INTRODUCTION

Mathematics in Cryptography: Securing the Digital World

In our increasingly interconnected world, where information flows freely across digital networks, the need for robust security measures has never been more critical. Cryptography, the art and science of securing communications and data, plays a central role in protecting our online interactions. But how does mathematics make this possible? Let's delve into the fascinating realm of mathematics in cryptography.

ABSTRACT

Cryptography is an art of developing techniques of writing messages in a secret way and ensuring the security in communication. Earlier, the use of cryptography was restricted to the safety of information in diplomatic and military areas. With the growth in e-commerce, ATM machines, e-mail and video conferencing through computers, the threat of unauthorized accessibility to the data became a serious concern. So, in order to secure the stored data and to communicate safely the need was to develop economical, efficient and safe cryptography systems. The intent of this paper is to discuss how mathematics plays a significant role in developing various techniques of cryptography.

INTRODUCTION

The term Cryptography, coined from the Greek language, is collaboration of two words 'kryptos'- ‘hidden’ and 'graphein'- ‘to write’. Cryptography came into picture when the use of physical locks was abandoned in communication. The first recorded use of cryptography comes from Julius Caesar, a Roman army commander,around 50 B.C.

Some of the important terms related to cryptography are following:

Encryption: Encryption is part of cryptography used to hide information by converting it into an illegible code. It uses a particular parameter or key to perform the information conversion. Decryption is the reverse of Encryption.

Plaintext: It is the information to be encrypted.

Ciphertext: It is the output of the encryption.

Cipher: Cipher is an algorithm used for encrypting and decrypting messages. It is the set of transformations to convert plaintext into ciphertext. Cipher can be thought of as the virtual lock.

Cryptanalysis: The art of interpreting secret messages and discovering the method used for cryptography is
called cryptanalysis. It exposes the drawbacks in existing cryptography systems. Cryptographers invent hidden
codes and cryptanalysts try to break these codes.

II. ANCIENT CRYPTOGRAPHY TECHNIQUES

Caesar Cipher:

Julius Caesar solved the problem of secure communication with his army. He shifted each letter of his military commands to make the message meaningless. Caesar used three ‘3’ as the key to his cipher system. All the alphabets represent one-to-one correspondence with numbers 0 to 25.Polyalphabetic Cipher:
In the mid of 15th century, Cryptography progressed towards Polyalphabetic Cipher to obtain more security than Caesar Cipher.The objective was to flatten the distribution of letter frequencies which acted as a major breakthrough for Caesar Cipher. Unlike Caesar Cipher, here multiple shifts were used (In Caeser Cipher a

single shift key was used). A sequence of n letters was used as a key. This key was repeated several times for
encrypting the message. The mathematical formulation was as follows: The plaintext message can be considered in the form of blocks each of length n.

One-time Pad:

Polyalphabetic Cipher continued for almost 400 years. In 1882, Frank Miller invented cipher called as One-time Pad. In One-time Pad, a key was selected whose length was same as that of the plaintext message.The shifts in the plaintext never followed a repetitive pattern and the encrypted message had uniform frequency distribution, thereby providing no leakage of the information. The number of possible keys was so this cipher was theoretically unbreakable for large values of n.

III. MODERN CRYPTOGRAPHY TECHNIQUES

Public Key Cryptosystems:

With the advancement of internet in 20th century, the need of cryptography became public. So, the aim was to develop a technique which would not need the two parties to share the secret key.

Diffie-Hellman Key Exchange: In 1976, Whitfield Diffie and Martin Hellman proposed a new technique called as Public key cryptosystem. They devised an amazing trick to produce a one-way function that was easy on the one side and difficult on the other side. The mathematical method that served their purpose was modula arithmetic. Given a generator ‘g’, an exponent ‘e’ and prime modulus ‘p’ it is easy to calculate x (an integer between 0 to p-1).

Elliptic Curve Cryptography:

Elliptic Curve Cryptography (ECC) is a move towards the encryption that uses the nature of elliptic curves infinite fields. ECC utilizes the methods of Diffie-Hellman Key Exchange and RSA Encryption. The only difference is that in ECC, the prime numbers are selected with the help of elliptic curve in a finite field. The advantage of this usage is that key sizes may become smaller while maintaining the same level of security. This provides more efficient cryptography method.

RSA Encryption:

RSA (Rivest-Shamir-Adleman) encryption is a widely used public-key cryptosystem that enables secure communication over unsecured networks like the internet.

Now, the task is to find d. It is easy to multiply two prime numbers and quite hard to factorize a given number into prime factors. To make this encryption possible the work done by Swiss mathematician, Leonard Euler, was used. The trick was to use a function that depends upon the prime factorisation of n. Euler's totient function Φ served the purpose. If n = p × q, then Φ being multiplicative function, Φ(n) = (p-1)(q-1). Euler's theorem was used to connect Φ function to modular exponentiation (1) as follows:

$$m^{\Phi(n)} \equiv 1 \bmod n;\ \text{where m and n are co-prime.}$$

$$m^{k*\Phi(n)} \equiv 1 \bmod n;\ \text{where k is any number.}$$

$$m^{1+k*\Phi(n)} \equiv m \bmod n$$

From (1) and (2), the derived equation was:

$$e * d = 1+k*\Phi(n)$$

$$d = \frac{1+k*\Phi(n)}{e}$$

Now the person who knew the prime factorisation of n could solve easily and further calculate d. Without knowing the prime factorisation of n it was very difficult and time consuming to find the trapdoor information d. To ensure the security of RSA encryption system p and q must be 100 digit long primes or even more than that .

Number Theory

Number theory, a branch of mathematics focused on the properties and relationships of numbers, plays a crucial role in cryptography, the science of secure communication.

Key Concepts in Number Theory for Cryptography:

1.Primes and Composite Numbers: Primes are integers greater than 1 with no divisors other than 1 and themselves. Composite numbers can be factored into primes, and this property is foundational to cryptographic systems like RSA.

2.Modular Arithmetic: Also known as "clock arithmetic," this concept deals with numbers wrapping around after reaching a certain value, the modulus. In cryptography, modular arithmetic enables operations within a fixed range, which is essential for encryption and decryption.

3.Greatest Common Divisor (GCD): The GCD of two integers is the largest number that divides both without leaving a remainder. It's used to check if two numbers are relatively prime, which has implications for key generation and algorithm security.

4.Euler's Totient Function: This function counts the number of integers from 1 to (n - 1) that are relatively prime to (n). It's a key component in RSA, as it's used to determine the number of elements in a finite group, important for generating keys.

5.Modular Exponentiation: This involves raising a number to a power within a modulo. It allows efficient calculations even with large numbers, crucial for encryption algorithms that rely on such operations.

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Cryptography Applications Leveraging Number Theory:

1.Public-Key Cryptography: Systems like RSA and Diffie-Hellman use principles from number theory to create secure encryption and key exchange protocols. RSA relies on the difficulty of factoring large composite numbers, while Diffie-Hellman uses discrete logarithms to establish shared secrets.

2.Digital Signatures: These use number theory to create a unique signature that verifies the authenticity of a message. Algorithms like the Digital Signature Algorithm (DSA) and Elliptic Curve Digital Signature Algorithm (ECDSA) leverage concepts like modular arithmetic and prime numbers.

3.Elliptic Curve Cryptography (ECC):ECC uses elliptic curves over finite fields to create secure cryptographic systems with shorter key lengths, making them efficient for mobile devices and resource-constrained environments. ECC is based on the difficulty of the elliptic curve discrete logarithm problem.

4.Cryptographic Primitives: Hash functions, pseudorandom number generators, and other cryptographic components often use concepts from number theory to ensure security and unpredictability.

Security Considerations: the security of cryptographic systems often relies on hard mathematical problems. As computational power grows, certain key sizes or algorithms might become vulnerable. It's crucial to stay updated with the latest cryptographic research and best practices to ensure secure implementations. Robust number theory-based encryption requires careful key management, padding schemes, and protection against side-channel attacks.Number theory has had a profound impact on cryptography, providing the mathematical backbone for secure communications, digital signatures, and key exchange protocols. The continuous interplay between these two fields drives advances in secure data protection and encryption.

Divisibility and Modular Arithmetic

Section Summary

Division

Division Algorithm

Modular Arithmetic

Division

Definition: If a and b are integers with a ≠ 0, then a divides b if there exists an integer c such that b = ac.

* When a divides b we say that a is a factor or divisor of b
and that b is a multiple of a.

* The notation a | b denotes that a divides b.

* If a | b, then b/a is an integer.

* If a does not divide b, we write a ∤ b.

Example: Determine whether 3 | 7 and whether
3 | 12

Properties of Divisibility :

Theorem 1: Let a, b, and c be integers, where a ≠0.
i. If a | b and a | c, then a | (b + c);
ii. If a | b, then a | bc for all integers c;
iii. If a | b and b | c, then a | c.
Proof:(i) Suppose a | b and a | c, then it follows that there are
integers s and t with b = as and c = at. Hence, b + c = as + at = a(s + t). Hence, a | (b + c)
(Exercises **3** and **4** ask for proofs of parts **(ii)** and **(iii)**.)

Corollary: Let a, b, and c be integers, where a ≠0, such that a | b and a | c, then a | mb + nc whenever m and n are integers.

Division Algorithm

When an integer is divided by a positive integer, there is a quotient and a remainder. This is traditionally called the “Division Algorithm,” but is really a theorem.

Division Algorithm: If a is an integer and d a positive integer, then there are unique integers q and r, with $0 \leq r < d$, such that $a = dq + r$ (proved in Section 5.2).
* d is called the divisor.
* a is called the dividend.
* q is called the quotient.
* r is called the remainder.

