Library reals_uniqueness

reals_uniqueness

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Require Export reals_axioms.

Section real_numbers_uniqueness.

Variable R_str1 : Real_Struct.
Variable RA1 : Real_Axioms R_str1.
Variable R_str2 : Real_Struct.
Variable RA2 : Real_Axioms R_str2.

Let R1 := @R R_str1.
Let R2 := @R R_str2.
Let N1 := N R_str1.
Let N2 := N R_str2.
Let Z1 := Z R_str1.
Let Z2 := Z R_str2.
Let Q1 := Q R_str1.
Let Q2 := Q R_str2.
Let Sup1a := Sup1 R_str1.
Let Sup1b := Sup1 R_str2.

Delimit Scope R1_scope with r1.
Delimit Scope R2_scope with r2.

Notation "0" := (@zeroR R_str1).
Notation "0 '" := (@zeroR R_str2).

Notation "1" := (@oneR R_str1).
Notation "1 '" := (@oneR R_str2).

Notation "x + y" := ((@fp R_str1)[[x, y]])
    (at level 50, left associativity) : R1_scope.
Notation "x + y" := ((@fp R_str2)[[x, y]])
    (at level 50, left associativity) : R2_scope.

Notation "x · y" := ((@fm R_str1)[[x, y]])(at level 40) : R1_scope.
Notation "x · y" := ((@fm R_str2)[[x, y]])(at level 40) : R2_scope.

Notation "- a" := (∩(\{ λ u, u ∈ R1 /\ (u + a)%r1 = 0 \}))
    (at level 35, right associativity) : R1_scope.
Notation "- a" := (∩(\{ λ u, u ∈ R2 /\ (u + a)%r2 = 0' \}))
    (at level 35, right associativity) : R2_scope.

Notation "x - y" := (x + (-y))%r1 : R1_scope.
Notation "x - y" := (x + (-y))%r2 : R2_scope.

Notation "a -" := (∩(\{ λ u, u ∈ (R1 ~ [0]) /\ (u · a)%r1 = 1 \}))
    (at level 30) : R1_scope.
Notation "a -" := (∩(\{ λ u, u ∈ (R2 ~ [0']) /\ (u · a)%r2 = 1' \}))
    (at level 30) : R2_scope.

Notation "m / n" := (m · (n-))%r1 : R1_scope.
Notation "m / n" := (m · (n-))%r2 : R2_scope.

Notation "x ≤ y" := ([x,y] ∈ (@Leq R_str1))(at level 60) : R1_scope.
Notation "x ≤ y" := ([x,y] ∈ (@Leq R_str2))(at level 60) : R2_scope.
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Notation "x < y" := (x ≤ y /\ x <> y)%r1 : R1_scope.
Notation "x < y" := (x ≤ y /\ x <> y)%r2 : R2_scope.

