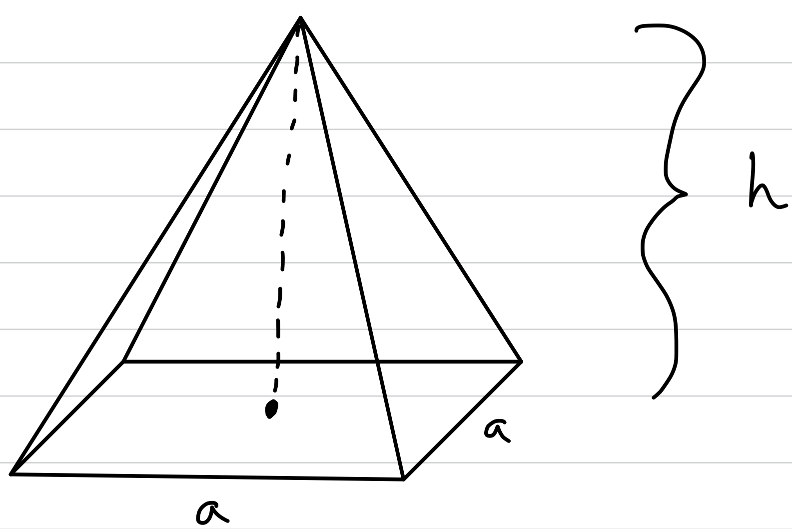
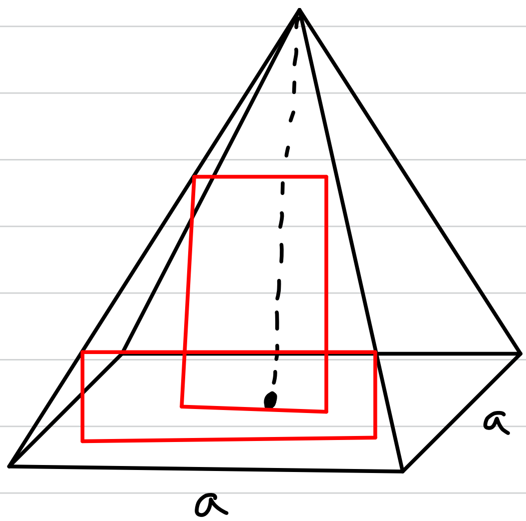
Solids of Revolution

If you have a 3-dimensional object, and its cross-sectional area is $A(x)$, then we can find the volume by integrating $A(x)$.

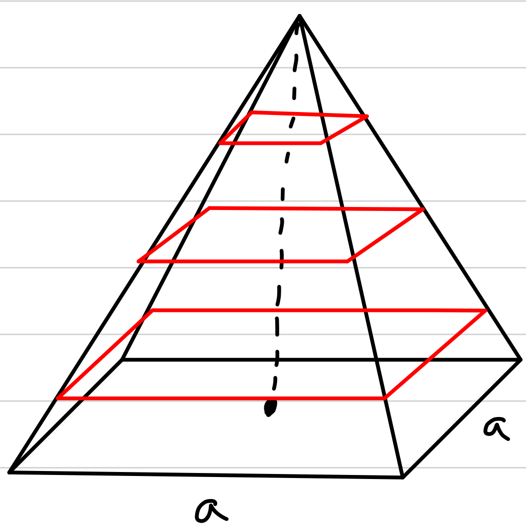
Ex: Find the volume of a pyramid with height h and side length a



We want to find $A(x)$, which is the cross-sectional area of a section at height x. You can take these cross-sections however you like, but you need to be able to find the area.

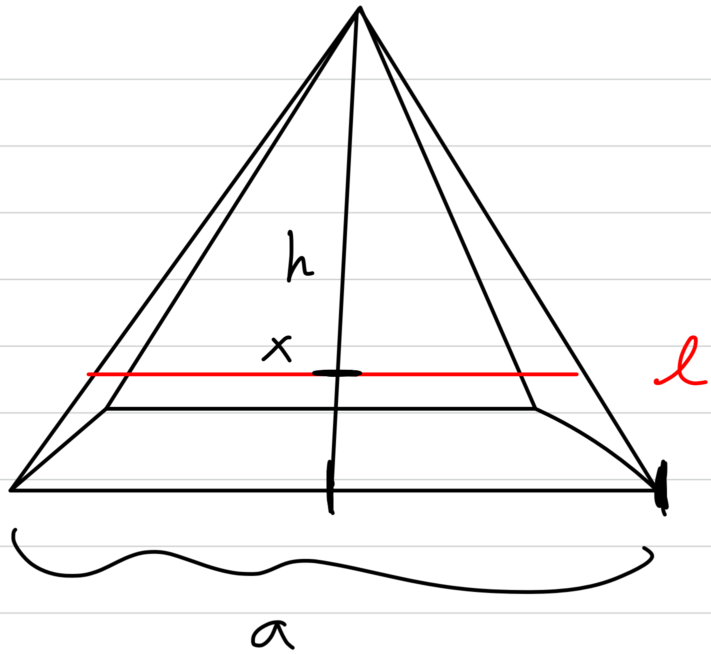


These red cross-sections are not a good idea b/c we can't easily find their areas

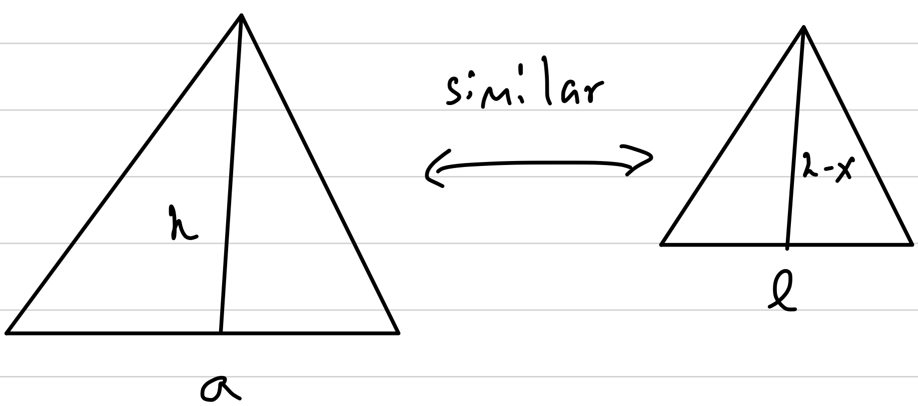


We can find these squares' areas

We need to find the side length of a square at height x.



