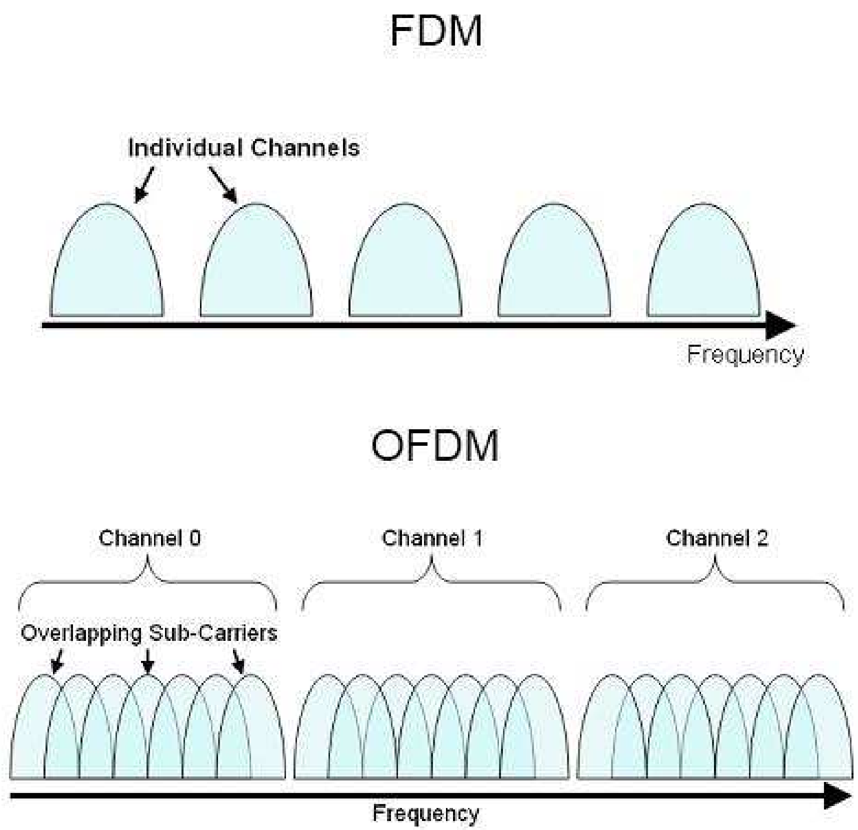
Frequency Division Multiplexing

- Many stations/channels broadcast simultaneously by each having a different carrier frequencies and bandwidths that don't overlap.



Discrete Fourier Transform

Let a sequence of N equally spaced samples over the interval $(0, NT)$ be represented by

$$x(kT) = x(0), x(T), x(2T), \ldots, x[(N-1)T]$$

The *discrete Fourier transform* is defined as the sequence of N (complex-valued) samples given by,

$$X(n\Omega) = \sum_{k=0}^{N-1} x(kT)\mathrm{e}^{-\mathrm{j}\Omega T n k}$$

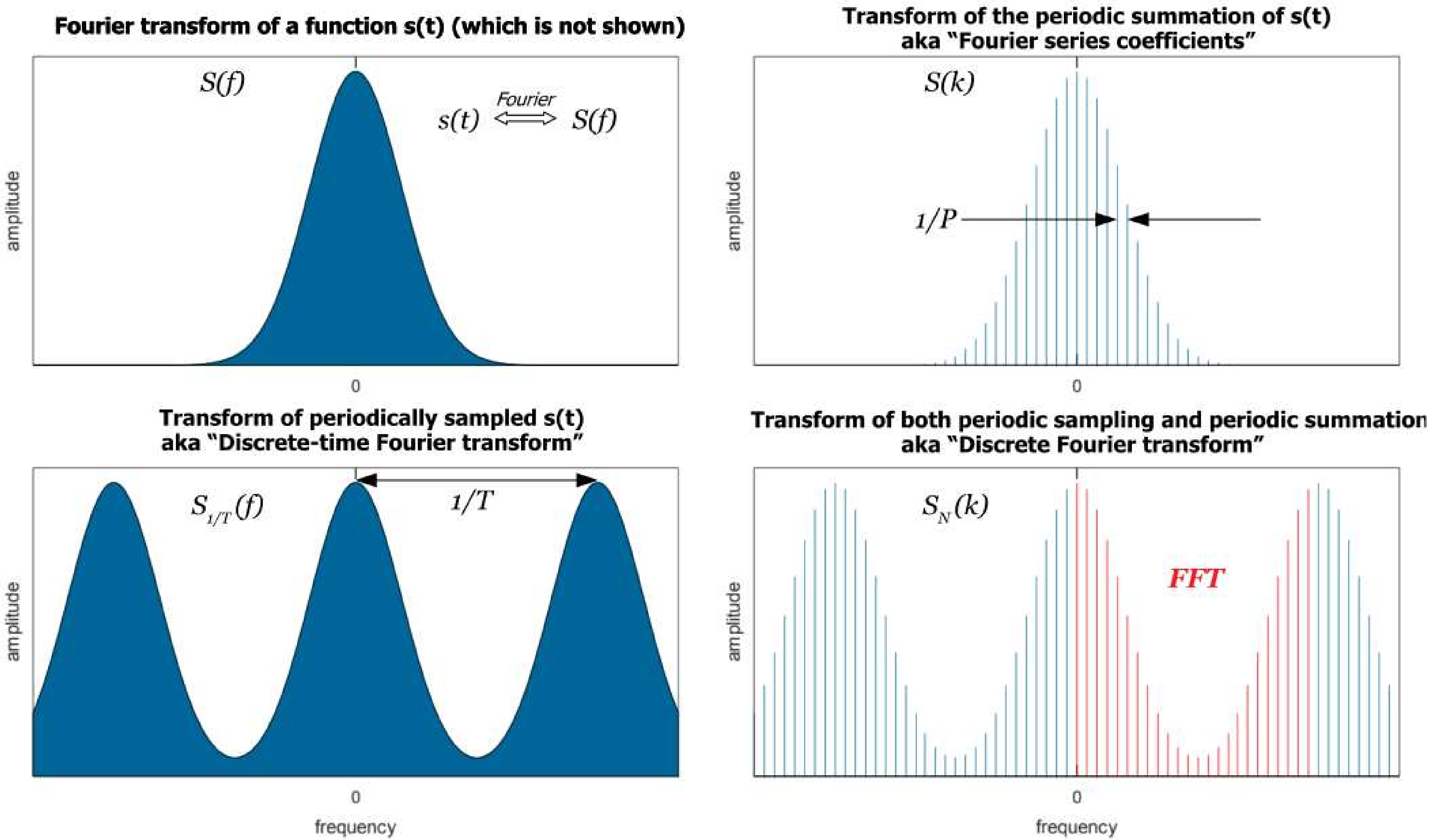
for $n = 0, 1, \ldots, N-1$ and where $\Omega = 2\pi/(NT)$.

