

# Random Numbers

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**Abstract**—This manual provides solutions to the Assignment on Random Numbers

## 1 UNIFORM RANDOM NUMBERS

Let  $U$  be a uniform random variable between 0 and 1.

1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat .

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/1.1.c
```

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

**Solution:** The following code plots Fig. 1.2

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/1.2.py
```

Download the above files and execute the following commands to produce Fig.1.2

```
$ python3 1.2.py
```

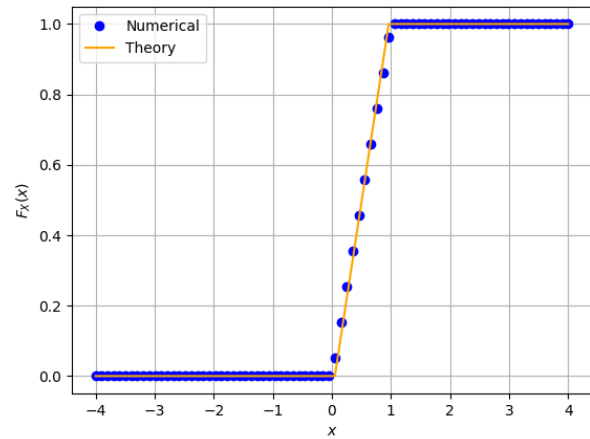


Fig. 1.2: The CDF of  $U$

1.3 Find a theoretical expression for  $F_U(x)$ .

**Solution:** Given  $U$  is a uniform Random Variable

$$p_U(x) = 1 \text{ for } 0 < x < 1 \quad (1.2)$$

$$F_U(x) = \int_{-\infty}^{\infty} p_U(x) dx \quad (1.3)$$

$$\Rightarrow F_U(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases} \quad (1.4)$$

1.4 The mean of  $U$  is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of  $U$ .

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/1.4.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.4.c
$ ./a.out
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

**Solution:**

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.8)$$

$$\Rightarrow \text{var}[U] = E[U^2] - E[U]^2 \quad (1.9)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.10)$$

$$E[U] = \int_0^1 x \quad (1.11)$$

$$\Rightarrow E[U] = \frac{1}{2} \quad (1.12)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.13)$$

$$E[U^2] = \int_0^1 x^2 dF_U(x) \quad (1.14)$$

$$\Rightarrow E[U^2] = \frac{1}{3} \quad (1.15)$$

$$\Rightarrow \text{var}[U] = \frac{1}{12} = 0.0833 \quad (1.16)$$

## 2 CENTRAL LIMIT THEOREM

2.1 Generate  $10^6$  samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where  $U_i, i = 1, 2, \dots, 12$  are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/2.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of  $X$  using the samples in gau.dat. What properties does a CDF have?

**Solution:** The CDF of  $X$  is plotted in Fig. 2.2 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/2.2.py
```

Download the above files and execute the following commands to produce Fig.2.2

```
$ python3 2.2.py
```

Some of the properties of CDF

- a)  $\lim_{x \rightarrow \infty} F_X(x) = 1$
- b)  $F_X(x)$  is non decreasing function.
- c) Symmetric about one point.

2.3 Load gau.dat in python and plot the empirical PDF of  $X$  using the samples in gau.dat. The PDF of  $X$  is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

**Solution:** The PDF of  $X$  is plotted in Fig. 2.3 using the code below

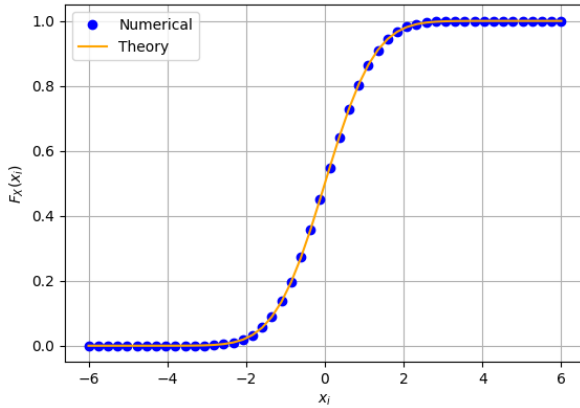


Fig. 2.2: The CDF of X

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/2.3.py
```

