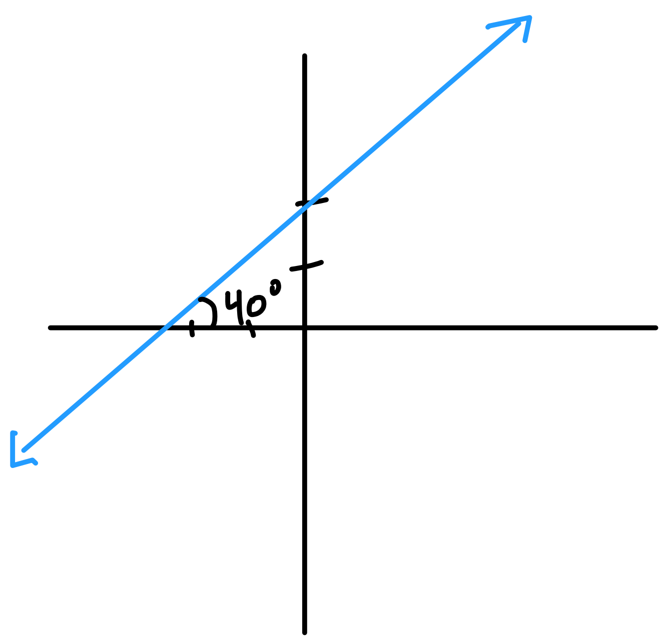
Comment: Recall that the tangent function is $\tan\theta = \frac{\sin\theta}{\cos\theta}$. It's also the slope of the line passing through $(0,0)$ and the point on the unit circle with angle θ.

Ex: Find the equation of the following line:

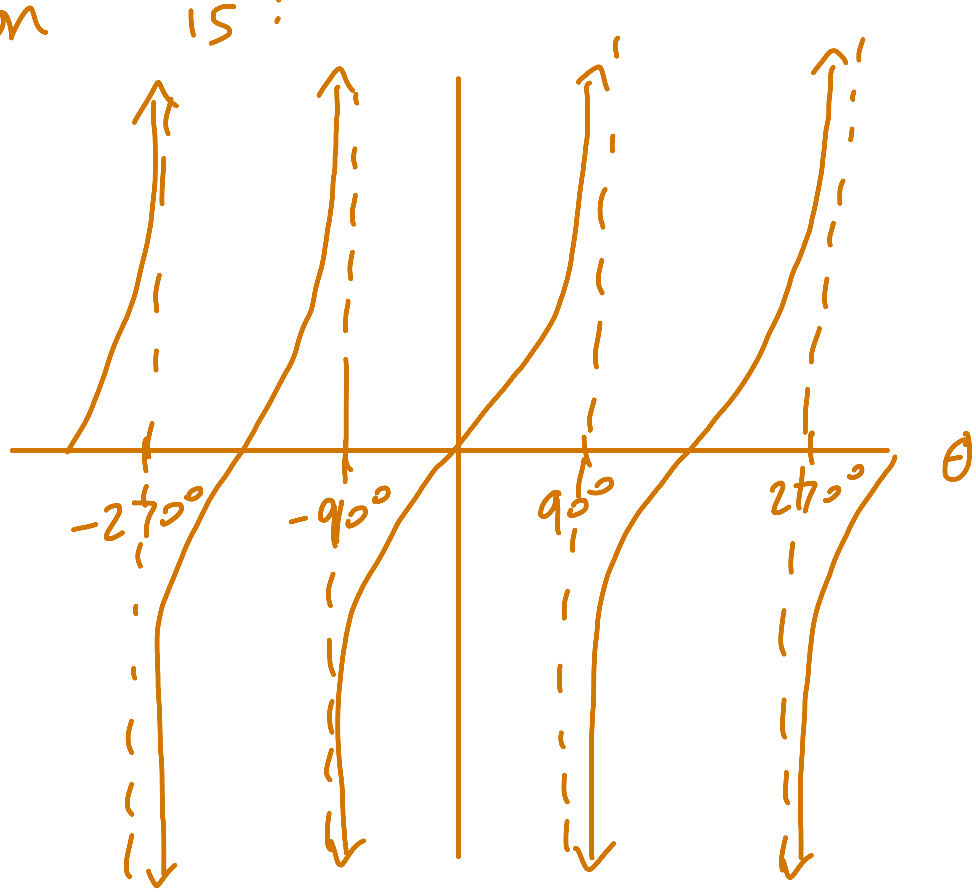


We know the equation is $y = mx + 2$, since 2 is the y-intercept. Now

$\tan 40° = .839$, so the slope is $m = .839$. Thus the equation is

$y = .839x + 2$.

