Ficha 4

1. $\int_1^4 f(x)dx = 3 \qquad \int_2^4 f(x)dx = 5$

a) $\int_1^4 f(t)\,dt = \int_1^4 f(x)dx = 3$

b) $\int_4^2 f(t)dt = -\int_2^4 f(t)dt = -5$

c) $\int_1^2 f(x)dx = \int_1^4 f(x)dx + \int_4^2 f(x)dx =$

$= \int_1^4 f(x)dx - \int_2^4 f(x)dx = 3 - 5 = -2$

d) $\int_{1/2}^2 f(2x)\,dx$ Fazendo a mudança de variável $2x = t \Leftrightarrow x = \frac{t}{2}$

$\circ\ dx = \frac{dt}{2}$

$x = 1/2 \xrightarrow{2x=t} 2 \times \frac{1}{2} = t \Leftrightarrow t = 1$

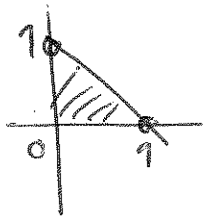
$x = 2 \Rightarrow 2 \times 2 = t \Leftrightarrow t = 4$

$\int_{1/2}^2 f(2x)dx = \int_1^4 f(t)\frac{dt}{2} = \frac{1}{2}\int_1^4 f(t)dt = \frac{1}{2} \times 3 = \frac{3}{2}$.

2. $\int_0^1 f(x)dx$. Como $f(x) \geq 0$, para $x \in [0,1]$ então $\int_0^1 f(x)dx$ dá o valor da área limitada pelo eixo Ox, pelo gráfico de $y = f(x)$ e pelas rectas verticais $x = 0$ e $x = 1$.

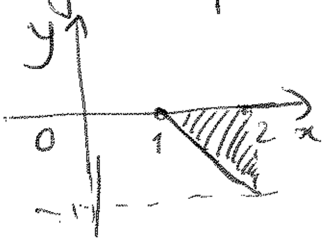
Ficha 4

2. Assim $\int_0^1 f(x)\,dx$ é área do triângulo



$$\int_0^1 f(x)\,dx = \frac{1\times 1}{2} = \frac{1}{2}$$

$\int_1^2 f(x)\,dx$. Como $f(x)\leq 0$, para $x\in[1,2]$ então o integral representa o simétrico da área da região a sombreado



$$\int_1^2 f(x)\,dx = -\frac{1}{2}$$

$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx.$$

Temos que $\int_0^1 f(x)\,dx = \int_4^5 f(x)\,dx$

e $\int_1^2 f(x)\,dx = \int_3^4 f(x)\,dx$

e que $\int_1^2 f(x)\,dx = -\int_0^1 f(x)\,dx$.

Assim $\int_0^5 f(x)\,dx = \int_2^3 f(x)\,dx = -1.$

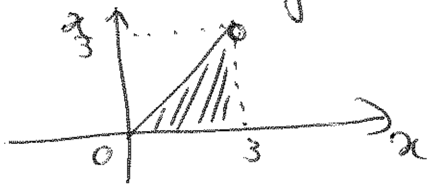
Ficha 4

3. $\int_0^5 f(x)\,dx$. Como $f(x)$ tem um nº finito de descontinuidades (2 descontinuidades), f é integrável e

$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx$$

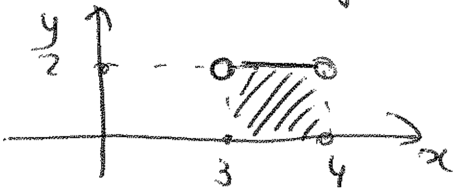
e em cada um dos integrais, $f(x) \geq 0$ no intervalo indicado, logo posso dizer que

$\int_0^3 f(x)\,dx$ tem o valor da área do triângulo



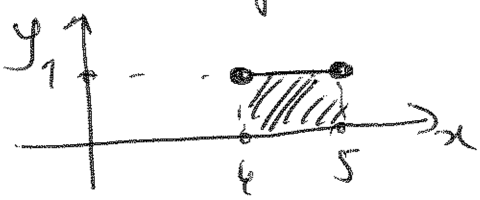
$$\int_0^3 f(x)\,dx = \frac{3\times 3}{2} = \frac{9}{2}$$

$\int_3^4 f(x)\,dx$ tem o valor da área do retângulo



$$\int_3^4 f(x)\,dx = 1\times 2 = 2$$

$\int_4^5 f(x)\,dx$ tem o valor da área do retângulo



$$\int_4^5 f(x)\,dx = 1\times 1 = 1$$

Assim $\int_0^5 f(x)\,dx = \frac{9}{2} + 2 + 1 = \frac{9}{2} + 3 = \frac{15}{2}$.

4. $F(3) - F(0) = \int_0^3 f(x)\,dx$ onde $F'(x) = f(x)$

Assim $F(3) - F(0)$ é o valor da área da região limitada pelo gráfico de $y = f(x)$, ~~pelo~~ entre $x = 0$ e $x = 3$ e pelo eixo Ox.

~~Assim~~ $\therefore F(3) - F(0) = \int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx =$

$= 2\times 2 + \frac{1\times 2}{2} = 4 + 1 = 5.$

Ficha 4

5. $F(x) = \int_0^{x^2} f(t)\,dt$

$F(\sqrt{3}) = \int_0^3 f(t)\,dt$ ~~...~~ que é o valor da área da região limitada pelo gráfico de $y = f(x)$, $x \in [0,1]$ e pelo eixo Ox pois $f(x) \geq 0$ nesse intervalo mais o simétrico do valor da área da região limitada pelo gráfico de $y = f(x)$, $x \in [1,3]$ e pelo eixo Ox pois $f(x) \leq 0$ nesse intervalo.

