Random Numbers

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Contents

***Abstract*—This manual provides solutions to the Assignment of Random Numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/1.1.c
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/1.2.py
```

Download the above files and execute the following commands to produce Fig.1.2

```
$ python3 1.2.py
```

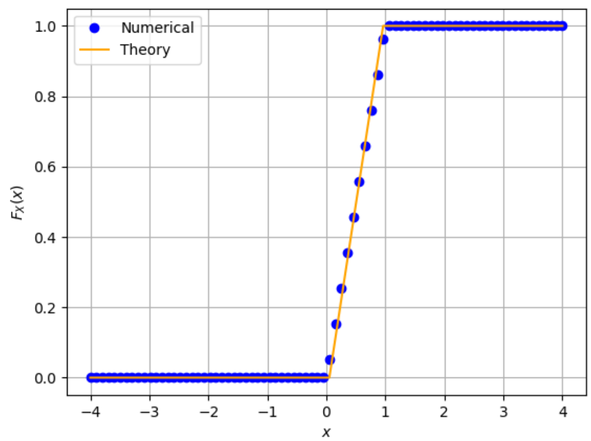

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniform Random Variable

$$p_U(x) = 1 \text{ for} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{\infty} p_U(x)dx \tag{1.3}$$

$$\boxed{\implies F_U(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/1.4.c
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.4.c
$ ./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution:

$$\text{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2 \tag{1.8}$$

$$\implies \text{var}\left[U\right] = E\left[U^2\right] - E\left[U\right]^2 \tag{1.9}$$

$$E\left[U\right] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.10}$$

$$E\left[U\right] = \int_{0}^{1} x \tag{1.11}$$

$$\implies \boxed{E\left[U\right] = \frac{1}{2}} \tag{1.12}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.13}$$

$$E\left[U^2\right] = \int_{0}^{1} x^2 dF_U(x) \tag{1.14}$$

$$\implies E\left[U^2\right] = \frac{1}{3} \tag{1.15}$$

$$\implies \boxed{\text{var}\left[U\right] = \frac{1}{12} = 0.0833} \tag{1.16}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/2.1.c
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/2.2.py
```

Download the above files and execute the following commands to produce Fig.2.2

```
$ python3 2.2.py
```

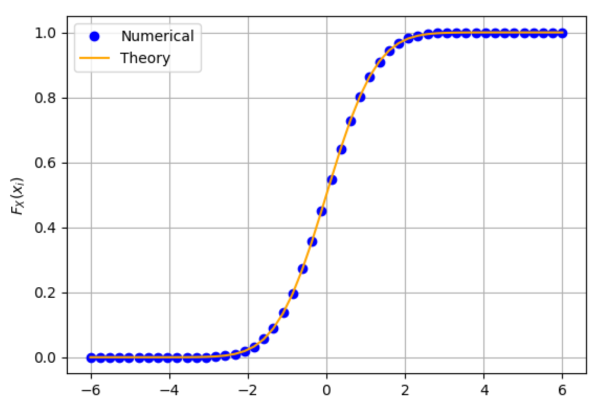

Fig. 2.2: The CDF of X

Some of the properties of CDF

a) $\lim_{x\to\infty} F_X(x) = 1$

b) $F_X(x)$ is non decreasing function.
c) Symmetric about one point.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/2.3.py
```

Download the above files and execute the following commands to produce Fig.2.3

```
$ python3 2.3.py
```

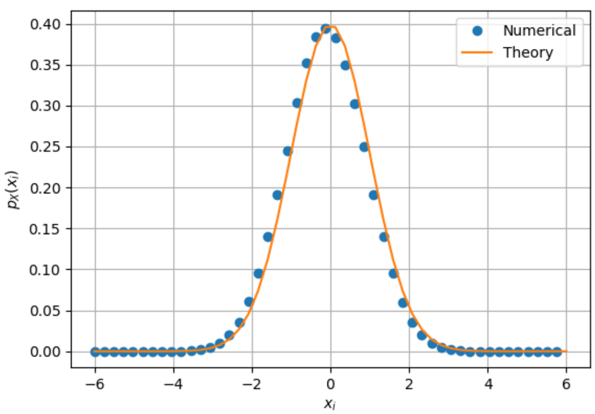

Fig. 2.3: The PDF of X

Some of the properties of the PDF:
a) Symmetric about $x = \mu$ in this case
b) Decreasing function for $x > \mu$ and increasing for $x < \mu$ and attains maximum at $x = \mu$
c) Area under the curve is unity.

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/2.4.c
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.4.c
$ ./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.

Solution:

1) CDF is given by

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x)dx \tag{2.4}$$

$$\boxed{F_X(x) = 1} \tag{2.5}$$

2) Mean is given by

$$E(x) = \int_{-\infty}^{\infty} x p_X(x)dx \tag{2.6}$$

$$\implies \boxed{E(x) = 0} \tag{2.7}$$

3) Variance is given by

$$\text{var}\,[U] = E(U^2) - (E(U))^2 \tag{2.8}$$

$$\implies \boxed{\text{var}\,[U] = \sqrt{2}} \tag{2.9}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/3.1.c
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 3.1.c -lm
$ ./a.out
```

