multiple dimensions!

Sigurd Fyhn Sørensen

Agenda:

- Root finding
- Derivative and Integrals 102
- Series
  - Taylor Series.

Recap:

Derivatives

- **f(x):** Y value/place given our placement on the x axis.
- **f'(x):** the speed of change in position
- **f''(x):** The change in speed of speed/acceleration at a position.

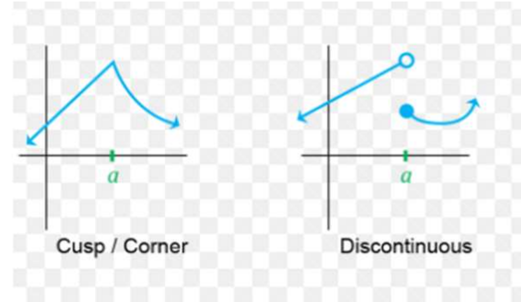
Integrals

- Definite integrals and indefinite integrals.
  - **Definite integrals** an "area" under the curve given some bounds.
  - **Indefinite** the function which describes the "area" under the curve. Unbound for all x's defined/allowed.

Limits

- $\lim_{x \to a} \frac{x^2}{10} e^x = \frac{a^2}{10} e^a$
- What happens with our expression when it reaches a specified limit.
- **L'Hospital** $\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}$

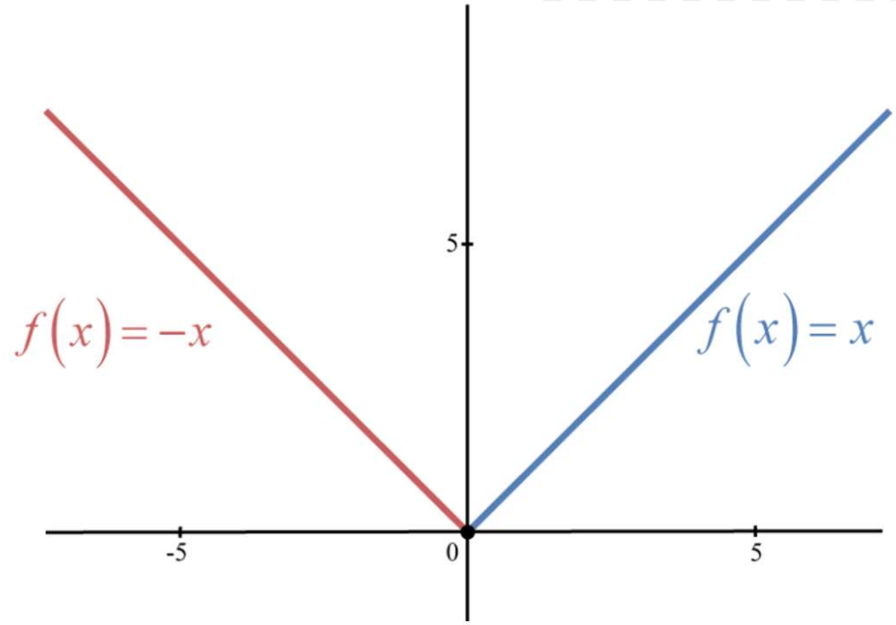
differentiable means the derivative exists at every point in its domain. Consequently, the only way for the derivative to exist is if the function also exists *(i.e., is continuous)* on its domain. Thus, a differentiable function is also a continuous function.



Recap

When is something differentiable:

- Our tangent should be the same at point **A** no matter which direction we approach from.
- X = 0 is an issue, but why?



- Knowledge of limits:
  - If we approach from + or -.

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{(-(x+h))-(-x)}{h} = \lim_{h\to 0^-} \frac{-x-h+x}{h} \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-}(-1) = -1$$

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{((x+h))-(x)}{h} = \lim_{h\to 0^+} \frac{x+h-x}{h} \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+}(1) = 1$$

Derivatives

<u>**Higher dimensional spaces and derivatives.**</u>

- When is something differentiable in $\mathbb{R}^2$?

<u>**Criteria:**</u>

f defined in an open space $\underline{A}$ is a C^1 function(differentiable) if all its partial derivatives exist in $\underline{A}$ and is continuous.

- Let's untangle this.