Download the above files and execute the following commands to produce Fig.2.3

```
$ python3 2.3.py
```

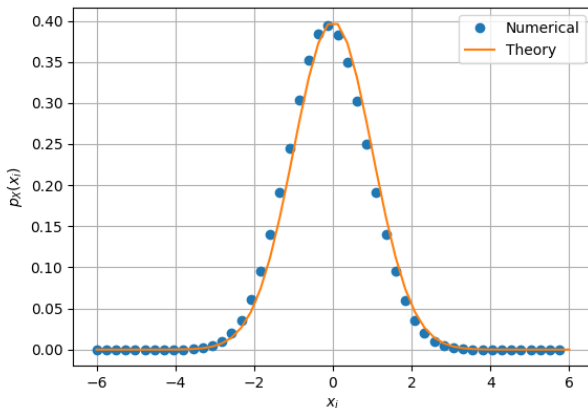


Fig. 2.3: The PDF of X

Some of the properties of the PDF:

- Symmetric about  $x = \mu$  in this case
- Decreasing function for  $x > \mu$  and increasing for  $x < \mu$  and attains maximum at  $x = \mu$
- Area under the curve is unity.

2.4 Find the mean and variance of X by writing a C program.

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/2.4.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.4.c
$ ./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

**Solution:**

1) CDF is given by

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x) dx \quad (2.4)$$

$$F_X(x) = 1 \quad (2.5)$$

2) Mean is given by

$$E(x) = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.6)$$

$$\Rightarrow E(x) = 0 \quad (2.7)$$

3) Variance is given by

$$\text{var}[U] = E(U^2) - (E(U))^2 \quad (2.8)$$

$$E x^2 = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.9)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 \exp\left(-\frac{x^2}{2}\right) dx \quad (2.10)$$

$$= \frac{1}{\sqrt{2\pi}} \left( x \int \exp\left(-\frac{x^2}{2}\right) dx \right) \quad (2.11)$$

$$- \frac{1}{\sqrt{2\pi}} \int \int \left( x \exp\left(-\frac{x^2}{2}\right) \right) dx dx \quad (2.12)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.13)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \quad (2.14)$$

$$= 1 \Rightarrow \text{var}[U] = 1 \quad (2.15)$$

### 3 FROM UNIFORM TO OTHER

#### 3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/3.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 3.1.c -lm
$ ./a.out
```

The CDF of  $V$  is plotted in Fig. 3.1 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/3.1pyth.py
```

Download the above files and execute the following commands to produce Fig.3.1

```
$ python3 3.1pyth.py
```

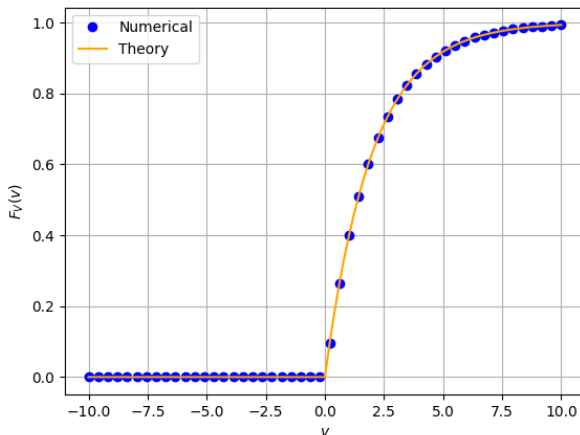


Fig. 3.1: The CDF of  $V$

#### 3.2 Find a theoretical expression for $F_V(x)$ .

**Solution:** If  $Y = g(X)$ , we know that  $F_Y(y) = F_X(g^{-1}(y))$ , here

$$V = -2 \ln(1 - U) \quad (3.2)$$

$$1 - U = e^{-\frac{V}{2}} \quad (3.3)$$

$$U = 1 - e^{-\frac{V}{2}} \quad (3.4)$$

$$F_V(x) = F_U(1 - e^{-\frac{x}{2}}) \quad (3.5)$$

$$\Rightarrow F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x}{2}} & x \geq 0 \end{cases} \quad (3.6)$$

