Random Numbers

Aryan Sharan Redddy
BT21BTECH11002

Contents

1 **Uniform Random Numbers** 1

2 **Central Limit Theorem** 2

3 **Uniform to Other** 4

4 **Triangular Distribution** 4

5 **Maximum Likelihood** 5

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/1_1.c
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/1_1.h
```

and compile and execute the C program using

```
$ gcc 1_1.c −lm
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.1

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/1_2.py
```

It is executed with

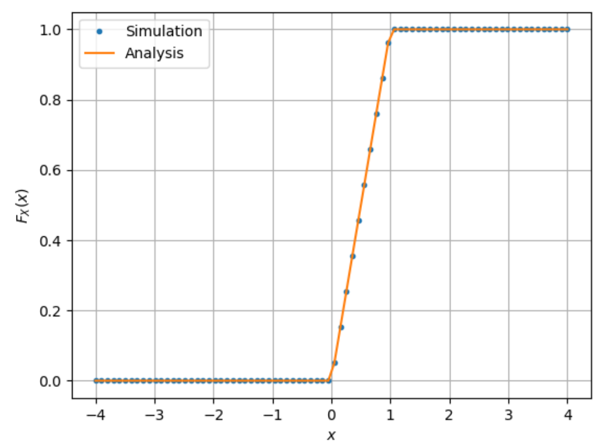

Fig. 1.1: The CDF of U

```
$ python3 1_2.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

We now have three cases:

a) $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \leq x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{1.3}$$

c) $x \geq 1$: Put $x = 1$ in (1.3) as U is uniform in $[0, 1]$ to get $F_U(x) = 1$.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{1.4}$$

This is verified in Figure (1.1)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution: The C program can be downloaded using

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/1_4.c
```

and compiled and executed with

```
$ gcc 1_4.c −lm
$ ./a.out
```

The calculated mean is 0.500007 and the calculated variance is 0.083301.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \tag{1.7}$$

Solution: We write

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.8}$$
$$= \int_{-\infty}^{\infty} x^2 p_U(x)dx \tag{1.9}$$
$$= \int_0^1 x^2 dx = \frac{1}{3} \tag{1.10}$$

and

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.11}$$
$$= \int_{-\infty}^{\infty} x p_U(x)dx \tag{1.12}$$
$$= \int_0^1 x dx = \frac{1}{2} \tag{1.13}$$

which checks out with the empirical mean on 0.500007. Now, using linearity of expectation,

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.14}$$
$$= E\left[U^2 - 2UE[U] + (E[U])^2\right] \tag{1.15}$$
$$= E\left[U^2\right] - 2(E[U])^2 + (E[U])^2 \tag{1.16}$$
$$= E\left[U^2\right] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.17}$$

and this checks out with the empirical variance 0.083301 of the sample data.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: The sample data is generated by the C file in Question 1.1.

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have? **Solution:** The CDF of X is plotted in Fig. 2.1 Download the

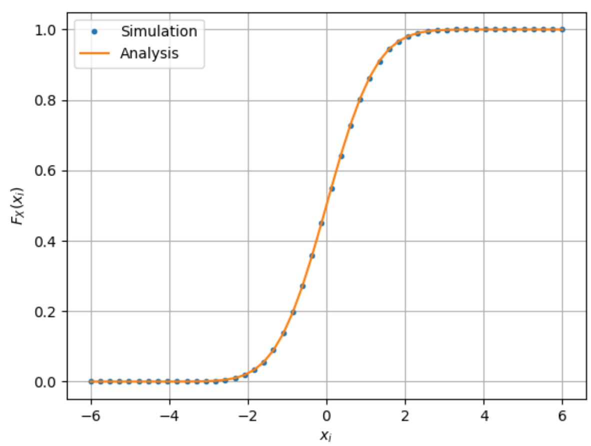

Fig. 2.1: The CDF of X

Python code using

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/2_2.py
```

and execute it with

```
$ python3 2_2.py
```

The CDF of a probability distribution has the following properties:

a) It is non-decreasing
b) It is right-continuous
c) $\lim_{x\to-\infty} F_X(x) = 0$
d) $\lim_{x\to\infty} F_X(x) = 1$