Similar triangles: two triangles with the same angles have proportional sides.



$$\Rightarrow \quad \frac{l}{a} = \frac{h-x}{h} \qquad\qquad l = a\left(\frac{h-x}{h}\right)$$

$$A(x) = \ell^2 = a^2\left(\frac{h-x}{h}\right)^2$$

area of a square with side length ℓ

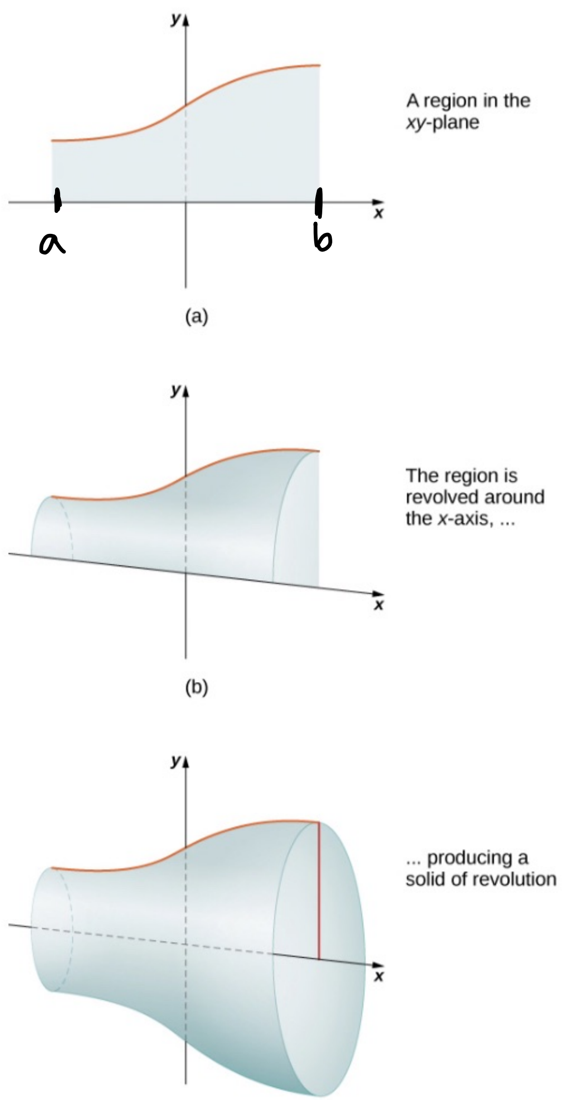
So volume $= \displaystyle\int_0^h a^2\left(\frac{h-x}{h}\right)^2 dx$

$= \frac{1}{3}a^2h$

Thm: Let S be a solid and let $A(x)$ be the cross-sectional area of a slice at x. Then the volume of S between $x=a$ and $x=b$ is $\int_a^b A(x)\,dx$.

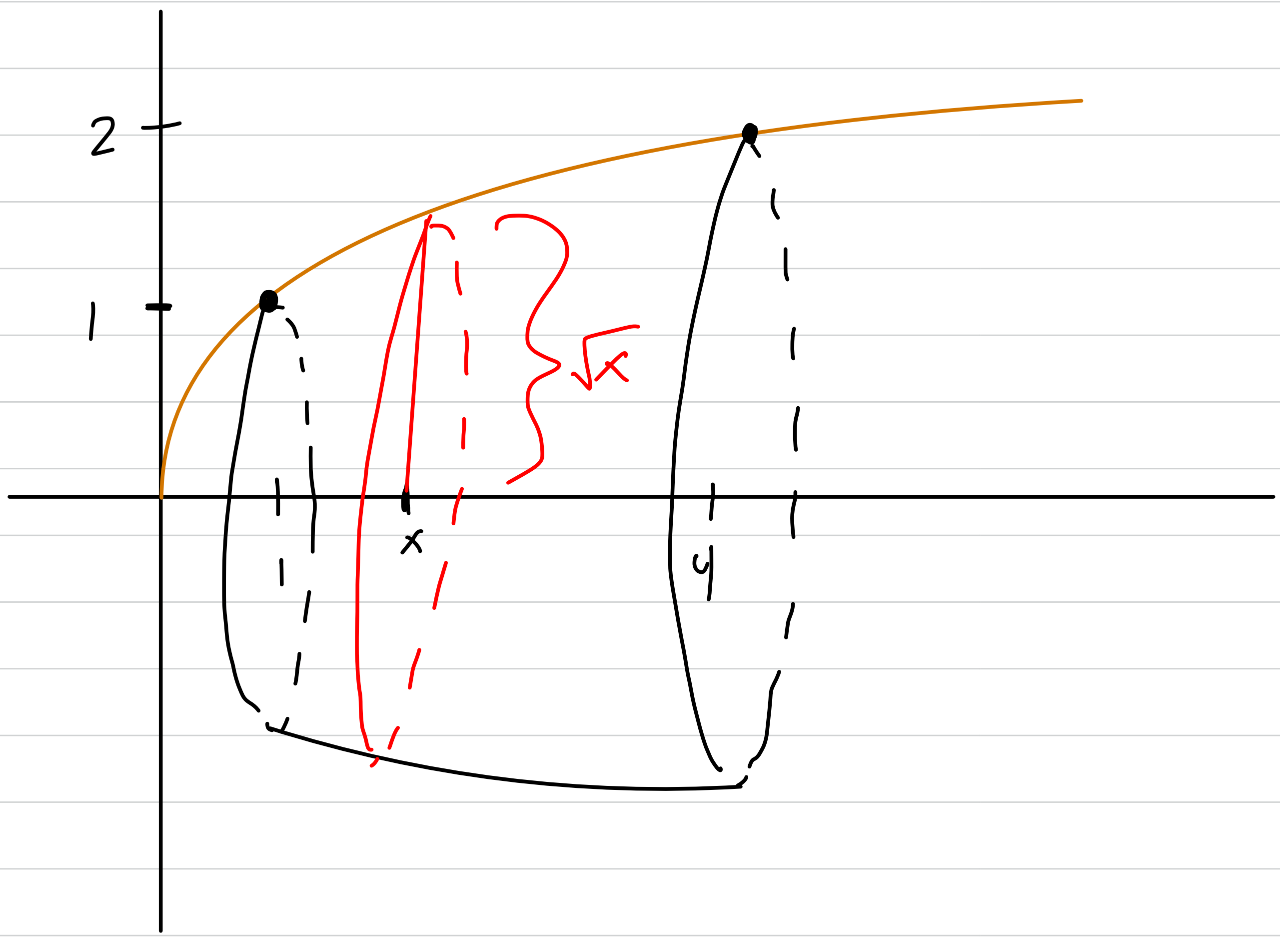
Def: Let $f(x)$ be a positive function on $[a,b]$. The solid of revolution of f about the x-axis is the 3-dimensional shape formed by rotating the graph of f about the x-axis

(Similarly for the y-axis)



The cross-sections of these things are always circles!

Ex: The graph of $y = \sqrt{x}$ is rotated about the x-axis. Find the volume between $x = 1$ and $x = 4$.