Examples:

* What are the quotient and remainder when 101 is divided by 11?
* **Solution:** The quotient when 101 is divided by 11 is 9 = 101 div 11, and the remainder is 2 = 101 mod 11.
* What are the quotient and remainder when −11 is divided by 3?
* **Solution:** The quotient when −11 is divided by 3 is −4 = −11 div 3, and the remainder
is 1 = −11 mod 3

Congruence Relation

Definition: If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides a − b.
* The notation $a \equiv b \pmod{m}$ says that a is congruent to b modulo m.
* We say that $a \equiv b \pmod{m}$ is a congruence and that m is its modulus.
* Two integers are congruent mod m if and only if they have the same remainder when divided by m.
* If a is not congruent to b modulo m, we write $a \not\equiv b \pmod{m}$

Example: Determine whether 17 is congruent to 5 modulo 6 and whether 24 and 14 are congruent modulo 6.

Solution:

* $17 \equiv 5 \pmod 6$ because 6 divides $17 - 5 = 12$.
* $24 \not\equiv 14 \pmod 6$ since $24 - 14 = 10$ is not divisible by 6.

More on Congruences :

Theorem 4: Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

Proof:

* If $a \equiv b \pmod m$, then (by the definition of congruence) $m \mid a - b$. Hence, there is an integer k such that $a - b = km$ and equivalently $a = b + km$.
* Conversely, if there is an integer k such that $a = b + km$, then $km = a - b$. Hence, $m \mid a - b$ and $a \equiv b \pmod m$.

The Relationship between (mod m) and mod m Notations

The use of “mod” in $a \equiv b \pmod m$ and $a \bmod m = b$ are different.
* $a \equiv b \pmod m$ is a relation on the set of integers.
* In $a \bmod m = b$, the notation mod denotes a function.

The relationship between these notations is made clear in this theorem.

Theorem 3: Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod m$ if and only if $a \bmod m = b \bmod m$.

Congruences of Sums and Products

Theorem 5: Let m be a positive integer. If $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then $a + c \equiv b + d \pmod m$ and $ac \equiv bd \pmod m$

Proof:

* Because $a \equiv b \pmod m$ and $c \equiv d \pmod m$, by Theorem 4 there are integers s and t with $b = a + sm$ and $d = c + tm$.
* Therefore,
* $b + d = (a + sm) + (c + tm) = (a + c) + m(s + t)$ and
* $b\,d = (a + sm)(c + tm) = ac + m(at + cs + stm)$.
* Hence, $a + c \equiv b + d \pmod m$ and $ac \equiv bd \pmod m$.

Example: Because $7 \equiv 2 \pmod 5$ and $11 \equiv 1 \pmod 5$, it follows from Theorem 5 that

$18 = 7 + 11 \equiv 2 + 1 = 3 \pmod 5$
$77 = 7 \cdot 11 \equiv 2 \cdot 1 = 2 \pmod 5$

Algebraic Manipulation of Congruences

Multiplying both sides of a valid congruence by an integer preserves validity. If $a \equiv b \pmod m$ holds then $c \cdot a \equiv c \cdot b \pmod m$, where c is any integer, holds by Theorem 5 with d = c.Adding an integer to both sides of a valid congruence preserves validity.If $a \equiv b \pmod m$ holds then $c + a \equiv c + b \pmod m$, where c is any integer, holds by Theorem 5 with d = c.Dividing a congruence by an integer does not always produce a valid congruence.

Example: The congruence $14 \equiv 8 \pmod 6$ holds. But dividing both sides by 2 does not produce a valid congruence since 14/2 = 7 and 8/2 = 4, but $7 \not\equiv 4 \pmod 6$.

Computing the mod m Function of Products and Sums

We use the following corollary to Theorem 5 to compute the remainder of the product or sum of two integers when divided by m from the remainders when each is divided by m.Corollary: Let m be a positive integer and let aand b be integers. Then $(a + b) \pmod m = ((a \bmod m) + (b \bmod m)) \bmod m$ and $ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$.

Arithmetic Modulo m1

Definitions: Let Zm be the set of nonnegative integers less than m: {0,1, …., m−1}
* The operation $+_m$ is defined as $a +_m b = (a + b) \bmod m$. This is addition modulo m.
* The operation $\cdot_m$ is defined as $a \cdot_m b = (a \cdot b) \bmod m$. This is multiplication modulo m.
* Using these operations is said to be doing arithmetic modulo m.

Example: Find $7 +_{11} 9$ and $7 \cdot_{11} 9$.

Solution: Using the definitions above:
* $7 +_{11} 9 = (7 + 9) \bmod 11 = 16 \bmod 11 = 5$
* $7 \cdot_{11} 9 = (7 \cdot 9) \bmod 11 = 63 \bmod 11 = 8$

Arithmetic Modulo m2

The operations +m and ·m satisfy many of the same properties as ordinary addition and multiplication.

* **Closure:** If a and b belong to Zm , then a +m b and a ·m b belong to Zm .
* **Associativity:** If a, b, and c belong to Zm , then (a +m b) +m c = a +m (b +m c) and (a ·m b) ·m c = a ·m (b ·m c).
* **Commutativity:** If a and b belong to Zm , then a +m b = b +m a and a ·m b = b ·m a.
* **Identity elements:** The elements 0 and 1 are identity elements for addition and multiplication modulo m, respectively.
* If a belongs to Zm , then a +m 0 = a and a ·m 1 = a.

Arithmetic Modulo m3

* **Additive inverses:** If a≠ 0 belongs to Zm , then m− a is the additive inverse of modulo m and 0 is its own additive inverse.
* a +m (m− a) = 0 and 0 +m 0 = 0
* **Distributivity:** If a, b, and c belong to Zm , then
* a ·m (b +m c) = (a ·m b) +m (a ·m c) and (a +m b) ·m c = (a ·m c) +m (b ·m c).

Multiplicatative inverses have not been included since they do not always exist. For example, there is no multiplicative inverse of 2 modulo 6.

(optional) Using the terminology of abstract algebra, Zm with +m is a commutative group and Zm with +m and ·m is a commutative ring.

Integer Representations and Algorithms

Section Summary

Integer Representations

- Base b Expansions
- Binary Expansions
- Octal Expansions
- Hexadecimal Expansions

Base Conversion Algorithm Algorithms for Integer Operations

Representations of Integers

In the modern world, we use decimal, or base 10, notation to represent integers. For example when we write 965, we mean $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0$. We can represent numbers using any base b, where b is a positive integer greater than 1. The bases b = 2 (binary), b = 8 (octal) , and b = 16 (hexadecimal) are important for computing and communications The ancient Mayans used base 20 and the ancient Babylonians used base 60.

Base b Representations

We can use positive integer b greater than 1 as a base, because of this theorem:
Theorem 1: Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form:

$$n = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0$$

where k is a nonnegative integer, $a_0, a_1, \ldots a_k$ are nonnegative integers less than b, and $a_k \neq 0$. The a_j , j = 0,…,k are called the baseb digits of the representation. (We will prove this using mathematical induction in Section 5.1.)

The representation of n given in Theorem 1 is called the base b expansion of n and is denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$. We usually omit the subscript 10 for base 10 expansions
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Binary Expansions

Most computers represent integers and do arithmetic with binary (base 2) expansions of integers. In these expansions, the only digits used are 0 and 1.

Example: What is the decimal expansion of the integer that has (1 0101 1111)2 as its binary expansion?

Solution: (1 0101 1111)2 = 1·28 + 0·27 + 1·26 + 0·25 + 1·24 + 1·23 + 1·22 + 1·21 + 1·20 =351.

Example: What is the decimal expansion of the integer that has (11011)2 as its binary expansion?

Solution: (11011)2 = 1 ·24 + 1·23 + 0·22 + 1·21 + 1·20 =27.

Octal Expansions

The octal expansion (base 8) uses the digits {0,1,2,3,4,5,6,7}.

Example: What is the decimal expansion of the number with octal expansion (7016)8 ?

Solution: 7·83 + 0·82 + 1·81 + 6·80 =3598

Example: What is the decimal expansion of the number with octal expansion (111)8 ?

Solution: 1·82 + 1·81 + 1·80 = 64 + 8 + 1 = 73

Hexadecimal Expansions

The hexadecimal expansion needs 16 digits, but our decimal system provides only 10. So letters are used for the additional symbols. The hexadecimal system uses the digits {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}. The letters A through F represent the decimal numbers 10 through 15.

Example: What is the decimal expansion of the number with hexadecimal expansion (2AE0B)16 ?

Solution: $2 \cdot 16^4 + 10 \cdot 16^3 + 14 \cdot 16^2 + 0 \cdot 16^1 + 11 \cdot 16^0 = 175627$

Example: What is the decimal expansion of the number with hexadecimal expansion (E5)16 ?

Solution: $14 \cdot 16^1 + 5 \cdot 16^0 = 224 + 5 = 229$

Base Conversion

To construct the base b expansion of an integer n:

• Divide n by b to obtain a quotient and remainder. $n = bq_0 + a_0$ $0 \leq a_0 \leq b$

• The remainder, a_0 , is the rightmost digit in the base b expansion of n. Next, divide q_0 by b. $q_0 = bq_1 + a_1$ $0 \leq a_1 \leq b$

• The remainder, a_1 , is the second digit from the right in the base b expansion of n.

• Continue by successively dividing the quotients by b, obtaining the additional base b digits as the remainder. The process terminates when the quotient is 0 .

Algorithm for Base b Expansions

procedure base b expansion (n, b: positive integers with b > 1)

q := n

k := 0

while (q ≠ 0)

ak := q mod b

q := q div b

k := k + 1

return (ak-1 ,…, a1 ,a0) {(ak-1 … a1a0)b is base b expansion of n}

q represents the quotient obtained by successive divisions by b, starting with q = n. The digits in the base b expansion are the remainders of the division given by q mod b. The algorithm terminates when q = 0 is reached.

Base Conversion

Example: Find the octal expansion of (12345)10

Solution: Successively dividing by 8 gives:

- 12345 = 8 · 1543 + 1
- 1543 = 8 · 192 + 7
- 192 = 8 · 24 + 0
- 24 = 8 · 3 + 0
- 3 = 8 · 0 + 3

The remainders are the digits from right to left yielding (30071).

