Partial Equilibrium in Mathematica

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Much of microeconomic analysis is based on the concept of a partial equilibrium. By using computational tools– such as Mathematica– we can quickly derive, plot and adapt analytical models of consumer and firm behaviour, and how the intersection of the two fix market equilibrium.

1 Utility and Production Functions

The basis of all economic analysis is the consumer and their goal of utility-maximization. The most common numerical repersentation of two-input production and consumer utility are the Cobb-Douglas and Leontief functions. These forms satisfy the following properties depending on if they are used to repersent production or utility, respectively:

1. monotonicly increasing/decreasing
2. concave/convex
3. nonintersecting

The general Cobb-Douglas function takes the form

$$Y = L^{\beta} K^{\alpha} \tag{1}$$

Where Y is production, L is labor, K is capital, and α, β are the elasticities of labor and capital respectively. The Cobb-Douglass (**Figure 1**) function assumes that continous substitution between goods x_1 and x_2 in production/utility.

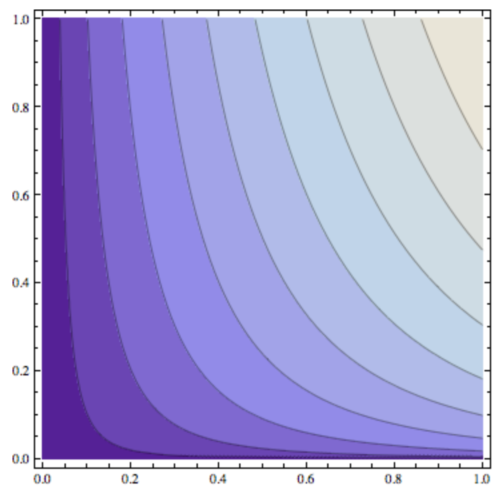

Figure 1: A Cobb-Douglas Production Function ($\rho = .7$)

$$\alpha + \beta = 1 \tag{2}$$

The above equation implies a special conditon when the production function experinces constant returns to scale.

Cobb-Douglas production is a special case of the constant elasticity of substitution (CES) production function as described by Solow (Solow, 1956) when $\lim \gamma \to 0$. The two-factor CES function (**Figure 2**) is of the form:

$$Y = \alpha K^{\gamma} + (1-\alpha) L^{\gamma})^{\frac{1}{\gamma}} \tag{3}$$

A Leontief function is a special case of a Cobb-Douglas function in which there is no substitutability between goods x_1 and x_2 and thus there exists a predefined proportion of goods x_1 and x_2 used in the production of Y.

$$Y(x_1, x_2) = min(c_1 x_1, c_2 x_2) \tag{4}$$

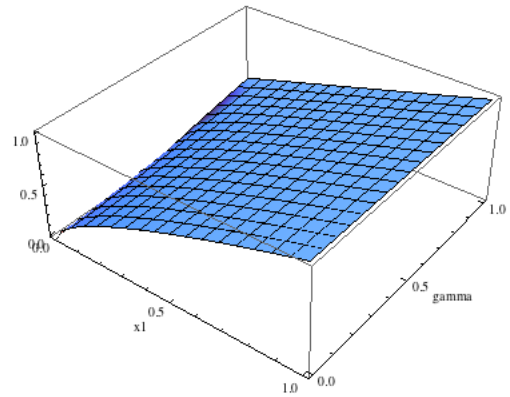

Figure 2: CES Function ($\gamma = [.01, 1], x_1 = [0, 1], x_2 = 1$)

When the proportion

$$\frac{c_1 x_1}{c_2 x_2} = 1$$

the Leontief has a 45°line through the contour plot as seen in **Figure 3**:

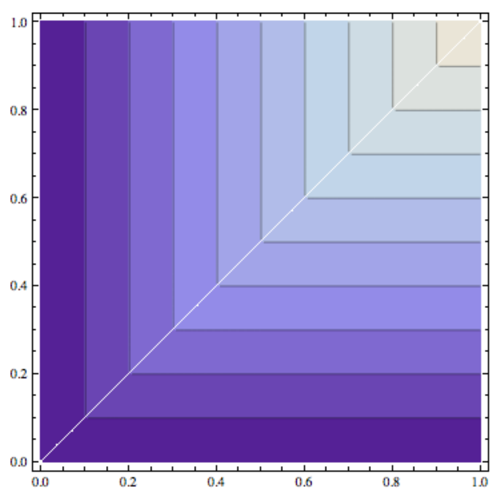

Figure 3: A Leontief Production Function

2 Consumer Theory

Traditionally, consumer theory is modeled as agents maximizing their utility (strictly a function of consumption, C) subject to their budget constraint, (m). This implies that given some bundle of goods x_1, x_2 at prices p_1, p_2, the rational consumer will purchase the bundle that maximizes their total utility, U. The optimal bundle will allways include an nonzero amount of both goods since utility is subject to the principle of diminishing marginal returns. A Cobb-Douglas utlilty function with elasticities α will take the form:

$$max : U = x_1^{\alpha} x_2^{1-\alpha} \tag{5}$$

$$subjectto : m = p_1 x_1 + p_2 x_2 \tag{6}$$

Via logarithims, this equation is equivelent to:

$$log(U) = \alpha log(x_1) + (1-\alpha)log(x_2) \tag{7}$$

To find the optimal bundle of goods, the Lagrangian of the system of equations is modeled in *Mathematica*. The Lagrangian, $\mathcal{L}$ is written as follows:

$$\mathcal{L}(x_1, x_2, \lambda) = \alpha log(x_1) + (1-\alpha)log(x_2) + \lambda(m - p_1x_1 - p_2x_2) \tag{8}$$

