If you have $\int f(g(x))\, g'(x)\, dx$, then

① Set $u = g(x)$.

② Write $\frac{du}{dx} = g'(x)$ as $du = g'(x)\, dx$

looks weird, but it's fine, I promise

③ Rewrite the integral as $\int f(u)\, du$ and integrate to get $F(u) + C$

④ Substitute $u = g(x)$ to get $F(g(x)) + C$

$$\Rightarrow \int f(g(x))\, g'(x)\, dx = F(g(x)) + C$$

Ex: $\int 6x\,(3x^2+4)^4\, dx$

Could foil out $(3x^2+4)^4$, but that's more work than necessary. Instead, use u-sub, because

$$u = 3x^2 + 4$$

$$du = (6x)\,dx$$

$$\int 6x\,(3x^2+4)^4\,dx = \int u^4\,du = u^5/5 + C$$

$$= (3x^2+4)^5/5 + C.$$

Notice: this is just the chain rule backward:

$$\frac{d}{dx}\left[\frac{(3x^2+4)^5}{5} + C\right] = \frac{5\,(3x^2+4)^4}{5} \cdot 6x$$

$$= (3x^2+4)^4\,6x$$

Comment: u-sub only works in these very specific cases. For example,

$$\int 6x^2 (3x^2+4)^4 \, dx$$

Try: $u = 3x^2 + 4$

$du = 6x \, dx$

$$\int u^4 \underline{x} \, du$$ ← can't integrate b/c it's not all u

Ex: $$\int_3^4 2z \sqrt{z^2-5} \, dz$$

$u = z^2 - 5$

$du = 2z \, dz$

$$= \int_3^4 \sqrt{u} \, du$$

$$= \int_3^4 u^{1/2}\, du$$

$$= \left[\frac{u^{3/2}}{3/2} \right] \Big|_3^4$$

Warning: these are z-limits, not u-limits. Must sub back in for z before evaluating.

$$= \left[\frac{(z^2-5)^{3/2}}{3/2} \right] \Big|_3^4$$

$$= \frac{(16-5)^{3/2}}{3/2} - \frac{(9-5)^{3/2}}{3/2}.$$

Ex: $\int_0^1 x e^{4x^2+3}\, dx$

$u = 4x^2 + 3$

$du = 8x\, dx$ ← we have $x\, dx$ in the integral

$\frac{1}{8} du = x\, dx$ ← only works with multiplication by constant

$$\int_0^1 e^u \frac{1}{8}\,du$$

$$= \left[\frac{1}{8}e^u\right]\Big|_0^1$$

$$= \left[\frac{1}{8}e^{4x^2+3}\right]\Big|_0^1$$

$$= \frac{1}{8}e^{4(1)^2+3} - \frac{1}{8}e^{4(0)^2+3}$$

$$= \frac{1}{8}\left(e^7 - e^3\right).$$

Def: $\cos^2(\theta) = \cos(\theta)^2$

$\sin^2(\theta) = \sin(\theta)^2$

Prop: $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$

$\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$

Ex $\int_0^{\pi/2} 3\cos^2(\theta)\,d\theta$

$$= \int_0^{\pi/2} 3\left(\frac{1+\cos(2\theta)}{2}\right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{3}{2} + \frac{3}{2}\cos(2\theta)\right) d\theta$$

$$= \underbrace{\int_0^{\pi/2} \frac{3}{2}\,d\theta}_{} + \underbrace{\int_0^{\pi/2} \frac{3}{2}\cos(2\theta)\,d\theta}_{}$$

$$\left[\frac{3}{2}\theta\right]\Big|_0^{\pi/2}$$

$$= \frac{3}{2}\left(\frac{\pi}{2}\right)$$

$u = 2\theta$

$du = 2\,d\theta$

$\frac{1}{2}du = d\theta$

$$\int_0^{\pi/2} \frac{3}{2}\cos(u)\,\frac{1}{2}\,du$$

$$= \left[\frac{3}{4}\sin(u)\right]\Big|_0^{\pi/2}$$

$$= \left[\frac{3}{4}\sin(2\theta)\right]\Big|_0^{\pi/2}$$

$$= \frac{3}{4}\cancel{\sin(\pi)}^{0} - \frac{3}{4}\cancel{\sin(0)}^{0}$$

$$= 0$$

In total, $\int_0^{\pi/2} 3\cos^2(\theta)\, d\theta = \frac{3}{2}\left(\frac{\pi}{2}\right) = \frac{3\pi}{4}$