Example: Find the binary expansion of (1693)

Solution: Successively dividing by 2 gives:

dividend	quotient	remainder
1693	846	1
846	423	0
423	211	1
211	105	1
105	52	1
52	26	0
26	13	0
13	6	1
6	3	0
3	1	1
1	0	1

remainders in reverse order = 110100111012

Comparison of Hexadecimal, Octal, and Binary Representations

TABLE 1 Hexadecimal, Octal, and Binary Representation of the Integers 0 through 15.

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

Initial 0s are not shown

Each octal digit corresponds to a block of 3 binary digits. Each hexadecimal digit corresponds to a block of 4 binary digits. So, conversion between binary, octal, and hexadecimal is easy.

Conversion Between Binary, Octal, and Hexadecimal Expansions

Example:

Find the octal and hexadecimal expansions of (11 1110 1011 1100)2 .

Solution:

• To convert to octal, we group the digits into blocks of three (011 111 010 111 100)2 , adding initial 0s as needed. The blocks from left to right correspond to the digits 3,7,2,7, and 4. Hence, the solution is (37274)8 .

• To convert to hexadecimal, we group the digits into blocks of four (0011 1110 1011 1100)2 , adding initial 0s as needed. The blocks from left to right correspond to the digits 3,E,B, and C. Hence, the solution is (3EBC)

Binary Addition of Integers

Algorithms for performing operations with integers using their binary expansions are important as computer chips work with binary numbers. Each digit is called a bit.

procedure add (a, b: positive integers)

{the binary expansions of a and b are (an-1 ,an-2 ,…,a0)2 and (bn-1 ,bn-2 ,…,b0)2 , respectively}

```
 c := 0
 for j := 0 to n − 1
d := ⌊(aj + bj + c)/2⌋
 sj := aj + bj + c − 2d
 c := d
sn := c
return (s0 ,s1 ,…, sn ) {the binary expansion of the sum is (sn ,sn-1 ,…,s0 )2 }
```

The number of additions of bits used by the algorithm to add two n-bit integers is O(n).

Binary Multiplication of Integers

Algorithm for computing the product of two n bit integers.

procedure multiply (a, b: positive integers)
{the binary expansions of a and b are (an-1 ,an-2 ,…,a0)2 and (bn−1 ,bn-2 ,…,b0)2 , respectively}
for j := 0 to n − 1

if bj = 1 **then** cj = a shifted j places

else cj := 0

{co ,c1 ,…, cn-1 are the partial products} p := 0

for j := 0 to n − 1

p := p + cj

return p {p is the value of ab}

The number of additions of bits used by the algorithm to multiply two n-bit integers is O(n 2).

Binary Modular Exponentiation

In cryptography, it is important to be able to find b n mod m efficiently, where b, n, and m are large integers. Use the binary expansion of n, n = (ak-1 ,…,a1 ,a0)2 , to compute b n .

$$b^n = b^{a_{k-1}} \cdot 2^{k-1} + \cdots + a_1 \cdot 2 + a_0 = b^{a_{k-1}} \cdot 2^{k-1} \cdots b^{a_1 .2} \cdot b^{a_0}$$

Therefore, to compute b n , we need only compute the values of b, b 2 , (b 2) 2 = b 4 , (b 4) 2 = b 8 , …, and then multiply the terms in this list, where aj = 1.

Example: Compute 3 11 using this method.

Solution: Note that 11 = (1011)2 so that 3 11 = 3 8 3 2 3 1 = ((32) 2) 2 3 2 3 1 = (92) 2 · 9 ·3 = (81)2 · 9 ·3 = 6561 · 9 ·3 =117,147

Binary Modular Exponentiation Algorithm

Algorithm for computing the product of two n bit integers.

procedure modular exponentiation (b: integer, n = (ak-1ak-2…a1a0)2 , m: positive integers)
x := 1

power := b **mod** m

for i := 0 to k − 1

if ai= 1 **then** x := (x· power) **mod** m

power := (power· power) **mod** m

return x {x equals b n mod m }

O((log m) 2 log n) bit operations are used to find b n mod m.

Primes and Greatest Common Divisors

Section Summary

Prime Numbers and their Properties

Conjectures and Open Problems About Primes

Greatest Common Divisors and Least Common Multiples

The Euclidian Algorithm

gcds as Linear Combinations

Primes

Definition: A positive integer p greater than 1 is called prime if the only positive factors of p are 1 and p. A positive integer that is greater than 1 and is not prime is called composite.

Example: The integer 7 is prime because its only positive factors are 1 and 7, but 9 is composite because it is divisible by 3.

The Fundamental Theorem of Arithmetic

Theorem: Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Examples:

- 100 = 2 · 2 · 5 · 5 = 22 · 52
- 641 = 641
- 999 = 3 · 3 · 3 · 37 = 3^3 · 37
- 1024 = 2 · 2 · 2 · 2 · 2 · 2 · 2 · 2 · 2 · 2 = 2^10

The Sieve of Eratosthenes

The Sieve of Eratosthenes can be used to find all primes not exceeding a specified positive integer. For example, begin with the list of integers between 1 and 100.s

a. Delete all the integers, other than 2, divisible by 2.

b. Delete all the integers, other than 3, divisible by 3.

c. Next, delete all the integers, other than 5, divisible by 5.

d. Next, delete all the integers, other than 7, divisible by 7.

e. Since all the remaining integers are not divisible by any of the previous integers, other than 1, the primes are:

{2,3,5,7,11,15,1719,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89, 97}

TABLE 1 The Sieve of Eratosthenes.

Integers divisible by 2 other than 2 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

If an integer n is a composite integer, then it has a prime divisor less than or equal to √n. To see this, note that if n = ab, then a ≤ √n or b ≤√n. Trial division, a very inefficient method of determining if a number n is prime, is to try every integer i ≤√n and see if n is divisible by i.

Infinitude of Primes

Theorem: There are infinitely many primes. (Euclid)

Proof: Assume finitely many primes: p1 , p2 , ….., pn
• Let q = p1p2 ··· pn + 1

• Either q is prime or by the fundamental theorem of arithmetic, it is a product of primes.

• But none of the primes pj divides q since if pj | q, then pj divides q − p1p2 ··· pn = 1

. • Hence, there is a prime not on the list p1 , p2 , ….., pn . It is either q, or if q is composite, it is a prime factor of q. This contradicts the assumption that p1 , p2 , ….., pn are all the primes.

• Consequently, there are infinitely many primes.

This proof was given by Euclid in The Elements. The proof is considered to be one of the most beautiful in all mathematics. It is the first proof in The Book, inspired by the famous mathematician Paul Erdős' imagined collection of perfect proofs maintained by God.

Representing Functions

Definition: Prime numbers of the form $2^p - 1$, where p is prime, are called Mersenne primes.

• $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, and $2^7 - 1 = 127$ are Mersenne primes.

• $2^{11} - 1 = 2047$ is not a Mersenne prime since $2047 = 23 \cdot 89$.

• There is an efficient test for determining if $2^p - 1$ is prime. • The largest known prime numbers are Mersenne primes.

• As of mid 2011, 47 Mersenne primes were known, the largest is $2^{43,112,609} - 1$, which has nearly 13 million decimal digits.

• The Great Internet Mersenne Prime Search (GIMPS) is a distributed computing project to search for new Mersenne Primes.

Distribution of Primes

Mathematicians have been interested in the distribution of prime numbers among the positive integers. In the nineteenth century, the prime number theorem was proved which gives an asymptotic estimate for the number of primes not exceeding x.

Prime Number Theorem: The ratio of the number of primes not exceeding x and $x/\ln x$ approaches 1 as x grows without bound. ($\ln x$ is the natural logarithm of x)

• The theorem tells us that the number of primes not exceeding x, can be approximated by $x/\ln x$.

• The odds that a randomly selected positive integer less than n is prime are approximately $(n/\ln n)/n = 1/\ln n$

Primes and Arithmetic Progressions

Euclid's proof that there are infinitely many primes can be easily adapted to show that there are infinitely many primes in the following 4k + 3, k = 1,2,… In the 19th century G. Lejuenne Dirichlet showed that every arithmetic progression ka + b, k = 1,2, …, where a and b have no common factor greater than 1 contains infinitely many primes. (The proof is beyond the scope of the text.)
Are there long arithmetic progressions made up entirely of primes?

• 5,11, 17, 23, 29 is an arithmetic progression of five primes.

• 199, 409, 619, 829, 1039,1249,1459,1669,1879,2089 is an arithmetic progression of ten primes.

In the 1930s, Paul Erdős conjectured that for every positive integer n greater than 1, there is an arithmetic progression of length n made up entirely of primes. This was proven in 2006, by Ben Green and Terrence Tau

Generating Primes

The problem of generating large primes is of both theoretical and practical interest. We will see that finding large primes with hundreds of digits is important in cryptography.

So far, no useful closed formula that always produces primes has been found. There is no simple function f(n) such that f(n) is prime for all positive integers n. But f(n) = n 2 − n + 41 is prime for all integers 1,2,…, 40. Because of this, we might conjecture that f(n) is prime for all positive integers n. But f(41) = 412 is not prime.

More generally, there is no polynomial with integer coefficients such that f(n) is prime for all positive integers n.Fortunately, we can generate large integers which are almost certainly primes.

Conjectures about Primes

Even though primes have been studied extensively for centuries, many conjectures about them are unresolved, including: Goldbach's Conjecture: Every even integer n, n > 2, is the sum of two primes. It has been verified by computer for all positive even integers up to 1.6 ·1018. The conjecture is believed to be true by most mathematicians. There are infinitely many primes of the form n 2 + 1, where n is a positive integer. But it has been shown that there are infinitely many primes of the form n 2 + 1, where n is a positive integer or the product of at most two primes.

The Twin Prime Conjecture: The twin prime conjecture is that there are infinitely many pairs of twin primes. Twin primes are pairs of primes that differ by 2. Examples are 3 and 5, 5 and 7, 11 and 13, etc. The current world's record for twin primes (as of mid 2011) consists of numbers 65,516,468,355·2333,333 ±1, which have 100,355 decimal digits.

