} + a_{\boldsymbol{p}} b_{-\boldsymbol{p}} + a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} + b^\dagger_{-\boldsymbol{p}} b_{-\boldsymbol{p}} + b^\dagger_{-\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} \right)$$

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} \left[b_{\boldsymbol{p}} a_{-\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} + a_{\boldsymbol{p}} b_{\boldsymbol{p}} + b^\dagger_{-\boldsymbol{p}} a^\dagger_{-\boldsymbol{p}} + [a_{\boldsymbol{p}}, a^\dagger_{-\boldsymbol{p}}] + [b^\dagger_{-\boldsymbol{p}}, b_{\boldsymbol{p}}] \right] \tag{5.2.17}$$

which is Hamiltonian corresponding to two sets of particles with mass m.

Conserved Charge

$$Q = \int d^3x \frac{i}{2} (\phi^* \pi^* - \pi \phi) \tag{5.2.18}$$

$$\begin{aligned} Q &= \int d^3x \frac{1}{2} \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\boldsymbol{p}+\boldsymbol{p'})\cdot\boldsymbol{x}} \left[(b_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}})(a_{\boldsymbol{p'}} - b^\dagger_{-\boldsymbol{p'}}) - (b_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}})(a_{\boldsymbol{p'}} + b^\dagger_{-\boldsymbol{p'}}) \right] \\ &= \int d^3x \frac{1}{2} \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\boldsymbol{p}+\boldsymbol{p'})\cdot\boldsymbol{x}} \left[b_{\boldsymbol{p}} a_{\boldsymbol{p'}} + a^\dagger_{-\boldsymbol{p}} a_{\boldsymbol{p'}} - b_{\boldsymbol{p}} b^\dagger_{-\boldsymbol{p'}} - a^\dagger_{-\boldsymbol{p}} b^\dagger_{-\boldsymbol{p'}} - b_{\boldsymbol{p}} a_{\boldsymbol{p'}} + a^\dagger_{-\boldsymbol{p}} a_{\boldsymbol{p'}} - b_{\boldsymbol{p}} b^\dagger_{-\boldsymbol{p'}} + a^\dagger_{-\boldsymbol{p}} b^\dagger_{-\boldsymbol{p'}} \right] \end{aligned} \tag{5.2.19}$$

Integrating over x and p' and simplifying

$$Q = \int \frac{d^3p}{(2\pi)^2} \left[a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} - b_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} \right] \tag{5.2.20}$$

5.3. The Dirac Field

The key difference between the quantization procedure for Dirac fields as compared to Klein-Gordon Field is the use of *anti*-commutators in place of the commutators.
The equal time commutation relations are defined as

$$\begin{aligned} \{\psi_a(\boldsymbol{x}), \psi^\dagger_b(\boldsymbol{y})\} &= \delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\delta_{ab} \\ \{\psi_a(\boldsymbol{x}), \psi_b(\boldsymbol{y})\} = \{\psi^\dagger_a(\boldsymbol{x}), &\psi^\dagger_b(\boldsymbol{y})\} = 0 \end{aligned} \tag{5.3.1}$$

The creation and annihilation operators satisfy

$$\{a^r_{\boldsymbol{p}}, a^{s\,\dagger}_{\boldsymbol{q}}\} = \{b^r_{\boldsymbol{p}}, b^{s\,\dagger}_{\boldsymbol{q}}\} = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q})\delta^{rs} \tag{5.3.2}$$

with all other anti-commutators equal to zero. Here, $b^{s\,\dagger}_{\boldsymbol{q}}$ creates negative energy solutions. However, by the virtue of the symmetry of the anti-commutators, we can define

$$\tilde{b}^s_{\boldsymbol{p}} b^{s\,\dagger}_{\boldsymbol{p}} \ ; \ \tilde{b}^{s\,\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}}$$