Theorem: The graph of the tangent function is:



It is periodic with period 180°.

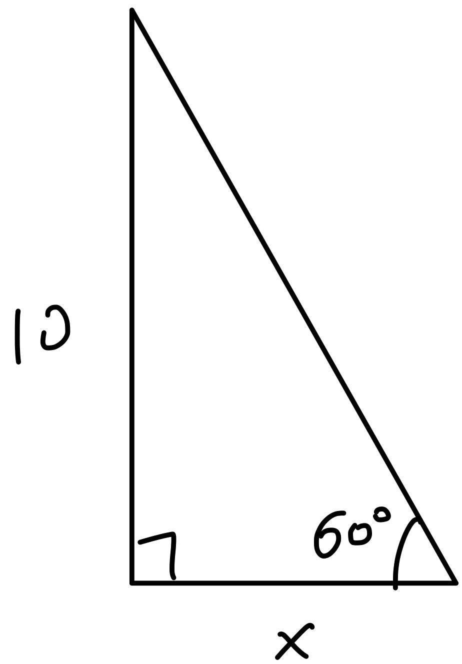
It's an odd function, it has asymptotes at

$180° n + 90°$ for every integer n.

The domain of $\tan\theta$ is

$$\cdots \cup (-270°, -90°) \cup (-90°, 90°) \cup (90°, 270°) \cup \cdots$$

Ex: A ladder is leaning up against a wall. It reaches 10 ft up, and it makes an angle of $60°$ with the ground. How far away from the wall is the base of the ladder? (do this without finding the length of the ladder!)

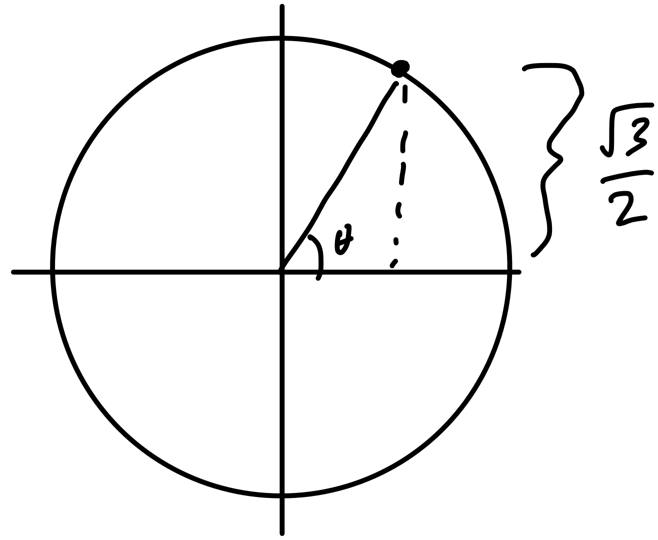


$\tan 60° = \frac{10}{x}$, so $\frac{\sin 60°}{\cos 60°} = \frac{\sqrt{3}/2}{1/2} = \frac{10}{x}$.

Then $\sqrt{3} = \frac{10}{x}$, and so $x = \frac{10}{\sqrt{3}} \approx 5.77$.

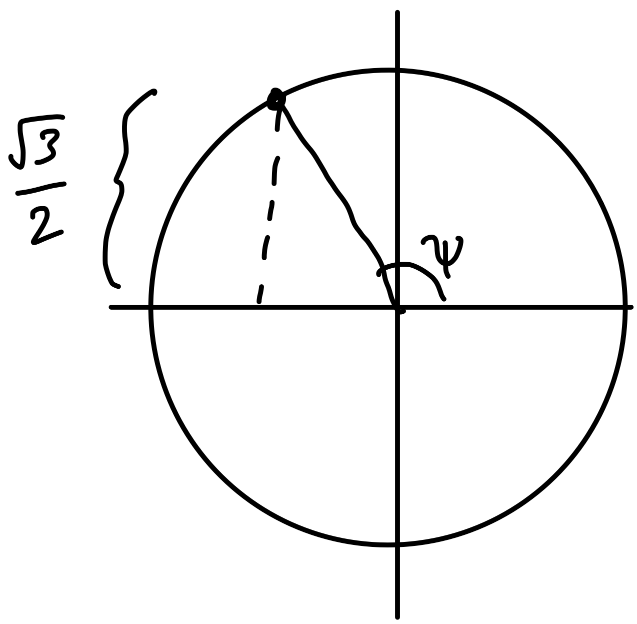
Inverse Functions

Ex: What is θ ?



