Linear Regression

Linear regression problem statement:

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$$\hat{y} = w_0 + \sum_{k=1}^{p} x_k \cdot w_k = //\mathbf{x} = [1, x_1, x_2, \ldots, x_p]// = \mathbf{x}^T \mathbf{w}$$

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we added an additional column of 1’s to the design matrix to simplify the formulas

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- Least squares method (MSE minimization) provides a solution:

$$\hat{\mathbf{w}} = \arg\min_{\mathbf{w}} \|Y - \hat{Y}\|_2^2 = \arg\min_{\mathbf{w}} \|Y - X\mathbf{w}\|_2^2$$

Analytical solution

Denote quadratic loss function:

$$Q(\mathbf{w}) = (Y - X\mathbf{w})^T(Y - X\mathbf{w}) = \|Y - X\mathbf{w}\|_2^2 \text{ ,}$$

where $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n], \quad \mathbf{x}_i \in \mathbb{R}^p \;\; Y = [y_1, \ldots, y_n], \quad y_i \in \mathbb{R}$.

To find optimal solution let's equal to zero the derivative of the equation above:

$$\nabla_{\mathbf{w}} Q(\mathbf{w}) = \nabla_{\mathbf{w}}[Y^TY - Y^TX\mathbf{w} - \mathbf{w}^TX^TY + \mathbf{w}^TX^TX\mathbf{w}] =$$

$$= 0 - X^TY - X^TY + (X^TX + X^TX)\mathbf{w} = 0$$

$$\hat{\mathbf{w}} = \boxed{(X^TX)}^{-1}X^TY$$

what if this matrix is *singular*?

Analytical solution

$$\hat{\mathbf{w}} = \boxed{(X^T X)}^{-1} X^T Y$$

what if this matrix is *singular?*

Unstable solution

In case of multicollinear features the matrix $X^T X$ is almost singular .

