Prop: For a 2-dim object, $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$.

Ex: There is a 2 kg mass at (-1, 3), a 6 kg mass at (1, 1), and a 4 kg mass at (2, -2). Find the center of mass.

$m_1 = 2 \qquad x_1 = -1 \qquad y_1 = 3$

$m_2 = 6 \qquad x_2 = 1 \qquad y_2 = 1$

$m_3 = 4 \qquad x_3 = 2 \qquad y_3 = -2$

$$M_x = 2 \cdot 3 + 6 \cdot 1 + 4(-2) = 4$$

$$M_y = 2(-1) + 6 \cdot 1 + 4 \cdot 2 = 12$$

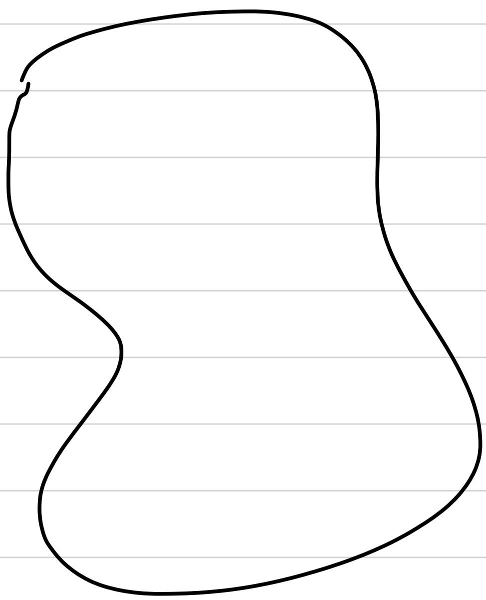
$$m = 2 + 6 + 4 = 12$$

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1$$

$$\bar{y} = \frac{M_x}{m} = \frac{4}{12} = 1/3$$

Def: A lamina is a 2-dimensional sheet with constant density.

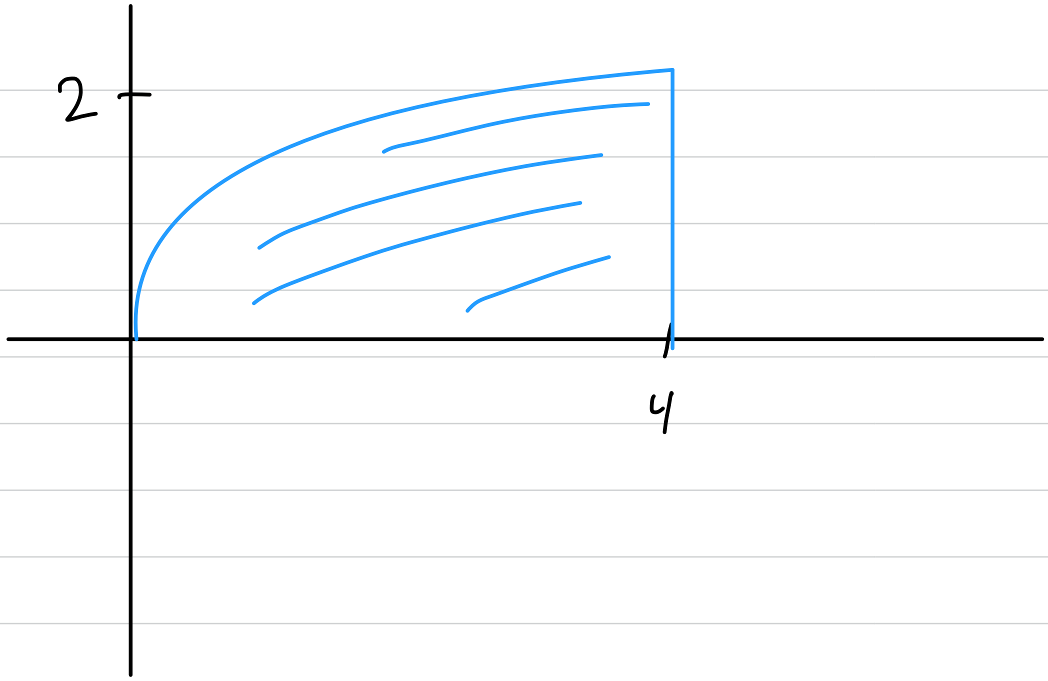
Ex:



$\rho = 3$

Comment: We will be discussing lamina that are given by areas under curves

Ex: The region under the graph of $y = \sqrt{x}$ on $[0,4]$ is a lamina.