which obey the exact same commutation relations. By making the choice of vacuum $|0\rangle$ such that it is annihilated by $a^s_{\boldsymbol{p}}$, $\tilde{b}^s_{\boldsymbol{p}}$ then both $a^{s\,\dagger}_{\boldsymbol{p}}$ and $b^{s\,\dagger}_{\boldsymbol{p}}$ produce positive energy particles.
The field operators are defined as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left(a^s_{\boldsymbol{p}} u^s(p) e^{-ip\cdot x} + b^{s\,\dagger}_{\boldsymbol{p}} v^s(p) e^{ip\cdot x} \right) \tag{5.3.3}$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left(b^s_{\boldsymbol{p}} \bar{v}^s(p) e^{-ip\cdot x} + a^{s\,\dagger}_{\boldsymbol{p}} \bar{u}^s(p) e^{ip\cdot x} \right) \tag{5.3.4}$$

and the Hamiltonian is found to be

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\boldsymbol{p}} \left(a^{s\,\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} + b^{s\,\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}} \right) \tag{5.3.5}$$

Momentum operator is

$$\boldsymbol{P} = \int \frac{d^3p}{(2\pi)^3} \sum_s \boldsymbol{p} \left(a^{s\,\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} + b^{s\,\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}} \right) \tag{5.3.6}$$

The particles created by $a_{\boldsymbol{p}}^{s\,\dagger}$ are called *fermions* and the ones created by $b_{\boldsymbol{p}}^{s\,\dagger}$ are called *anti-fermions*.
The one-particle states are defined with the appropriate normalization

$$|\boldsymbol{p}, s\rangle \equiv \sqrt{2E_{\boldsymbol{p}}}a_{\boldsymbol{p}}^{s\,\dagger}|0\rangle \tag{5.3.7}$$

such that the inner product

$$\langle \boldsymbol{q}, r|\boldsymbol{p}, s\rangle = 2E_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs} \tag{5.3.8}$$

is Lorentz covariant.
In Dirac theory, the current is $j^\mu = \bar{\psi}\gamma^\mu\psi$. The charge associated with this current

$$Q = \int d^3\psi^\dagger(x)\psi(x) = \int \frac{d^3p}{(2\pi)^3}\sum_s \left(a_{\boldsymbol{p}}^{s\,\dagger}a_{\boldsymbol{p}}^s + b_{-\boldsymbol{p}}^s b_{-\boldsymbol{p}}^{s\,\dagger} +\right)$$

which can be rewritten as

$$Q = \int \frac{d^3p}{(2\pi)^3}\sum_s \left(a_{\boldsymbol{p}}^{s\,\dagger}a_{\boldsymbol{p}}^s + b_{\boldsymbol{p}}^{s\,\dagger}b_{\boldsymbol{p}}^s +\right) \tag{5.3.9}$$

where another infinite term due to the commutator has been ignored. 5.3.9 implies that $a_{\boldsymbol{p}}^{s\,\dagger}$ creates a particle with charge $+1$ and $b_{\boldsymbol{p}}^{s\,\dagger}$ creates a particle with charge -1.

The Dirac Propagator

Similar to the procedure for the Klein-Gordon equation, we find the retarded Green's function for the Dirac equation to be

$$S_R^{ab}(x-y) = \theta(x^0-y^0)\langle 0|\{\psi_a(x), \bar{\psi}_b(y)\}|0\rangle \tag{5.3.10}$$

$$\Rightarrow S_R(x-y) = (i\partial\!\!\!/_x + m)D_R(x-y) \tag{5.3.11}$$

It can be verified that this S_R is the Green's function of the Dirac operator:

$$(i\partial\!\!\!/_x - m)S_R(x-y) = i\delta^{(4)}(x-y)\cdot \mathbb{1}_{4\times 4}$$

With Feynman boundary conditions this is defined as

$$\begin{aligned} S_F(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{i(p\!\!\!/ + m)}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)} \\ &= \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle & \text{for } x^0 > y^0 \\ -\langle 0|\bar{\psi}(y)\psi(x)|0\rangle & \text{for } x^0 < y^0 \end{cases} \\ &\equiv \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle \end{aligned} \tag{5.3.12}$$

