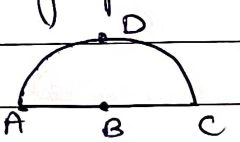
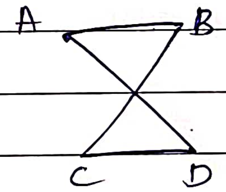
Statistics and discrete mathematics

A graph is a pair (V, E) where V is non empty set known as vertices or pointers or nodes. E is a set of unordered pairs of elements taken from V called edges or lines the graph is denoted by G (V, E) or (V, E) or G

The vertex V is referred as vertex set which has to be non empty set E is known as x set which can be empty of graph.

A graph which contains no edges is called null graph.
e.g., a null graph with n vertices is denoted by N_n



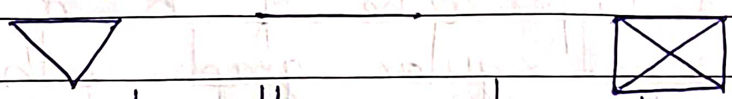


AB, BC, BD, DA

Both the graphs have same number of vertices ABCD and same edges {AB} {BC} {CD} & {DA}
Two diagrams look different yet they represent the same graph since each conveys the same information.
The number of vertices in a finite graph is known as order of the graph and the number of edges in it is known as its size

★ A simple graph of order greater than or equal to 2 in which there is an edge between every pair of vertices is known as complete graph (full graph).

e.g.,



A complete graph with n vertices has $\frac{n(n-1)}{2}$ no. of edges.

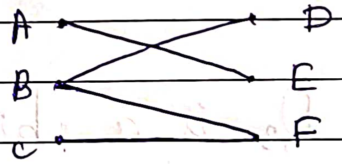
★ A simple graph G is said to be BiPartite if vertex V is union of two disjoint non-empty subsets of vertices V_1 and V_2 such that no two vertices are adjacent in same set.

e.g., If $V_1 = \{A \;\; B \;\; C\}$

$V_2 = \{D \;\; E \;\; F\}$

$V_1 \cup V_2 = \{A \;\; B \;\; C \;\; D \;\; E \;\; F\}$

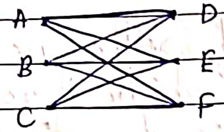
BiPartite graph



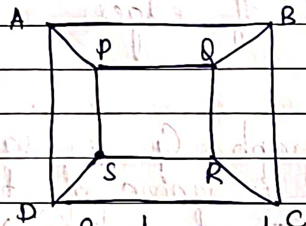
A bipartite graph is called complete bipartite graph if there is an edge between every vertex in V_1 to every vertex in V_2.

In general, if V_1 has R vertices and V_2 S no. of vertices then complete bipartite graph is denoted by $K_{R,S}$ has $R+S$ no. of vertices and $R \times S$ no. of edges.

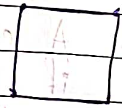
e.g.,



* A hyper cube of K dimensional Q_K is a regular graph with $\underline{2^K}$ no. of vertices.



Q_3 - hypercube

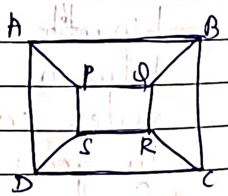


Q_2

* Sum of degrees = 2 × no. of edges.
Sum of degrees of all vertices in a graph is an even number and this is equal to twice no. of edges in a graph.
i.e., $\boxed{\sum d(V) = 2|E|}$ → hand-shaking ppty

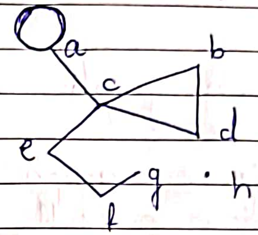
1) Show that Q_3 is a bipartite which is not complete.

Sol:- Taking
$V_1 = \{A\ C\ S\ Q\}$,
$V_2 = \{B\ D\ P\ R\}$



Q_3 is a bipartite
$V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$
It is not complete as AR, CP, etc are not joined or $R \times S \neq$ no. of edges.

2) Verify hand-shaking property for the graph



$d(a) = 3$, $d(b) = 2$, $d(c) = 4$, $d(d) = 2$
$d(e) = 2$, $d(f) = 2$, $d(g) = 1$, $d(h) = 0$

Sum of degrees = 16

$16 = 2 \times |E|$
$16 = 2 \times 8$
Hence, HSP is verified.

3) Prove that K dimentional hypercube has $K.2^{K-1}$ no. of edges.

Sol:- Q_K - 2^K ~~edges~~ vertices
Applying HSP,
$\sum d(V) = 2|E|$
$K\,2^K = 2 \cdot 2|E|$
$|E| = K \cdot 2^{K-1}$

4) Can there be a graph with 12 vertices such that 2 of the vertices have degree 3 each and remaining 10 vertices have degree 4 each.

Sol:- $46 = 2(|E|)$

$|E| = 23$

There can exist such graph with 23 edges

5) Show that there is no graph with 28 edges and 12 vertices in foll cases -

i) degree of vertices is either 3 or 4

ii) degree of vertices is either 3 or 6

Sol:- i) If P is the vertex of 3 degree and Q is the " " 4

Sum of degree of all vertices $= 3P + 4Q$

$= 3P + 4(12-P)$

$= 3P + 48 - 4P$

$= -P + 48$

By hand-shaking property,

$$\Sigma_i d(v) = 2|E|$$

$$-P + 48 = 2(28)$$

$$P = -8$$

No. of vertices cannot be negative

$\therefore$ Such a graph is not possible

ii) Let P be no. of vertices of degree 3 and Q be no. of vertices of degree 6

Sum of degrees $= 3P + 6Q$

$= 3P + 6(12-P)$

$= 3P + 72 - 6P = -3P + 72$

By hand-shaking property,

$$\Sigma_i d(v) = 2|E|$$

$$-3P + 72 = 2(28)$$

$$P = \frac{16}{3}$$

No. of vertices cannot be fraction.

$\therefore$ Such a graph cannot exist.

6) For a graph with n vertices and m edges, if δ is minimum & Δ is maximum of degree of vertices, show that $\delta \leq \frac{2m}{n} \leq \Delta$

Sol:- Let $d_1, d_2 \ldots d_m$ be the degrees

$d_1 + d_2 + \ldots + d_m = 2m$

δ - minimum

$d_1 \geq \delta, \; d_2 \geq \delta \; \ldots \; d_m \geq \delta$

$d_1 + d_2 + \ldots + d_m \geq n\delta$ — ①

$2m \geq n\delta$

Δ - maximum

$d_1 \leq \Delta, \; d_2 \leq \Delta \; \ldots \; d_m \leq \Delta$

$d_1 + d_2 + \ldots + d_m \leq n\Delta$ — ②

$2m \leq n\Delta$

$$n\delta \leq 2m \leq n\Delta$$

$$\delta \leq \frac{2m}{n} \leq n\Delta$$

→ no of vertices

7) If G be a graph of order 9 such that each vertex has degree 5 or 6. Prove that atleast 5 vertices has degree 6 or atleast 6 vertices has degree 5.