### 4 TRIANGULAR DISTRIBUTION

#### 4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/4.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 4.1.c
$ ./a.out
```

#### 4.2 Find the CDF of $T$ .

**Solution:** The CDF of  $T$  is plotted in Fig. 4.2 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/4.5cdf.py
```

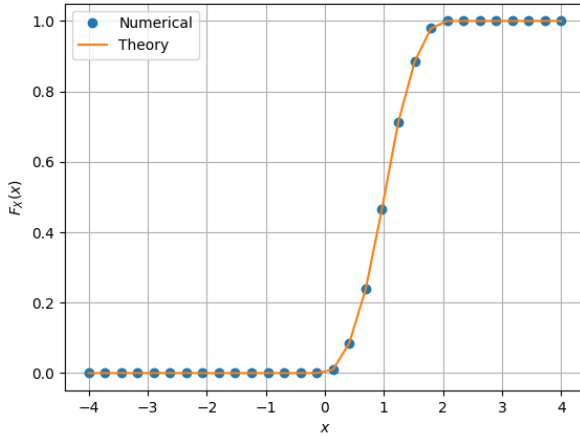
Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.5cdf.py
```

#### 4.3 Find the PDF of $T$ .

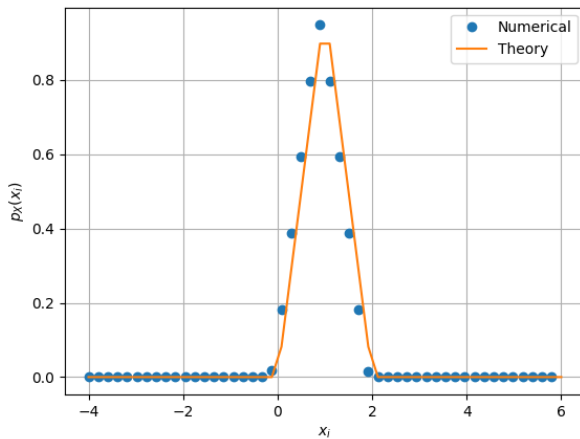
**Solution:** The PDF of  $T$  is plotted in Fig. 4.2 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/4.5pdf.py
```

Fig. 4.2: The CDF of  $T$ 

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.5pdf.py
```

Fig. 4.3: The PDF of  $T$ 

4.4 Find the Theoretical Expression for the PDF and CDF of  $T$

**Solution:**

$$T = U_1 + U_2 \quad (4.2)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_{U1}(x)p_{U2}(y)dx \quad (4.3)$$

$$\text{As, } p_{U1}(x) = p_{U1}(y) = p_U(u) \quad (4.4)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_U(u)p_U(t-u)du \quad (4.5)$$

a) Theoretical PDF

i)  $t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \quad (4.6)$$

$$\Rightarrow p_T(t) = \int_0^t du = t \quad (4.7)$$

ii)  $t > 1$

$$p_T(t) = \int_0^1 p_U(t-u)du \quad (4.8)$$

$$\Rightarrow p_T(t) = \int_{t-1}^1 du = 2-t \quad (4.9)$$

$$\Rightarrow P_T(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t < 0 \text{ or } t > 2 \end{cases}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^t p_T(u)du \quad (4.10)$$

$$\Rightarrow F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t-1-\frac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot

**Solution:** The Results are verified in the plots Fig 4.2 and Fig 4.3

## 5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable  $X \in \{1, -1\}$ .

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/5.1.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 5.1.c
$ ./a.out
```

## 5.2 Generate

$$Y = AX + N, \quad (5.1)$$

Where  $A=5$  dB and  $N \sim \mathcal{N}(0, 1)$

**Solution:** :

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/5.2.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 5.2.c
$ ./a.out
```

5.3 Plot  $Y$  using a scatter plot.

**Solution:** The CDF of  $V$  is plotted in Fig. 5.5 using the code below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/5.3.py
```

Download the above files and execute the following commands to produce Fig.5.5

```
$ python3 5.3.py
```

