

Random Numbers

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Abstract—This manual provides solutions to the Assignment on Random Numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/1.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/1.2.py
```

Download the above files and execute the following commands to produce Fig.1.2

```
$ python3 1.2.py
```

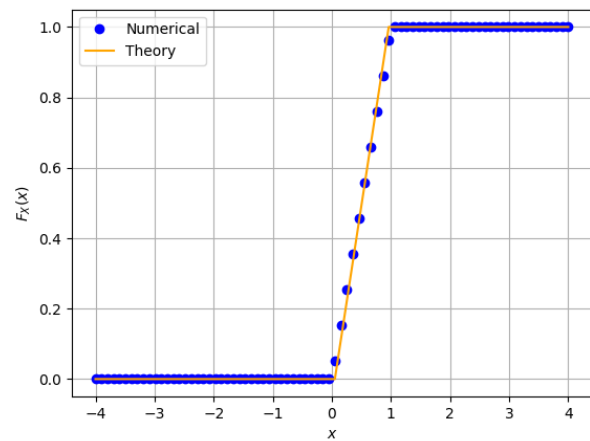


Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.

Solution: Given U is a uniform Random Variable

$$p_U(x) = 1 \text{ for } 0 < x < 1 \quad (1.2)$$

$$F_U(x) = \int_{-\infty}^{\infty} p_U(x) dx \quad (1.3)$$

$$\Rightarrow F_U(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases} \quad (1.4)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of U .

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/1.4.c
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.4.c
$ ./a.out
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

Solution:

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.8)$$

$$\Rightarrow \text{var}[U] = E[U^2] - E[U]^2 \quad (1.9)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.10)$$

$$E[U] = \int_0^1 x \quad (1.11)$$

$$\Rightarrow E[U] = \frac{1}{2} \quad (1.12)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.13)$$

$$E[U^2] = \int_0^1 x^2 dF_U(x) \quad (1.14)$$

$$\Rightarrow E[U^2] = \frac{1}{3} \quad (1.15)$$

$$\Rightarrow \text{var}[U] = \frac{1}{12} = 0.0833 \quad (1.16)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/2.1.c
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/2.2.py
```

Download the above files and execute the following commands to produce Fig.2.2

```
$ python3 2.2.py
```

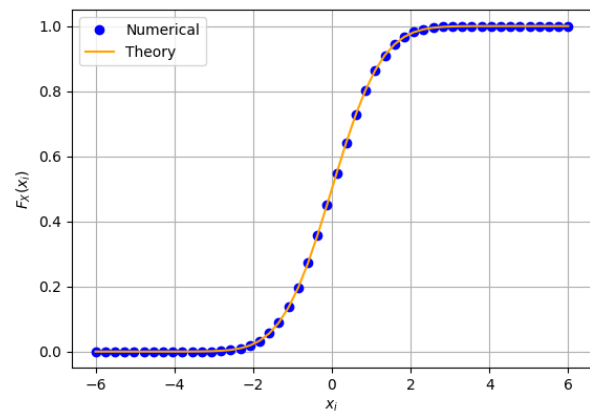


Fig. 2.2: The CDF of X

Some of the properties of CDF

a) $\lim_{x \rightarrow \infty} F_X(x) = 1$

- b) $F_X(x)$ is non decreasing function.
- c) Symmetric about one point.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/2.3.py
```

Download the above files and execute the following commands to produce Fig.2.3

```
$ python3 2.3.py
```

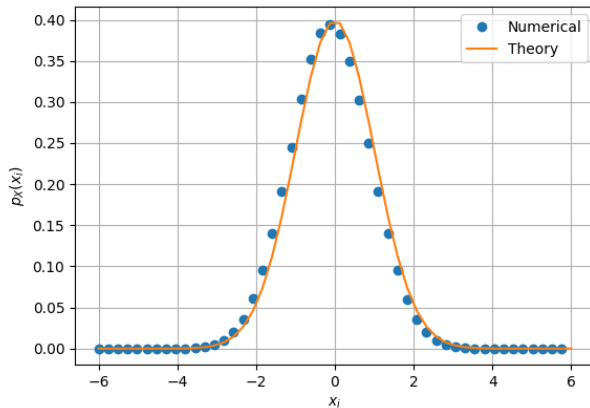


Fig. 2.3: The PDF of X

Some of the properties of the PDF:

- a) Symmetric about $x = \mu$ in this case
- b) Decreasing function for $x > \mu$ and increasing for $x < \mu$ and attains maximum at $x = \mu$
- c) Area under the curve is unity.

2.4 Find the mean and variance of X by writing a C program.

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/2.4.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.4.c
$ ./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution:

1) CDF is given by

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x) dx \quad (2.4)$$

$$\boxed{F_X(x) = 1} \quad (2.5)$$

2) Mean is given by

$$E(x) = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.6)$$

$$\Rightarrow \boxed{E(x) = 0} \quad (2.7)$$

3) Variance is given by

$$\text{var}[U] = E(U^2) - (E(U))^2 \quad (2.8)$$

$$E x^2 = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.9)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 \exp\left(-\frac{x^2}{2}\right) dx \quad (2.10)$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int \exp\left(-\frac{x^2}{2}\right) dx \right) \quad (2.11)$$

$$- \frac{1}{\sqrt{2\pi}} \int \int \left(x \exp\left(-\frac{x^2}{2}\right) \right) dx \cdot dx \quad (2.12)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.13)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \quad (2.14)$$

$$= 1 \Rightarrow \boxed{\text{var}[U] = 1} \quad (2.15)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/3.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 3.1.c -lm
$ ./a.out
```

