Decision Tree

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Motivation

- PCA identifies one or more orthogonal directions that capture the greatest amount of variance in a feature matrix $X \in \mathbb{R}^{m \times n}$.
- Assuming zero-mean feature matrix $X \in \mathbb{R}^{m \times n}$, the variance of the samples' projections onto aunit vector$v$ is given by:

$$\mathrm{Var}(Xv) = \mathbb{E}[(Xv - \mathbb{E}(Xv))^2] = \frac{1}{m} \sum_{i=1}^{m} (x_i^t v)^2 = \frac{1}{m} \|Xv\|^2 = \frac{1}{m} v^t X^t X v$$

- In light of this consideration, we define the first desired vector v_1 as the solution to the constrained optimization problem:

$$\max_{\|v\|_2 = 1} v^t X^t X v$$

- We convert this constrained optimization problem into an unconstrained one by writing down its Lagrangian:

$$\mathcal{L}(v) := v^t X^t X v - \lambda(v^t v - 1)$$

First PC

- First-order necessary conditions for optimal value imply that:

$$0 = \nabla \mathcal{L}(v_1) = 2X^t X v_1 - 2\lambda v_1$$

- Since $X^t X v_1 = \lambda v_1$, v_1 is an eigenvector of $X^t X$ with eigenvalue λ.
- Since we constrain $\|v_1\|_2^2 = v_1^t v_1 = 1$, the value of the objective is precisely:

$$v_1^t X^t X v_1 = v_1^t (\lambda v_1) = \lambda v_1^t v_1 = \lambda$$

- The optimal value is $\lambda = \lambda_{\max}(X^t X)$, which is achieved when v_1 is a unit eigenvector of $X^t X$ corresponding to its largest eigenvalue.

More PCs?

- How to find more direction with the desired property?
  - Ideally, the subsequent directions found should also be directions of high variance.
  - They should be orthogonal to the existing ones in order to minimize redundancy.

- We define the k-th loading vector v_k as the solution to the constrained optimization problem:

$$\max_{v} v^t X^t X v \quad \text{subject to} \quad v^t v = 1, v^t v_i = 0, \quad i = 1, \ldots, k-1$$

- **Claim**: v_k is a unit eigenvector of $X^t X$ corresponding to its k-th largest eigenvalue.

- The unit vector that defines the k-th axis is called the k-th principal component (PC).

Evaluation of PCs

- Assuming the singular value decompositon of feature matrix X as follows:

$$X = U\Sigma V^T = [u_1, u_2, \cdots, u_r][\begin{matrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_r \end{matrix}][\begin{matrix} v_1^T \\ v_2^T \\ \vdots \\ v_r^T \end{matrix}]$$

  The first k PCs are $W_k = [v_1, v_2, \cdots, v_k]$.

- Explained Variance Ratioexplains the proportion of the dataset's variance that lies along the axis of each PC.

- PCA can also be viewed as the projection of the sample points to the subspace with the minimum perpendicular distance.

Other Derivation?

Definition

For a matrix X, operator 2-norm is defined as

$$\|X\|_2 = \sup \frac{\|Xv\|_2}{\|v\|_2} = \max(s_i)$$

and Frobenius norm as

$$\|X\|_F = \sqrt{\sum_{ij} X_{ij}^2} = \sqrt{\operatorname{tr}(X^t X)} = \sqrt{\sum \sigma_i^2}$$

where σ_i are singular values of X, i.e. diagonal elements of Σ in the singular value decomposition $X = U\Sigma V^t$

Other Derivation?

- PCA is given by the same singular value decomposition when the data are centered.
- $U\Sigma$ are principal components, and V are principal axes, i.e. eigenvectors of the covariance matrix.
- The reconstruction of X with only the k principal components corresponding to the k largest singular values is given by $X_k = U_k \Sigma_k V_k^\top$.
- The Eckart-Young theorem says that X_k is the matrix minimizing the norm of the reconstruction error $\|X - A\|$ among all matrices A of rank k.
- This is true for both, Frobenius norm and the operator 2-norm

Thank You!

Any Question?