Sol:- Let P be no. of vertices of degree 5
Q be no. of vertices of degree 6

$P+Q=9$

By hand-shaking property

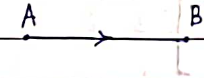
$$\sum d(V) = 2|E|$$
$$5P+6Q = 2E$$
$$5P+6(9-P) = 2E$$
$$-P+54 = 2E$$
$$2E = \underbrace{54-P}_{\text{even}}$$

$0 \le P \le 9 \qquad 0 \le Q \le 9$

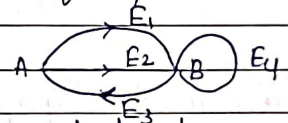
0	9
2	7
4	5
6	3
8	1

Directed graph

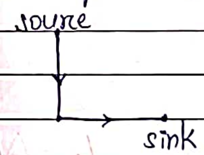
A directed graph is a pair of (V, E) where V is vertex set and elements of E are directed edge



This directed edge is AB where A is called initial vertex and B is called terminal vertex. A edge set with same initial & terminal vertex is known as self loop.



E_1 & E_2 are parallel edges & E_1 & E_3 are non-parallel edges. E_4 is self loop.

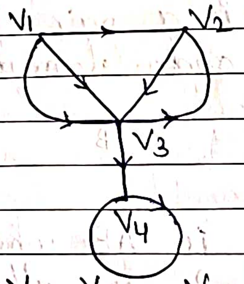


A non-isolated vertex which is not a terminal vertex for any directed edge is known as source. A non-isolated vertex which is not a initial vertex for any directed edge is known as sink.

- No. of edges for which V is the initial vertex is called out-degree of vertex V i.e., od(V) d^{+}(V)
- No. of edges for which V is terminal vertex is called in-degree of V and denoted by i.e. id(V) or $d^{-}(V)$
- In every digraph, sum of out degrees of all vertices is equal to sum of in degrees of all vertices and each sum is equal to

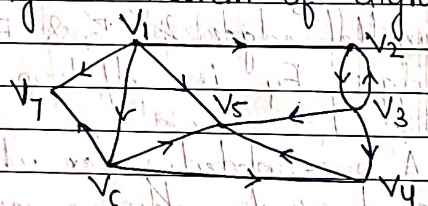
no. of edges in digraph.

e.g.,



	V_1	V_2	V_3	V_4	Sum	
Indegree	0	2	3	2	7	} no. of edges
Outdegree	3	1	2	1	7	

1) Verify the theorem of digraph



	V_1	V_2	V_3	V_4	V_5	V_6	V_7	Sum
Indegree	0	2	1	2	4	1	2	12
Outdegree	4	1	3	1	0	3	0	12

No. of edges = 12

Sum of indegree = Sum of outdegree

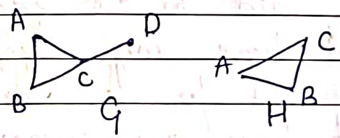
SubGraph

A graph H is said to be subgraph of graph G if all vertices and all edges of H are in G and each edge of H has same n vertices in H as in G.

If H is a sub-graph of G, then G is a supergraph of H.

★ Every graph is its own subgraph.

e.g.,

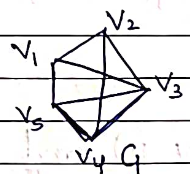


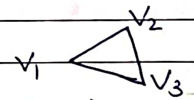
A subgraph H of G is spanning subgraph of G whenever H contains all vertices of G. Thus graph and its spanning subgraph has same vertex set.

Induced SubGraph

Given a graph G (V, E), suppose that there is a graph G_1 (V_1, E_1) such that every edge {A, B} of G where $A, B \in V_1$ is an edge of G_1 is called induced subgraph of G induced by V_1 and dinoted by $\langle V_1 \rangle$

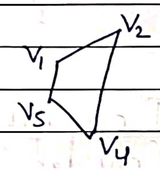
e.g.,





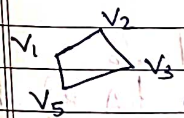
$V = \{V_1, V_2, V_3\}$

induced



$V = \{V_1, V_2, V_4, V_5\}$

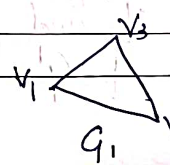
induced

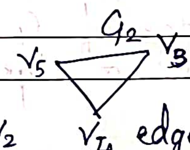


non-induced

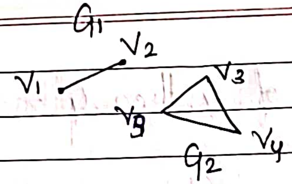
$V = \{V_1, V_2, V_3, V_5\}$

* $\{V_1, V_3\}$ edge is there in G but not in G_3





edge-disjoint



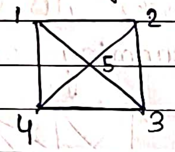
vextex-disjoint

Let G be a graph and G_1 and G_2 be sub-graphs of G then

i) G_1 and G_2 are said to be edge-disjoint if they do not have any edge in common

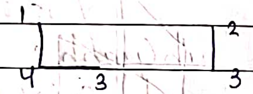
ii) G_1 and G_2 are said to be vertex-disjoint if they do not have any common vertex or common edge.

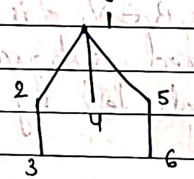
e.g.,



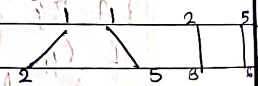
edge disjoint

vertex disjoint

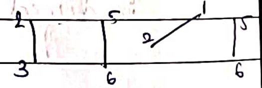




edge disjoint



vertex disjoint



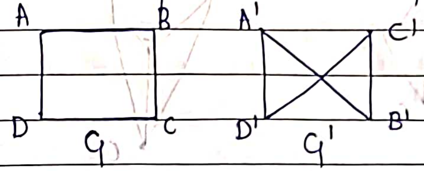
Isomorphism -

Consider two graphs $G(V, E)$ and $G'(V', E')$ suppose there exist a function $f: V \to V'$ such that

i) f is a one to one correspondance

ii) for all vertices of $\{A, B\}$ of G, $\{A, B\}$ is an edge of G if and only if $\{f(A), f(B)\}$ is an edge of G', then f is called an isomorphism between G and G' and say that G and G' are isomorphic graphs.