- Think of the DFT as an approximation to the regular Fourier transform with the variable changes $\omega \to n\Omega$, $t \to kT$ and $\mathrm{d}t \to T$ and with a periodic extension.
- In analogy to the continuous case the *inverse discrete Fourier transform* (IDFT) is

$$x(kT) = \frac{1}{N}\sum_{n=0}^{N-1} X(n\Omega)\mathrm{e}^{\mathrm{j}\Omega T kn}$$

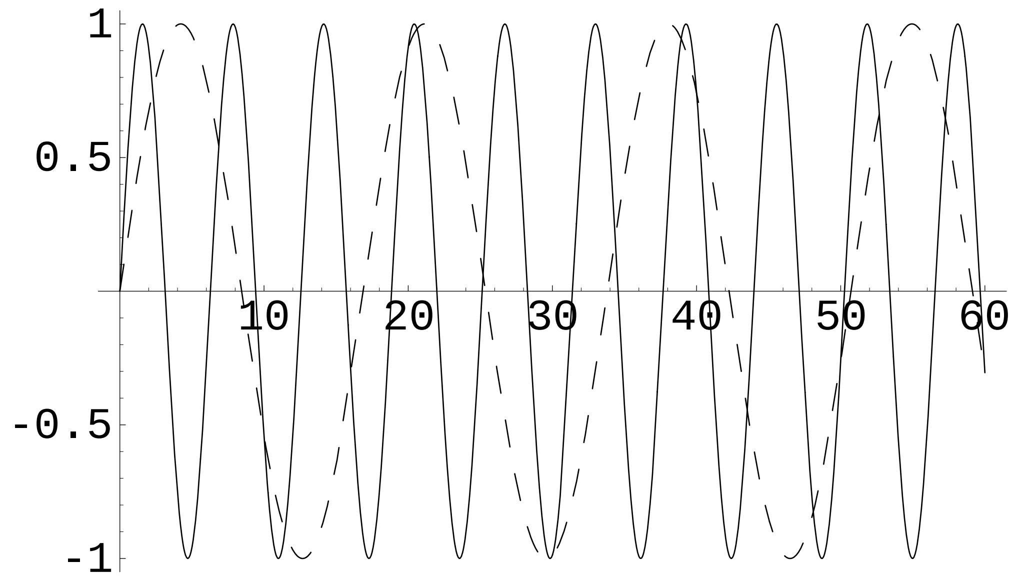
- DFT and IDFT form an **exact transform pair**.

Discrete Fourier Transform



Aliasing

Aliasing occurs if the function is not sampled at a high enough rate. As an example, $\sin t$ has the same sampled values as $\sin(1 - 2\pi/10)t$ with an interval $T = 10$.



sampling theorem states: a real-valued band-limited signal having no spectral components above a frequency of B Hz is determined uniquely by its values at uniform intervals spaced no greater than $1/(2B)$ seconds apart.

The Fast Fourier Transform

- Computation of the DFT requires N^2 multiplications ($0 \leq k < N$ and $0 \leq n < N$) and can take an excessive time for large N.
- The Fast Fourier Transform (FFT) uses the symmetry of the complex exponential to enable the DFT to be computed more efficiently.
- the most commonly used FFT is the Cooley–Tukey algorithm. This is a divide-and-conquer algorithm that recursively breaks down a DFT of any composite size $N = N_1 N_2$ into many smaller DFTs.
- The best known use of the Cooley–Tukey algorithm is to divide the transform into two pieces of size $N/2$ at each step, and is therefore limited to power-of-two sizes.
- Require the number of points $N = 2^r$ to be a power of 2; computation time is proportional to $N \log_2 N$.

