

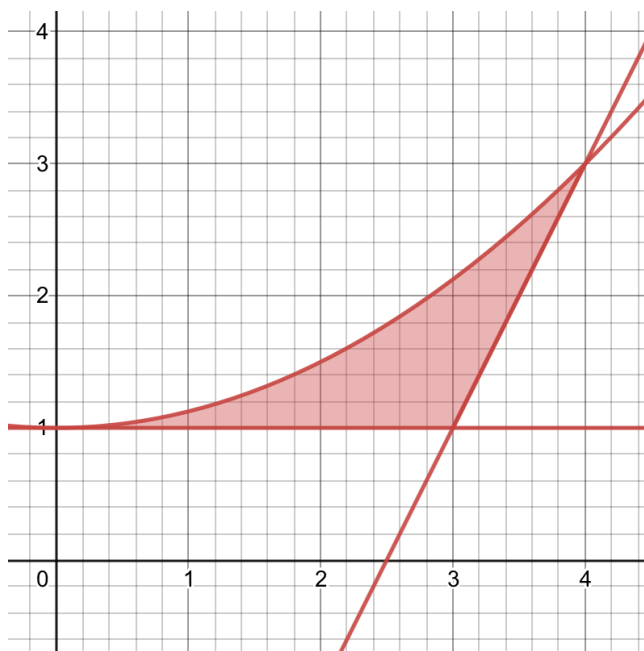
# Practice Midterm 2

Math 252

**Exercise 1:** Let  $f(x)$  be a differentiable function with a continuous derivative. What is the arc length of  $f$  between  $x = 2$  and  $x = 5$ ?

$$\int_2^5 \sqrt{1 + (f'(x))^2} \, dx.$$

**Exercise 2:** The shaded region below is bounded by the curves  $y = 1$ ,  $y = 2x - 5$ , and  $y = \frac{1}{8}x^2 + 1$ . Find the shaded area.



Let's slice with respect to  $y$ , since then we don't have to split up the interval at  $x = 3$ . Then we need to solve for  $y$ :  $x = \frac{y+5}{2}$  and  $x = \sqrt{8y-8}$  (we want the positive square root since  $y > 0$  in the graph. Our limits are now  $y = 1$  to  $y = 3$ , and since the linear function is the rightmost one, we have that the area is

$$\begin{aligned}
\int_1^3 \left( \frac{y+5}{2} - \sqrt{8y-8} \right) dy &= \int_1^3 \left( \frac{y}{2} + \frac{5}{2} - (8y-8)^{1/2} \right) dy \\
&= \left[ \frac{y^2}{4} + \frac{5}{2}y - \frac{(8y-8)^{3/2}}{3/2} \cdot \frac{1}{8} \right]_1^3 \\
&= \left( \frac{9}{4} + \frac{15}{2} - \frac{(16)^{3/2}}{3/2} \cdot \frac{1}{8} \right) - \left( \frac{1}{4} + \frac{5}{2} - \frac{(0)^{3/2}}{3/2} \cdot \frac{1}{8} \right) \\
&= \frac{5}{3}.
\end{aligned}$$

**Exercise 3:** A 5-meter long rope is hanging straight down from a platform.  $x$  meters **below the platform**, the weight density of the rope is  $100 - \sqrt{x}$  Newtons per meter. What is the total work done by pulling the rope up onto the platform? Drawing a picture might be helpful.

Since we're given the weight density at  $x$  meters below the platform, it will be easiest to integrate with respect to that variable. A slice of rope  $x$  meters down from the top gets lifted  $x$  meters and has a force of  $100 - \sqrt{x}$  applied to it, so the total work done is

$$\int_0^5 (100 - \sqrt{x})x \, dx = \left[ 50x^2 - \frac{x^{5/2}}{5/2} \right]_0^5 \approx 1227.639.$$

**Exercise 4:** Let  $R$  be the region bounded by  $\sin(x)$  and  $\frac{4}{\pi^2}x^2$  on  $[0, \frac{\pi}{2}]$ . Find the volume of the solid of revolution given by rotating  $R$  about the  $x$ -axis (you may use any method you like).

Note: this is a long problem. On an actual exam, I would likely only ask you to set up the integral and not solve it.

Here we go. We'll use disks — then the volume is

$$\pi \int_0^{\pi/2} \sin^2(x) \, dx - \pi \int_0^{\pi/2} \left( \frac{4}{\pi^2} x^2 \right)^2 \, dx = \pi \int_0^{\pi/2} \sin^2(x) \, dx - \pi \int_0^{\pi/2} \frac{16}{\pi^4} x^4 \, dx.$$

Let's handle these terms one at a time. First,

$$\int_0^{\pi/2} \frac{16}{\pi^4} x^4 \, dx = \left[ \frac{16}{5\pi^4} x^5 \right]_0^{\pi/2} = \frac{16}{5\pi^4} \cdot \frac{\pi^5}{32} = \frac{\pi}{10}.$$

Then we also have

$$\begin{aligned}
\int_0^{\pi/2} \sin^2(x) \, dx &= \int_0^{\pi/2} \left( \frac{1}{2} - \frac{\cos(2x)}{2} \right) dx \\
&= \left[ \frac{1}{2}x - \frac{1}{4} \sin(2x) \right]_0^{\pi/2} \\
&= \frac{\pi}{4}, \text{ since } \sin(0) = \sin(\pi) = 0.
\end{aligned}$$

Putting it all together, we have that the area is

$$\pi \cdot \frac{\pi}{4} - \pi \cdot \frac{\pi}{10}.$$

**Exercise 5:** Set up, but do not solve, the integral for the surface area of the solid of revolution given by rotating  $\ln(x)$  for  $2 \leq x \leq 5$  about the  $y$ -axis.

Since we're revolving about the  $y$ -axis, we need to write this as a function of  $y$  and have  $y$ -limits. We have  $x = e^y$  and  $y = \ln(2)$  to  $y = \ln(5)$ . Thus the integral is

$$\int_{\ln(2)}^{\ln(5)} 2\pi e^y \sqrt{1 + e^{2y}} \, dy.$$

**Exercise 6:** Let  $U$  be the region bounded by  $e^x$  and  $e^{-x^2}$ . Set up, but do not solve, the three integrals necessary to find the centroid of  $U$ .

The intersection occurs when  $x = -x^2$ , so  $x = 0$  or  $x = -1$ . Then the integrals are

$$M_x = \int_{-1}^0 \frac{1}{2} e^{-2x^2} \, dx - \int_{-1}^0 \frac{1}{2} e^{2x} \, dx$$

$$M_y = \int_{-1}^0 x e^{-x^2} \, dx - \int_{-1}^0 x e^x \, dx$$

$$m = \int_{-1}^0 e^{-x^2} \, dx - \int_{-1}^0 e^x \, dx$$