AI1110: Assignment 11

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Outline

1. Question (a)
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4. Solution (b)

Question (a)

Consider a stationary Markov chain $...x_n, x_{n+1}, x_{n+2}, ...$ with transition probabilities $\{p_{ij}\}$ and steady state probabilities q_{ij}. (a) Show that the reversed sequence $...x_n, x_{n-1}, x_{n-2}, ...$ is also a stationary Markov process with transition probabilities

$$\mathrm{P}\{\mathrm{x}_n = j | x_{n+1} = i\} \triangleq p_{ij}^* = \frac{q_j pji}{q_i}$$

and steady state probabilities $\{\mathrm{q}_i\}$.
A Markov chain is said to be time reversible if $p_{ij}^* = p_{ij}$ for all i, j.

Question (b)

Show that a necessary condition for time reversibility is that

$$p_{ij}p_{ik}p_{ki} = p_{ik}p_{kj}p_{ji} \text{ for all i, j, k}$$

which states that the transition $e_i \to e_j \to e_k \to e_i$ has the same probability as the reversed transition $e_i \to e_k \to e_j \to e_i$. In fact, for a reversible chain starting at any state e_i, any path back to e_i has the same probability as the reversed path.

Solution (a)

From Bayes' theorem

$$P\{x_n = j | x_{n+1} = i\} = \frac{P\{x_{n+1} = i | x_n = j\} P\{x_n = j\}}{P\{x_{n+1} = i\}} \tag{1}$$

$$= \frac{q_j p_{ji}}{q_i} = p^*_{ij}, \tag{2}$$

where we have assumed the chain to be in steady state

Solution (b)

Notice that time-reversibility is equivalent to

$$p_{ij}^* = p_{ij} \tag{3}$$

and using (2) this gives us,

$$p_{ij}^* = \frac{q_j p_{ji}}{q_i} = p_{ij} \tag{4}$$

or, for a time reversible chain, we get

$$q_j \times p_{ji} = q_i \times p_{ij} \tag{5}$$

Thus, using (3), we obtain, the desired result, by direct substitution

$$p_{ij} p_{ik} p_{ki} = \left(\frac{q_j}{q_i} p_{ji}\right)\left(\frac{q_k}{q_j} p_{kj}\right)\left(\frac{q_i}{q_k} p_{ik}\right) \tag{6}$$

$$= p_{ik} p_{kj} p_{ji} \tag{7}$$