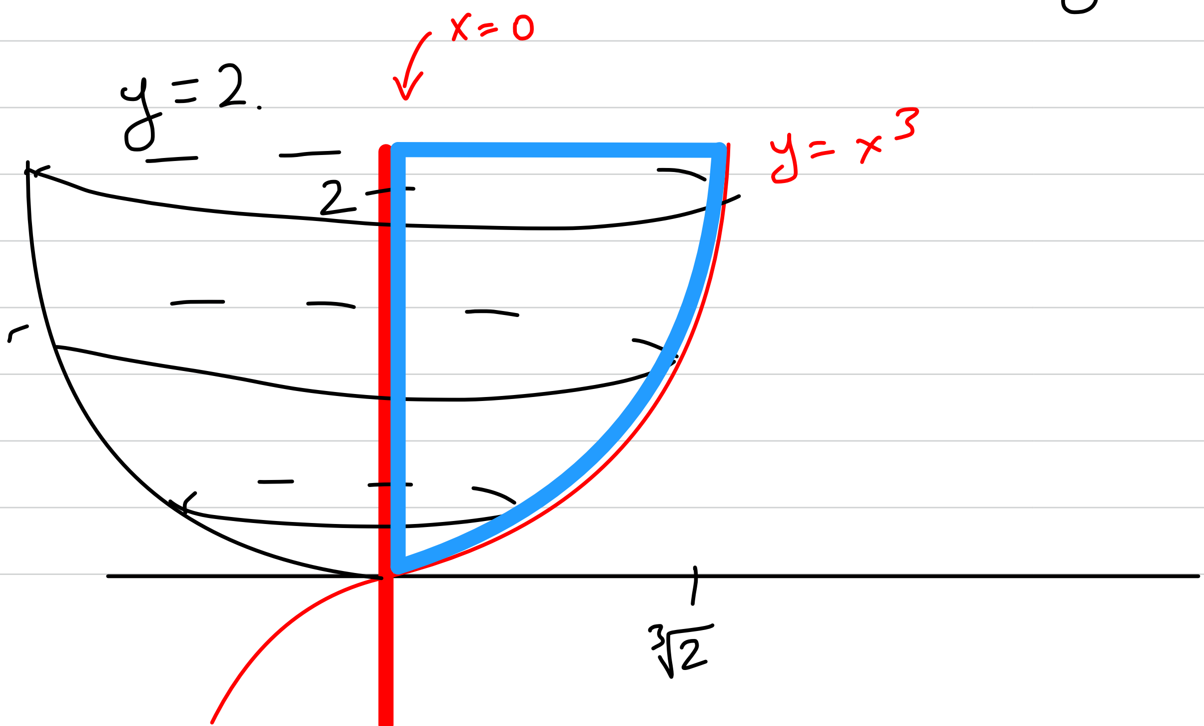
What is $A(x)$? The cross-sections are circles, so we need the radius. That is just $\sqrt{x}$, so $A(x) = \pi r^2 = \pi (\sqrt{x})^2 = \pi x$. So the volume is

$$\int_1^4 \pi x \, dx = \left[\pi \frac{x^2}{2}\right]\Bigg|_1^4 = \pi \frac{4^2}{2} - \pi \frac{1^2}{2}$$

$$= 8\pi - \pi/2$$

$$= 15\pi/2$$

Ex: The $\boxed{\text{area}}$ bounded by $y = x^3$ and $x = 0$ is rotated about the y-axis. Find the volume between $y = 0$ and $y = 2$.



$y=x^3$ tells you the height (y) at a given x-coordinate. We want the width (x) at a given y-coordinate. So we solve $y=x^3$ for x: $x=\sqrt[3]{y}$. Now $A(y)=\pi(\sqrt[3]{y})^2$,

$$\text{so vol} = \int_0^2 \pi(\sqrt[3]{y})^2\,dy$$

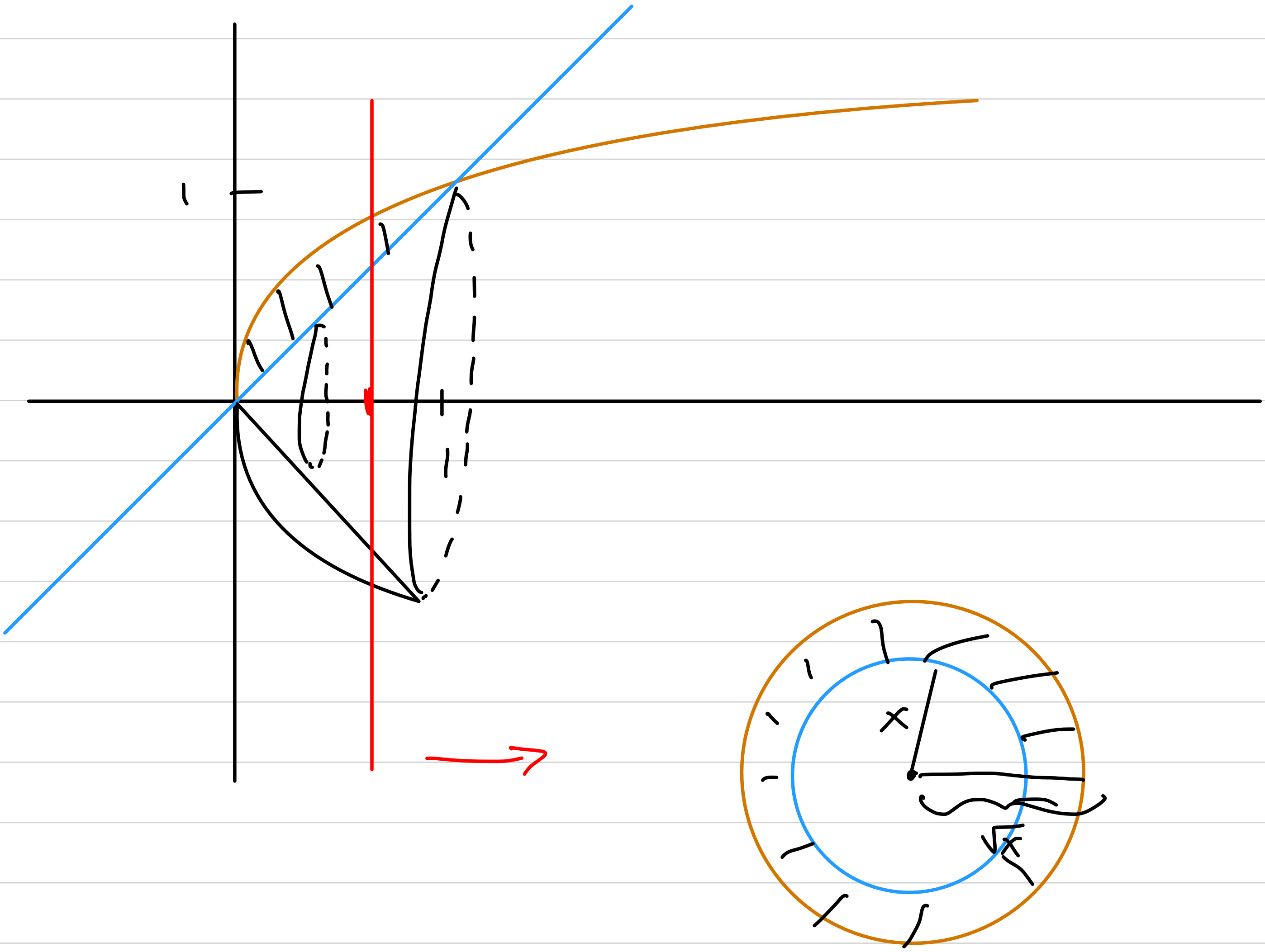
$$= \int_0^2 \pi\, y^{2/3}\,dy$$

$$= \left[\pi\,\frac{y^{5/3}}{5/3}\right]\Big|_0^2$$

$$= \pi\,\frac{2^{5/3}}{5/3} - \pi(0)$$

$$= \frac{3\pi}{5}\,2^{5/3}.$$

Ex: The area between $y=x$ and $y=\sqrt{x}$ between $x=0$ and $x=1$ is rotated about the x-axis. Find the volume.