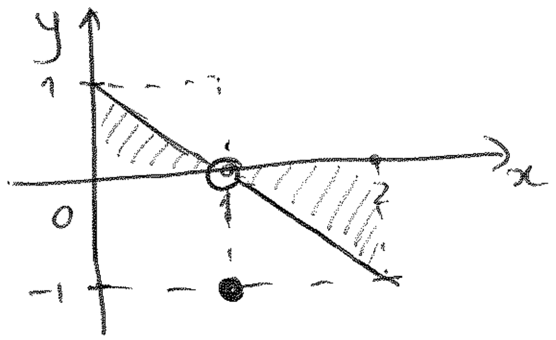
Assim $F(\sqrt{3}) = \int_0^3 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^3 f(t)\,dt =$

$= \frac{1}{2} - \frac{1}{2} - 1 = -1$

$F'(x) = \left(\int_0^{x^2} f(t)\,dt\right)' = (x^2)' \cdot f(x^2) = 2x \cdot f(x^2)$

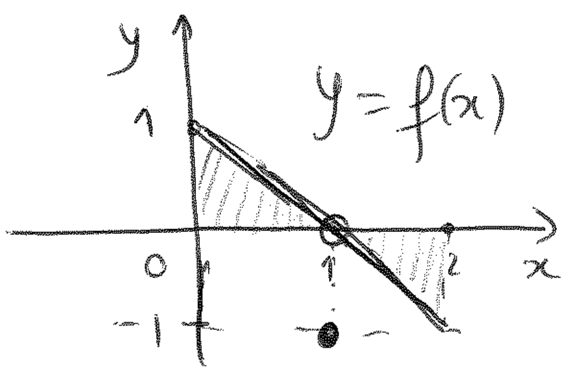
Assim $F'(\sqrt{3}) = 2 \cdot \sqrt{3} \cdot f(3) = 2\sqrt{3} \cdot (-1) = -2\sqrt{3}$.

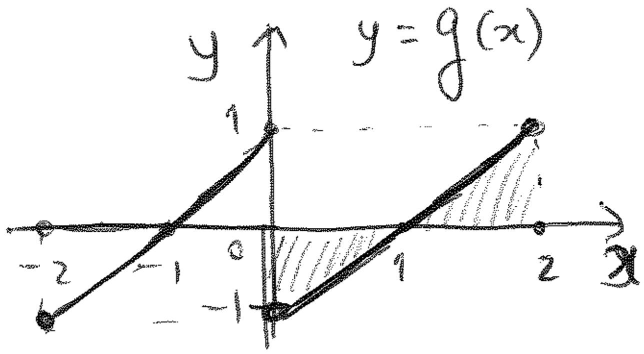
6. a) $f : [0,2] \to \mathbb{R}$ t.q. $\int_0^2 f(x)\,dx = 0$ e $f(x) \neq 0$, $\forall x \in [0,2]$.



$$f(x) = \begin{cases} y = -x+1 & \text{se } x \neq 1 \\ -1 & \text{se } x = 1 \end{cases}$$

b)





Ficha 4

7. a) $\int_0^2 (x+1)^2 dx = \left[\frac{(x+1)^3}{3}\right]_0^2 = \frac{3^3}{3} - \frac{1}{3} = \frac{26}{3}$

b) $\int_{-1}^1 \frac{1}{1+x^2} dx = [\text{arctg}\, x]_{-1}^1 = \text{arctg}\, 1 - \text{arctg}(-1) =$

$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

c) $\int_{-3}^2 \sqrt{|x|}\, dx = \int_{-3}^0 \sqrt{-x}\, dx + \int_0^2 \sqrt{x}\, dx =$

$= \left[-\frac{(-x)^{3/2}}{\frac{3}{2}}\right]_{-3}^0 + \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^2 = -\frac{2}{3}\left[0^{3/2} - (3)^{3/2}\right] + \frac{2}{3}\left[2^{3/2} - 0\right] =$

$= +\frac{2}{3} \cdot 3^{3/2} + \frac{2}{3} \cdot 2^{3/2} = \frac{2}{3} \cdot \sqrt{3^3} + \frac{2}{3}\sqrt{2^3} = \frac{2 \times 3\sqrt{3}}{3} + \frac{2 \times 2\sqrt{2}}{3} =$

$= 2\sqrt{3} + \frac{4\sqrt{2}}{3}$.

d) $\int_0^3 2-|x|\, dx = \int_0^3 (2-x)\, dx = \left[2x - \frac{x^2}{2}\right]_0^3 = 6 - \frac{9}{2} - 0 = \frac{3}{2}$.

e) $\int_{-1}^2 x|x|\, dx = \int_{-1}^0 x(-x)\, dx + \int_0^2 x.x\, dx = \int_{-1}^0 -x^2 dx + \int_0^2 x^2 dx$

$= -\left[\frac{x^3}{3}\right]_{-1}^0 + \left[\frac{x^3}{3}\right]_0^2 = -\left[0 - \frac{(-1)^3}{3}\right] + \frac{2^3}{3} = -\frac{1}{3} + \frac{8}{3} = \frac{7}{3}$.

f) $\int_0^{2\pi} |\cos x|\, dx = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\frac{3\pi}{2}} -\cos x\, dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\, dx =$

$= [\text{sen}\, x]_0^{\pi/2} - [\text{sen}\, x]_{\pi/2}^{3\pi/2} + [\text{sen}\, x]_{\frac{3\pi}{2}}^{2\pi} = [1-0] - [-1-1] + [0+1]$

$= 1+2+1 = 4$.

Ficha 4

g) $$\int_3^4 \frac{1-4x^3}{x-x^4}\,dx = \Big[\ln|x-x^4|\Big]_3^4 = \ln|4-4^4| - \ln|3-3^4| =$$

$$= \ln\frac{|4-4^4|}{|3-3^4|} = \ln\frac{|4-256|}{|3-81|} = \ln\frac{252}{78} = \ln\frac{42}{13}$$

h) $$\int_0^{\pi} x \operatorname{sen} x\,dx = \Big[-x\cos x\Big]_0^{\pi} + \int_0^{\pi}\cos x\,dx =$$

$$= \Big[-x\cos x\Big]_0^{\pi} + \Big[\operatorname{sen} x\Big]_0^{\pi} = -\pi\cos\pi + 0 + \operatorname{sen}\pi - \operatorname{sen} 0 =$$

$$= \pi.$$

i) $$\int_0^1 x \cdot \operatorname{arctg} x^2\,dx = \left[\frac{x^2}{2}\cdot \operatorname{arctg} x^2\right]_0^1 - \int_0^1 \frac{x^2}{2}\cdot\frac{2x}{1+x^4}\,dx$$

$$= \left[\frac{x^2}{2}\operatorname{arctg} x^2\right]_0^1 - \int_0^1 \frac{x^3}{1+x^4}\,dx = \left[\frac{x^2}{2}\operatorname{arctg} x^2\right]_0^1 - \frac{1}{4}\Big(\ln|1+x^4|\Big)_0^1 =$$

