Complex Analysis Notes

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I — The Complex Numbers

Definition 1.1: The **complex numbers** are the field $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$.

Definition 1.2: The **modulus** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.

Definition 1.3: The **distance** between z_1 and z_2 is $|z_2 - z_1|$.

Definition 1.4: The **conjugate** of $z = a + bi$ is $\overline{z} = z - bi$.

Proposition 1.5: Let $z, w \in \mathbb{C}$.

1. $\overline{z + w} = \overline{z} + \overline{w}$.
2. $\frac{z+\overline{z}}{2} = \operatorname{Re} z$ and $\frac{z-\overline{z}}{2i} = \operatorname{Im} z$.
3. $z\overline{z} = |z|^2$.

Proposition 1.6: Let $z \in \mathbb{C}^*$. Then there is a unique $r \in \mathbb{R}^+$ and $\theta \in [0, 2\pi)$ such that $z = re^{i\theta}$.

Definition 1.7: The **argument** of $z = re^{i\theta}$ is the set $\arg z = \{\theta + 2\pi k \mid k \in \mathbb{Z}\}$.

Definition 1.8: The **principal argument** of $z = re^{i\theta}$, denoted $\operatorname{Arg} z$, is the unique element of $\arg z$ lying in $(-\pi, \pi]$. If $\tau \in \mathbb{R}$, $\arg_\tau z$ is the unique element of $\arg z$ lying in $(\tau, \tau + 2\pi]$.

Proposition 1.9: For $\theta \in \mathbb{R}$, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

Proof: Let $z = e^{i\theta}$. Then $\operatorname{Re} z = \cos\theta = \frac{z+\overline{z}}{2} = \frac{e^{i\theta}+e^{-i\theta}}{2}$, and similarly for $\operatorname{Im} z$.

Proposition 1.10: For all $\theta \in \mathbb{R}$, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$.

Definition 1.11: The n distinct ***n*th roots of unity** are $1^{\frac{1}{n}} = e^{\frac{2\pi i k}{n}}$ for $k \in \{0, ..., n-1\}$.

Definition 1.12: The **primitive *n*th root of unity** is $\omega_n = e^{\frac{2\pi i}{n}}$.

Proposition 1.13: Let $z = re^{i\theta}$. Then $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta+2\pi i k}{n}}$ for $k \in \{0, ..., n-1\}$.

Definition 1.14: The **open disk** of radius r about z_0 is the set $\{z \in \mathbb{C} \mid |z - z_0| < r\}$.

Definition 1.15: An **interior point** of a set $S \subseteq \mathbb{C}$ is a point $z \in S$ such that some open disk about z lies entirely inside S.

Definition 1.16: A set S is **open** if every element of S is an interior point of S.

Definition 1.17: A set S is **connected** if any two points can be connected by a series of straight lines.

Definition 1.18: A **domain** is an open connected set.

Definition 1.19: A point $w \in \mathbb{C}$ is a **boundary point** of a set S if every open disk about w has points both in and out of S.

Definition 1.20: The **boundary** of a set S is the set of boundary points of S, denoted ∂S.

Definition 1.21: A set S is **closed** if $\partial S \subseteq S$.

Definition 1.22: The **Riemann Sphere** is the unit 2-sphere in $\mathbb{R}^3$ that is homeomorphic to $\mathbb{C}$, with the projection of $z \in \mathbb{C}$ onto the sphere given by the intersection with the line through $(0,0,1)$ and $z = (x,y,0)$.

Example: Find the projection of $a + bi$ onto the Riemann Sphere.

We have $x_1^2 + x_2^2 + x_3^2 = 1$, $x_1 = at$, $x_2 = bt$, and $x_3 = 1 - t$ for $t \in [0,1]$. Then $a^2t^2 + b^2t^2 + t^2 - 2t + 1 = 1$, so $a^2t + b^2t + t - 2 = 0$ (the case when $t = 0$ is trivial), and $t = \frac{2}{a^2+b^2+1}$. Thus the point of intersection is

$$\left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right), \text{ or equivalently, } \left(\frac{2a}{|z|^2+1}, \frac{2b}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

II — Complex Limits and Derivatives

Definition 2.1: A sequence (z_n) of complex numbers **converges** to $z \in \mathbb{C}$ if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|z_n - z| < \varepsilon$.

Definition 2.2: Let f be a complex function defined in a neighborhood of z_0. Then $\lim\limits_{z \to z_0} f(z) = w$ if for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |z - z_0| < \delta$, then $|f(z) - w| < \varepsilon$.

Proposition 2.3: Suppose $\lim\limits_{z \to z_0} f(z) = L$ and $\lim\limits_{z \to z_0} g(z) = M$. Then

1. $\lim\limits_{z \to z_0} (f(z) + g(z)) = L + M$.
2. $\lim\limits_{z \to z_0} (f(z)g(z)) = LM$.
3. $\lim\limits_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$ if $M \neq 0$.

Definition 2.4: A function f is **continuous** at z_0 if $f(z)$ exists, $\lim\limits_{z \to z_0} f(z)$ exists, and $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

Definition 2.5: Let $f : \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\} \longrightarrow \mathbb{C}$. The **derivative** of f at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided the limit exists.

Proposition 2.6: Let f and g be differentiable at z_0. Then

1. $(f + g)'(z_0) = f'(z_0) + g'(z_0)$.
2. $(cf)'(z_0) = cf'(z_0)$.
3. $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.
4. $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$.
5. If g is differentiable at $f(z_0)$, then $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$.

Definition 2.7: A function f is **analytic** on an open set G if $f'(z_0)$ exists for every $z_0 \in G$.

Definition 2.8: A function f is **entire** if f is analytic on $\mathbb{C}$.

Proposition 2.9: For sufficiently small ε-neighborhoods of z_0, $|f'(z_0)|$ is the scaling factor of the neighborhood's image and $\arg f'(z_0)$ is the rotation factor.

Proof:
Since $f'(z_0) = \lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$, for $z \approx z_0$ (i.e. $|z - z_0| < \varepsilon$), we have $|f(z) - f(z_0)| \approx |f'(z_0)||z - z_0|$ and $\arg(f(z) - f(z_0)) - \arg(z - z_0) \approx \arg f'(z_0)$.

Theorem 2.10: (The Cauchy-Riemann Equations) If $f'(z_0)$ exists, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at z_0, where $f(z) = f(x + iy) = u(x, y) + iv(x, y)$.

Proof: Since $f'(z_0)$ exists, $\lim\limits_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ exists. In particular, it has the same value if the limit is taken along the real or imaginary axes. Along the real axis, we have

$$
\begin{aligned}
&\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
=& \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\
=& \lim_{\Delta x \to 0} \left(\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \right) + i \lim_{\Delta x \to 0} \left(\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right) \\
=& \frac{\partial u}{\partial x}(z_0) + i \frac{\partial v}{\partial x}(z_0)
\end{aligned}
$$

Similarly, approaching on the imaginary axis gives us $f'(z) = \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0)$, so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at z_0.

Proposition 2.11: If $f' = 0$ on a domain G, then f is constant on G.

Proof: Let $z_1, z_2 \in G$. Then $0 = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}$, so $u(x_1, y_1) = u(x_2, y_2)$ and $v(x_1, y_1) = v(x_2, y_2)$. Thus $f(z_1) = f(z_2)$.

Proposition 2.12: If f is analytic on a domain G and $\operatorname{Im} f$ is constant on G, then f is constant on G.

Proof: If $f = u + iv$, then $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$, so f is constant.

Proposition 2.13: If f is analytic on a domain G and $|f|$ is constant, then f is constant.

Theorem 2.14: If f is defined on a domain G containing z_0, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ are defined on all of G and are continuous at z_0, and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at z_0, then $f'(z_0)$ exists.

Definition 2.15: A real-valued function φ is **harmonic** on a domain G if all of φ's second-order partials are continuous on G and $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$.

Theorem 2.16: If $f = u + iv$ is analytic on a domain G, then u and v are harmonic on G.

Example: Create an analytic function f with $\operatorname{Re} f(x+iy) = xy - x + y$.

The request is not impossible, since $xy - x + y$ is harmonic. Since f is to be analytic, $\frac{\partial u}{\partial x} = y - 1 = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = x + 1 = -\frac{\partial v}{\partial x}$. Thus $v(x,y) = \frac{y^2}{2} - \frac{x^2}{2} - y - x + C$. This v is called the **harmonic conjugate** of u.

III — Complex Elementary Functions

Proposition 3.1: $(e^z)' = e^z$.

Proof: $(e^z)' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^z$.

Definition 3.2: Let $z \in \mathbb{C}$. The **sine** and **cosine** of z are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Proposition 3.3: Let $z, w \in \mathbb{C}$.

1. Both sin and cos are entire.
2. $\frac{d}{dz}[\sin z] = \cos z$ and $\frac{d}{dz}[\cos z] = \sin z$.
3. $\sin^2 z + \cos^2 z = 1$.
4. $\sin^2 z = \frac{1-\cos 2z}{2}$ and $\cos^2 z = \frac{1+\cos 2z}{2}$.
5. $\sin 2z = 2\sin z \cos z$.
6. $\sin(z + w) = \sin z \cos w + \cos z \sin w$.

Proposition 3.4: The only roots of $\sin z$ are πk for $k \in \mathbb{Z}$.

Proof: $\sin z = 0$ if and only if $e^{iz} = e^{-iz}$, if and only if $iz = -iz + 2\pi k$, if and only if $z = \pi k$.

Example: Solve $e^z = 2 + 2i$.

$2 + 2i = s\sqrt{2}\frac{1+i}{2} = 2\sqrt{2}e^{i\frac{\pi}{4}} = e^{\log(2\sqrt{2})+i\frac{\pi}{4}}$. Thus $z = \log(2 + 2i) = \log(2\sqrt{2}) + i(\frac{\pi}{4} + 2\pi k)$ for $k \in \mathbb{Z}$. In particular, log is multivalued.