Greatest Common Divisor

Definition: Let a and b be integers, not both zero. The largest integer d such that d | a and also d | b is called the greatest common divisor of a and b. The greatest common divisor of a and b is denoted by gcd(a,b).

One can find greatest common divisors of small numbers by inspection.

Example: What is the greatest common divisor of 24 and 36?

Solution: gcd(24, 36) = 12

Example: What is the greatest common divisor of 17 and 22?

Solution: gcd(17,22) = 1

Definition: The integers a and b are relatively prime if their greatest common divisor is 1.

Example: 17 and 22

Definition: The integers a1, a2, …, an are pairwise relatively prime if gcd(ai, aj) = 1 whenever $1 \leq i<j \leq n$.

Example: Determine whether the integers 10, 17 and 21 are pairwise

relatively prime.

Solution: Because gcd(10,17) = 1, gcd(10,21) = 1, and gcd(17,21) = 1, 10, 17, and 21 are pairwise relatively prime.

Example: Determine whether the integers 10, 19, and 24 are pairwise relatively prime.

Solution: Because gcd(10,24) = 2, 10, 19, and 24 are not pairwise relatively prime.

Finding the Greatest Common Divisor Using Prime Factorizations

Suppose the prime factorizations of a and b are:

$$a = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}, \qquad b = p_1^{b_1} p_2^{b_2} \dots p_n^{b_n},$$

where each exponent is a nonnegative integer, and where all primes occurring in either prime factorization are included in both. Then:

$$\gcd(a,b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \dots p_n^{\min(a_n,bn)},$$

This formula is valid since the integer on the right (of the equals sign) divides both a and b. No larger integer can divide both a and b.

Example: 120 = 2 3 ·3 ·5 500 = 2 2 ·53

gcd(120,500) = 2 min(3,2) ·3min(1,0) ·5min(1,3) = 2 2 ·30 ·51 = 20

Finding the gcd of two positive integers using their prime factorizations is not efficient because there is no efficient algorithm for finding the prime factorization of a positive integer.

Least Common Multiple

Definition: The least common multiple of the positive integers a and b is the smallest positive integer that is divisible by both a and b. It is denoted by lcm(a,b).

The least common multiple can also be computed from the prime factorizations. ‘

$$\operatorname{lcm}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \cdots p_n^{\max(a_n,bn)},$$

This number is divided by both a and b and no smaller number is divided by a and b.

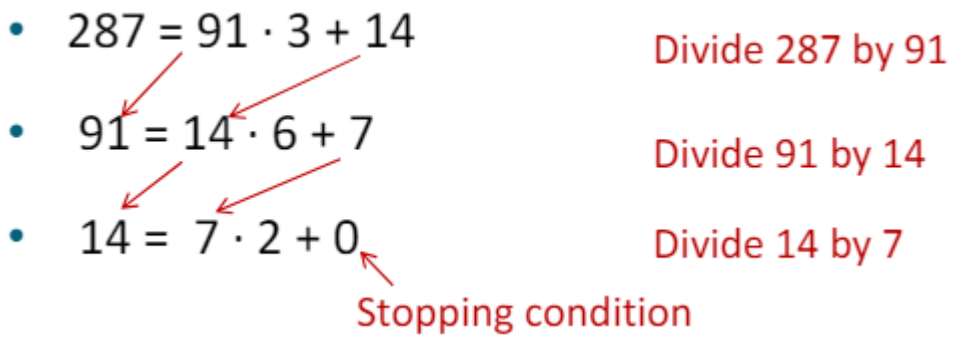
Example: $\operatorname{lcm}(2^3 3^5 7^2, 2^4 3^3) = 2^{\max(3,4)}\, 3^{\max(5,3)}\, 7^{\max(2,0)} = 2^4\, 3^5\, 7^2$

The greatest common divisor and the least common multiple of two integers are related by:

Theorem 5: Let a and b be positive integers. Then ab = gcd(a,b) ·lcm(a,b)

Euclidean Algorithm

The Euclidian algorithm is an efficient method for computing the greatest common divisor of two integers. It is based on the idea that gcd(a,b) is equal to gcd(a,c) when a > b and c is the remainder when a is divided by b.



gcd(287, 91) = gcd(91, 14) = gcd(14, 7) = 7

Euclidean Algorithm

The Euclidean algorithm expressed in pseudocode is:

procedure gcd (a, b: positive integers)

x := a

y := b

while y ≠ 0

r := x mod y

x := y

y := r

return x {gcd(a,b) is x}

we'll see that the time complexity of the algorithm is O (log b), where a > b.

Correctness of Euclidean Algorithm

Lemma 1: Let a = bq + r, where a, b, q, and r are integers. Then gcd(a,b) = gcd(b,r).

Proof:

• Suppose that d divides both a and b. Then d also divides a − bq = r (by Theorem 1 of Section 4.1). Hence, any common divisor of a and b must also be any common divisor of b and r.

• Suppose that d divides both b and r. Then d also divides bq + r = a. Hence, any common divisor of a and b must also be a common divisor of b and r.

• Therefore, gcd(a,b) = gcd(b,r).

Correctness of Euclidean Algorithm2

Suppose that a and b are positive integers with a ≥ b. Let r0 = a and r1 = b. Successive applications of the division algorithm yields:

Eventually, a remainder of zero occurs in the sequence of terms: a = r0 > r1 > r2 > · · · ≥ 0. The sequence can't contain more than a terms.

By Lemma 1 gcd(a,b) = gcd(r0 ,r1) = · · · = gcd(rn-1 ,rn) = gcd(rn , 0) = rn . Hence the greatest common divisor is the last nonzero remainder in the sequence of divisions.

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2 && 0 \le r_2 < r_1, \\
r_1 &= r_2 q_2 + r_3 && 0 \le r_3 < r_2, \\
&\vdots \\
r_{n-2} &= r_{n-1} q_{n-1} + r_2 && 0 \le r_n < r_{n-1}, \\
r_{n-1} &= r_n q_n.
\end{aligned}
$$

gcd's as Linear Combinations

Bézout's Theorem: If a and b are positive integers, then there exist integers s and t such that gcd(a,b) = sa + tb.

Definition: If a and b are positive integers, then integers s and t such that gcd(a,b) = sa + tb are called Bézout coefficients of a and b. The equation gcd(a,b) = sa + tb is called Bézout's identity.

By Bézout's Theorem, the gcd of integers a and b can be expressed in the form sa + tb where s and t are integers. This is a linear combination with integer coefficients of a and b.

• gcd(6,14) = (−2)·6 + 1·14

Example: Express gcd(252,198) = 18 as a linear combination of 252 and 198.

Solution: First use the Euclidean algorithm to show gcd(252,198) = 18

i. 252 = 1·198 + 54

ii. 198 = 3 ·54 + 36

iii. 54 = 1 ·36 + 18

iv. 36 = 2 ·18

• Now working backwards, from iii and i above

• 18 = 54 − 1 ·36
• 36 = 198 − 3 ·54

• Substituting the 2nd equation into the 1st yields:

• 18 = 54 − 1 ·(198 − 3 ·54)= 4 ·54 − 1 ·198

• Substituting 54 = 252 − 1 ·198 (from i)) yields: • 18 = 4 ·(252 − 1 ·198) − 1 ·198 = 4 ·252 − 5 ·198 This method illustrated above is a two pass method. It first uses the Euclidian algorithm to find the gcd and then works backwards to express the gcd as a linear combination of the original two integers. A one pass method, called the extended Euclidean algorithm, is developed in the exercises.

Consequences of Bézout's Theorem*

Lemma 2: If a, b, and c are positive integers such that gcd(a, b) = 1 and a | bc, then a | c.

Proof: Assume gcd(a, b) = 1 and a | bc

• Since gcd(a, b) = 1, by Bézout's Theorem there are integers s and t such that sa + tb = 1.

• Multiplying both sides of the equation by c, yields sac + tbc = c.

• From Theorem 1 of Section 4.1: a | tbc (part ii) and a divides sac + tbc since a | sac and a|tbc (part i) • We conclude a | c, since sac + tbc = c.

Lemma 3: If p is prime and p | a1a2 ···an , then p | ai for some i. (proof uses mathematical induction; see Exercise 64 of Section 5.1) Lemma 3 is crucial in the proof of the uniqueness of prime factorizations

Uniqueness of Prime Factorization*

We will prove that a prime factorization of a positive integer where the primes are in nondecreasing order is unique. (This part of the fundamental theorem of arithmetic. The other part, which asserts that every positive integer has a prime factorization into primes, will be proved in Section 5.2.)

Proof: (by contradiction) Suppose that the positive integer n can be written as a product of primes in two distinct ways:

n = p1p2 ··· ps and n = q1q2 ··· pt .

• Remove all common primes from the factorizations to get

• By Lemma 3, it follows that divides, for some k, contradicting the assumption that and are distinct primes.

• Hence, there can be at most one factorization of n into primes in nondecreasing order.

Dividing Congruences by an Integer

Dividing both sides of a valid congruence by an integer does not always produce a valid congruence . But dividing by an integer relatively prime to the modulus does produce a valid congruence:

Theorem 7: Let m be a positive integer and let a, b, and c be integers. If $ac \equiv bc \pmod{m}$ and gcd(c,m) = 1, then $a \equiv b \pmod{m}$.

Proof: Since $ac \equiv bc \pmod{m}$, $m \mid ac - bc = c(a - b)$ by Lemma 2 and the fact that gcd(c,m) = 1, it follows that $m \mid a - b$. Hence, $a \equiv b \pmod{m}$.