To see how the FFT works, let us look at the case $N = 4$.

$$
\begin{aligned}
X_0 &= x_0 \mathrm{e}^0 + x_1 \mathrm{e}^0 + x_2 \mathrm{e}^0 + x_3 \mathrm{e}^0 \\
X_1 &= x_0 \mathrm{e}^0 + x_1 \mathrm{e}^{-\mathrm{j}\pi/2} + x_2 \mathrm{e}^{-\mathrm{j}\pi} + x_3 \mathrm{e}^{-\mathrm{j}3\pi/2} \\
X_2 &= x_0 \mathrm{e}^0 + x_1 \mathrm{e}^{-\mathrm{j}\pi} + x_2 \mathrm{e}^{-\mathrm{j}2\pi} + x_3 \mathrm{e}^{-\mathrm{j}3\pi} \\
X_3 &= x_0 \mathrm{e}^0 + x_1 \mathrm{e}^{-\mathrm{j}3\pi/2} + x_2 \mathrm{e}^{-\mathrm{j}3\pi} + x_3 \mathrm{e}^{-\mathrm{j}9\pi/2}
\end{aligned}
$$

This requires $N^2 = 16$ multiplications.

$N = 4$ FFT

Using the fact that the complex exponential is 2π periodic, we can rewrite this as

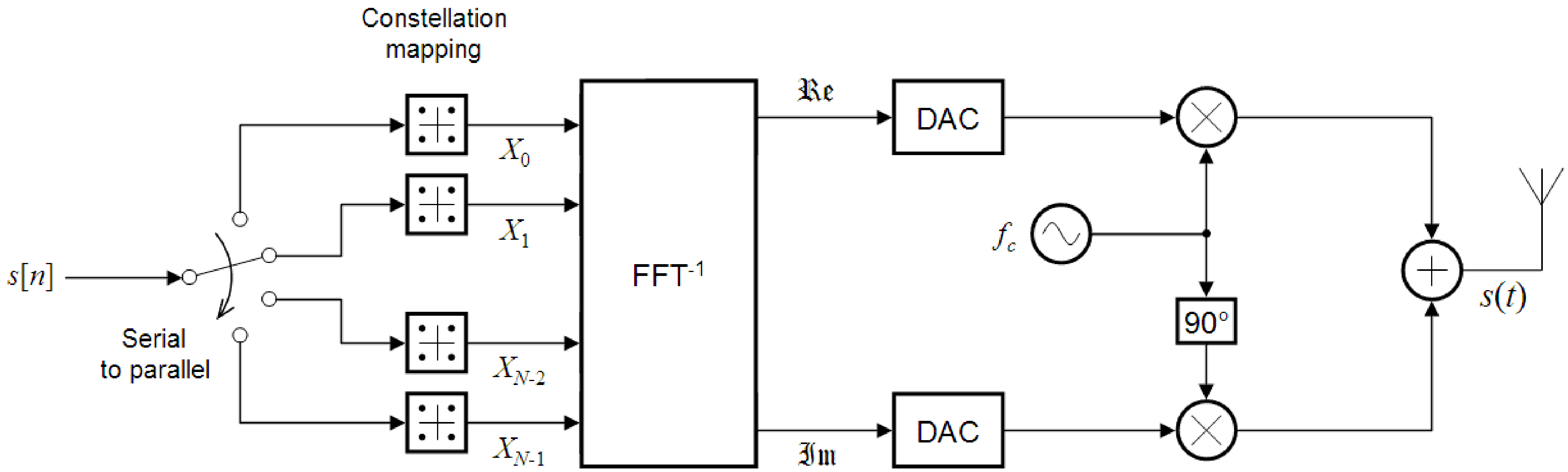
$$
\begin{aligned}
X_0 &= (x_0 + x_2 \mathrm{e}^{0}) + (x_1 + x_3 \mathrm{e}^{0})\mathrm{e}^{0} \\
X_1 &= (x_0 + x_2 \mathrm{e}^{-\mathrm{j}\pi}) + (x_1 + x_3 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}\pi/2} \\
X_2 &= (x_0 + x_2 \mathrm{e}^{0}) + (x_1 + x_3 \mathrm{e}^{0})\mathrm{e}^{-\mathrm{j}\pi} \\
X_3 &= (x_0 + x_2 \mathrm{e}^{-\mathrm{j}\pi}) + (x_1 + x_3 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}3\pi/2}
\end{aligned}
$$

By computing intermediate quantities: $(x_0 + x_2 \mathrm{e}^{0})$, $(x_0 + x_2 \mathrm{e}^{-\mathrm{j}\pi})$, $(x_1 + x_3 \mathrm{e}^{0})$ and $(x_1 + x_3 \mathrm{e}^{-\mathrm{j}\pi})$ we cut down the required total multiplications to $N \log_2 N = 8$.