e.g.,



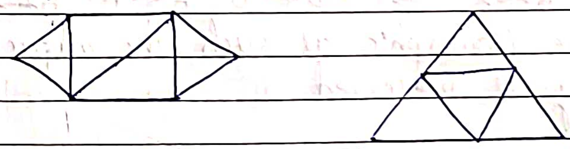
$f: V \to V'$

Both graphs have 4 vertices and 4 edges. One to one correspondance between vertices and edges are:

$A \leftrightarrow A'$	$\{A, B\} \leftrightarrow \{A', B'\}$
$B \leftrightarrow B'$	$\{B, C\} \leftrightarrow \{B', C'\}$
$C \leftrightarrow C'$	$\{C, D\} \leftrightarrow \{C', D'\}$
$D \leftrightarrow D'$	$\{A, D\} \leftrightarrow \{A', D'\}$

There is one to one correspondance between vertices and edges as such adjacency property is preserved, hence, two graphs are isomorphic

ii) Verify whether the graphs are isomorphic

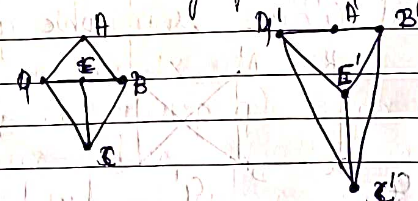


degrees of vertices do not match

Sol:- They are not isomorphic
There are 6 vertices and 9 edges in both edges but there is no one to one correspondance since the degrees of vertices do not match.

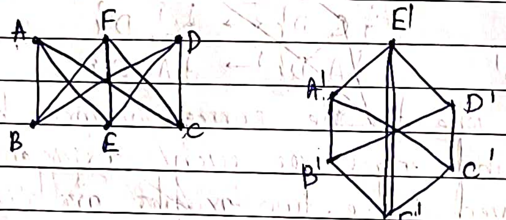
2) Verify whether the graphs are isomorphic.

Sol:-



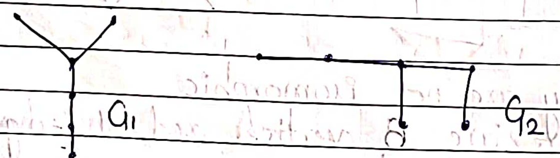
There are 5 vertices and 7 edges in both graphs.
One to one correspondance b/w vertices & edges -
$A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$, $D \leftrightarrow D'$, $E \leftrightarrow E'$.
They are isomorphic.

3)



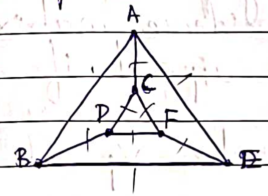
Sol:- There are 6 vertices and 9 edges in both graphs.
One to one correspondance b/w vertices & edges are
$A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$, $D \leftrightarrow D'$, $E \leftrightarrow E'$, $F \leftrightarrow F'$
They are isomorphic as such the adjacency property is preserved.

4)



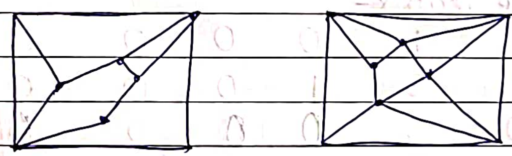
Sol:- They have 6 vertices and 5 edges in both graphs.
They are not isomorphic. since, in G_1 two vertices of degree 1 are adjacent to vertex of degree 3 but in G_2 one vertex of degree 1 is adjacent to vertex of 2 & other vertex of degree 1 is adjacent to vertex of degree 3
Thus, there does not exist one to one correspondance between vertices and edges.

5)



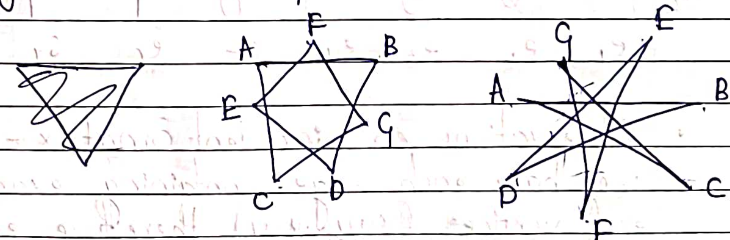
Sol:- They have 6 vertices and 9 edges in both graphs
They are not isomorphic. Each vertex is of degree 3 but graphs are not isomorphic as there is no one to one correspondance b/w edges and vertices as adcancy ppty of vertices are not preserved. Since in G_2 there are 3 vertices are connected in G_2 but not in G_1.

6)



Sol:- They are not isomophic.

7)



Sol:- They are isomorphic.
There are 7 vertices & 7 edges in both graphs
since they have one to one correspondance b/w edges & vertices.

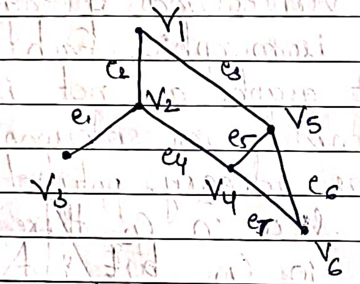
Matrix Representation

Incidence matrix –

Let G be a (p,q) graph without self loops, we define an incidence matrix $A = [a_{ij}]_{p \times q}$ where $a_{ij} = 1$ if an edge e_i is incident on vertex v_j

$a_{ij} = 0$, otherwise.

ex:



Incidence matrix –

$$
\begin{array}{c}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ \\
\end{array}
\begin{array}{c}
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}_{6 \times 7} \\
\;e_1 \;\; e_2 \;\; e_3 \;\; e_4 \;\; e_5 \;\; e_6 \;\; e_7
\end{array}
$$

- Observations on incident matrix –

* we have each edge containing exactly two end vertices (say x, y) therefore each column contains one's in exactly two places (in rows x and y).

* The number 1 in each row represents the edge incident from the vertex corresponding to the row, therefore, the sum of one's in each row represent the degree of a vertex corresponding to the row

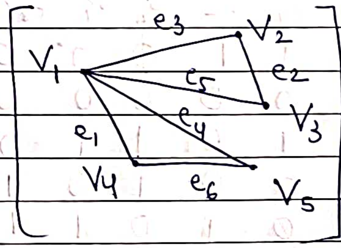
* A row with all zeros represents an isolated vertex

* Two identical columns in an incidence matrix corresponds to the parallel edges.