$$= \frac{1}{2}\operatorname{arctg} 1 - 0 - \frac{1}{4}\Big[\ln 2 - \ln 1\Big] = \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{4}\ln 2 = \frac{\pi}{8} - \frac{\ln 2}{4}.$$

j) $$\int_0^{\sqrt{2}/2} \operatorname{arcsen} x\,dx = \Big[x \operatorname{arcsen} x\Big]_0^{\sqrt{2}/2} - \int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1-x^2}}\,dx =$$

$$= \Big[x \operatorname{arcsen} x\Big]_0^{\sqrt{2}/2} - \int_0^{\sqrt{2}/2} x\,(1-x^2)^{-1/2}\,dx = \Big[x \operatorname{arcsen} x\Big]_0^{\sqrt{2}/2} + \frac{1}{2}\left[\frac{(1-x^2)^{1/2}}{1/2}\right]_0^{\sqrt{2}/2} =$$

$$= \Big[x \operatorname{arcsen} x\Big]_0^{\sqrt{2}/2} + \Big[(1-x^2)^{1/2}\Big]_0^{\sqrt{2}/2} = \frac{\sqrt{2}}{2}\cdot \operatorname{arcsen}\frac{\sqrt{2}}{2} + \left[\left(1-\frac{1}{2}\right)^{1/2} - 1^{1/2}\right] =$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\pi}{4} + \sqrt{\frac{1}{2}} - 1 = \frac{\sqrt{2}}{2}\left(\frac{\pi}{4}+1\right) - 1.$$

k) $\displaystyle\int_0^2 \frac{2x-1}{(x-3)(x+1)}\,dx.$

C.A.: $\displaystyle\frac{2x-1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$$2x-1 = A(x+1) + B(x-3)$$

$x=-1 \Rightarrow -2-1 = B(-4) \Leftrightarrow B = \frac{3}{4}$

$x = 3 \Rightarrow 5 = A(4) \Leftrightarrow A = 5/4$

$$\int_0^2 \frac{2x-1}{(x-3)(x+1)}\,dx = \frac{5}{4}\int_0^2 \frac{dx}{x-3} + \frac{3}{4}\int_0^2 \frac{dx}{x+1} =$$

$$= \frac{5}{4}\Big[\ln|x-3|\Big]_0^2 + \frac{3}{4}\Big[\ln|x+1|\Big]_0^2 =$$

$$= \frac{5}{4}\Big[\ln|2-3| - \ln|0-3|\Big] + \frac{3}{4}\Big[\ln|2+1| - \ln|1|\Big] =$$

$$= \frac{5}{4}\Big[\ln 1 - \ln 3\Big] + \frac{3}{4}\Big[\ln 3 - \ln 1\Big] = -\frac{5}{4}\ln 3 + \frac{3}{4}\ln 3 = -\frac{1}{2}\ln 3.$$

$$= \ln 3^{-1/2} = \ln\left(\frac{1}{\sqrt{3}}\right) = \ln\left(\frac{\sqrt{3}}{3}\right).$$

l) $\displaystyle\int_e^{e^2} \frac{\ln(\ln x^2)}{x}\,dx = \int_e^{e^2} \frac{1}{x}\cdot\ln(2\ln x)\,dx = \Big[\ln|x|\,\ln(2\ln x)\Big]_e^{e^2} -$

$$-\int_e^{e^2} \ln\cancel{x}\cdot\frac{\frac{2}{x}}{2\cancel{\ln x}}\,dx = \Big[\ln|x|.\ln(2\ln x)\Big]_e^{e^2} - \int_e^{e^2}\frac{1}{x}\,dx$$

$$= \Big[\ln|x|.\ln(2\ln x)\Big]_e^{e^2} - \Big[\ln|x|\Big]_e^{e^2} = 2\ln 4 - \ln 2 - 1 = 3\ln 2 - 1.$$

Ficha 4

l) $\cancel{= \ln(2\times 2) - \ln 2 = \ln 2 + \ln 1 = \ln 4 - 2\ln 2 = 0}$

m) $\int_0^1 \ln(x^2+1)\,dx = \left[x\ln(x^2+1)\right]_0^1 - \int_0^1 x\cdot\frac{2x}{x^2+1}\,dx =$

$= \left[x\ln(x^2+1)\right]_0^1 - \int_0^1 \frac{2x^2}{x^2+1}\,dx = \left[x\ln(x^2+1)\right]_0^1 - \int_0^1 \left[2 - \frac{2}{x^2+1}\right]dx$

$= \left[x\ln(x^2+1)\right]_0^1 - \left[2x - 2\,\mathrm{arctg}\,x\right]_0^1 =$

$= \ln 2 - 0 - \left[2 - 2\,\mathrm{arctg}\,1 - 0 + 2\cdot\mathrm{arctg}\,0\right] =$

$= \ln 2 - 2 - \frac{2\pi}{4} + 0 \quad = \ln 2 - 2 - \frac{\pi}{2}$.

n) $\int_0^{\pi/2} \mathrm{sen}(2x)\cdot\cos(5x)\,dx = \frac{1}{5}\left[\mathrm{sen}(2x)\,\mathrm{sen}(5x)\right]_0^{\pi/2} - \frac{1}{5}\int_0^{\pi/2} 2\cos(2x)\cdot\mathrm{sen}(5x)\,dx$

$= \frac{1}{5}\left[\mathrm{sen}(2x)\,\mathrm{sen}(5x)\right]_0^{\pi/2} - \frac{2}{5}\left[-\frac{1}{5}\cos(5x)\cdot\cos(2x) - \frac{2}{5}\int_0^{\pi/2}\cos(5x)\,\mathrm{sen}(2x)\,dx\right]$

$= \frac{1}{5}\left[\mathrm{sen}(2x)\cdot\mathrm{sen}(5x)\right]_0^{\pi/2} + \frac{2}{25}\left[\cos(5x)\cdot\cos(2x)\right]_0^{\pi/2} + \frac{4}{25}\int_0^{\pi/2}\cos(5x)\cdot\mathrm{sen}(2x)\,dx$.

