

Random Numbers

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Abstract—This manual provides solutions to the Assignment on Random Numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/1.1.c
```

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/1.2.py
```

Download the above files and execute the following commands to produce Fig.1.2

```
$ python3 1.2.py
```

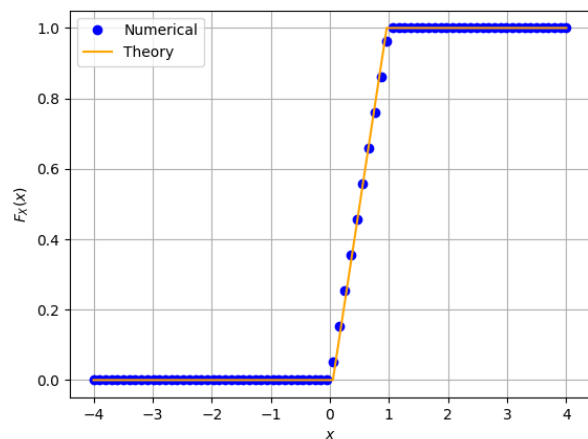


Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.

Solution: Given U is a uniform Random Variable

$$p_U(x) = 1 \text{ for } 0 < x < 1 \quad (1.2)$$

$$F_U(x) = \int_{-\infty}^{\infty} p_U(x) dx \quad (1.3)$$

$$\Rightarrow F_U(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases} \quad (1.4)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of U .

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/1.4.c
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.4.c
$ ./a.out
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

Solution:

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.8)$$

$$\Rightarrow \text{var}[U] = E[U^2] - E[U]^2 \quad (1.9)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.10)$$

$$E[U] = \int_0^1 x \quad (1.11)$$

$$\Rightarrow E[U] = \frac{1}{2} \quad (1.12)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.13)$$

$$E[U^2] = \int_0^1 x^2 dF_U(x) \quad (1.14)$$

$$\Rightarrow E[U^2] = \frac{1}{3} \quad (1.15)$$

$$\Rightarrow \text{var}[U] = \frac{1}{12} = 0.0833 \quad (1.16)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/2.1.c
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/GovindaRohith/
  Assignments/blob/main/Randomnum/
  codes/2.2.py
```

Download the above files and execute the following commands to produce Fig.2.2

```
$ python3 2.2.py
```

Some of the properties of CDF

- a) $\lim_{x \rightarrow \infty} F_X(x) = 1$
- b) $F_X(x)$ is non decreasing function.
- c) Symmetric about one point.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

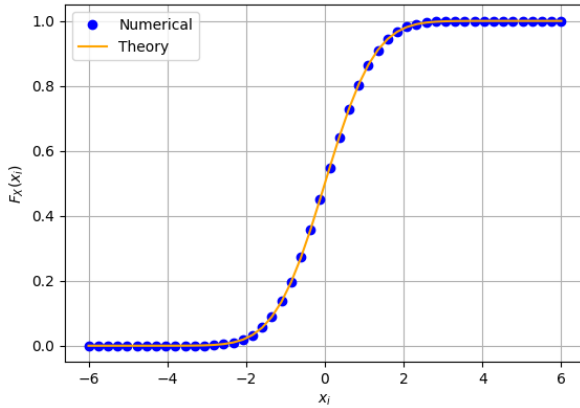


Fig. 2.2: The CDF of X

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/2.3.py
```

Download the above files and execute the following commands to produce Fig.2.3

```
$ python3 2.3.py
```

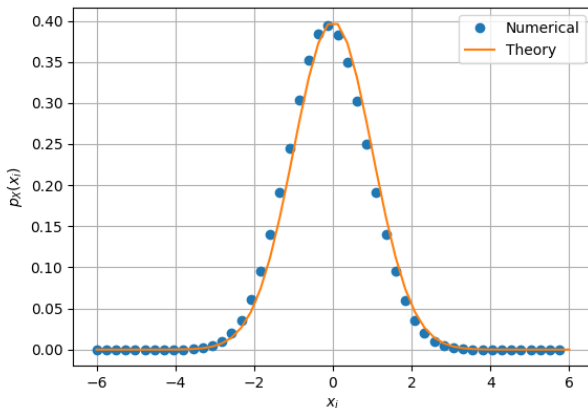


Fig. 2.3: The PDF of X

Some of the properties of the PDF:

- Symmetric about $x = \mu$ in this case
- Decreasing function for $x > \mu$ and increasing for $x < \mu$ and attains maximum at $x = \mu$
- Area under the curve is unity.