Partial Derivatives

Partial derivatives.

$$\frac{\partial}{\partial x_n} \; for \; \mathbb{R}^n, n > 2$$

- Speed? Acceleration? But in which direction?
  - Is it $\frac{\partial}{\partial x}$? $\frac{\partial}{\partial y}$?
  - Differentiate with respect to x or y and let the rest be a constant.
- Let's visualize!

Example:

$$f(x,y) = x^2 + y^2$$

Increase of f(x,y) in the x-direction:

$$\frac{\partial f}{\partial x} = 2x$$

Increase of f(x,y) in the y-direction:

$$\frac{\partial f}{\partial y} = 2y$$

Example:

$$f(x, y) = x^2 * y^2$$

Partial deriv f(x,y) of x: $\frac{\partial f}{\partial x} = 2x * y^2$

Partial deriv f(x,y) of y: $\frac{\partial f}{\partial y} = x^2 \cdot 2y$

Gradient

For a point:

*The gradient vector can be interpreted as the "direction and rate of fastest increase". If the gradient of a function is non-zero at a point p, the **direction** of the gradient is the direction in which the function increases most quickly from p, and the **magnitude** of the gradient is the rate of increase in that direction*

$$\nabla f(x,y) = \left(\frac{\partial}{\partial x}(x,y), \frac{\partial}{\partial y}(x,y)\right)$$

- **Interpretation:** A vector (magnitude, direction) in the direction the function grows the fastest.

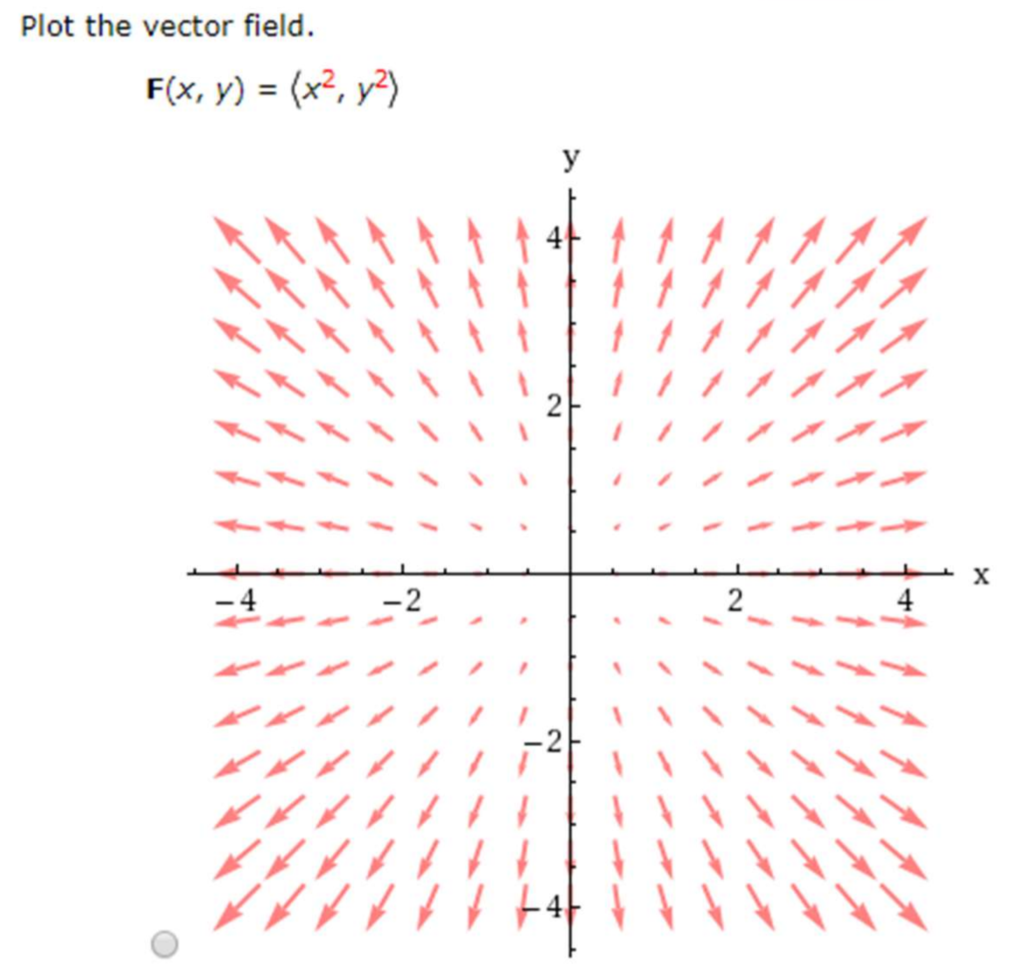
- $f(x,y) = x^2 + y^2, \quad \nabla f(x,y) = (2x, 2y)$
- In point (2,1)

$$\nabla f(2,1) = (4,2)$$

Gradient

- Do the operation for every possible spot in space **A** which f(x,y) is an inner part of, and we get vector field.