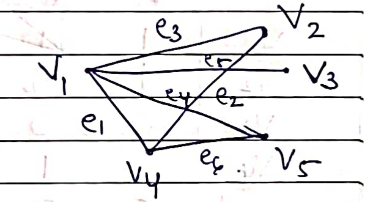
1) Write down graph corresponding to foll incidence matrix –

$$
\begin{array}{c}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ \\
\end{array}
\begin{array}{c}
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}_{5 \times 6} \\
\;e_1 \;\; e_2 \;\; e_3 \;\; e_4 \;\; e_5 \;\; e_6
\end{array}
$$

Sol:–



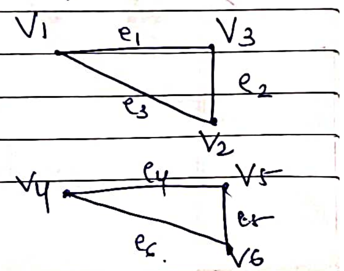
or



2) Write down graph and write any 3 obsn

$$
\begin{array}{c}
\\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6
\end{array}
\begin{array}{c}
\;e_1 \;\; e_2 \;\; e_3 \;\; e_4 \;\; e_5 \;\; e_6 \\
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

Sol:–



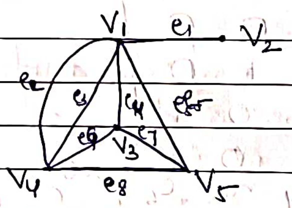
Obⁿ–

- Incidence matrix of every disconnected graph can be written as–

$$A(G) = \begin{bmatrix} A(G_1) & 0 & 0 & 0 \\ 0 & A(G_2) & 0 & 0 \\ 0 & 0 & A(G_3) & 0 \\ 0 & 0 & 0 & A(G_4) \end{bmatrix}$$

where $A(G_i)$ represents the incidence matrix of ~~higher~~ i^{th} component of G, such representation is called block diagram of $A(G)$.

3) Write the incidence matrix for the foll:



$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Adjacency matrix

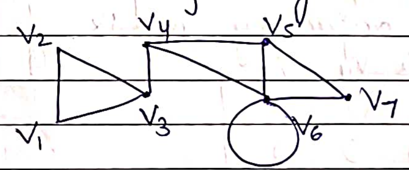
Let G be a (p, q) graph having no parallel edges, then adjacency matrix $X = [x_{ij}]_{p \times q}$

where $x_{ij} = 1$ if $v_{ij} \in E(G)$
$= 0$, otherwise

ex–

$$X = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}_{5 \times 5} \end{array}$$

1) Form the adjacency matrix of foll:



$$X = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}_{7 \times 7} \end{array}$$

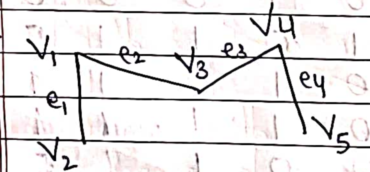
Observations on adjacency matrix –

- The adjacency matrix of a graph is a symmetric binary matrix
- Diagonal elements of adjacency matrix are all zero if and only if G does not have a loop.
- Number of one's in a row or in a column gives the degree of the vertex corresponding to the row/column counting diagonal elements twice.
- Number of vertices having self loop equal to number of non-zero diagonal entries.

2) Construct incidence matrix and write the graph for the foll adjacency matrix.

$$\begin{array}{c|c} & \begin{matrix} V_1 & V_2 & V_3 & V_4 & V_5 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Sol:–



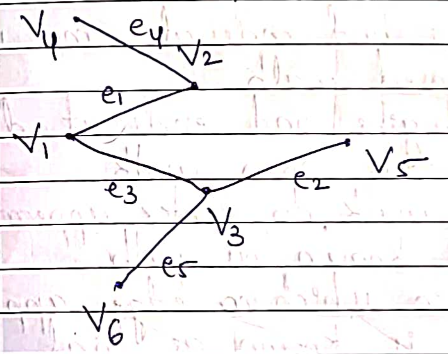
incidence matrix –

$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

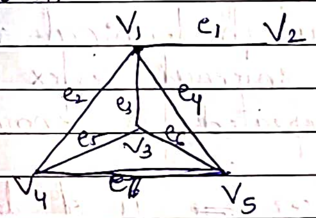
3) Draw the graph for the incidence matrix A(G) and write any 3 observations on it.

$$A(G) = \begin{array}{c|c} \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \end{array}$$

Sol:–



4) For the given graph, write adjacency & incidence matrix.



Sol:–

$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

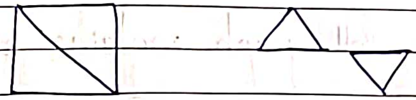
incidence

$$\begin{array}{c|c} & \begin{matrix} V_1 & V_2 & V_3 & V_4 & V_5 \end{matrix} \\ \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

adjacency

- Walk : A walk in graph is defined as a infinite alternative sequence of vertices and edges beginning and ending with vertices.

- No. of edges present in a walk is known as its length.
- End vertices of a walk are called terminal vertices.
- Walk that begins and ends at same vertex is known as closed walk.
- A walk that begins and ends at different vertex is known as open walk.
- If in an open walk, no vertex appears more than once, it is known as path.
- An open walk in which no edge appears more than once is known as trial.
- A closed walk in which no edge appears more than once is known as circuit.
- A closed walk in which terminal vertex does not appear as internal vertex and no internal vertex is repeated is known as cycle.
- A path with n vertices is of length n-1.
- A cycle with n vertices is of length n.
- The degree of every vertex in a cycle is 2.
- A graph G is said to be connected if there is atleast one path between every pair of vertexes in G, otherwise G is a disconnected graph.

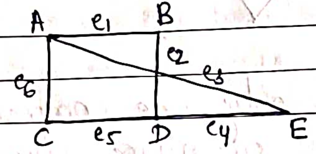


- Connected graph has only one component, disconnected has two or more components. Each of these components are connected graph.

Euler graph

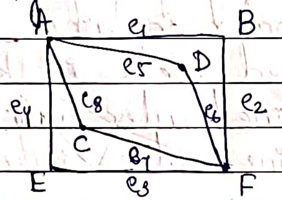
- Let G be a connected graph, if there is a closed walk in G that contains all edges of G, then that closed walk is known as euler circuit/ euler line/ euler tour.
- A graph that contains euler circuit is known as euler graph.
- If there is an open walk in a connected graph G that contains all the edges of G then that open walk is called euler trial and graph G is called semi euler graph.
- A connected graph G has euler circuit if and only if all vertices of G are of even degree.
- A connected graph has an euler line if and only if G can be decomposed into edge disjoint cycles.

1) Show that the foll graph contains euler circuit



Sol:- The given graph is a connected graph. The degree of vertex A and D are odd since all vertices are of not even degrees. This graph is not euler graph.

2) Verify whether foll graph is euler ~~graph~~ circuit



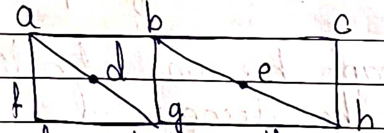
If yes, obtain the euler line.

Sol:- edge-disjoint cycles can be formed
(Vertices can be repeated)

euler line - A e_4 E e_3 F e_7 C e_8 A e_5 D e_6 F e_2 B e_1 A

The given graph is a connected graph. The degree of all vertices are even.