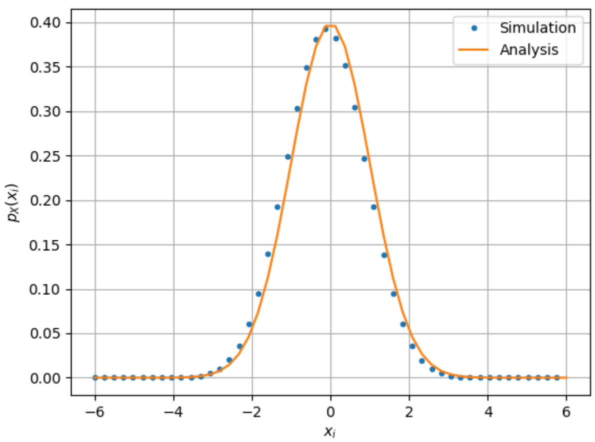

Fig. 2.2: The PDF of X

The CDF of the normal distribution is expressed in terms of the Q-function as $F_X(x) = 1 - Q(x)$.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have? **Solution:** The PDF of X is plotted in Fig. 2.2 using the code below

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/2_3.py
```

The figure is generated using

```
$ python3 2_3.py
```

The properties of a PDF are as follows:

a) $\forall x \in \mathbb{R},\ p_X(x) \geq 0$
b) $\int_{-\infty}^{\infty} p_X(x)dx = 1$
c) For $a < b,\ a, b \in \mathbb{R}$

$$\Pr(a < X < b) = \Pr(a \leq X \leq b) \tag{2.3}$$
$$= \int_a^b p_X(x)dx \tag{2.4}$$

If we take $a = b$, then we get $\Pr(X = a) = 0$.

2.4 Find the mean and variance of X by writing a C program.
Solution: The mean and variance have been calculated using (1.5) and (1.6) respectively. The C program can be downloaded using wget

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/2_4.c
```

and executed with the following commands

```
$ gcc 2_4.c −lm
$ ./a.out
```

The calculated mean is 0.000326 and the calculated variance is 1.000906.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution: The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) = 0 \tag{2.6}$$

as the integrand is odd. This checks out with the empirical mean of 0.000326. The variance is given by

$$\text{var}[X] = E\left[X^2\right] - (E[X])^2 \tag{2.7}$$
$$= \int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \tag{2.8}$$
$$= \int_0^{\infty} \frac{2}{\sqrt{2\pi}}\sqrt{2t}e^{-t}dt \tag{2.9}$$
$$= \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) \tag{2.10}$$
$$= \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = 1 \tag{2.11}$$

where we have used $t = \frac{x^2}{2}$ and so $dt = xdx$. We have also used the gamma function defined as

$$\Gamma(n) = \int_{-\infty}^{\infty} x^{n-1}e^{-x}dx \tag{2.12}$$
$$\Gamma(n) = (n-1)\Gamma(n-1) \text{ for } n > 1 \tag{2.13}$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$. This agrees with the empirical variance of 1.000906.

3 Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.

Solution: The relevant python code is at

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/3_1.py
```

and can be executed with

```
$ python3 3_1.py
```

The CDF is plotted in Figure (3.1).

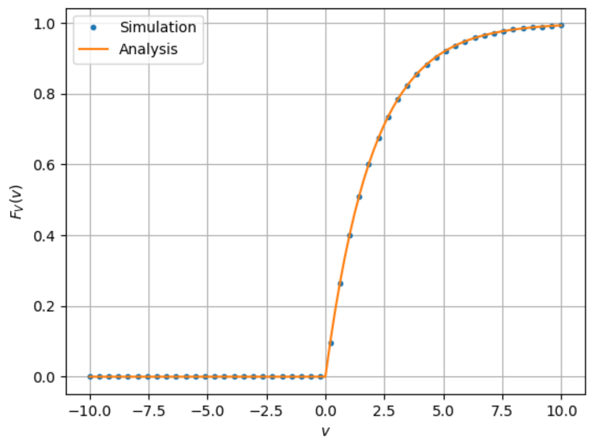

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution: Note that the function

$$v = f(u) = -2\ln(1-u) \tag{3.2}$$

is monotonically increasing in [0, 1] and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \tag{3.3}$$

Therefore, from the monotonicity of v, and using (1.4),

$$F_V(v) = F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \tag{3.4}$$

$$\implies F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \tag{3.5}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: The samples are generated in the C file exrand.c in 1.1 as the file tri.dat.

4.2 Find the CDF of T.
Solution: The Python code for the figure is at

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/4_2.py
```