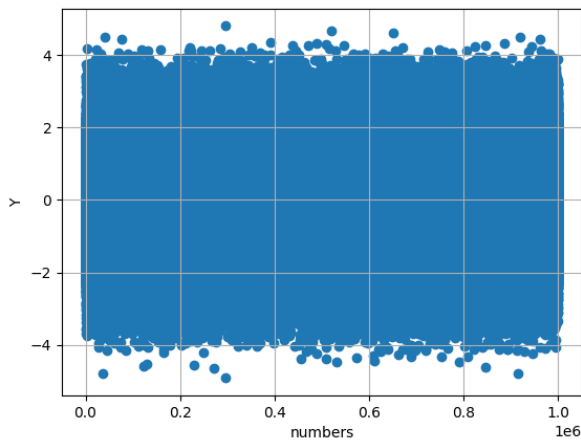


Fig. 5.5: The Scatter Plot

5.4 Guess how to estimate  $X$  from  $Y$ .

**Solution:** :

- i) If  $Y < 0$  then probably  $X = -1$
- ii) If  $Y > 0$  then probably  $X = 1$

## 5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.3)$$

**Solution:** :

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/5.5.c
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/source.h
```

Download the above files and execute the following commands to get the result

```
$ gcc 5.5.c
$ ./a.out
```

$$P_{e|0} = 0.499033 \quad (5.4)$$

$$P_{e|1} = 0.500138 \quad (5.5)$$

5.6 Find  $P_e$  assuming that  $X$  has equiprobable symbols.

**Solution:**

$$P_e = P(X = 1)P_{e|0} + P(X = -1)P_{e|1} \quad (5.6)$$

Since  $X$  is equiprobable

$$P(X = 1) = P(X = -1) = 0.5 \quad (5.7)$$

$$\Rightarrow P_e = \frac{P_{e|0} + P_{e|1}}{2} \quad (5.8)$$

$$\Rightarrow P_e = 0.499585 \quad (5.9)$$

5.7 Verify by plotting the theoretical  $P_e$  with respect to  $A$  from 0 to 10 dB.

**Solution:** :

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.10)$$

$$P_{e|0} = \Pr(AX + N < 0|X = 1) \quad (5.11)$$

$$P_{e|0} = \Pr(N < -A) \quad (5.12)$$

$$P_{e|0} = \int_{-\infty}^{-A} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \quad (5.13)$$

$$P_{e|0} = \int_A^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \quad (5.14)$$

$$P_{e|0} = Q_N(A) \quad (5.15)$$

$$\text{Similarly, } P_{e|1} = Q_N(A) \quad (5.16)$$

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/5.6.py
```

Download the above files and execute the following commands to produce Fig.5.5

```
$ python3 5.6.py
```

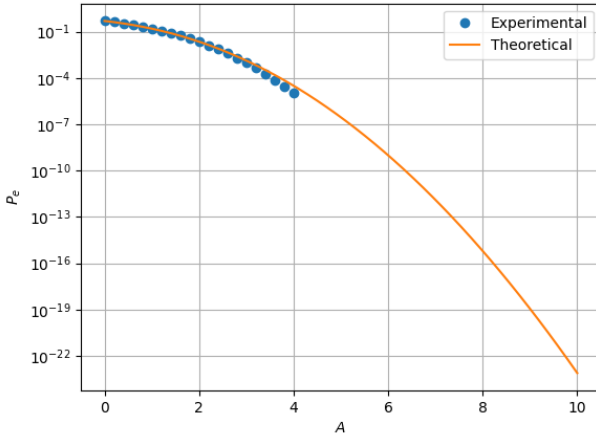


Fig. 5.5:  $P_e(A)$  with semilog-y axis

5.8 Now, consider a threshold  $\delta$  while estimating  $X$  from  $Y$ . Find the value of  $\delta$  that maximizes the theoretical  $P_e$ .