3)



Verify whether the graph is euler or semi euler.

Sol:- Given graph is a connected graph. All the vertices are not of even degree hence the graph is not euler graph

abchgfadgbeh - open walk

cycle - closed walk
path - open walk

Hamilton Graph

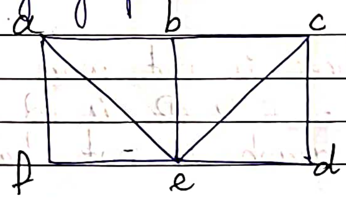
- Let G be a connected graph, if there is a cycle in G that contains all vertices in G then that cycle is known as hamilton cycle
- A graph that contains hamilton cycle is known as hamilton graph

{Cycle must include all vertices but not necessarily all edges}

- Let G be a connected graph, if there is a path in 'G' that contains all vertices of G then that path is known as hamilton path

- Length of hamilton cycle in a connected graph of n vertices is n.
- Length of hamilton ~~graph~~ path in a connected graph of n vertices is n-1

1) Verify graph is eular or hamilton graph.



→ hamilton cycle

Sol:- The given graph is a connected graph. It is not a euler graph. (since the degree of all vertices are not even)

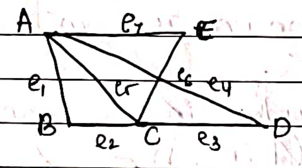
It is a hamilton graph.

→ abcdef

- A simple connected graph with n vertices $(n \geq 3)$ is hamiltonian cycle, if sum of degrees of every pair of non-adjacent vertices is greater than or equal to n
- A simple connected graph with n vertices $(n \geq 3)$ is hamiltonian if degree of every vertex is greater than or equal to $\frac{n}{2}$.

2) Verify whether the given graphs are euler or hamiltonian.

i)



Sol:- Given graph is a connected graph. All the vertices have even degree since it is an euler graph

Euler circuit – $A e_1 B e_2 C e_3 D e_4 A e_5 C e_6 E e_7 A$

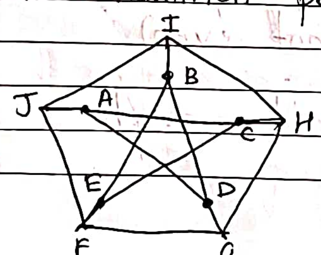
$\frac{n}{2} = 2.5$

Degree of all vertices is not more than 2.5 since degree of B, E & D is 2

Hence the given graph is not hamiltonian

2) Show that peterson graph has no hamilton cycle but has hamilton path.

Sol:-



Peterson graph is a connected graph.

No of vertices = 10

No of edges = 15

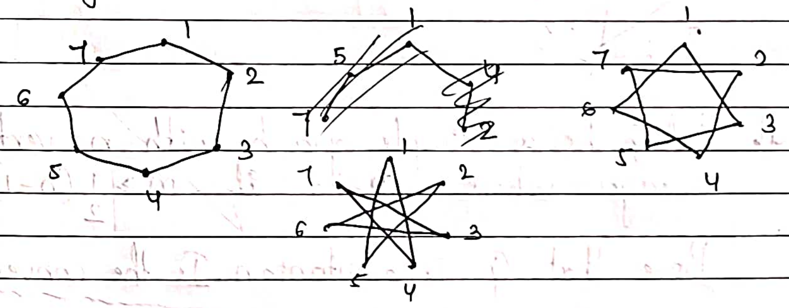
Degree of all vertices is 3. Since the degree of vertices are not greater than or equal to $\frac{n}{2} = 5$, Hence, it is not hamiltonian graph

Hamilton path is ACEBDGFJIH

★ Note: If n is odd, there are $\frac{n-1}{2}$ edge-disjoint hamilton cycles.

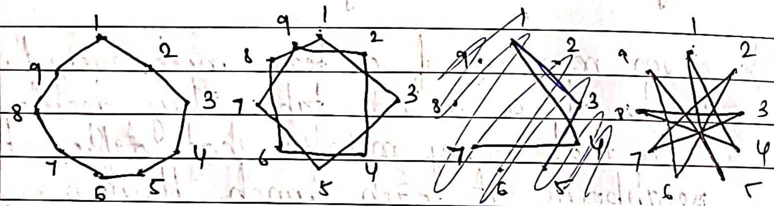
3) Seven members of a team meet everyday for lunch at a round table. They decide to sit such that every member has different neighbours at each lunch. How many days can this arrangement last and list the arrangements.

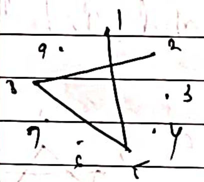
Sol:-



There can be 3 arrangements as shown. It can be represented by 7 vertices such that vertices represent members and edge joining represents they are sitting next to each other. Since every member is allowed to sit next to every other member, graph is a connected graph. Every arrangement is hamilton cycle since n is odd there are $\frac{n-1}{2}$ edge disjoint hamilton cycles. Thus there are three disjoint cycles and the arrangement lasts for three days. The listing of arrangements for lunch for three days are 12345671, 13572461, 14736251

* If there are 9 members





4) Let G be a simple graph with n vertices & m edges where $n \geq 3$ if $m \geq \frac{1}{2}(n-1)(n-2)+2$

Prove that G is hamiltonian. Is the converse true?

Sol:- Let G be a simple graph with n vertices and m edges. Let u & v be any two non-adjacent vertices in G. Let x & y be their respective degrees. If we delete u & v from G, we get a subgraph with $n-2$ vertices.

Let Q be the no of edges of this subgraph. Then, $Q \leq \frac{1}{2}(n-2)(n-3)$

[$\because$ we know that no. of edges in simple graph of order $n \leq \frac{1}{2}n(n-1)$]

Since u and v are not adjacent, there is no edge between u & v.

Total no. of edges in G is

$$m = x + y + Q$$

$$m - Q = x + y$$

Given $m \geq \frac{1}{2}(n-1)(n-2)+2$

and $Q \leq \frac{1}{2}(n-2)(n-3)$

$$-Q \geq -\frac{1}{2}(n-2)(n-3)$$

$$m - Q \geq \frac{1}{2}(n-1)(n+2)+2 - \frac{1}{2}(n-2)(n-3)$$

$$\geq \frac{1}{2}(2n)$$

$$m - Q \geq n$$

$$x + y \geq n$$

The sum of degrees of non-adjacent vertices is greater than or equal to n, thus, by the theorem, the graph is hamiltonian.

And the converse need not be true.

The hamiltonian graph with 5 vertices & 5 edges does not hold the inequality $m \geq \frac{1}{2}(n-1)(n-2)+2$

5) Let G be a disconnected graph of even order with two components each of which is complete. Prove that G has $\frac{n(n-2)}{4}$ minimum edges.

