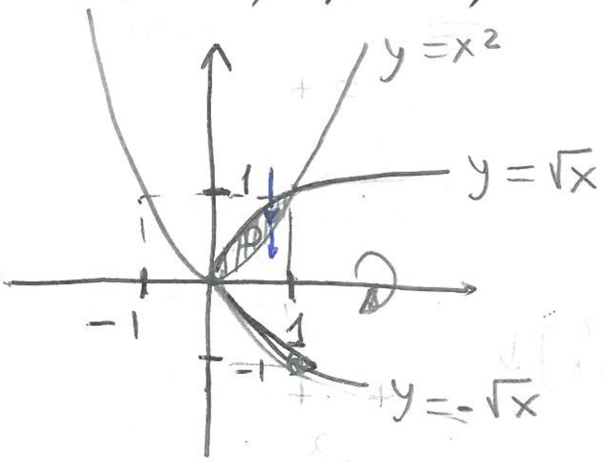
Ficha 6 $\quad x = y^2$ parábola

1. $\underset{\text{parábola}}{y = x^2}$; $y = \sqrt{x}$; $0 \le x \le 1$



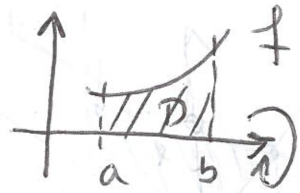
$$Vol(S) = \int_0^1 \pi \left[(\sqrt{x})^2 - (x^2)^2 \right] dx$$

$$= \int_0^1 \pi \left[x - x^4 \right] dx = \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1$$

$$= \pi \left[\underset{(5)}{\frac{1}{2}} - \underset{(2)}{\frac{1}{5}} \right] = \frac{3\pi}{10}$$

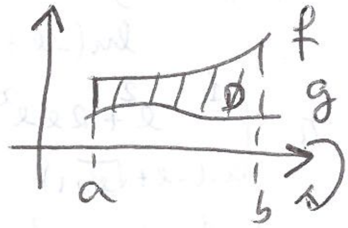
Volume de um sólido de Revolução (1)

1)



$$Vol(S) = \int_a^b \pi (f(x))^2 dx$$

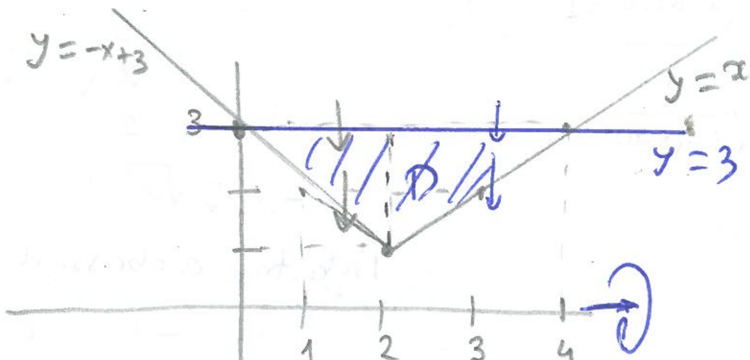
2)



$$Vol(S) = \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right]$$

2.

a) $R = \{ (x,y) \in \mathbb{R}^2 : |x-2| + 1 \le y \le 3 \}$



$$Vol(S) = 2 \times \int_0^2 \pi \left[3^2 - (-x+3)^2 \right] dx +$$

$$= 2\pi \int_0^2 (\cancel{9} - x^2 + 6x - \cancel{9}) dx$$

$$= 2\pi \left[-\frac{x^3}{3} + \frac{6x^2}{2} \right]_0^2 = 2\pi \left[-\frac{8}{3} + 3 \times 4 \right]$$

$$= 2\pi \left[\frac{36-8}{3} \right] = \frac{56\pi}{3}$$

C.A.:

$y = |x-2| + 1$

$$y = \begin{cases} x-2+1 & \text{se } x-2 \ge 0 \\ -x+2+1 & \text{se } x-2 < 0 \end{cases}$$

$$y = \begin{cases} x-1 & \text{se } x \ge 2 \\ -x+3 & \text{se } x < 2 \end{cases}$$