It leads to unstable solution:

```
w_true
```

```
array([ 2.68647887, -0.52184084, -1.12776533])
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w_star = np.linalg.inv(X.T.dot(X)).dot(X.T).dot(Y)
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corresponding features are almost collinear

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the coefficients are huge and sum up to almost 0

To make the matrix nonsingular, we can add a diagonal matrix:

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where $I = \mathrm{diag}[1_1, \ldots, 1_p]$.

Regularization

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where $I = \mathrm{diag}[1_1, \ldots, 1_p]$.

Actually, it's a solution for the following loss function:

$$Q(\mathbf{w}) = \|Y - X\mathbf{w}\|_2^2 + \lambda^2 \|\mathbf{w}\|_2^2$$

exercise: derive it by yourself

Gauss-Markov theorem

Gauss-Markov theorem

Suppose target values are expressed in following form:

$$Y = X\mathbf{w} + \boldsymbol{\varepsilon}$$, where $\boldsymbol{\varepsilon} = [\varepsilon_1, \ldots, \varepsilon_N]$

are random variables

Gauss–Markov assumptions:

- $\mathbb{E}(\varepsilon_i) = 0 \quad \forall i$
- $\mathrm{Var}(\varepsilon_i) = \sigma^2 < \inf \quad \forall i$
- $\mathrm{Cov}(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j$

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$$\boxed{\hat{\mathbf{w}} = (X^T X)^{-1} X^T Y}$$

delivers **B**est **L**inear **U**nbiased **E**stimator

Loss functions in regression

$$MSE(\mathbf{y}, \hat{\mathbf{y}}) = \frac{1}{N}\|\mathbf{y} - \hat{\mathbf{y}}\|_2^2 = \frac{1}{N}\sum_i (y_i - \hat{y}_i)^2$$

$$MAE(\mathbf{y}, \hat{\mathbf{y}}) = \frac{1}{N}\|\mathbf{y} - \hat{\mathbf{y}}\|_1 = \frac{1}{N}\sum_i |y_i - \hat{y}_i|$$

