Time Value of Money

In general, the relationship between a current or present value (PV) and future value (FV) of a cash flow, where r is the stated discount rate per period and t is the number of compounding periods, is as follows:

$$FV_t = PV(1+r)^t \tag{1}$$

If the number of compounding periods t is very large, that is, $t \to \infty$, we compound the initial cash flow on a continuous basis as follows:

$$FV_t = PVe^{rt} \tag{2}$$

Conversely, present values can be expressed in future value terms, which requires recasting Equation 1 as follows:

$$\begin{aligned} FV_t &= PV(1+r)^t \\ PV &= FV_t\left[\frac{1}{(1+r)^t}\right] \\ PV &= FV_t(1+r)^{-t} \qquad (3) \end{aligned}$$

The continuous time equivalent expression of Equation 3 is as follows:

$$PV_t = FVe^{-rt} \tag{4}$$

The present value (PV) calculation for a discount bond with principal (FV) paid at time t with a market discount rate of r per period is:

$$PV(DiscountBond) = FV_t/(1+r)^t \tag{5}$$

The investor's sole source of return is the difference between the price paid (PV) and full principal (FV) received at maturity. This type of bond is often referred to as a zero-coupon bond given the lack of intermediate interest cash flows, which for bonds are generally referred to as coupons.

Pricing a coupon bond extends the single cash flow calculation for a discount bond to a general formula for calculating a bond's price (PV) given the market discount rate on a coupon date, as follows:

PV(Coupon Bond)

$$= \frac{PMT_1}{(1+r)^1} + \frac{PMT_2}{(1+r)^2} + \cdots + \frac{PMT_N + FV_N}{(1+r)^N} \tag{6}$$

A perpetual bond is a less common type of coupon bond with no stated maturity date. Most perpetual bonds are issued by companies to obtain equity-like financing and often include redemption features. As $N \to \infty$ in Equation 6, we can simplify this to solve for the present value of a perpetuity (or perpetual fixed periodic cash flow without early redemption), where $r > 0$, as follows:

$$PV_{\text{Perpetual Bond}} = \frac{PMT}{r} \tag{7}$$

We may calculate the periodic annuity cash flow (A), which occurs at the end of each respective period, as follows:

$$A = \frac{r(PV)}{1-(1+r)^{-t}} \tag{8}$$

where:

A = periodic cash flow,

r = market interest rate per period,

PV = present value or principal amount of loan or bond, and

t = number of payment periods.

The price of a preferred or common share expected to pay a constant periodic dividend is an infinite series that simplifies to the formula for the present value of a perpetuity shown and is similar to the valuation of a perpetual bond that we encountered earlier. Specifically, the valuation in Equation 7:

$$PV_t = \sum\nolimits_{i=1}^{\infty} \frac{D_t}{(1+r)^i}, \text{ and} \tag{9}$$

$$PV_t = \frac{D_t}{r} \tag{10}$$

If dividends grow at a rate of g per period and are paid at the end of each period, the next dividend (at time $t+1$) may be shown as follows:

$$D_{t+1} = D_t(1+g) \tag{11}$$

or generally in i periods as:

$$D_{t+i} = D_t(1+g)^i \tag{12}$$

If dividend cash flows continue to grow at g indefinitely, then we may rewrite Equation 10 as follows:

$$PV_t = \sum\nolimits_{i=1}^{\infty} \frac{D_t(1+g)^i}{(1+r)^i} \tag{13}$$

which simplifies to:

$$PV_t = \frac{D_t(1+g)}{r-g} = \frac{D_{t+1}}{r-g} \tag{14}$$

where $r - g > 0$.

The example in Exhibit 6 shows a company with an initial higher short-term dividend growth of g_s for the first three years, followed by lower long-term growth (g_l, where $g_s > g_l$) indefinitely thereafter. If we generalize the initial growth phase to n periods followed by indefinite slower growth at g_l, we obtain a modified version of Equation 14 as follows:

$$PV_t = \sum\nolimits_{i=1}^{n} \frac{D_t(1+g_s)^i}{(1+r)^i} + \sum\nolimits_{j=n+1}^{\infty} \frac{D_{t+n}(1+g_l)^j}{(1+r)^j} \tag{15}$$

