Abstract

When we design the ultimate brownie pan for a rectangular oven, space utilization rate (SUR) and even heat distribution are two essential concerns. Intuitively, rectangular pans have higher SUR but the heat distribution is not even, and circular pans are on the contrary. In the paper, we introduce rounded rectangles as transition shapes between rectangle and circle to seek the optimum shape of pan by changing the parameters of the rounded rectangles.

This problem can be solved in two steps: First, we build the Heat Distribution Model (use software *COMSOL Multiphysics 4.3*). Second, we build the Shape Evaluation Model to select the best shape of pan.

In the Heat Distribution Model, we solve the 3-D heat equation — a partial differential equation. We study the process of heat transfer as (a) convection between air and pan (and the top face of brownies) and (b) conduction between pan and brownie. Considering the incomplete contact of the pan and brownies, we introduce thermal contact conduction to simplify and unify the boundary condition and make the heat equation solvable.

There are two sub-problems in the Shape Evaluation Model — First, we must find a proper parameter to describe the SUR. Due to the area-sensitivity of such problem, we set a continuous range of the area of racks, and evaluate the mean SUR s_1 within the range. Second, known the heat distribution, we calculate s_2, the variance of the temperature gradient, to describe the inequality degree of heat distribution. Then we normalize s_1 and s_2 and assign the weights p and $(1-p)$ to them and add up to get the comprehensive score.

Through data analysis, we draw two important conclusions:

- (a) We prefer to choose the pan with width-to-length ratio (WLR) that is closest to the oven's WLR.
- (b) When p is close to 1, we prefer to choose circular pans; when p is close to 0, we choose rectangular pans; in other cases, we choose rounded rectangular pans.

Optimize the Shape For Brownie Pan
MCM 2013 Problem A

Contents

1 Introduction

Space utilization rate and even heat distribution are two essential concerns when people choose baking pans. When baking in a rectangular pan heat is concentrated in the 4 corners and the product gets overcooked at the corners (and to a lesser extent at the edges). In a round pan the heat is distributed evenly over the entire outer edge and the product is not overcooked at the edges. However, since most ovens are rectangular in shape using round pans is not efficient with respect to using the space in an oven.

In conclusion, rectangular pans have higher space utilization rate while round pans take the advantage of even heat distribution. How to balance this two factors is critical for the design of baking pans.

2 Assumptions

The problem gives us the following assumptions:

- The shape of the oven is a rectangular with a width to length ratio of W/L.
- Each pan must have an area of A. In our model, we set $A = 0.06\,\mathrm{m}$.
- Initially two racks in the oven, evenly spaced.

Besides, we make these assumptions in our model:

- The racks have the same width to length ratio W/L.
- Two racks of the oven share the same shape. So the best strategy to arrange most pans on them is also the same. The SUR of two racks is equal to that of one rack so that we only need to consider brownie pans on one rack.
- Before baking starts, the oven has been well-preheated so that the temperature of inside air and the racks have all reached the set temperature $T_{\text{ext}} = 450\,\mathrm{K}$.
- The side faces of the pan are perpendicular to the bottom. and the height of the pan is fixed at $b = 0.05\,\mathrm{m}$ while the shape of the bottom varies.
- The pan is made of material that heats and cools quickly, so we can ignore the time for the pan to heat up and assume the temperature of the pan is the same with the oven's, i.e., T_{ext}, initially.

- The heat radiation is neglected.
- Pans are full of the material for making brownies, and when fully baked, the brownies still has the same height.
- The material for making brownies is isotropic.
- All chemical reactions during the baking process should be neglected.

3 Terms and Description

All terms in the paper are defined as followed:

Description	Symbol	Unit
The width of oven	W	m
The length of oven	L	m
The width of pan	w	m
The length of pan	l	m
The radius of circle	r	m
Coefficient of heat transfer	k	$\mathrm{W/(m\cdot K)}$
Density	ρ	$\mathrm{kg/m^3}$
Thermal capacitance	c	$\mathrm{J/(kg\cdot K)}$
Temperature in brownie	u	K
Temperature of outer space	T_{ext}	K
Heat flux	$\mathbf{q}$	$\mathrm{J/(m\cdot s)}$
Outward unit normal of the surface	$\mathbf{n}_0$	/
Convection heat transfer coefficient	h	$\mathrm{W/(m^2\cdot K)}$
The area of rack	x	$\mathrm{m^2}$
The largest number of pans that can be fitted on a rack for a certain shape of pan	n_0	/
The largest number of pans that can be fitted on a rack for all shapes of pan	n_{max}	/
Space utilization score of a certain shape	s_1	/
Variance of gradient of a certain shape	s_2	$\mathrm{K^2/m^2}$

Table 1: Terms and Descriptions

4 Model for Heat Distribution

As we all know, there are three ways for heat exchange — conduction, convection and radiation. According to our assumption, radiation is neglected. Our model is with concern of conduction and convention, which can both be depicted by the heat equation mentioned below.

4.1 Heat Equation

We begin with the 3-dimensional heat equation

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{q_s}{\rho c} \tag{1}$$

where

$$\alpha = \frac{k}{\rho c}$$

and q_s is the source heat per unit volume. Since there is no heat source inside pans or brownies, we have

$$q_s = 0.$$

Thus Eq. (1) can be rewritten as

$$\rho c\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0. \tag{2}$$

The descriptions of these symbols can be found in Table 1.

