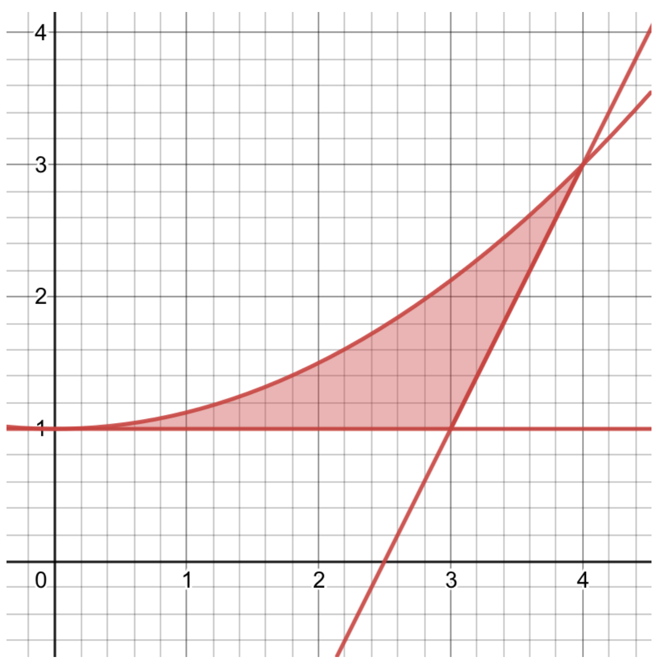
Practice Midterm 2

Math 252

Exercise 1: Let $f(x)$ be a differentiable function with a continuous derivative. What is the arc length of f between $x = 2$ and $x = 5$?

$\int_2^5 \sqrt{1 + (f'(x))^2}\, dx.$

Exercise 2: The shaded region below is bounded by the curves $y = 1$, $y = 2x - 5$, and $y = \frac{1}{8}x^2 + 1$. Find the shaded area.



Let's slice with respect to y, since then we don't have to split up the interval at $x = 3$. Then we need to solve for y: $x = \frac{y+5}{2}$ and $x = \sqrt{8y - 8}$ (we want the positive square root since $y > 0$ in the graph. Our limits are now $y = 1$ to $y = 3$, and since the linear function is the rightmost one, we have that the area is

$$\int_1^3 \left(\frac{y+5}{2} - \sqrt{8y-8}\right)\, dy = \int_1^3 \left(\frac{y}{2} + \frac{5}{2} - (8y-8)^{1/2}\right)\, dy$$
$$= \left[\frac{y^2}{4} + \frac{5}{2}y - \frac{(8y-8)^{3/2}}{3/2}\cdot\frac{1}{8}\right]\Bigg|_1^3$$
$$= \left(\frac{9}{4} + \frac{15}{2} - \frac{(16)^{3/2}}{3/2}\cdot\frac{1}{8}\right) - \left(\frac{1}{4} + \frac{5}{2} - \frac{(0)^{3/2}}{3/2}\cdot\frac{1}{8}\right)$$
$$= \frac{5}{3}.$$

Exercise 3: A 5-meter long rope is hanging straight down from a platform. x meters **below the platform**, the weight density of the rope is $100 - \sqrt{x}$ Newtons per meter. What is the total work done by pulling the rope up onto the platform? Drawing a picture might be helpful.

Since we're given the weight density at x meters below the platform, it will be easiest to integrate with respect to that variable. A slice of rope x meters down from the top gets lifted x meters and has a force of $100 - \sqrt{x}$ applied to it, so the total work done is

$$\int_0^5 (100 - \sqrt{x})x\, dx = \left[50x^2 - \frac{x^{5/2}}{5/2}\right]\Bigg|_0^5 \approx 1227.639.$$

Exercise 4: Let R be the region bounded by $\sin(x)$ and $\frac{4}{\pi^2}x^2$ on $[0, \frac{\pi}{2}]$. Find the volume of the solid of revolution given by rotating R about the x-axis (you may use any method you like).

Note: this is a long problem. On an actual exam, I would likely only ask you to set up the integral and not solve it.