The partial derivative of the Lagrangian with respect to each variable is taken and set equal to zero.

$$\frac{d\mathcal{L}}{dx_1} = -p_1\lambda + \frac{\alpha}{x_1} == 0 \tag{9}$$

$$\frac{d\mathcal{L}}{dx_2} = -p_2\lambda + \frac{1-\alpha}{x_2} == 0 \tag{10}$$

$$\frac{d\mathcal{L}}{dx_\lambda} = m - p_1x_1 - p_2x_2 == 0 \tag{11}$$

This system of equations can be solved for the demand curves of x_1 and x_2, the results of which are plotted below.

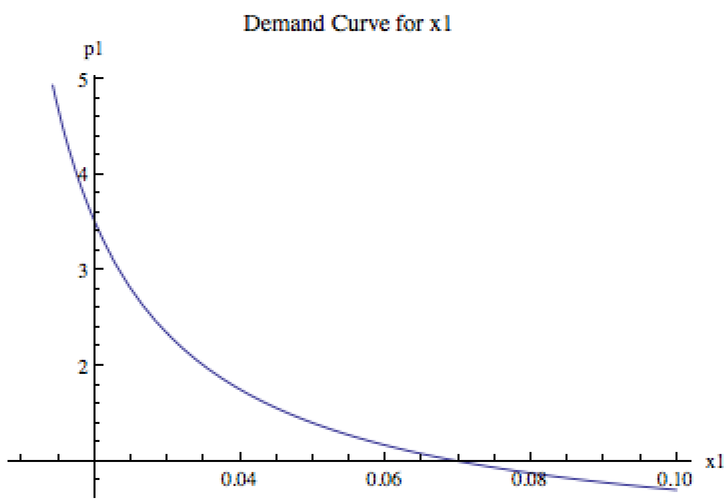

Figure 4: Consumer Demand Function, x_1 vs p_1

3 Firms

In opposition to consumers goal of utilitiy-maximization, firms are modeled as stictly profit-maximizing entities. In the original model, the firm's profit-maximization function is expressed mathematically as:

$$max : \pi = p_1x_1 - wL \tag{12}$$

$$subjectto : x_1 = TL^\beta \tag{13}$$

Where w are wages, L is labor supply, T is the given technology level, and β is the production's elasticity to labor. By substituting the profit function π into the production function and setting the resulting function's derivative with respect to labor (L) to zero, we establish the labor demand function.

$$L = \frac{w}{\beta p_1 T})^{\frac{1}{-1+\beta}} \tag{14}$$

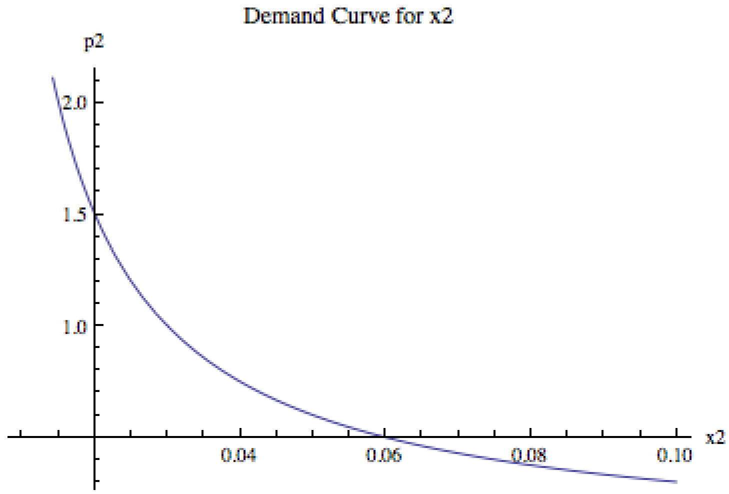

Figure 5: Labor Demand Curve ($\beta = .4$, T = 1, $p_1 = 1$)

Substituting the labor demand function (**6**) into the production function (**13**) we find the supply curve.