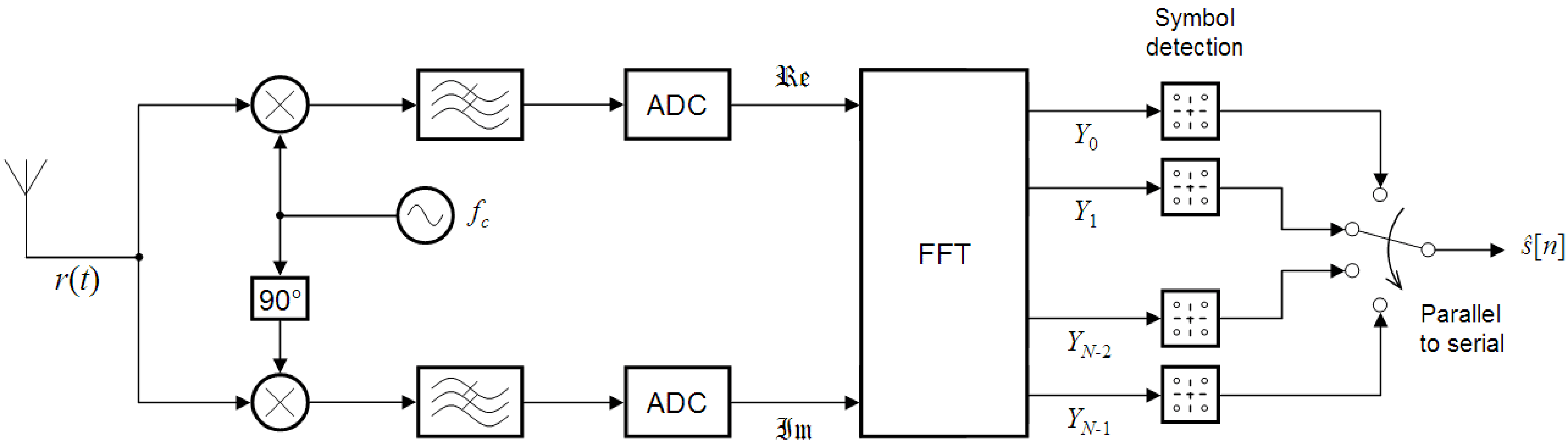
$N = 8$ FFT

$$
\begin{aligned}
X_0 &= [(x_0 + x_4 \mathrm{e}^{0}) + (x_2 + x_6 \mathrm{e}^{0})\mathrm{e}^{0}] + [(x_1 + x_5 \mathrm{e}^{0}) + (x_3 + x_7 \mathrm{e}^{0})\mathrm{e}^{0}]\mathrm{e}^{0} \\
X_1 &= [(x_0 + x_4 \mathrm{e}^{-\mathrm{j}\pi}) + (x_2 + x_6 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}\pi/2}] + [(x_1 + x_5 \mathrm{e}^{-\mathrm{j}\pi}) + (x_3 + x_7 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}\pi/2}]\mathrm{e}^{-\mathrm{j}\pi/4} \\
X_2 &= [(x_0 + x_4 \mathrm{e}^{0}) + (x_2 + x_6 \mathrm{e}^{0})\mathrm{e}^{-\mathrm{j}\pi}] + [(x_1 + x_5 \mathrm{e}^{0}) + (x_3 + x_7 \mathrm{e}^{0})\mathrm{e}^{-\mathrm{j}\pi}]\mathrm{e}^{-\mathrm{j}\pi/2} \\
X_3 &= [(x_0 + x_4 \mathrm{e}^{-\mathrm{j}\pi}) + (x_2 + x_6 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}3\pi/2}] + [(x_1 + x_5 \mathrm{e}^{-\mathrm{j}\pi}) + (x_3 + x_7 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}3\pi/2}]\mathrm{e}^{-\mathrm{j}3\pi/4} \\
X_4 &= [(x_0 + x_4 \mathrm{e}^{0}) + (x_2 + x_6 \mathrm{e}^{0})\mathrm{e}^{0}] + [(x_1 + x_5 \mathrm{e}^{0}) + (x_3 + x_7 \mathrm{e}^{0})\mathrm{e}^{0}]\mathrm{e}^{-\mathrm{j}\pi} \\
X_5 &= [(x_0 + x_4 \mathrm{e}^{-\mathrm{j}\pi}) + (x_2 + x_6 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}\pi/2}] + [(x_1 + x_5 \mathrm{e}^{-\mathrm{j}\pi}) + (x_3 + x_7 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}\pi/2}]\mathrm{e}^{-\mathrm{j}5\pi/4} \\
X_6 &= [(x_0 + x_4 \mathrm{e}^{0}) + (x_2 + x_6 \mathrm{e}^{0})\mathrm{e}^{-\mathrm{j}\pi}] + [(x_1 + x_5 \mathrm{e}^{0}) + (x_3 + x_7 \mathrm{e}^{0})\mathrm{e}^{-\mathrm{j}\pi}]\mathrm{e}^{-\mathrm{j}3\pi/2} \\
X_7 &= [(x_0 + x_4 \mathrm{e}^{-\mathrm{j}\pi}) + (x_2 + x_6 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}3\pi/2}] + [(x_1 + x_5 \mathrm{e}^{-\mathrm{j}\pi}) + (x_3 + x_7 \mathrm{e}^{-\mathrm{j}\pi})\mathrm{e}^{-\mathrm{j}3\pi/2}]\mathrm{e}^{-\mathrm{j}7\pi/4}
\end{aligned}
$$

Quantities in brackets $[\ldots]$ and $(\ldots)$ are used in multiple instances.



use modulation methods, e.g. PSK, QAM, to obtain complex subcarrier frequency coefficients from multiple bit values.