4.2 Initial Condition

At $t = 0$, the temperature of pan (and brownies) is equal to the room temperature. That gives the initial condition

$$u|_{t=0} = 300. \tag{3}$$

4.3 Boundary Conditions

The boundary conditions are more complicated, because both the pan and the brownies are in touch with different kinds of materials. We regard the heat transfer as three processes:

- between air and pans,
- between the pan and the brownie, and
- inside the brownie.

4.3.1 Between Air and Pans

As the pan has a low heat capacity c and high thermal conductance k, the temperature increase very fast. Thus we assume that the pans can reach the oven temperature T_{ext} soon and keep this temperature during our baking.

4.3.2 The Top Face of the Brownie

The top face of the brownie is also exposed to air, where convection plays a major role in heat exchange. Newton's cooling law indicates the heat flux is proportional to the difference of temperature $(T_{\text{ext}} - u)$. By Fourier's law, we have

$$\mathbf{q} = -k\nabla u$$

where $\mathbf{q}$ is the heat flux and ∇u is the temperature gradient. That gives the boundary condition on the top:

$$-\mathbf{n}_0 \cdot \mathbf{q} = \mathbf{n}_0 \cdot (k_b \nabla u) = h(T_{\text{ext}} - u) \tag{4}$$

where $\mathbf{n}_0$ is the outward unit normal of the top face, and h is the convection heat transfer coefficient.

4.3.3 Between Pan and Brownie

By intuition, we know the cake cannot reach the pan's temperature. We regard this case as conduction with thermal contact. Given both surfaces are not actually smooth, the heat flux cannot completely result in the increase of temperature due to the incomplete contact between pans and food. This can be treated as an abrupt change in temperature Δu in the interface. [5] (See Figure 1) The abrupt temperature change Δu can be derived as

$$\Delta u = \frac{q}{h_c} \tag{5}$$

where h_c is the thermal contact conductance.

The pan keeps transfering heat to the brownie and in the mean time, receiving heat flux $\mathbf{q}$ from the air. As we assumed, the pans always keeps the temperature of T_{ext}. Thus we can say $\mathbf{q}$ is generated for heating the brownie.

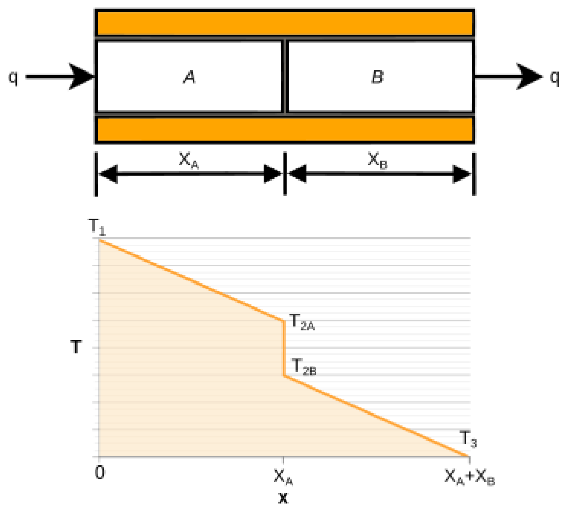


Figure 1: The abrupt change in temperature, Δu

From Eq. (5) and Fourier's law, we obtain

$$-\mathbf{n}_0 \cdot \mathbf{q} = \mathbf{n}_0 \cdot (k_b \nabla u) = h_c \Delta u = h_c (T_{\text{ext}} - u) \tag{6}$$

which is the boundary condition for the interfaces of the pan and brownie.

4.3.4 Inside the Brownie

The heat transfer inside the brownie follows the heat equation (2), with initial condition (3) and boundary conditions – Eq. (4) for the top face, and Eq. (6) for the other faces of the brownies. That is, we have known all the conditions needed to solve the heat distribution.

4.4 Solve the Model

Given the experimental value of thermal conductivity [1], we assume the properties as in Table 2.

Description	Symbol	Value	Unit
Oven temperature	T_{ext}	450	K
Mass Density			
– pan	ρ_p	7.9×10^3	kg/m^3
– brownie	ρ_b	650	kg/m^3
Heat Capacity			
– pan	c_p	0.46×10^3	$\text{J/(kg} \cdot \text{K)}$
– brownie	c_b	1.5×10^3	$\text{J/(kg} \cdot \text{K)}$
Thermal conductivity			
– pan	k_p	40	$\text{W/(m} \cdot \text{K)}$
– brownie	k_b	0.3	$\text{W/(m} \cdot \text{K)}$
Convection coefficient	h	15	$\text{W/(m}^2 \cdot \text{K)}$
Thermal contact conductance	h_c	350	$\text{W/(m}^2 \cdot \text{K)}$

Table 2: Properties used to solve the model

4.5 The Performance of the Model

4.5.1 Four typical shapes

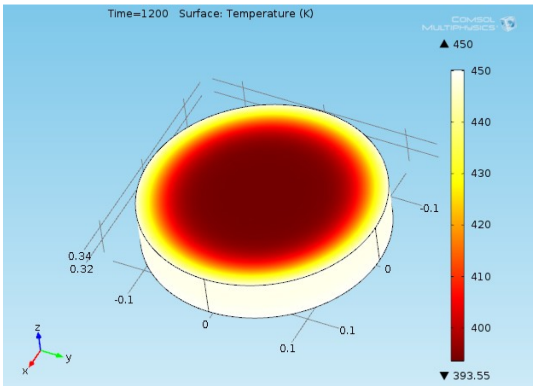
Applying the model we have developed in the former section, we can simulate the temperature distribution of brownie. As is mentioned before, we will use the software *COMSOL Multiphysics 4.3* to analyze the performance of pans of four representative shapes. The parameters are as follows:

Shape	w/l	r	baking time(second)
rectangle	2/3	0	1200
rounded rectangle I	2/3	0.216	1200
rounded rectangle II	1	0.086	1200
circle	1	0.138	1200

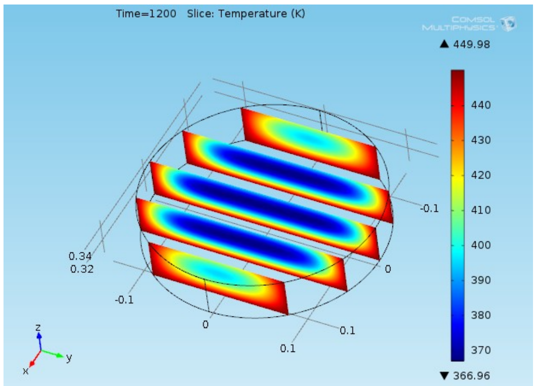
where w/l is the width to length ratio, r is radius of the circle on the corner.

4.5.2 Heat Distribution Figure and a Preliminary Analysis

For each pan of the four shapes, we present a surface heat distribution figure and a slice heat distribution figure.

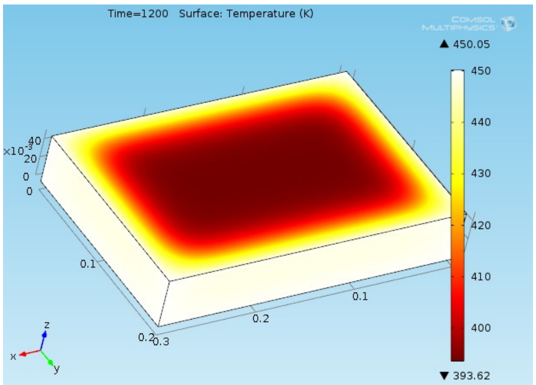


(a) Surface

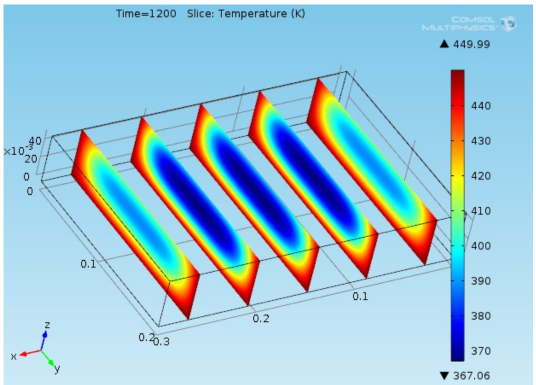


(b) Slice

Figure 2: Round pan

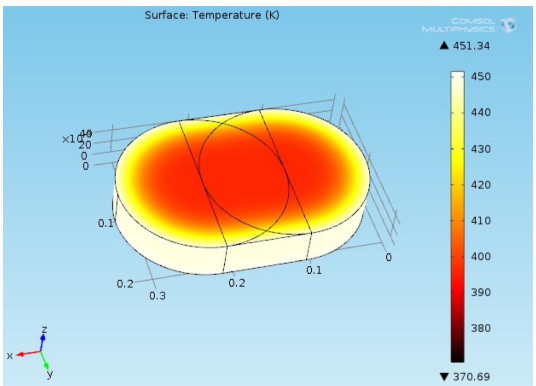


(a) Surface

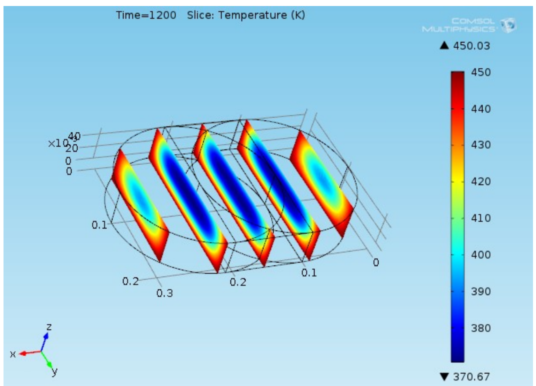


(b) Slice

Figure 3: Rectangular pan

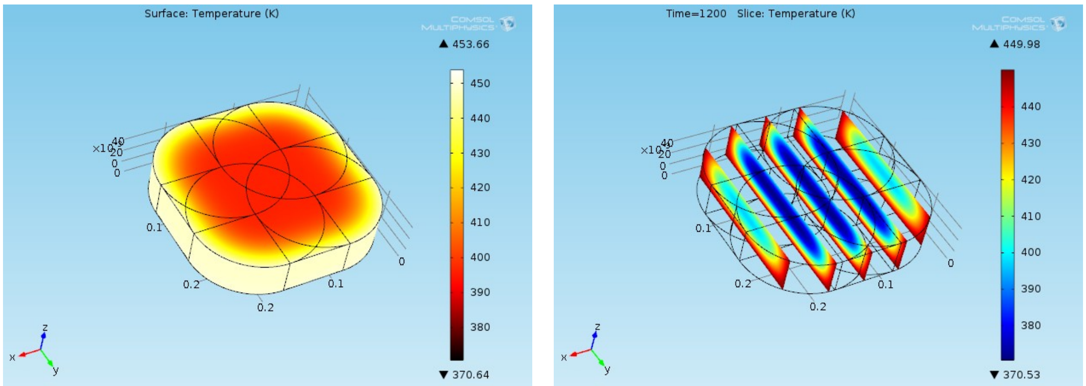


(a) Surface



(b) Slice

Figure 4: Rounded rectangular pan (Type I)



(a) Surface (b) Slice

Figure 5: Rounded rectangular pan (Type II)

Observing the figures above, our model successfully simulates the heat transform process of a brownie in a oven. As we expected, heat accumulates most heavily at the corners of the rectangular pan and to a lesser extent at the edges.