5.4. Interacting Fields and Feynman Diagrams

Wick's Theorem

The time ordered product of a set of operators can be decomposed into a sum of corresponding normal ordered product:

$$T(\phi_1\phi_2\phi_3\ldots\phi_m) = :\phi_1\phi_2\phi_3\ldots\phi_m: + \text{ all possible contractions of operator pairs} \tag{5.4.1}$$

Feynman Diagrams

Consider the time ordered product $\langle 0| T\phi_1\phi_2\phi_3\phi_4 |0\rangle$. Applying Wick's Theorem:

$$\begin{aligned} T\phi_1\phi_2\phi_3\phi_4 = {}&:\phi_1\phi_2\phi_3\phi_4: + :\wick{\c1\phi_1\c1\phi_2}\phi_3\phi_4: + :\wick{\c1\phi_1\phi_2\c1\phi_3}\phi_4: + :\wick{\c1\phi_1\phi_2\phi_3\c1\phi_4}: \\ &+ :\phi_1\wick{\c1\phi_2\c1\phi_3}\phi_4: + :\phi_1\wick{\c1\phi_2\phi_3\c1\phi_4}: + :\phi_1\phi_2\wick{\c1\phi_3\c1\phi_4}: \\ &+ :\wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4}: + :\wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4}: + :\wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}: \end{aligned} \tag{5.4.2}$$

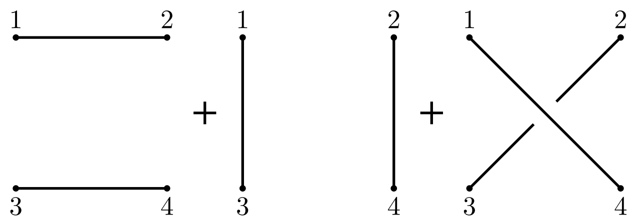
where

$$\begin{aligned} :\wick{\c1\phi_1\c1\phi_2}\phi_3\phi_4: &\equiv :\phi_3\phi_4: \langle 0| T\phi_1\phi_2 |0\rangle \\ &= :\phi_3\phi_4: D_F(x_1 - x_2) \end{aligned}$$

We know for any O, $\langle 0| :\mathcal{O}: |0\rangle = 0$. Therefore, 5.4.2 reduces to

$$\begin{aligned} \langle 0| T\phi_1\phi_2\phi_3\phi_4 |0\rangle = {}&D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) \\ &+ D_F(x_1 - x_4)D_F(x_2 - x_3) \end{aligned} \tag{5.4.3}$$

which can be diagrammatically represented as



In the presence of interaction terms in the Hamiltonian, we define the ground states to be $|\Omega\rangle$. Then evaluating the two point function

$$\langle \Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\left\{\phi_I(x)\phi_I(y)exp\left[-i\int_{-T}^{T} dtH_{int}(t)\right]\right\} |0\rangle}{\langle 0| T\left\{exp\left[-i\int_{-T}^{T} dtH_{int}(t)\right]\right\} |0\rangle} \tag{5.4.4}$$

where the numerator represents the sum of all possible diagrams with two external points. On further analysis we see that the two point correlation function $\langle 0| T\left\{\phi_I(x)\phi_I(y)exp\left[-i\int_{-T}^{T} dtH_{int}(t)\right]\right\} |0\rangle$ can be factorized such that

$$\langle 0| T\left\{\phi_I(x)\phi_I(y)exp\left[-i\int_{-T}^{T} dtH_{int}(t)\right]\right\} |0\rangle = \left(\sum \text{connected}\right) \times \left(\sum_i V_i\right)$$

where V_i are all the *disconnected diagrams*, i.e., the diagrams that are internally connected but disconnected from the external points. Similarly,

$$\langle 0| T\left\{exp\left[-i\int_{-T}^{T} dtH_{int}(t)\right]\right\} |0\rangle = \left(\sum_i V_i\right)$$

$$\langle \Omega| T\{\phi(x)\phi(y)\} |\Omega\rangle = \text{sum of all connected diagrams with two external points} \tag{5.4.5}$$