Idea: take orange volume − blue volume

$$= \int_0^1 \pi (\sqrt{x})^2 \, dx - \int_0^1 \pi x^2 \, dx$$

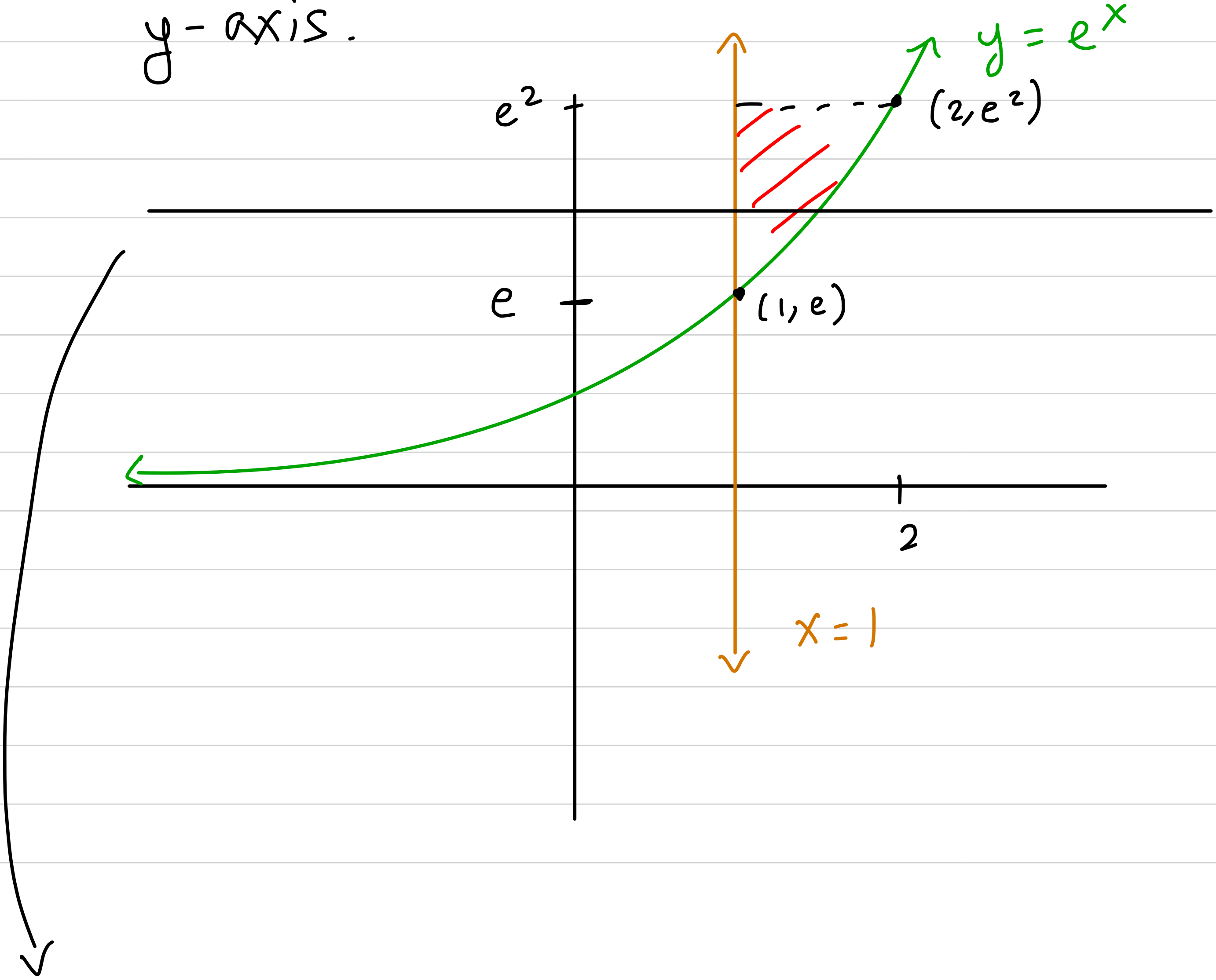
$$= \int_0^1 \pi x \, dx - \int_0^1 \pi x^2 \, dx$$

$$= \left[\pi \frac{x^2}{2}\right]\Big|_0^1 - \left[\pi \frac{x^3}{3}\right]\Big|_0^1$$

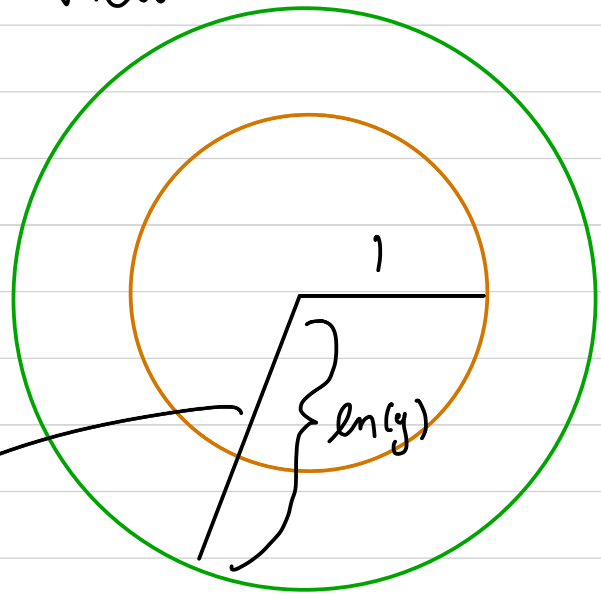
$$= \frac{\pi}{2} - \frac{\pi}{3}$$

$$= \frac{\pi}{6}$$

<u>Ex</u>: Find the volume of the shape bounded by $y=e^x$ and $x=1$ between $y=e$ and $y=e^2$ rotated about the y-axis.