Symbol
detection
$\Re$e
$\Im$m
ADC
ADC
FFT
90°
f_c
$r(t)$
Y_0
Y_1
Y_{N-2}
Y_{N-1}
$\hat{s}[n]$
Parallel
to serial

Baseband Modulation

- subcarriers complex amplitudes are independent of each other and therefore OFDM is not a DSB signal.
- lowest carrier frequency possible is $B/2$ (Nyquist frequency), with OFDM spectrum covering $0 - B$.
- can produce a sampled OFDM baseband signal, but the sample rate has to be doubled to incorporate both **Real** and **Imag** components.
- Time series is given by

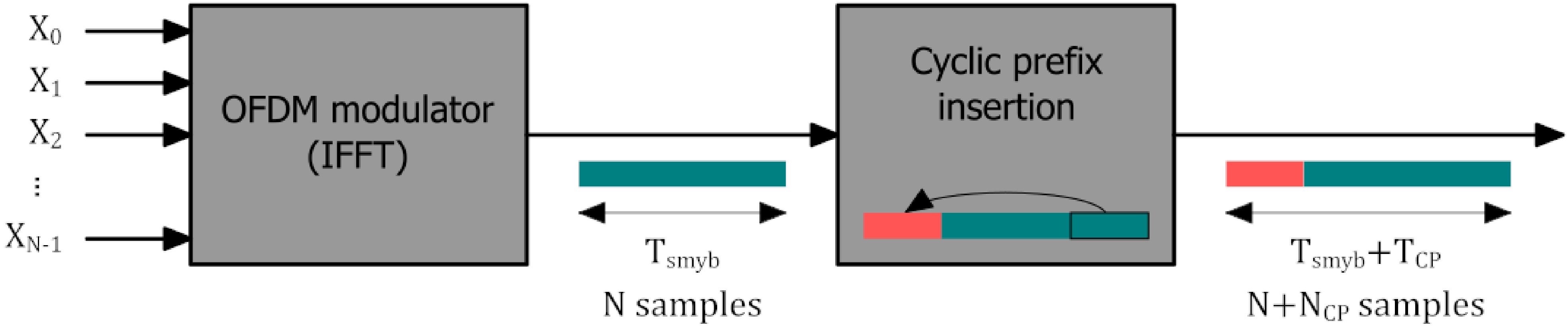
$$\operatorname{Re} x(0), \operatorname{Im} x(0), -\operatorname{Re} x(1), -\operatorname{Im} x(1), \operatorname{Re} x(2), \operatorname{Im} x(2), \ldots, \\ \operatorname{Re} x(N-2), \operatorname{Im} x(N-2), -\operatorname{Re} x(N-1), -\operatorname{Im} x(N-1)$$

Guard Interval/Cyclic Prefix

- In a mobile system a multipath radio channel might have significant paths arriving a few to 10s of microsecond later than the first path. That means the OFDM symbols would start to overlap each other if we sent them one after the other.
- To handle this problem we normally add a gap between symbols to allow time for late copies of one symbol to arrive before the first copies of the next symbol. We call this gap the guard interval, e.g. $g = 1/4$.
- As DFT corresponds to a periodic function, it is straightforward to repeat the last section of the OFDM symbol as a so-called cyclic prefix, i.e. $x(0:N-1) \rightarrow x((1-g)N:N-1), x(0:N-1)$
- Cyclic prefix allows coarse timing synchronisation by maximising Schmidl & Cox metric, using samples T_{symb} apart.