Sol:- Let one component of graph have x ~~edges~~ vertices and other component of graph will have $(n-x)$ ~~edges~~ vertices.

Total no. of edges in first component $= \frac{1}{2}(x)(x-1)$

No. of edges in second component $= \frac{1}{2}(n-x)(n-x-1)$

Total no. of edges

$$m = \frac{1}{2}x(x-1) + \frac{1}{2}(n-x)(n-x-1)$$

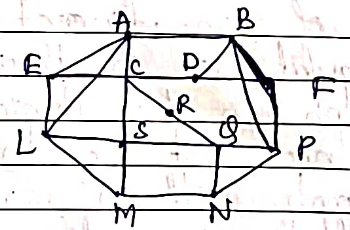
$$m = \frac{1}{2}\left[x^2 - x + n^2 - nx - n - nx + x^2 + x\right]$$

$$\frac{dm}{dx} = \frac{1}{2}\left[4x - 2n\right] = 0$$

$$\boxed{x = \frac{n}{2}}$$

$$\frac{d^2m}{dx^2} = 2 > 0$$

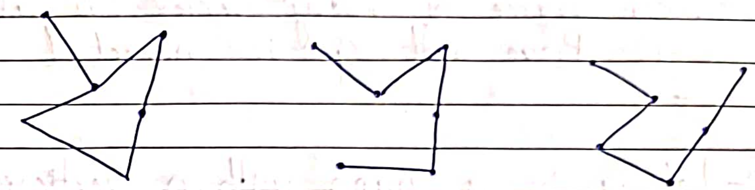
6) Find the hamilltonian cycle.



BAELMNQRCDFPB

Hamilton cycle

Tree



* If a graph has exactly 2 vertices of odd degrees then there must be a path connecting these vertices.
* A simple graph with n vertices and K component can have atmost $\frac{1}{2}(n-K)(n-K+1)$ no. of edges.
* A connected graph with n vertices has atleast $(n-1)$ edges.

1) Let G be a graph with n vertices where n is even and greater than 2. If the degree of every vertex in G is $\frac{1}{2}(n-2)$. ~~Di~~ Prove that G is connected.

Sol:- Let G be a graph with 6 vertices, each vertex is of degree 2 also $\frac{n-2}{2} = \frac{4}{2} = 2$ holds good for all the vertex. Hence, the graph is not a connected graph.

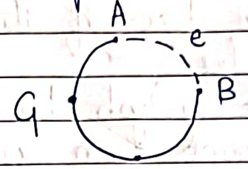
2) If G is a simple graph with n vertices in which degree of every vertex is atleast $\frac{(n-1)}{2}$. Prove that G is connected.

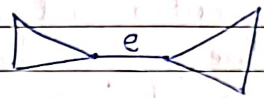
Sol:- Let G be a graph with n vertices. Let u and v be any two vertices. If they are adjacent, then graph is connected. Similarly, if they are not adjacent, we have to prove graph is G is connected.
Given degree of every vertex is atleast $\frac{n-1}{2}$ hence both u and v taken together have atleast: $\frac{n-1}{2} + \frac{n-1}{2} = (n-1)$ neighbours.

But G has total of n vertices, total no of neighbours which u and v together can have is $(n-2)$. Therefore atleast one vertex say x is neighbour of both u & v. Thus, there is a path between u and v as such G must be connected.

3) Prove that the connected graph G remains connected after removing an edge e from G if and only if e is a part of some cycle in G.

Sol:- Suppose e is a part of some cycle in G, then the end vertices of e are joined by atleast two paths, one of which is e and other is $G-e$. Hence the removal of e from G because even after removal of e, end vertices of e remain connected through the path $(G-1)$.



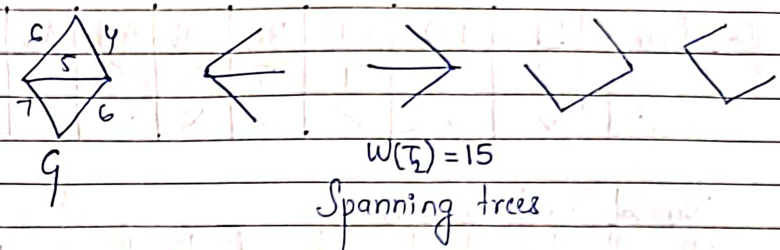


Conversely, suppose e is not a part of any cycle in G, then the vertices of e are connected by atmost one path. Hence, the removal of e from G disconnects these end points. This means that $G-1$ is disconnected graph. Thus, if e is not a part of any cycle in G, then $G-e$ is disconnected. This is equivalent to saying that if $G-e$ is connected then e belongs to some cycle in G. Thus, the required result.

Tree

- A graph is said to be a tree if its connected and has no cycles
- Tree has to be a simple graph because parallel loops and self loops form cycles.
- A pendant vertex of a tree is known as leaf.
- In a disconnected graph, if each component is a tree, such a graph is known as forest
- In a tree, there is only one path between every pair of vertices.
- A tree with n vertices has (n−1) edges
- A connected graph with n vertices and (n−1) edges is a tree.
- Let G be a connected graph, a subgraph P of G is known as spanning tree of G if P is a tree and contains all vertices of G.
- If T is a tree then edges of G which are not in T are called chords of G.
- Let G be a graph, suppose there is a +ve real no associated with each edge of G, then G is called a weighted graph.
- Let T be a spanning tree of this graph, then every branch of T has an edge in G with corresponding weights.
- Sum of these weights is known as weight of T.
- The spanning tree whose weight is least is known as minimal spanning tree of the graph.

ex –



→ Minimal Spanning Tree in $W(T_2)$

Kruskal's Algorithm: (Minimal Spanning Tree)

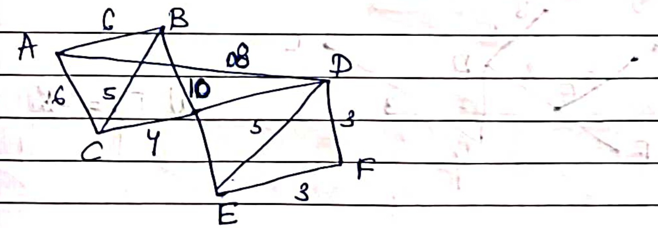
Step 1: Given a connected weighted graph G with n vertices, list edges of G in increasing order of weights.

Step 2: Starting with smallest weighted edge, proceed sequentially by selecting one edge at a time such that no cycle is formed.

Step 3: Stop process of step 2, when (n−1) edges are selected. These (n−1) edges constitute a minimal spanning tree of graph G.