and can be run using

```
$ python3 4_2.py
```

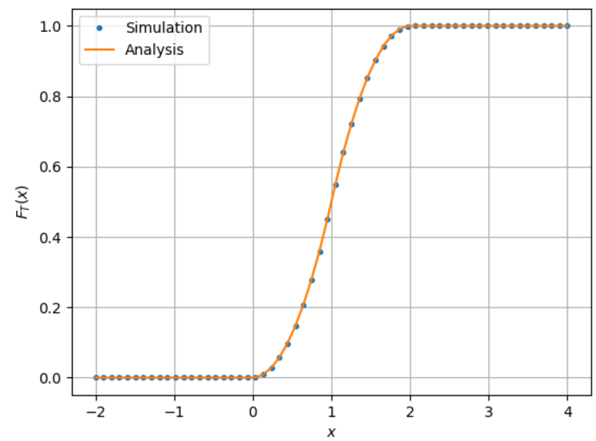

Fig. 4.1: The CDF of T

4.3 Find the PDF of T.
Solution: The Python code for the figure can be downloaded using

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/4_3.py
```

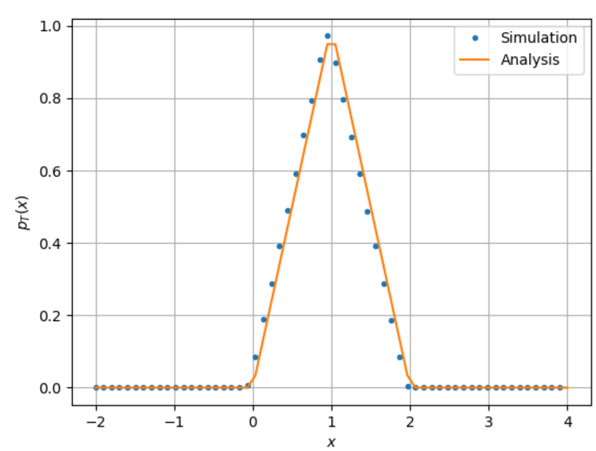

Fig. 4.2: The PDF of T

and run using

```
$ python3 4_3.py
```

4.4 Find the theoretical PDF and CDF of T.
Solution: We write,

$$F_T(t) = \Pr(U_1 + U_2 \le t) \tag{4.2}$$
$$= \Pr(U_1 \le t - U_2) \tag{4.3}$$
$$= \int_0^1 F_{U_1}(t-x)p_{U_2}(x)dx \tag{4.4}$$

where U_1 and U_2 are uniform i.i.d. random variables in [0, 1]. Then, $0 \le U_1 + U_2 \le 2$. We have three cases:

a) $t < 0$: Using Equation 1.4, $F_T(t) = 0$.
b) $0 \le t < 1$: We have,

$$F_T(t) = \int_0^t (t-x)dx = \frac{t^2}{2} \tag{4.5}$$

c) $1 \le t < 2$: Here, we get

$$F_T(t) = \int_0^{t-1} dx + \int_{t-1}^1 (t-x)dx \tag{4.6}$$
$$= t - 1 + t(2-t) - \frac{1-(t-1)^2}{2} \tag{4.7}$$
$$= -\frac{t^2}{2} + 2t - 1 \tag{4.8}$$

d) $t \ge 2$: Here, $F_T(t) = 1$.

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t < 1 \\ -\frac{t^2}{2} + 2t - 1 & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases} \tag{4.9}$$

Using Equation 2.2,

$$p_T(t) = \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

4.5 Verify your results through a plot.
Solution: This has been done in the plots (4.1) and (4.2).

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: The C file in Question 1.1 generates samples of X in the file data/ber.dat.

5.2 Generate

$$Y = AX + N \tag{5.1}$$

where $A = 5$ dB, $X \in \{1, -1\}$ is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
Solution: The C file in Question 1.1 generates the numbers in the file data/gau_ber.dat

5.3 Plot Y using a scatter plot.
Solution: The Python code can be downloaded using

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/5_2.py
```