Definition 3.5: Let $z \in \mathbb{C}$. Log z = Log $|z| + i$Arg z, where Log $x = \ln x$ for $x \in \mathbb{R}$.

Definition 3.6: Let $z \in \mathbb{C}$. $\log z$ = Log $|z| + i($Arg $z + 2\pi k)$ for $k \in \mathbb{Z}$.

Proposition 3.7: Log z is continuous on $\mathbb{C}^* \setminus \mathbb{R}^-$.

Theorem 3.8: Log z is analytic on $\mathbb{C}^* \smallsetminus \mathbb{R}^-$, and $\frac{d}{dz}[\text{Log } z] = \frac{1}{z}$.

Proof: Let $z_0 \in \mathbb{C}^* \smallsetminus \mathbb{R}^-$. Then

$$\begin{aligned}\frac{d}{dz}[\text{Log } z]|_{z=z_0} &= \lim_{z\to z_0} \frac{\text{Log } z - \text{Log } z_0}{z - z_0} \\ &= \lim_{z\to z_0} \frac{w - w_0}{z - z_0} \\ &= \lim_{z\to z_0} \frac{1}{\frac{z-z_0}{w-w_0}}.\end{aligned}$$

This last step is valid, since if $w = w_0$, then $e^w = e^{w_0}$, so $z = z_0$, since Log is branch-cut. Continuing,

$$\begin{aligned}\frac{d}{dz}[\text{Log } z]|_{z=z_0} &= \lim_{z\to z_0} \frac{1}{\frac{e^w - e^{w_0}}{w-w_0}} \\ &= \frac{1}{\frac{d}{dz}[e^w]|_{w=w_0}} \\ &= \frac{1}{e^{w_0}} \\ &= \frac{1}{z}.\end{aligned}$$

Comment: In general, Log $(z_1 z_2) \neq$ Log z_1 + Log z_2 and Log $e^z \neq z$, but $\log(z_1 z_2) = \log z_1 + \log z_2$ and $e^{\text{Log } z} = z$.

Definition 3.9: Let $\tau \in \mathbb{R}$. $\mathcal{L}_\tau(z) = \text{Log } |z| + i \arg_\tau(z)$.

Proposition 3.10: $\mathcal{L}_\tau$ is continuous on all of $\mathbb{C}^*$ except for the ray from 0 with angle τ, analytic where it is continuous, and has derivative $\frac{d}{dz}[\mathcal{L}_\tau(z)] = \frac{1}{z}$.

Definition 3.11: Let $\alpha \in \mathbb{C}$ and $z \neq 0$. $z^\alpha = e^{\alpha \log z}$.

Proposition 3.12: Let $z, \alpha \in \mathbb{C}^*$. Then z^α is single-valued if $\alpha \in \mathbb{Z}$, finitely-valued if $\alpha \in \mathbb{Q} \smallsetminus \mathbb{Z}$, and infintely-valued if $\alpha \in \mathbb{C} \smallsetminus \mathbb{Q}$.

Definition 3.13: The **principal branch** of z^α is $e^{\alpha \text{Log } z}$.

Proposition 3.14: The principal branch of z^α is analytic on $\mathbb{C}^* \smallsetminus \mathbb{R}^-$, and for this branch, $\frac{d}{dz}[z^\alpha] = \frac{d}{dz}\left[e^{\alpha \text{Log } z}\right] = \left(e^{\alpha \text{Log } z}\right)\left(\frac{\alpha}{z}\right) = \alpha z^{\alpha - 1}$.

IV — Complex Integration

Definition 4.1: A **curve** from z_1 to z_2 is a parametric function of the form $z(t) = x(t) + iy(t)$.

Definition 4.2: Let $z(t)$ be a parametric curve. $z'(t) = x'(t) + iy'(t)$.

Definition 4.3: A **smooth arc** is a curve γ parameterized by $z(t) = x(t) + iy(t)$ for $t \in [a, b]$ such that $z'(t)$ is continuous on $[a, b]$, $z'[t] \neq 0$ for all $t \in [a, b]$, and z is injective on $[a, b]$.

Definition 4.4: A **smooth closed curve** is a smooth arc such that z is injective on $[a, b)$, $z(a) = z(b)$, and $z'(a) = z'(b)$.

Definition 4.5: A **contour** is a collection Γ of smooth, directed arcs such that the terminal point of γ_k is the initial point of γ_{k+1} for all k. We write $\Gamma = \gamma_1 + \cdots + \gamma_n$.

Definition 4.6: Let f be defined on a smooth directed arc γ. The **Riemann sum of** f is $S = \sum\limits_{k=1}^{n} f(c_k)(z_k - z_{k-1})$ for $z_0, ..., z_n \in \gamma$ successively and $c_k \in [z_{k-1}, z_k]$ along γ.

Definition 4.7: A function f is **integrable** on γ if there is an $L \in \mathbb{C}$ such that $\lim\limits_{m(P)\to 0} S(P, \{c_k\}) = L$, where P is the partition of γ given by $z_0, ..., z_n$ and $m(P)$ is the mesh of P, given by $m(P) = \max\{|z_k - z_{k-1}|\}$.

Theorem 4.8: If f is continuous on γ, then f is integrable on γ.

Proposition 4.9: If γ is parameterized by $z(t)$ for $t \in [a, b]$, then $\int_\gamma f(z)\ \mathrm{d}z = \int_a^b f(z(t))z'(t)\ \mathrm{d}t$.

Theorem 4.10: Let D be a domain, f a continuous function on D, F an analytic function on D with $F'(z) = f(z)$, and Γ a contour in D with initial point z_a and terminal point z_b. Then

$$\int_\Gamma f(z)\ \mathrm{d}z = F(z_b) - F(z_a).$$

Proof: For each smooth arc γ in Γ, suppose γ is parameterized by $z(t)$ for $t \in [a, b]$. Then $\int_\gamma f(z)\ \mathrm{d}z = \int_a^b f(z(t))z'(t)\ \mathrm{d}t$. But $\frac{d}{dt}[F(z(t))] = F'(z(t))z'(t) = f(z(t))z'(t)$, so $\int_a^b f(z(t))z'(t)\ \mathrm{d}t = \int_a^b \frac{d}{dt}[F(z(t))]\ \mathrm{d}t = F(z(b)) - F(z(a))$. Thus if $\Gamma = \gamma_1 + \cdots + \gamma_n$, we have $\int_\gamma f(z)\ \mathrm{d}z = F(z_b) - F(z_a)$.

Definition 4.11: A **continuous deformation** of a contour Γ_0 into a contour Γ_1 is a function $z(s,t) : [0,1] \times [0,1] \longrightarrow D$, where D is a domain on which both Γ_0 and Γ_1 are defined, such that $z(0,t)$ for $t \in [0,1]$ is a parameterization for Γ_0, $z(1,t)$ for $t \in [0,1]$ is a parameterization for Γ_1, and z is continuous.

Example: $z(s,t) = (s+1)e^{2\pi i t}$ is a continuous deformation of the unit circle into a circle of radius 2.

Theorem 4.12: Suppose f is analytic on D and that the closed loop Γ_0 can be continuously deformed into the closed loop Γ_1, where both are in D. Then $\int_{\Gamma_0} f(z)\ \mathrm{d}z = \int_{\Gamma_1} f(z)\ \mathrm{d}z$.

Theorem 4.13: If f is analytic on a simply connected domain D and $\Gamma \subseteq D$ is a closed loop, then $\int_\Gamma f(z)\ \mathrm{d}z = 0$, f has an antiderivative on D, and f is path-independent.

Theorem 4.14: Suppose f is analytic on a domain D, $z_0 \in D$, and $\Gamma \subseteq D$ is a closed loop. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\ \mathrm{d}z = f(z_0).$$

Proof: For any $\Gamma_0 \subseteq \mathbb{C}^*$ that loops 0,

$$\begin{aligned}
\int_{\Gamma_0} \frac{1}{z}\ \mathrm{d}z &= \int_{|z|=1} \frac{1}{z}\ \mathrm{d}z \\
&= \int_0^1 \frac{1}{e^{2\pi i t}} (2\pi i) e^{2\pi i t}\ \mathrm{d}t \\
&= \int_0^1 2\pi i\ \mathrm{d}t \\
&= 2\pi i.
\end{aligned}$$

Now if C_r is the circle of radius r centered at z_0, then

$$\begin{aligned}
\int_{\Gamma_0} \frac{f(z)}{z - z_0}\ \mathrm{d}z &= \int_{C_r} \frac{f(z)}{z - z_0}\ \mathrm{d}z \\
&= \int_{C_r} \frac{f(z_0 + f(z) - f(z_0)}{z - z_0}\ \mathrm{d}z \\
&= \int_{C_r} \frac{f(z_0)}{z - z_0}\ \mathrm{d}z + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\ \mathrm{d}z \\
&= 2\pi i f(z_0) + \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\ \mathrm{d}z.
\end{aligned}$$

Now

$$\begin{aligned}
\left| \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\ \mathrm{d}z \right| &\le \max_{z \in C_r} \left\{ \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \right\} (2\pi r) \\
&= \max_{z \in C_r} \{ |f(z) - f(z_0)| \} (2\pi),
\end{aligned}$$

and as $r \to 0$, $\max\limits_{z \in C_r} \{ |f(z) - f(z_0)| \} (2\pi) \to 0$. Since the deformation is continuous,

$$\int_\Gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z = 2\pi i f(z_0).$$

Theorem 4.15: Let f be analytic on and inside a positively-oriented, simple closed loop Γ. Then for any z inside Γ,

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,\mathrm{d}\zeta = f^{(n)}(z).$$

Corollary 4.15.1: If f is analytic on a domain D, then $f^{(n)}$ exists and is analytic on D for all $n \in \mathbb{N}$.