Ex: $\int \frac{x}{\sqrt{x-1}}\, dx$

$u = x - 1 \quad | \quad x = u + 1$

$du = dx$

$\int \frac{x}{\sqrt{u}}\, du$ ← this looks like it didn't work, but we can solve for x in terms of u

$= \int \frac{u+1}{\sqrt{u}}\, du$

$$= \int \left(\frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}} \right) du$$

$$= \int \left(u^{1/2} + u^{-1/2} \right) du$$

$$= \frac{u^{3/2}}{3/2} + \frac{u^{1/2}}{1/2} + C$$

$$= \frac{(x-1)^{3/2}}{3/2} + \frac{(x-1)^{1/2}}{1/2} + C.$$

$f(g(\theta))$, where $g(\theta) = \cos(\theta)$, $f(\theta) = \theta^3$

<u>Ex</u>: $\int \cos^3(\theta) \sin(\theta)\, d\theta$

$u = \sin(\theta)$

$du = \cos(\theta)\, d\theta$

$\downarrow$

this u didn't work

$\int u\, (du)^3$

meaningless

Instead, try $u = \cos(\theta)$

$du = -\sin(\theta)\, d\theta$

$-du = \sin(\theta)\, d\theta$

$\int -u^3\, du$

Exponential and Logarithmic Integrals

Ex: $\int e^x \, dx = e^x + C.$

Ex: $\int e^{2x} \, dx$

$u = g(x) = 2x$

$du = g'(x)dx = 2\,dx$

$\frac{1}{2} du = dx$

$\frac{du}{dx} = u'$

$du = u'\,dx$

$\int \frac{1}{2} e^u \, du$

$= \frac{1}{2}\left[e^u + C\right]$

$= \frac{1}{2} e^u + C$

$= \frac{1}{2} e^{2x} + C$

Check : $\frac{d}{dx}\left[\frac{1}{2}e^{2x}+C\right]=e^{2x}$. ✓

Ex: $\int_0^4 e^x\sqrt{1+e^x}\,dx$

$u=1+e^x$

$du=e^x\,dx$

$\int_0^4 \sqrt{u}\,du$

$=\int_0^4 u^{1/2}\,du$

$\int x^p = \frac{x^{p+1}}{p+1}+C$

$=\left[\frac{u^{3/2}}{3/2}\right]\Big|_0^4$

$=\left[\frac{(1+e^x)^{3/2}}{3/2}\right]\Big|_0^4$

b/c $e^0=1$

$=\frac{(1+e^4)^{3/2}}{3/2}-\frac{2^{3/2}}{3/2}$

Ex: Find the price-demand function for toothpaste when the demand is 50 tubes per week at \$2.35 per tube, given that the marginal price-demand function for x tubes per week is given by $-.015e^{-.01x}$. If you're selling 100 tubes per week, what price do you set?

#1: $\int$(marginal function) = function

#2: price-demand: $p(x)$ for x = # tubes per week sold and p = unit price.

So, $p(x) = \int -.015e^{-.01x}\,dx = -.015\int e^{-.01x}\,dx$

$u = -.01x$

$du = -.01\,dx \quad \Big| \ \frac{1}{-.01}\,du = dx$

$$-.015 \int \frac{1}{-.01} e^u \, du$$

$$-.015 \frac{1}{-.01} e^u + C$$

$$-.015(-100) e^{-.01x} + C$$

$$p(x) = 1.5 e^{-.01x} + C$$

$$p(50) = 2.35$$

$$2.35 = 1.5 e^{-.01(50)} + C$$

$$2.35 = .9098 + C$$

$$1.4402 = C$$

$$\Rightarrow p(x) = 1.5e^{-.01x} + 1.4402$$

$$p(100) = 1.5 e^{-1} + 1.4402 = 1.992 \approx 2$$

Could optimize : maximize $x\,p(x)$ = revenue

Ex: A certain strain of bacteria grow at 3^t new bacteria per hour. If the population starts at 100 bacteria, how many will there be after 6 hours?