and run using

```
$ python3 5_2.py
```

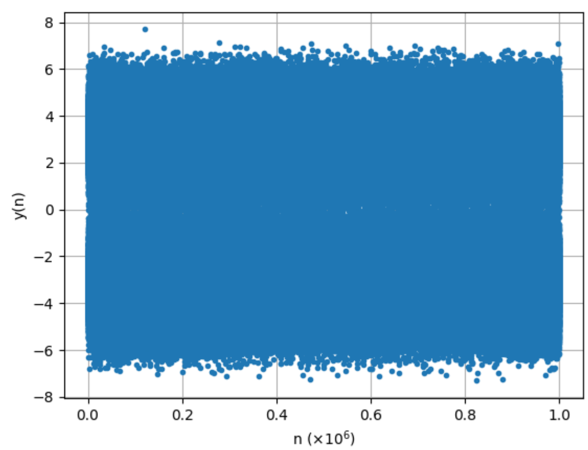

Fig. 5.1: Plot of Y

5.4 Guess how to estimate X from Y.
Solution: From the plot of Y, we see that the estimate model can be written as

$$\hat{X} = \begin{cases} 1 & Y > 0 \\ 0 & Y < 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution: Letting $X = 1$ and $X = -1$ respectively, we see the number of mismatched data points to compute the error probabilities. The simulation is coded in

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/5_4.py
```

and can be run by typing

```
$ python3 5_4.py
```

The results are

$$P_{e|0} = 5.33 \times 10^{-4} \tag{5.5}$$
$$P_{e|1} = 5.57 \times 10^{-4} \tag{5.6}$$

5.6 Find P_e assuming that X has equiprobable symbols. **Solution:** Here, $\Pr(X = 1) = \Pr(X = -1) = 0.5$. Thus,

$$P_e = \Pr(X = 1) P_{e|1} + \Pr(X = -1) P_{e|0} \tag{5.7}$$
$$= \frac{1}{2}\left(P_{e|0} + P_{e|1}\right) = 5.45 \times 10^{-4} \tag{5.8}$$

5.7 Verify by plotting the theoretical P_e wrt A from 0 dB to 10 dB. **Solution:** We note that

$$P_{e|0} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.9}$$
$$= \Pr(Y > 0 | X = -1) \tag{5.10}$$
$$= \Pr(AX + N > 0 | X = -1) \tag{5.11}$$
$$= \Pr(N > A | X = -1) = Q(A) \tag{5.12}$$

since X and N are independent. Writing a similar expression for $P_{e|1}$ and noting that

$$\Pr(N < -A) = \Pr(N > A) = Q(A) \tag{5.13}$$

it follows that $P_e = Q(A)$. This is the idea used to plot the theoretical P_e. The plot is coded both in the rectangular axes and the semilog-y axes. Download the relevant codes using

```
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/5_6.py
$ wget https://github.com/1Aryan8/
    AI1110_bt21btech11002/blob/main/
    randNums(sim)/codes/5_6_semilog.py
```

and execute them using

```
$ python3 5_6.py
$ python3 5_6_semilogy.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