Solving Congruences

Section Summary

Linear Congruences

The Chinese Remainder Theorem

Computer Arithmetic with Large Integers (not currently included in slides, see text)

Fermat's Little Theorem Pseudoprimes

Primitive Roots and Discrete Logarithms

Linear Congruences

Definition: A congruence of the form ax ≡ b (mod m), where m is a positive integer, a and b are integers, and x is a variable, is called a linear congruence. The solutions to a linear congruence ax ≡ b(mod m) are all integers x that satisfy the congruence.

Definition: An integer ā such that āa ≡ 1(mod m) is said to be an inverse of a modulo m.

Example: 5 is an inverse of 3 modulo 7 since 5·3 = 15 ≡ 1(mod 7) One method of solving linear congruences makes use of an inverse ā, if it exists. Although we can not divide both sides of the congruence by a, we can multiply by ā to solve for x

Inverse of a modulo m

The following theorem guarantees that an inverse of a modulo m exists whenever a and m are relatively prime. Two integers a and b are relatively prime when gcd(a,b) = 1. \

Theorem 1: If a and m are relatively prime integers and m > 1, then an inverse of a modulo m exists. Furthermore, this inverse is unique modulo m. (This means that there is a unique positive integer ā less than m that is an inverse of a modulo m and every other inverse of a modulo m is congruent to ā modulo m.)

Proof: Since gcd(a,m) = 1, by Theorem 6 of Section 4.3, there are integers s and t such that sa + tm = 1

. • Hence, sa + tm ≡ 1 (mod m).

• Since tm ≡ 0 (mod m), it follows that sa ≡ 1 (mod m)

• Consequently, s is an inverse of a modulo m.

• The uniqueness of the inverse

Finding Inverses

The Euclidean algorithm and Bézout coefficients gives us a systematic approaches to finding inverses.

Example: Find an inverse of 3 modulo 7.

Solution: Because gcd(3,7) = 1, by Theorem 1, an inverse of 3 modulo 7 exists.

• Using the Euclidian algorithm: 7 = 2·3 + 1.

• From this equation, we get −2·3 + 1·7 = 1, and see that −2 and 1 are Bézout coefficients of 3 and 7

. • Hence, −2 is an inverse of 3 modulo 7.

• Also every integer congruent to −2 modulo 7 is an inverse of 3 modulo 7, i.e., 5, −9, 12, etc.

Example: Find an inverse of 101 modulo 4620.

Solution: First use the Euclidian algorithm to show that gcd(101,4620) = 1.

$42620 = 45 \cdot 101 + 75$

$101 = 1 \cdot 75 + 26$

$75 = 2 \cdot 26 + 23$

$26 = 1 \cdot 23 + 3$

$23 = 7 \cdot 3 + 2$

$3 = 1 \cdot 2 + 1$

$2 = 2 \cdot 1$

Since the last nonzero remainder is 1, gcd(101,4260) = 1

Working Backwards:

$1 = 3 - 1 \cdot 2$

$1 = 3 - 1 \cdot (23 - 7 \cdot 3) = -1 \cdot 23 + 8 \cdot 3$

$1 = -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) = 8 \cdot 26 - 9 \cdot 23$

$1 = 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) = 26 \cdot 26 - 9 \cdot 75$

$1 = 26 \cdot (101 - 1 \cdot 75) - 9 \cdot 75$

$= 26 \cdot 101 - 35 \cdot 75$

$1 = 26 \cdot 101 - 35 \cdot (42620 - 45 \cdot 101)$

$= -35 \cdot 42620 + 1601 \cdot 101$

Bézout coefficients : − 35 and 1601

1601 is an inverse of 101 modulo 42620

Using Inverses to Solve Congruences

We can solve the congruence $ax \equiv b \pmod{m}$ by multiplying both sides by $\bar{a}$.

Example: What are the solutions of the congruence $3x \equiv 4 \pmod{7}$.

Solution: We found that -2 is an inverse of 3 modulo 7 (two slides back). We multiply both sides of the congruence by -2 giving $-2 \cdot 3x \equiv -2 \cdot 4 \pmod{7}$. Because $-6 \equiv 1 \pmod{7}$ and $-8 \equiv 6 \pmod{7}$, it follows that if x is a solution, then $x \equiv -8 \equiv 6 \pmod{7}$ We need to determine if every x with $x \equiv 6 \pmod{7}$ is a solution. Assume that $x \equiv 6 \pmod{7}$.

By Theorem 5 , it follows that $3x \equiv 3 \cdot 6 = 18 \equiv 4 \pmod{7}$ which shows that all such x satisfy the congruence. The solutions are the integers x such that $x \equiv 6 \pmod{7}$, namely, 6,13,20 … and −1, − 8, − 15,…

The Chinese Remainder Theorem

In the first century, the Chinese mathematician Sun-Tsu asked: There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?

This puzzle can be translated into the solution of the system of congruences:

$x \equiv 2 \pmod 3$,

$x \equiv 3 \pmod 5$,

$x \equiv 2 \pmod 7$?

We'll see how the theorem that is known as the Chinese Remainder Theorem can be used to solve Sun-Tsu's problem

Theorem 2: (The Chinese Remainder Theorem) Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than one and $a_1, a_2, \ldots, a_n$ an arbitrary integers. Then the system

$x \equiv a_1 \pmod{m_1}$

$x \equiv a_2 \pmod{m_2}$

$x \equiv a_n \pmod{m_n}$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$

.

(That is, there is a solution x with $0 \leq x < m$ and all other solutions are congruent modulo m to this solution.)

Proof: We'll show that a solution exists by describing a way to construct the solution. Showing that the solution is unique modulo m is

To construct a solution first let $M_k = m/m_k$ for $k = 1,2,\ldots,n$ and $m = m_1 m_2 \cdots m_n$. Since $\gcd(m_k, M_k) = 1$, by Theorem 1, there is an integer y_k, an inverse of M_k modulo m_k, such that

$M_k y_k \equiv 1 \pmod{m_k}$.

Form the sum

$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots + a_n M_n y_n$.

Note that because $M_j \equiv 0 \pmod{m_k}$ whenever $j \neq k$, all terms except the kth term in this sum are congruent to 0 modulo m_k .

Because $M_k y_k \equiv 1 \pmod{m_k}$, we see that $x \equiv a_k M_k y_k \equiv a_k \pmod{m_k}$, for k = 1,2,…,n.

Hence, x is a simultaneous solution to the n congruences.

$x \equiv a_1 \pmod{m_1}$

$x \equiv a_2 \pmod{m_2}$
.
.
.
$x \equiv a_n \pmod{m_n}$

Example: Consider the 3 congruences from Sun-Tsu's

problem: $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 2 \pmod 7$

. • Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_3 = m/5 = 21$, $M_3 = m/7 = 15$.

• We see that

• 2 is an inverse of $M_1 = 35$ modulo 3 since $35 \cdot 2 \equiv 2 \cdot 2 \equiv 1 \pmod 3$

• 1 is an inverse of $M_2 = 21$ modulo 5 since $21 \equiv 1 \pmod 5$

• 1 is an inverse of $M_3 = 15$ modulo 7 since $15 \equiv 1 \pmod 7$

• Hence, $x = a_1M_1y_1 + a_2M_2y_2 + a_3M_3y_3 = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 = 233 \equiv 23 \pmod{105}$

• We have shown that 23 is the smallest positive integer that is a simultaneous solution. Check it!

Back Substitution

We can also solve systems of linear congruences with pairwise relatively prime moduli by rewriting a congruences as an equality using Theorem 4 in Section 4.1, substituting the value for the variable into another congruence, and continuing the process until we have worked through all the congruences. This method is known as back substitution.

Example: Use the method of back substitution to find all integers x such that $x \equiv 1 \pmod 5$, $x \equiv 2 \pmod 6$, and $x \equiv 3 \pmod 7$.

Solution: By Theorem 4 in Section 4.1, the first congruence can be rewritten as $x = 5t + 1$, where t is an integer.

- Substituting into the second congruence yields $5t + 1 \equiv 2 \pmod 6$.
- Solving this tells us that $t \equiv 5 \pmod 6$.
- Using Theorem 4 again gives $t = 6u + 5$ where u is an integer.
- Substituting this back into $x = 5t + 1$, gives $x = 5(6u + 5) + 1 = 30u + 26$.
- Inserting this into the third equation gives $30u + 26 \equiv 3 \pmod 7$.
- Solving this congruence tells us that $u \equiv 6 \pmod 7$
- By Theorem 4, $u = 7v + 6$, where v is an integer.
- Substituting this expression for u into $x = 30u + 26$, tells us that $x = 30(7v + 6) + 26 = 210u + 206$.

Translating this back into a congruence we find the solution $x \equiv 206 \pmod{210}$.

Fermat's Little Theorem

Theorem 3: (Fermat's Little Theorem) If p is prime and a is an integer not divisible by p, then a p-1 ≡ 1 (mod p) Furthermore, for every integer a we have a p ≡ a (mod p) Fermat's little theorem is useful in computing the remainders modulo p of large powers of integers.

Example: Find 7^222 mod 11.

By Fermat's little theorem, we know that 7 10 ≡ 1 (mod 11), and so (7 10) k ≡ 1 (mod 11), for every positive integer k. Therefore,

7 ^222 = 7 ^ 22·10 + 2 = (7^10) ^22 7^2 ≡ (1) 22 ·49 ≡ 5 (mod 11).

Hence, 7 ^ 222 **mod** 11 = 5.

Pseudoprimes

By Fermat's little theorem n > 2 is prime, where 2 n-1 ≡ 1 (mod n). But if this congruence holds, n may not be prime. Composite integers n such that 2 n-1 ≡ 1 (mod n) are called pseudoprimes to the base 2.

Example: The integer 341 is a pseudoprime to the base 2. 341 = 11 · 31 2 340 ≡ 1 (mod 341) (see in Exercise 37) We can replace 2 by any integer b ≥ 2.

Definition: Let b be a positive integer. If n is a composite integer, and b n-1 ≡ 1 (mod n), then n is called a pseudoprime to the base b

Given a positive integer n, such that 2 n-1 ≡ 1 (mod n):

- If n does not satisfy the congruence, it is composite.
- If n does satisfy the congruence, it is either prime or a pseudoprime to the base 2.