2.4 Find the mean and variance of X by writing a C program.

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/2.4.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.4.c
$ ./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution:

1) CDF is given by

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x) dx \quad (2.4)$$

$$F_X(x) = 1 \quad (2.5)$$

2) Mean is given by

$$E(x) = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.6)$$

$$\Rightarrow E(x) = 0 \quad (2.7)$$

3) Variance is given by

$$\text{var}[U] = E(U^2) - (E(U))^2 \quad (2.8)$$

$$E x^2 = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.9)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 \exp\left(-\frac{x^2}{2}\right) dx \quad (2.10)$$

$$= \frac{1}{\sqrt{2\pi}} \left(x \int \exp\left(-\frac{x^2}{2}\right) dx \right) \quad (2.11)$$

$$= \frac{1}{\sqrt{2\pi}} \int \int \left(x \exp\left(-\frac{x^2}{2}\right) \right) dx dx \quad (2.12)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.13)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \quad (2.14)$$

$$= 1 \Rightarrow \text{var}[U] = 1 \quad (2.15)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/3.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 3.1.c -lm
$ ./a.out
```

The CDF of V is plotted in Fig. 3.1 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/3.1pyth.py
```

Download the above files and execute the following commands to produce Fig.3.1

```
$ python3 3.1pyth.py
```

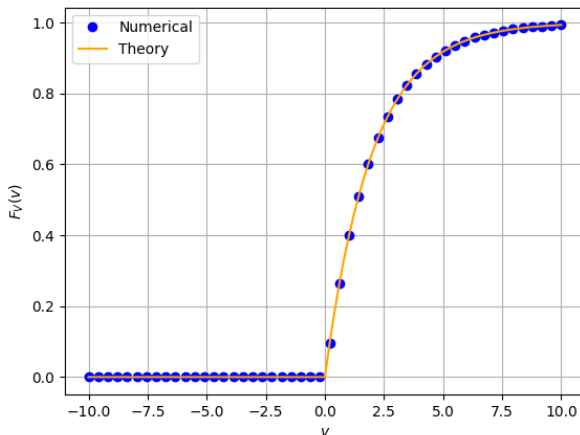


Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.

Solution: If $Y = g(X)$, we know that $F_Y(y) = F_X(g^{-1}(y))$, here

$$V = -2 \ln(1 - U) \quad (3.2)$$

$$1 - U = e^{-\frac{V}{2}} \quad (3.3)$$

$$U = 1 - e^{-\frac{V}{2}} \quad (3.4)$$

$$F_V(x) = F_U(1 - e^{-\frac{x}{2}}) \quad (3.5)$$

$$\Rightarrow F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x}{2}} & x \geq 0 \end{cases} \quad (3.6)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/4.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 4.1.c
$ ./a.out
```

4.2 Find the CDF of T .

Solution: The CDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/4.5cdf.py
```

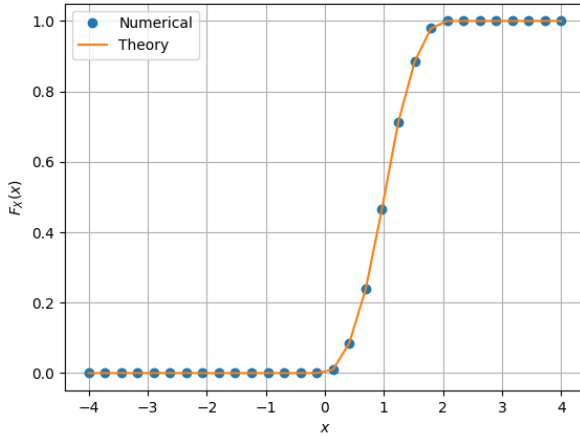
Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.5cdf.py
```

4.3 Find the PDF of T .

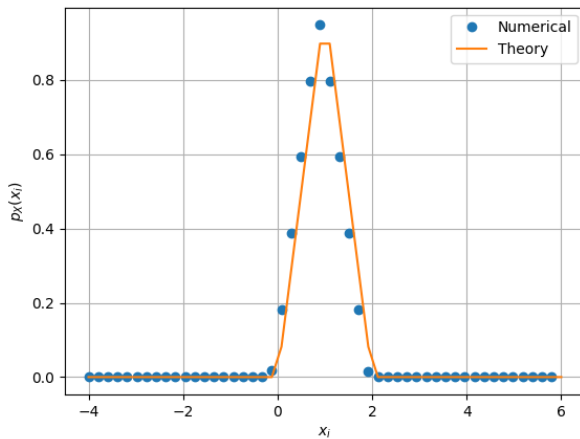
Solution: The PDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/4.5pdf.py
```