Top-down view



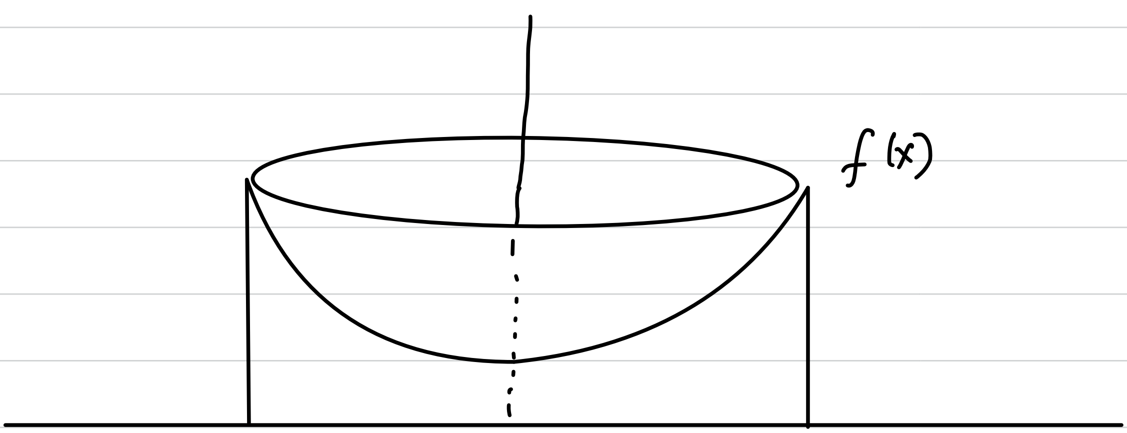
Need to solve $y = e^x$ for x

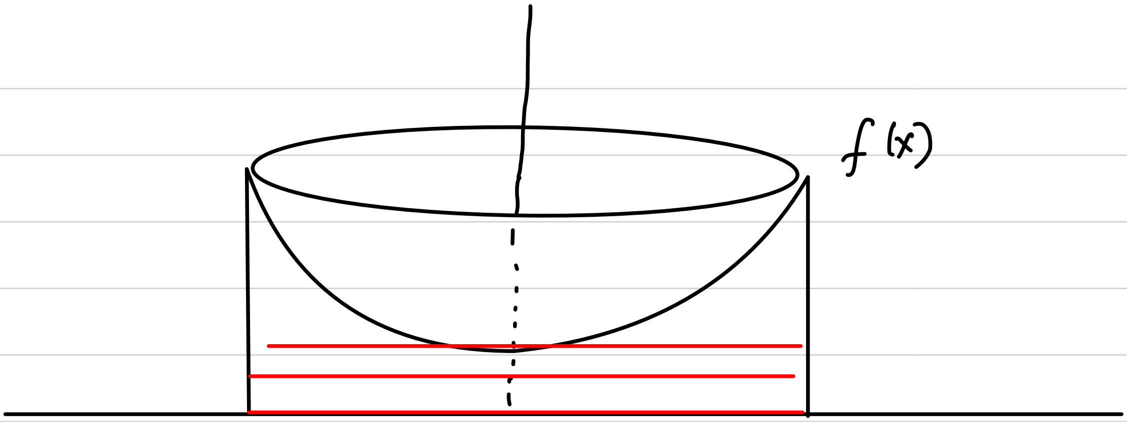
$x = \ln(y)$

$$\text{volume} = \underbrace{\int_e^{e^2} \pi \ln(y)^2 \, dy}_{\text{we can't do this yet}} - \int_e^{e^2} \pi (1)^2 \, dy$$

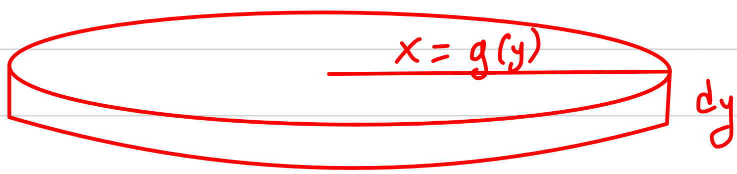
This approach failed! But there are other ways to find this volume that can sometimes succeed where this one failed.

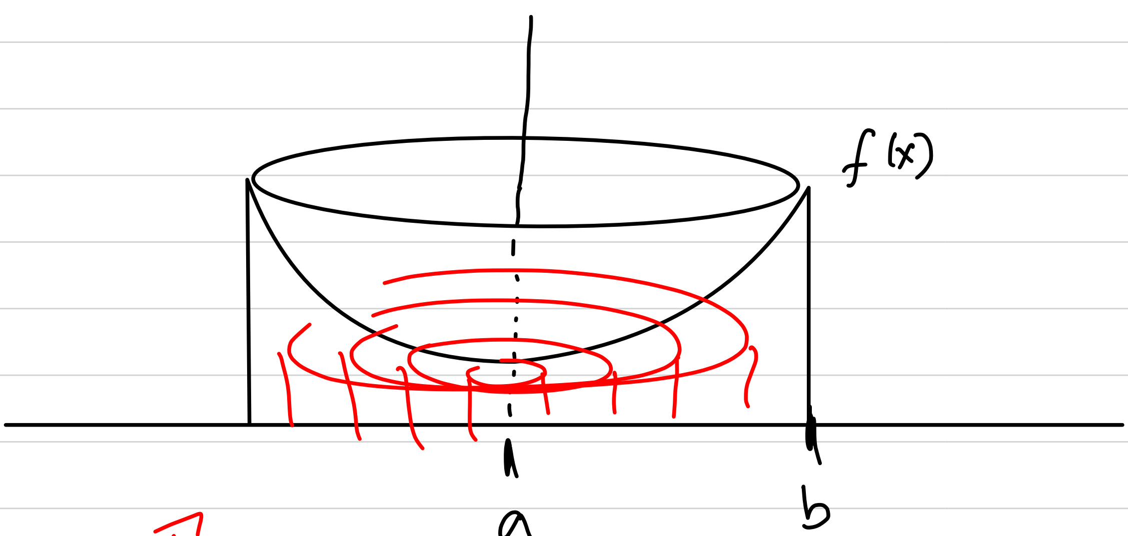
Comment: These methods (sometimes called the disc and washer methods) find the volume of a region rotated about the x-axis or y-axis by integrating with respect to the variable whose axis we rotated about. In section 2.3, we'll develop a method to find the same volume by integrating with respect to the other variable.





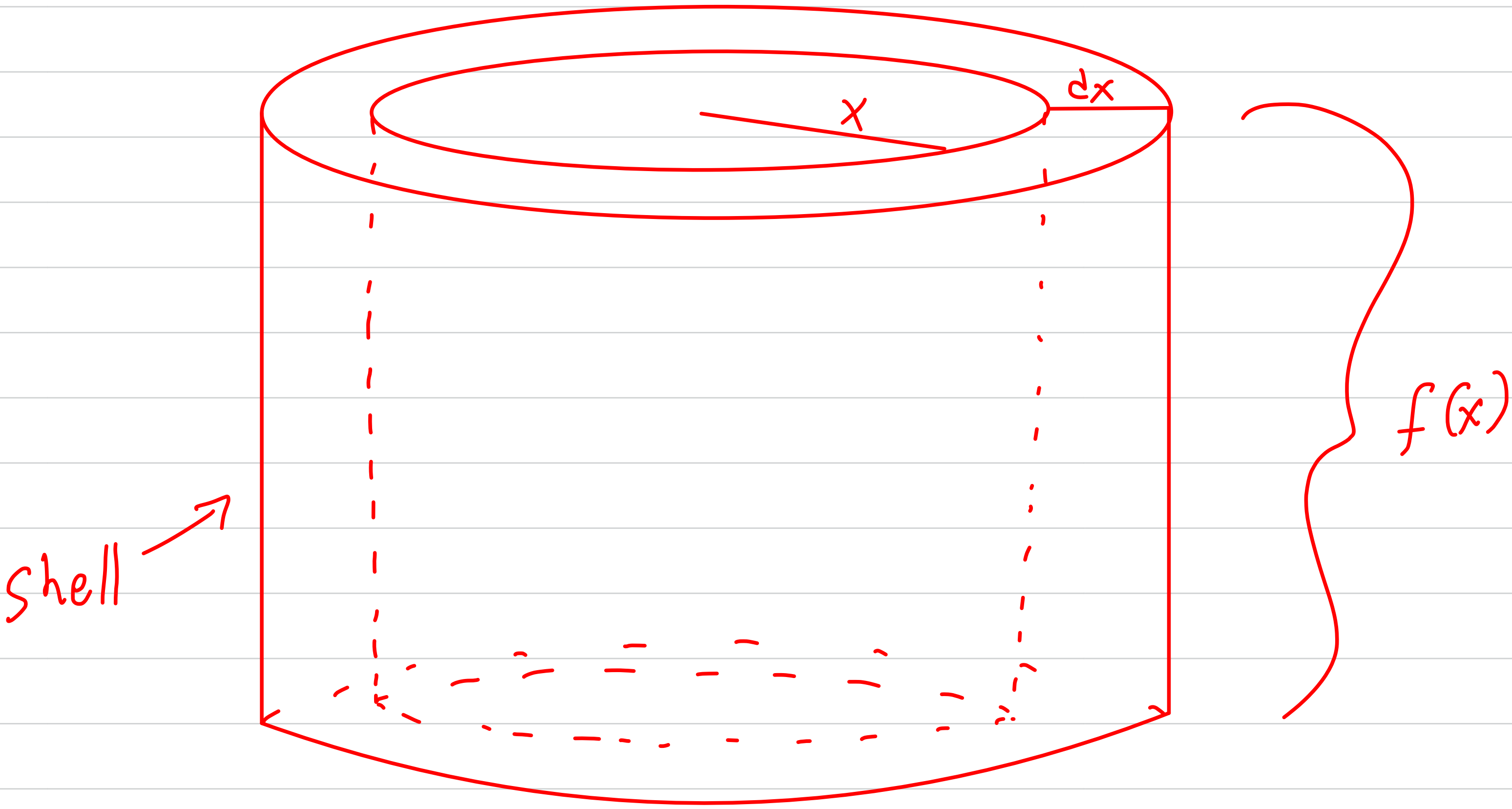
integrating with respect to y: slices look like discs:





integrate with respect to the x-axis:

slices look like



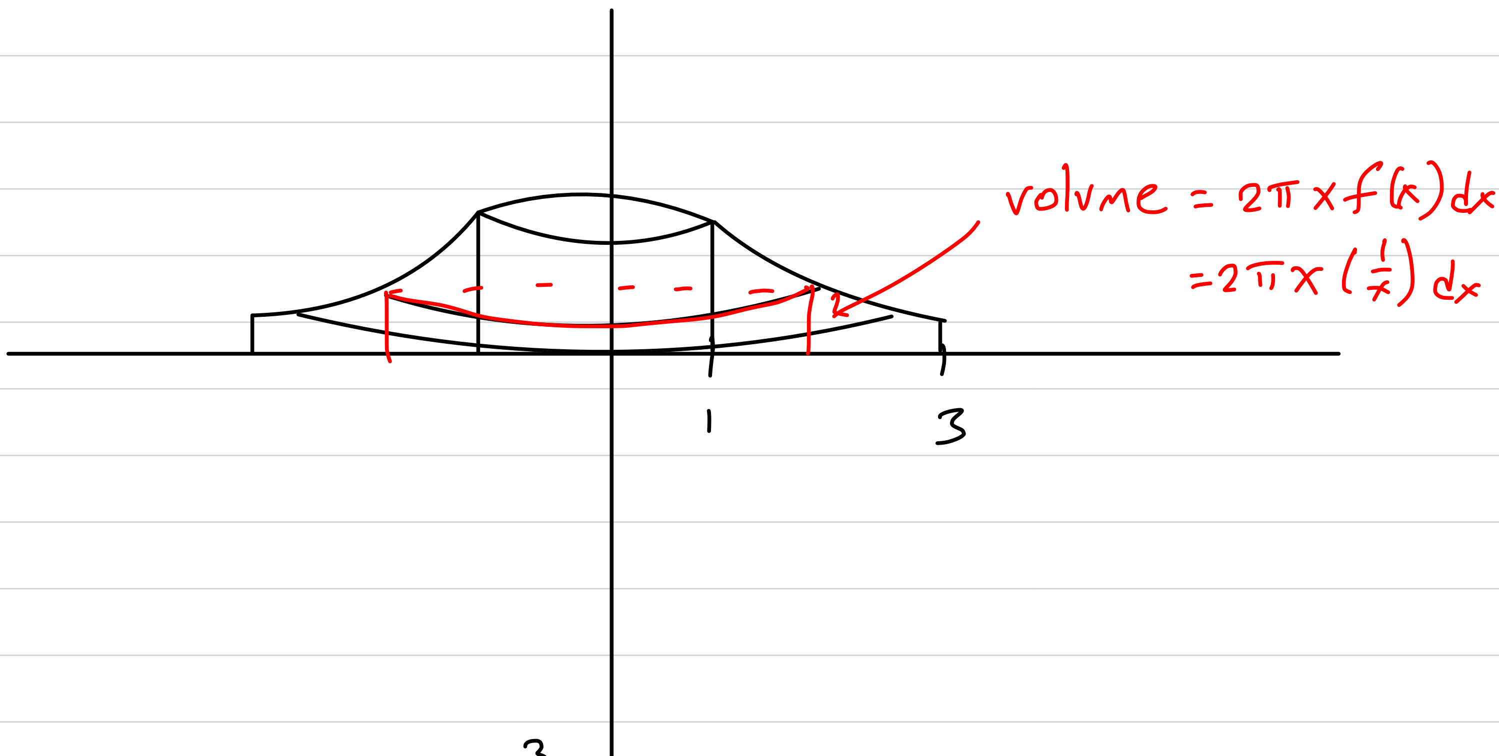
$$\text{volume} = 2\pi x\, f(x)\, dx$$

So volume of the solid of revolution is

$$\int_a^b 2\pi x\, f(x)\, dx$$

(think of x as the radius, so pick a and b so that x ranges over all radii)

Ex: Find the volume of the solid generated by rotating the region bounded by $y = 1/x$ and $y = 0$ on $[1,3]$ about the y-axis.