$\Leftrightarrow$

$\left(1 - \frac{4}{25}\right)\int_0^{\pi/2}\cos(5x)\cdot\mathrm{sen}(2x)\,dx = \frac{1}{5}\left[\mathrm{sen}(2x)\,\mathrm{sen}(5x)\right]_0^{\pi/2} + \frac{2}{25}\left[\cos(5x)\cdot\cos(2x)\right]_0^{\pi/2}$

$\int_0^{\pi/2}\cos(5x)\,\mathrm{sen}(2x)\,dx = \frac{25}{21}\left[\frac{1}{5}\left(\mathrm{sen}\,\pi\cdot\mathrm{sen}\,\frac{5\pi}{2}\right) + \frac{2}{25}\left(\cos\frac{5\pi}{2}\cdot\cos\pi - \cos 0\right)\right]$

$= \frac{2}{21}$

Ficha 4

e) $\int_0^1 g(x)\,dx = \int_0^{1/2} x\,dx + \int_{1/2}^1 -x\,dx = \left[\frac{x^2}{2}\right]_0^{1/2} - \left[\frac{x^2}{2}\right]_{1/2}^1 =$

$= \frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}.$

8. $\int_0^c x(1-x)\,dx = 0 \Leftrightarrow \int_0^c (x - x^2)\,dx = 0 \Leftrightarrow$

$\Leftrightarrow \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^c = 0 \Leftrightarrow \left(\frac{c^2}{2} - \frac{c^3}{3}\right) = 0 \Leftrightarrow c^2\left(\frac{1}{2} - \frac{c}{3}\right) = 0$

$c = 0 \vee \frac{1}{2} - \frac{c}{3} = 0 \Leftrightarrow c = 0 \vee \frac{c}{3} = 1/2 \Leftrightarrow c = 0 \vee c = 3/2.$

9. $p(x)$ polinómio quadrático $\Rightarrow p(x) = ax^2 + bx + c$
$a, b, c \in \mathbb{R}.$

$p(0) = 0 \Leftrightarrow c = 0$

$p(1) = 0 \Leftrightarrow a + b = 0 \Leftrightarrow b = -a$

$p(x) = ax^2 - ax =$

$\int_0^1 p(t)\,dt = 1 \Leftrightarrow \int_0^1 at^2 - at\,dt = 1 \Leftrightarrow$

$\left[a\frac{t^3}{3} - a\frac{t^2}{2}\right]_0^1 = 1 \Leftrightarrow \frac{a}{3} - \frac{a}{2} = 1 \Leftrightarrow -\frac{a}{6} = 1 \Leftrightarrow$

$a = -6 \qquad p(x) = -6x^2 + 6x$

10. $I = \int_0^1 \sqrt{1-x^2}\, dx$.

Como $\sqrt{1-x^2} \geq 0$, $\forall x \in [0,1]$ tem-se que

$$\int_0^1 \sqrt{1-x^2}\, dx > 0.$$

$$J = \int_{2\pi}^{3\pi/2} \operatorname{sen}^2 x\, dx.$$

Como $\operatorname{sen}^2 x \geq 0$, $\forall x \in \left[\frac{3\pi}{2}, 2\pi\right]$, tem-se que

$$\int_{\frac{3\pi}{2}}^{2\pi} \operatorname{sen}^2 x\, dx > 0 \quad e \quad \int_{2\pi}^{3\pi/2} \operatorname{sen}^2 x\, dx < 0.$$

11. Mostra que $0 \leq \int_0^1 \frac{dx}{1+x^3} < 1$:

Como $x^3 > 0$, $\forall x \in [0,1]$

então $1+x^3 > 1$, $\forall x \in [0,1)$

e $\frac{1}{1+x^3} < 1$, $\forall x \in [0,1]$.

Assim,
$$\int_0^1 \frac{dx}{1+x^3} < \int_0^1 1 . dx = 1.$$

Como $x^3 < x^2$, $\forall x \in [0,1]$

$1+x^3 < 1+x^2$

e $\frac{1}{1+x^3} > \frac{1}{1+x^2}$, $\forall x \in [0,1]$.

Logo $\int_0^1 \frac{1}{1+x^3}dx > \int_0^1 \frac{dx}{1+x^2}$

e $\int_0^1 \frac{dx}{1+x^2} = [\text{arctg}\, x]_0^1 = \text{arctg}\, 1 - \text{arctg}\, 0 = \frac{\pi}{4}$.

Conclui-se que $\frac{\pi}{4} < \int_0^1 \frac{dx}{1+x^3} < 1$.

Mostrar que $0 < \int_0^{\pi} \text{sen}^2 x\, dx < 2$

Como $\text{sen}^2 x > 0$ então $\int_0^{\pi} \text{sen}^2 x > \int_0^{\pi} 0 = 0$.

A função $1 \geq \text{sen}\, x \geq 0$, $\forall x \in [0,\pi]$, logo

$$\text{sen}^2 x \leq \text{sen}\, x \quad , \forall x \in [0,\pi].$$

e $\int_0^{\pi} \text{sen}^2 x\, dx \leq \int_0^{\pi} \text{sen}\, x\, dx = [-\cos x]_0^{\pi} =$

$= -\cos \pi + \cos 0 = 2$.

Logo $0 < \int_0^{\pi} \text{sen}^2 x\, dx < 2$.

Ficha 4

12. $\int_0^x f(t)\,dt = \frac{4}{3} + 3x^2 + \operatorname{sen}(2x)$

$$\left(\int_0^x f(t)\,dt\right)' = \left(\frac{4}{3} + 3x^2 + \operatorname{sen}(2x)\right)' \Leftrightarrow$$

$$\Leftrightarrow \quad f(x) = 6x + 2\cos(2x)$$

$$\text{e} \quad f\left(\frac{\pi}{2}\right) = \frac{6\pi}{2} + 2\cos\pi = 3\pi - 2.$$

○ $f'(x) = 6 - 4\operatorname{sen}(2x)$. e $f'\left(\frac{\pi}{4}\right) = 6 - 4\operatorname{sen}\frac{\pi}{2} = 6 - 4 = 2.$

13. Há uma gralha na ficha. $y = \int_0^x \sqrt{1-t^2}\,dt$

$y' = \sqrt{1-x^2}$ e $y'' = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$

Tem-se que $y' \cdot y'' = \sqrt{1-x^2} \cdot \left(\frac{-x}{\sqrt{1-x^2}}\right) = -x$. c.q.d.

e $y(0) = \int_0^0 \sqrt{1-t^2}\,dt = 0$.