Let $N(t)$ = # bacteria after t hours, so $N(0) = 100$. Then $N'(t) = 3^t$, so

$$N(t) = \int 3^t \, dt$$

Recall: $\frac{d}{dt}(b^t) = b^t \ln(b)$, so

$$\int b^t \, dt = \frac{b^t}{\ln(b)} + C.$$

$$N(t) = \frac{3^t}{\ln(3)} + C$$

$N(0) = 100$, so
$$100 = \frac{3^0}{\ln(3)} + C$$
$$= \frac{1}{\ln(3)} + C$$

$$C = 100 - \frac{1}{\ln(3)} = 99.09$$

$$\Rightarrow N(t) = \frac{3^t}{\ln(3)} + 99.09$$

Want $N(6) = \frac{3^6}{\ln(3)} + 99.09 = \boxed{762.7}$

Ex: $\int \frac{2^{1/x}}{x^2}\, dx$

$$u = \frac{1}{x} \Rightarrow \frac{du}{dx} = \left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -1x^{-2} = -\frac{1}{x^2}$$

$$du = -\frac{1}{x^2}\, dx, \text{ so } -du = \frac{1}{x^2}\, dx$$

$$\int -2^u\, du$$

$$= -\left[\frac{2^u}{\ln 2} + C\right]$$

$$= -\frac{2^{1/x}}{\ln 2} + C$$

Prop: ① $\int \frac{1}{x}\,dx = \ln|x| + C$ (b/c $(\ln|x|)' = \frac{1}{x}$)

② $\int \ln(x)\,dx = x\ln(x) - x + C$ (not clear why)

③ $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

Comment: $\log_a(x)$ = the number you raise a to in order to get x, i.e. $a^{\log_a(x)} = x$

Ex: $\log_2(8) = 3$ b/c $2^3 = 8$

$\log_{10}(.01) = -2$ b/c $10^{-2} = \frac{1}{10^2} = .01$.

$\ln(x) = \log_e(x)$.

$\log_a(x)$ is to a^x as x^a is to $\sqrt[a]{x}$.

Ex: $\int \frac{1}{3x}\,dx$

$u = 3x$

$du = 3\,dx$, so $\frac{1}{3}du = dx$

$\int \frac{1}{u}\cdot\frac{1}{3}\,du$

$= \frac{1}{3}\int \frac{1}{u}\,du$

$= \frac{1}{3}\left[\ln|u| + C\right]$

$= \frac{1}{3}\ln|3x| + C$.

Ex: $\int \frac{3}{y-10}\,dy$

$u = y-10 \longrightarrow (y-10)' = (y')' - (10)'$

$(y')' \to 1y^0 \to 1$, $(10)' \to 0$

$du = dy$

$\int \frac{3}{u}\,du$

$= 3 \ln |u| + C$

$= 3 \ln |y - 10| + C.$

Ex: $\displaystyle\int \frac{2x^3 + 3x}{x^4 + 3x^2}\, dx$

Try $u = x^4 + 3x^2$

$du = (4x^3 + 6x)\, dx$

$du = 2(2x^3 + 3x)\, dx$

$\frac{1}{2}\, du = (2x^3 + 3x)\, dx$

$\displaystyle\int \frac{1}{2} \frac{1}{u}\, du = \frac{1}{2} \ln |u| + C$

$= \frac{1}{2} \ln |x^4 + 3x^2| + C$

Ex: $\displaystyle\int_2^3 \log_2 (5x)\, dx$

$= \displaystyle\int_2^3 \frac{\ln(5x)}{\ln(2)}\, dx$

$$= \frac{1}{\ln(2)} \int_2^3 \ln(5x)\, dx$$

$u = 5x$

$du = 5\, dx \mid \frac{1}{5} du = dx$

$$= \frac{1}{\ln(2)} \int_2^3 \ln(u)\, \frac{1}{5} du$$

$$= \frac{1}{\ln(2)} \cdot \frac{1}{5} \left[u \ln(u) - u \right] \Big|_2^3$$

$$= \frac{1}{\ln(2)} \cdot \frac{1}{5} \left[5x \ln(5x) - 5x \right] \Big|_2^3$$

$$= \frac{1}{\ln(2)} \cdot \frac{1}{5} \Big(15 \ln(15) - 15 - 10 \ln(10) + 10 \Big).$$

Comment: The integrals we know

$$\int x^p\, dx = \frac{x^{p+1}}{p+1} + C$$

$$\int \sin(x)\, dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int e^x\,dx = e^x + C$$

$$\int b^x\,dx = \frac{b^x}{\ln(b)} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \ln(x) = x\ln(x) - x + C$$

$$\int \log_a(x)\,dx = \frac{1}{\ln(a)}\left(x\ln(x) - x\right) + C$$

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C \quad \text{for any } f, g$$

(u-sub)

Ex: person burns $300-50t$ cal/hr via treadmill consumes $100t$ cal during hour t. Find net decrease in cal after 3 hours.

Let $C(t)$ = change in calories after t hours.