Fig. 4.2: The CDF of T

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.5pdf.py
```

Fig. 4.3: The PDF of T

4.4 Find the Theoretical Expression for the PDF and CDF of T

Solution:

$$T = U_1 + U_2 \quad (4.2)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_{U1}(x)p_{U2}(y)dx \quad (4.3)$$

$$\text{As, } p_{U1}(x) = p_{U1}(y) = p_U(u) \quad (4.4)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_U(u)p_U(t-u)du \quad (4.5)$$

a) Theoretical PDF

i) $t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \quad (4.6)$$

$$\Rightarrow p_T(t) = \int_0^t du = t \quad (4.7)$$

ii) $t > 1$

$$p_T(t) = \int_0^1 p_U(t-u)du \quad (4.8)$$

$$\Rightarrow p_T(t) = \int_{t-1}^1 du = 2-t \quad (4.9)$$

$$\Rightarrow P_T(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t < 0 \text{ or } t > 2 \end{cases}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^t p_T(u)du \quad (4.10)$$

$$\Rightarrow F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t-1-\frac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot

Solution: The Results are verified in the plots Fig 4.2 and Fig 4.3

5 GUASSIAN TO OTHER

5.1 Generate equiprobable $X \in \{1, -1\}$. **Solution:** Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/5.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 5.1.c
$ ./a.out
```

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

Where $A=5$ dB and $N \sim \mathcal{N}(0, 1)$

Solution: :

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/5.2.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 5.2.c
$ ./a.out
```

5.3 Plot Y using a scatter plot.

Solution: The CDF of V is plotted in Fig. 5.5 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/5.3.py
```

Download the above files and execute the following commands to produce Fig.5.5

```
$ python3 5.3.py
```

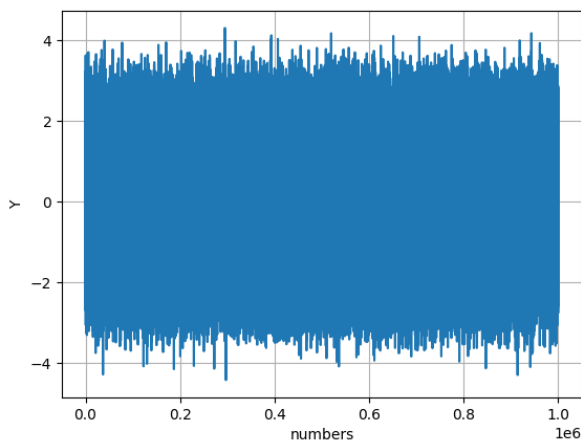


Fig. 5.5: The Scatter Plot

5.4 Guess how to estimate X from Y .

Solution: :

- i) If $Y < 0$ then probably $X = -1$
- ii) If $Y > 0$ then probably $X = 1$

Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.3)$$

Solution: :

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/5.5.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands to get the result

```
$ gcc 5.5.c
$ ./a.out
```

$$P_{e|0} = 0.499035 \quad (5.4)$$

$$P_{e|1} = 0.500176 \quad (5.5)$$

5.5 Find P_e assuming that X has equiprobable symbols.

Solution:

$$P_e = P(X = 1)P_{e|0} + P(X = -1)P_{e|1} \quad (5.6)$$

Since X is equiprobable

$$P(X = 1) = P(X = -1) = 0.5 \quad (5.7)$$

$$\Rightarrow P_e = \frac{P_{e|0} + P_{e|1}}{2} \quad (5.8)$$

$$\Rightarrow P_e = Q_N(A) \quad (5.9)$$

5.6 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

5.7 Now, consider a threshold δ while estimating X from Y . Find the value of δ that maximizes the theoretical P_e .

5.8 Repeat the above exercise when

$$p_X(0) = p \quad (5.10)$$

5.9 Repeat the above exercise using the MAP criterion.

6 GUASSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

Solution:

The CDF and PDF of V is plotted in Fig. 6.5 and 6.5 using the codes below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/6.1cdf.py
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/6.1pdf.py
```

Download the above files and execute the following commands to produce Fig.6.5 and 6.5

```
$ python3 6.1cdf.py
$ python3 6.1pdf.py
```

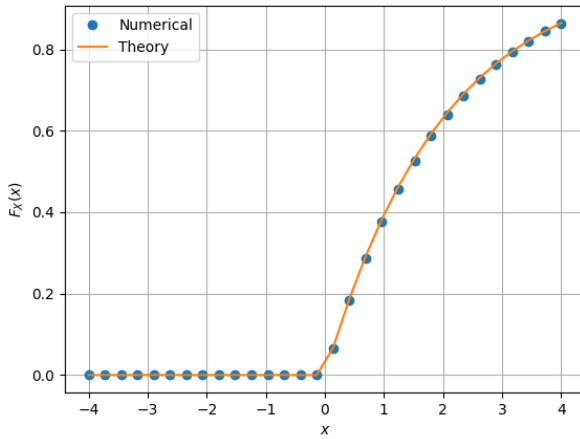


Fig. 6.5: The CDF of V

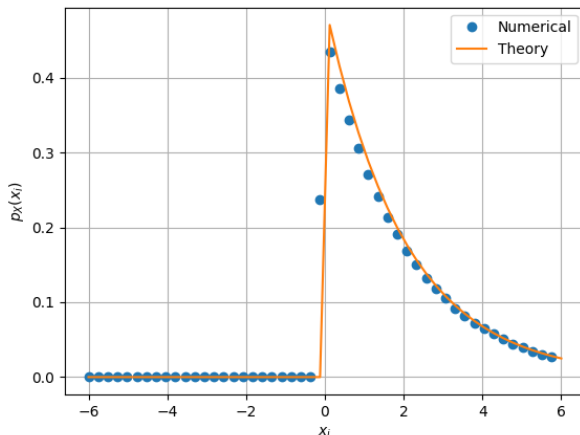


Fig. 6.5: The PDF of V

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α .

Solution: Assuming X_1 and X_2 are i.i.d

$$X_1 = r \cos \Theta \quad (6.3)$$

$$X_2 = r \sin \Theta \quad (6.4)$$

The Jacobian matrix transforming r, Θ to X_1, X_2 is defined as

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial r} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial r} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \quad (6.5)$$

$$\Rightarrow \mathbf{J} = \begin{pmatrix} \cos \Theta & -r \sin \Theta \\ \sin \Theta & r \cos \Theta \end{pmatrix} \quad (6.6)$$

$$\Rightarrow |\mathbf{J}| = r \quad (6.7)$$

$$(6.8)$$

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2) \quad (6.9)$$

$$\Rightarrow f_{X_1, X_2}(x_1, x_2) = \frac{e^{\frac{-x_1^2 - x_2^2}{2}}}{2\pi} \quad (6.10)$$

$$\Rightarrow f_{X_1, X_2}(x_1, x_2) = \frac{e^{\frac{-r^2}{2}}}{2\pi} \quad (6.11)$$

$$f_{R, \Theta}(r, \theta) = f_{X_1, X_2}(x_1, x_2) |\mathbf{J}| \quad (6.12)$$

$$\Rightarrow f_{R, \Theta}(r, \theta) = \frac{r e^{\frac{-r^2}{2}}}{2\pi} \quad (6.13)$$

$$f_R(r) = \int_0^{2\pi} f_{R, \Theta}(r, \theta) d\theta \quad (6.14)$$

$$f_R(r) = \int_0^{2\pi} \frac{r e^{\frac{-r^2}{2}}}{2\pi} d\theta \quad (6.15)$$