We know that $\sin\theta = \frac{\sqrt{3}}{2}$, so $\theta = 60°$.

Ex:



Here, $\sin\psi = \frac{\sqrt{3}}{2}$, so $\psi = 120°$, since it's in the second quadrant.

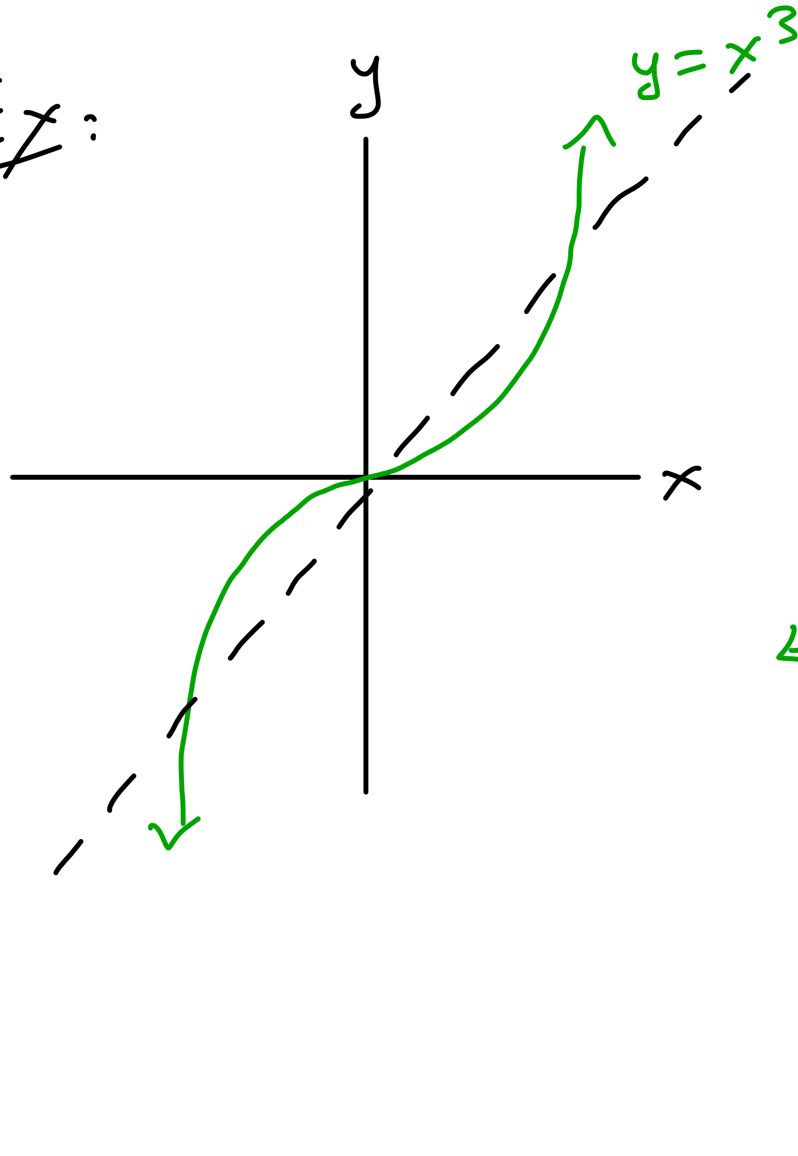
Comment: Recall inverse functions from 111: if $y=f(x)$, then f^{-1} is the function that takes in a y-value and outputs the x-value that f would take to that y-value.

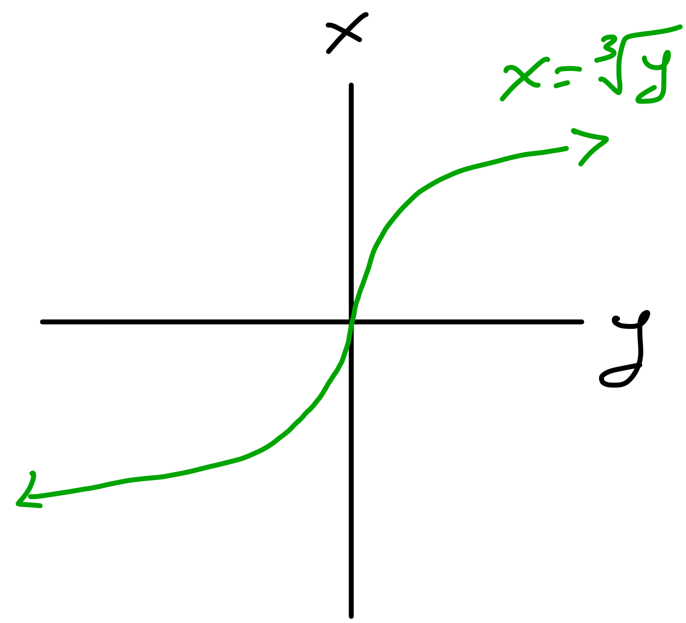
Ex.: If $y=f(x)=x^3$, then $f(2)=8$, so $f^{-1}(8)=2$. In general, $f^{-1}(y)$ $=\sqrt[3]{y}$.