Doing similar tests with additional bases b, provides more evidence as to whether n is prime.

Among the positive integers not exceeding a positive real number x, compared to primes, there are relatively few pseudoprimes to the base b.

• For example, among the positive integers less than 1010 there are 455,052,512 primes, but only 14,884 pseudoprimes to the base 2.

Carmichael Numbers

There are composite integers n that pass all tests with bases b such that gcd(b,n) = 1.

Definition: A composite integer n that satisfies the congruence b n-1 ≡ 1 (mod n) for all positive integers b with gcd(b,n) = 1 is called a Carmichael number.

Example: The integer 561 is a Carmichael number To see this:

• 561 is composite, since 561 = 3 · 11 · 13.

• If gcd(b, 561) = 1, then gcd(b, 3) = 1, then gcd(b, 11) = gcd(b, 17) =1.

• Using Fermat's Little Theorem: b 2 ≡ 1 (mod 3), b 10 ≡ 1 (mod 11), b 16 ≡ 1 (mod 17)

$$b^{560} = \left(b^2\right)^{280} \equiv 1 \pmod 3,$$
$$b^{560} = \left(b^{10}\right)^{56} \equiv 1 \pmod{11},$$
$$b^{560} = \left(b^{16}\right)^{35} \equiv 1 \pmod{17}.$$

It follows that b 560 ≡ 1 (mod 561) for all positive integers b with gcd(b,561) = 1. Hence, 561 is a Carmichael number.

Even though there are infinitely many Carmichael numbers, there are other tests (described in the exercises) that form the basis for efficient probabilistic primality testing.

Primitive Roots

Definition: A primitive root modulo a prime p is an integer r in Zp such that every nonzero element of Zp is a power of r.

Example: Since every element of Z11 is a power of 2, 2 is a primitive root of 11.

Powers of 2 modulo 11: 21 = 2, 22 = 4, 23 = 8, 24 = 5, 25 = 10, 26 = 9, 2 7 = 7, 28 = 3, 210 = 2.
Example: Since not all elements of Z11 are powers of 3, 3 is not a primitive root of 11.

Powers of 3 modulo 11: 31 = 3, 32 = 9, 33 = 5, 34 = 4, 35 = 1, and the pattern repeats for higher powers.

Important Fact: There is a primitive root modulo p for every prime number p.

Discrete Logarithms

Suppose p is prime and r is a primitive root modulo p. If a is an integer between 1 and p −1, that is an element of Zp , there is a unique exponent e such that r e = a in Zp , that is, r e mod p = a.

Definition: Suppose that p is prime, r is a primitive root modulo p, and a is an integer between 1 and p −1, inclusive. If r e mod p = a and 1 ≤ e ≤ p − 1, we say that e is the discrete logarithm of a modulo p to the base r and we write logr a = e (where the prime p is understood).

Example 1: We write log2 3 = 8 since the discrete logarithm of 3 modulo 11 to the base 2 is 8 as 28 = 3 modulo 11.

Example 2: We write log2 5 = 4 since the discrete logarithm of 5 modulo 11 to the base 2 is 4 as 24 = 5 modulo 11.

There is no known polynomial time algorithm for computing the discrete logarithm of a modulo p to the base r (when given the prime p, a root r modulo p, and a positive integer a ∈Zp). The problem plays a role in cryptography as will be discussed.

Applications of Congruences

Section Summary

Hashing Functions

Pseudorandom Numbers

Check Digits

Hashing Functions

Definition: A hashing function h assigns memory location h(k) to the record that has k as its key.

• A common hashing function is h(k) = k mod m, where m is the number of memory locations.

• Because this hashing function is onto, all memory locations are possible.

Example: Let h(k) = k mod 111. This hashing function assigns the records of customers with social security numbers as keys to memory locations in the following manner:

h(064212848) = 064212848 mod 111 = 14

h(037149212) = 037149212 mod 111 = 65

h(107405723) = 107405723 mod 111 = 14,

but since location 14 is already occupied, the record is assigned to the next available position, which is 15.

The hashing function is not one-to-one as there are many more possible keys than memory locations. When more than one record is assigned to the same location, we say a collision occurs. Here a collision has been resolved by assigning the record to the first free location.

For collision resolution, we can use a linear probing function:

$h(k,i) = (h(k) + i) \bmod m$, where i runs from 0 to $m - 1$.

There are many other methods of handling with collisions. You may cover these in a later CS course

Pseudorandom Numbers

Randomly chosen numbers are needed for many purposes, including computer simulations.

Pseudorandom numbers are not truly random since they are generated by systematic methods.

The linear congruential method is one commonly used procedure for generating pseudorandom numbers

Four integers are needed: the modulus m, the multiplier a, the increment c, and seed x_0 , with $2 \le a < m$, $0 \le c < m$, $0 \le x_0 < m$.

We generate a sequence of pseudorandom numbers $\{x_n\}$, with $0 \le x_n < m$ for all n, by successively using the recursively defined function

If psuedorandom numbers between 0 and 1 are needed, then the generated numbers are divided by the modulus, x_n /m.

Example: Find the sequence of pseudorandom numbers generated by the linear congruential method with modulus m = 9, multiplier a = 7, increment c = 4, and seed $x_0 = 3$.

Solution: Compute the terms of the sequence by successively using the congruence

$$x_{n+1} = (7x_n + 4) \bmod 9, \text{ with } x_0 = 3.$$

$$x_1 = 7x_0 + 4 \bmod 9 = 7 \cdot 3 + 4 \bmod 9 = 25 \bmod 9 = 7,$$
$$x_2 = 7x_1 + 4 \bmod 9 = 7 \cdot 7 + 4 \bmod 9 = 53 \bmod 9 = 8,$$
$$x_3 = 7x_2 + 4 \bmod 9 = 7 \cdot 8 + 4 \bmod 9 = 60 \bmod 9 = 6,$$
$$x_4 = 7x_3 + 4 \bmod 9 = 7 \cdot 6 + 4 \bmod 9 = 46 \bmod 9 = 1,$$
$$x_5 = 7x_4 + 4 \bmod 9 = 7 \cdot 1 + 4 \bmod 9 = 11 \bmod 9 = 2,$$
$$x_6 = 7x_5 + 4 \bmod 9 = 7 \cdot 2 + 4 \bmod 9 = 18 \bmod 9 = 0,$$
$$x_7 = 7x_6 + 4 \bmod 9 = 7 \cdot 0 + 4 \bmod 9 = 4 \bmod 9 = 4,$$
$$x_8 = 7x_7 + 4 \bmod 9 = 7 \cdot 4 + 4 \bmod 9 = 32 \bmod 9 = 5,$$
$$x_9 = 7x_8 + 4 \bmod 9 = 7 \cdot 5 + 4 \bmod 9 = 39 \bmod 9 = 3.$$

The sequence generated is 3,7,8,6,1,2,0,4,5,3,7,8,6,1,2,0,4,5,3,…

It repeats after generating 9 terms.

Commonly, computers use a linear congruential generator with increment c = 0. This is called a pure multiplicative generator. Such a generator with modulus 2 31 − 1 and multiplier 7 5 = 16,807 generates 231 − 2 numbers before repeating.

Check Digits: UPCs

A common method of detecting errors in strings of digits is to add an extra digit at the end, which is evaluated using a function. If the final digit is not correct, then the string is assumed not to be correct.

Example: Retail products are identified by their Universal Product Codes (UPCs). Usually these have 12 decimal digits, the last one being the check digit. The check digit is determined by the congruence:

3x1 + x2 + 3x3 + x4 + 3x5 + x6 + 3x7 + x8 + 3x9 + x10 + 3x11 + x12 ≡ 0 (mod 10).

a. Suppose that the first 11 digits of the UPC are 79357343104. What is the check digit?

b. Is 041331021641 a valid UPC

Solution:

a. 3·7 + 9 + 3·3 + 5 + 3·7 + 3 + 3·4 + 3 + 3·1 + 0 + 3·4 + x12 ≡ 0 (mod 10) 21 + 9 + 9 + 5 + 21 + 3 + 12+ 3 + 3 + 0 + 12 + x12 ≡ 0 (mod 10) 98 + x12 ≡ 0 (mod 10)
x12 ≡ 2 (mod 10) So, the check digit is 2.

b. 3·0 + 4 + 3·1 + 3 + 3·3 + 1 + 3·0 + 2 + 3·1 + 6 + 3·4 + 1 ≡ 0 (mod 10) 0 + 4 + 3 + 3 + 9 + 1 + 0+ 2 + 3 + 6 + 12 + 1 = 44 ≡ 4 ≠ 0 (mod 10)
Hence, 041331021641 is not a valid UPC.

Check Digits:ISBNs

Books are identified by an International Standard Book Number (ISBN-10), a 10 digit code. The first 9 digits identify the language, the publisher, and the book. The tenth digit is a check digit, which is determined by the following congruence.

$$x_{10} \equiv \sum_{i=1}^{9} i x_i \pmod{11}.$$

The validity of an ISBN-10 number can be evaluated with the equivalent

a. Suppose that the first 9 digits of the ISBN-10 are 007288008. What is the check digit?
b. Is 084930149X a valid ISBN10?

Solution:

a. X10 ≡ 1·0 + 2·0 + 3·7 + 4·2 + 5·8 + 6·8 + 7· 0 + 8·0 + 9·8 (mod 11). X10 ≡ 0 + 0 + 21 + 8 + 40 + 48 + 0 + 0 + 72 (mod 11). X10 ≡ 189 ≡ 2 (mod 11). Hence, X10 = 2.

b. 1·0 + 2·8 + 3·4 + 4·9 + 5·3 + 6·0 + 7· 1 + 8·4 + 9·9 + 10·10 = 0 + 16 + 12 + 36 + 15 + 0 + 7 + 32 + 81 + 100 = 299 ≡ 2 ≠ 0 (mod 11)

Hence, 084930149X is not a valid ISBN-10.

