AI1110: Assignment 7

Aryan Sharan Reddy
BT21BTECH11002

June 5, 2022

Outline

Question

The random variable x has a Poisson distribution with mean θ. We wish to find the Bayesian estimate $\hat{\theta}$ of θ under the assumption that θ is the value of a random variable θ with prior density $f_\theta(\theta) \sim \theta^b e^{-c\theta} U(\theta)$. Show that

$$\hat{\theta} = \frac{n\bar{x} + b + 1}{n + c}$$

Solution

The sum n is a Poisson random variable with mean nθ.
In the context of Bayesian estimation this means that

$$f_{\bar{x}}(\bar{x}|\theta) = e^{-n\theta}\frac{(n\theta)^k}{k!} \qquad k = n\bar{x} = 0, 1, ... \tag{1}$$

Also, we know that

$$f_\theta(\theta|X) = \frac{f(X|\theta)}{f(X)} f_\theta(\theta) \qquad X = [x_1, ..., x_n] \tag{2}$$

Here, $f(X|\theta)$ is the conditional density of the n random variables x_i assuming $\theta = \theta$.

Solution(Contd..)

Inserting (1) into (2), we obtain

$$f_\theta(\theta|\bar{x}) = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+a)}\theta^{n\bar{x}+b}e^{-(n+c)\theta} \tag{3}$$

Also, from

$$\hat{\theta} = E\{\theta|X\} = \int_{-\infty}^{\infty} \theta f_\theta(\theta|X)d\theta \tag{4}$$

we obtain

$$\hat{\theta} = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+1)} \times \frac{\Gamma(n\bar{x}+b+2)}{(n+c)^{n\bar{x}+b+2}} \tag{5}$$

$$= \frac{n\bar{x}+b+1}{n+c} \tag{6}$$

$$\therefore \lim_{n\to\infty} \hat{\theta} = \bar{x} \tag{7}$$