Theorem 4.16: If f is continuous on a domain D and $\int_\Gamma f(z)\,\mathrm{d}z = 0$ for all closed loops Γ in D, then f is analytic.

Theorem 4.17: If f is entire and bounded, then f is constant.

Proof: Since f is bounded, there is an $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be arbitrary and let C_r be the positively-oriented circle of radius r centered at z. Then

$$f'(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{(\zeta-z)^2}\,\mathrm{d}\zeta,$$

so

$$|f'(z)| = \frac{1}{2\pi}\left|\int_{C_r} \frac{f(\zeta)}{(\zeta-z)^2}\,\mathrm{d}\zeta\right| \leq \left(\frac{1}{2\pi}\right)\left(\frac{M}{r^2}\right)(2\pi r) = \frac{M}{r}.$$

Since this holds for all $r \in \mathbb{R}^+$ and $\frac{M}{r} \to 0$ as $r \to \infty$, $f'(z) = 0$. Since z was arbitrary, $f' = 0$, so f is constant.

Theorem 4.18: (The Fundamental Theorem of Algebra) Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof: Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_n \neq 0$ and $n \geq 1$. Suppose $p(z) \neq 0$ for any $z \in \mathbb{C}$. Now $\frac{p(z)}{z^n} = a_n + a_{n-1}\left(\frac{1}{z}\right) + \cdots + a_1\left(\frac{1}{z^{n-1}}\right) + a_0\left(\frac{1}{z^n}\right)$, so $\lim\limits_{z\to\infty} \frac{p(z)}{z^n} = a_n$. Thus there is a $\rho \in \mathbb{R}$ such that if $|z| > \rho$, then $\left|\frac{p(z)}{z_n}\right| > \frac{|a_n|}{2}$. Since $p(z) \neq 0$, $\frac{1}{p(z)}$ is defined for all $z \in \mathbb{C}$. If $|z| > \rho$, then $\left|\frac{1}{p(z)}\right| < \frac{2}{|z^n|\,|a_n|} < \frac{2}{\rho^n|a_n|}$, so in particular, $\frac{1}{p(z)}$ is bounded. If $|z| \leq \rho$, then $\frac{1}{p(z)}$ is still bounded, since $\frac{1}{p(z)}$ is continuous on $|z| \leq \rho$, a closed set. Thus $\frac{1}{p(z)}$ is entire and bounded, so it is constant, and therefore $p(z)$ is constant too. $\lightning$

Theorem 4.19: Let f be analytic on and inside a circle Γ centered at z_0. If $\max\{|f(z)| \mid z \text{ inside } \Gamma\} = |f(z_0)|$, then f is constant.

Theorem 4.20: Let Γ be a smooth closed curve and f a function analytic on and inside Γ. Then $|f(z)|$ is maximized for z on Γ.

V — Series Representations

Definition 5.1: A **series** of complex numbers is a formal expression $\sum_{j=1}^{\infty} b_j$. The series $\sum_{j=1}^{\infty} b_j$ **converges** to $B \in \mathbb{C}$ if $\lim_{n\to\infty} \sum_{j=1}^{n} b_j = B$. Otherwise, $\sum_{j=1}^{\infty} b_j$ **diverges**.

Proposition 5.2: If $|b_j| \leq M_j$ for all $j \in \mathbb{N}$ and $\sum M_j$ converges, then $\sum b_j$ converges.

Proposition 5.3: If $\lim_{j\to\infty} \left|\frac{b_{j+1}}{b_j}\right| < 1$, then $\sum b_j$ converges, and if $\lim_{j\to\infty} \left|\frac{b_{j+1}}{b_j}\right| > 1$, then $\sum b_j$ diverges.

Definition 5.4: A series $\sum f_n(z)$ converges **uniformly** to $f(z)$ if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq \mathbb{N}$, then $|f(z) - f_n(z)| < \varepsilon$ for all $z \in \mathbb{C}$.

Definition 5.5: Let f be analytic at z_0. The **Taylor series** for f about z_0 is

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j.$$

Theorem 5.6: Let f be analytic on the open disk of radius R centered at z_0. Then for each z such that $|z - z_0| < R$,

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j.$$

Moreover, the convergence is uniform on any closed subdisk of radius $r < R$.

Theorem 5.7: A power series $\sum a_j(z - z_0)^j$ either

1. converges only for $z = z_0$,
2. converges for $|z - z_0| < R$ for some $R \in (0, \infty)$, or
3. converges for all $z \in \mathbb{C}$.

Theorem 5.8: If $(f_n) \to f$ uniformly and the f_n are continuous, then f is continuous.

Theorem 5.9: If $(f_n) \to f$ uniformly on $D \subseteq \mathbb{C}$, the f_n are continuous, and Γ is a contour in D, then

$$\left(\int_\Gamma f_n \ \mathrm{d}z\right) \to \int_\Gamma f \ \mathrm{d}z.$$

Proof: Let $\varepsilon > 0$. Since $(f_n) \to f$ uniformly on D, there is an $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \frac{\varepsilon}{\lambda(\Gamma)}$ for all $n \geq N$ and $z \in D$, where $\lambda(\Gamma)$ is the length of Γ. Thus if $n \geq N$,

$$\begin{aligned}
\left|\int_\Gamma f_n(z) \ \mathrm{d}z - \int_\Gamma f(z) \ \mathrm{d}z\right| &= \left|\int_\Gamma f_n(z) - f(z) \ \mathrm{d}z\right| \\
&\leq \int_\Gamma |f_n(z) - f(z)| \ \mathrm{d}z \\
&< \left(\frac{\varepsilon}{\lambda(\Gamma)}\right)\lambda(\Gamma) \\
&= \varepsilon,
\end{aligned}$$

so $\left(\int_\Gamma f_n \ \mathrm{d}z\right) \to \int_\Gamma f \ \mathrm{d}z$.c

Theorem 5.10: If $(f_n) \to f$ uniformly on a simply connected domain D and the f_n are analytic on D, then f is analytic on D.

Proof: Let Γ be a closed loop in D. Then $\int_\Gamma f_n(z) \ \mathrm{d}z = 0$ for all $n \in \mathbb{N}$, since each f_n is analytic. Since $(\int_\Gamma f_n(z) \ \mathrm{d}z) = (0) \to (\int_\Gamma f(z) \ \mathrm{d}z)$, $\int_\Gamma f(z) \ \mathrm{d}z = 0$. Since Γ was arbitrary, $\int_\Gamma f(z) \ \mathrm{d}z = 0$ for every closed loop in D. Thus f is analytic on D.

Theorem 5.11: If $f = \sum a_j(z - z_0)^j$, then f is analytic wherever it converges, and f is equal to its oen Taylor series (i.e. $a_j = \frac{f^{(j)}(z_0)}{j!}$).

Example: If we want to express $f(z) = \frac{1}{z-3}$ as a series centered at 2 and valid at $5i$, we cannot use a Taylor series, since $\frac{1}{z-3} = -\frac{1}{1-(z-2)} = -\sum(z-2)^j$, $|z-2| < 1$, which does not converge at $z = 5i$. Instead, we can express f as a series by noticing that

$$\begin{aligned}
\frac{1}{z-3} &= \frac{1}{(z-2)-1} \\
&= \left(\frac{1}{z-2}\right)\left(\frac{1}{1-\frac{1}{z-2}}\right) \\
&= \frac{1}{z-2}\sum_{j=0}^{\infty}\left(\frac{1}{z-2}\right)^j \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{z-2}\right)^{j+1}, \ |z-2| > 1.
\end{aligned}$$

Building a series in this way will always result in one convergent on an annulus $r < |z - z_0| < R$.

Definition 5.12: A **Laurent series** is a series of the form

$$\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j.$$

Theorem 5.13: Let D be the annulus $r < |z-z_0| < R$ and let f be analytic on D. Then on D, f is expressible as $f(z) = \sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$, where the series is converges on D and uniformly on any closed subannulus $r < \rho_1 \leq |z-z_0| \leq \rho_2 < R$. Moreover,

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0}^{j+1} \,\mathrm{d}\zeta,$$

where C is any contour in D with z_0 in its interior.

Theorem 5.14: If $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ is valid for $|z-z_0| < R$, $\sum_{j=-\infty}^{-1} a_j(z-z_0)^j$ is valid for $r < |z-z_0|$, and $r < R$, then $\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$ is valid and analytic on the annulus $r < |z-z_0| < R$.

Definition 5.15: A point z_0 is a **zero of order m** of a function f if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$, but $f^{(m)}(z_0) \neq 0$.

Proposition 5.16: Let f be analytic at z_0. Then z_0 is a zero of order m if and only if $f(z) = (z-z_0)^m g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.

Proof: ($\Rightarrow$) Since f is analytic at z_0, $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$. But $a_j = \frac{f^{(j)}(z_0)}{j!}$, so $a_j = 0$ for $0 \leq j < m$. Thus $f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots = (z-z_0)^m(a_m + a_{m+1}(z-z_0) + \cdots) = (z-z_0)^m g(z)$.

Theorem 5.17: Suppose f is analytic at z_0 and $f(z_0) = 0$. Then either $f(z) = 0$ on a disk centered at z_0, or $f(z)$ is never zero on some punctured disk $0 < |z-z_0| < R$.

Definition 5.18: A point z_0 is an **isolated singularity** of a function f if f is analytic on a punctured disk centered z_0, but not at z_0 itself.

Definition 5.19: Let $\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$ be the Laurent series for f, valid on $0 < |z-z_0| < R$.