x	$y = x-1$
2	+1
3	2
4	3

P1

x	$y = -x+3$
2	1
1	2
0	3

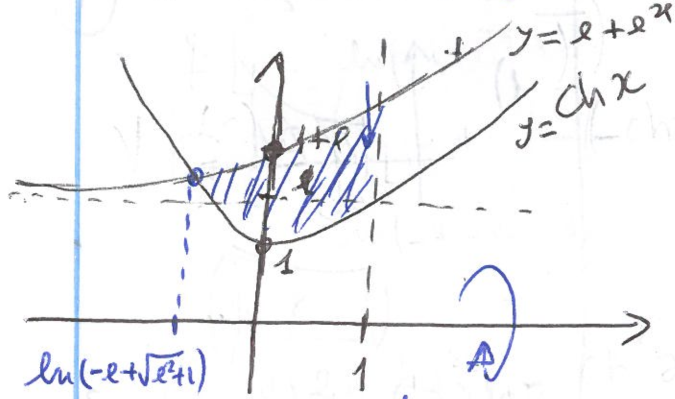
P2

Nota: P.I.: Pontos de Intersecção

$P_1 \begin{cases} y = 3 \\ y = x-1 \end{cases} \Rightarrow x = 4$

$P_2 \begin{cases} y = 3 \\ y = -x+3 \end{cases} \Rightarrow x = 0$

b) $R=\{(x,y)\in\mathbb{R}^2 : +\cosh x \le y \le e+e^x \wedge x\le 1\}$ (2)



$$Vol(S)=\int_{\ln(-e+\sqrt{e^2+1})}^{1}\pi\left[(e+e^x)^2-\cosh^2 x\right]dx$$

$$=\pi\int_{\ln(-e+\sqrt{e^2+1})}^{1} e^2+2ee^x+e^{2x}-\cosh^2 x\,dx$$

$$=\pi\left(\int_{\ln(-e+\sqrt{e^2+1})}^{1} e^2+2ee^x+e^{2x}\,dx-\int_{\ln(-e+\sqrt{e^2+1})}^{1}\underbrace{\cosh^2 x}\,dx\right)$$

$$=\pi\left[e^2x+2ee^x+\frac{1}{2}e^{2x}\right]_{\ln(-e+\sqrt{e^2+1})}^{1}-\int_{\ln(-e+\sqrt{e^2+1})}^{1}\frac{1+\cosh(2x)}{2}dx$$

$= \ldots$

C.A.: Ponto de Intersecção entre as curvas:

$$\begin{cases} y=+\cosh x \\ y=e+e^x \end{cases}\Leftrightarrow$$

$\cosh x = e+e^x \Leftrightarrow$

$\dfrac{e^x+e^{-x}}{2}=e+e^x$

$\Leftrightarrow +e^x+e^{-x}=2e+2e^x$

$\Leftrightarrow e^x-e^{-x}+2e=0$

$\Leftrightarrow e^x-\dfrac{1}{e^x}+2e=0$
$\quad(e^x)\qquad\qquad(e^x)$

$\Leftrightarrow \dfrac{e^{2x}-1+2ee^x}{e^x}=0$

$\Rightarrow \underset{a}{1}e^{2x}+\underset{b}{2e}\,e^x\underset{c}{-1}=0$

Fórmula resolvente:

$e^x=\dfrac{-2e\pm\sqrt{4e^2+4}}{2}$

$e^x=\dfrac{-2e\pm\sqrt{4(e^2+1)}}{2}$

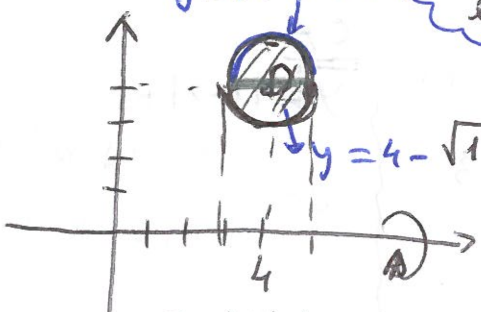
$e^x=\dfrac{-2e\pm2\sqrt{e^2+1}}{2}$

$e^x=-e\,\cancel{\pm}\sqrt{e^2+1}\quad(e^x>0)$

$\Rightarrow$ Portanto, a abcissa do ponto inter. é:
$x=\ln\left(-e+\sqrt{e^2+1}\right)$

c) $R=\{(x,y)\in\mathbb{R}^2 : \underbrace{(x-4)^2+(y-4)^2\le 1}\}$

$\odot$ de centro $(4,4)$ e raio $=1$



$3\le x\le 5$

$$Vol(S)=\pi\int_3^5\left[\left(4+\sqrt{1-(x-4)^2}\right)^2-\left(4-\sqrt{1-(x-4)^2}\right)^2\right]dx$$

$$=\pi\int_3^5\left[\left(16+8\sqrt{1-(x-4)^2}+|1-(x-4)^2|\right)-\left(16-8\sqrt{1-(x-4)^2}+|1-(x-4)^2|\right)\right]dx$$

$$=\pi\int_3^5 16\sqrt{1-(x-4)^2}\,dx=16\pi\int_3^5\underbrace{\sqrt{1-(x-4)^2}}\,dx=\ldots$$