If we rank the ability of distributing heat evenly over the entire outer edge, it's straightforward for us to conclude that

(consider a cross-section of the brownie and neglect the difference in temperature along vertical direction.)

- The heat is distributed evenly over the entire outer edge for a circular pan, but not evenly for other shapes.
- Rounded rectangular pan performs better than rectangular pan.
- Rounded rectangular II performs better than Rounded rectangular I.

Besides, there are other data that intrigues us. For instance, the temperature of the center point and in each figure is quite steady at approximately $370K$ while the temperature of the very edge of brownie in each figure can always reach $440K$ or higher. These results imply that we can't simply use the maximum temperature difference inside the brownie to decide whether a pan of a certain shape provides a good heat distribution since the maximum temperature difference is not much influenced by the shape of the pan. How to develop a criterion to measure the ability of heating the brownie evenly over the entire outer edge is the main problem we want to solve in Section 5.

5 Shape Evaluation Model

A good pan should be evenly heated so that the cake would not be overcooked, as well as space-saving in the oven.

Before continuing, we define the parameters of rectangle, circle and rounded rectangle in a unified form, as in Figure 6. A rectangle can be regarded as $r = 0$ and a circle can be regarded as $w = l = 2r$. Thus in our model, a set of (w, l, r) can determine both the size and the shape of the pan.

Now we study these features in detail. In this and following sections, all the lengths are in meters (m), the areas in m^2, unless otherwise specified.

5.1 To Maximize the Number of Pans

In this section, we study the maximum number of shape-fixed pans that can fit in the oven, which can be identified as a packing problem.

Maximizing the number of pans that can be fitted in a rectangular rack can be abstracted as parking problems.[2]

In order to fit the pans in an actual oven, besides the conditions above, we must know the specific W and L, as well as the size and shape of pans w, l and r.

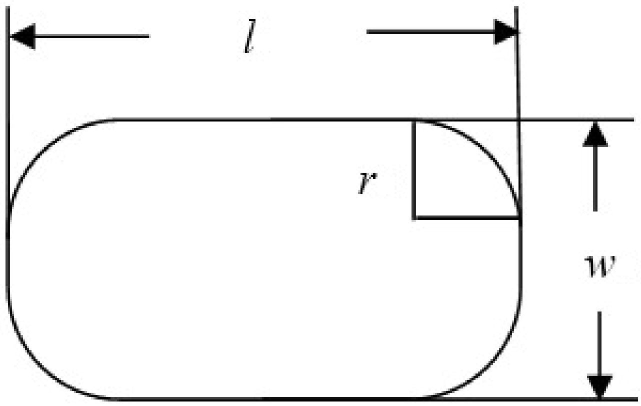


Figure 6: Definition of length and width for rounded rectangles

We set the size of the racks as $L = 0.75$ by $W = 0.5$ (i.e., WLR=$2:3$). The size of pans w and l will be set in the next section.

Let n_0 denote the largest number of pans that can be fitted on *one* rack, so the total number of pans would be $N = 2n_0$. Assume the area of the rack is the same with the oven's. Naturally n_0 has an upper bound given by the following inequality

$$n_0 \leq n_{\max} = \lfloor W \cdot\ L/A \rfloor$$

where $\lfloor a \rfloor$ is the largest integer no more than a. It is an inequality instead of equation, because the arrangement of the pans must be considered in most cases.

Now let $A = 0.06$. The max possible number would be

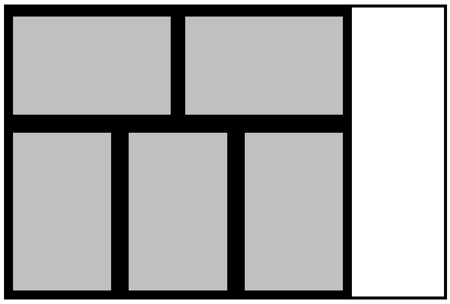
$$n_{\max} = \lfloor (0.5 \times 0.75)/0.06 \rfloor = 6.$$

Now we study the different shapes of the pans.

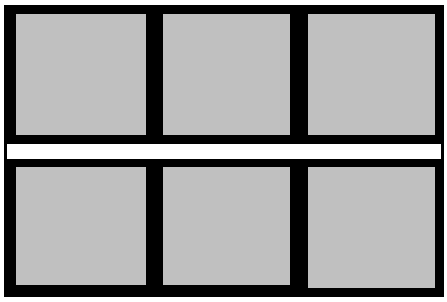
5.1.1 Arrange Rectangular Pans

Packing rectangles into a rectangle is apparently easy, but it is claimed as an NP-hard problem [4], which is hard when n_0 is big. Fortunately, as pans are not that small compared to the size of racks, n_0 is small enough so that we can fit the pans manually.

Example: If we set $w = 0.2$ and $l = 0.3$ (which is the case in Section 4, we can only reach the maximum of $n_0 = 5$ instead of 6 (Figure 7(a)). We cannot arrange the 6th pan anywhere. However, we can reach $n_0 = 6$ when we set $w = 0.24$ and $l = 0.25$ and arrange the pans as in Figure 7(b).



(a) Size: 0.3×0.2



(b) Size: 0.25×0.24

Figure 7: Two arrangements for rectangular pans

5.1.2 Arrange Round Pans

The situation gets more complex when we use pans other than rectangular ones. There will always be vacant space left between pans if we adopt circular pans or partially arc-shaped pans.

