Name: _____ Homework 1 | Math 342 | Cruz Godar

Due Wednesday of Week 2 at the start of class

Complete the following problems and submit them as a pdf to Canvas. 8 points are awarded for thoroughly attempting every problem, and I'll select three problems to grade on correctness for 4 points each. Enough work should be shown that there is no question about the mathematical process used to obtain your answers.

Section 0

In problems 1–3, do the following:

a) Write the system of equations in the form $A\vec{x} = \vec{b}$ for a matrix A and a vector $\vec{b}$ of constants, and a vector $\vec{x}$ of variables.

b) Augment A with $\vec{b}$ and row reduce the system to solve for $\vec{x}$.

c) If A is square, find $\det A$.

d) Let T be the linear transformation corresponding to the matrix A. Write down the domain and codomain for T.

e) Find a basis for $\ker T$ and use it to find a basis for $\operatorname{image} T$ using the fundamental theorem of linear algebra.

f) Determine if T is one-to-one, onto, both, or neither. If it's both one-to-one and onto, find a formula for T^{-1}.

1.

$$x + y = 1$$
$$x - y = 3$$

2.

$$x + 2y - z = -1$$

$$2x + 3y - w = -2$$

3.

$$x + y + 2z = 4$$

$$2x + y - z = 2$$

$$-3x - y + 4z = 0$$

In problems 4–7, do the following:

a) The given sets V and W are vector spaces. Determine whether the subset X of V is a subspace. If it is, show it satisfies all three subspace properties, and if not, give a specific example showing one of the properties fails.

b) Determine whether the function $T : V \to W$ is a linear transformation. If it is, show it satisfies the two properties, and if not, give a specific example showing one of them fails.

c) If T is a linear transformation, find the matrix for T with respect to the standard bases for V and W.

d) If T is a linear transformation, find a basis for $\ker T$ and use it to find bases for V and $\operatorname{image} T$ using the fundamental theorem of linear algebra. Then extend the basis for $\operatorname{image} T$ to a basis for W, and find the matrix for T with respect to the bases for V and W you found.

4. $V = \mathbb{R}^4$, $W = M_{2\times 2}(\mathbb{R})$, X is the set of vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$ such that $x + 2y = w$, and $T : V \to W$ is defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 0 & x \\ y+z & 2w \end{bmatrix}.$$

5. $V = M_{2\times 2}(\mathbb{R})$, $W = M_{2\times 2}(\mathbb{R})$, X is the set of matrices A such that $\det A = 0$, and $T : V \to W$ is defined by

$$T(A) = A^T,$$

where A^T is the **transpose** of A, defined by $(A^T)_{ij} = A_{ji}$ (it effectively flips A across its diagonal).

6. V is the space of polynomials with degree at most 3, $W = \mathbb{R}^2$, X is the set of polynomials $p(x)$ such that $p'(x) = 1$, and $T : V \to W$ is defined by

$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a+b \\ c+d \end{bmatrix}.$$

7. $V = \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$, $W = M_{2\times 2}(\mathbb{R})$, X is the set of linear transformations $S : \mathbb{R} \to \mathbb{R}^2$ such that $S(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $T : V \to W$ is defined by

$$T(S) = \begin{bmatrix} | & | \\ S(1) & S(-1) \\ | & | \end{bmatrix},$$

i.e. the outputs of S are placed as columns in a 2×2 matrix.

Section 1

In problems 8–10, find the eigenvalues and eigenvectors of the matrix.

8. $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$.

9. $B = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$.

10. $C = \begin{bmatrix} 2 & 2 & -2 \\ -3 & 7 & 3 \\ -5 & 5 & 5 \end{bmatrix}$.

11. Let $A = [a_{ij}]$ be an $n \times n$ matrix that is upper triangular — that is, $a_{ij} = 0$ whenever $i > j$. What are the eigenvalues of A in terms of the entries a_{ij}? Justify your answer.

12. Let A be an $n \times n$ matrix with eigenvectors $\vec{v_1}, ..., \vec{v_n}$ and corresponding eigenvalues $\lambda_1, ..., \lambda_n$. If A is invertible, what are the eigenvectors and eigenvalues of A^{-1}? Justify your answer.

13. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$. What are the eigenvalues of A^T? Justify your answer.