14. a) $\int_0^1 \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

$$x = t^6$$
$$dx = 6t^5\,dt$$

$$x=1 \xRightarrow{x=t^6} t=1$$
$$x=0 \Rightarrow t=0$$

$$\int_0^1 \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{\sqrt{t^6}}{1+\sqrt[3]{t^6}}\,6t^5\,dt = \int_0^1 \frac{t^3\cdot 6\cdot t^5}{1+t^2}\,dt$$

$$= 6\int_0^1 \frac{t^8}{1+t^2}\,dt$$

$$\begin{array}{l|l}
t^8 & t^2+1 \\ \hline
-t^8-t^6 & t^6-t^4+t^2-1 \\ \hline
-\cancel{t^6} & \\
\cancel{t^6}+t^4 & \\ \hline
\cancel{t^4} & \\
-\cancel{t^4}-t^2 & \\ \hline
-\cancel{t^2} & \\
\cancel{t^2}-1 & \\ \hline
+1 &
\end{array}$$

$$= 6\int_0^1 \left(t^6-t^4+t^2-1+\frac{1}{1+t^2}\right)dt$$

$$= 6\left[\frac{t^7}{7}-\frac{t^5}{5}+\frac{t^3}{3}-t+\operatorname{arctg} t\right]_0^1$$

$$= 6\left[\frac{1}{7}-\frac{1}{5}+\frac{1}{3}-1+\operatorname{arctg} 1\right] = 6\left[-\frac{2}{35}-\frac{2}{3}+\frac{\pi}{4}\right] = 6\left[-\frac{76}{105}+\frac{\pi}{4}\right]$$

b) $\int_{-5}^0 2x\sqrt{4-x}\,dx$

$$\sqrt{4-x} = t$$
$$4-x = t^2 \Leftrightarrow x = 4-t^2$$
$$dx = -2t\,dt$$

$$x=-5 \xRightarrow{\sqrt{4-x}=t} \sqrt{9}=t \Leftrightarrow t=3$$
$$x=0 \Rightarrow \sqrt{4}=t \Leftrightarrow t=2.$$

$$\int_{-5}^0 2x\sqrt{4-x}\,dx = \int_3^2 2(4-t^2)\cdot t\cdot(-2t)\,dt = -4\int_3^2 (4t^2-t^4)\,dt$$

$$= -4\left[\frac{4t^3}{3}-\frac{t^5}{5}\right]_3^2 = -4\left[\frac{4\times 2^3}{3}-\frac{2^5}{5}-\frac{4\times 3^3}{3}+\frac{3^5}{5}\right] = -4\left[\frac{-4\times 19}{3}+\frac{3^5-2^5}{5}\right]$$

Ficha 4

c) $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$

Substituição: $\sqrt{1+x} = t$

$\Downarrow$

$1+x = t^2$

$x = t^2 - 1$

- $dx = 2t\,dt$
- $x = 0 \xrightarrow{\sqrt{1+x}=t} \sqrt{1+0} = t \Leftrightarrow t = 1$
- $x = 3 \Rightarrow \sqrt{1+3} = t \Leftrightarrow t = 2$

$$\int_0^3 \frac{x}{\sqrt{1+x}}\,dx = \int_1^2 \frac{t^2-1}{\cancel{t}} \cdot 2\cancel{t}\,.dt = 2\int_1^2 (t^2-1)\,dt =$$

$$= 2\left[\frac{t^3}{3} - t\right]_1^2 = 2\left[\frac{2^3}{3} - 2 - \frac{1}{3} + 1\right] = 2\left[\frac{8}{3} - \frac{1}{3} - 1\right] =$$

$$= 2\left[\frac{7}{3} - 1\right] = 2\left[\frac{4}{3}\right] = \frac{8}{3}.$$

d) $\int_0^1 \frac{e^x}{1+e^{3x}}\,dx$

<u>Substituição</u>: $e^x = t$

$\Downarrow$

$x = \ln t$

- $dx = \frac{1}{t}\,.dt$
- $x = 0 \xrightarrow{e^x = t} e^0 = t \Leftrightarrow t = 1$
- $x = 1 \Rightarrow e^1 = t \Leftrightarrow t = e$

$$\int_0^1 \frac{e^x}{1+e^{3x}}\,dx = \int_1^e \frac{\cancel{t}}{1+t^3} \cdot \frac{1}{\cancel{t}}\,.dt = \int_1^e \frac{1}{1+t^3}\,.dt$$

$t^3 + 1 = (t+1)\,p(t)$. Determinar o polinómio $p(t)$ usando a regra de Ruffini:

	1	0	0	1
-1		-1	1	-1
	1	-1	1	0

$p(t) = t^2 - t + 1.$

Ficha 4

14.d) Determinar os zeros de $f(t)=t^2-t+1$ pela fórmula resolvente

$$t^2-t+1=0 \Leftrightarrow t=\frac{1\pm\sqrt{1-4}}{2} \rightarrow \text{não há zeros reais.}$$

Assim, a fracção

$$\frac{1}{1+t^3}=\frac{A}{t+1}+\frac{Bt+C}{t^2-t+1}$$

Determinar A, B, C:

$$1=A(t^2-t+1)+(Bt+C)(t+1)$$

$t=-1 \Rightarrow$

$$1=A(1+1+1)+0$$

$$1=3A \Leftrightarrow \boxed{A=1/3}$$

$t=0$

$$1=A(0-0+1)+C(1)$$

$$1=A+C \Leftrightarrow 1=\frac{1}{3}+C \Leftrightarrow \boxed{C=2/3}$$

$t=+1$

$$1=A(1-1+1)+(B+C)(2)$$

$$1=A+2B+2C \Leftrightarrow 1=\frac{1}{3}+2B+\frac{4}{3} \Leftrightarrow$$

$$1=\frac{5}{3}+2B \Leftrightarrow 1-\frac{5}{3}=2B \Leftrightarrow -\frac{2}{3}=2B \Leftrightarrow \boxed{B=-\frac{1}{3}}$$

Então,

$$\frac{1}{1+t^3}=\frac{1}{3}\cdot\frac{1}{t+1}+\frac{1}{3}\,\frac{-t+2}{t^2-t+1}$$

e

$$P\frac{1}{1+t^3}=\frac{1}{3}P\frac{1}{t+1}+\frac{1}{3}P\frac{-t+2}{t^2-t+1}$$

$$=\frac{1}{3}\ln|t+1|-\frac{1}{3}P\frac{t-2}{t^2-t+1}$$

14.d) $P\dfrac{1}{1+t^3} = \dfrac{1}{3}\ln|t+1| - \dfrac{1}{6} P \dfrac{2t-4}{t^2-t+1} =$