Let zero_in_R_str1 := @zero_in_R R_str1 RA1.
Let zero_in_R_str2 := @zero_in_R R_str2 RA2.
Let one_in_R_Co_str1 := @one_in_R_Co R_str1 RA1.
Let one_in_R_Co_str2 := @one_in_R_Co R_str2 RA2.
Let Plus_close_str1 := @Plus_close R_str1 RA1.
Let Plus_close_str2 := @Plus_close R_str2 RA2.
Let Mult_close_str1 := @Mult_close R_str1 RA1.
Let Mult_close_str2 := @Mult_close R_str2 RA2.
Let Plus_neg1a_str1 := @Plus_neg1a R_str1 RA1.
Let Plus_neg1a_str2 := @Plus_neg1a R_str2 RA2.
Let Plus_neg1b_str1 := @Plus_neg1b R_str1 RA1.
Let Plus_neg1b_str2 := @Plus_neg1b R_str2 RA2.
Let Plus_neg2_str1 := @Plus_neg2 R_str1 RA1.
Let Plus_neg2_str2 := @Plus_neg2 R_str2 RA2.
Let Minus_P1_str1 := @Minus_P1 R_str1 RA1.
Let Minus_P1_str2 := @Minus_P1 R_str2 RA2.
Let Minus_P2_str1 := @Minus_P2 R_str1 RA1.
Let Minus_P2_str2 := @Minus_P2 R_str2 RA2.
Let Mult_inv1_str1 := @Mult_inv1 R_str1 RA1.
Let Mult_inv1_str2 := @Mult_inv1 R_str2 RA2.
Let Mult_inv2_str1 := @Mult_inv2 R_str1 RA1.
Let Mult_inv2_str2 := @Mult_inv2 R_str2 RA2.
Let Divide_P1_str1 := @Divide_P1 R_str1 RA1.
Let Divide_P1_str2 := @Divide_P1 R_str2 RA2.
Let Divide_P2_str1 := @Divide_P2 R_str1 RA1.
Let Divide_P2_str2 := @Divide_P2 R_str2 RA2.
Let OrderPM_Co9_str1 := @OrderPM_Co9 R_str1 RA1.
Let OrderPM_Co9_str2 := @OrderPM_Co9 R_str2 RA2.
Let OrderPM_Co10_str1 := @OrderPM_Co10 R_str1 RA1.
Let OrderPM_Co10_str2 := @OrderPM_Co10 R_str2 RA2.
Let N_Subset_R_str1 := @N_Subset_R R_str1 RA1.
Let N_Subset_R_str2 := @N_Subset_R R_str2 RA2.
Let one_in_N_str1 := @one_in_N R_str1 RA1.
Let one_in_N_str2 := @one_in_N R_str2 RA2.
Let Nat_P1a_str1 := @Nat_P1a R_str1 RA1.
Let Nat_P1a_str2 := @Nat_P1a R_str2 RA2.
Let Nat_P1b_str1 := @Nat_P1b R_str1 RA1.
Let Nat_P1b_str2 := @Nat_P1b R_str2 RA2.
Let N_Subset_Z_str1 := @N_Subset_Z R_str1.
Let N_Subset_Z_str2 := @N_Subset_Z R_str2.
Let Z_Subset_R_str1 := @Z_Subset_R R_str1 RA1.
Let Z_Subset_R_str2 := @Z_Subset_R R_str2 RA2.
Let Int_P1a_str1 := @Int_P1a R_str1 RA1.
Let Int_P1a_str2 := @Int_P1a R_str2 RA2.
Let Int_P1b_str1 := @Int_P1b R_str1 RA1.
Let Int_P1b_str2 := @Int_P1b R_str2 RA2.
Let Z_Subset_Q_str1 := @Z_Subset_Q R_str1 RA1.
Let Z_Subset_Q_str2 := @Z_Subset_Q R_str2 RA2.
Let Q_Subset_R_str1 := @Q_Subset_R R_str1 RA1.
Let Q_Subset_R_str2 := @Q_Subset_R R_str2 RA2.
Let Rat_P1a_str1 := @Rat_P1a R_str1 RA1.
Let Rat_P1a_str2 := @Rat_P1a R_str2 RA2.
Let Rat_P1b_str1 := @Rat_P1b R_str1 RA1.
Let Rat_P1b_str2 := @Rat_P1b R_str2 RA2.
Let Abs_in_R_str1 := @Abs_in_R R_str1 RA1.
Let Abs_in_R_str2 := @Abs_in_R R_str2 RA2.

Local Hint Resolve
    zero_in_R_str1 zero_in_R_str2 one_in_R_Co_str1 one_in_R_Co_str2
    Plus_close_str1 Plus_close_str2 Mult_close_str1 Mult_close_str2
    Plus_neg1a_str1 Plus_neg1a_str2 Plus_neg1b_str1 Plus_neg1b_str2
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    Plus_neg2_str1 Plus_neg2_str2 Minus_P1_str1 Minus_P1_str2
    Minus_P2_str1 Minus_P2_str2 Mult_inv1_str1 Mult_inv1_str2
    Mult_inv2_str1 Mult_inv2_str2 Divide_P1_str1 Divide_P1_str2
    Divide_P2_str1 Divide_P2_str2 OrderPM_Co9_str1 OrderPM_Co9_str2
    OrderPM_Co10_str1 OrderPM_Co10_str2 N_Subset_R_str1 N_Subset_R_str2
    one_in_N_str1 one_in_N_str2 Nat_P1a_str1 Nat_P1a_str2
    Nat_P1b_str1 Nat_P1b_str2 N_Subset_Z_str1 N_Subset_Z_str2
    Z_Subset_R_str1 Z_Subset_R_str2 Int_P1a_str1 Int_P1a_str2
    Int_P1b_str1 Int_P1b_str2 Z_Subset_Q_str1 Z_Subset_Q_str2
    Q_Subset_R_str1 Q_Subset_R_str2 Rat_P1a_str1 Rat_P1a_str2
    Rat_P1b_str1 Rat_P1b_str2 Abs_in_R_str1 Abs_in_R_str2 : real.

Theorem UniqueT1 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> f[0] = 0' /\ (ran(f) <> [0'] -> f[1] = 1').

Lemma UniqueT2_Lemma1 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0']
    -> (∀ m, m ∈ N1 -> f[m] ∈ N2).

Lemma UniqueT2_Lemma2 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0']
    -> f[(-(1))%r1] = (-f[1%r1])%r2.

Lemma UniqueT2_Lemma3 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0']
    -> (∀ r, r ∈ R1 -> f[(-r)%r1] = (-f[r])%r2).