Different norms

Once more: loss functions:

$$\boxed{MSE = \frac{1}{n}\|\mathbf{x}^T\mathbf{w} - \mathbf{y}\|_2^2}$$

only works for Gauss-Markov theorem

$$MAE = \frac{1}{n}\|\mathbf{x}^T\mathbf{w} - \mathbf{y}\|_1$$

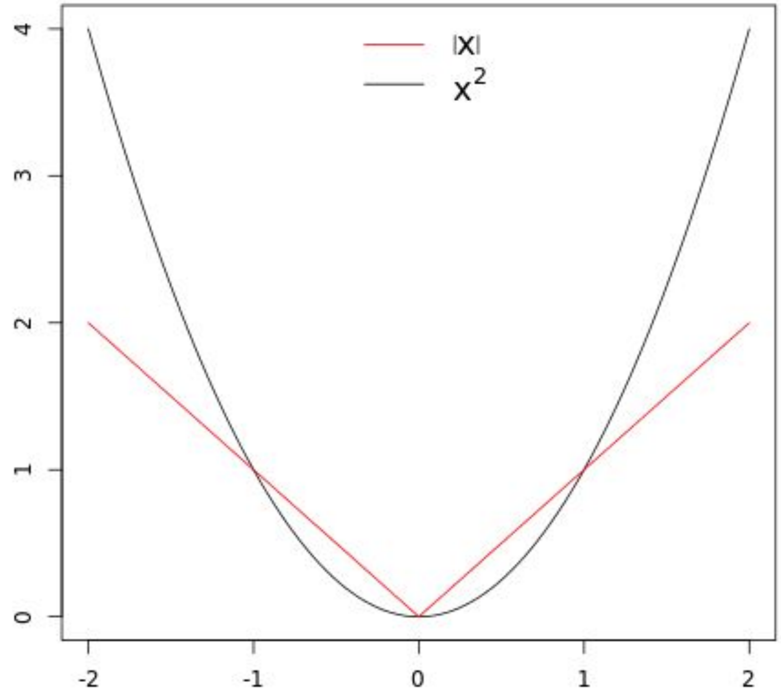
Regularization terms:

- $L_2 : \|\mathbf{w}\|_2^2$
- $L_1 : \|\mathbf{w}\|_1$

What’s the difference?

- MSE (L_2)
  - delivers BLUE according to Gauss-Markov theorem
  - differentiable
  - sensitive to noise

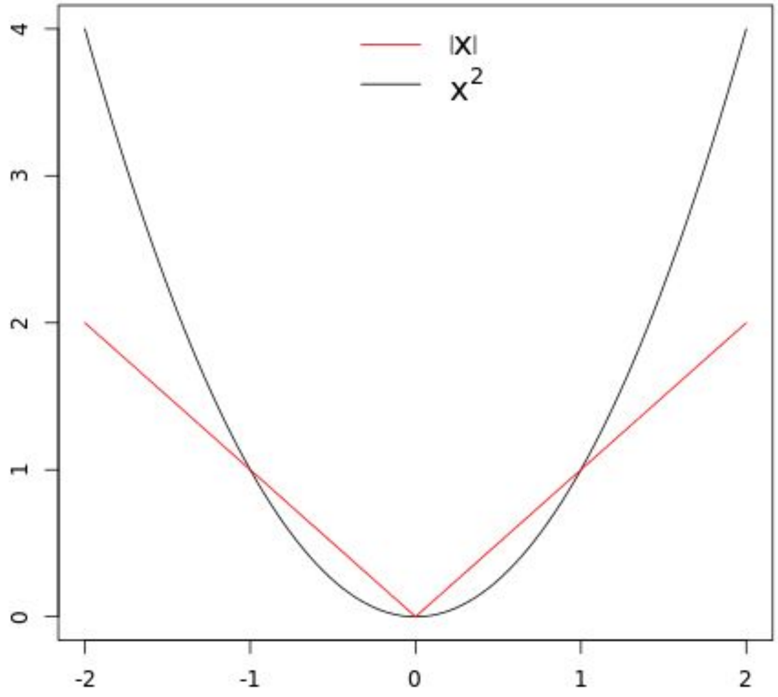
- MAE (L_1)
  - non-differentiable
    - not a problem
  - much more prone to noise



What’s the difference?

- L_2 regularization
  - constraints weights
  - delivers more stable solution
  - differentiable

- L_1 regularization
  - non-differentiable
    - not a problem
  - selects features



Loss functions in regression

Other functions to measure the quality in regression:

- R2 score
- MAPE
- SMAPE
- ...

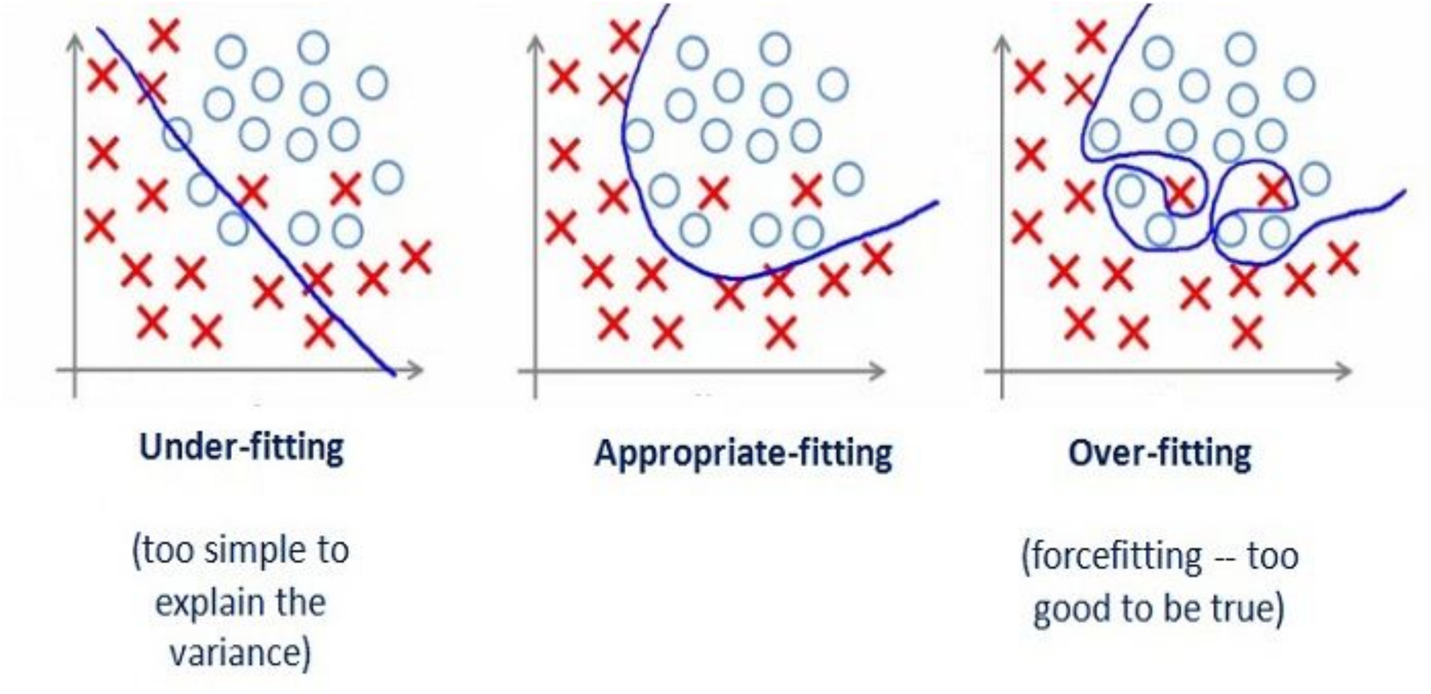
Model validation and evaluation

Supervised learning problem statement

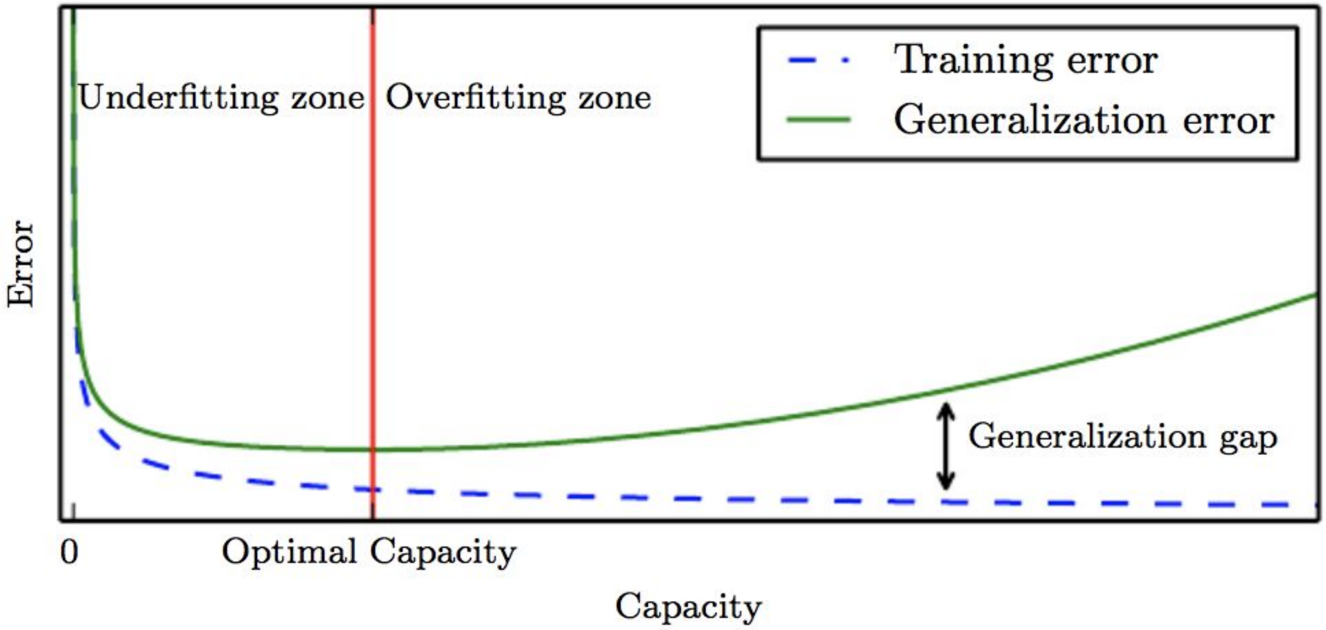
Let's denote:

- Training set $\mathcal{L} = \{\mathbf{x}_i, y_i\}_{i=1}^n$, where
  - $(\mathbf{x} \in \mathbb{R}^p,\ y \in \mathbb{R}\ )$ for regression
  - $\mathbf{x}_i \in \mathbb{R}^p$, $y_i \in \{+1, -1\}$ for binary classification
- Model $f(\mathbf{x})$ predicts some value for every object
- Loss function $Q(\mathbf{x}, y, f)$ that should be minimized

Overfitting vs. underfitting



Overfitting vs. underfitting



Overfitting vs. underfitting

- We can control overfitting / underfitting by altering model’s capacity (ability to fit a wide variety of functions):
- select appropriate hypothesis space