$= \dfrac{1}{3}\ln|t+1| - \dfrac{1}{6} P \dfrac{2t-1}{t^2-t+1} - \dfrac{1}{6} P \dfrac{-3}{t^2-t+1} =$

$= \dfrac{1}{3}\ln|t+1| - \dfrac{1}{6}\ln|t^2-t+1| - \dfrac{1}{6} P \dfrac{-3}{t^2-t+\frac{1}{4}-\frac{1}{4}+1} =$

$=$ " " $- \dfrac{1}{6} P \dfrac{-3}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}$ •

$-\dfrac{1}{6} P \dfrac{-3}{\frac{3}{4}\left[\frac{4}{3}\left(t-\frac{1}{2}\right)^2+1\right]} = \dfrac{1}{2}\cdot\dfrac{4}{3}\cdot\dfrac{\sqrt{3}}{2} P \dfrac{\frac{2}{\sqrt{3}}}{\frac{4}{3}\left(t-\frac{1}{2}\right)^2+1} =$

$u^2 = \dfrac{4}{3}\left(t-\dfrac{1}{2}\right)^2$

$= \dfrac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\dfrac{2}{\sqrt{3}}\left(t-\dfrac{1}{2}\right)\right)$.

$u = \dfrac{2}{\sqrt{3}}\left(t-\dfrac{1}{2}\right)$

Assim, $\displaystyle\int_1^e \frac{1}{1+t^3}\,dt = \left[\frac{1}{3}\ln|t+1| - \frac{1}{6}\ln|t^2-t+1| + \frac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)\right)\right]_1^e =$

$= \dfrac{1}{3}\ln|e+1| - \dfrac{1}{6}\ln|e^2-e+1| + \dfrac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\dfrac{2}{\sqrt{3}}\left(e-\dfrac{1}{2}\right)\right) -$

$- \dfrac{1}{3}\ln 2 - \dfrac{1}{6}\ln|1| - \dfrac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\dfrac{2}{\sqrt{3}}\left(\dfrac{1}{2}\right)\right) =$

$= \ln\dfrac{\sqrt[3]{e+1}}{\sqrt[6]{e^2-e+1}\cdot\sqrt[3]{2}} + \dfrac{\sqrt{3}}{3}\left[\mathrm{arctg}\left(\dfrac{2}{\sqrt{3}}\left(e-\dfrac{1}{2}\right)\right) - \dfrac{\pi}{6}\right]$.

Ficha 4

e) $\int_1^e \frac{\sqrt{\ln x}}{x}\,dx$

substituição: $\sqrt{\ln x} = t$

$$\Downarrow$$

$$\ln x = t^2$$

$$x = e^{t^2}$$

- $dx = 2t \cdot e^{t^2} \cdot dt$
- $x = 1 \xrightarrow{\sqrt{\ln x} = t} \sqrt{\ln 1} = t \Leftrightarrow t = 0$
- $x = e \Longrightarrow \sqrt{\ln e} = t \Leftrightarrow t = 1$

$$\int_1^e \frac{\sqrt{\ln x}}{x}\,dx = \int_0^1 \frac{t}{\cancel{e^{t^2}}} \cdot 2t \cdot \cancel{e^{t^2}} \cdot dt = 2\int_0^1 t^2\,dt =$$

$$= 2\left[\frac{t^3}{3}\right]_0^1 = 2 \times \frac{1}{3} = \frac{2}{3}.$$

f) $\int_0^3 \sqrt{9-x^2}\,dx$

substituição: $x = 3\,\text{sen}\,t$

- $dx = 3\cos t \cdot dt$
- $x = 0 \xrightarrow{x = 3\,\text{sen}\,t} 0 = 3\,\text{sen}\,t \Leftrightarrow t = 0$
- $x = 3 \Longrightarrow 3 = 3 \cdot \text{sen}\,t \Leftrightarrow t = \frac{\pi}{2}$

$$\int_0^3 \sqrt{9-x^2}\,dx = \int_0^{\pi/2} \sqrt{9-9\,\text{sen}^2 t} \cdot 3\cos t \cdot dt =$$

$$= \int_0^{\pi/2} \sqrt{9(1-\text{sen}^2 t)} \cdot 3\cos t \cdot dt = \int_0^{\pi/2} 3 \cdot \sqrt{\cos^2 t} \cdot 3\cos t \cdot dt =$$

$$= 9\int_0^{\pi/2} \cos^2 t \cdot dt = 9\int_0^{\pi/2} \frac{1+\cos(2t)}{2}\,dt =$$

$$= \frac{9}{2}\int_0^{\pi/2} (1+\cos(2t))\,dt = \frac{9}{2}\left[t + \frac{1}{2}\text{sen}(2t)\right]_0^{\pi/2} =$$

$$= \frac{9}{2}\left[\frac{\pi}{2} + \frac{1}{2}\,\text{sen}\,\pi - 0 - \frac{1}{2}\,\text{sen}\,0\right] = \frac{9\pi}{4}.$$

g) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$ substituição: $x = 2\,\text{sen}\,t$.

$\bullet\ dx = 2\cos t \,.\, dt$

$\bullet\ x=0 \xRightarrow{x=2\,\text{sen}\,t} 0 = 2\,\text{sen}\,t \Rightarrow t=0$

$\bullet\ x=1 \Rightarrow 1 = 2\,\text{sen}\,t \Leftrightarrow t = \pi/6$

$$\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\pi/6} \frac{4\,\text{sen}^2 t}{\sqrt{4-4\,\text{sen}^2 t}} \cdot 2\cos t \,.\, dt =$$

$$= \int_0^{\pi/6} \frac{4\,\text{sen}^2 t}{\sqrt{4}\sqrt{1-\text{sen}^2 t}}\, 2\cos t \,.\, dt = \int_0^{\pi/6} \frac{4\,\text{sen}^2 t \,.\, \cancel{\cos t}}{\cancel{\cos t}}\, dt =$$

$$= 4\int_0^{\pi/6} \text{sen}^2 t \,.\, dt = 4\int_0^{\pi/6} \frac{1-\cos(2t)}{2}\,dt = 2\int_0^{\pi/6}\left(1-\cos(2t)\right) =$$

$$= 2\left[t - \frac{1}{2}\,\text{sen}\,(2t)\right]_0^{\pi/6} = 2\left[\frac{\pi}{6} - \frac{1}{2}\,\text{sen}\,\frac{\pi}{3} - 0 - \frac{1}{2}\,\text{sen}\,0\right] = 2\left[\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right]$$

$$= 2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] = \frac{\pi}{3} - \frac{\sqrt{3}}{2}.$$

h) $\displaystyle\int_{3/4}^{4/3} \frac{1}{x^2\sqrt{1+x^2}}\,dx$ Substituição $\boxed{x = \text{sh}\,t}$