1) Find minimal spanning tree using kruskal's algorithm for the foll:

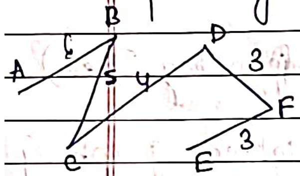
i)



Sol:-

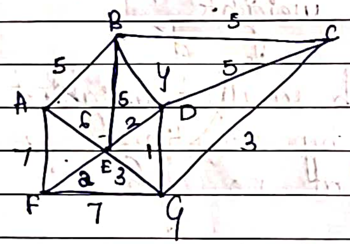
edges	EF	FD	CD	BC	DE	AB	AC	AD	BE
weights	3	3	4	5	5	6	6	8	10
selected	✓	✓	✓	✓	✗	✓			

No. of vertices = 6
No. of edges to be selected to form minimal spanning tree = 5



W(T) = 21

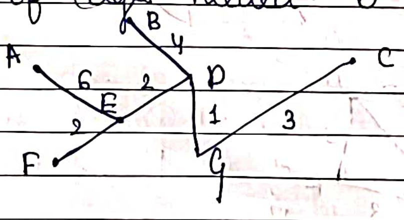
ii)



Sol:-

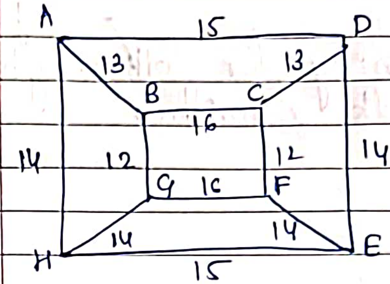
edges	DG	EF	DE	EG	GC	BD	BE	BC	DC	AE	AF	FG
weights	1	2	2	3	3	4	5	5	5	6	7	7
	✓	✓	✓	∝	✓	✓	∝	∝	✗			

No. of vertices = 7
No of edges needed = 6



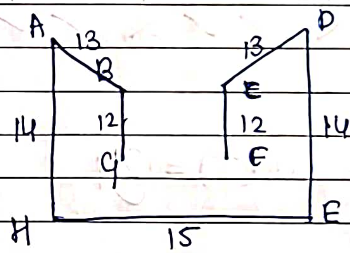
W(T) = ~~18~~ 17

iii)



Sol:-

edges	BG	FC	AB	CD	AH	DE	EF	GH	AD	HE	BC
weights	12	12	13	13	14	14	14	14	15	16	16
	✓	✓	✓	✓	✓	✓	∝	∝	∝	✓	



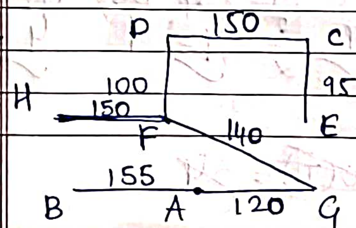
W(T) = 93

2) 8 cities ABCDEFGH are required to form a new railway network. Possible track can cost are summarised in the foll data. Determine railway network of minimum cost that connect all the cities?

Track	CE	AG	FG	AD	CD	FH	AB	GH
cost	95	120	140	145	150	150	155	160
	✓	✓	✓	∝	✓	✓	✓	

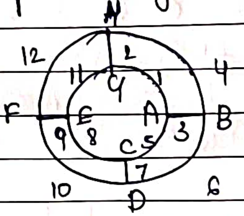
DF
100 ✓

Sol:-



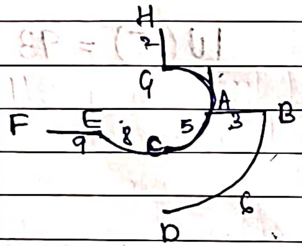
W(T) = 910

3) Find minimal spanning for foll^owing weighted graph using kruskal's algorithm.



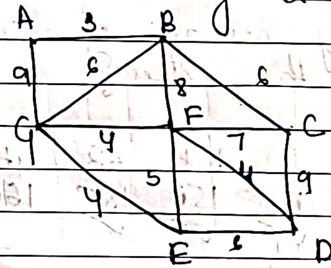
Sol:-

edges	AG	GH	AB	HB	AC	BD	DC	CE	EF	FD	EG	FH
weights	1	2	3	4	5	6	7	8	9	10	11	12
	✓	✓	✓	✗	✓	✓	✗	✓	✓			



$W(T) = 24$

4) Using kruskal's algorithm, find minimal spanning tree for the weighted graph.

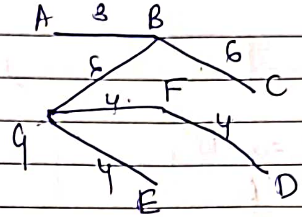


Sol:-

edges	AB	FD	FE	BC	FC	BF	AG	DC	GE	FG
weights	3	4	5	6	7	8	9	9	4	4
	✓	✓	✗		✓	✗	✓		✓	✓

BG
6

~~W(T) = 34~~



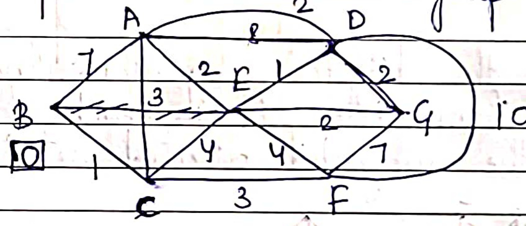
$W(T) = 27$

~~Digs~~

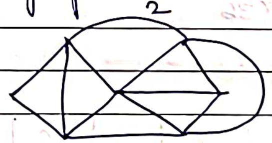
Dijkstra's Algorithm

To find shortest distance

1) Find shortest path from vertex B to vertex G for the foll graph:



Sol:- Simple graph - Remove self loop & parallel edge



Min(Temporary label, Permanent label + weight)

	A	B	C	D	E	F	G	
B	∞	$\boxed{0}$	∞	∞	∞	∞	∞	B
C	7	$\boxed{0}$	$\boxed{1}$	∞	∞	∞	∞	↑
A	$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	∞	5	4	∞	C
F	$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	6	5	$\boxed{4}$	∞	↑
E	$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	6	$\boxed{5}$	$\boxed{4}$	11	E
D	$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	$\boxed{6}$	$\boxed{5}$	$\boxed{4}$	7	↑
G	$\boxed{4}$	$\boxed{0}$	$\boxed{1}$	$\boxed{6}$	$\boxed{5}$	$\boxed{4}$	$\boxed{7}$	G

7

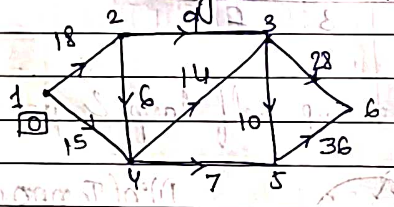
Dr. DSC

A = Min(∞, 0+7)
A = 7
C = Min(∞, 0+1) = 1
A = min(7, 1+4) = 4
E = min(∞, 1+4) = 5
F = min(∞, 1+3) = 4

Shortest path = BCEG

* Note - Shortest path from vertex B to all other vertices are listed in the table.