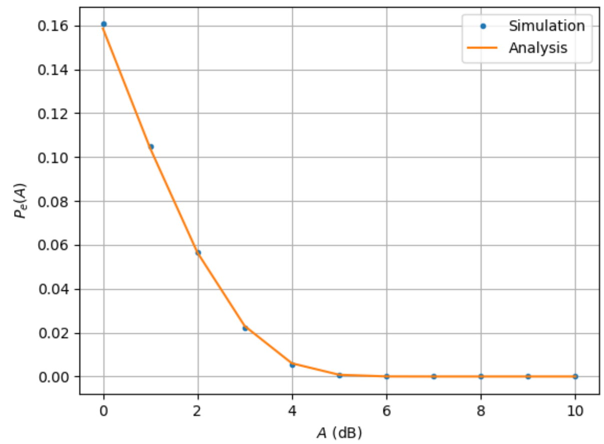

Fig. 5.2: $P_e(A)$ (rectangular axes)

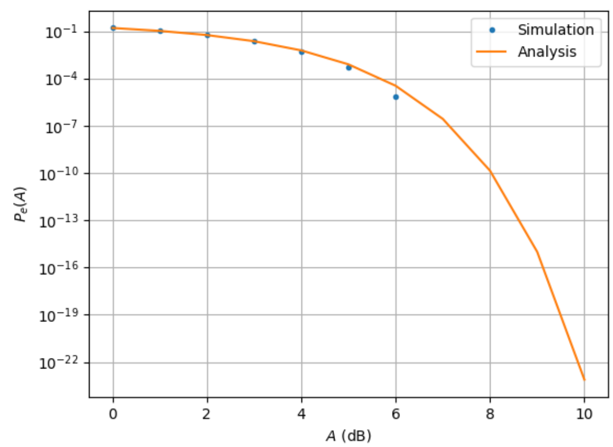

Fig. 5.3: $P_e(A)$ (semilog-y axes)

Solution: Replacing the 0 in (5.11) with δ and performing a similar operation for $P_{e|1}$, we get

$$P_e = \Pr(X = -1) Q(A + \delta)$$
$$+ \Pr(X = 1) Q(A - \delta) \tag{5.14}$$
$$= \frac{1}{2}\left(Q(A + \delta) + Q(A - \delta)\right) \tag{5.15}$$

Differentiating with respect to δ leads to the equation (here f_N denotes standard normal distribution)

$$f_N(A + \delta) = f_N(A - \delta) \tag{5.16}$$

which implies that for $A \neq 0,\ \delta = 0$ and for $A = 0,\ \delta \in \mathbb{R}$.

5.9 Repeat the above exercise when

$$p_X(1) = p \tag{5.17}$$

Solution: Using (5.14) and following a similar procedure as in the previous question, we see that

$$p f_N(A-\delta) = (1-p) f_N(A+\delta) \tag{5.18}$$

$$p e^{-\frac{(A-\delta)^2}{2}} = (1-p) e^{-\frac{(A+\delta)^2}{2}} \tag{5.19}$$

$$\implies \delta = \frac{1}{2A} \ln\left(\frac{1-p}{p}\right) \tag{5.20}$$

5.10 Repeat the above exercise using MAP criterion.
Solution: Using Bayes' Theorem, we get

$$\Pr(X = 1|Y = y)$$

$$= \frac{\Pr(N = y - A|X = 1)\Pr(X = 1)}{p_Y(y)} \tag{5.21}$$

$$= \frac{p f_N(y-A)}{p f_N(y-A) + (1-p) f_N(y+A)} \tag{5.22}$$

$$= \frac{p}{p + (1-p) e^{-2yA}} \tag{5.23}$$

and

$$\Pr(X = -1|Y = y)$$

$$= \frac{\Pr(N = y + A|X = -1)\Pr(X = -1)}{p_Y(y)} \tag{5.24}$$

$$= \frac{(1-p) f_N(y+A)}{p f_N(y-A) + (1-p) f_N(y+A)} \tag{5.25}$$

$$= \frac{1-p}{(1-p) + p e^{2yA}} \tag{5.26}$$

Hence,

$$\frac{p}{p + (1-p) e^{-2yA}} \gtrless \frac{1-p}{(1-p) + p e^{2yA}} \tag{5.27}$$

$$\implies p^2 e^{2yA} \gtrless (1-p)^2 e^{-2yA} \tag{5.28}$$

$$\implies y \gtrless \frac{1}{2A} \ln\left(\frac{1-p}{p}\right) \tag{5.29}$$