$\bullet\ dx = \text{ch}\,t \,.\, dt$

$\bullet\ x = \frac{3}{4} \xRightarrow{x=\text{sh}\,t} \frac{3}{4} = \text{sh}\,t \Leftrightarrow t = \text{argsh}\,\frac{3}{4} \approx 0{,}7$

$\bullet\ x = \frac{4}{3} \xRightarrow{x=\text{sh}\,t} \frac{4}{3} = \text{sh}\,t \Leftrightarrow t = \text{argsh}\,\frac{4}{3} \approx 1{,}1$

h) $$\int_{3/4}^{4/3} \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int_{\operatorname{argsh}\frac{3}{4}}^{\operatorname{argsh}\frac{4}{3}} \frac{1}{sh^2t\,\sqrt{1+sh^2t}}\,.ch\,t.dt =$$

$$= \int_{\operatorname{argsh}\frac{3}{4}}^{\operatorname{argsh}4/3} \frac{1}{sh^2t\,\sqrt{\cancel{ch^2t}}}\,\cancel{cht}.dt = \int_{\operatorname{argsh}3/4}^{\operatorname{argsh}4/3} \frac{1}{sh^2t}\,dt.$$

$$= \Big[-\coth t\Big]_{\operatorname{argsh}3/4}^{\operatorname{argsh}4/3} = -\coth(\underbrace{\operatorname{argsh}4/3}_{x}) + \coth\left(\operatorname{argsh}\frac{3}{4}\right)$$

Da fórmula $ch^2x - sh^2x = 1$ obtemos a fórmula $\boxed{\coth^2 x - 1 = \frac{1}{sh^2x}}$

Pretendo determinar $\coth(\operatorname{argsh}4/3)$. Se representar $\operatorname{argsh}4/3 = x$, tenho $\coth^2 x - 1 = \frac{1}{sh^2 x} \Leftrightarrow \coth^2(\operatorname{argsh}4/3) - 1 = \frac{1}{sh^2(\operatorname{argsh}4/3)} \Leftrightarrow$

$$\coth^2(\operatorname{argsh}4/3) - 1 = \frac{1}{\left(\frac{4}{3}\right)^2} \Leftrightarrow \coth^2(\operatorname{argsh}4/3) = 1 + \left(\frac{3}{4}\right)^2$$

$$\coth^2\left(\operatorname{argsh}\frac{4}{3}\right) = \frac{25}{16} \Rightarrow \coth\left(\operatorname{argsh}\frac{4}{3}\right) = \frac{5}{4}$$ (considerando a restrição do $\coth x$)

Do mesmo modo,

$$\coth^2\left(\operatorname{argsh}\frac{3}{4}\right) = 1 + \frac{1}{sh^2\left(\operatorname{argsh}\frac{3}{4}\right)} = 1 + \frac{16}{9} = \frac{25}{9}$$

$$\Rightarrow \coth\left(\operatorname{argsh}\frac{3}{4}\right) = \frac{5}{3}.$$

Assim, $$\int_{3/4}^{4/3} \frac{1}{x^2\sqrt{1+x^2}}\,dx = -\frac{5}{4} + \frac{5}{3} = \frac{-15+20}{12} = \frac{5}{12}$$ //

i) $\displaystyle\int_0^{3/8} \sqrt{1+4x^2}\, dx$

substituição

$$x = \frac{1}{2}\,\text{sh}\, t$$

$$\circ\ dx = \frac{1}{2}\,\text{ch}\, t\, dt$$

$$\circ\ x=0 \Rightarrow 0 = \frac{1}{2}\,\text{sh}\, t \Leftrightarrow t=0$$

$$\circ\ x = \frac{3}{8} \Rightarrow \frac{3}{8} = \frac{1}{2}\,\text{sh}\, t \Rightarrow \frac{3}{4} = \text{sh}\, t \Leftrightarrow t = \text{argsh}\,\frac{3}{4}$$

Assim,

$$\int_0^{3/8} \sqrt{1+4x^2}\, dx = \int_0^{\text{argsh}\frac{3}{4}} \sqrt{1+4\left(\frac{1}{4}\,\text{sh}^2 t\right)} \cdot \frac{1}{2}\,\text{ch}\, t\, dt =$$

$$= \frac{1}{2}\int_0^{\text{argsh}\, 3/4} \sqrt{1+\text{sh}^2 t}\,.\,\text{ch}\, t\, dt = \frac{1}{2}\int_0^{\text{argsh}\, 3/4} \text{ch}^2 t\,.\,dt =$$

$$= \frac{1}{2}\int_0^{\text{argsh}\, 3/4} \frac{\text{ch}(2t)+1}{2}\, dt = \frac{1}{4}\int_0^{\text{argsh}\, 3/4} (\text{ch}(2t)+1)\, dt =$$

$$= \frac{1}{4}\left[\frac{1}{2}\,\text{sh}(2t) + t\right]_0^{\text{argsh}\, 3/4} = \frac{1}{4}\left[\frac{1}{2}\,\text{sh}\underbrace{\left(2\,\text{argsh}\,\frac{3}{4}\right)} + \text{argsh}\,\frac{3}{4}\right]$$

onde

$$\text{sh}\left(2\,\text{argsh}\,\frac{3}{4}\right) = \text{sh}\left(\text{argsh}\,\frac{3}{4}\right)\cdot \text{ch}\left(\text{argsh}\,\frac{3}{4}\right) = \frac{3}{4}\cdot \underbrace{\text{ch}\left(\text{argsh}\,\frac{3}{4}\right)}.$$

e $\displaystyle \text{ch}^2\left(\text{argsh}\,\frac{3}{4}\right) = 1 + \text{sh}^2\left(\text{argsh}\,\frac{3}{4}\right) = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow$

$$\Rightarrow \text{ch}\left(\text{argsh}\,\frac{3}{4}\right) = \frac{5}{4}.$$

Assim, $\displaystyle\int_0^{3/8} \sqrt{1+4x^2}\, dx = \frac{1}{4}\left[\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{4} + \text{argsh}\,\frac{3}{4}\right] = \frac{1}{4}\left[\frac{15}{32} + \text{argsh}\,\frac{3}{4}\right].$

j) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\cos x}\, dx$

substituição universal (ver caso 10.d) do formulário).