A function f is only invertible if it's one-to-one: for all a and b, if $f(a)=f(b)$, then $a=b$. This just means that every output comes from only one input.

Finally, the graph of an inverse function is the graph of the original function flipped over the line $y = x$.

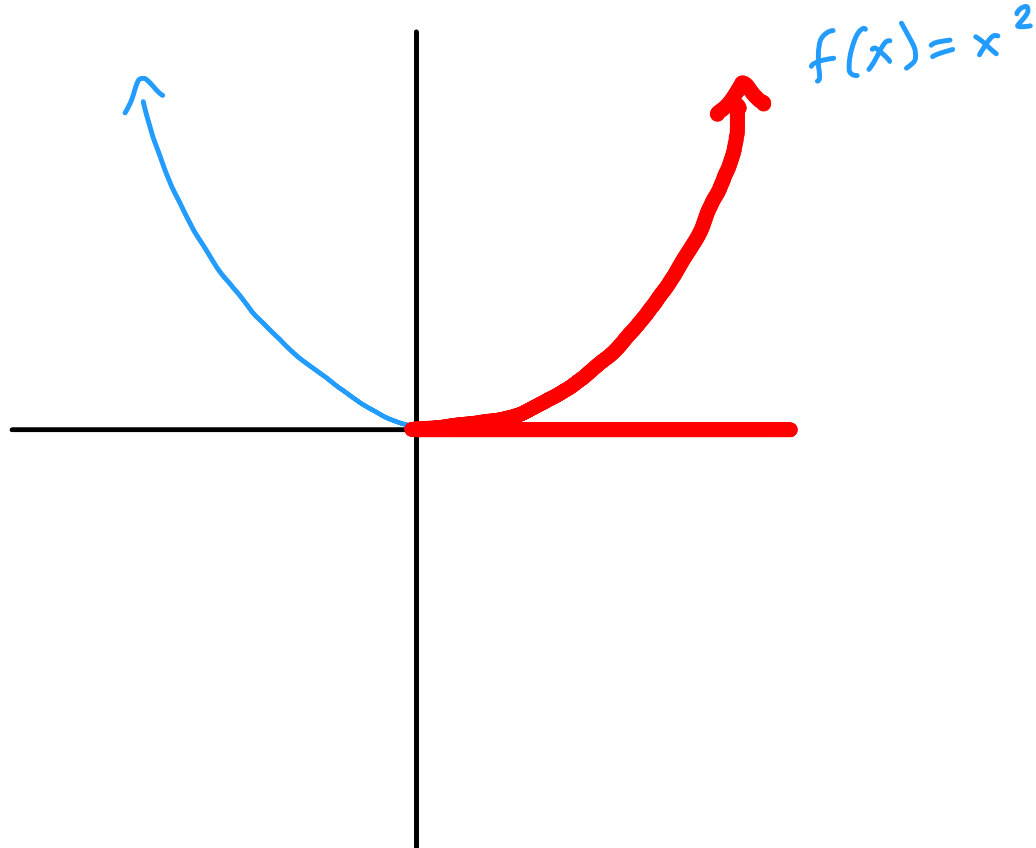
Ex:





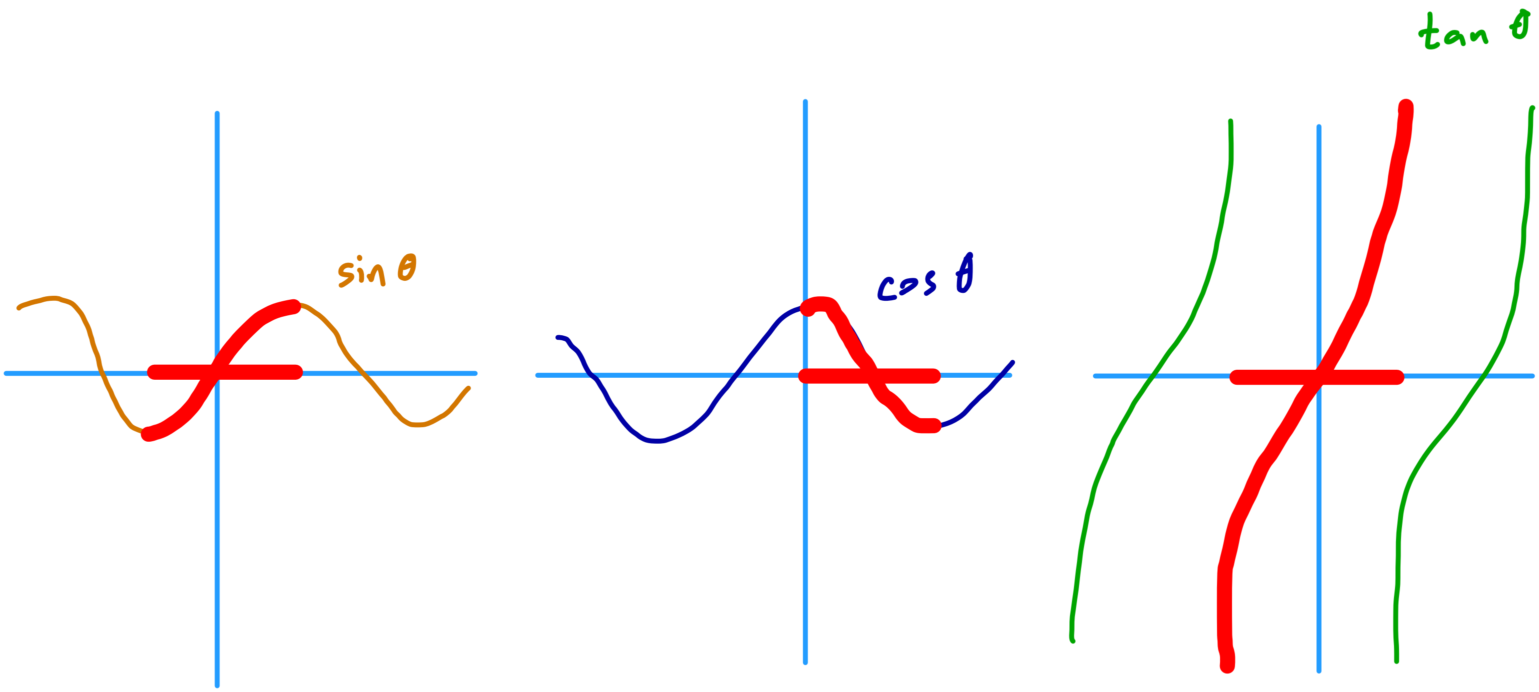
Comment: We'd like to define inverses of the trig functions, but they're not one-to-one.

Ex: The function $f(x) = x^2$ isn't one-to-one (since, for example, $(-2)^2 = 2^2$).



But f is one-to-one on $[0, \infty)$, so it's invertible there. And on that domain, $f^{-1}(y) = \sqrt{y}$.

Comment: We can similarly restrict the domains of $\sin\theta$, $\cos\theta$, and $\tan\theta$ to make them invertible.



Def: Let x be in $[-1,1]$. The arcsine of x is $\arcsin(x)=\theta$, where θ is the angle in $[-90^\circ, 90^\circ]$ such that $\sin\theta = x$.

Def: Let x be in $[-1, 1]$. The arccosine of x is $\arccos(x) = \theta$, where θ is the angle in $[0°, 180°]$ such that $\cos\theta = x$.

Def: Let x be in $(-\infty, \infty)$. The arctangent of x is $\arctan(x) = \theta$, where θ is angle in $[-90°, 90°]$ such that $\tan\theta = x$.

Comment: These functions take in distances (or, in the case of arctan, slopes), and they output one possible angle that could be fed into their non-arc counterpart to get that distance or slope.