C.A.: $(x-4)^2+(y-4)^2=1\Leftrightarrow(y-4)^2=1-(x-4)^2$

$\Leftrightarrow y-4=\pm\sqrt{1-(x-4)^2}$

$\Leftrightarrow y=4\pm\sqrt{1-(x-4)^2}$

(*) Mudança variável: $x-4=\operatorname{sen} t$

3. Curvas: $\underbrace{y = x}_{\text{recta}}$; $\underbrace{x = 4y - y^2}_{\text{parábola}}$

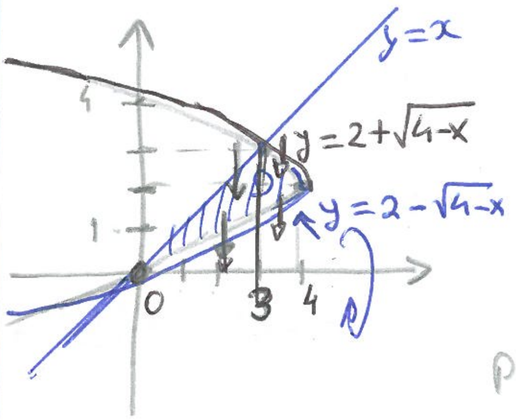
C.A.: parábola:

• zeros: $x = 0 \Leftrightarrow 4y - y^2 = 0 \Leftrightarrow y(4-y) = 0$
$\Leftrightarrow y = 0 \vee y = 4$

• vértice: $x' = 0 \Leftrightarrow (4y - y^2)' = 0 \Leftrightarrow$
$\Leftrightarrow 4 - 2y = 0 \Leftrightarrow y = \frac{-4}{-2} = 2$

Vértice: $(2, x(2)) = (2, 8-4) = (2,4)$

• concavidade: $\cap$



Ponto de intersecção entre as curvas:

$\begin{cases} y = x \\ x = 4y - y^2 \end{cases} \Leftrightarrow y = 4y - y^2 \Leftrightarrow$

$\Leftrightarrow y^2 - 3y = 0 \Leftrightarrow y(y-3) = 0 \Leftrightarrow y = 0 \vee y = 3$

$$Vol(S) = \pi \int_0^3 \left(x^2 - (2 - \sqrt{4-x})^2\right) dx$$

$$+ \pi \int_3^4 (2+\sqrt{4-x})^2 - (2-\sqrt{4-x})^2 dx$$

$$= \pi \int_0^3 \left(x^2 - (4 - 4\sqrt{4-x} + 4 - x)\right) dx$$

$$+ \pi \int_3^4 4 + 2\sqrt{4-x} + 4 - x - (4 - 2\sqrt{4-x} + 4 - x)$$

C.A.2:

$x = 4y - y^2 \Leftrightarrow \underset{a}{y^2} \underset{b}{- 4y} \underset{c}{+ x} = 0$

Fórmula resolvente:

$y = \frac{4 \pm \sqrt{16 - 4x}}{2} = \frac{4 \pm \sqrt{4(4-x)}}{2}$

$y = \frac{4 \pm 2\sqrt{4-x}}{2} = 2 \pm \sqrt{4-x}$

$$= \pi \int_0^3 (x^2 - 4 + 4\sqrt{4-x} - 4 + x) dx + \pi \int_3^4 4\sqrt{4-x}\, dx$$

$$= \pi \int_0^3 (x^2 + x - 8 + 4\sqrt{4-x}) dx + 4\pi \int_3^4 \sqrt{4-x}\, dx$$

$$= \pi \left[\frac{x^3}{3} + \frac{x^2}{2} - 8x\right]_0^3 + 4\pi \underbrace{\int_0^3 \sqrt{4-x}\, dx} + 4\pi \underbrace{\int_3^4 \sqrt{4-x}\, dx}$$

$$= \pi \left[\frac{3^{\cancel{3}\,2}}{\cancel{3}} + \frac{9}{2} - 24\right] +$$

Mud. var:
$4 - x = t^2 \Leftrightarrow x = \underbrace{4 - t^2}_{g(t)}$, $t > 0$

$dx = g'(t)dt = -2t\, dt$

Repor intervalo: $t = \sqrt{4-x}$

$x = 0 \rightarrow t = \sqrt{4} = 2$; $x = 4 \rightarrow t = \sqrt{0} = 0$

$x = 3 \rightarrow t = \sqrt{1} = 1$

$$= \pi\left[9+\frac{9}{2}-24\right] + 4\pi\int_2^1 \sqrt{t^2}\cdot(-2t)\,dt + 4\pi\int_1^0 \sqrt{t^2}\cdot(-2t)\,dt$$