- Different ways of illustrating them.
  - Size/color (Magnitude)
  - Direction: Direction of arrow.



Here are the possible vector fields:

(a) $\mathbf{F}(x,y) = \langle 1, x\rangle$

(b) $\mathbf{F}(x,y) = \langle -y, x\rangle$

(c) $\mathbf{F}(x,y) = \langle y, x\rangle$

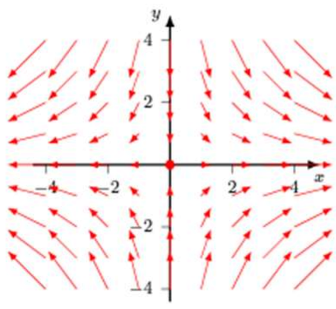
(d) $\mathbf{F}(x,y) = \langle 2x, -2y\rangle$

(e) ∇f, where $f(x,y) = x^2 + y^2$

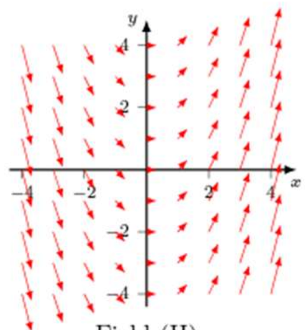
(f) ∇f, where $f(x,y) = \sqrt{x^2+y^2}$

(g) ∇f, where $f(x,y) = xy$

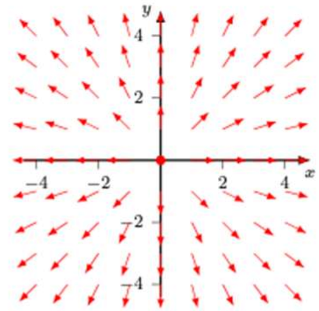
(h) ∇f, where $f(x,y) = x^2 - y^2$



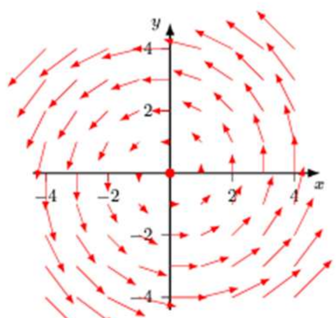
Field (I)



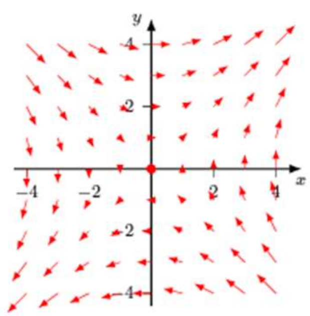
Field (II)



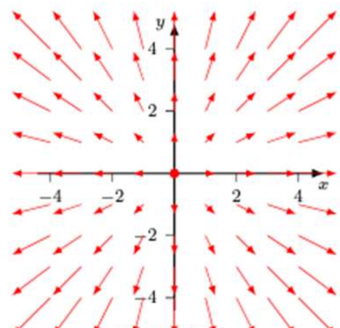
Field (III)



Field (IV)



Field (V)