These properties can eventually be used to deduce the Feynman rules for Quantum Electrodynamics given in Section 3.2

6
Summary

This report has can best be viewed as a condensed description of the route from Non-relativistic Quantum Mechanics and Special Relativity, two theories developed in the early 20th century, to Quantum Field Theory, while also discussing cross-section calculations and elements of Quantum Chromodynamics along the way.

The Klein-Gordon and Dirac Equations served as the first steps along the way. They predicted the existence of antiparticles and their solutions were discussed. Moreover, the properties of Dirac equation introduced the Gamma matrices and their properties which were instrumental in cross-section calculations using Feynman Diagrams.

This was followed by an overview of Group Theory and the interpretation of the fundamental representation of SU(3) group as the three quarks: up, down and strange. These could then be used to geometrically construct the meson and hadron octets and served as an illustration of how the Eightfold Way come about and could be used to predict the existence of particles, as it successfully did in the case of Ω^-.

The last chapter discussed canonical quantization of the Klein-Gordon and Dirac Fields by raising the dynamical variables (fields the corresponding conjugate momenta) to operators and introducing the creation and annihilation operators. Finally, Feynman Diagrams were derived from Hamiltonians containing interaction terms.

The next report will contain discussions on Renormalization, Ultraviolet divergences and Quantum Chromodynamics as well as the project/research work on Infrared Divergences.

References

[1] D.J. Griffiths. *Introduction to Elemtary Particles*. Wiley-VCH, 2008.

[2] F. Halzen and A.D. Martin. *Quarks and Leptons*. Wiley India, 2008.

[3] M.E. Peskin and D.V. Schroeder. *An Introcuction to Quantum Field Theory*. CRC Press, 2018.

[4] J.J. Sakurai. *Advanced Quantum Mechanics*. Pearson, 2002.

[5] F. Schwabl. *Advanced Quantum Mehcanics*. Springer, 2008.

A

Trace Theorems and Identities

A.1. Trace Theorems

1. $\mathrm{Tr}(\mathbb{1}) = 4$
2. Trace of an odd number of γ matrices is always zero.
3. $\mathrm{Tr}(\not{a}\not{b}) = 4a \cdot b$

   Proof:

$$\begin{aligned} Tr(\not{a}\not{b}) &= a_\mu b_\nu Tr(\gamma^\mu \gamma^\nu) \\ &= \frac{1}{2} a_\mu b_\nu Tr(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \\ &= a_\mu b_\nu g^{\mu\nu} Tr(\mathbb{1}) \\ &= 4a \cdot b \end{aligned}$$

   Also, $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$.

4. $\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$
5. $\mathrm{Tr}(\gamma_5 \not{a}\not{b}) = 0$
6. $\mathrm{Tr}(\gamma_5 \not{a}\not{b}\not{c}\not{d}) = 4i\epsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma$

A.2. Identities

1. $\gamma_\mu \gamma^\mu = 4\mathbb{1}$
2. $\gamma_\mu \not{a} \gamma^\mu = -\not{a}$
3. $\gamma_\mu \not{a}\not{b} \gamma^\mu = 4a \cdot b$
4. $\gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{a}\not{b}\not{c}$

B

Group Theory

A group $\mathcal{G} = \{a, b, c, \dots\}$ is a set of objects with a composition rule which satisfies

1. Closure: if $a, b \in \mathcal{G}$ then $c = ab \in \mathcal{G}$
2. Associativity: if $a, b, c \in \mathcal{G}$ then $(ab)c = a(bc)$
3. Existence of Identity: $\exists\ e \in \mathcal{G}$ such that $ae = ea = a$, $\forall a \in \mathcal{G}$
4. Existence of Inverse: $\forall a \in \mathcal{G}$, $\exists a^{-1} \in \mathcal{G}$ such that $aa^{-1} = a^{-1}a = \mathbb{1}$

Note: A group $\mathcal{G} = \{R_i\}$ is said to be an *Abelian* group if $R_i R_j = R_j R_i$, $\forall R_i, R_j \in \mathcal{G}$.