Here we go. We'll use disks — then the volume is

$$\pi\int_0^{\pi/2} \left(\sin(x) - \frac{4}{\pi^2}x^2\right)^2\, dx = \pi\int_0^{\pi/2}\left(\sin^2(x) - \frac{8}{\pi^2}x^2\sin(x) + \frac{16}{\pi^4}x^4\right)\, dx.$$

Let's handle these terms one at a time. First,

$$\int_0^{\pi/2} \frac{16}{\pi^4}x^4\, dx = \left[\frac{16}{5\pi^4}x^5\right]\Bigg|_0^{\pi/2} = \frac{16}{5\pi^4}\cdot\frac{\pi^5}{32} = \frac{\pi}{10}.$$

For the first term,

$$
\begin{aligned}
\int_0^{\pi/2} \sin^2(x)\ dx &= \int_0^{\pi/2} \left(\frac{1}{2} - \frac{\cos(2x)}{2}\right)\ dx \\
&= \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]\Big|_0^{\pi/2} \\
&= \frac{\pi}{4},\ \text{since}\ \sin(0) = \sin(\pi) = 0.
\end{aligned}
$$

Finally, the middle term is a product and doesn't work with u-sub, so we apply integration by parts: factor out the $-\frac{8}{\pi^2}$ and set $u = x^2$ and $dv = \sin(x)dx$. Then $du = 2xdx$ and $v = -\cos(x)\ dx$. Now the integral becomes

$$
\begin{aligned}
-\frac{8}{\pi^2}\int_0^{\pi/2} x^2 \sin(x)\ dx &= -\frac{8}{\pi^2}\left[-x^2\cos(x) - \int -2x\cos(x)\ dx\right]\Big|_0^{\pi/2} \\
&= -\frac{8}{\pi^2}\left[-x^2\cos(x) + 2\int x\cos(x)\ dx\right]\Big|_0^{\pi/2}
\end{aligned}
$$

Once again, the inner integral requires integration by parts. Let $u = x$ and $dv = \cos(x)$. Then $du = dx$ and $v = \sin(x)$, and we get

$$
\begin{aligned}
-\frac{8}{\pi^2}\int_0^{\pi/2} x^2 \sin(x)\ dx &= -\frac{8}{\pi^2}\left[-x^2\cos(x) + 2\left(x\sin(x) - \int \sin(x)\ dx\right)\right]\Big|_0^{\pi/2} \\
&= -\frac{8}{\pi^2}\left[-x^2\cos(x) + 2\left(x\sin(x) + \cos(x)\right)\right]\Big|_0^{\pi/2} \\
&= -\frac{8}{\pi^2}\left(\left(-\frac{\pi^2}{4}\cdot 0 + 2\left(\frac{\pi}{2}\cdot 1 + 0\right)\right) - \left(-\frac{\pi^2}{4}\cdot 1 + 2\left(\frac{\pi}{2}\cdot 0 + 1\right)\right)\right) \\
&= -\frac{8}{\pi^2}\left(\pi + \frac{\pi^2}{4} + 2\right) \\
&= -\frac{8}{\pi} - 2 - \frac{16}{\pi^2}.
\end{aligned}
$$

Putting it all together, we have

$$
\pi\left(\frac{\pi}{2} + \frac{\pi}{10} - \frac{8}{\pi} - 2 - \frac{16}{\pi^2}\right).
$$

Exercise 5: Set up, but do not solve, the integral for the surface area of the solid of revolution given by rotating $\ln(x)$ for $2 \le x \le 5$ about the y-axis.

Since we're revolving about the y-axis, we need to write this as a function of y and have y-limits. We have $x = e^y$ and $y = \ln(2)$ to $y = \ln(5)$. Thus the integral is

$$\int_{\ln(2)}^{\ln(5)} 2\pi e^{y}\sqrt{1+e^{2y}}\ dy.$$

Exercise 6: Let U be the region bounded by e^x and e^{-x^2}. Set up, but do not solve, the three integrals necessary to find the centroid of U.

The intersection occurs when $x = -x^2$, so $x = 0$ or $x = -1$. Then the integrals are

$$M_x = \int_{-1}^{0} \frac{1}{2}e^{-2x^2}\ dx - \int_{-1}^{0} \frac{1}{2}e^{2x}\ dx$$

$$M_y = \int_{-1}^{0} xe^{-x^2}\ dx - \int_{-1}^{0} xe^{x}\ dx$$

$$m = \int_{-1}^{0} e^{-x^2}\ dx - \int_{-1}^{0} e^{x}\ dx$$