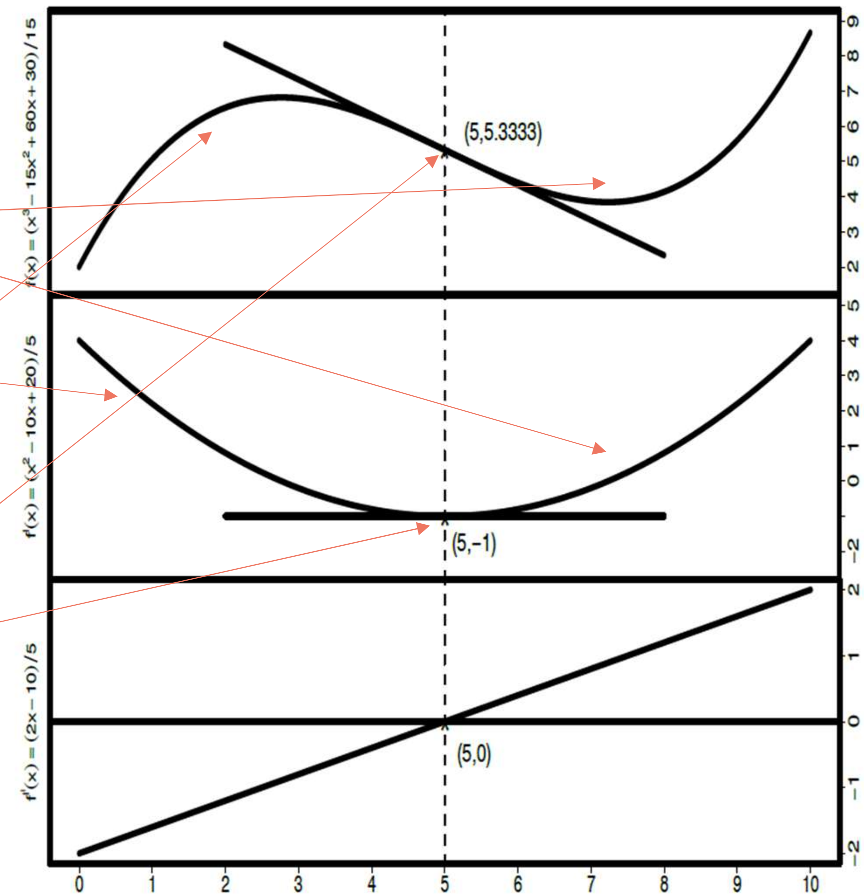


Field (VI)

Concave (up/down)

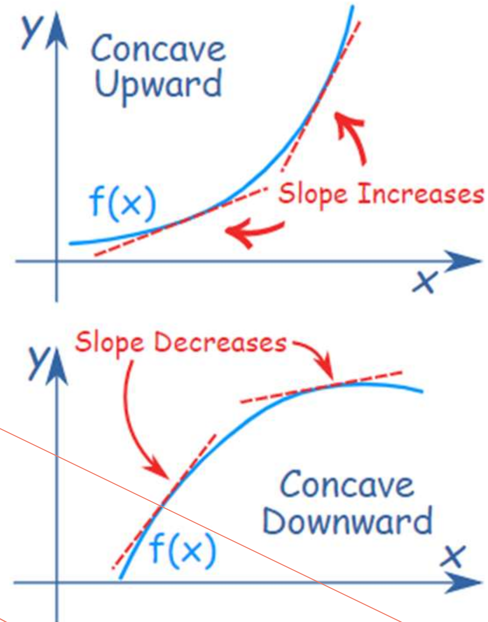
- When the function $y = f(x)$ is concave up, the graph of its derivative $y = f'(x)$ is increasing.
- When the function $y = f(x)$ is concave down, the graph of its derivative $y = f'(x)$ is decreasing.
- When the function $y = f(x)$ has a point of inflection (changes from concave up to concave down), the graph of its derivative $y = f'(x)$ has a maximum or minimum (and so changes from increasing to decreasing or decreasing to increasing respectively).

Fig. 6.1. Illustrating the Inflection Point



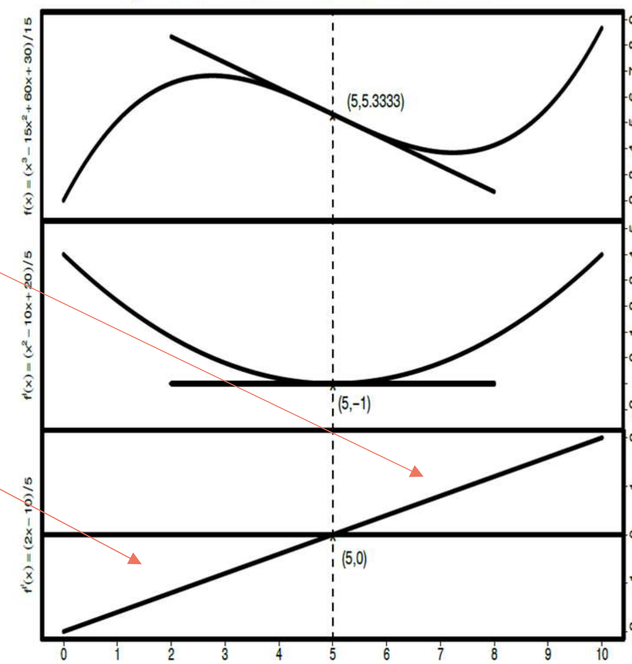
Inflection Points:

- Concave upward:
  - $f''(x) = \frac{d^2}{d^2x} = positive$
- Concave downward:
  - $f''(x) = \frac{d^2}{d^2x} = negative$



- **Inflection points** is where the concave changes i.e.. f''(x) changes **+/-**
- **Inflection points:** f''(x) = 0

Fig. 6.1. Illustrating the Inflection Point

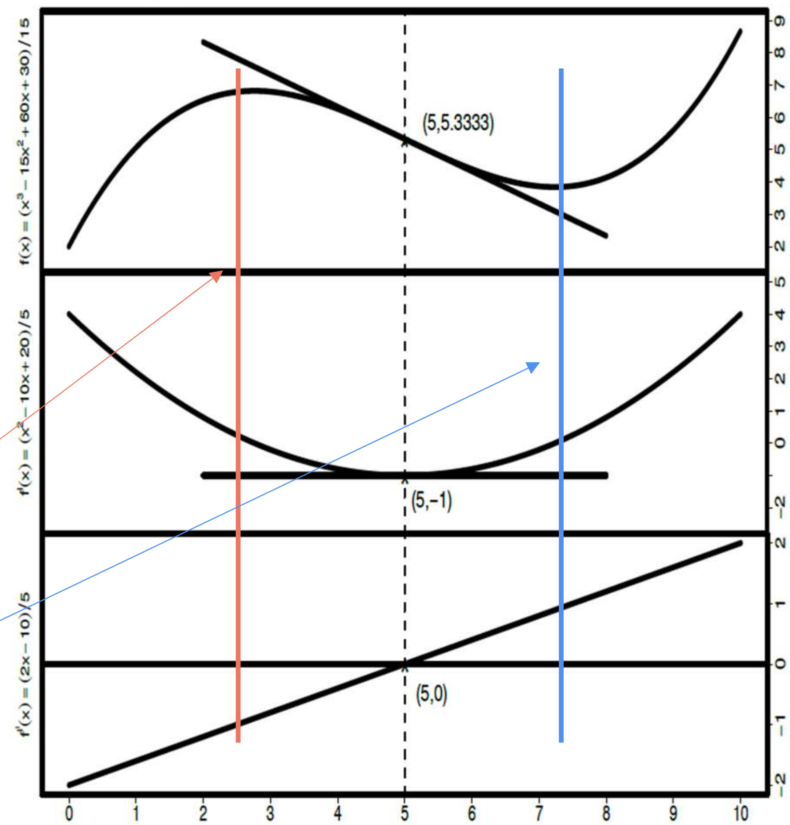


Second Derivative test:

- If f′(x) = 0 then we have local minimum or maximum?
  - We could find f′(x+1) and f′(x-1) to graph it, but…

- Local maximum iff f′(x) = 0 & f′′(x) = negative
- Local minimum iff f′(x) = 0 & f′′(x) = positive

