Name: ________________________ Homework 5 | Math 342 | Cruz Godar

Due Wednesday of Week 6 at the start of class

Complete the following problems and submit them as a pdf to Canvas. 8 points are awarded for thoroughly attempting every problem, and I'll select three problems to grade on correctness for 4 points each. Enough work should be shown that there is no question about the mathematical process used to obtain your answers.

Section 6

In problems 1–5, compute the determinant.

1. $A = \begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix}$.

2. $B = \begin{bmatrix} 2 & 3 & 0 \\ -3 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

3. $C = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 3 \\ 2 & -1 & -15 \end{bmatrix}$.

4. $D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

5. $E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -1 & 0 & 3 \\ 2 & 0 & 1 & 2 \\ -1 & -3 & -7 & 2 \end{bmatrix}$.

6. For each of the matrices A–E in problems 1–5, classify it as invertible or noninvertible based on its determinant.

7. Let A be the matrix from problem 1. Sketch a picture of the unit square in $\mathbb{R}^2$ and its image under the linear operator corresponding to A. Verify that the area of that image is $|\det A|$ times the area of the unit

square (i.e. 1).

8. We can use the multiplicativity of the determinant to show some nice facts about the determinants of inverse matrices.

a) What is $\det I$?

b) Let A be an invertible matrix. Using part a), find $\det A^{-1}$ in terms of $\det A$.

9. **Cramer's Rule** is a method for computing the inverse of a matrix without row reduction. In this problem, we'll work through an example application of it.

a) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 5 & 6 \\ 0 & -1 & 2 \end{bmatrix}$ and let c_{ij} be the determinant of the minor given by removing row i and column j from A. Find all nine c_{ij} and form a matrix C whose entry in row i and column j is c_{ij}.

b) Form a new matrix D by applying the checkerboard signs to C:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

a) If all went well, the matrix $E = \frac{1}{\det A} D$ should be equal to $(A^{-1})^T$. Compute E and check that it is in fact the transpose of A^{-1} (Note: the matrix A appeared in homework 2).

Eigenvectors and Eigenvalues

Let A be an $n \times n$ matrix. When A only scales a nonzero vector and doesn't multiply it — i.e $A\vec{v} = \lambda\vec{v}$ for a vector $\vec{v}$ and a constant λ — we say that $\vec{v}$ is an **eigenvector** of A with **eigenvalue** λ. You'll see more on these in the next course if you take it, but for now, we'll work through a few basic examples.

10. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$. Show that $\vec{v_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of A and find their eigenvalues.

11. Let $B = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$.

a) If $B\vec{v} = \lambda\vec{v}$ for a nonzero vector $\vec{v}$, then $B\vec{v} = \lambda I\vec{v}$, so $(B - \lambda I)\vec{v} = \vec{0}$. That means $B - \lambda I$ is not one-to-one, so $\det(B - \lambda I) = 0$ (the left side is called the **characteristic polynomial** of B). Find that determinant and solve it for λ.

b) The values of λ in part a) are the eigenvalues of B. For each value of λ, we want to solve $(B - \lambda I)\vec{v} = 0$, so augment $B - \lambda I$ with $\vec{0}$ and row reduce. In total, what are the eigenvectors and eigenvalues of B?

12. Let $C = \begin{bmatrix} 2 & 2 & -2 \\ -3 & 7 & 3 \\ -5 & 5 & 5 \end{bmatrix}$. Find the eigenvectors and eigenvalues of C in the same manner as the previous problem.

13. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.

a) What is the characteristic polynomial $\det(A - \lambda I)$ of A?

b) By setting $\lambda = 0$ in part a), express $\det A$ in terms of the λ_i.

14. One application of eigenvalues is to systems of differential equations. If $\vec{v_1}$ and $\vec{v_2}$ are eigenvectors of a 2×2 matrix A with eigenvalues λ_1 and λ_2, then the solution to the system

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2}$$

for any value of c_1 and c_2.

a) Write the general solutions to the systems

$$x'(t) = 2x(t) - y(t)$$
$$y'(t) = 3x(t) - 2y(t)$$

and

$$x'(t) = -x(t) + 2y(t)$$
$$y'(t) = -3y(t).$$

a) We can plot solutions to systems of differential equations as **vector fields**: every point (x, y) has a velocity (x', y'), so if we fill an area with particles and move them according to that velocity, we can see the entire effect of the system. Using a vector field applet with the generating functions <code>(2x - y, 2x - 2y)</code> and <code>(-x + 2y, -3y)</code>, plot the systems from part a).