$$\operatorname{tg}\frac{x}{2} = t \Rightarrow \operatorname{sen} x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\Downarrow$$

$$\frac{x}{2} = \operatorname{arctg} t$$

- $dx = \dfrac{2}{1+t^2}\, dt$

- $x = 0 \Rightarrow \operatorname{tg}\frac{x}{2} = t$

  $\operatorname{tg} 0 = t \Rightarrow t = 0$

  $x = \frac{\pi}{2} \Longrightarrow \operatorname{tg}\frac{\pi}{4} = t \Leftrightarrow t = 1.$

$$\int_0^{\pi/2} \frac{\cos x}{1+\cos x}\, dx = \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt =$$

$$= 2\int_0^1 \frac{1-t^2}{(1+t^2+1-t^2)(1+t^2)}\, dt = \cancel{2}\int_0^1 \frac{1-t^2}{\cancel{2}(1+t^2)}\, dt =$$

$$= \int_0^1 \frac{1-t^2}{1+t^2}\, dt$$

$$\begin{array}{r|l} \cancel{-t^2}+1 & t^2+1 \\ \cancel{+t^2}+1 & \overline{-1} \\ \hline 2 & \end{array}$$

$$= \int_0^1 -1 + \frac{2}{1+t^2}\, dt$$

$$= \left[-t + 2\operatorname{arctg} t\right]_0^1 = -1 + 2\operatorname{arctg} 1 - 0 = -1 + 2\frac{\pi}{4} = -1 + \frac{\pi}{2}.$$

k) $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{dx}{2+\cos x}$

substituição universal
(Ver caso 10.d) no formulário)

$$\operatorname{tg}\frac{x}{2} = t \Rightarrow \operatorname{sen} x = \frac{2t}{1+t^2}\ ,\ \cos x = \frac{1-t^2}{1+t^2}$$

$$\Updownarrow$$

$$\frac{x}{2} = \operatorname{arctg} t$$

- $dx = \dfrac{2}{1+t^2}\,dt$

- $x = \dfrac{\pi}{2} \Rightarrow \cancel{\frac{\pi}{4} = \operatorname{arctg}}\ \operatorname{tg}\dfrac{\pi}{4} = t = 1$

- $x = 2\dfrac{\pi}{3} \Rightarrow \operatorname{tg}\left(\dfrac{2\pi}{6}\right) = t = \dfrac{\sqrt{3}}{\cancel{3}}$.

$$\int_{\pi/2}^{\pi/3} \frac{dx}{1+\cos x} = \int_1^{\sqrt{3}/\cancel{3}} \frac{\frac{2}{1+t^2}\,dt}{1+\frac{1-t^2}{1+t^2}} = 2\int_1^{\sqrt{3}} \frac{dt}{1+t^2+1-t^2} =$$

$$= 2\int_1^{\sqrt{3}} \frac{dt}{2} = \left[t\right]_1^{\sqrt{3}} = \sqrt{3} - 1 .$$

15.a) $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx.$

Se f é par, $f(x) = f(-x)$.

Assim, se no 1º integral $\left(\int_{-a}^{0} f(x)\,dx\right)$ fizer a mudança de variável $x = -t$, $\Rightarrow dx = -dt$

$x = -a \Rightarrow t = a$

$x = 0 \Rightarrow t = 0$

e $\displaystyle\int_{-a}^{0} f(x)\,dx = -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt$

e finalmente,

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(t)\,dt + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

b) $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$.

se f é ímpar, $f(-x) = -f(x)$. Se fizer a mesma substituição de variável $(x=-t)$ no 1º integral, obtém-se

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-t)dt = +\int_{a}^{0} f(t)dt = -\int_{0}^{a} f(t)dt.$$

Finalmente,

$$\int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(t)dt + \int_{0}^{a} f(x)dx = 0.$$

16. $F(x) = \int_{0}^{x} f(t)\,dt$.

a) se f é par, mostrar que $F(x) = -F(-x)$.

Tem-se que $F(-x) = \int_{0}^{-x} f(t)\,dt$.

se fizer a mudança de variável $t=-u \Rightarrow dt = -du$

e $t=-x \Rightarrow u=x$

$t=0 \Rightarrow u=0$.

Assim

$$F(-x) = \int_{0}^{-x} f(t)dt = -\int_{0}^{x} f(-u)\,du = -\int_{0}^{x} f(u)\,du = -F(x).$$

pois f é par

Logo F é ímpar.

b) Mostrar que $\left(F(x) = F(-x)\right)$ se f é ímpar.

Do mesmo modo que na alínea a), faz-se a mesma mudança de variável $t=-u \Rightarrow dt=-du$

$t=-x \Rightarrow u=x$

$t=0 \Rightarrow u=0$.

16.b)

$$F(-x) = \int_0^{-x} f(t)\,dt = -\int_0^{x} f(-u)\,du$$

$$= \int_0^{x} f(u)\,du = F(x)$$

pois f é ímpar

Logo F é par.

17. $\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}$ ← Sabendo isto, calcular

$\int_{-a}^{a} \sqrt{a^2-x^2}\,dx$ usando a mudança de variável

$x = at$

- $dx = a\,dt$
- $x = -a \Rightarrow -a = at \Rightarrow t = -1$
- $x = a \Rightarrow a = at \Rightarrow t = 1$.

$$\int_{-a}^{a} \sqrt{a^2-x^2}\,dx = \int_{-1}^{1} \sqrt{a^2-a^2t^2}\,.\,a\,dt = a\int_{-1}^{1} a\sqrt{1-t^2}\,dt =$$

$$= a^2\int_{-1}^{1} \sqrt{1-t^2}\,dt = a^2 \cdot \frac{\pi}{2}$$ porque $\int_{-1}^{1} \sqrt{1-t^2}\,dt = \pi/2$.