Round pans packing in a L by W rectangular racks can be abstracted as points at the center of pans. The points can be spread in an $(L - 2r)$ by $(W - 2r)$ "refined rectangular", with a minimum distance of $2r$ for the every two points. It is obvious that a pan can fit in the rack if and only if the pan center is inside the refined rectangular.

The most efficient way of packing is hexagonal packing[2], where the nearest abstracted points make equilateral triangles (See Figure 8). So we try to fit as many lattice points within the refined rectangle as possible.

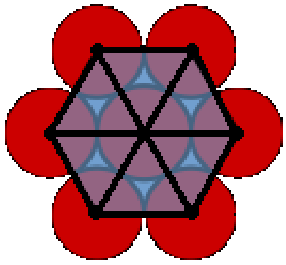


Figure 8: Hexagonal packing

Since each triangle has area

$$A_{\text{tri}} = \sqrt{3}r^2,$$

the number of lattice points can not exceed

$$\bar{n} = \begin{cases} 1, & L < 4r, \\ 2, & L \geq 4r \text{ and } (L-2r)(W-2r) < \sqrt{3}r^2, \\ 1 + \lfloor \dfrac{(L-2r)(W-2r)}{\sqrt{3}r^2} \rfloor, & \text{otherwise.} \end{cases}$$

depending on whether the refined rectangle is big enough to contain the lattice points.

Example: Given the area of pan $A = 0.06$, oven size $L = 0.75$ and $W = 0.5$, we have

$$r = \sqrt{A/\pi} \approx 0.138,$$

$$\bar{n} = 1 + \lfloor \frac{(L-2r)(W-2r)}{\sqrt{3}r^2} \rfloor = 1 + \lfloor 3.2 \rfloor = 4.$$

Actually we can only arrange 3 pans on each rack in the given oven, as in Figure 9. Generally, to make $n_0 = \bar{n}$, the WLR of oven W/L must satisfy certain condition. For instance, we could fit a 4th pan in only if the lattice point marked 'x' were inside the refined rectangle, i.e., W cannot be less than $2r + \sqrt{3}r \approx 0.516$.

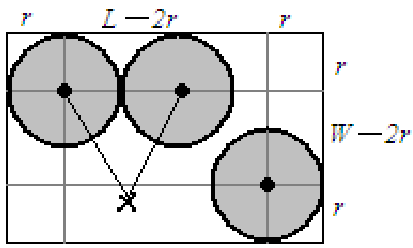


Figure 9: The arrangement for round pans

5.1.3 Arrange Rounded Rectangular Pans

As rounded rectangle has straight-line segments, it is not efficient to let the arc segments contact like hexagonal packing, because that would constrain the direction of pans to some extent, leading to more waste of space. Generally we would consider fitting the pans as arrange rectangles with length l and width w as defined in Figure 6. However, there are situations where an alternative optimizing way can be applied — by "inserting" a pan as shown in Figure 10. Such contacts will save the length/width by

$$(2 - 2\cos\theta)r$$

at the cost of an increase of

$$(4\sin\theta - 2)r + (z - 2r)$$

in width/length, where r is the radius of arc segments, $\theta \in (0, \pi/2)$ is the contact angle, and z is either length or width of the rounded rectangle, depending on the direction the pan is inserted. Here is an example of how we use this optimization method.

Example: Determine the smallest rack size (with $W/L = 2/3$) required to fit in 3 pans with shape $l = w = 3r$.

According to our discussion, the contact angle θ is the key to find the answer. As the the WLR of oven (also the rack, as assumed) is fixed, we have the following programming problem:

$$\mathbf{min}\ L,$$
$$\text{s.t.} \begin{cases} L \geq 2 \cdot 3r + (4\sin\theta - 2)r + (3r - 2r) \\ W = \dfrac{2}{3}L \geq 2 \cdot 3r - (2 - 2\cos\theta)r. \end{cases}$$

Solve it and get $\theta \approx 48.4°$.
That is, $L = (5 + 4\sin\theta)r \approx 7.99r$ and $W \approx 5.33r$.

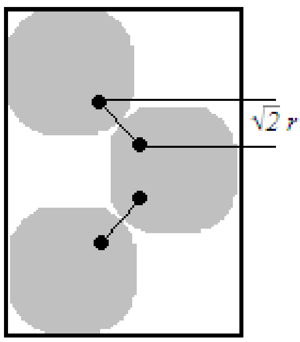


Figure 10: "Insert" a rounded rectangular pan

5.1.4 Evaluate the Performance

As we know from the discussion above, the area of the rack can affect the space utility even if W/L is fixed. To quantify our model, for a given WLR of oven W/L and a certain shape of pans, we define

$$f(x) = n_0(x)/n_{\max}(x).$$

where $x = W \cdot L$ is the area of each rack, To find out the performance of different shapes of pans in a oven of given W/L, we let x vary in a small range from x_a to x_b and define the mean s_1, as "space utilization score"

$$s_1 = \frac{1}{x_b - x_a} \int_{x_a}^{x_b} f(x)\, dx.$$

According to the family-use ovens in real life, we set $x_a = 0.250$ and $x_b = 0.400$ in our model.

Example: Study the performance for rectangular pans with $w/l = 2/3$, assuming $W/L = 2/3$ and $A = 0.06$.

We first calculate the pan size from w/l and A, yielding $w = 0.2$ and $l = 0.3$. At the condition of $x = x_a = 0.250$, which is the smallest possible oven area, the oven size would be $0.408(W) \times 0.612(L)$. We can fit 4 pans at most in each rack(2 by 2), and therefore $n_0(0.25) = 4$.

