

Tide-predicting machine

- The first tide-predicting machine was conceived by Sir William Thomson (who later became Lord Kelvin).
- Thomson had introduced the method of harmonic analysis of tidal patterns in the 1860s
- photograph shows 10-component tide-predicting machine of 1872-3



Fourier Series (revision)

A function $f(t)$ is periodic if there is a number T such that

$$f(t + T) = f(t)$$

The smallest positive value of T is called the *period* of $f(t)$.

(Fundamental) frequency $\nu_0 = \frac{1}{T}$ (Hz)

and corresponding circular frequency $\omega_0 = 2\pi\nu_0 = \frac{2\pi}{T}$ (rad s⁻¹)

Fourier series exploit the orthogonality of a suitable set of basis functions. In the trigonometric form we have the set

$$\{1, \cos \omega_0 t, \sin \omega_0 t, \cos 2\omega_0 t, \sin 2\omega_0 t, \dots, \cos n\omega_0 t, \sin n\omega_0 t, \dots\}$$

For example, using the identity

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B),$$

$$\begin{aligned} \int_0^T \sin m\omega_0 t \sin n\omega_0 t \, dt &= \int_0^T \frac{1}{2} [\cos(m - n)\omega_0 t - \cos(m + n)\omega_0 t] \, dt \\ &= \frac{1}{2} \left[\frac{\sin(m - n)\omega_0 t}{(m - n)\omega_0} - \frac{\sin(m + n)\omega_0 t}{(m + n)\omega_0} \right]_0^T = 0 \end{aligned}$$

unless $m = n$ when,

$$\begin{aligned} \int_0^T \sin n\omega_0 t \sin n\omega_0 t \, dt &= \int_0^T \frac{1}{2} [1 - \cos 2n\omega_0 t] \, dt \\ &= \frac{1}{2} \left[t - \frac{1}{2n\omega_0} \sin 2n\omega_0 t \right]_0^T = \frac{T}{2} \end{aligned}$$

Similarly, using the identity

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B),$$

$$\int_0^T \cos m\omega_0 t \cos n\omega_0 t \, dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

and, using

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B),$$

$$\int_0^T \cos m\omega_0 t \sin n\omega_0 t \, dt = 0$$

Expanding $f(t)$ as a trigonometric Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

The coefficients are given by the Euler formulae

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t \, dt \quad , \quad a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \, dt$$
$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t \, dt$$

Usually the integration is over $0 < t < T$ or $-T/2 < t < T/2$.

Even functions

- A function $f(t)$ is said to be *even* if $f(-t) = f(t)$
 - $f(t)$ has reflection symmetry in the vertical axis.
- The Fourier series of an even function contains no sin terms,
 - i.e. $b_n = 0$.
- In addition we can reduce the integration to half the period,

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt = \frac{4}{T} \int_0^{T/2} f(t) dt$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t dt = \frac{4}{T} \int_0^{T/2} f(t) \cos n\omega_0 t dt$$

Odd functions

- A function $f(t)$ is said to be *odd* if $f(-t) = -f(t)$
 - $f(t)$ has 180° rotation symmetry about the origin.
- The Fourier series of an odd function contains no constant or cos terms,
 - i.e. $a_0, a_n = 0$.
- In addition we can reduce the integration to half the period,

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t \, dt = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega_0 t \, dt$$

Example: full-wave rectifier

Find the Fourier series of the full-wave rectified sine wave, $f(t) = |\sin t|$.

Even function, $T = \pi$, and hence $\omega_0 = 2$.

$$a_0 = \frac{4}{\pi} \int_0^{\pi/2} \sin t \, dt = \frac{4}{\pi}$$

$$a_n = \frac{4}{\pi} \int_0^{\pi/2} \sin t \cos 2nt \, dt = -\frac{4}{\pi(4n^2 - 1)}$$

$$|\sin t| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2 - 1} = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \dots \right)$$

This Fourier Series example also calculated using computer algebra on Moodle.

Parseval's theorem

Instantaneous power dissipated in a resistor R

$$p(t) = |v(t)|^2/R = |i(t)|^2 R \propto |f(t)|^2$$

Average power by integrating over one period (mean square)

$$\begin{aligned}\overline{P} &= \frac{1}{T} \int_{t_0}^{t_0+T} |f(t)|^2 dt \\ &= \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\end{aligned}$$

exploiting the orthogonality of the basis functions.

In our rectified example

$$\begin{aligned}\frac{1}{\pi} \int_0^{\pi} \sin^2 t \, dt &= \frac{1}{2} \\ &= \frac{4}{\pi^2} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)^2}\end{aligned}$$

with the DC order giving $4/\pi^2 \simeq 0.405$, i.e. ideal full-wave rectifier is 81% efficient.

Complex Exponential Fourier series

For complex exponential Fourier series use the set of orthogonality basis functions

$$\{\dots e^{-nj\omega_0 t}, \dots, e^{-2j\omega_0 t}, e^{-j\omega_0 t}, 1, e^{j\omega_0 t}, e^{2j\omega_0 t}, \dots, e^{nj\omega_0 t}, \dots\}$$

$$\int_0^T e^{nj\omega_0 t} e^{-mj\omega_0 t} dt = \frac{1}{(n-m)j\omega_0} \left[e^{(n-m)j\omega_0 t} \right]_0^T = 0$$

unless $n = m$, where the integral evaluates to T .

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(jn\omega_0 t)$$

with an integer counter n running from $-\infty$ to ∞ .

Complex Exponential Fourier series

Since we can express $\exp jx = \cos x + j \sin x$,

$$c_0 \equiv \frac{a_0}{2}$$

$$c_n \equiv \frac{1}{2} (a_n - jb_n) = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \exp(-jn\omega_0 t) dt$$

$$c_{-n} = c_n^* \equiv \frac{1}{2} (a_n + jb_n)$$

assuming $f(t)$ is real.

■ for *even* functions

☐ c_n are real

☐ $c_{-n} = c_n$

■ for *odd* functions

☐ c_n are imaginary

☐ $c_{-n} = -c_n, c_0 = 0$

assuming $f(t)$ is real.

Parseval's theorem for exponential Fourier series

$$P = \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Example: find the exponential Fourier series for the square wave

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

- YACRS (yet another classroom response system) is a tool that allows students to use their smartphones, tablets or laptops to answer questions during a class.
- Either use your data connection, or the University's wi-fi by following instructions at:
 - ☐ <http://www.gla.ac.uk/myglasgow/it/eduroam>
- Connect your web browser to
 - ☐ <https://classresponse.gla.ac.uk>
- Login to YACRS with your GUID (student number + initial)
- Join the *session* by typing the session number indicated by the teacher (number shown on-screen beside YACRS control box).
- Once you have joined the session, it will show that there is no active question, or a question to answer.

3. This function has symmetry

- A** even
- B** odd
- C** neither even nor odd
- D** both even and odd

4. The period of this function is

- A** 2
- B** 1
- C** π
- D** 2π