1. If $a_j = 0$ for all $j < 0$, then z_0 is a **removable singularity**.
2. If $a_{-m} \neq 0$ for some $m \in \mathbb{N}$ but $a_j = 0$ for all $j < -m$, then z_0 is a **pole of order m**.

3. Otherwise, z_0 is an **essential singularity**.

Example:

1. $f(z) = \frac{z^2-1}{z-1}$ has a removable singularity at $z = 1$, since its Laurent expression there is $2 + (z-1)$, which has no negative powers.
2. $g(z) = \frac{\cos z}{z^3}$ has a pole of order 3 at $z = 0$, since its Laurent series there is $\frac{1}{z^3}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)$, which has z^{-3} as its smallest exponent of z.
3. $e^{\frac{1}{z}}$ has an essential singularity at $z = 0$, since its Laurent expansion is $1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$, which has no smallest exponent.

Proposition 5.20: If f has a removable singularity at z_0, then $\lim\limits_{z \to z_0} f(z) = a_0$, f is bounded near z_0, and defining $f(z_0) = a_0$ makes f analytic at z_0.

Proposition 5.21: Let f have a pole of order m at z_0. Then $f(z) = \frac{g(z)}{(z-z_0)^m}$ for some g analytic at z_0 such that $g(z_0) \neq 0$.

Proof: Since f is analytic on $0 < |z - z_0| < R$, $f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots = \frac{1}{(z-z_0)^m}(g(z))$. Since z_0 is a pole of order m, $a_{-m} \neq 0$, so $g(z_0) \neq 0$.

Proposition 5.22: If f has a pole of order m at z_0, then $\lim\limits_{z \to z_0} |f(z)| = \infty$ and $(z - z_0)^m f(z)$ has a removable singularity at z_0.

Theorem 5.23: (Picard) Suppose f has an essential singularity at z_0. Then in any disk centered at z_0, f achieves every complex value, with possibly one exception per disk.

Definition 5.24: The **extended complex numbers** are the set $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 5.25: A **neighborhood of infinity** is an annulus $|z| > r$.

Definition 5.26: A function f is **analytic at infinity** if $g(w) = f(\frac{1}{w})$ is analytic or has a removable singularity at $w = 0$.

Definition 5.27: A function f has a **singularity at infinity** if $f(\frac{1}{w})$ has the same type of singularity at $w = 0$.

Example: Let $f(z) = \frac{3z-1}{z+2}$. If we define $f(-2) = \lim_{z \to -2} \frac{3z-1}{z+2} = \infty$ and $f(\infty) = \lim_{z\to\infty} \frac{3z-1}{z+2} = 3$, then $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is analytic on $\mathbb{C}$. And f is analytic at ∞, since $f(\frac{1}{w}) = \frac{\frac{3}{w}-1}{\frac{1}{w}+2} = \frac{3-w}{1+2w}$, which is analytic at $w = 0$. Thus f is analytic on $\hat{\mathbb{C}}$.

VI — Residue Theory

Proposition 6.1: Let Γ be a smooth closed curve enclosing z_0. Then

$$\int_\Gamma (z - z_0)^n \, \mathrm{d}z = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1 \end{cases}.$$

Proof: Deform Γ to a circle C of radius 1 centered at z_0. We can parameterize C by $z = z_0 + e^{it}$ and $dz = ie^{it}dt$ for $t \in [0, 2\pi]$. Then

$$\int_\Gamma (z - z_0)^n \, \mathrm{d}z = \int_0^{2\pi} ie^{(n+1)it} \, \mathrm{d}t.$$

If $n \neq -1$, this integral evaluates to $\left[\frac{i}{n+2} e^{(n+1)it}\right]\Big|_0^{2\pi} = 0$. Otherwise, we are integrating i from 0 to 2π, which results in $2\pi i$.

Proposition 6.2: Let f be analytic on and inside a contour Γ, except at z_0. Then

$$\int_\Gamma f(z) \, \mathrm{d}z = 2\pi i a_{-1},$$

where a_{-1} is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion for f about z_0.

Proof: We have

$$\int_\Gamma f(z) \, \mathrm{d}z = \int_\Gamma \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j \, \mathrm{d}z = 2\pi i a_{-1}.$$

Definition 6.3: Let f have the Laurent expansion $\sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$. The **residue** of f at z_0 is $\operatorname{Res}(f; z) = \operatorname{Res} z_0 = a_{-1}$.

Proposition 6.4: Suppose f has a simple pole at z_0. Then $\operatorname{Res} z_0 = \lim_{z \to z_0} (z - z_0) f(z)$.

Theorem 6.5: Suppose f has a pole of order m at z_0. Then

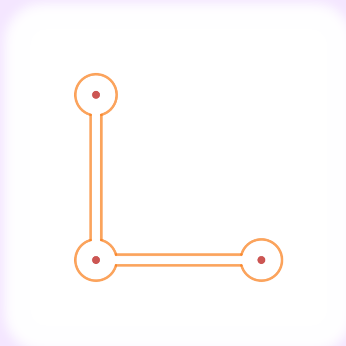
$$\operatorname{Res} z_0 = \lim_{z \to z_0} \left(\frac{1}{(m-1)!}\right) \left(\frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]\right).$$

Proposition 6.6: Suppose $f(z) = \frac{g(z)}{h(z)}$, where g and h are analytic at z_0, $g(z_0) \neq 0$, and $h(z_0) = 0$ is simple zero. Then $\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}$.

Theorem 6.7: (The Residue Theorem) Let Γ be a simple, positively-oriented closed contour and f a function analytic on and inside Γ, except at $z_1, ..., z_n$. Then

$$\int_\Gamma f(z)\ \mathrm{d}z = 2\pi i \sum_{j=1}^{n} \text{Res } z_j.$$

Proof: Continuously deform Γ into a contour that surrounds each singularity in an arc that limits to a circle, where each is sufficiently small that no two intersect. Connect the circles with pairs of oppositely-oriented parallel lines that limit to coinciding. For example,



In the limit case, the line segments will cancel one another out when integrated, since they are oriented in opposite directions. The arcs will limit to circles, and then the single-singularity Residue Theorem applies to each.

Example: Compute

$$\int_0^{2\pi} \frac{1}{(3+2\cos\theta)^2}\ \mathrm{d}\theta.$$

Consider the unit circle C, parameterized by $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta = izd\theta$ for $\theta \in [0, 2\pi]$. Then $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$, so we have

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{(3+2\cos\theta)^2}\ \mathrm{d}\theta &= \int_C \frac{1}{\left(3+z+\frac{1}{z}\right)^2}\frac{1}{iz}\ \mathrm{d}z \\
&= \frac{1}{i}\int_C \frac{1}{\left(\frac{3z+z^2+1}{z}\right)^2}\frac{1}{z}\ \mathrm{d}z \\
&= \frac{1}{i}\int_C \frac{z}{(3z+z^2+1)^2}\ \mathrm{d}z \\
&= \frac{1}{i}\int_C \frac{z}{(z-z_1)^2(z-z_2)^2}\ \mathrm{d}z,
\end{aligned}$$

where $z_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}$ and $z_2 = -\frac{3}{2} - \frac{\sqrt{5}}{2}$. Since only z_1 is inside C,

$$
\begin{aligned}
\frac{1}{i}\int_C \frac{z}{(z-z_1)^2(z-z_2)^2}\ \mathrm{d}z &= \left(\frac{1}{i}\right)(2\pi i)\text{Res } z_1 \\
&= 2\pi \lim_{z\to z_1}\left(\left(\frac{1}{1!}\right)\frac{d}{dz}\left[\frac{z}{(z-z_2)^2}\right]\right) \\
&= 2\pi \lim_{z\to z_1}\left(-\frac{z+z_2}{(z-z_2)^3}\right) \\
&= 2\pi\left(\frac{3}{\sqrt{5}^3}\right) \\
&= \frac{6\pi}{5\sqrt{5}}.
\end{aligned}
$$

Definition 6.8: The **principal value** of $\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x$ is

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\ \mathrm{d}x.$$

Example: Compute

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\ \mathrm{d}x.$$

Let C_ρ^+ be the upper half of the circle centered at 0 with radius ρ, let γ_ρ be the line segment from $-\rho$ to ρ, and let $\Gamma_\rho = C_\rho^+ + \gamma_\rho$. Then

$$
\begin{aligned}
\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\ \mathrm{d}x &= \lim_{\rho\to\infty}\int_{\gamma_\rho} \frac{z^2+1}{z^4+1}\ \mathrm{d}z \\
&= \lim_{\rho\to\infty}\int_{\Gamma_\rho} \frac{z^2+1}{z^4+1}\ \mathrm{d}z - \lim_{\rho\to\infty}\int_{C_\rho^+} \frac{z^2+1}{z^4+1}\ \mathrm{d}z.
\end{aligned}
$$

Now we tackle each integral separately. For the first, around Γ_ρ, we use the Residue Theorem. The roots of z^4+1 are $e^{i\frac{\pi}{4}}$, $e^{i\frac{3\pi}{4}}$, $e^{i\frac{5\pi}{4}}$, and $e^{i\frac{7\pi}{4}}$, but only the first two will eventually be inside Γ_ρ. Therefore,

$$
\begin{aligned}
\lim_{\rho\to\infty}\int_{\Gamma_\rho} \frac{z^2+1}{z^4+1}\ \mathrm{d}z &= 2\pi i(\text{Res } e^{i\frac{\pi}{4}} + \text{Res } e^{i\frac{3\pi}{4}}) \\
&= 2\pi i\left(-\frac{i}{2\sqrt{2}} - \frac{i}{2\sqrt{2}}\right) \\
&= \pi\sqrt{2}.
\end{aligned}
$$

For the second integral, notice that

$$
\begin{aligned}
\int_{C_\rho^+} \frac{z^2+1}{z^4+1}\ \mathrm{d}x &\le \left(\max_{z\in C_\rho^+}\left|\frac{z^2+1}{z^4+1}\right|\right)(\pi\rho) \\
&= \left(\frac{\rho^2+1}{\rho^4+1}\right)(\pi\rho) \to 0 \text{ as } \rho\to\infty.
\end{aligned}
$$

Thus we finally conclude that

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\ \mathrm{d}x = \pi\sqrt{2}.$$

Theorem 6.9: Let $p(z)$ and $q(z)$ be polynomials in $\mathbb{C}$ with $\deg q \geq (\deg p) + 2$. Then

$$\lim_{\rho\to\infty} \int_{C_\rho^+} \frac{p(z)}{q(z)}\ \mathrm{d}z = 0.$$

The theorem also holds for C_ρ^-.