Then we find the minimum area x required to fit in more than 4 pans. The results are

n_0	Minimum x	As seen in
5	0.375	Figure 7(a)
6	0.427	–

Therefore

$$n_0(x) = \begin{cases} 4, & x \in [0.250, 0.375), \\ 5, & x \in [0.375, 0.400], \\ \text{Don't care}, & \text{otherwise}. \end{cases}$$

Also, we can easily find that

$$n_{\max}(x) = \begin{cases} 4, & x \in [0.250, 0.300), \\ 5, & x \in [0.300, 0.360), \\ 6, & x \in [0.360, 0.400], \\ \text{Don't care}, & \text{otherwise}. \end{cases}$$

Now we can get $f(x)$, shown in Figure 11. Integral $f(x)$ and we get the final score

$$s_1 = \frac{1}{0.400 - 0.250} \int_{0.250}^{0.400} f(x)\, dx = 0.859.$$

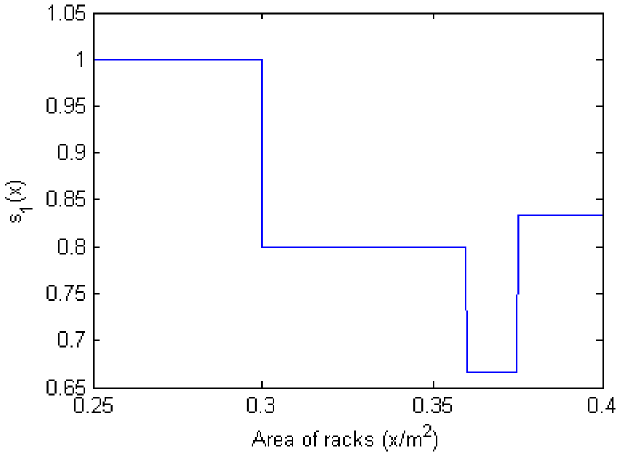


Figure 11: f as a function of x

5.2 To Maximize Even Distribution of Heat

In this section, we turn back to our first model in section ?, now we want to develop criteria to measure the ability of heating the brownie evenly over the entire outer edge.

5.2.1 Criteria

In order to rate each shape and assign a reasonable score , the criteria should meet the following requirements:

- Based on practical physical implication and reflect the heat distribution explicitly.
- The score should be significantly influenced by the shape of the pan and varies in a proper range.

As mentioned in section 4.6, we find that the temperature of outer edge of brownie almost reaches $450K$ (the outer space temperature we have assumed), so it is difficult for us to evaluate the even distribution rate by calculating the maximum and minimum temperature of the edge. We find calculating the temperature gradient a good method.

For a certain shape of pan, the outer edge temperature maybe almost the same, but the temperature gradient must be different if the heat distribution is not even. In low-temperature-gradient area like at corners, there are more chunks of cake have a high temperature, so that brownie will be overcooked. If the temperature gradient is all the same along the edge, the outer edge of the brownie will be well-cooked at a same moment, however, if the temperature gradient differs a lot along the edge, the brownie will be overcooked at low-temperature-gradient area while other parts are still not well-cooked.

With this method, we can evaluate the heat evenly distributed rate by the following steps:

- Choose a series of points which are uniformly distributed on the edge.

- Calculate the temperature gradient of every point in direction of X axis and Y axis.
- Calculate the angle between X axis and the vector of the point heading to center and the angle between Y axis and the vector, then calculate the temperature gradient in direction of the vector by the equation:

$$s_2 = Grad_i = Grad_x * \cos(\theta_x) + Grad_y * \cos(\theta_y) \tag{7}$$

- Calculate the variance of the temperature gradient s_2 on the edge. Obviously, the bigger the variance is, the less even the heat distribution is.

5.2.2 The Variance of Four Typical Shapes

We have calculate the variance of the following four shapes:

Shape	w/l	r	baking time(second)	variance s_2
rectangle	0.67	0	1200	2.46e+05
rounded rectangle I	0.67	0.216	1200	4.30e+04
rounded rectangle II	1	0.086	1200	3.61e+04
circle	1	0.138	1200	3.66e+02

Table 3: The variance of typical shapes

In section 4, we have found out that

- The heat is distributed evenly over the entire outer edge for circular pan, but not evenly for other shapes.
- Rounded rectangular pans perform better than rectangular pan
- Rounded rectangular I performs better than Rounded rectangular II.

From the data in Table 3, we can see that our former conclusion judging from Figure 2 to 5 is reasonable.

5.3 Optimize a Shape Based on Two Conditions

5.3.1 Score Algorithm

In practice, we want to evaluate a pan by the space utilization and the heat distribution in the meantime, that is, we need make the score criteria based on a combination of condition I and condition II. From the former model we can calculate the space utilization score s_1 for condition I and the variance of gradient s_2 for condition II. So we first normalize the two scores s_1, s_2 to $score_1$ and $score_2$ and then assign weights p and $(1-p)$ to $score_1$ and $score_2$ respectively.

Since the numeric values of s_1 and s_2 vary in different orders of magnitude ,we need to normalize s_1 and s_2 to $score_1$ and $score_2$.

