Name: ______________________

Final Exam

Math 253

Fall 2022

You have 2 hours to complete this exam and turn it in. You may use a scientific calculator, but not a graphing one, and you may not consult the internet or other people. If you have a question, don't hesitate to ask — I just may not be able to answer it. **Enough work should be shown that there is no question about the mathematical process used to obtain your answers.**

1. (16 points) Multiple choice. You don't need to show your work.

a) (4 points) Which of the following series converges?

A) $\sum_{n=1}^{\infty} \ln(n)$.

B) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}+1/4}$.

C) $\sum_{n=1}^{\infty} \frac{1}{n}$.

D) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$: since $\cos(\pi n) = (-1)^n$, this is the (negative) alternating Harmonic series. The others all diverge.

b) (4 points) Evaluate $\sum_{n=0}^{\infty} (-1)^n \frac{4^n}{(2n)!}$.

A) $\ln(2)$.

B) $\cos(2)$: this is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ with $x = 2$.

C) 1.

D) The sum diverges.

c) (4 points) Which power series has the largest interval of convergence?

A) $\sum_{n=1}^{\infty} n!x^n$.

B) $\sum_{n=1}^{\infty} \frac{x^n}{n}$: this one by a hair. It converges on $[-1,1)$, whereas the others converge on $(-1,1)$ or only for $x = 0$.

C) $\sum_{n=1}^{\infty} x^n$.

D) $\sum_{n=1}^{\infty} x$.

d) (4 points) The series $\sum_{k=1}^{\infty} \frac{(-2)^k}{3^k+1}$

A) converges absolutely: the absolute value of this series can be limit compared to $\sum_{k=1}^{\infty} \frac{2^k}{3^k}$, which converges by the root test.

B) converges conditionally.

C) diverges.

2. (48 points) Short-answer. Explain your reasoning and/or show your work for each question.

a) (8 points) Does the series $\sum_{n=0}^{\infty} \frac{1}{n^2+n+1}$ converge or diverge?

For large n, the denominator is roughly n^2, so the series is roughly $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges. To make that roughness precise, we need to use a comparison test. The regular comparison test would require that each term of our series is less than $\frac{1}{n^2}$, which is true since all the denominators are larger. Therefore, the comparison test guarantees this series converges.

b) (8 points) The Harmonic series diverges because it is a p-series with $p = 1$. Show that it diverges using another test.

There are a few ways to do this, but the easiest is likely with the integral test. The corresponding integral is

$$\begin{aligned}
\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty} \left[\ln(x)\right]\Big|_1^b \\
&= \lim_{b\to\infty} \left(\ln(b) - \ln(1)\right) \\
&= \infty,
\end{aligned}$$

so the series diverges.

c) (8 points) Estimate $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2+1}$ to within 0.1 of its actual value.

This is an alternating series with decreasing terms that limit to zero, so we can apply the alternating series remainder estimate:

$$\begin{aligned}
R_N \le \frac{1}{(N+1)^2+1} &\le 0.1 \\
\frac{1}{(N+1)^2+1} &\le 0.1 \\
(N+1)^2+1 &\ge 10 \\
(N+1)^2 &\ge 9 \\
N+1 &\ge 3 \\
N &\ge 2.
\end{aligned}$$

With $N = 2$, the partial sum S_2 is $\sum_{n=1}^{2}(-1)^n \frac{1}{n^2+1} = -\frac{1}{2} + \frac{1}{5} = -\frac{3}{10}$.

d) (8 points) Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$. Find $f'\left(\frac{1}{2}\right)$.

We first need to find $f'(x)$, which we can do by differentiating the series term-by-term:

$$\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n} \\
&= \sum_{n=1}^{\infty} x^{n-1} \\
&= \sum_{n=0}^{\infty} x^n.
\end{aligned}$$

When $x = \frac{1}{2}$, that's just equal to $\frac{1}{1-1/2} = 2$.

e) (8 points) Find the Maclaurin series for $x\sin(x^2)$.

The Maclaurin series for $\sin(x)$ is $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$, so the Maclaurin series for $x\sin(x^2)$ is $\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(2n+1)!}$ (plug in x^2 for x and multiply everything by another copy of x.)

f) (8 points) Give an example of a power series with an interval of convergence of exactly $(-2, 2)$. Show that your answer is correct.