Fig. 6.1. Illustrating the Inflection Point

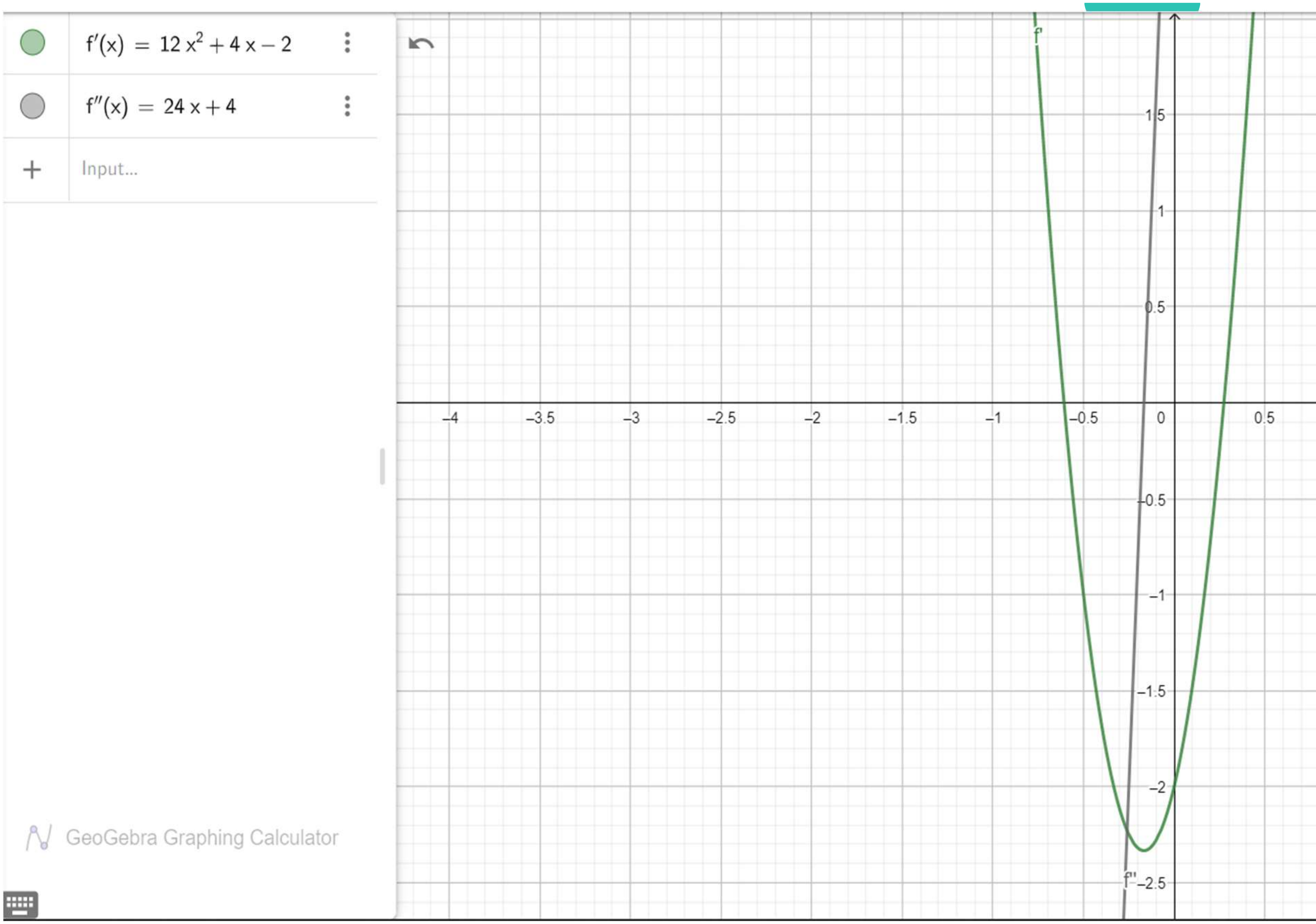


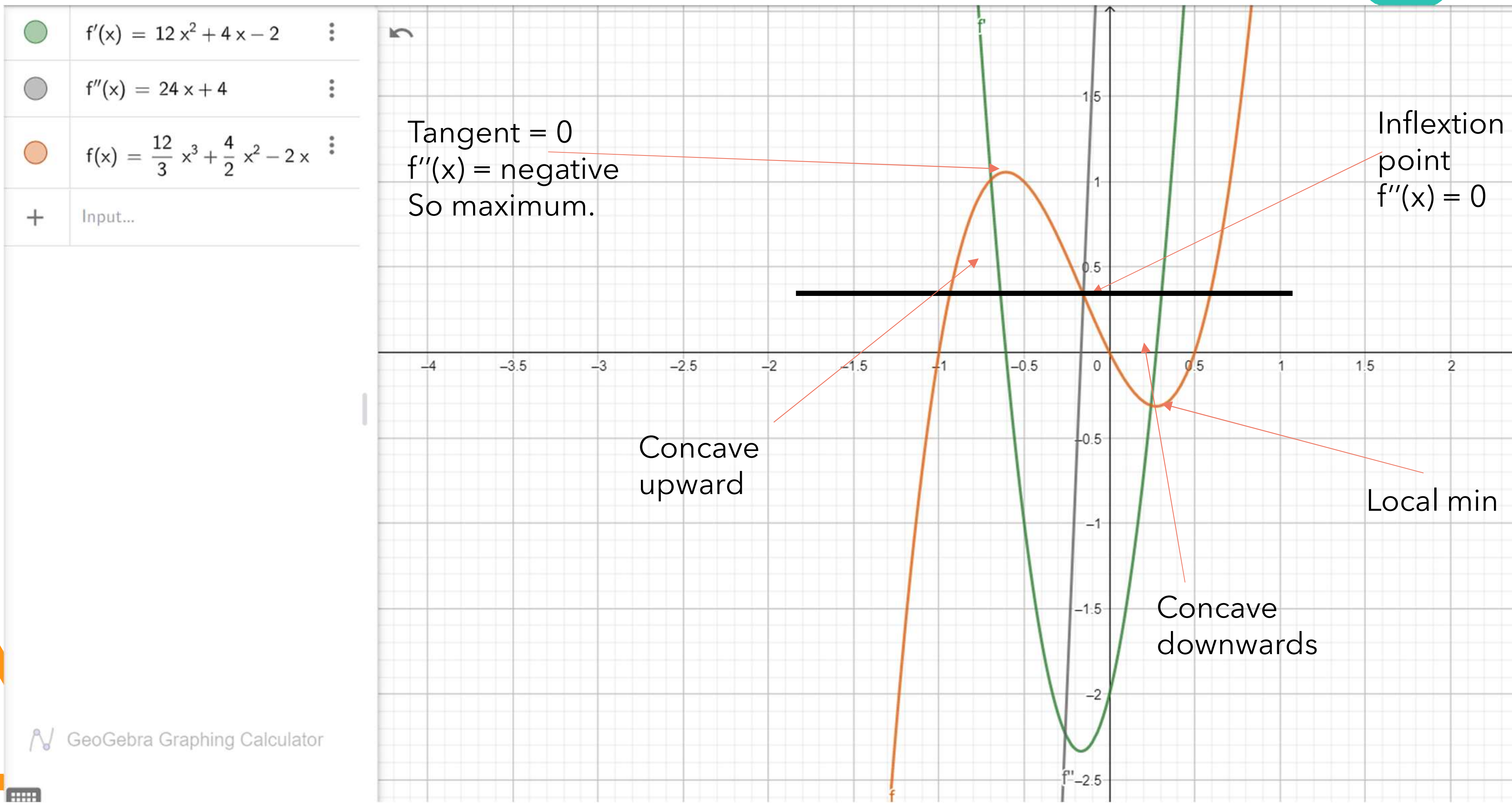
What can we say?!

You know f′(x) and f′′(x)

Find:

- Maximum and minimum of f(x)
- Inflection Points of f(x)
- Concave Upwards and Concave downwards.
- Finally find f(x)




f′(x) = 12 x² + 4 x − 2
f″(x) = 24 x + 4
f(x) = 12/3 x³ + 4/2 x² − 2 x
Input...
GeoGebra Graphing Calculator
Tangent = 0
f″(x) = negative
So maximum.
Inflextion
point
f″(x) = 0
Concave
upward
Local min
Concave
downwards

SS3

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SS3 Sigurd Sørensen, 3/28/2022

Multidimensional integrals.

$$V = \int_a^b \int_c^d f(x,y)\,dy\,dx,$$

Example:

$$\int_2^3 \int_0^1 x^2 y^3 dydx = \int_2^3 \left[\frac{1}{4} x^2 y^4 \Big|_{y=0}^{y=1} \right] dx$$

$$= \int_2^3 \left[\frac{1}{4} x^2 (1)^4 - \frac{1}{4} x^2 (0)^4 \right] dx$$

$$= \int_2^3 \frac{1}{4} x^2 dx = \frac{1}{12} x^3 \Big|_{x=2}^{x=3} = \frac{19}{12}.$$

$$\frac{1}{12} 3^3 - \frac{1}{12} 2^3 = \frac{27}{12} - \frac{8}{12} = \frac{19}{12}$$

Exercises:

- Continue if you didn't finish those from last week.

- $f(x,y) = \frac{10x^2 \cdot e^y}{2y}$
- $g(x,y) = \ln(x) + 10y$
- $j(x,y) = x^2 + y^2$
- $h(x,y) = \frac{x^2+y}{e^x+y}$
- Determine if f, g, j & h are c^1 $functions\ (differentaible)$

- $f(x,y) = 10x^2 + e^y$
  - Find $\nabla \mathrm{f(x,y)}$
  - Choose 5 coordinate sets and draw the vector fields for the resulting vectors.
  - Try and graph the function and see if your vector fields makes sense.

- Exercise 6.4,
- Exercise 6.1 , 6.11 (for the brave souls)