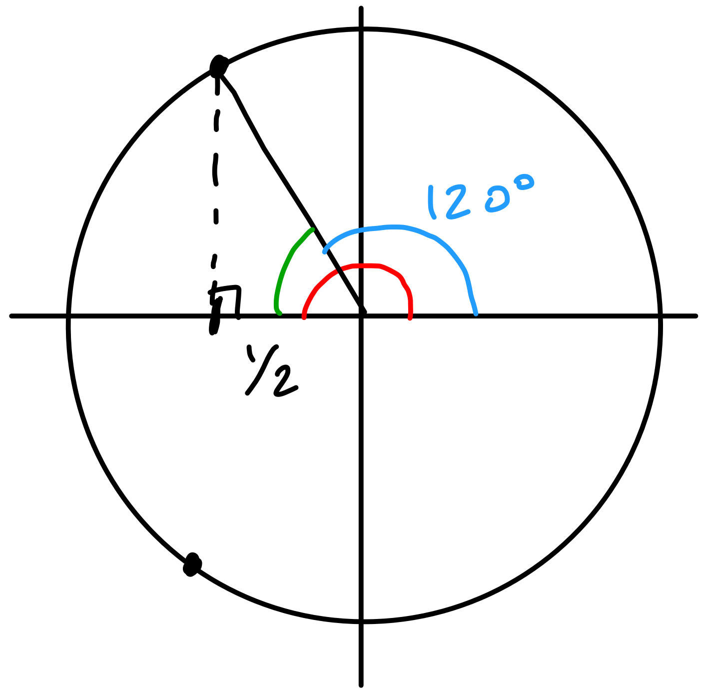
Ex: $\arccos\left(\sqrt{3}/2\right) = 30^\circ$, since $\cos 30^\circ = \frac{\sqrt{3}}{2}$ and 30° is in $[0^\circ, 180^\circ]$.

$\arcsin\left(-\sqrt{2}/2\right) = -45^\circ$, since $\sin(-45^\circ) = -\sqrt{2}/2$ and -45° is in $[-90^\circ, 90^\circ]$.

$\arctan(0) = 0^\circ$, since $\tan 0^\circ = 0$ and 0° is in $[-90^\circ, 90^\circ]$.

Comment: To find these, draw a circle and draw a point with the proper x-value / y-value / slope.

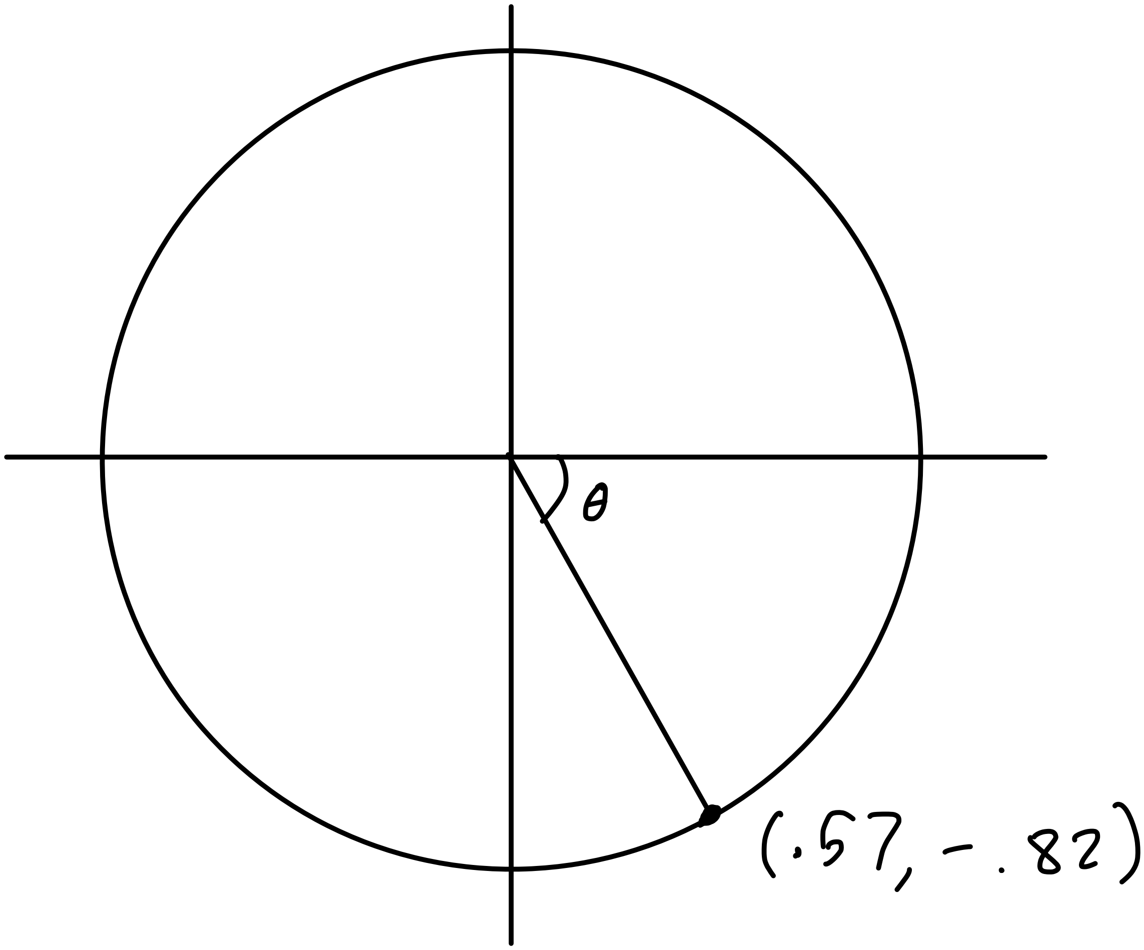
Ex: Find arccos$(-1/2)$.