Lemma UniqueT2_Lemma4 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0']
    -> (∀ m, m ∈ R1 -> m <> 0 -> f[m] <> 0').

Lemma UniqueT2_Lemma5 : ∀ n, n ∈ Z1 -> (0 < n)%r1 -> n ∈ N1.

Theorem UniqueT2 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0']
    -> (∀ m, m ∈ Z1 -> f[m] ∈ Z2)
        /\ Function1_1 (f|(Z1)) /\ dom(f|(Z1)) = Z1 /\ ran(f|(Z1)) = Z2
        /\ (∀ x y, x ∈ Z1 -> y ∈ Z1 -> (x ≤ y)%r1 -> (f[x] ≤ f[y])%r2).

Lemma UniqueT3_Lemma1 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0']
    -> (∀ r, r ∈ (R1 ~ [0]) -> f[(r⁻)%r1] = ((f[r])⁻)%r2).

Lemma UniqueT3_Lemma2 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0']
    -> (∀ r, r ∈ Q1 -> (0 < r)%r1 -> (0' < f[r])%r2).

Theorem UniqueT3 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
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        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0’]
    -> (∀ m n, m ∈ Z1 -> n ∈ (Z1 ~ [0]) -> f[(m/n)%r1] = (f[m]/f[n])%r2)
        /\ Function1_1 (f|(Q1)) /\ dom(f|(Q1)) = Q1 /\ ran(f|(Q1)) = Q2
        /\ (∀ x y, x ∈ Q1 -> y ∈ Q1 -> (x ≤ y)%r1 -> (f[x] ≤ f[y])%r2).

Lemma UniqueT4_Lemma1 : ∀ r, r ∈ R2 -> Sup1b \{ λ u, u ∈ Q2 /\ (u ≤ r)%r2 \} r.

Lemma UniqueT4_Lemma2 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0’]
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> (x < y)%r1 <-> (f[x] < f[y])%r2).

Lemma UniqueT4_Lemma3 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0’]
    -> (∀ A r, A ⊂ Q1 -> r ∈ R1 -> Sup1a A r -> Sup1b ran(f|(A)) f[r]).

Theorem UniqueT4 : ∀ f, Function f -> dom(f) = R1 -> ran(f) ⊂ R2
    -> (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2
        /\ f[(x · y)%r1] = (f[x] · f[y])%r2)
    -> ran(f) <> [0’]
    -> Function1_1 f /\ dom(f) = R1 /\ ran(f) = R2
        /\ (∀ x y, x ∈ R1 -> y ∈ R1 -> (x ≤ y)%r1 -> (f[x] ≤ f[y])%r2).

Lemma UniqueT5_Lemma1 : ∃ f, Function1_1 f /\ dom(f) = N1 /\ ran(f) = N2
    /\ (∀ m n, m ∈ N1 -> n ∈ N1 -> (m < n)%r1 <-> (f[m] < f[n])%r2).

Lemma UniqueT5_Lemma2 : ∀ f, Function1_1 f -> dom(f) = N1 -> ran(f) = N2
    -> (∀ m n, m ∈ N1 -> n ∈ N1 -> (m < n)%r1 <-> (f[m] < f[n])%r2)
    -> (∀ m n, m ∈ N1 -> n ∈ N1
              -> f[(m + n)%r1] = (f[m] + f[n])%r2 /\ f[(m · n)%r1] = (f[m] · f[n])%r2
                  /\ ((m < n)%r1 -> f[(n - m)%r1] = (f[n] - f[m])%r2))
          /\ f[1] = 1’.

Lemma UniqueT5_Lemma3 : ∃ f, Function f /\ dom(f) = Z1 /\ ran(f) ⊂ Z2
    /\ f[0] = 0’ /\ f[1] = 1’
    /\ (∀ m n, m ∈ Z1 -> n ∈ Z1
          -> f[(m + n)%r1] = (f[m] + f[n])%r2 /\ f[(m · n)%r1] = (f[m] · f[n])%r2)
    /\ (∀ m, m ∈ Z1 -> m <> 0 -> f[m] <> 0’)
    /\ (∀ m, m ∈ Z1 -> (0 < m)%r1 -> (0’ < f[m])%r2).

Lemma UniqueT5_Lemma4 : ∃ f, Function f /\ dom(f) = Q1 /\ ran(f) ⊂ Q2
    /\ f[0] = 0’ /\ f[1] = 1’
    /\ (∀ a b, a ∈ Q1 -> b ∈ Q1
          -> f[(a + b)%r1] = (f[a] + f[b])%r2 /\ f[(a · b)%r1] = (f[a] · f[b])%r2)
    /\ (∀ a b, a ∈ Q1 -> b ∈ Q1 -> (a < b)%r1 -> (f[a] < f[b])%r2).