Note that the second expression in Equation 15 involves constant growth starting in n periods, for which we can substitute the geometric series simplification:

$$PV_t = \sum\nolimits_{i=1}^{n} \frac{D_t(1+g_s)^i}{(1+r)^i} + \frac{E(S_{t+n})}{(1+r)^n} \tag{16}$$

where the stock value of the stock in n periods ($E(S_{t+n})$ is referred to as the terminal value) and is equal to the following:

$$E(S_{t+n}) = \frac{D_{t+n+1}}{r-g_l} \tag{17}$$

We may rearrange Equation 5 from earlier to solve for the implied periodic return earned over the life of the instrument (t periods):

$$r = \sqrt[t]{\frac{FV_t}{PV}} - 1 = \left(\frac{FV_t}{PV}\right)^{\frac{1}{t}} - 1 \tag{18}$$

The YTM assumes an investor expects to receive all promised cash flows through maturity and reinvest any cash received at the same YTM. For coupon bonds, this involves periodic interest payments only, while for level payment instruments such as mortgages, the calculation assumes both interest and amortized principal may be invested at the same rate. Like other internal rates of return, the YTM cannot be solved using an equation, but it may be calculated using iteration with a spreadsheet or calculator, a process that solves for r in Equation 19, as follows:

$$PV(Coupon\ Bond) = \frac{PMT_1}{(1+r)^1} + \frac{PMT_2}{(1+r)^2} + \cdots + \frac{PMT_N + FV_N}{(1+r)^N} \tag{19}$$

where FV equals a bond's principal and N is the number of periods to maturity.

The Microsoft Excel or Google Sheets $YIELD$ function can be used to calculate YTM for fixed-income instruments with periodic interest and full principal payment at maturity:

= YIELD (settlement, maturity, rate, pr, redemption, frequency, [basis])

where:

settlement = settlement date entered using the DATE function;

maturity = maturity date entered using the DATE function;

rate = semi-annual (or periodic) coupon;

pr = price per 100 face value;

redemption = future value at maturity;

frequency = number of coupons per year; and

[basis] = day count convention, 1 through 5 for US bonds (30/360 day count).

Equity Instruments, Implied Return, and Implied Growth

As noted in the discussion of calculating the present value of an equity investment, the price of a share of stock reflects not only the required return but also the growth of cash flows. If we begin with an assumption of constant growth of dividends from Equation 14, we can rearrange the formula as follows:

$$r - g = \frac{D_t(1+g)}{PV_t} = \frac{D_{t+1}}{PV_t} \tag{20}$$

The left-hand side of Equation 20 simply reflects the difference between the required return and the constant growth rate, and the right-hand side is the dividend yield of the stock based on expected dividends over the next period. Thus, the implied return on a stock given its expected dividend yield and growth is given by Equation 21, as follows:

$$r = \frac{D_t(1+g)}{PV_t} + g = \frac{D_{t+1}}{PV_t} + g \tag{21}$$

Simply put, if we assume that a stock's dividend grows at a constant rate in perpetuity, the stock's implied return is equal to its expected dividend yield plus the constant growth rate.

Alternatively, we may be interested in solving for a stock's implied growth rate, and this relation is given by Equation 22:

$$g = \frac{r * PV_t - D_t}{PV_t + D_t} = r - \frac{D_{t+1}}{PV_t} \tag{22}$$

Implied Return and Growth

Price-to-earnings ratios not only are used for individual stocks but also are a val- uation metric for stock indexes, such S&P 500, FTSE 100, or Nikkei 225. Here, the stock index value is divided by a weighted sum of the index constituents' earnings per share. This will be explored in depth later in the curriculum, but we can relate the price-to-earnings ratio to our earlier discussion of relating a stock's price (PV) to expected future cash flows to make some useful observations. First, recall Equation 14:

$$PV_t = \frac{D_t(1+g)}{r-g}$$

We can divide both sides by E_t, earnings per share for period t, to obtain:

$$\frac{PV_t}{E_t} = \frac{\frac{D_t}{E_t} \times (1+g)}{r-g} \tag{23}$$

The left-hand side of Equation 23 is the price-to-earnings ratio, whereas the first term in the numerator on the right is the proportion of earnings distributed to shareholders as dividends known as the **dividend payout ratio**.

In practice, the **forward price-to-earnings ratio** or ratio of its share price to an estimate of its next period (t+1) earnings per share is commonly used. With it, we can simplify the previous equation as follows:

$$\frac{PV_t}{E_{t+1}} = \frac{\frac{D_{t+1}}{E_{t+1}}}{r - g} \tag{24}$$

From Equation 24, we can see that forward price-to-earnings ratio is positively related to higher expected dividend payout ratio and higher expected growth but is negatively related to the required return.

Under the cash flow additivity principle, a risk-neutral investor would be indifferent between strategies 1 and 2 under the following condition:

$$FV_2 = PV_0 \times (1 + r_2)^2 = PV_0 \times (1 + r_1)(1 + F_{1,1}) \tag{25}$$