$-1/2$ is the x-coordinate of the point we're trying to find.

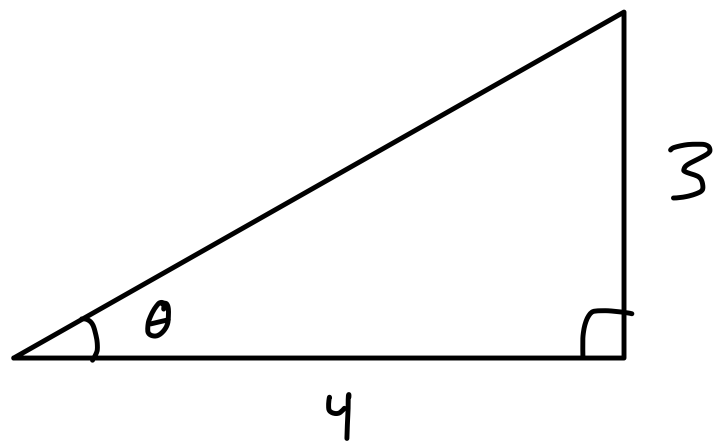
Now remember that arccos only gives angles in $[0°, 180°]$. Therefore, we want the top point, and it has an angle of 120° by a reference angle argument.

Ex: Find θ.



We're trying to find an angle given coordinates, so we'll use arcsin or arccos. But arccos gives angles in $[0°, 180°]$, which θ is not. But arcsin outputs angles in $[-90°, 90°]$, which θ is. So $\theta = \arcsin(-.82) = -55.1°$ by a calculator.

Ex: Find θ.



$\tan\theta = \frac{3}{4}$, so $\arctan\left(\frac{3}{4}\right) = \theta$

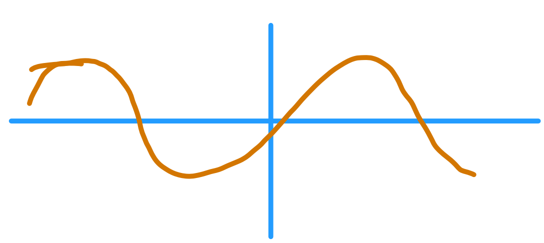
since θ is in $[-90°, 90°]$. And

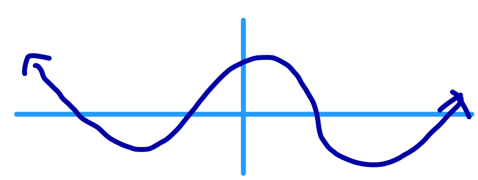
$\arctan(3/4) = 36.9°$, so $\theta = 36.9°$.

Recap: - sin and cos give the y- and x-coordinates of a point on the unit circle with a certain angle. In right triangles, they give ratios of side lengths.

- $0°$, $30°$, $45°$, $60°$, $90°$, and any angle with one of those as a reference angle are called special angles, and we can find the values of sin and cos of these angles exactly.

- The graphs of sin and cos are waves.





- The tangent function is $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and gives the slope of a line passing through $(0,0)$ and a point on the unit circle with angle θ.

- The inverse trig functions take in distances / slopes and output one possible angle that their non-arc counterpart would send to that distance / slope.