- learning algorithm’s effective capacity may be less than the representational capacity of the model family

Evaluating the quality

Image credit: Joseph Nelson @josephofiowa

Evaluating the quality

Is it good enough?

Image credit: Joseph Nelson [@josephofiowa]

Evaluating the quality

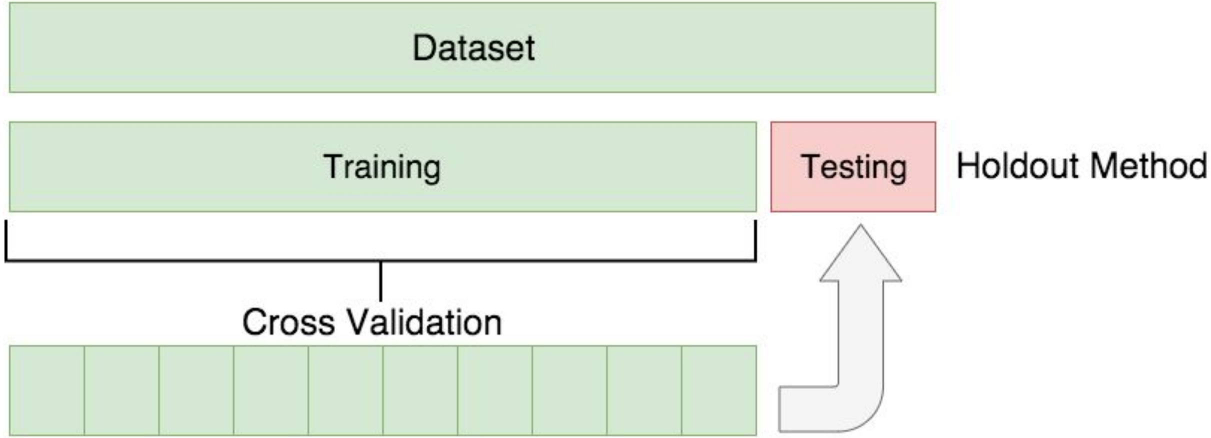


Image credit: Joseph Nelson [@josephofiowa]

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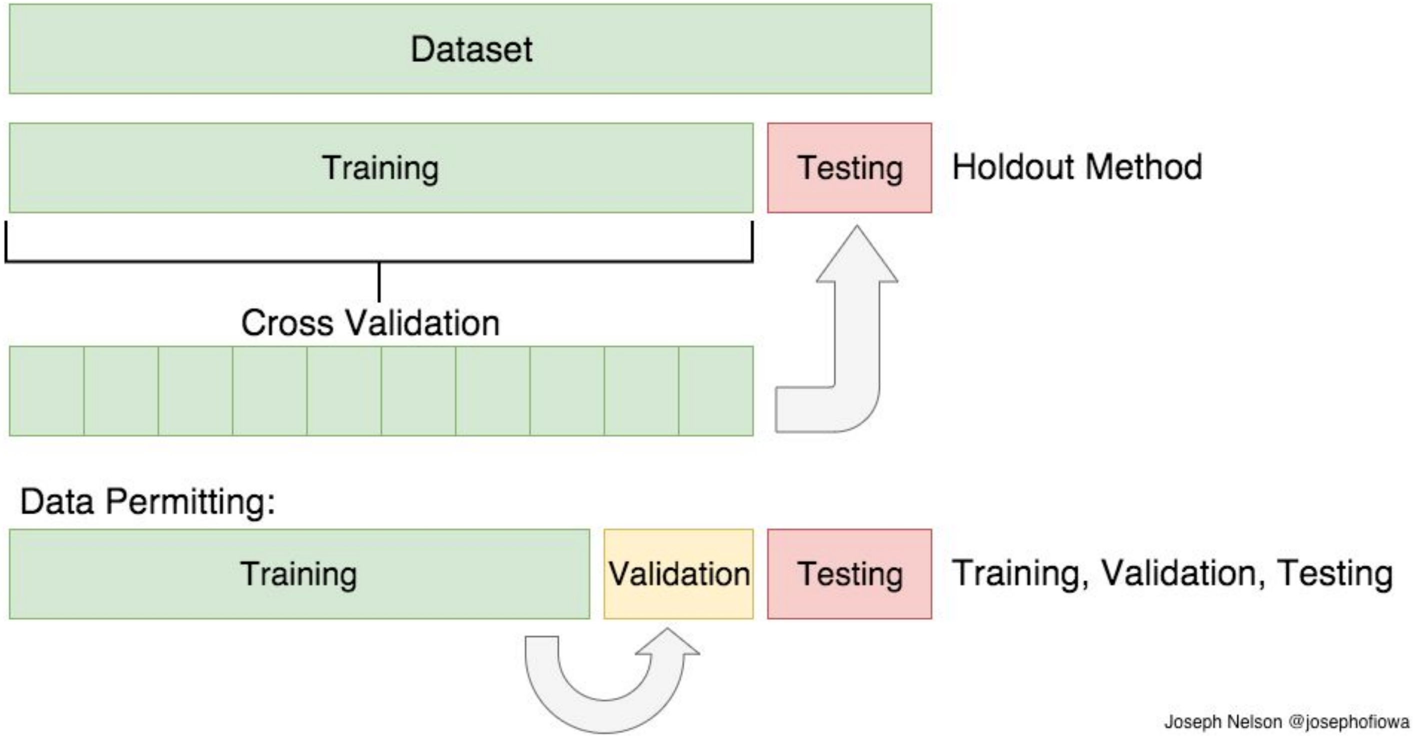

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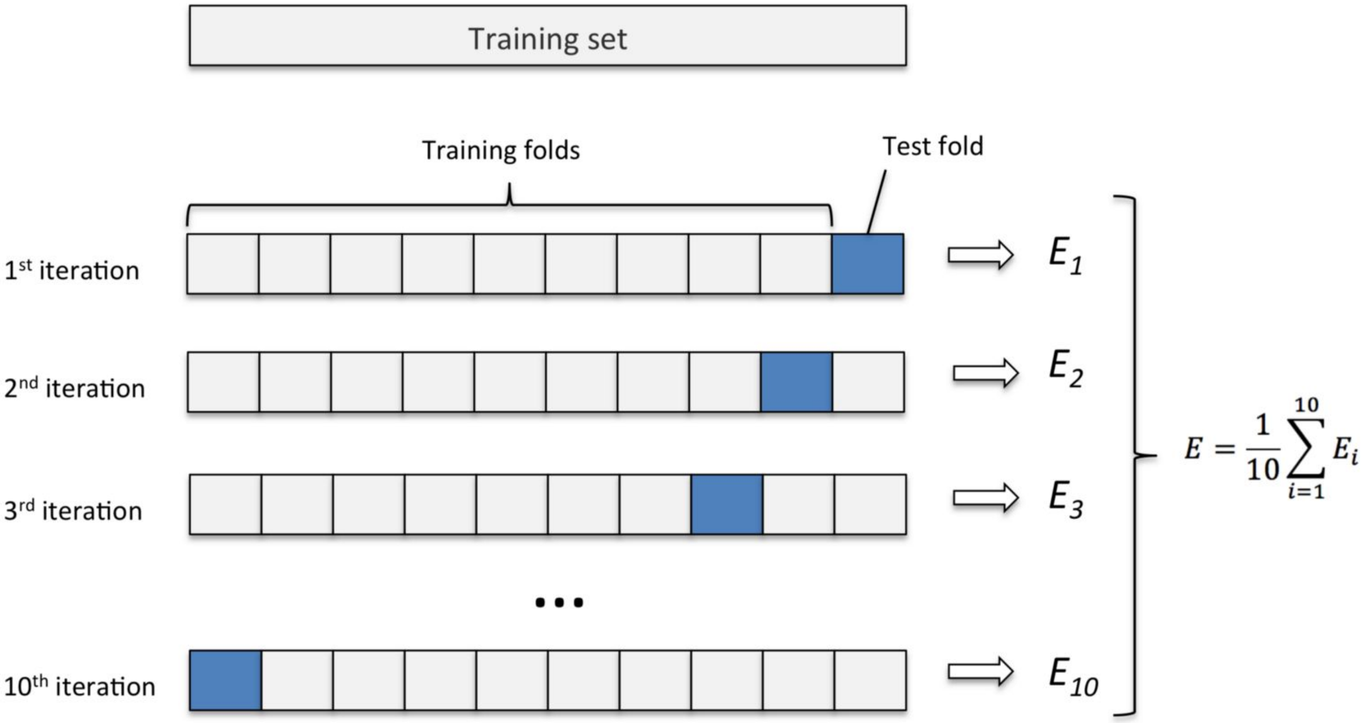
Cross-validation



Outro

- Linear models are simple yet quite effective models
- Regularization incorporates some prior assumptions/additional constraints
- Trust your validation

Backup

Linear models

$$Y = \underbrace{X_1 + X_2 + X_3}$$

Dependent Variable	Independent Variable
Outcome Variable	Predictor Variable
Response Variable	Explanatory Variable