total volume $= \int_1^3 2\pi x \left(\frac{1}{x}\right) dx$

$= \int_1^3 2\pi \, dx$

$= \left[2\pi x\right]\Big|_1^3$

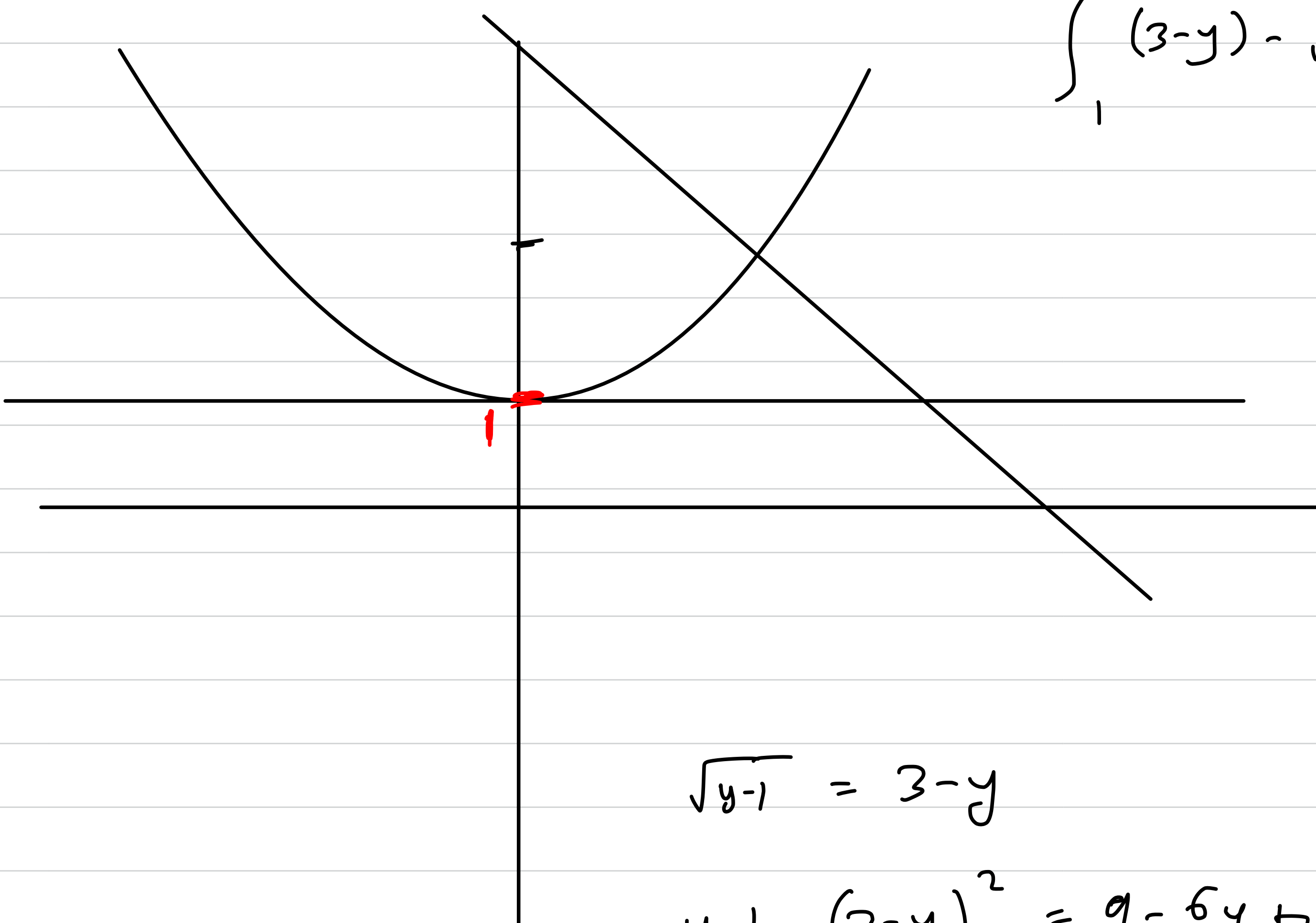
$= 6\pi - 2\pi$

$= 4\pi$

Quiz: 2.1, 2.2

$x = \sqrt{y-1}$ $\qquad x = 3-y$

$y = 1 + x^2, \quad y = 3 - x, \quad y = 1$



$$\int_1^2 (3-y) - \sqrt{y-1} \, dy$$

$\sqrt{y-1} = 3-y$

$y - 1 = (3-y)^2 = 9 - 6y + y^2$

$y^2 - 7y + 10 = 0$

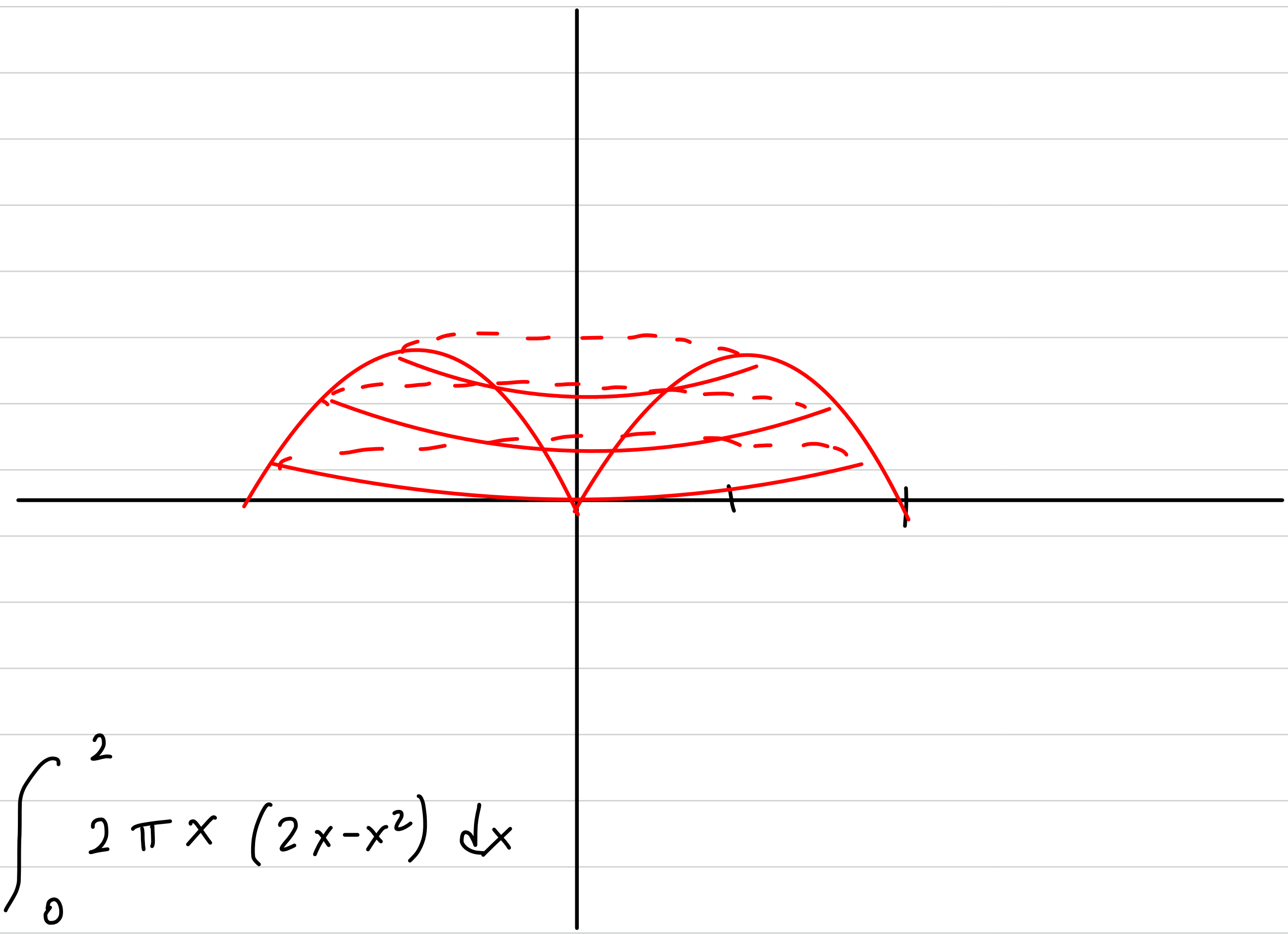
$(y-5)(y-2) = 0$

$y = 5$ or $y = 2$

$y = 2$

Ex: Find the volume of the solid generated by rotating $y = 2x - x^2$ on $[0, 2]$ about the y-axis.

