Name: ______________________ Homework 3 | Math 341 | Cruz Godar

Due Wednesday of Week 4 at the start of class

Complete the following problems and submit them as a pdf to Canvas. 8 points are awarded for thoroughly attempting every problem, and I'll select three problems to grade on correctness for 4 points each. Enough work should be shown that there is no question about the mathematical process used to obtain your answers.

Section 3

1. Let $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

   a) Draw $\vec{v}$ and $\vec{w}$ in the plane.

   b) Draw $2\vec{v} + \vec{w}$ geometrically and verify that it matches the algebraic definition (i.e. adding the entries of $\vec{v}$ and $\vec{w}$).

   c) Draw a vector linearly dependent with $\vec{v}$, but linearly independent with $\vec{w}$.

   d) Draw a vector linearly dependent with both $\vec{v}$ and $\vec{w}$, but not with either of them alone.

In problems 2–5, determine if the vectors are linearly dependent or independent. If they are dependent, find a linear combination equal to $\vec{0}$.

2. $\vec{v_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

3. $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$.

4. $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

5. $\vec{v_1} = \begin{bmatrix} 1 \\ 5 \\ 6 \\ -10 \\ \frac{17}{2} \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 1 \\ 7 \\ -100 \\ 0 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 0 \\ 89 \end{bmatrix}.$

In problems 6–7, express $\vec{v}$ as a linear combination of the $\vec{u}_i$ or show it's impossible.

6. $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{u_1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \vec{u_2} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \vec{u_3} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$

7. $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{u_1} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \vec{u_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u_3} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}.$

8. Let $\vec{w_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w_2} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \vec{w_3} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$

a) Show that $\vec{w_1}$, $\vec{w_2}$, and $\vec{w_3}$ are linearly dependent.

b) Show that just $\vec{w_1}$ and $\vec{w_2}$ on their own are linearly independent. (Hint: You should be able to modify the last step in the previous part to get this result without starting over).

c) Using the previous two parts, write a sentence explaining why

$$\text{span}\{\vec{w_1}, \vec{w_2}\} = \text{span}\{\vec{w_1}, \vec{w_2}, \vec{w_3}\}.$$

a) The span of $\vec{w_1}$ and $\vec{w_2}$ is the set consisting of all $\vec{w} = c_1\vec{w_1} + c_2\vec{w_2}$. Renaming $c_1 = u$ and $c_2 = v$, write the generic vector $\vec{w}$ in the span. Your answer should depend on u and v.

b) Math3D is a capable 3D grapher that can handle parametric surfaces. Open the linked example and replace the span expression with the one you found in the previous part — if all went well, you should see the three vectors lying *in* that plane. This plane is the span, and the fact that the three vectors are contained in a two-dimensional surface is their linear dependence.

Section 4

9. Linear transformations are related to typical linear functions like $y = mx + b$, but they're not quite the same. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$ is a linear function, but not a linear transformation. Pick two inputs a and b and show that $f(a + b) \neq f(a) + f(b)$.

In problems 10–12, find the matrix for the linear transformation T and use it to evaluate $T(\vec{v})$.

10. $T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix}$, $T\left(\begin{bmatrix}0\\0\\3\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}$, and $\vec{v} = \begin{bmatrix}2\\3\\2\end{bmatrix}$.

11. $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$, $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}$, and $\vec{v} = \begin{bmatrix}2\\3\end{bmatrix}$.

12. $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 2$, $T\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = 7$, and $\vec{v} = \begin{bmatrix}2\\1\end{bmatrix}$.

13. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x+3y\\x+2y\\3x+z\end{bmatrix}.$$

a) Express T as a 3×3 matrix A.

b) Find A^{-1}.

c) Write a linear transformation S whose matrix is A^{-1}.

d) Since $A^{-1}A = I$ and matrix multiplication is equivalent to function composition, we should expect

$S \circ T = id$, the identity function. Evaluate $S \circ T$ as functions by using the output of T as the input to S, and show this is in fact the case.