Example: Compute

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\ \mathrm{d}x.$$

Since $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$,

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\ \mathrm{d}x = \frac{1}{2}\ \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\ \mathrm{d}x + \frac{1}{2}\ \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{-ix}}{(x^2+1)^2}\ \mathrm{d}x.$$

The first integral can be solved by using the contour $\Gamma_1 = C_\rho^+ + \gamma_\rho$ to split the integral into two more, the first of which can be solved by the Residue Theorem and the second by showing it vanishes in the limit with a method similar to the previous example. The second is identical, except that it uses the contour $\Gamma_2 = C_\rho^- + \gamma_\rho$.

Theorem 6.10: (Jordan's Lemma) If $P(z)$ and $Q(z)$ are polynomials, Q has no real zeros, $\deg Q > (\deg P)+1$, and $m > 0$, then

$$\lim_{\rho\to\infty} \int_{C_\rho^+} e^{miz} \frac{P(z)}{Q(z)}\ \mathrm{d}z = 0.$$

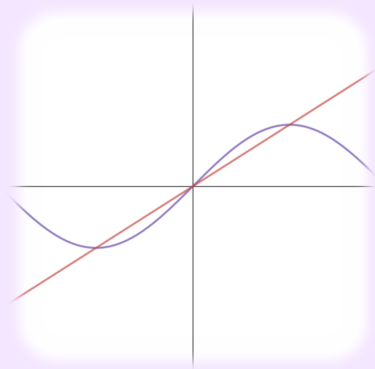
Proof: We can parameterize C_ρ^+ by $z = \rho e^{it}$ for $t \in [0, \pi]$. Then

$$\begin{aligned}
\left| \int_{C_\rho^+} e^{miz} \frac{P(z)}{Q(z)}\ \mathrm{d}z \right| &= \left| \int_0^\pi e^{mi\rho e^{it}} \frac{P(\rho e^{it})}{Q(\rho e^{it})} i\rho e^{it}\ \mathrm{d}t \right| \\
&\leq \int_0^\pi \left| e^{mi\rho e^{it}} \frac{P(\rho e^{it})}{Q(\rho e^{it})} i\rho e^{it} \right|\ \mathrm{d}t.
\end{aligned}$$

Now $\left|e^{mi\rho e^{it}}\right| = \left|e^{mi\rho(\cos t + i\sin t)}\right| = \left|e^{-m\rho\sin t}\right| = e^{-m\rho \sin t}$, and $\left|\frac{P(\rho e^{it})}{Q(\rho e^{it})} i\rho e^{it}\right| = \left|\frac{\rho P(\rho e^{it})}{Q(\rho e^{it})}\right| \leq K$ for some $K \in \mathbb{R}$ as $\rho \to \infty$, since $\deg \rho P(\rho e^{it}) = (\deg P(\rho e^{it})) + 1 < \deg Q(\rho e^{it})$. Thus

$$\begin{aligned}
\int_0^\pi \left| e^{mi\rho e^{it}} \frac{P(\rho e^{it})}{Q(\rho e^{it})} i\rho e^{it} \right|\ \mathrm{d}t &\leq K \int_0^\pi e^{-m\rho\sin t}\ \mathrm{d}t \\
&= 2K \int_0^{\frac{\pi}{2}} e^{-m\rho\sin t}\ \mathrm{d}t \\
&\leq 2K \int_0^{\frac{\pi}{2}} e^{-m\rho\frac{2}{\pi}t}\ \mathrm{d}t
\end{aligned}$$

This last inequality is due to the fact that $\sin t \geq \frac{2}{\pi} t$ for $t \in [0, \frac{2}{\pi}]$, as the following plot shows.



Continuing, we have

$$\begin{aligned}\le 2K\int_0^{\frac{\pi}{2}} e^{-m\rho\frac{2}{\pi}t}\ \mathrm{d}t &= 2K\left[-\frac{\pi}{2m\rho}e^{-m\rho\frac{2}{\pi}t}\right]\Bigg|_0^{\frac{\pi}{2}}\\ &= -\frac{\pi K}{m\rho}(e^{-m\rho}-1)\\ &< \frac{\pi K}{m\rho}\to 0 \text{ as } \rho\to\infty.\end{aligned}$$

Theorem 6.11: (The Arc Lemma) Let f have a simple pole at $c \in \mathbb{R}$ and let T_r be the arc given by $z = c + re^{i\theta}$ for $\theta \in [\theta_1, \theta_2]$. Then

$$\lim_{r\to 0}\int_{T_r} f(z)\ \mathrm{d}z = i(\theta_2-\theta_1)\mathrm{Res}(f;c).$$

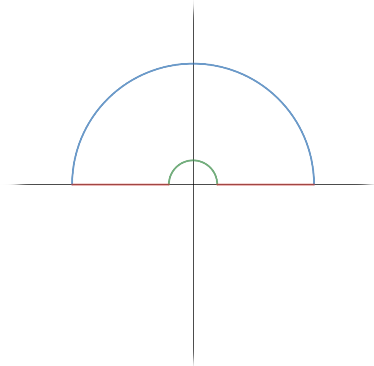
Proof: Since f has a simple pole at c, $f(z) = \frac{a_{-1}}{z-c} + g(z)$, where $g(z) = \sum_{j=0}^{\infty} a_j(z-c)^j$ is analytic at c and therefore bounded around it, say by M. Then

$$\begin{aligned}\int_{T_r} f(z)\ \mathrm{d}z &= \int_{T_r}\frac{a_{-1}}{z-c}\ \mathrm{d}z + \int_{T_r} g(z)\ \mathrm{d}z\\ &= a_{-1}\int_{\theta_1}^{\theta_2}\frac{1}{re^{i\theta}}ire^{i\theta}\ \mathrm{d}\theta + \int_{T_r} g(z)\ \mathrm{d}z\\ &= i(\theta_2-\theta_1)\mathrm{Res}(f;c) + \int_{T_r} g(z)\ \mathrm{d}z\\ &\le i(\theta_2-\theta_1)\mathrm{Res}(f;c) + \left(\max_{z\in T_r} g(z)\right)(\theta_2-\theta_1)r\\ &\le i(\theta_2-\theta_1)\mathrm{Res}(f;c) + M\left(\frac{\theta_2-\theta_1}{2\pi}r\right)\to i(\theta_2-\theta_1)\mathrm{Res}(f;c) \text{ as } r\to 0.\end{aligned}$$

Example: Compute

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\ \mathrm{d}x.$$

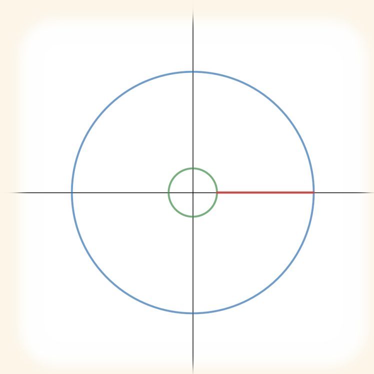
Because there is a singularity at 0, along the path of integration, we will need to use a more intricate contour and the Arc Lemma.



Let C_ρ^+ be the upper half-circle of radius ρ, oriented counter-clockwise and S_r^+ the upper half-circle of radius r, oriented clockwise. Then

$$\begin{aligned}\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \,\mathrm{d}x &= \lim_{\substack{\rho\to\infty \\ r\to 0}} \left(\int_{C_\rho^+} \frac{e^{iz}}{z} \,\mathrm{d}z - \int_{S_r^+} \frac{e^{iz}}{z} \,\mathrm{d}z \right) \\ &= \lim_{r\to 0} \int_{S_r^+} \frac{e^{iz}}{z} \,\mathrm{d}z \\ &= -i(0-\pi)\mathrm{Res}\left(\frac{e^{iz}}{z}; 0 \right) \\ &= i\pi. \end{aligned}$$

Theorem 6.12: (The Upgraded Residue Theorem) Let Γ be the following contour. The inner circle has radius ε, the outer one ρ, and the red line is a traced twice, once from ε to ρ at argument 0 (γ_1) and back from ρ to ε at argument 2π (γ_2).



Let $f(z) = z^\alpha \frac{P(z)}{Q(z)}$, where $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, Q has no zeros on Γ, and we take the branch of z^α with $0 < \mathrm{Arg}\, z \leq 2\pi$. Then

$$\int_\Gamma f(z) \,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f; z_j),$$

where the z_j are the singularities of f inside Γ.

Example: Compute

$$\text{p.v.} \int_0^\infty \frac{x^\alpha}{(x+9)^2}\ \mathrm{d}x,$$

where $\alpha \in (-1,1) \setminus \{0\}$.