Thm: Let L be a lamina given by the region under the graph of $f(x)$ on $[a,b]$. Suppose L has density ρ. Then:

$$m = \rho \int_a^b f(x)\,dx$$

$$M_x = \rho \int_a^b \frac{1}{2}(f(x))^2\,dx$$

$$M_y = \rho \int_a^b x\,f(x)\,dx$$

Ex: Find the center of mass of the lamina given by the region under the curve of $\sqrt{x}$ from $x=0$ to $x=4$. Note that although we're not given density, it doesn't affect the center of mass and will therefore cancel out.

$$m = \rho \int_0^4 \sqrt{x}\, dx$$

$$= \rho \left[\frac{x^{3/2}}{3/2} \right] \Big|_0^4$$

$$= \rho \, \frac{4^{3/2}}{3/2}$$

$$= \rho \cdot 8 \cdot \frac{2}{3}$$

$$= \frac{16}{3} \rho.$$

$$M_x = \rho \int_a^b \frac{1}{2} (f(x))^2 \, dx$$

$$= \rho \int_0^4 \frac{1}{2} x \, dx$$

$$= \rho \left[\frac{x^2}{4} \right] \Big|_0^4$$

$$= \rho \cdot 4$$

$$= 4\rho.$$

$$x^{3/2}$$

$$M_y = \rho \int_0^4 x\sqrt{x} \, dx$$

$$= \rho \left[\frac{x^{5/2}}{5/2} \right] \Big|_0^4$$

$$= \rho \cdot 32 \cdot \frac{2}{5}$$

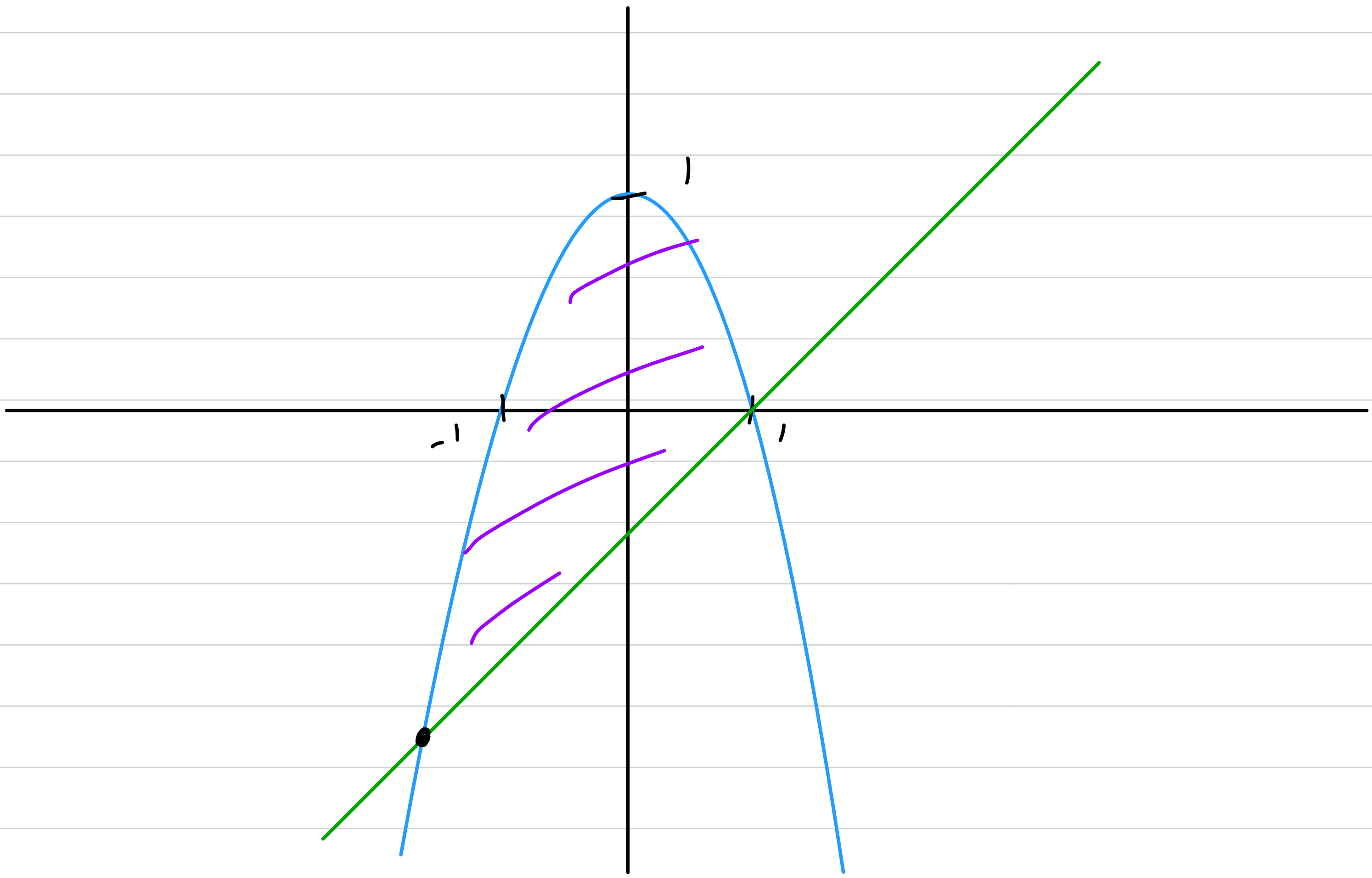
$$= \frac{64}{5}\rho.$$

$$\bar{x} = \frac{M_y}{m} = \frac{64/5 \ \rho}{16/3 \ \rho} = \frac{64}{5} \cdot \frac{3}{16} = \frac{12}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4\rho}{16/3 \ \rho} = 4 \cdot \frac{3}{16} = \frac{3}{4}$$

So the center of mass is $\left(\frac{12}{5}, \frac{3}{4}\right)$.