$$P(i) = \sum_{k=0}^{gN-1} x^*(i - gN + k)x(i + (1-g)N + k)$$

Cyclic Prefix Insertion



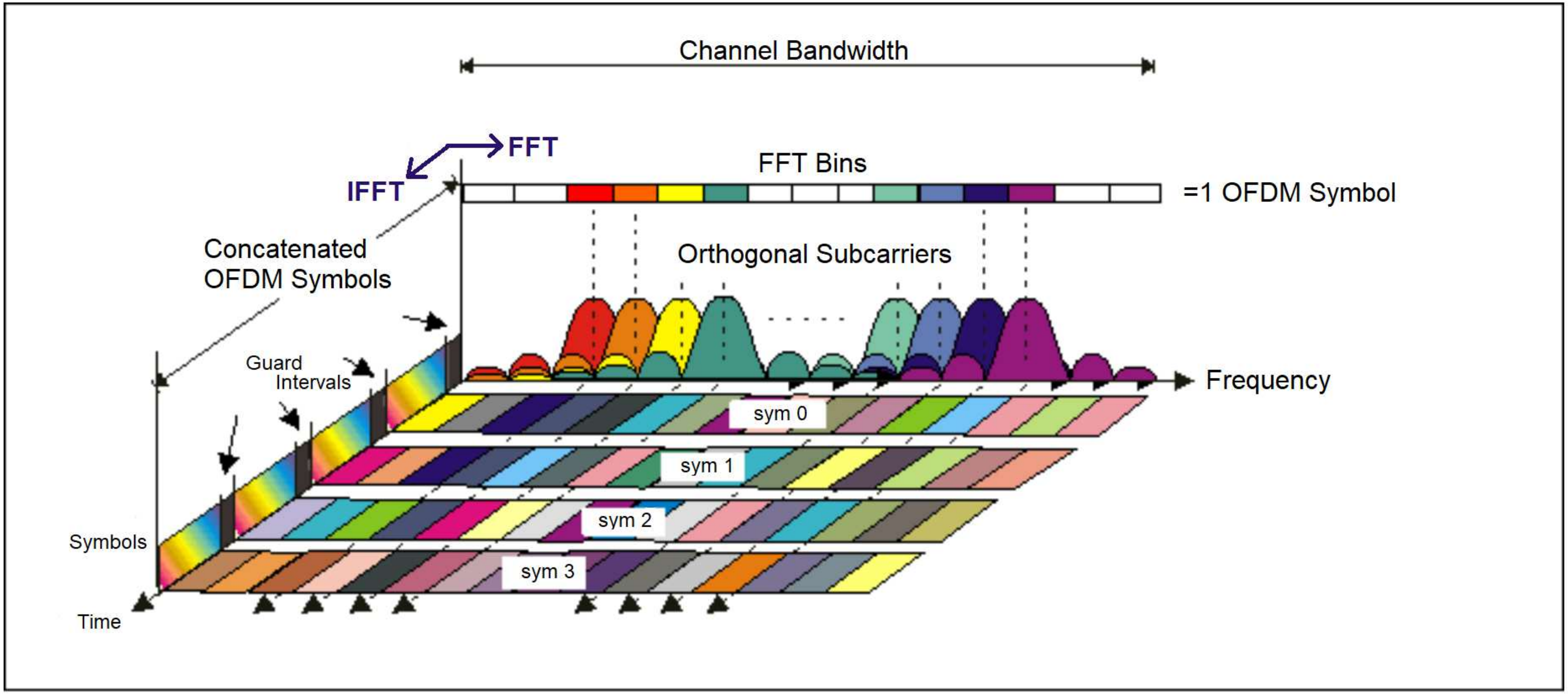
Randomising data for cyclic prefix detection

- Detecting cyclic prefix can be problematic if data is too regular.
- Solution is to use a known random data sequence and XOR with input.
- Data is retrieved at receiver by again XOR with known random sequence.
- Share seed with pseudo-random number generator at transmitter and receiver.
- Alternatively described as **Energy dispersal**, as it avoids energy being concentrated in a few subcarriers.

Pilot Tones

- OFDM symbols often have constant amplitude and phase pilot tones inserted at specific subcarriers.
- Pilot tones can be employed for measurement of the channel conditions, i.e. the equalizer gain and phase shift for each subcarrier.
- Pilot tones also facilitate fine timing synchronisation. Use real-valued X for pilot tones. For correct symbol start point, the FFT will result in real-valued Y for received pilot tones, i.e. zero imaginary parts.
- So for example, look to minimise (at zero) the sum of the square of the imaginary parts of the received pilot tones $\sum_{\text{pilots}} (\operatorname{Im} Y_i)^2$ as a function of start index.

OFDM in frequency and time



Frequency-Time Representative of an OFDM signal

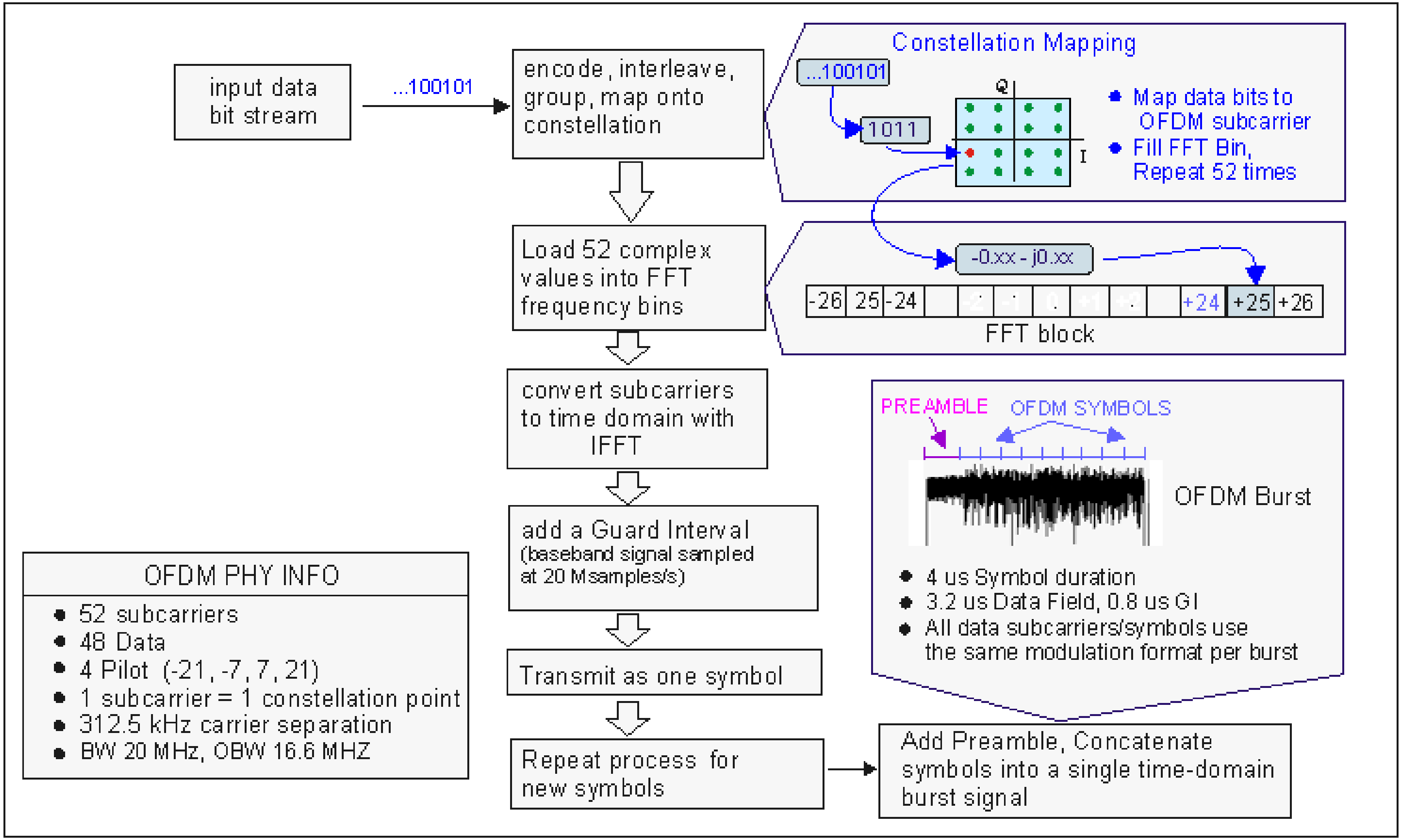
Example: 802.11g OFDM Signal Implementation