Lemma UniqueT5_Lemma5a : ∀ a b c, a ∈ Q1 -> b ∈ R1 -> c ∈ R1
    -> (0 < a)%r1 -> (0 < b)%r1 -> (0 < c)%r1 -> (a < (b + c))%r1
    -> ∃ a1 a2, a1 ∈ Q1 /\ a2 ∈ Q1 /\ (0 < a1)%r1 /\ (a1 < b)%r1
        /\ (0 < a2)%r1 /\ (a2 < c)%r1 /\ (a1 + a2)%r1 = a.

Lemma UniqueT5_Lemma5b : ∀ a b c, a ∈ R2 -> b ∈ R2 -> c ∈ R2
    -> (0’ < a)%r2 -> (0’ < b)%r2 -> (0’ < c)%r2 -> (a < (b + c))%r2
    -> ∃ a1 a2, a1 ∈ R2 /\ a2 ∈ R2 /\ (0’ < a1)%r2 /\ (a1 < b)%r2
        /\ (0’ < a2)%r2 /\ (a2 < c)%r2 /\ (a1 + a2)%r2 = a.

Lemma UniqueT5_Lemma6a : ∀ a b c, a ∈ Q1 -> b ∈ R1 -> c ∈ R1
    -> (0 < a)%r1 -> (0 < b)%r1 -> (0 < c)%r1 -> (a < (b · c))%r1
    -> ∃ a1 a2, a1 ∈ Q1 /\ a2 ∈ Q1 /\ (0 < a1)%r1 /\ (a1 < b)%r1
        /\ (0 < a2)%r1 /\ (a2 < c)%r1 /\ (a1 · a2)%r1 = a.
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Lemma UniqueT5_Lemma6b : ∀ a b c, a ∈ R2 -> b ∈ R2 -> c ∈ R2
    -> (0' < a)%r2 -> (0' < b)%r2 -> (0' < c)%r2 -> (a < (b · c))%r2
    -> ∃ a1 a2, a1 ∈ R2 /\ a2 ∈ R2 /\ (0' < a1)%r2 /\ (a1 < b)%r2
        /\ (0' < a2)%r2 /\ (a2 < c)%r2 /\ (a1 · a2)%r2 = a.

Lemma UniqueT5_Lemma7 : ∃ f, Function f
    /\ dom(f) = \{ λ u, u ∈ R1 /\ (0 < u)%r1 \} /\ ran(f) ⊂ R2
    /\ (∀ m n, m ∈ dom(f) -> n ∈ dom(f) -> f[(m + n)%r1] = (f[m] + f[n])%r2)
    /\ (∀ m n, m ∈ dom(f) -> n ∈ dom(f) -> f[(m · n)%r1] = (f[m] · f[n])%r2)
    /\ (∀ m n, m ∈ dom(f) -> n ∈ dom(f) -> (m < n)%r1
        -> f[(n - m)%r1] = (f[n] - f[m])%r2)
    /\ f[1] = 1'.

Lemma UniqueT5_Lemma8 : ∃ f, Function f
    /\ dom(f) = R1 /\ ran(f) ⊂ R2 /\ ran(f) <> [0']
    /\ (∀ m n, m ∈ dom(f) -> n ∈ dom(f) -> f[(m + n)%r1] = (f[m] + f[n])%r2)
    /\ (∀ m n, m ∈ dom(f) -> n ∈ dom(f) -> f[(m · n)%r1] = (f[m] · f[n])%r2).

Definition Reals_Isomorphism := ∃ f, Function1_1 f /\ dom(f) = R1 /\ ran(f) = R2
    /\ (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x + y)%r1] = (f[x] + f[y])%r2)
    /\ (∀ x y, x ∈ R1 -> y ∈ R1 -> f[(x · y)%r1] = (f[x] · f[y])%r2)
    /\ (∀ x y, x ∈ R1 -> y ∈ R1 -> (x ≤ y)%r1 <-> (f[x] ≤ f[y])%r2).

Theorem UniqueT5 : Reals_Isomorphism.

End real_numbers_uniqueness.
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Global Index A B C D E F G H I J K L M N O P Q R S T U V W X Y Z _ other (1 entry)
Library Index A B C D E F G H I J K L M N O P Q R S T U V W X Y Z _ other (1 entry)

Global Index

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reals_uniqueness [library]

Library Index

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Global Index A B C D E F G H I J K L M N O P Q R S T U V W X Y Z _ other (1 entry)
Library Index A B C D E F G H I J K L M N O P Q R S T U V W X Y Z _ other (1 entry)