We have

$$\begin{aligned}
\lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \int_\varepsilon^\rho \frac{x^\alpha}{(x+9)^2}\ \mathrm{d}x &= \lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \int_{\gamma_1} \frac{z^\alpha}{(z+9)^2}\ \mathrm{d}z \\
&= \lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \left(\left(\int_\Gamma - \int_{C_\rho} - \int_{C_\varepsilon} - \int_{\gamma_1} \right) \left(\frac{z^\alpha}{(z+9)^2} \right)\ \mathrm{d}z \right).
\end{aligned}$$

Now both $\int_{C_\rho}$ and $\int_{C_\varepsilon}$ limit to 0, and $\int_\Gamma$ is solved with the Upgraded Residue Theorem, so we need only examine $\int_{\gamma_2}$. We have

$$\begin{aligned}
\int_{\gamma_2} \frac{z^\alpha}{(z+9)^2}\ \mathrm{d}z &= \int_{\gamma_2} \frac{e^{\alpha \log z}}{(z+9)^2}\ \mathrm{d}z \\
&= \int_{\gamma_2} \frac{e^{\alpha(\mathrm{Log}\ |z|+2\pi i)}}{(z+9)^2}\ \mathrm{d}z \\
&= \int_\rho^\varepsilon \frac{e^{\alpha \mathrm{Log}\ |z|} e^{2\pi i\alpha}}{(z+9)^2}\ \mathrm{d}z \\
&= -e^{2\pi i\alpha} \int_\varepsilon^\rho \frac{e^{\alpha \mathrm{Log}\ |z|}}{(z+9)^2}\ \mathrm{d}z \\
&= -e^{2\pi i\alpha} \int_{\gamma_1} \frac{z^\alpha}{(z+9)^2}\ \mathrm{d}z.
\end{aligned}$$

Now we can solve our original integral:

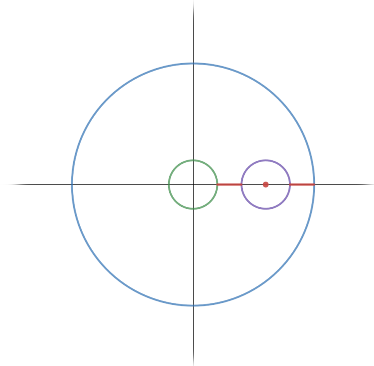
$$\lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \int_{\gamma_1} \frac{z^\alpha}{(z+9)^2}\ \mathrm{d}z = \left(\frac{1}{1-e^{2\pi i\alpha}} \right) \lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \int_\Gamma \frac{z^\alpha}{(z+9)^2}\ \mathrm{d}z,$$

which, after much calculation, simplifies to $\frac{9^{\alpha-1}\pi\alpha}{\sin \pi\alpha}$.

Example: Compute

$$\text{p.v.} \int_0^\infty \frac{x^{\frac{1}{3}}}{x^2-4}\ \mathrm{d}x.$$

Because of the singularity on the positive real axis, we will need to modify our contour.



If we label the new circle C_δ, then we have a truly gargantuan computation:

$$\int_\Gamma = \int_{C_\varepsilon} + \int_\varepsilon^\delta + \int_{C_\delta^+} + \int_\delta^\rho + \int_{C_\rho} + \int_\rho^\delta + \int_{C_\delta^-} + \int_\delta^\varepsilon .$$

The only integrals we have not seen are $\int_{C_\delta^+}$ and $\int_{C_\delta^-}$. Both can be solved with the Arc Lemma: on C_δ^+, for example, $\frac{z^{\frac{1}{3}}}{z^2-4} = \frac{e^{\frac{1}{3}\text{Log } z}}{z^2-4}$, so

$$\lim_{\delta\to 0}\int_{C_\delta^+}\frac{z^{\frac{1}{3}}}{z^2-4}\,\mathrm{d}z = -(i\pi)\text{Res}\left(\frac{e^{\frac{1}{3}\text{Log } z}}{z^2-4};2\right).$$

Definition 6.13: A function is **meromorphic** if is analytic, except possibly at poles.

Theorem 6.14: (The Argument Principle) Let C be a simple, closed, positively-oriented contour and f a nonzero function analytic on C and meromorphic inside it. Then

$$\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i(N_0(f) - N_p(f)),$$

where $N_0(f)$ is the number of zeros of f inside C and $N_p(f)$ is the number of poles of f inside C, counting multiplicity.

Proof: By the Residue Theorem,

$$\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i\sum_{j=1}^n \text{Res}\left(\frac{f'}{f};z_j\right).$$

Now all the poles of $\frac{f'}{f}$ occur at either zeros of f or poles of f'. If z_0 is a zero of f of order m, then $f(z) = (z-z_0)^m g(z)$. Then

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}.$$

Since $\frac{g'z}{g(z)}$ is analytic at z_0, since both g and g' are and $g(z_0) \neq 0$, $\text{Res}\left(\frac{f'}{f};z_0\right) = \text{Res}\left(\frac{m}{z-z_0};z_0\right) = m$.

For a pole z_p of f (not f') of order k, $f(z) = \frac{g(z)}{(z-z_p)^k}$, so

$$\frac{f'(z)}{f(z)} = \frac{(z-z_p)^k}{g(z)}\left(\frac{g'(z)(z-z_p)^k - g(z)\left(k(z-z_p)^{k-1}\right)}{(z-z_p)^{2k}}\right) = \frac{g'(z)}{g(z)} - \frac{k}{z-z_p}.$$

Then $\text{Res}\left(\frac{f'}{f};z_p\right) = \text{Res}\left(-\frac{k}{z-z_p};z_p\right) = -k$. Since these are all the poles of $\frac{f'}{f}$,

$$\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i(N_0(f) - N_p(f)).$$

Theorem 6.15: (Rouché) Suppose f and h are analytic on and inside a simple, closed, positively-oriented curve C, and that $|h(z)| < |f(z)|$ for all $z \in C$. Then f and $f + h$ ave the same number of zeros inside C.

Proof: Since $0 \geq |h(z)| < |f(z)|$, $f(z) \neq 0$ for any $z \in C$. And since $|h(z)| < |f(z)|$, $f(z) + h(z) \neq 0$ either. Let $F(z) = \frac{h(z)}{f(z)}$. Then $h(z) = F(z)f(z)$, so

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{f'(z)+h'(z)}{f(z)+h(z)}\,\mathrm{d}z &= \frac{1}{2\pi i}\int_C \frac{f'(z)+(F(z)f(z))'}{f(z)+F(z)f(z)}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_C \frac{f'(z)+F'(z)f(z)+F(z)f'(z)}{f(z)(1+F(z))}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_C \frac{f'(z)(1+F(z))}{f(z)(+F(z))} + \frac{F'(z)}{1+F(z)}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z,
\end{aligned}$$

since $1 + F(z) = \frac{f(z)+h(z)}{h(z)} \neq 0$. By The argument principle, f and $f + h$ ave the same number of zeros inside C.

Example: How many zeros of $g(z) = z^6 + 4z - 3$ are inside $|z| = 3$?

Let $f(z) = z^6$ and $h(z) = 4z - 3$. On $|z| = 3$, $|h(z)| = |4z - 3| \leq |4z| + 3 = 15 \leq |f(z)| = 3^6$. Thus $g = f + h$ has as many zeros as f inside $|z| = 3$ — six. This method gives an alternate proof of the Fundamental Theorem of Algebra.

VII — Spaces of Analytic Functions

Definition 7.1: An **inner product** on a complex vector space V is a mapping $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$ such that for all $x, y, z \in \mathbb{C}$ and $\lambda \in \mathbb{C}$,

1. $\langle x, x \rangle > 0$ unless $x = 0$.
2. $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$.
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

Example: The set $C[0,1] = \{f : [0,1] \longrightarrow \mathbb{C} \mid f \text{ is continuous}\}$ with the inner product $\langle f, g \rangle = \int_0^1 f\overline{g}$ is an inner product space.

Proposition 7.2: Let V be an inner product space, $x, y, z \in V$, and $\lambda \in \mathbb{C}$. Then

1. $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$.
2. $\langle x, y + z \rangle = \langle x, y \rangle \langle x, z \rangle$.
3. $\langle x, 0 \rangle = \langle 0, x \rangle = 0$.
4. If $\langle w, y \rangle = \langle w, z \rangle$ for all $w \in V$, then $y = z$.

Definition 7.3: Let $x \in V$. The **norm** of x is $||x|| = \sqrt{\langle x, x \rangle}$.

Proposition 7.4: Let $x \in V$ and $\lambda \in \mathbb{C}$. Then

1. $||x|| > 0$.
2. $||x|| = 0$ if and only if $x = 0$.
3. $||\lambda x|| = |\lambda| \cdot ||x||$.

Theorem 7.5: (The Cauchy-Schwarz Inequality) Let $x, y \in V$. Then $|\langle x, y \rangle| \leq ||x|| \cdot ||y||$, with equality if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C}$.

Proof: If $x + \lambda y \neq 0$ for any $\lambda \in \mathbb{C}$, then for all $\lambda \in \mathbb{C}$,

$$\begin{aligned} 0 &< \langle x + \lambda y, x + \lambda y \rangle \\ &= \langle x, x \rangle + \langle x, \lambda y \rangle + \langle \lambda y, x \rangle + \langle \lambda y, \lambda y \rangle \\ &= ||x||^2 + \overline{\lambda} \langle x, y \rangle + \overline{\overline{\lambda} \langle x, y \rangle} + |\lambda|^2 ||y||^2 \\ &= ||x||^2 + 2\mathrm{Re}\left(\overline{\lambda} \langle x, y \rangle\right) + |\lambda|^2 ||y||^2. \end{aligned}$$

Consider the line in $\mathbb{C}$ passing through 0 and $\langle x, y \rangle$. There is a $\theta \in [0, \pi)$ such that every point on the line is of the form $te^{i\theta}$ for some $t \in \mathbb{R}$. Let $\lambda = te^{i\theta}$ be a function of t. Then

$$\begin{aligned} 0 &< \|x\|^2 + 2\mathrm{Re}\left(te^{-i\theta}|\langle x,y\rangle|e^{i\theta}\right) + |te^{i\theta}|^2\|y\|^2 \\ &= \|x\|^2 + 2t|\langle x,y\rangle| + t^2\|y\|^2. \end{aligned}$$

As a function of t, this is a quadratic with no roots, so the discriminant is negative. Thus $4|\langle x,y\rangle|^2 - 4\|y\|^2\|x\|^2 < 0$, so $|\langle x,y\rangle| < \|x\| \cdot \|y\|$.

