Tide-predicting machine

- The first tide-predicting machine was conceived by Sir William Thomson (who later became Lord Kelvin).
- Thomson had introduced the method of harmonic analysis of tidal patterns in the 1860s
- photograph shows 10-component tide-predicting machine of 1872-3

Fourier Series (revision)

A function $f(t)$ is periodic if there is a number T such that

$$f(t+T) = f(t)$$

The smallest positive value of T is called the *period* of $f(t)$.
(Fundamental) frequency $\nu_0 = \frac{1}{T}$ (Hz)
and corresponding circular frequency $\omega_0 = 2\pi\nu_0 = \frac{2\pi}{T}$ (rad s^{-1})

Fourier series exploit the orthogonality of a suitable set of basis functions. In the trigonometric form we have the set

$$\{1, \cos\omega_0 t, \sin\omega_0 t, \cos 2\omega_0 t, \sin 2\omega_0 t, \ldots, \cos n\omega_0 t, \sin n\omega_0 t, \ldots\}$$

For example, using the identity
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$,

$$\int_0^T \sin m\omega_0 t \sin n\omega_0 t \, \mathrm{d}t = \int_0^T \frac{1}{2}\left[\cos(m-n)\omega_0 t - \cos(m+n)\omega_0 t\right] \mathrm{d}t$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)\omega_0 t}{(m-n)\omega_0} - \frac{\sin(m+n)\omega_0 t}{(m+n)\omega_0}\right]_0^T = 0$$

unless $m = n$ when,

$$\int_0^T \sin n\omega_0 t \sin n\omega_0 t \, \mathrm{d}t = \int_0^T \frac{1}{2}\left[1 - \cos 2n\omega_0 t\right] \mathrm{d}t$$

$$= \frac{1}{2}\left[t - \frac{1}{2n\omega_0}\sin 2n\omega_0 t\right]_0^T = \frac{T}{2}$$

Similarly, using the identity
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$,

$$\int_0^T \cos m\omega_0 t \cos n\omega_0 t \,\mathrm{d}t = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

and, using
$2\cos A \sin B = \sin(A+B) - \sin(A-B)$,

$$\int_0^T \cos m\omega_0 t \sin n\omega_0 t \,\mathrm{d}t = 0$$

Expanding $f(t)$ as a trigonometric Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

The coefficients are given by the Euler formulae

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t \,\mathrm{d}t \quad , \quad a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t \,\mathrm{d}t$$

Usually the integration is over $0 < t < T$ or $-T/2 < t < T/2$.

Even functions

- A function $f(t)$ is said to be *even* if $f(-t) = f(t)$
  - $f(t)$ has reflection symmetry in the vertical axis.
- The Fourier series of an even function contains no $\sin$ terms,
  - i.e. $b_n = 0$.
- In addition we can reduce the integration to half the period,

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t = \frac{4}{T}\int_{0}^{T/2} f(t)\,\mathrm{d}t$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,\mathrm{d}t = \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega_0 t\,\mathrm{d}t$$

Odd functions

- A function $f(t)$ is said to be *odd* if $f(-t) = -f(t)$
  - $f(t)$ has 180° rotation symmetry about the origin.
- The Fourier series of an odd function contains no constant or $\cos$ terms,
  - i.e. a_0, $a_n = 0$.
- In addition we can reduce the integration to half the period,

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,\mathrm{d}t = \frac{4}{T}\int_{0}^{T/2} f(t)\sin n\omega_0 t\,\mathrm{d}t$$

Example: full-wave rectifier

Find the Fourier series of the full-wave rectified sine wave, $f(t) = |\sin t|$.

Even function, $T = \pi$, and hence $\omega_0 = 2$.

$$a_0 = \frac{4}{\pi}\int_0^{\pi/2} \sin t \,\mathrm{d}t = \frac{4}{\pi}$$

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} \sin t \cos 2nt \,\mathrm{d}t = -\frac{4}{\pi(4n^2-1)}$$

$$|\sin t| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt}{4n^2-1} = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \dots\right)$$

This Fourier Series example also calculated using computer algebra on Moodle.

Parseval's theorem

Instantaneous power dissipated in a resistor R

$$p(t) = |v(t)|^2/R = |i(t)|^2 R \propto |f(t)|^2$$

Average power by integrating over one period (mean square)

$$\begin{aligned}\overline{P} &= \frac{1}{T}\int_{t_0}^{t_0+T} |f(t)|^2 \, \mathrm{d}t \\ &= \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\end{aligned}$$

exploiting the orthogonality of the basis functions.

In our rectified example

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi} \sin^2 t \,\mathrm{d}t &= \frac{1}{2} \\ &= \frac{4}{\pi^2} + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}\end{aligned}$$

with the DC order giving $4/\pi^2 \simeq 0.405$, i.e. ideal full-wave rectifier is 81% efficient.

Complex Exponential Fourier series

For complex exponential Fourier series use the set of orthogonality basis functions

$$\{\ldots \mathrm{e}^{-nj\omega_0 t}, \ldots, \mathrm{e}^{-2j\omega_0 t}, \mathrm{e}^{-j\omega_0 t}, 1, \mathrm{e}^{j\omega_0 t}, \mathrm{e}^{2j\omega_0 t}, \ldots, \mathrm{e}^{nj\omega_0 t}, \ldots\}$$

$$\int_0^T \mathrm{e}^{nj\omega_0 t}\mathrm{e}^{-mj\omega_0 t}\,\mathrm{d}t = \frac{1}{(n-m)j\omega_0}\left[\mathrm{e}^{(n-m)j\omega_0 t}\right]_0^T = 0$$

unless $n = m$, where the integral evaluates to T.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(jn\omega_0 t)$$

with an integer counter n running from $-\infty$ to ∞.

Complex Exponential Fourier series

Since we can express $\exp \mathrm{j}x = \cos x + \mathrm{j} \sin x$,

$$
\begin{aligned}
c_0 &\equiv \frac{a_0}{2} \\
c_n &\equiv \frac{1}{2}\left(a_n - \mathrm{j}b_n\right) = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\exp(-\mathrm{j}n\omega_0 t)\,\mathrm{d}t \\
c_{-n} = c_n^* &\equiv \frac{1}{2}\left(a_n + \mathrm{j}b_n\right)
\end{aligned}
$$

assuming $f(t)$ is real.

- for *even* functions
  - c_n are real
  - $c_{-n} = c_n$
- for *odd* functions
  - c_n are imaginary
  - $c_{-n} = -c_n$, $c_0 = 0$

assuming $f(t)$ is real.

Parseval's theorem for exponential Fourier series

$$P = \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Example: find the exponential Fourier series for the square wave

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

YACRS

- YACRS (yet another classroom response system) is a tool that allows students to use their smartphones, tablets or laptops to answer questions during a class.
- Either use your data connection, or the University's wi-fi by following instructions at:
  - `http://www.gla.ac.uk/myglasgow/it/eduroam`
- Connect your web browser to
  - `https://classresponse.gla.ac.uk`
- Login to YACRS with your GUID (student number + initial)
- Join the *session* by typing the session number indicated by the teacher (number shown on-screen beside YACRS control box).
- Once you have joined the session, it will show that there is no active question, or a question to answer.

YACRS questions

3. This function has symmetry

   A even
   B odd
   C neither even nor odd
   D both even and odd

4. The period of this function is

   A 2
   B 1
   C π
   D 2π