A single error is an error in one digit of an identification number and a transposition error is the accidental interchanging of two digits. Both of these kinds of errors can be detected by the check digit for ISBN-10.

CRYPTOGRAPHY

Section Summary

Classical Cryptography

Cryptosystems

Public Key Cryptography

RSA Cryptosystem

Cryptographic Protocols

Primitive Roots and Discrete Logarithms

Caesar Cipher

Julius Caesar created secret messages by shifting each letter three letters forward in the alphabet (sending the last three letters to the first three letters.) For example, the letter B is replaced by E and the letter X is replaced by A. This process of making a message secret is an example of encryption.

Here is how the encryption process works:

• Replace each letter by an integer from Z26, that is an integer from 0 to 25 representing one less than its position in the alphabet.

• The encryption function is $f(p) = (p + 3) \bmod 26$. It replaces each integer p in the set {0,1,2,…,25} by f(p) in the set {0,1,2,…,25} .

• Replace each integer p by the letter with the position p + 1 in the alphabet.

Example: Encrypt the message “MEET YOU IN THE PARK” using the Caesar cipher.

Solution: 12 4 4 19 24 14 20 8 13 19 7 4 15 0 17 10.

Now replace each of these numbers p by $f(p) = (p + 3) \bmod 26$.

15 7 7 22 1 17 23 11 16 22 10 7 18 3 20 13.

Translating the numbers back to letters produces the encrypted message “PHHW BRX LQ WKH SDUN.”

To recover the original message, use $f^{-1}(p) = (p-3) \bmod 26$. So, each letter in the coded message is shifted back three letters in the alphabet, with the first three letters sent to the last three letters. This process of recovering the original message from the encrypted message is called decryption.

The Caesar cipher is one of a family of ciphers called shift ciphers. Letters can be shifted by an integer k, with 3 being just one possibility. The encryption function is

$f(p) = (p + k) \bmod 26$

and the decryption function is

$f^{-1}(p) = (p-k) \bmod 26$

The integer k is called a key.

Shift Cipher

Example 1: Encrypt the message “STOP GLOBAL WARMING” using the shift cipher with k = 11.

Solution: Replace each letter with the corresponding element of Z.

18 19 14 15 6 11 14 1 0 11 22 0 17 12 8 13 6.

Apply the shift f(p) = (p + 11) **mod** 26, yielding

3 4 25 0 17 22 25 12 11 22 7 11 2 23 19 2417

Translating the numbers back to letters produces the ciphertext

“DEZA RWZMLW HLCXTYR”

Example 2: Decrypt the message “LEWLYPLUJL PZ H NYLHA ALHJOLY” that was encrypted using the shift cipher with k = 7.

Solution: Replace each letter with the corresponding element of Z

11 4 22 11 24 15 11 20 9 1 15 25 7 13 24 11 7 0 0 11 7 9 14 11 24.

Shift each of the numbers by −k = −7 modulo 26, yielding

4 23 15 4 17 8 4 13 2 4 8 18 0 6 17 4 0 19 19 4 0 2 7 4 17.

Translating the numbers back to letters produces the decrypted message

“EXPERIENCE IS A GREAT TEACHER.”

Affine Ciphers

Shift ciphers are a special case of affine ciphers which use functions of the form f(p) = (ap + b) mod 26,

where a and b are integers, chosen so that f is a bijection.

The function is a bijection if and only if gcd(a,26) = 1.

Example: What letter replaces the letter K when the function f(p) = (7p + 3) mod 26 is used for encryption.

Solution: Since 10 represents K, f(10) = (7·10 + 3) mod 26 =21, which is then replaced by V.

To decrypt a message encrypted by a shift cipher, the congruence c ≡ ap + b (mod 26) needs to be solved for p.

• Subtract b from both sides to obtain c− b ≡ ap (mod 26).

• Multiply both sides by the inverse of a modulo 26, which exists since gcd(a,26) = 1.

• ā(c− b) ≡ āap (mod 26), which simplifies to ā(c− b) ≡ p (mod 26).

• p ≡ ā(c− b) (mod 26) is used to determine p in Z

Cryptanalysis of Affine Ciphers

The process of recovering plaintext from ciphertext without knowledge both of the encryption method and the key is known as cryptanalysis or breaking codes. An important tool for cryptanalyzing ciphertext produced with a affine ciphers is the relative frequencies of letters. The nine most common letters in the English texts are E 13%, T 9%, A 8%, O 8%, I 7%, N 7%, S 7%, H 6%, and R 6%.

To analyze ciphertext:
• Find the frequency of the letters in the ciphertext.
• Hypothesize that the most frequent letter is produced by encrypting E.

• If the value of the shift from E to the most frequent letter is k, shift the ciphertext by −k and see if it makes sense.
• If not, try T as a hypothesis and continue.

Example: We intercepted the message “ZNK KGXRE HOXJ MKZY ZNK CUXS” that we know was produced by a shift cipher. Let’s try to cryptanalyze.

Solution: The most common letter in the ciphertext is K. So perhaps the letters were shifted by 6 since this would then map E to K. Shifting the entire message by −6 gives us “THE EARLY BIRD GETS THE WORM.”

Block Ciphers

Ciphers that replace each letter of the alphabet by another letter are called character or monoalphabetic ciphers.

They are vulnerable to cryptanalysis based on letter frequency. Block ciphers avoid this problem, by replacing blocks of letters with other blocks of letters.

A simple type of block cipher is called the transposition cipher. The key is a permutation σ of the set {1,2,…,m}, where m is an integer, that is a one-to-one function from {1,2,…,m} to itself.

To encrypt a message, split the letters into blocks of size m, adding additional letters to fill out the final block. We encrypt $p_1, p_2, \ldots, p_m$ as $c_1, c_2, \ldots, c_m = p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(m)}$.

To decrypt the $c_1, c_2, \ldots, c_m$ transpose the letters using the inverse permutation σ^{-1} .

Example: Using the transposition cipher based on the permutation σ of the set {1,2,3,4} with σ(1) = 3, σ(2) = 1, σ(3) = 4, σ(4) = 2,

a. Encrypt the plaintext PIRATE ATTACK

b. Decrypt the ciphertext message SWUE TRAEOEHS, which was encrypted using the same cipher

Solution:

a. Split into four blocks PIRA TEAT TACK. Apply the permutation σ giving IAPR ETTA AKTC.

b. $\sigma -1$: $\sigma -1(1) = 2$, $\sigma -1(2) = 4$, $\sigma -1(3) = 1$, $\sigma -1(4) = 3$

Apply the permutation $\sigma -1$ giving USE WATER HOSE. Split into words to obtain USE WATER HOSE.

Cryptosystems

Definition: A cryptosystem is a five-tuple (P,C,K,E,D), where

- P is the set of plaintext strings,
- C is the set of ciphertext strings,
- K is the keyspace (set of all possible keys),
- E is the set of encryption functions, and
- D is the set of decryption functions.

The encryption function in E corresponding to the key k is denoted by Ek and the description function in D that decrypts cipher text encrypted using Ek is denoted by Dk . Therefore:

Dk (Ek (p)) = p, for all plaintext strings p.

Example: Describe the family of shift ciphers as a cryptosystem

Solution: Assume the messages are strings consisting of elements in Z

P is the set of strings of elements in Z26,

- C is the set of strings of elements in Z26,
- K = Z26,
- E consists of functions of the form Ek (p) = (p + k) mod 26 , and
- D is the same as E where Dk (p) = (p − k) mod 26

Public Key Cryptography

All classical ciphers, including shift and affine ciphers, are private key cryptosystems. Knowing the encryption key allows one to quickly determine the decryption key.

All parties who wish to communicate using a private key cryptosystem must share the key and keep it a secret.

In public key cryptosystems, first invented in the 1970s, knowing how to encrypt a message does not help one to decrypt the message. Therefore, everyone can have a publicly known encryption key. The only key that needs to be kept secret is the decryption key.

The Rsa Cryptosystem

A public key cryptosystem, now known as the RSA system was introduced in 1976 by three researchers at MIT.

Ronald Rivest (Born 1948)

Adi Shamir (Born 1952)

Leonard Adelman (Born 1945)

It is now known that the method was discovered earlier by Clifford Cocks, working secretly for the UK government.

The public encryption key is (n,e), where n = pq (the modulus) is the product of two large (200 digits) primes p and q, and an exponent e that is relatively prime to (p−1)(q −1). The two large primes can be quickly found using probabilistic primality tests, discussed earlier. But n = pq, with approximately 400 digits, cannot be factored in a reasonable length of time

Rsa Encryption

To encrypt a message using RSA using a key (n,e) :

i. Translate the plaintext message M into sequences of two digit integers representing the letters. Use 00 for A, 01 for B, etc.

ii. Concatenate the two digit integers into strings of digits.

iii. Divide this string into equally sized blocks of 2N digits where 2N is the largest even number 2525…25 with 2N digits that does not exceed n.

iv. The plaintext message M is now a sequence of integers m1 ,m2 ,…,mk .

v. Each block (an integer) is encrypted using the function C = Me mod n

Example: Encrypt the message STOP using the RSA cryptosystem with key(2537,13).

• 2537 = 43· 59,

• p = 43 and q = 59 are primes and gcd(e,(p−1)(q −1)) = gcd(13, 42· 58) = 1.

Solution: Translate the letters in STOP to their numerical equivalents 18 19 14 15.

Divide into blocks of four digits (because 2525 < 2537 < 252525) to obtain 1819 1415.

• Encrypt each block using the mapping C = M13 mod 2537.

• Since 181913 mod 2537 = 2081 and 141513 mod 2537 = 2182, the encrypted message is 2081 2182.

Rsa Decryption

To decrypt a RSA ciphertext message, the decryption key d, an inverse of e modulo (p−1)(q −1) is needed. The inverse exists since gcd(e,(p−1)(q −1)) = gcd(13, 42· 58) = 1.