Example: Let $f : [0,1] \longrightarrow \mathbb{C}$ be continuous. Then

$$\left|\int_0^1 f(t)\sin(\pi t)\ \mathrm{d}t\right| \le \frac{1}{\sqrt{2}}\sqrt{\int_0^1 |f(t)|^2\ \mathrm{d}t}.$$

Theorem 7.6: (The Triangle Inequality) For all $x, y \in V$, $\|x+y\| \le \|x\| + \|y\|$.

Proof: We have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + 2\mathrm{Re}\left(\langle x,y\rangle\right) + \|y\|^2 \\ &\le \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \\ &\le \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Proposition 7.7: (The Parallelogram Law) For all $x, y \in V$, $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Proposition 7.8: (The Polarization Identity) For all $x, y \in V$, $4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2 + \|x+iy\|^2 - \|x-iy\|^2$.

Definition 7.9: The set

$$RL^2 = \left\{ f(z) = \frac{p(z)}{q(z)} \;\middle|\; p \text{ and } q \text{ are polynomials in } \mathbb{C}, \text{ and } q(z) \ne 0 \text{ on } |z| = 1 \right\}$$

equipped with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi i}\int_{|z|=1} f(z)\overline{g(z)}\left(\frac{1}{z}\right)\ \mathrm{d}z$$

is an inner product space. The set $RH^2 = \{f \in RL^2 \mid f \text{ is analytic on } |z| < 1\}$ is also one, using the same inner product.

Example: Show that in RL^2, $\left\langle \frac{1}{z-\alpha}, \frac{1}{z-\beta}\right\rangle = \frac{1}{1-\alpha\overline{\beta}}$, where $|\alpha|, |\beta| < 1$.

We have

$$\left\langle \frac{1}{z-\alpha}, \frac{1}{z-\beta} \right\rangle = \frac{1}{2\pi i} \int_{|z|=1} \left(\frac{1}{z-\alpha}\right)\left(\frac{1}{\overline{z}-\overline{\beta}}\right)\left(\frac{1}{z}\right) \, \mathrm{d}z$$
$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{(1-z\overline{\beta})(z-\alpha)} \, \mathrm{d}z$$
$$= \frac{1}{1-\alpha\overline{\beta}}.$$

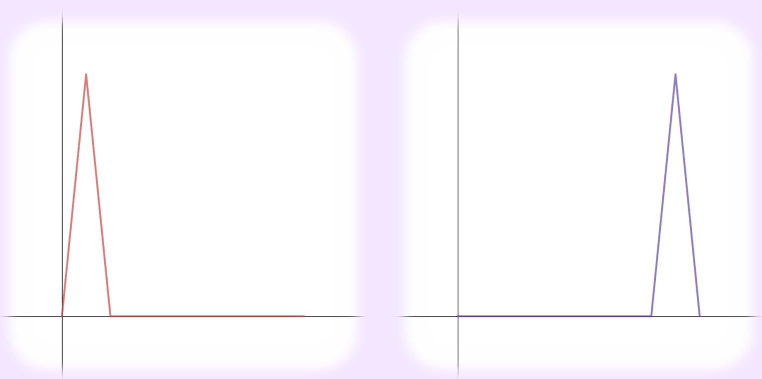
Definition 7.10: Let V be a vector space. A **norm** on V is a function $\|\cdot\| : V \longrightarrow [0,\infty)$ such that for all $x, y \in V$ and $\lambda \in \mathbb{C}$,

1. $\|x\| > 0$ unless $x = 0$.
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$.
3. $\|x + y\| \le \|x\| + \|y\|$.

Example: A common norm equipped on $C[0,1]$ is the **sup norm**, given by $\|f\|_\infty = \sup\{|f(x)| \mid x \in [0,1]\}$. Notice that the sup norm is not the same as the norm induced by the standard inner product on $C[0,1]$: for example, $\|\sin \pi t\|_\infty = 1$, but $\|\sin \pi t\| = \frac{1}{\sqrt{2}}$.

Proposition 7.11: Not every norm induces an inner product.

Proof: Consider the following two functions f and g in $C[0,1]$:



Then $\|f+g\|_\infty = \|f-g\|_\infty = \|f\|_\infty = \|g\|_\infty = 1$. If $\|\cdot\|$ corresponded to an inner product, then by the parallelogram law, $\|f+g\|_\infty + \|f-g\|_\infty = 2\|f\|_\infty + 2\|g\|_\infty$, but one is 2 and the other 4. $\lightning$

Definition 7.12: A sequence $(x_n) \subseteq V$ **converges** to $x \in V$, written $(x_n) \to x$, if $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.

Definition 7.13: A sequence $(x_n) \subseteq V$ is **Cauchy** if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n, m \ge N$, then $\|x_n - x_m\| < \varepsilon$.

Definition 7.14: A normed vector space V is **complete** if every Cauchy sequence in V converges to a limit in V.

Example: Neither $\mathbb{Q}$ nor $C[0,1]$ is complete, at least with $|\cdot|$ and $\int_0^1 f\overline{g}$.

Definition 7.15: A **Banach space** is a complete normed vector space.

Definition 7.16: A **Hilbert space** is a complete inner product space.

Definition 7.17: Let $\mathcal{H}$ be an inner product space. Two vectors $x, y \in \mathcal{H}$ are **orthogonal**, written $x \perp y$, if $\langle x, y \rangle = 0$.

Example: In RH^2 with the standard inner product,

$$\begin{aligned}\left\langle \frac{3z-1}{z^2+4}, \frac{3}{3-z} \right\rangle &= \frac{1}{2\pi i} \int_{|z|=1} \frac{(3z-1)(3)}{(z^4+4)(3-\overline{z})(z)} \, \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_{|z|=1} \frac{9z-3}{(z^4+4)(3z-1)} \, \mathrm{d}z \\ &= \operatorname{Res}\left(\frac{1}{3}\right) \\ &= 0,\end{aligned}$$

so $f \perp g$.

Theorem 7.18: (The Pythagorean Theorem) Let $x, y \in \mathcal{H}$ with $x \perp y$. Then $||x+y||^2 = ||x||^2 + ||y||^2$.

Proof: We have $||x+y||^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle = \langle x, x \rangle + \langle y, y \rangle = ||x||^2 + ||y||^2$.

Corollary 7.18.1: Let $x_1, ..., x_n \in \mathcal{H}$ with $x_i \perp x_j$ for all $i \neq j$. Then $||x_1 + \cdots + x_n||^2 = ||x_1||^2 + \cdots + ||x_n||^2$.

Definition 7.19: Vectors $e_1, ..., e_n \in \mathcal{H}$ are **orthonormal** if $||e_i|| = 1$ for all i and $e_i \perp e_j$ for all $i \neq j$.

Example: $(1,0)$ and $(0,i)$ are orthonormal in $\mathbb{R}^2$ over $\mathbb{C}$.

Theorem 7.20: (Bessel's Inequality) Let $e_1, e_2, \ldots \in \mathcal{H}$ be orthonormal and let $x \in \mathcal{H}$. Then

$$||x||^2 \geq \sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2.$$

Proof: We have

$$\begin{aligned}
0 &\leq \left\| x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\|^2 \\
&= ||x||^2 - 2\mathrm{Re}\left(\sum_{j=1}^{n} \langle x, \langle x, e_j \rangle e_j \rangle \right) + \sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \\
&= ||x||^2 - 2\sum_{j=1}^{n} |\langle x, e_j \rangle|^2 + \sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \\
&= ||x||^2 - \sum_{j=1}^{n} |\langle x, e_j \rangle|^2.
\end{aligned}$$

Thus

$$||x||^2 \geq \sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2.$$

Theorem 7.21: Let $e_1, e_2, \ldots \in \mathcal{H}$ be orthonormal in a Hilbert space $\mathcal{H}$. Then the following two statements are equivalent:

1. If $\langle x, e_j \rangle = 0$ for all j, then $x = 0$.
2. For all $x \in \mathcal{H}$, $||x||^2 = \sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2$.

Proof: ($\Rightarrow$) Let $x \in \mathcal{H}$ and let $y = x - \sum \langle x, e_j \rangle e_j$. Then $\langle y, e_j \rangle = 0$ for all j, so $y = 0$.

($\Leftarrow$) If $\langle x, e_j \rangle = 0$ for all j, then $||x||^2 = 0$, so $x = 0$.

Theorem 7.22: Let $f = \sum_{j=-\infty}^{\infty} a_j z^j \in RL^2 \setminus RH^2$. Then $\sum_{j=0}^{\infty} a_j z^j$ is the closest element of RH^2 to f.

Definition 7.23: The open unit disk is $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Definition 7.24: Hardy space is

$$H^2 = \left\{ f : \mathbb{D} \longrightarrow \mathbb{C} \;\middle|\; f \text{ is analytic and } \sum_{j=0}^{\infty} |a_j|^2 \text{ converges, where } f(z) = \sum_{j=0}^{\infty} a_j z^j \right\}.$$

Proposition 7.25: H^2 is an inner product space with the inner product

$$\langle f, g \rangle = \left\langle \sum_{j=0}^{\infty} a_j z^j, \sum_{j=0}^{\infty} b_j z^j \right\rangle = \sum_{j=0}^{\infty} a_j \overline{b_j}.$$

Example: 0, $z + 2z^5$, $\frac{3}{3-z}$, and $\frac{1-2z}{2-z}$ are elements of H^2, but $\frac{1}{6z}$ and $\frac{1}{1-z}$ are not.