B.1. Representation of a Group

Representation involves mapping of the elements of a group $\mathcal{G} = \{g_i\}$ on a set of linear operators/matrices $D(\mathcal{G})$ with the properties

(i) $D(e) = \mathbb{1}$

(ii) $D(g_1)D(g_2) = D(g_1 g_2)$

A representation is said to be *Faithful* if each element of $\mathcal{G}$ is mapped to a unique element of $D(\mathcal{G})$.

Reducible and Irreducible Representations

If a representation can be brought to a block diagonal form by

$$D_{\text{diagonal}} = MD(a)M^{-1}$$

where M is a non-singular matrix then $D(a)$ is called a *Reducible* representation.
If this decomposition is not possible, the representation is said to *Irreducible*

B.2. Lie Groups

Consider a continuous group

$$a(\vec{\theta}) = a(\theta_1, \theta_2, \dots, \theta_n)$$

where

$$e = a(\vec{0}) = a(0, 0, \dots, 0)$$

then multiplying two elements

$$a(\vec{\theta})a(\vec{\phi}) = a(\vec{\xi})$$

where $\vec{\xi} = f(\vec{\theta}, \vec{\phi})$ and satisfies the following properties

(i)
$$f(\vec{0},\vec{\phi}) = \vec{\phi}$$

(ii)
$$f(\vec{\theta},\vec{0}) = \vec{\theta}$$

(iii)
$$f(\vec{\theta}, f(\vec{\phi},\vec{\xi})) = f(\vec{\theta,\vec{\phi}},\vec{\xi})$$

(iv) Corresponding to every $\vec{\theta}$, $\exists\vec{\theta}'$ such that
$$f(\vec{\theta},\vec{\theta}') = 0$$
and $[a(\vec{\theta})]^{-1} = a(\vec{\theta}')$

A group is called a *Lie Group* if the function
$$\vec{\xi} = f(\vec{\theta},\vec{\phi})$$
is an analytic function, i.e., it is continuously differentiable with respect to $\vec{\theta},\vec{\phi}$.

Unitary Representation of Lie Groups

For a lie group $\mathcal{G} = \{g(\alpha)\}$,
$$g(\alpha)|_{\alpha=0} = e$$
and its representation $D(\alpha)$
$$D(\alpha)|_{\alpha=0} = \mathbb{1}$$
For $d\alpha \to 0$,
$$D(\alpha) = \mathbb{1} + iX_a d\alpha_a$$
where $X_a = \left.\frac{\partial D(\alpha)}{\partial \alpha_a}\right|_{\alpha=0}$ Repeatedly applying this infinitesimal transformation takes us finitely far away from $\mathbb{1}$ in the direction 'a'. To do so, we raise the infinitesimal group element to a large power and let $\alpha \to \left(\frac{\alpha_a}{k}\right) k$. Then
$$D(\alpha) = \lim_{k\to\inf}\left[\mathbb{1} + \frac{i\alpha_a X_a}{k}\right] = e^{i\alpha_a X_a} \tag{B.2.1}$$

Lie Algebra

The generators (eg. X_a in the above derivation) of Lie Groups satisfy commutation relations called Lie Algebra which is in general given by
$$[X_a, X_b] = ifabcX_c$$
where $fabc = -fabc$ is the structure constant. This satisfies the Jacobi Identity
$$[X_a,[X_b,X_c]] + [X_b,[X_c,X_a]] + [X_c,[X_a,X_b]] = 0 \tag{B.2.2}$$
$$\Rightarrow f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} = 0 \tag{B.2.3}$$