$$\Rightarrow f_R(r) = r e^{\frac{-r^2}{2}} \quad (6.16)$$

$$F_V(r) = F_{X_1^2 + X_2^2}(r) \quad (6.17)$$

$$\Rightarrow F_V(r) = F_{R^2}(r) \quad (6.18)$$

$$\Rightarrow F_V(r) = \Pr(R^2 \leq x) \quad (6.19)$$

$$\Rightarrow F_V(r) = \Pr(R \leq \sqrt{x}) \quad (6.20)$$

$$\text{But } F_V(x) = \Pr(R \leq x) \quad (6.21)$$

$$\Rightarrow F_V(x) = \int_0^x f_R(r) dr \quad (6.22)$$

$$F_V(x) = 1 - e^{-\frac{x}{2}} \quad (6.23)$$

$$\Rightarrow F_V(r) = \begin{cases} 0 & r < 0 \\ 1 - e^{-\frac{r}{2}} & r \geq 0 \end{cases} \quad (6.24)$$

$$\Rightarrow \boxed{\alpha = \frac{1}{2}} \quad (6.25)$$

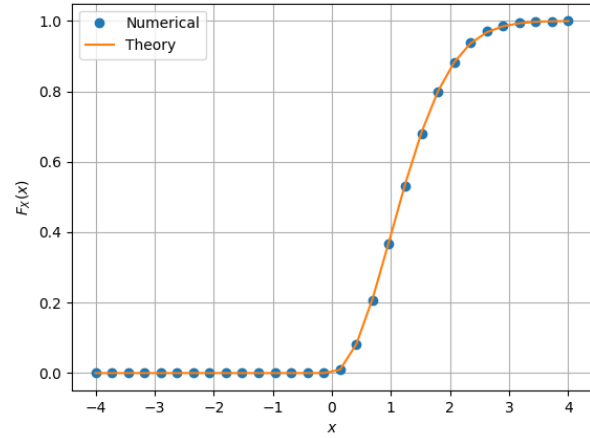


Fig. 6.5: The CDF of V

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.26)$$

Solution: : For $r \geq 1$

$$F_A(r) = \Pr(A \leq r) \quad (6.27)$$

$$= \Pr(\sqrt{V} \leq r) \quad (6.28)$$

$$= \Pr(V \leq r^2) \quad (6.29)$$

$$= F_V(r^2) = 1 - e^{-\frac{r^2}{2}} \quad (6.30)$$

and so on differentiating

$$f_A(r) = re^{-\frac{r^2}{2}} \quad (6.31)$$

Thus, the CDF and PDF of A is given by

$$F_V(r) = \begin{cases} 1 - e^{-\frac{r^2}{2}} & r \geq 0 \\ 0 & r < 0 \end{cases} \quad (6.32)$$

$$p_V(r) = \begin{cases} re^{-\frac{r^2}{2}} & r \geq 0 \\ 0 & r < 0 \end{cases} \quad (6.33)$$

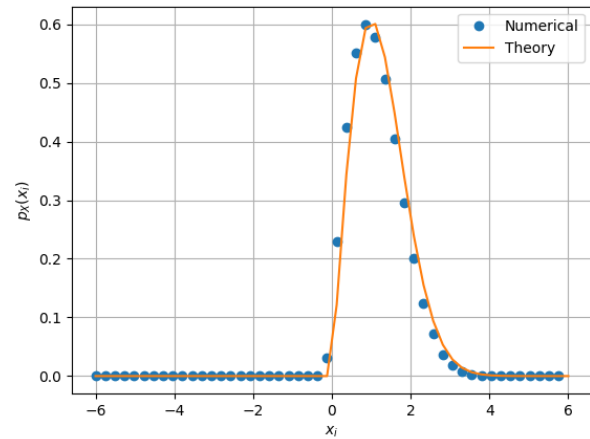


Fig. 6.5: The PDF of V

The CDF and PDF of A is plotted in Fig. 6.5 and 6.5 using the codes below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/6.3cdf.py
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/6.3pdf.py
```

Download the above files and execute the following commands to produce Fig.6.5 and 6.5

```
$ python3 6.3cdf.py
$ python3 6.3pdf.py
```