With the decryption key d, we can decrypt each block with the computation M = C d mod p·q. (see text for full derivation)

RSA works as a public key system since the only known method of finding d is based on a factorization of n into primes. There is currently no known feasible method for factoring large numbers into primes.

Example: The message 0981 0461 is received. What is the decrypted message if it was encrypted using the RSA cipher from the previous example.

Solution: The message was encrypted with n = 43· 59 and exponent 13. An inverse of 13 modulo 42· 58 = 2436 (exercise 2 in Section 4.4) is d = 937.

To decrypt a block C, M = C 937 mod 2537. • Since 0981937 mod 2537 = 0704 and 0461937 mod 2537 = 1115, the decrypted message is 0704 1115. Translating back to English letters, the message is HELP.

Cryptographic Protocols: Key Exchange

Cryptographic protocols are exchanges of messages carried out by two or more parties to achieve a particular security goal.

Key exchange is a protocol by which two parties can exchange a secret key over an insecure channel without having any past shared secret information. Here the DiffeHellman key agreement protocol is described by example.

i. Suppose that Alice and Bob want to share a common key.

ii. Alice and Bob agree to use a prime p and a primitive root a of p.
iii. Alice chooses a secret integer k1 and sends a k1 mod p to Bob.

iv. Bob chooses a secret integer k2 and sends a k2 mod p to Alice.

v. Alice computes (a k2) k1 mod p.

vi. Bob computes (a k1) k2 mod p

At the end of the protocol, Alice and Bob have their shared key (a k2) k1 mod p = (a k1) k2 mod p.

To find the secret information from the public information would require the adversary to find k1 and k2 from a k1 mod p and a k2 mod p respectively. This is an instance of the discrete logarithm problem, considered to be computationally infeasible when p and a are sufficiently large.

Cryptographic Protocols: Digital Signatures

Adding a digital signature to a message is a way of ensuring the recipient that the message came from the purported sender.

Suppose that Alice's RSA public key is (n,e) and her private key is d. Alice encrypts a plain text message x using E(n,e) (x)= x d mod n. She decrypts a ciphertext message y using D(n,e) (y)= y d mod n.

Alice wants to send a message M so that everyone who receives the message knows that it came from her.

1. She translates the message to numerical equivalents and splits into blocks, just as in RSA encryption.

2. She then applies her decryption function D(n,e) to the blocks and sends the results to all intended recipients.

3. The recipients apply Alice's encryption function and the result is the original plain text since E(n,e) (D(n,e) (x))= x.

Everyone who receives the message can then be certain that it came from Alice.

Example: Suppose Alice's RSA cryptosystem is the same as in the earlier example with key(2537,13), 2537 = 43· 59, p = 43 and q = 59 are primes and gcd(e,(p−1)(q −1)) = gcd(13, 42· 58) = 1.

Her decryption key is d = 937.

She wants to send the message "MEET AT NOON" to her friends so that they can be certain that the message is from her.

Solution: Alice translates the message into blocks of digits 1204 0419 0019 1314 1413.

1. She then applies her decryption transformation D(2537,13) (x)= x 937 mod 2537 to each block.

2. She finds (using her laptop, programming skills, and knowledge of discrete mathematics) that 1204937 mod 2537 = 817, 419937 mod 2537 = 555 , 19937 mod 2537 = 1310, 1314937 mod 2537 = 2173, and 1413937 mod 2537 = 1026.

3. She sends 0817 0555 1310 2173 1026.

When one of her friends receive the message, they apply Alice's encryption transformation E(2537,13) to each block. They then obtain the original message which they translate back to English letters.

PREFACE

The research Project on A study of Number theory and its application in cryptography has been prepared with the help of standard books, some useful websites and the knowledge given by my teachers.

Cryptography is the study of encrypted techniques of communication, which only enable the sender and the intended recipient to see their information. The concept comes from the Greek word kryptos, meaning secret.

Chapter1: It is on Introduction In this chapter we will discuss Mathematics in cryptography, Abstract, Introduction, Ancient cryptography techniques and Modern cryptography techniques.

Chapter2: It is on Number theory In this chapter we will discuss key concepts in cryptography, cryptography application Leveraging Number theory and security consideration, divisibility and modular arithmetic ,integer representations & algorithms, primes and greatest common divisors , solving congruences and applications of congruences

Chapter3:It is on Cryptography in this chapter we will discuss classical cryptography ,crypto systems ,public key cryptography,RSA crypto systems, cryptography protocols, primitive rules and discrete logarithms

Chapter4:Application of cryptography in industries in this chapter we will discuss about banking and finance,health,telecommunication , IT & cyber security ,government & defence, e -commerce and retail and iot.

I am profoundly grateful to my project advisor **Dr. Gyanvendra Pratap Singh, department of mathematics and statistics, Deen Dayal Upadhyaya Gorakhapur University** for his help, valuable suggestions and support in this research project.

APPLICATION OF CRYPTOGRAPHY IN INDUSTRIES

1. Banking and Finance

The banking and finance sector has always been a prime target for cyberattacks, given the sensitive nature of financial transactions and the value of the data involved. Cryptography plays a crucial role in securing this industry through various mechanisms:

Secure Transactions: Cryptographic techniques like Secure Sockets Layer (SSL) and Transport Layer Security (TLS) are employed to protect online banking transactions and payment systems. This ensures that data transmitted over the network is encrypted and inaccessible to unauthorized parties.

Digital Signatures: Financial institutions use digital signatures to verify the authenticity of documents and transactions. This cryptographic mechanism ensures that a document has not been tampered with since it was signed and verifies the identity of the signer.

Encryption for Data at Rest and in Transit: Financial data, such as customer account information, loan details, and transaction records, is often encrypted both when stored (data at rest) and when transmitted over networks (data in transit). This protects against data breaches and unauthorized access.

Multi-factor Authentication (MFA): Cryptography is used to implement MFA, which requires multiple forms of authentication for users to access their accounts. This approach significantly reduces the risk of unauthorized access.

Blockchain and Cryptocurrencies: Cryptography is fundamental to blockchain technology, enabling secure, decentralized transactions. Cryptocurrencies like Bitcoin use cryptographic techniques to ensure transaction integrity, authenticity, and anonymity.

2. Healthcare

The healthcare sector handles sensitive patient information and is subject to strict regulations to protect patient privacy and confidentiality. Cryptography helps ensure compliance and safeguard data:

Protection of Patient Data: Healthcare providers use cryptography to secure electronic health records (EHRs), ensuring that patient data is confidential and compliant with regulations like the Health Insurance Portability and Accountability Act (HIPAA).

Medical Devices Security: As medical devices become more connected, cryptography is used to ensure they are secure from unauthorized access and manipulation. This is especially critical for devices like pacemakers and insulin pumps, where security breaches could be life-threatening.

3.Telecommunications:

The telecommunications industry relies on cryptography to secure voice and data transmissions and protect customer information:

Secure Voice and Data Transmission: Cryptography ensures that phone calls and data communications are encrypted, preventing eavesdropping and unauthorized interception.

SIM Card Security: Cryptographic techniques are used to secure SIM cards, preventing cloning and unauthorized access to mobile networks.

Secure Mobile Communication: With the proliferation of mobile devices, cryptography helps secure communication between mobile phones and the network infrastructure. Technologies like LTE and 5G use cryptographic algorithms to ensure secure connections.

4. Information Technology and Cybersecurity

Cryptography plays a central role in information technology and cybersecurity, providing mechanisms to ensure data integrity, confidentiality, and authenticity:

Authentication and Authorization: Cryptography is used in various forms of authentication, including password hashing, MFA, and public key infrastructure (PKI). These methods ensure that only authorized users have access to sensitive systems and data.

Data Encryption: Sensitive corporate information, customer data, and intellectual property are often encrypted to protect against unauthorized access and data breaches.

Secure Software Development: Cryptography is used to secure software applications, preventing vulnerabilities and exploits. Secure coding practices and cryptographic libraries help developers build secure software.

5. Government and Defense

Governments and defense agencies handle classified information and require robust cryptographic methods to ensure security:

Protection of Classified Information: Governments use advanced cryptographic techniques to protect national secrets and classified information from unauthorized access.

Secure Communication Channels: Cryptography is essential for securing communication networks used by military and governmental organizations, ensuring confidentiality and integrity.

Digital Forensics: Cryptography plays a role in digital forensics, where investigators analyze encrypted data to uncover evidence during criminal investigations.

6. E-commerce and Retail

The e-commerce and retail industries use cryptography to secure online transactions and protect customer information:

Secure Online Transactions: Cryptography ensures that online shopping and payment systems are secure, preventing unauthorized access and data breaches. Technologies like SSL/TLS create secure connections between e-commerce websites and customers' browsers.

Digital Certificates and Trust: E-commerce platforms use digital certificates to establish trust with customers. These certificates ensure that websites are legitimate and secure.

Secure Payment Systems: Payment systems rely on cryptography to ensure secure transactions, protecting credit card information and other sensitive data.

7. IoT (Internet of Things)

The Internet of Things (IoT) connects devices to the internet, creating new security challenges that cryptography helps address:

Device Security: Cryptography is used to secure IoT devices, protecting them from unauthorized access and data breaches. As IoT devices often operate in distributed environments, cryptographic techniques like public key cryptography ensure secure communication.

Secure Communication: IoT devices communicate with each other and with centralized servers. Cryptography ensures that this communication is secure, preventing unauthorized interception and tampering.

Secure Firmware Updates: Cryptography is used to ensure that firmware updates for IoT devices are secure, preventing unauthorized code from being installed on devices.

Conclusion

Cryptography is an indispensable tool for securing data and ensuring privacy across a wide range of industries. It underpins the security of financial transactions, protects patient data in healthcare, secures telecommunications, and enables secure communication in government and defense. Additionally, cryptography is crucial for the security of e-commerce platforms, IoT devices, and software applications. As technology continues to evolve, cryptography will remain a critical component of modern security, ensuring that sensitive information remains protected in a rapidly changing digital landscape.

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