$$= \pi\left[-15+\frac{9}{2}\right] + 4\pi\int_2^1 -2t^2\,dt + 4\pi\int_1^0 -2t^2\,dt$$

$$= \pi\left[-\frac{21}{2}\right] - 8\pi\int_2^1 t^2\,dt - 8\pi\int_1^0 t^2\,dt$$

$$= -\frac{21}{2}\pi - 8\pi\left[\frac{t^3}{3}\right]_2^1 - 8\pi\left[\frac{t^3}{3}\right]_1^0$$

$$= -\frac{21}{2}\pi - 8\pi\left[\frac{1}{3}-\frac{8}{3}\right] - 8\pi\left[0-\frac{1}{3}\right] = -\frac{21}{2}\pi + \frac{56\pi}{3} + \frac{8\pi}{3}$$

$$= -\underset{(2)}{\frac{21}{2}}\pi + \underset{(3)}{\frac{64\pi}{3}} = \frac{150}{6}\pi = 25\pi$$

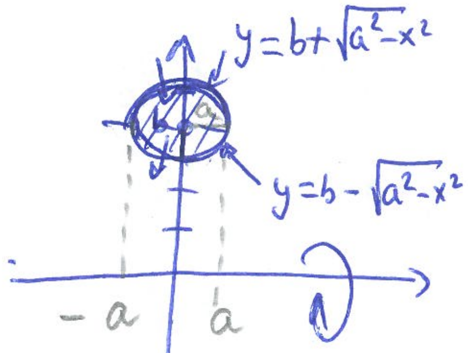
4. $b > a > 0 \rightarrow \odot$ centro $(0,b)$ e raio $= a$

$$x^2 + (y-b)^2 = a^2$$

$$\Leftrightarrow (y-b)^2 = a^2 - x^2$$

$$\Leftrightarrow (y-b) = \pm\sqrt{a^2-x^2}$$

$$\Leftrightarrow y = b \pm \sqrt{a^2-x^2}$$



$-a \leq x \leq a$

$$Vol(S) = \pi\int_{-a}^{a}\left[\left(b+\sqrt{a^2-x^2}\right)^2 - \left(b-\sqrt{a^2-x^2}\right)^2\right]dx$$

$$= \pi\int_{-a}^{a}\left[\left(b^2+2b\sqrt{a^2-b^2}+a^2-x^2\right)-\left(b^2-2b\sqrt{a^2-x^2}+a^2-x^2\right)\right]dx$$

$$= \pi\int_{-a}^{a}\left[\cancel{b^2}+2b\sqrt{a^2-b^2}+\cancel{a^2}-\cancel{x^2}-\cancel{b^2}+2b\sqrt{a^2-x^2}-\cancel{a^2}+\cancel{x^2}\right]dx$$

$$= 4b\pi\underbrace{\int_{-a}^{a}\sqrt{a^2-x^2}\,dx}$$

↑ mud. variável: $x = \underbrace{a\,\text{sen}\,t}_{g(t)}$

$$dx = g'(t)\,dt = a\cos t\,dt$$

$$= 4b\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \operatorname{sen}^2 t} \cdot a\cos t \, dt$$

C.A.:
Novo intervalo para "t":
$\frac{x}{a} = \operatorname{sen} t \to t = \arcsin\left(\frac{x}{a}\right)$

$x = a \to t = \arcsin 1 = \frac{\pi}{2}$
$x = -a \to t = \arcsin -1 = -\frac{\pi}{2}$

$$= 4b\pi \int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{a^2 \underbrace{(1 - \operatorname{sen}^2 t)}_{= \cos^2 t}} \cdot a\cos t \, dt$$

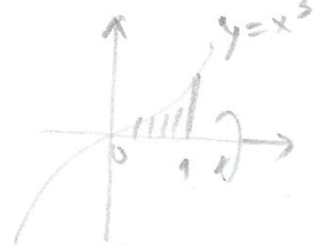
$$= 4b\pi \int_{-\frac{\pi}{2}}^{\pi/2} a^2 \cos^2 t \, dt = 4a^2 b \pi \int_{-\frac{\pi}{2}}^{\pi/2} \frac{1 + \cos(2t)}{2} \, dt$$

$$= 4a^2 b\pi \left(\frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dt + \frac{1}{2} \times \frac{1}{2} \int_{-\pi/2}^{\pi/2} 2\cos(2t) \, dt \right)$$