- An 802.11g OFDM carrier signal (burst type) is the sum of one or more OFDM symbols each comprised of 52 orthogonal subcarriers, with baseband data on each subcarrier being independently modulated using QAM. This composite baseband signal is used to modulate a main RF carrier.
- the input data bit stream is encoded with convolutional coding and Interleaving. Each data stream is divided into groups of bits and converted into complex numbers $(I + jQ)$ representing the mapped constellation point. Note that the bit-rate will be different depending on the modulation format, a 64-QAM constellation (6 bits at a time) can have a bit rate of 54 Mbps while a QPSK constellation (2 bits at time) may only be 12 Mbps.
- Then 52 bins of the IFFT block are loaded. 48 bins contain the constellation points which are mapped into frequency offset indexes ranging from -26 to +26, skipping the 4 Pilot and zero bins. There are 4 Pilot subcarriers inserted into frequency offset index locations -21, -7, +7, and +21. The zero bin is the Null or DC subcarrier and is not used; it contains a 0 value $(0 + j0)$.

Example: 802.11g OFDM Signal Implementation

- the Inverse FFT is computed, giving a set of complex time-domain samples representing the combined OFDM subcarrier waveform. The samples are clocked out at 20 Msps to create a $3.2\,\mu s$ (20Msps/64) duration OFDM waveform. To complete the OFDM symbol, a $0.8\,\mu s$ duration Guard Interval (GI) is then added to the beginning of the OFDM waveform. This produces a "single" OFDM symbol with a time duration of $4\,\mu s$ in length, $(3.2\,\mu s + 0.8\,\mu s)$.
- To complete the OFDM frame structure, the single OFDM symbols are concatenated together and then appended to a $16\,\mu s$ Preamble (used for synchronization) and a $4\,\mu s$ SIGNAL symbol (provides Rate and Length information).

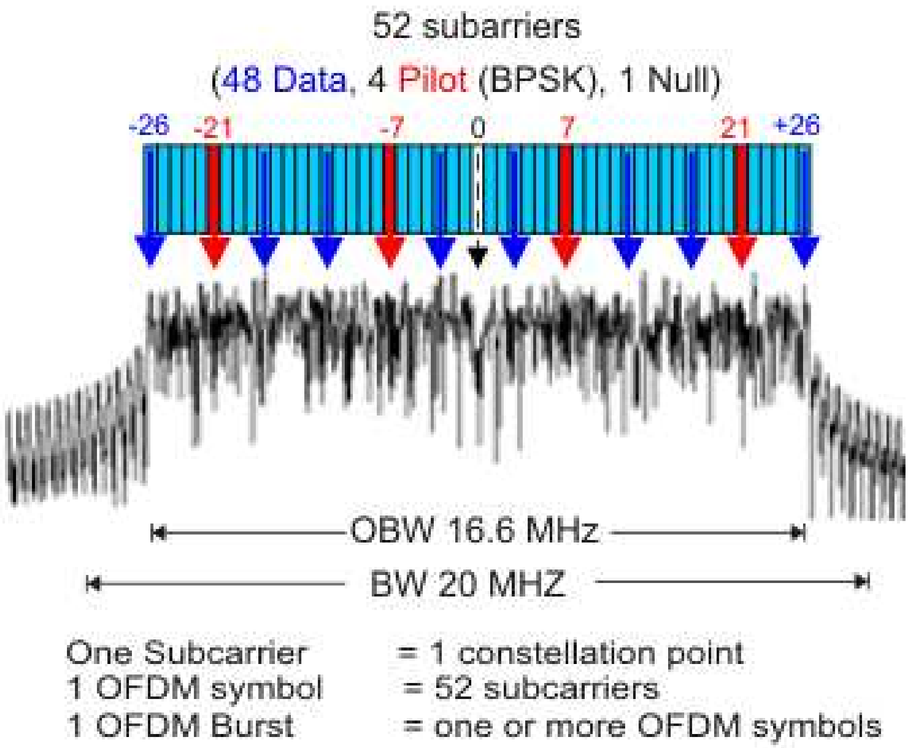
Example: 802.11g OFDM Signal Implementation