$C'(t) = ($cals consumed per hr $-$ cals burned per hr$)$ at time t

$$C'(t) = 100t - (300-50t) = 150t - 300$$

$$C(t) = \int C'(t)\,dt = \int (150t - 300)\,dt$$

$$= 150\frac{t^2}{2} - 300t + D$$

usually use C but it's taken

$$= 75t^2 - 300t + D$$

Want net decrease, so we want

$$\int_0^3 (150t - 300)\,dt = \left[75t^2 - 300t\right]\Big|_0^3$$

$$= 75(9) - 300(3) = \boxed{-225}$$

Net decrease $= 225$ cal

Want $C(3) - C(0)$

Inverse Trig Functions

Comment: Recall

① $\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$

② $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{x^2+1}$

③ $$\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{x\sqrt{x^2-1}}$$

Prop:

① $$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

② $$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

③ $$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

Comment: The inverse trig functions take in outputs of the corresponding standard trig functions and output the angles that produce those outputs.

Ex: $\sin(\pi/6) = 1/2$ $\qquad$ $\sin^{-1}(1/2) = \pi/6$

$\tan(\pi/3) = \sqrt{3}$ $\qquad$ $\tan^{-1}(\sqrt{3}) = \pi/3$

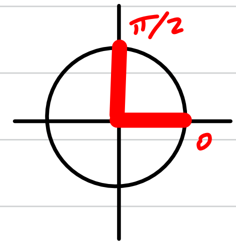
Ex:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1}(x)\right]\Big|_0^1$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \pi/2 - 0$$

$$= \pi/2$$



Ex:

$$\int \frac{1}{\sqrt{4-9x^2}}\,dx = \int \frac{1}{\sqrt{4-(3x)^2}}\,dx$$

$$u = 3x$$

$$du = 3\,dx \quad \Big| \quad \frac{1}{3}\,du = dx$$

$$= \int \frac{1}{\sqrt{4-u^2}} \cdot \frac{1}{3}\,du$$

$$2^2 = 4$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + C.$$

Ex: $\int \frac{1}{1+4y^2}\,dy = \int \frac{1}{1+(2y)^2}\,dy$

$u = 2y$

$du = 2\,dy \quad | \quad dy = \frac{1}{2}\,du$

$= \int \frac{1}{1+u^2} \cdot \frac{1}{2}\,du$

$= \frac{1}{2}\tan^{-1}(u) + C$

$= \frac{1}{2}\tan^{-1}(2y) + C.$

Ex: $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{1}{9+x^2}\,dx$

$= \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]\Bigg|_{\sqrt{3}/3}^{\sqrt{3}}$

$= \frac{1}{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{\sqrt{3}}{9}\right)$

$$= \frac{1}{3} \cdot \frac{\pi}{6} - \frac{1}{3}(.1901).$$

<u>Ex</u>: $$\int_0^1 \frac{1}{3x\sqrt{9x^2-9}}\,dx = \int_0^1 \frac{1}{(3x)\sqrt{(3x)^2 - 3^2}}\,dx$$

$$u = 3x$$

$$du = 3\,dx \quad \Big| \quad dx = \frac{1}{3}\,du$$

$$\int_0^1 \frac{1}{u\sqrt{u^2-9}} \cdot \frac{1}{3}\,du$$

$$= \frac{1}{3}\int_0^1 \frac{1}{u\sqrt{u^2-9}}\,du$$

$$= \frac{1}{3}\left[\frac{1}{3}\sec^{-1}\left(\frac{u}{3}\right)\right]\Bigg|_0^1$$

$$= \frac{1}{3}\left[\frac{1}{3}\sec^{-1}(x)\right]\Bigg|_0^1$$

$$= \frac{1}{3} \cdot \left(\frac{1}{3}\sec^{-1}(1) - \frac{1}{3}\sec^{-1}(0)\right)$$

$$= \frac{1}{3}\left(0 - \text{undefined!}\right)$$

Recall: $\sec(x) = \dfrac{1}{\cos(x)}$

$\sec^{-1}(0)$ = value of x so that $\sec(x) = 0$

But, $\dfrac{1}{\cos(x)} \neq 0$ for any x.

Right now, this means we can't compute the integral. But, we'll soon have a way to get around this.

Comment:

x	$\sin(x)$	$\tan(x)$	$\sec(x)$
0	0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/3$	$2/\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	1	$2/\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$\sqrt{3}$	2
$\pi/2$	1	undefined	undefined

So use this for the inverse functions:

for example, $\sec^{-1}(2) = \pi/3$ or $\tan^{-1}(1) = \pi/4$.