The CDF of V is plotted in Fig. 3.1 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/3.1pyth.py
```

Download the above files and execute the following commands to produce Fig.3.1

```
$ python3 3.1pyth.py
```

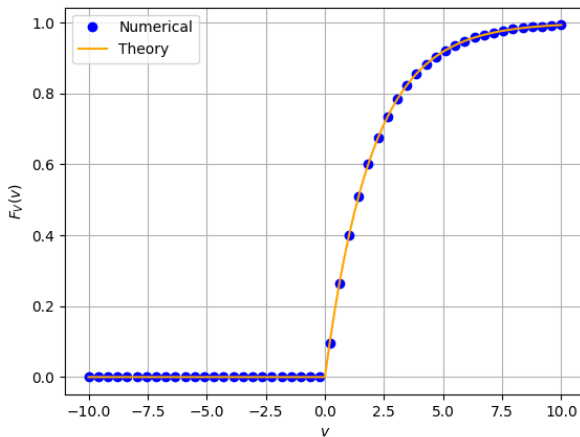


Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.

Solution: If $Y = g(X)$, we know that $F_Y(y) = F_X(g^{-1}(y))$, here

$$V = -2 \ln(1 - U) \quad (3.2)$$

$$1 - U = e^{\frac{-V}{2}} \quad (3.3)$$

$$U = 1 - e^{\frac{-V}{2}} \quad (3.4)$$

$$F_V(x) = F_U(1 - e^{\frac{-x}{2}}) \quad (3.5)$$

$$\Rightarrow F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{\frac{-x}{2}} & x \geq 0 \end{cases} \quad (3.6)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/4.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 4.1.c
$ ./a.out
```

4.2 Find the CDF of T .

Solution: The CDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/4.5cdf.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.5cdf.py
```

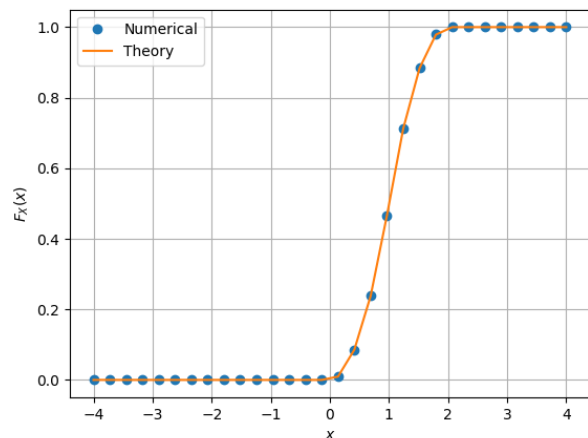


Fig. 4.2: The CDF of T

4.3 Find the PDF of T .

Solution: The PDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/4.5pdf.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.5pdf.py
```

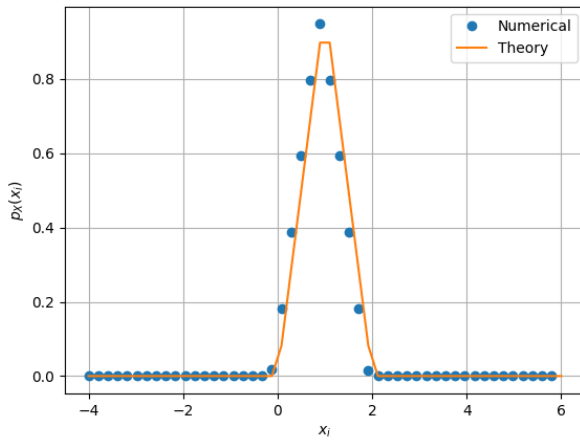


Fig. 4.3: The PDF of T

4.4 Find the Theoretical Expression for the PDF and CDF of T

Solution:

$$T = U_1 + U_2 \quad (4.2)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_{U1}(x)p_{U2}(y)dx \quad (4.3)$$

$$\text{As, } p_{U1}(x) = p_{U1}(y) = p_U(u) \quad (4.4)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_U(u)p_U(t-u)du \quad (4.5)$$

a) Theoretical PDF

i) $t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \quad (4.6)$$

$$\Rightarrow p_T(t) = \int_0^t du = t \quad (4.7)$$

ii) $t > 1$

$$p_T(t) = \int_0^1 p_U(t-u)du \quad (4.8)$$

$$\Rightarrow p_T(t) = \int_{t-1}^1 du = 2-t \quad (4.9)$$

$$\Rightarrow P_T(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t < 0 \text{ or } t > 2 \end{cases}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^t p_T(u)du \quad (4.10)$$

$$\Rightarrow F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - 1 - \frac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot

Solution: The Results are verified in the plots Fig 4.2 and Fig 4.3