**Solution:** To estimate  $X$  from  $Y$ , we now consider the following:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \quad (5.17)$$

Therefore,

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.18)$$

$$= \Pr(AX + N < \delta | X = 1) \quad (5.19)$$

$$\Rightarrow P_{e|0} = \Pr(N < \delta - A) \quad (5.20)$$

$$= \int_{-\infty}^{\delta-A} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (5.21)$$

$$= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (5.22)$$

$$\Rightarrow P_{e|0} = Q_N(A - \delta) \quad (5.23)$$

Similarly,

$$P_{e|1} = Q_N(A + \delta) \quad (5.24)$$

$$P_e = P_{e|0} \Pr(X = 1) + P_{e|1} \Pr(X = -1) \quad (5.25)$$

$$= \frac{Q_N(A - \delta) + Q_N(A + \delta)}{2} \quad (5.26)$$

Differentiating the above equation wrt  $\delta$ :

$$0 = \frac{d}{d\delta} \left( \frac{Q_N(A - \delta) + Q_N(A + \delta)}{2} \right) \quad (5.27)$$

$$= \frac{1}{2} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \right) \quad (5.28)$$

$$\Rightarrow (\delta - A)^2 = (A + \delta)^2 \quad (5.29)$$

$$\Rightarrow \boxed{\delta = 0} \quad (5.30)$$

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.31)$$

**Solution:** Using Eq. (5.25), we have:

$$P_e = P_{e|0}p + P_{e|1}(1 - p) \quad (5.32)$$

$$= pQ_N(A - \delta) + (1 - p)Q_N(A + \delta) \quad (5.33)$$

Differentiating as before, we get:

$$0 = p \frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - (1 - p) \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \quad (5.34)$$

$$e^{\frac{(\delta+A)^2 - ((\delta-A))^2}{2}} = \frac{1 - p}{p} \quad (5.35)$$

$$\Rightarrow \boxed{\delta = 0} \quad (5.36)$$

5.10 Repeat the above exercise using the MAP criterion.

**Solution:** Assume that

$$\Pr(X = -1) = p \quad (5.37)$$

$$\Pr(X = 1) = (1 - p) \quad (5.38)$$

From Total Probability Theorem, we have:

$$p_Y(y) = p_{Y|X=-1}(y|-1) \Pr(X = -1) + p_{Y|X=1}(y|1) \Pr(X = 1) \quad (5.39)$$

$$p_Y(y) = p \times p_{(-A+N)}(y) \quad (5.40)$$

$$+ (1 - p) \times p_{(A+N)}(y) \quad (5.41)$$

$$p_Y(y) = p \frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p) \frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}} \quad (5.42)$$

To use the MAP criterion, we must find  $p_{X|Y}(x|y)$ . To do this, we use the Theorem of Conditional Probability:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)} \quad (5.43)$$

When  $X = 1$ , we have:

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1) \times p_X(1)}{p_Y(y)} \quad (5.44)$$

$$= \frac{(1 - p) \frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}{p \frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p) \frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}} \quad (5.45)$$

$$= \frac{(1 - p) e^{2yA}}{p + (1 - p) e^{2yA}} \quad (5.46)$$

Similarly, when  $X = -1$ , we get:

$$p_{X|Y}(-1|y) = \frac{p}{p + (1 - p) e^{2yA}} \quad (5.47)$$

Therefore, when  $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$ , we have:

$$\frac{(1 - p) e^{2yA}}{p + (1 - p) e^{2yA}} > \frac{p}{p + (1 - p) e^{2yA}} \quad (5.48)$$

$$e^{2yA} > \frac{p}{(1 - p)} \quad (5.49)$$

$$\Rightarrow y > \frac{1}{2A} \ln \frac{p}{(1 - p)} \quad (5.50)$$

Therefore, when Eq. (5.50), we can assert that  $X = 1$ , and  $X = -1$  otherwise. Now, consider when  $p = \frac{1}{2}$ . We have:

$$y > \frac{1}{2A} \ln \frac{p}{(1 - p)} \quad (5.51)$$

$$= \frac{1}{2A} \ln 1 \quad (5.52)$$

$$= 0 \quad (5.53)$$

Therefore, when  $y > 0$ , we choose  $X = 1$ ,

and we choose  $X = -1$  otherwise.