$\quad = x(2-x)$



$$\int_0^2 2\pi x\,(2x - x^2)\,dx$$

$$= 2\pi \int_0^2 2x^2 - x^3\,dx$$

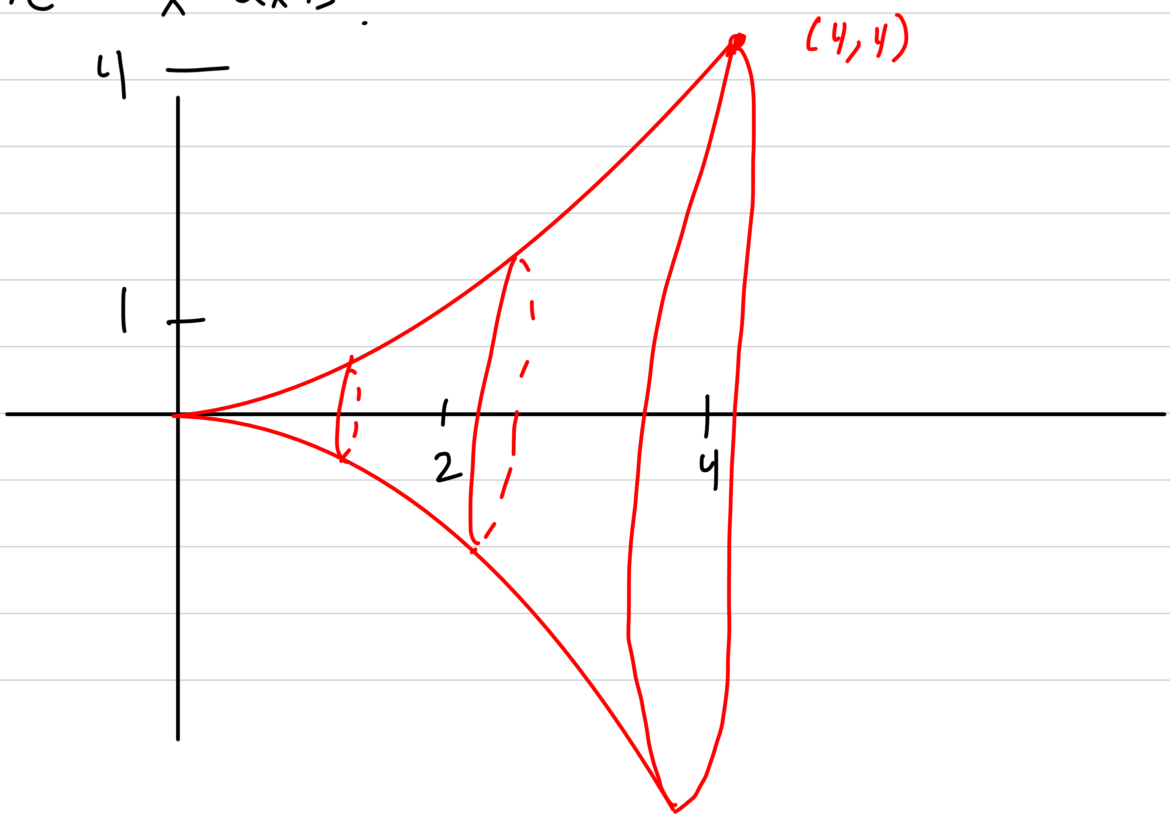
$$= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]\Bigg|_0^2$$

$$= 2\pi \left(\frac{16}{3} - \frac{16}{4} \right)$$

$$= 2\pi \left(\frac{16}{12} \right)$$

$$= \frac{8\pi}{3}.$$

Ex: Find the volume of the solid generated by rotating $y = \left(\frac{x}{2}\right)^2$ on $[0, 4]$ about the x-axis.



Could (and probably should) use discs here, but let's see how the shell method works.

$$y = (x/2)^2$$

$$\sqrt{y} = x/2$$

$$x = 2\sqrt{y}$$

$$\text{volume} = \int_0^4 2\pi y \, (2\sqrt{y}) \, dy = 4\pi \int_0^4 y^{3/2} \, dy$$

$$= 4\pi \left[\frac{y^{5/2}}{5/2} \right] \Big|_0^4$$

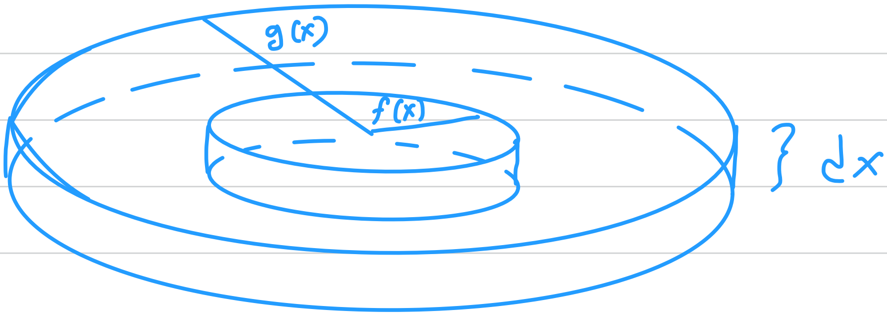
$$= 4\pi \, \frac{4^{5/2}}{5/2}$$

$$= 4\pi \, \frac{2^5}{5/2}$$

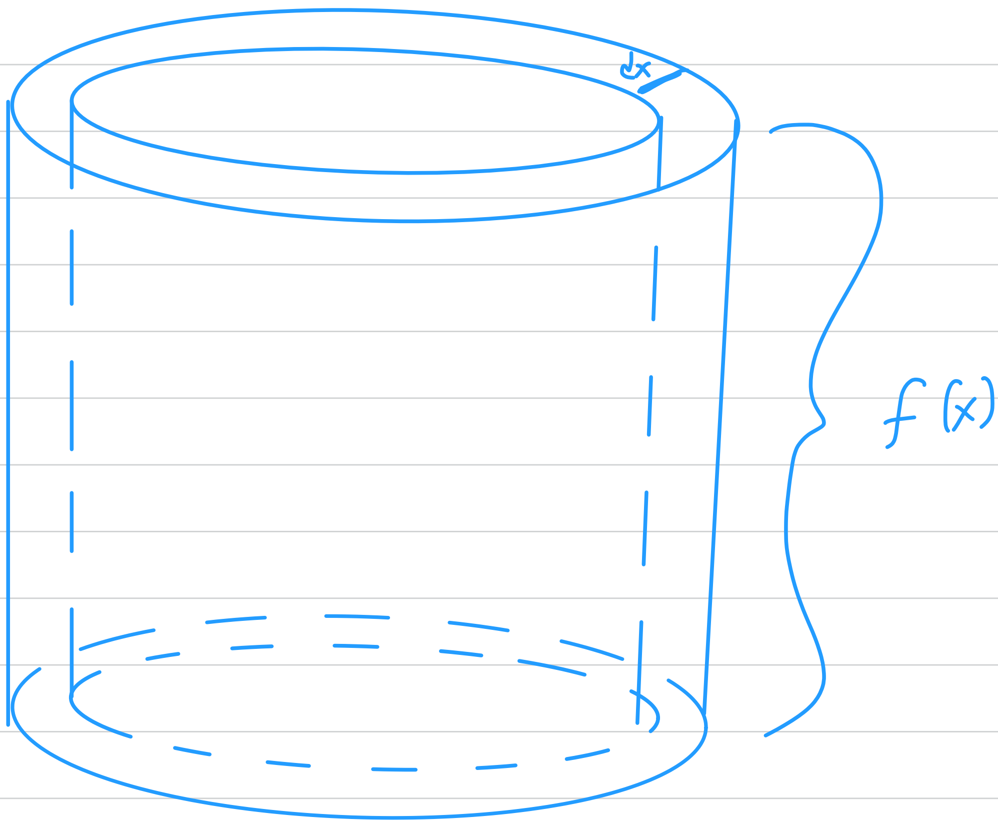
$$= 4\pi \, (32) \cdot \frac{2}{5}$$

$$= 256\pi/5$$

Comment: Don't confuse washers with shells!



washer



shell