$$S = T[(\frac{w}{\beta p_1 T})^{\frac{1}{-1+\beta}}]^{\beta} \tag{15}$$

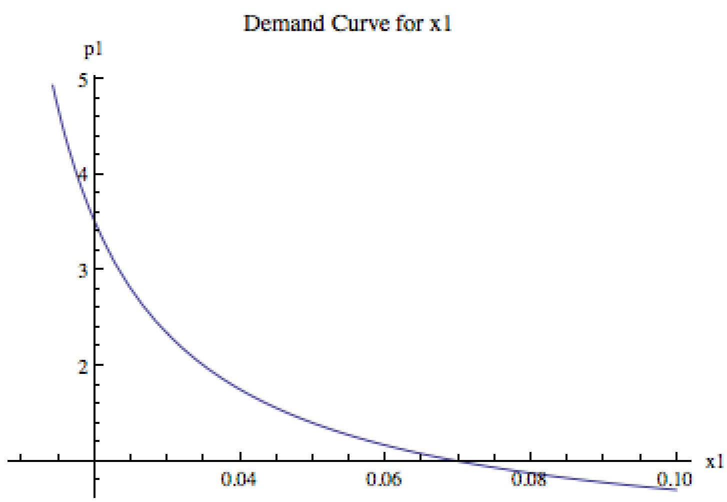

Figure 6: Supply Curve for x_1 ($\beta = .4$, T = 1, w = 100)

4 Market Equilibrium

Using the supply and demand curves established earlier, we can derive the equation for market equilbrium. The demand curve found in Part 2 and the supply function as described in Part 3 are set equal to one another to find market equilibrium.

$$\frac{\alpha m}{x_1} == w\frac{((\frac{x_1}{T})^{\frac{1}{\beta}})^{1-\beta}}{\beta T} \tag{16}$$

In **Figure 7** the effects of different levels of total factor productivity T are shown. TFP is used as a metric of the level of how efficiently the economy can convert inputs to outputs. Historically, this factor serves mainly as a variable that can be altered in curve-fitting the model to real world data.

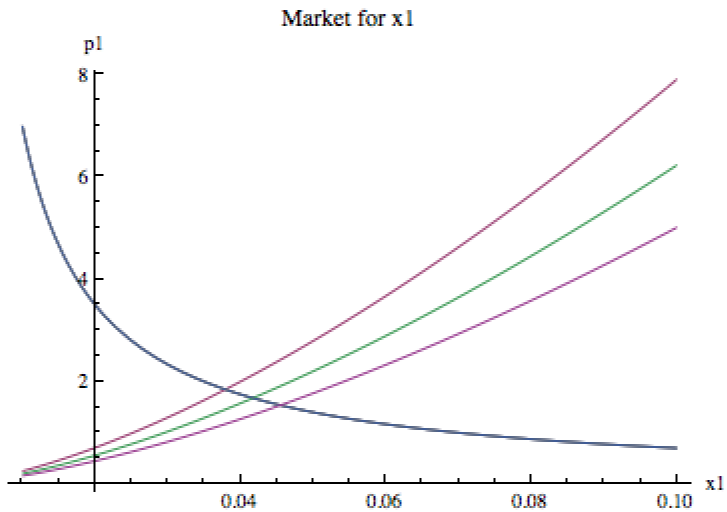

Figure 7: Market Equilibrium for x_1 as a function of T

The effects of changing consumer's elasticity α to good x_1 can be seen in **Figure 8**. Higher elasticity values correspond with higher curves as consumers are less reponsive to price changes of $x1$

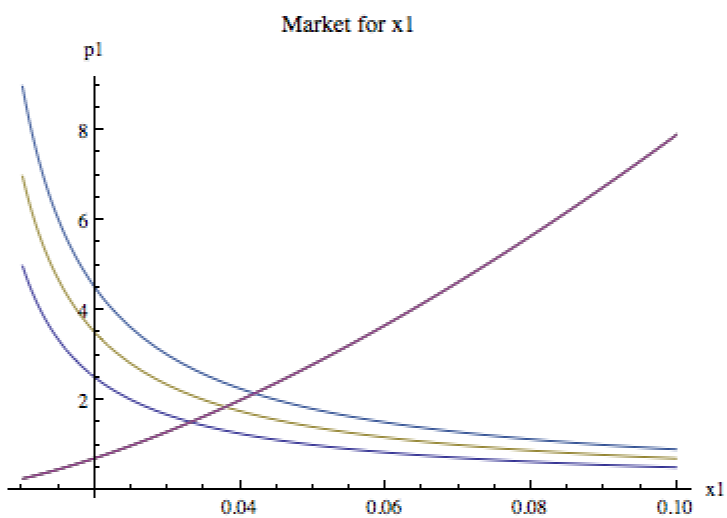

Figure 8: Market Equilibrium for x_1 as a Function on β

5 Future Improvements to the Model

The current market equilbrium model could be improved by adding the effects of taxes. However, simply adding a flat tax would only serve to shift the results above down as the budget constraint would be reduced. To make the model more interesting, a consumption tax could be added, such that the price of good x_1 would be dependent on the amount consumed. This could be implememted using the following psuedocode:

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\begin{center}
px1 = px1_0 + tauqx1
\end{center}
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6 Refrences

Jesus Filipe and Gerard Adams (2005). 'The Estimation of the Cobb Douglas Function'. Eastern Economic Journal 31 (3): 427445.