Proposition 7.26: In H^2, $1, z, z^2, \ldots$ are orthonormal.

Theorem 7.27: H^2 is complete.

Proof: Given $(f_n) \subseteq H^2$, $(f_n) = (a_{n,0} + a_{n,1} z + a_{n,2} z^2 + \cdots) \to A_0 + A_1 z + A_2 z^2 + \cdots \in H^2$.

Definition 7.28: A **reproducing kernel** is a function $K_\omega = \frac{1}{1-\omega z}$ for $|\omega| < 1$.

Theorem 7.29: Let $f \in H^2$ and $\omega \in \mathbb{D}$. Then

$$\left\langle f, \frac{1}{K_\omega} \right\rangle = f(\omega).$$

Proof: We have

$$\begin{aligned}
\langle f, K_\omega \rangle &= \left\langle f, \frac{1}{1 - \alpha z} \right\rangle \\
&= \left\langle \sum a_j z^j, \sum (\alpha z)^j \right\rangle \\
&= \sum a_j \alpha^j \\
&= f(\alpha).
\end{aligned}$$

Theorem 7.30: An alternate definition for H^2 is

$$H^2 = \left\{ f : \mathbb{D} \longrightarrow \mathbb{C} \text{ analytic } \middle| \lim_{r \to 1^-} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^2 \, \mathrm{d}\theta \right) < \infty \right\}.$$

Proof: Let $f(z) = \sum a_j z^j \in H^2$. Then

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^2\ \mathrm{d}\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{j=0}^{\infty} a_j\left(re^{i\theta}\right)^j\right)\overline{\left(\sum_{k=0}^{\infty} a_k\left(re^{i\theta}\right)^k\right)}\ \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_j\overline{a_k}r^{j+k}e^{i(j-k)\theta}\ \mathrm{d}\theta \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi} a_j\overline{a_k}r^{j+k}e^{i(j-k)\theta}\ \mathrm{d}\theta\right) \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}|a_j|^2 r^{2j}\ \mathrm{d}\theta\right) \\
&= \sum_{j=0}^{\infty}|a_j|^2 r^{2j}.
\end{aligned}
$$

Thus

$$
\|f\|^2 = \lim_{r\to 1^-}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^2\ \mathrm{d}\theta\right),
$$

so

$$
H^2 = \left\{f:\mathbb{D}\longrightarrow\mathbb{C}\text{ analytic}\ \middle|\ \lim_{r\to 1^-}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^2\ \mathrm{d}\theta\right)<\infty\right\}.
$$

Corollary 7.30.1: RH^2 is a subspace of H^2.

Definition 7.31: **Ackermann space** is

$$
A^2 = \left\{f:\mathbb{D}\longrightarrow\mathbb{C}\text{ analytic}\ \middle|\ \sum\frac{|a_j|^2}{j+1}<\infty\right\}.
$$

Definition 7.32: The set $\mathcal{D}$ is defined as

$$
\mathcal{D} = \left\{f:\mathbb{D}\longrightarrow\mathbb{C}\text{ analytic}\ \middle|\ \sum|a_j|^2(j+1)<\infty\right\}.
$$

Definition 7.33: Let $(\beta(j))$ be a sequence. The **weighted Hardy space** corresponding to $(\beta(j))$ is

$$
H^2(\beta) = \left\{f:\mathbb{D}\longrightarrow\mathbb{C}\text{ analytic}\ \middle|\ \sum|a_j|^2\beta(j)^2<\infty\right\},
$$

equipped with the inner product

$$
\left\langle\sum a_j z^j,\sum b_j z^j\right\rangle = \sum a_j\overline{b_j}\beta(j)^2.
$$

Definition 7.34: A linear operator $T:\mathcal{H}\longrightarrow\mathcal{K}$ between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is **bounded** if $\sup\limits_{x\neq 0}\frac{\|Tx\|}{\|x\|}<\infty$. If T is bounded, the **operator norm** of T is $\|T\| = \sup\limits_{x\neq 0}\frac{\|Tx\|}{\|x\|}$.

Proposition 7.35: If T is bounded, then it is continuous.

Definition 7.36: Let $z \in \mathbb{C}$. The **left multiplication operator** corresponding to z is $M_z : H^2 \longrightarrow H^2$, given by $M_z(f) = zf$. Since $\|M_z\| = 1$, M_z is continuous.

Definition 7.37: Let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be bounded. The **adjoint** of T is the linear operator $T^* : \mathcal{H} \longrightarrow \mathcal{H}$, defined such that $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for all $x, y \in \mathcal{H}$.

Example: For $f(z) = \sum a_j z^j$ and $g(z) = \sum b_j z^j$, $\langle M_z f, g\rangle = a_0\overline{b_1} + a_1\overline{b_2} + \cdots = \langle f, b_1 + b_2 z + \cdots\rangle$. Thus $M_z^*(b_0 + b_1 z + \cdots) = b_1 + b_2 z + \cdots$. Notice that $M_z^* M_z(a_0 + a_1 z + a_2 z^2 + \cdots) = a_0 + a_1 z + a_2 z^2 + \cdots$, but $M_z M_z^*(a_0 + a_1 z + a_2 z^2 + \cdots) = a_1 z + a_2 z^2 + \cdots \neq a_0 + a_1 z + a_2 z^2 + \cdots$.

Definition 7.38: Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic. The **multiplication operator** corresponding to φ is $M_\varphi : H^2 \longrightarrow H^2$, defined by $M_\varphi f(z) = \varphi(z) f(z)$.

Proposition 7.39: Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic and let $f \in H^2$. Then $\|M_\varphi f\| \leq \|f\|$.

Proof: We have

$$\begin{aligned}
\|M_\varphi f\|^2 &= \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \left|(M_\varphi f)\left(re^{i\theta}\right)^2\right| \, \mathrm{d}\theta \\
&= \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \left|\varphi\left(re^{i\theta}\right)\right|^2 \left|f\left(re^{i\theta}\right)\right|^2 \, \mathrm{d}\theta \\
&\leq \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 \, \mathrm{d}\theta \\
&= \|f\|^2.
\end{aligned}$$

Definition 7.40: Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic and let $f \in H^2$. The **composition operator** corresponding to φ is C_φ, defined by $C_\varphi f = f(\varphi)$. Note that it is not clear that C_φ maps into H^2.

Theorem 7.41: (Littlewood) If $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is analytic and $\varphi(0) = 0$, then $C_\varphi : H^2 \longrightarrow H^2$ and $\|C_\varphi f\| \leq \|f\|$ for all $f \in H^2$.

Proof: Let $f = \sum a_j z^j \in H^2$. Then $f(z) = f(0) + M_z M_z^* f$ and for all $n \in \mathbb{N}$, $\left((M_z^*)^n f\right)(0) = a_n$. Substituting $\varphi(z)$ for z in the first equation, we find that

$$\begin{aligned}
f(\varphi(z)) &= f(\varphi(0)) + M_{\varphi(z)}\left(M_{\varphi(z)}^*(f(\varphi))\right) \\
&= f(0) + M_\varphi\left(M_\varphi^*(f(\varphi))\right).
\end{aligned}$$

Therefore,

$$\begin{aligned} C_\varphi f &= f(0) + M_\varphi(M_\varphi^*(f(\varphi(z)))) \\ &= f(0) + M_\varphi(M_\varphi^*(C_\varphi f(z))) \\ &= f(0) + M_\varphi(C_\varphi(M_z^*(f(z))). \end{aligned}$$

Since $M_\varphi C_\varphi M_z^* f = M_\varphi M_\varphi^* f(\varphi)$ and $M_\varphi M_\varphi^* f(\varphi)$ has no constant term, $f(0)$ and $M_\varphi C_\varphi M_z^* f$ are orthogonal. Thus $||C_\varphi f||^2 = |f(0)|^2 + ||M_\varphi C_\varphi M_z^* f||^2 \le |f(0)|^2 + ||C_\varphi M_z^* f||^2$, since M_φ is a contraction mapping (that is, it only makes things smaller, since $|\varphi(z)| < 1$). Continuing similarly,

$$\begin{aligned} ||C_\varphi f||^2 &\le |f(0)|^2 + |(M_z^*(f))(0)|^2 + \cdots + |((M_z^*)^n(f))(0)|^2 + \left\|C_\varphi(M_z^*)^{n+1} f\right\|^2 \\ &= |a_0|^2 + |a_1|^2 + \cdots + |a_n|^2 + \left\|C_\varphi(M_z^*)^{n+1} f\right\|^2. \end{aligned}$$

Thus if f is a polynomial of degree n, $||C_\varphi||^2 \le ||f||^2 + 0$, so $||C_\varphi|| \le ||f||$. Since every element of H^2 is the uniformly convergent limit of polynomials, $||C_\varphi|| \le ||f||$ for all $f \in H^2$.

Theorem 7.42: If $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ is analytic, then $C_\varphi : H^2 \longrightarrow H^2$, and

$$\frac{1}{\sqrt{1 - |\varphi(0)|^2}} \le ||C_\varphi|| \le \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$

Comment: Computing $||C_\varphi||$ exactly can be extremely difficult — for example, $||C_\varphi||$ for $\varphi(z) = \frac{(3+3i)z-(9+i)}{4z-12}$ is not known.

Comment: We can find a lower bound for $||C_\varphi||$ by suping only over reproducing kernels: if we define

$$S_\varphi = \sup_{\omega \in \mathbb{D}} \frac{||C_\varphi K_\omega||}{||K_\omega||},$$

then $S_\varphi \le ||C_\varphi||$, and S_φ is far easier to calculate.

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