## 6 GUASSIAN TO OTHER

6.1 Let  $X_1 \sim \mathcal{N}(0, 1)$  and  $X_2 \sim \mathcal{N}(0, 1)$ . Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

### Solution:

The CDF and PDF of  $V$  is plotted in Fig. 6.5 and 6.5 using the codes below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/6.1cdf.py
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/6.1pdf.py
```

Download the above files and execute the following commands to produce Fig.6.5 and 6.5

```
$ python3 6.1cdf.py
$ python3 6.1pdf.py
```

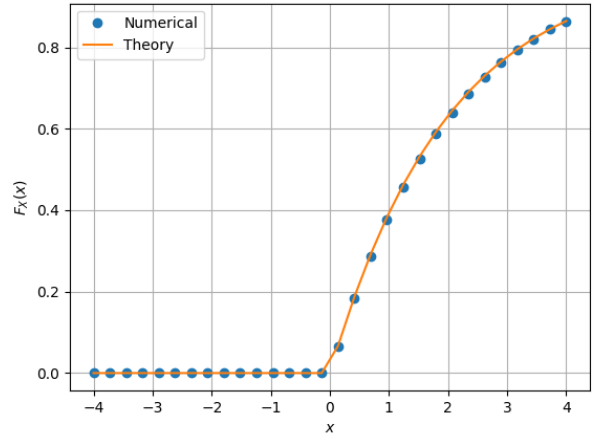


Fig. 6.5: The CDF of  $V$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find  $\alpha$ .

**Solution:** Assuming  $X_1$  and  $X_2$  are i.i.d

$$X_1 = r \cos \Theta \quad (6.3)$$

$$X_2 = r \sin \Theta \quad (6.4)$$



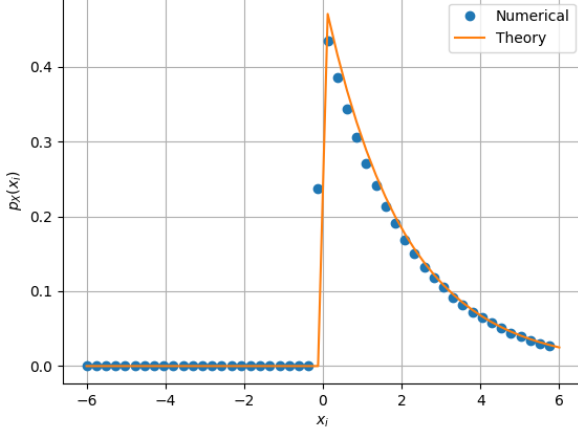


Fig. 6.5: The PDF of  $V$

The Jacobian matrix transforming  $r, \Theta$  to  $X_1, X_2$  is defined as

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial r} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial r} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \quad (6.5)$$

$$\Rightarrow \mathbf{J} = \begin{pmatrix} \cos \Theta & -r \sin \Theta \\ \sin \Theta & r \cos \Theta \end{pmatrix} \quad (6.6)$$

$$\Rightarrow |\mathbf{J}| = r \quad (6.7)$$

$$(6.8)$$

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2) \quad (6.9)$$

$$\Rightarrow f_{X_1, X_2}(x_1, x_2) = \frac{e^{-\frac{x_1^2 + x_2^2}{2}}}{2\pi} \quad (6.10)$$

$$\Rightarrow f_{X_1, X_2}(x_1, x_2) = \frac{e^{-\frac{r^2}{2}}}{2\pi} \quad (6.11)$$

$$f_{R, \Theta}(r, \theta) = f_{X_1, X_2}(x_1, x_2) |\mathbf{J}| \quad (6.12)$$

$$\Rightarrow f_{R, \Theta}(r, \theta) = \frac{r e^{-\frac{r^2}{2}}}{2\pi} \quad (6.13)$$

$$f_R(r) = \int_0^{2\pi} f_{R, \Theta}(r, \theta) d\theta \quad (6.14)$$

$$f_R(r) = \int_0^{2\pi} \frac{r e^{-\frac{r^2}{2}}}{2\pi} d\theta \quad (6.15)$$

$$\Rightarrow f_R(r) = r e^{-\frac{r^2}{2}} \quad (6.16)$$

$$F_V(r) = F_{X_1^2 + X_2^2}(r) \quad (6.17)$$

$$\Rightarrow F_V(r) = F_{R^2}(r) \quad (6.18)$$

$$\Rightarrow F_V(r) = \Pr(R^2 \leq x) \quad (6.19)$$

$$\Rightarrow F_V(r) = \Pr(R \leq \sqrt{x}) \quad (6.20)$$

$$\text{But } F_V(x) = \Pr(R \leq x) \quad (6.21)$$

$$\Rightarrow F_V(x) = \int_0^x f_R(r) dr \quad (6.22)$$

$$F_V(x) = 1 - e^{-\frac{x}{2}} \quad (6.23)$$

$$\Rightarrow F_V(r) = \begin{cases} 0 & r < 0 \\ 1 - e^{-\frac{r}{2}} & r \geq 0 \end{cases} \quad (6.24)$$

$$\Rightarrow \boxed{\alpha = \frac{1}{2}} \quad (6.25)$$

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.26)$$

**Solution:** : For  $r \geq 1$

$$F_A(r) = \Pr(A \leq r) \quad (6.27)$$

$$= \Pr(\sqrt{V} \leq r) \quad (6.28)$$

$$= \Pr(V \leq r^2) \quad (6.29)$$

$$= F_V(r^2) = 1 - e^{-\frac{r^2}{2}} \quad (6.30)$$

and so on differentiating

$$f_A(r) = r e^{-\frac{r^2}{2}} \quad (6.31)$$

Thus, the CDF and PDF of  $A$  is given by

$$F_V(r) = \begin{cases} 1 - e^{-\frac{r^2}{2}} & r \geq 0 \\ 0 & r < 0 \end{cases} \quad (6.32)$$

$$f_V(r) = \begin{cases} r e^{-\frac{r^2}{2}} & r \geq 0 \\ 0 & r < 0 \end{cases} \quad (6.33)$$

The CDF and PDF of  $A$  is plotted in Fig. 6.5 and 6.5 using the codes below