2) Using the dijkstra's algorithm, find the shortest path from vertex 1 to each of the vertices in the weighted directed network shown.



Sol:-

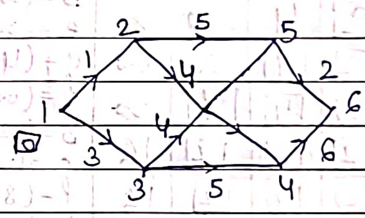
	1	2	3	4	5	6	
1	$\boxed{0}$	∞	∞	∞	∞	∞	2 = (∞, 0+18) = 18
4	$\boxed{0}$	18	∞	$\boxed{15}$	∞	∞	4 = (∞, 0+15) = 15
2	$\boxed{0}$	$\boxed{18}$	29	$\boxed{15}$	22	∞	3 = (∞, 15+14) = 29
5	$\boxed{0}$	$\boxed{18}$	29	$\boxed{15}$	$\boxed{22}$	∞	5 = (∞, 15+7) = 22
3	$\boxed{0}$	$\boxed{18}$	$\boxed{27}$	$\boxed{15}$	$\boxed{22}$	58	3 = (29, 18+9) = 27
6	$\boxed{0}$	$\boxed{18}$	$\boxed{27}$	$\boxed{15}$	$\boxed{22}$	$\boxed{55}$	6 = (∞, 22+36) = 58
							3 = (27, 22+10) = 27
							6 = (58, 27+28) = 55

min (Temp label, Permanent label + weight)

Shortest path from 1

to 2	1-2	18
to 3	1-2-3	27
to 4	1-4	15
to 5	1-4-5	22
to 6	1-2-3-6	55

3) Using dijkstra's algorithm, find shortest path from vertex 1 to each of the other vertices in the foll directed network.



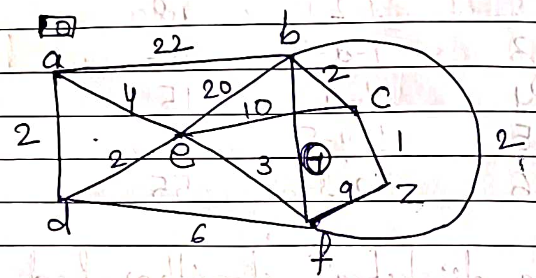
Sol:-

	1	2	3	4	5	6	
1	$\boxed{0}$	∞	∞	∞	∞	∞	2 = (∞, 0+1) = 1
2	$\boxed{0}$	$\boxed{1}$	3	∞	∞	∞	3 = (∞, 0+3) = 3
3	$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	5	6	∞	5 = (∞, 1+5) = 6
							4 = (∞, 1+4) = 5
4	$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	$\boxed{5}$	6	∞	4 = (5, 3+5) = 5
5	$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	11	5 = (6, 3+4) = 6
6	$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	$\boxed{8}$	6 = (∞, 5+6) = 11
							6 = (11, 6+2) = 8

Shortest path

1 to 2	1-2	1
1 to 3	1-3	3
1 to 4	1-2-4	5
1 to 5	1-2-5	6
1 to 6	1-2-5-6	8

4) Find the shortest path from A to z for the graph.



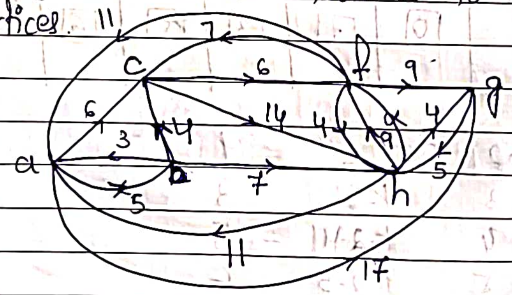
Sol:-

	a	b	c	d	e	f	z
a	$\boxed{0}$	∞	∞	∞	∞	∞	∞
d	$\boxed{0}$	22	∞	$\boxed{2}$	4	∞	∞
e	$\boxed{0}$	22	∞	$\boxed{2}$	$\boxed{4}$	8	∞
f	$\boxed{0}$	22	~~14~~	$\boxed{2}$	$\boxed{4}$	$\boxed{7}$	∞
b	$\boxed{0}$	$\boxed{9}$	14	$\boxed{2}$	$\boxed{4}$	$\boxed{7}$	16
c	$\boxed{0}$	$\boxed{9}$	$\boxed{11}$	$\boxed{2}$	$\boxed{4}$	$\boxed{7}$	16
z	$\boxed{0}$	$\boxed{9}$	$\boxed{11}$	$\boxed{2}$	$\boxed{4}$	$\boxed{7}$	$\boxed{12}$

$d = (\infty, 0+2) = 2$
$b = (\infty, 0+22) = 22$
$e = (\infty, 0+4) = 4$
$e = (4, 2+2) = 4$
$f = (\infty, 2+6) = 8$
$b = (22, 4+20) = 22$
$c = (\infty, 4+10) = 14$
$f = (8, 4+3) = 7$
$b = (22, 7+2) = 9$
$z = (\infty, 7+9) = 16$
$c = (14, 9+2) = 11$
$z = (16, 11+1) = 12$

Shortest path $= a \to e \to f \to b \to c \to z$
$= 12$

5) Find the shortest distance from c to each of other vertices.



Sol:-

	a	b	c	f	g	h
c	∞	∞	$\boxed{0}$	∞	∞	∞
f	∞	∞	$\boxed{0}$	$\boxed{6}$	∞	14
h	~~∞~~ 17	∞	$\boxed{0}$	$\boxed{6}$	15	$\boxed{10}$
g	17	∞	$\boxed{0}$	$\boxed{6}$	$\boxed{14}$	$\boxed{10}$
a	$\boxed{17}$	∞	$\boxed{0}$	$\boxed{6}$	$\boxed{14}$	$\boxed{10}$
b	$\boxed{17}$	$\boxed{22}$	$\boxed{0}$	$\boxed{6}$	$\boxed{14}$	$\boxed{10}$

$a = 6$
$h = (\infty, 0+14) = 14$
$f = (\infty, 0+6) = 6$
$g = (\infty, 6+9) = 15$
$c = 0,$
$h = (14, 6+4) = 10$
$h = (15, 10+4) = 14$
$a = (17, 10+11) = 17$
$b = (\cancel{10}, 17+5) = 22$

c-a	$c \to f \to a$	17
c-b	$b \to a \to f \to c$	22
c-f	$c \to f$	6
c-g	$c \to f \to g$	14
c-h	$c \to f \to h$	10