The series $\sum_{n=0}^{\infty} x^n$ converges on $(-1, 1)$, so the endpoints are the same style as what we're looking for. Instead of $|x| < 1$, we want $|x| < 2$, so $\left|\frac{x}{2}\right| < 1$. Therefore, we can take $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

3. (32 points) Define a sequence (a_n) by $a_0 = 1$ and $a_n = 3na_{n-1}$.

a) (8 points) Find a_1, a_2, and a_3.

Plugging these in, $a_1 = 3(1)(1) = 3$, $a_2 = 3(2)(3) = 3^2(2)$, and $a_3 = 3(3)(3^2(2)) = 3^3(3!)$.

b) (8 points) Find an explicit formula for (a_n). Check your answer by plugging in $n = 0$, $n = 1$, $n = 2$, and $n = 3$, and making sure they match.

Following the pattern, the 3s stack up each time to result in 3^n, and the ns stack up to $n!$. The result is $a_n = 3^n n!$.

c) (8 points) Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{a_n}$, where a_n is the same sequence from the previous parts. Determine the interval of convergence of f.

Applying the ratio test,

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{3^{n+1}(n+1)!}}{\frac{x^n}{3^n n!}} \right| &= \lim_{n\to\infty} \left| \frac{x^{n+1}3^n n!}{x^n 3^{n+1}(n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{x}{3(n+1)} \right| \\
&= 0.
\end{aligned}$$

Therefore, the series always converges, so its interval of convergence is $(-\infty, \infty)$.

d) (8 points) Find the exact value of $f(-1)$.

To do this, we need to express the series as a function, ideally with a Taylor series. It looks like e^x but with an extra 3^n. However, we can absorb that into the x:

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n n!} = \sum_{n=0}^{\infty} \frac{(x/3)^n}{n!}$$

This converges on $(-\infty, \infty)$, which includes -1, and so the value of $f(-1)$ is $e^{-1/3}$.

4. (32 points) Define a function g by $g(x) = \ln(x)$.

a) (8 points) For $n \geq 1$, find an expression for $g^{(n)}(x)$ (i.e. the nth derivative of g).

Looking at a few derivatives, $g'(x) = \frac{1}{x} = x^{-1}$, $g''(x) = -x^{-2}$, $g'''(x) = 2x^{-3}$, $g^{(4)}(x) = -6x^{-4}$, $g^{(5)}(x) = 24x^{-5}$, and so on. We keep getting factorials as coefficients, and the sign alternates: extrapolating, $g^{(n)}(x) = (-1)^{n+1}(n-1)!x^{-n}$.

b) (12 points) Find the Taylor series for g centered at 1 and determine its interval of convergence.

Plugging in 1 to all of these, $g^{(n)}(1) = (-1)^{n+1}(n-1)!$, so the Taylor series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n.$$

The $n = 0$ term is zero since $\ln(1) = 0$. We can find the interval of convergence with the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{n+1}(x-1)^{n+1}}{\frac{(-1)^{n+1}}{n}(x-1)^n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)(x-1)n}{n+1} \right| \\ &= |x-1|. \end{aligned}$$

The series converges when $|x-1| < 1$, so $0 < x < 2$. Now we just need to check the endpoints. At $x = 0$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-1)^n = -\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. At $x = 2$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1)^n$$

is the alternating Harmonic series, which converges. Therefore, the interval of convergence is $(0, 2]$.

c) (12 points) Approximate $g(1.1)$ with a degree-3 Taylor polynomial and determine the maximum error.

Plugging in $x = 1.1$,

$$\begin{aligned} g(1.1) &\approx \sum_{n=1}^{3} \frac{(-1)^{n+1}}{n} 0.1^n \\ &= 0.1 - \frac{0.1}{2} + \frac{0.1}{3}. \end{aligned}$$

We can bound $g^{(4)}(x)$ on $[0.9, 1.1]$ with

$$\left|g^{(4)}(x)\right| = \left|-6x^{-4}\right| \le 6(0.9)^{-4}.$$

Then

$$R_3(1.1) \le \frac{\left|6(0.9)^{-4}\right|}{4!}(1.1-1)^4.$$

d) (4 points extra credit) Give an example of a power series with an interval of convergence of **exactly** $[1, 2]$. Hint: try combining the Maclaurin series from this question with another.

One way to do this is to modify this power series into $\sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$: now it's centered at $x = 2$, and removing the minus sign flips the endpoints of the interval of convergence to $[1, 3)$. Adding the two together overlaps the interval of convergence to $[1, 2]$