```
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/6.3cdf.py
wget https://github.com/GovindaRohith/
Assignments/blob/main/Randomnum/
codes/6.3pdf.py
```

Download the above files and execute the following commands to produce Fig.6.5 and 6.5

```
$ python3 6.3cdf.py
$ python3 6.3pdf.py
```

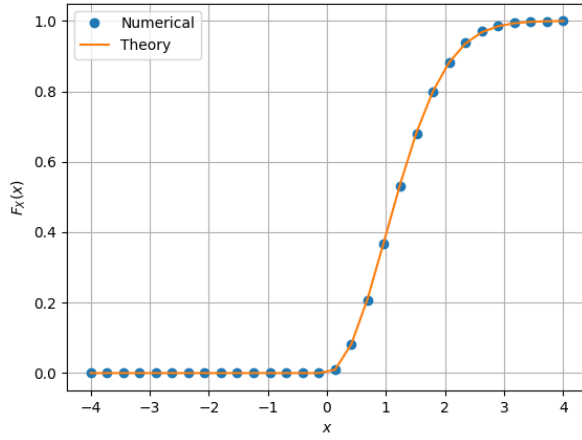


Fig. 6.5: The CDF of  $V$

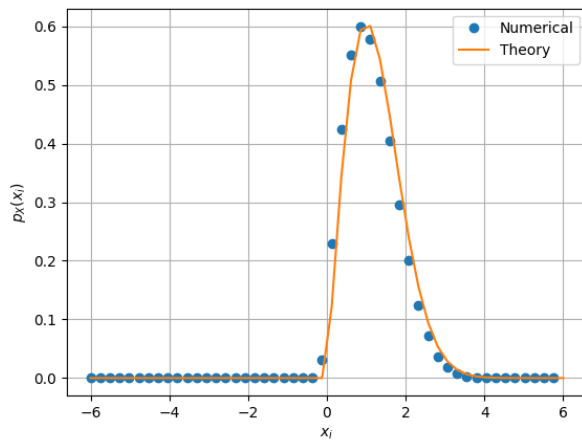


Fig. 6.5: The PDF of  $V$