To normalize s_1 to get $score_1$, we should follow the steps:

- Select the minimum and maximum in the s_1 list and denote them as s_{1min} and s_{1max}
- Calculate $score_1$ by the formula:

$$score_1 = \frac{s_1 - s_{1min}}{s_{1max} - s_{1min}} \tag{8}$$

To normalize s_2 to get $score_2$, we should follow the steps:

- Select the maximum in the s_2 list and denote it as s_{2max}
- Determine the parameter m by the formula:

$$m = |\log_{10}(\frac{s_2}{s_{2max}})| \tag{9}$$

  Then we can get one m for each s_2
- Select the maximum in the m list and denote it as m_{max}
- Calculate the $score_2$ by the formula:

$$score_2 = \sqrt{\frac{m}{m_{max}}} \tag{10}$$

Taking the weight factor p into account, the final score s should be:

$$s = p \cdot score_1 + (1 - p) \cdot score_2 \tag{11}$$

Thus, we developed a general score algorithm to evaluate all kinds of pans. Obviously, the higher the score is, the better the pan is designed.

5.3.2 How to Get a Relatively Optimal Shape in Practice

Even though the width to length radio W/L and the weight factor p have been fixed, there are still infinitely many kinds of shapes we can choose. To find out a Practical Highest Score, we must simplify our model one step further.

We select several representative values for the width to length ratio of the pan w/l and of the oven W/L respectively. Once w/l is settled, for rectangular pans, the shape is fixed, and for rounded rectangular pans, we again select several discrete representatives for the radius of circle on the corner r to decide the round rectangle. In this paper, we select three values for W/L, w/l, r and totally 27 combinations.

Symbol	Value1	Value2	Value3
The width to length ratio of pan w/l	1	3/4	2/3
The width to length ratio of oven W/L	3/4	2/3	1/2

Table 4: values of w/l and W/L

5.4 The Performance of the Model

5.4.1 Data

The performance of all calculated shapes is shown in the following table:

Here W/L means the WLR of oven, w means the width of pan, l means the length of pan, r means the radius of circle on the corner of rounded-rectangular pan, s_1 means space utilization score (applying the model in section 5.1), s_2 means variance of gradient (applying the model in section 5.2), s means the final score.

No.	W/L	w/l	l	w	r	s_1	s_2
1	1/2	1/1	0.245	0.245	0	0.590	2.86e+05
2			0.258	0.258	0.086	0.541	3.61e+04
3			0.276	0.276	0.138	0.482	3.66e+02
4		3/4	0.283	0.212	0	0.757	3.13e+05
5			0.294	0.220	0.0734	0.667	2.67e+04
6			0.309	0.232	0.116	0.624	2.24e+04
7		2/3	0.300	0.200	0	0.819	2.46e+05
8			0.310	0.207	0.0689	0.759	3.23e+04
9			0.324	0.216	0.108	0.683	4.30e+04
10	2/3	1/1	0.245	0.245	0	0.594	2.86e+05
11			0.258	0.258	0.086	0.523	3.61e+04
12			0.276	0.276	0.138	0.466	3.66e+02
13		3/4	0.283	0.212	0	0.802	3.13e+05
14			0.294	0.220	0.0734	0.701	2.67e+04
15			0.309	0.232	0.116	0.604	2.24e+04
16		2/3	0.300	0.200	0	0.859	2.46e+05
17			0.310	0.207	0.0689	0.809	3.23e+04
18			0.324	0.216	0.108	0.732	4.30e+04
19	3/4	1/1	0.245	0.245	0	0.611	2.86e+05
20			0.258	0.258	0.086	0.518	3.61e+04
21			0.276	0.276	0.138	0.517	3.66e+02
22		3/4	0.283	0.212	0	0.921	3.13e+05
23			0.294	0.220	0.0734	0.859	2.67e+04
24			0.309	0.232	0.116	0.719	2.24e+04
25		2/3	0.300	0.200	0	0.845	2.46e+05
26			0.310	0.207	0.0689	0.743	3.23e+04
27			0.324	0.216	0.108	0.638	4.30e+04

Table 5: Performance of the Model

No.	s_1	s_2	$s_{p=0}(score_2)$	$s_{p=0.25}$	$s_{p=0.5}$	$s_{p=0.75}$	$s_{p=1}(score_1)$
1	0.590	2.86e+05	0.116	0.155	0.194	0.234	0.273
2	0.541	3.61e+04	0.566	0.466	0.366	0.265	0.165
3	0.482	3.66e+02	1.000	0.759	0.518	0.276	0.035
4	0.757	3.13e+05	0.000	0.160	0.321	0.481	0.641
5	0.667	2.67e+04	0.604	0.563	0.523	0.482	0.442
6	0.624	2.24e+04	0.625	0.556	0.486	0.416	0.347
7	0.819	2.46e+05	0.189	0.336	0.483	0.630	0.778
8	0.759	3.23e+04	0.580	0.596	0.612	0.628	0.644
9	0.683	4.30e+04	0.542	0.526	0.510	0.494	0.478
10	0.594	2.86e+05	0.116	0.157	0.199	0.240	0.282
11	0.523	3.61e+04	0.566	0.455	0.345	0.235	0.125
12	0.466	3.66e+02	1.000	0.750	0.500	0.250	0.000
13	0.802	3.13e+05	0.000	0.185	0.370	0.555	0.740
14	0.701	2.67e+04	0.604	0.582	0.561	0.539	0.517
15	0.604	2.24e+04	0.625	0.545	0.464	0.384	0.303
16	0.859	2.46e+05	0.189	0.358	0.527	0.697	0.866
17	0.809	3.23e+04	0.580	0.624	0.668	0.711	0.755
18	0.732	4.30e+04	0.542	0.553	0.564	0.574	0.585
19	0.611	2.86e+05	0.116	0.167	0.218	0.269	0.320
20	0.518	3.61e+04	0.566	0.453	0.340	0.228	0.115
21	0.517	3.66e+02	1.000	0.778	0.556	0.334	0.112
22	0.921	3.13e+05	0.000	0.250	0.500	0.750	1.000
23	0.859	2.67e+04	0.604	0.670	0.735	0.800	0.866
24	0.719	2.24e+04	0.625	0.608	0.591	0.574	0.557
25	0.845	2.46e+05	0.189	0.350	0.511	0.672	0.834
26	0.743	3.23e+04	0.580	0.593	0.606	0.619	0.632
27	0.638	4.30e+04	0.542	0.501	0.460	0.419	0.378