Chapter 3: Trig in More Depth

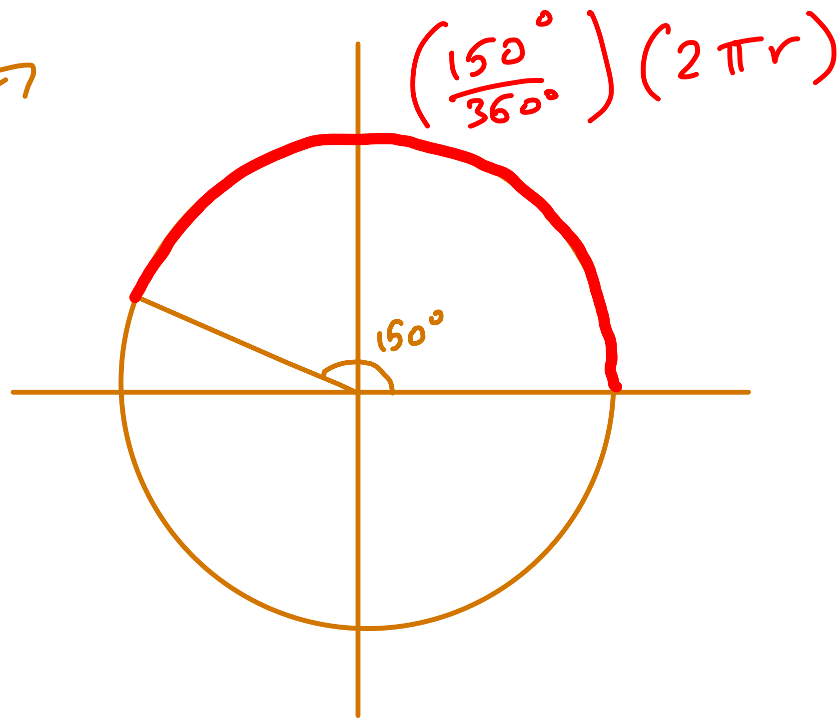
Radians

Comment: Degrees work fine for measuring angles, but the choice of 360° as a full circle is arbitrary. Is there a better angle measure we could choose?

<u>Prop</u>: A circle with radius r has area πr^2 and circumference $2\pi r$.

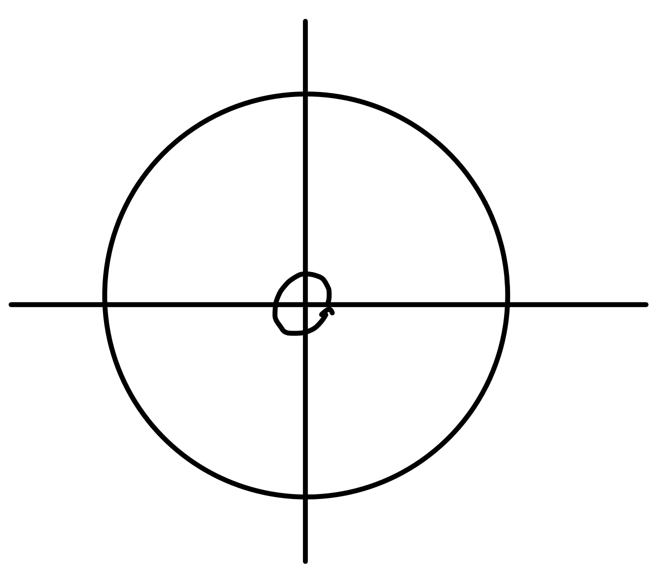
<u>Prop</u>: An arc of a circle with radius r that is θ degrees has arc length $\left(\frac{\theta}{360°}\right)(2\pi r)$.

don't use this



Def: Let θ be an angle. The radian measure of θ is equal to the arc length of an arc with angle θ in the unit circle.

Ex: $360° = 2\pi$ in radians, since the arc with angle $360°$ in the unit circle is the whole unit circle, and so has arc length 2π (since $r=1$).



Comment: Technically, radians have no unit. So we write $\theta = 360^\circ$ but $\theta = 2\pi$, not $\theta = 2\pi$ radians. Because of this, if you don't see a degree symbol, you should always assume that an angle is in radians.

Ex:

degrees	radians
360°	2π
180°	π
90°	$\pi/2$
45°	$\pi/4$
60°	$\pi/3$
30°	$\pi/6$
0°	0

don't get these confused!

Theorem: If θ is measured in degrees, then the radian measure of θ is $(\theta)\left(\frac{\pi}{180°}\right)$. If θ is measured in radians, then the degree measure of θ is $(\theta)\left(\frac{180°}{\pi}\right)$.

Ex: What is $120°$ in radians?

It's $(120°)\left(\frac{\pi}{180°}\right) = \frac{2}{3} \cdot \pi = \frac{2\pi}{3}$.

Ex: What is 5 radians in degrees?

It's $(5)\left(\frac{180°}{\pi}\right) = \frac{900°}{\pi} = 286.5°$.

Ex: Find all the quantities listed, with exact answers whenever possible

$\cos(\pi/3)$

$\sin(\pi/2)$

$\sin(3\pi/4)$

$\tan(3\pi/2)$

$\cos(-\pi/6)$

$\sin(35\pi/6)$

$\arcsin(1/2)$

$\arctan(-\sqrt{3})$

$\arccos(0)$