$$= 4a^2 b\pi \left[\frac{1}{2} (t) \right]_{-\pi/2}^{\pi/2} + \frac{1}{4} \underbrace{\left[-\operatorname{sen} 2t \right]_{-\pi/2}^{\pi/2}}_{0} = \frac{4a^2 b \pi}{2} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right)$$

$$= 2a^2 b \pi^2$$

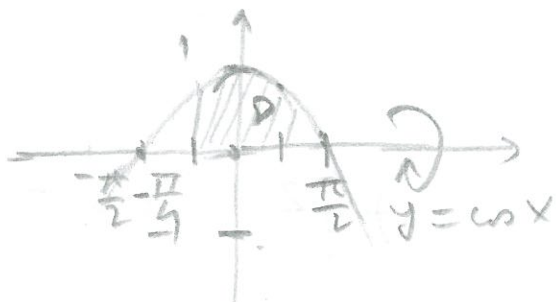
5. a) $y = x^3$; $x \in [0, 1]$



$$Vol(S) = \int_0^1 \pi \cdot (x^3)^2 \, dx$$

$$= \pi \int_0^1 x^6 \, dx = \pi \left[\frac{x^7}{7} \right]_0^1 = \frac{\pi}{7}$$

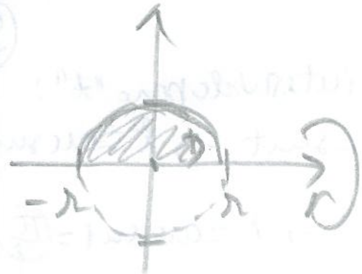
b) $y = \cos x$; $-\frac{\pi}{4} \leq y \leq \frac{\pi}{2}$



$$Vol(S) = \int_{-\frac{\pi}{4}}^{\pi/2} \pi (\cos x)^2 \, dx$$

c) $y = \sqrt{r^2 - x^2}$; $-r \leq x \leq r$

↑ Semi-circunferência, centro (0,0) e raio = R

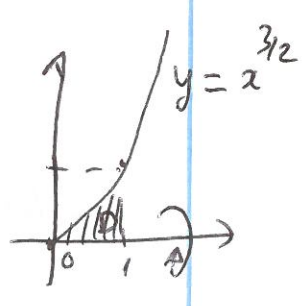


$$Vol(S) = \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \int_{-r}^{r} \pi \left(r^2 - x^2\right) dx = \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left[r^3 - \frac{r^3}{3} - \left(-r^3 + \frac{r^3}{3}\right)\right] = \pi\left[r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right]$$

$$= \pi\left[2r^3 - \frac{2}{3}r^3\right] = \frac{4}{3} r^3 \pi.$$

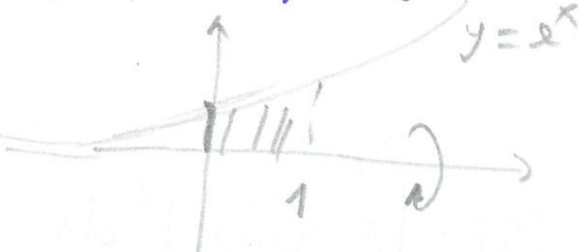


d) $R = \{(x,y) \in \mathbb{R}^2 : y = x^{3/2};\ 0 \leq x \leq 1\}$

$$Vol(S) = \int_0^1 \pi \left[x^{3/2}\right]^2 dx = \pi \int_0^1 x^3 dx$$

$$= \pi \left[\frac{x^4}{4}\right]_0^1 = \frac{\pi}{4}$$

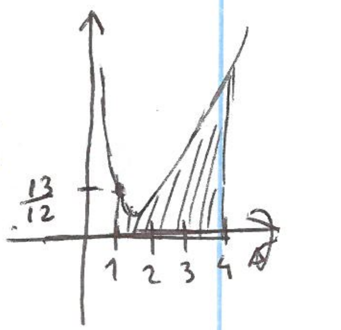
e) $R = \{(x,y) \in \mathbb{R}^2 : y = e^x;\ 0 \leq x \leq 1\}$



$$Vol(S) = \int_0^1 \pi \left[e^x\right]^2 dx$$

$$= \int_0^1 \pi e^{2x} dx = \pi \frac{1}{2} \int_0^1 2 e^{2x} dx$$

$$= \frac{\pi}{2} \left[e^{2x}\right]_0^1 = \frac{\pi}{2}\left[e^2 - 1\right]$$



f) $R = \{(x,y) \in \mathbb{R}^2 : y = \frac{x^3}{12} + \frac{1}{x};\ 1 \leq x \leq 4\}$

$$Vol(S) = \pi \int_1^4 \left(\frac{x^3}{12} + \frac{1}{x}\right)^2 dx = \dots$$