Table 6: Performance of the Model

5.4.2 Analysis

The parameters of pans are as follows:

Here W/L means the WLR of oven, w/l means the WLR of pan, w means the width of pan, l means the length of pan, r means the radius of circle on the corner of rounded-rectangular pan.

No.	w/l	l	w	r
1	1/1	0.245	0.245	0
2		0.258	0.258	0.086
3		0.276	0.276	0.138
4	3/4	0.283	0.212	0
5		0.294	0.220	0.0734
6		0.309	0.232	0.116
7	2/3	0.300	0.200	0
8		0.310	0.207	0.0689
9		0.324	0.216	0.108

Table 7: Parameters of pans

From the table, we can conclude the best shape for each oven:

p	$W/L = 1/2$	$W/L = 2/3$	$W/L = 3/4$
0	No.3(circle)	No.3(circle)	No.3(circle)
0.25	No.3(circle)	No.3(circle)	No.3(circle)
0.5	No.8(rounded rectangle)	No.8(rounded rectangle)	No.5(rounded rectangle)
0.75	No.8(rounded rectangle)	No.8(rounded rectangle)	No.5(rounded rectangle)
1	No.7(rectangle)	No.7(rectangle)	No.4(rectangle)

Table 8: The Best Shape for Each Oven

1. When the WLR of the oven W/L, the WLR of the pan w/l, the factor p are all fixed, $score_1$ monotonically decreases when the shape of pan transforms from a rectangle to a circle or the rounded rectangle with largest radius in the corner, and in contrast, $score_2$ increases. The distribution of $score_1$ and $score_2$ agree with the model we've generated and perform well.

2. When the WLR is fixed, we can see that the larger p is, the more important the condition I is and the less important the condition II is. When $p = 0$ and $p = 0.25$, the best choice is circular pan, which has the most even heat contribution. When $p = 1$, the best choice is rectangular pan, which has the largest space utilization score. And when $p = 0.5$ and $p = 0.75$, both two conditions are in significant effect and the rounded rectangular pan is the best choice. This conclusion agrees with the prediction we have made in the former section perfectly.

3. Let's consider those three pan shapes which share the same WLR as a group. When the factor p is fixed, we prefer to choose a pan whose WLR is closest to the WLR of oven, because in this circumstance $score_1$ of this group are always higher than any other groups. Obviously, the shapes in this group will achieve a higher final score since their $score_2$ are the same as other groups.

6 Strengths and Weaknesses

6.1 Strengths

- We derive a sophisticated Heat Distribution Model by the 3-D heat equation and show the detailed proof based on laws of thermodynamics.
- We develop several principles to optimize the arrangement and present a lot of specific examples to elaborate our algorithm on searching the maximum number of pans that can fit in the oven.
- Instead of using the temperature across the outer edge to describe the inequality degree of the heat distribution, we chose the the variance of the temperature gradient across the outer edge. In this way, we make use of more data and have a better understanding of the heat distribution.
- All parameters in our model are either from experimental values or reasonable estimation based on measurements or experience in real life. So our model is practical to some extent.

6.2 Weaknesses

- To simply the calculation, we have made an assumption that the temperature of the pan is the temperature of oven $450\,\mathrm{K}$ at the very beginning, as if the pans have been pre-heated too. It may accelerate the heating process, especially for a few seconds to minutes. However it may have little effect on the heat distribution when the baking time is long enough, yet not proven in our model.
- We assume everything in the oven (except the brownies) has nearly the same temperature, regardless of the difference of oven walls and racks. So we did not explore how the racks would effect our thermal model in detail.
- We only choose some discrete values as the parameters for the shapes of pans and ovens. Thus we might miss the exactly "best" shape for the certain oven.
- We solve the maximum number of shape-fixed pans that can fit into a shape-fixed oven by considering the real picture and every possible way to arrange most pans inside. This is a simple way to deal with the problem in this model. But it is not efficient when the numbers get larger.
- There are a bunch of other shapes between rectangle and circle, but we only choose the most practical and easily-calculated rounded rectangle as the representative shapes.

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  Size of oven: 45CM*60CM
  Recommended shape of pan: circular
  Parameters: radius=13.8CM

- Mr. White is a baker of a famous bakery in NewYork City. He finds that there is always some wasted space in the oven so that he could not bake as many brownies as possible. He is good at controlling baking time and temperature. We choose a rectangular pan for him, which can maximize the space utilization.
  Size of oven: 50CM*75CM
  Recommended shape of pan: rectangular
  Parameters: length=30CM, width=20CM

- Mrs. Black is a housewife raising three children. Her children are in favor of their mother’s brownie. She needs some pans that can work well in all conditions. Here we recommend a rounded rectangular pan for her.
  Size of oven: 40CM*80CM
  Recommended shape of pan: rounded rectangular
  Parameters: length=31.0CM, width=20.7CM, radius of the circle on the corner of rectangular=6.9CM

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