802.11a OFDM Signal Generation Process

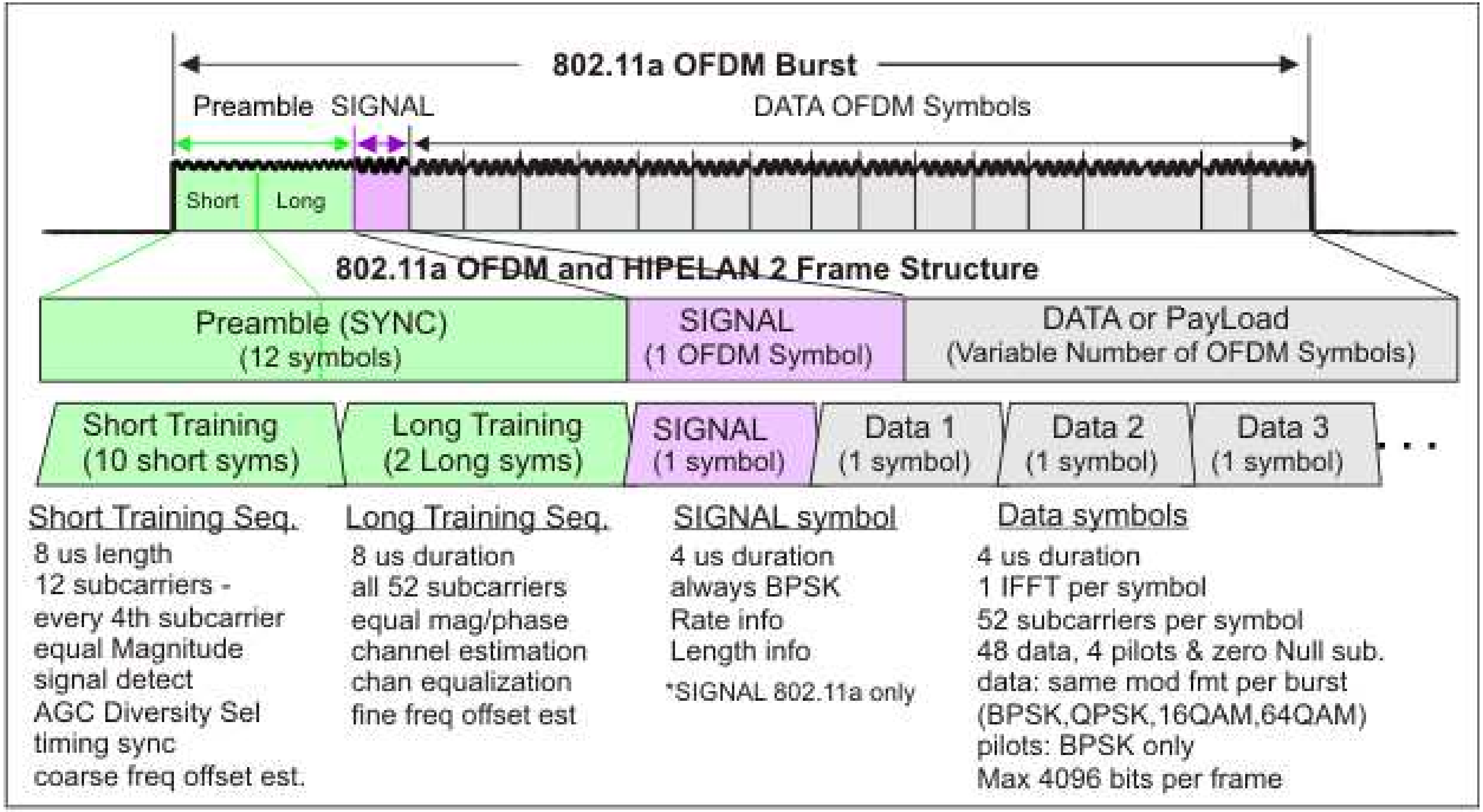
Example: 802.11g OFDM Signal Implementation

802.11a OFDM PHY Parameters	
BW	20 MHZ
OBW	16.6 MHZ
Subcarrer Spacing	312.5 Khz (20MHz/64 Pt FFT)
Information Rate	**6**/9/**12**/18/**24**/36/48/54 Mbits/s
Modulation	BPSK, QPSK, 16QAM, 64QAM
Coding Rate	1/2, 2/3, 3/4
Total Subcarriers	52 (Freq Index -26 to +26)
Data Subcarriers	48
Pilot Subcarriers*	4 (-21, -7, +7, +21) *Always BPSK
DC Subcarrier	Null (0 subcarrier)



802.11a OFDM Physical Parameters

Example: 802.11g OFDM Signal Implementation



802.11a and HIPERLAN/2 Frame Structure

$2.4\,\mathrm{GHz}$ WiFi channels