The CDF of V is plotted in Fig. 3.1 using the code below

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/3.1pyth.py
```

Download the above files and execute the following commands to produce Fig.3.1

```
$ python3 3.1pyth.py
```

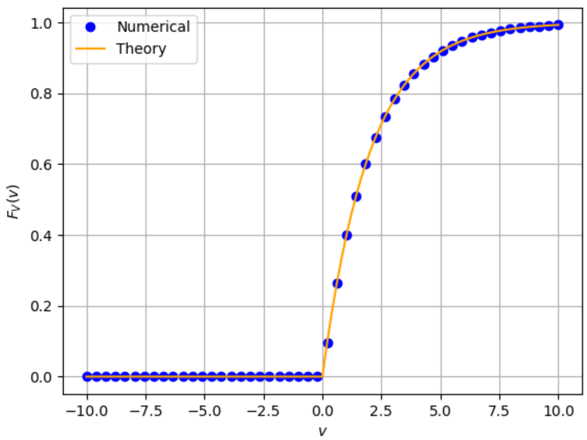

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution: If Y = g(X), we know that $F_Y(y) = F_X(g^{-1}(y))$, here

$$V = -2\ln(1 - U) \tag{3.2}$$

$$1 - U = e^{\frac{-V}{2}} \tag{3.3}$$

$$U = 1 - e^{\frac{-V}{2}} \tag{3.4}$$

$$F_V(x) = F_U(1 - e^{\frac{-x}{2}}) \tag{3.5}$$

$$\implies F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{3.6}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program.

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/4.1.c
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 4.1.c
$ ./a.out
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/4.5cdf.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.5cdf.py
```

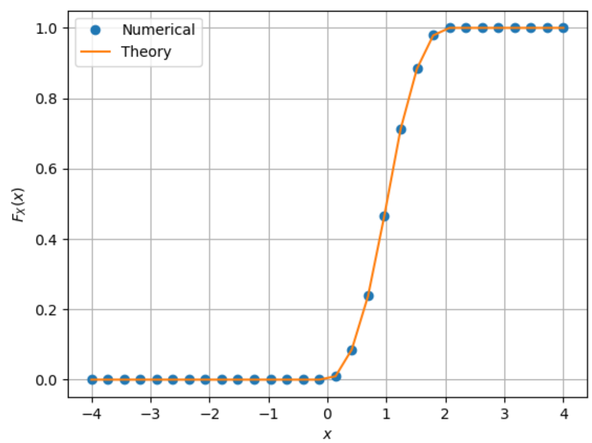

Fig. 4.2: The CDF of T

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/GovindaRohith/
    Assignments/blob/main/Randomnum/
    codes/4.5pdf.py
```

Download the above files and execute the following commands to produce Fig.4.2

```
$ python3 4.5pdf.py
```

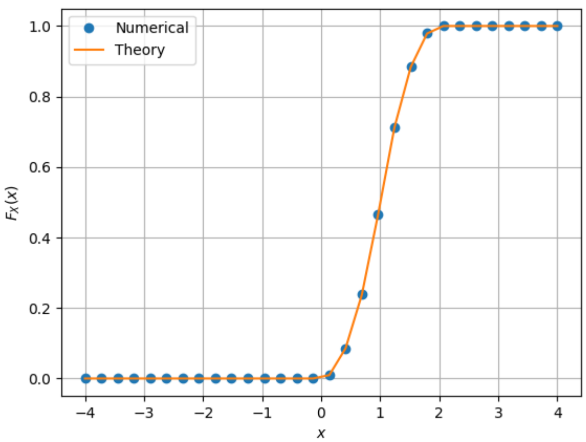

Fig. 4.3: The CDF of T

4.4 Find the Theoreotical Expression for the PDF and CDF of T
Solution:

$$T = U_1 + U_2 \tag{4.2}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_{U1}(x) p_{U2}(y) dx \tag{4.3}$$

$$\text{As}, p_{U1}(x) = p_{U1}(y) = p_U(u) \tag{4.4}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_U(u) p_U(t-u) du \tag{4.5}$$

a) Theoretical PDF
   i) $t \leq 1$

$$p_T(t) = \int_{0}^{t} p_U(t-u) du \tag{4.6}$$

$$\implies p_T(t) = \int_{0}^{t} du = t \tag{4.7}$$

   ii) $t > 1$

$$p_T(t) = \int_{0}^{1} p_U(t-u) du \tag{4.8}$$

$$\implies p_T(t) = \int_{t-1}^{1} du = 2 - t \tag{4.9}$$

$$\implies \boxed{P_T(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t < 0 \text{ or } t > 2 \end{cases}}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^{t} p_T(u) du \tag{4.10}$$

$$\implies \boxed{F_T(t) = \begin{cases} 0 & t < 0 \\ \dfrac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - 1 - \dfrac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}}$$

4.5 Verify your results through a plot
Solution: The Results are verfied in the plots Fig 4.2 and Fig 4.3