Ex: Find the x- and y-moments of a lamina with density 3 bounded above by $1-x^2$ and below by $x-1$.



First, find the intersection points:

$1-x^2 = x-1$

$x^2+x-2=0$

$(x-1)(x+2)=0$

$x=1$ or $x=-2$.

$$M_x = \underbrace{3\int_{-2}^{1} \tfrac{1}{2}(1-x^2)^2\,dx}_{\text{top } M_x} - \underbrace{3\int_{-2}^{1} \tfrac{1}{2}(x-1)^2\,dx}_{\text{bottom } M_x}$$

$$= \frac{3}{2}\int_{-2}^{1} 1-2x^2+x^4\,dx - \frac{3}{2}\int_{-2}^{1} x^2-2x+1\,dx$$

$$= \frac{3}{2}\left[x-\frac{2x^3}{3}+\frac{x^5}{5}\right]\Big|_{-2}^{1} - \frac{3}{2}\left[\frac{x^3}{3}-x^2+x\right]\Big|_{-2}^{1}$$

$$= -\frac{81}{10}$$

$$M_y = 3\int_{-2}^{1} x(1-x^2)\,dx - 3\int_{-2}^{1} x(x-1)\,dx$$

$$= 3\left[\frac{x^2}{2} - \frac{x^4}{4}\right]\Big|_{-2}^{1} - 3\left[\frac{x^3}{3} - \frac{x^2}{2}\right]\Big|_{-2}^{1}$$

$$= -\frac{27}{4}$$

Chapter II Review

Area between curves : $\int_a^b f(x) - g(x)\, dx$

f top

g bottom

$$\int_a^b f(y) - g(y)\, dy$$

f right

g left

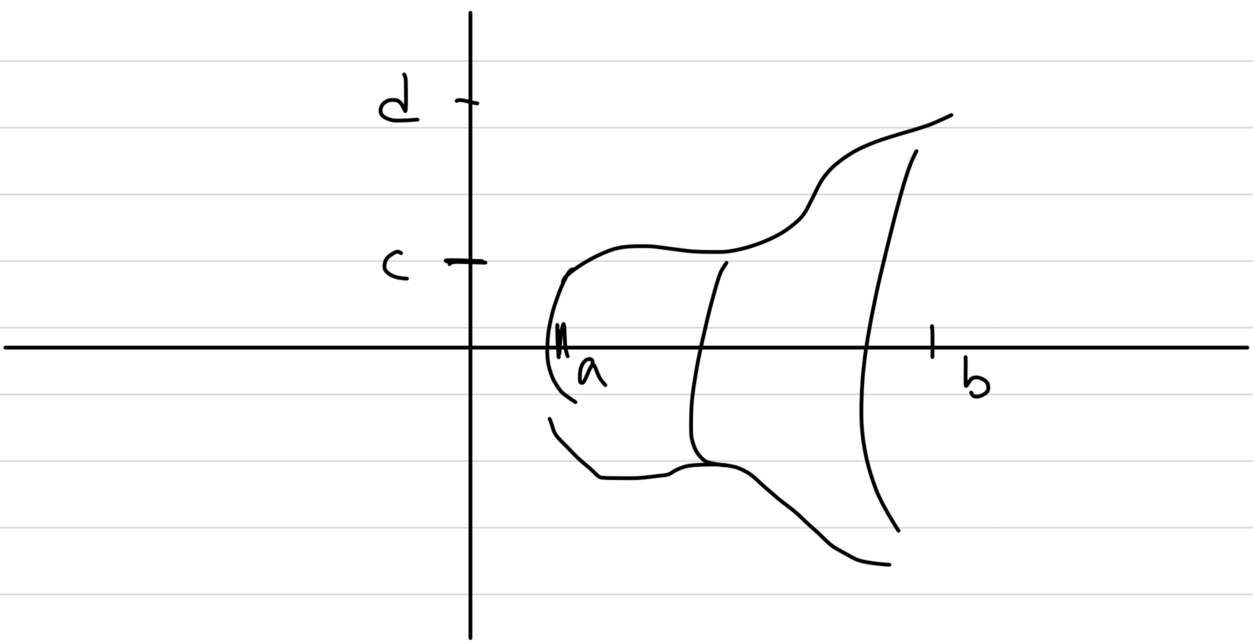
Revolution about x-axis : volume is

Disk method : $\int_a^b \pi (f(x))^2\, dx$

- Washers: $\int_a^b \pi (f(x))^2\,dx - \int_a^b \pi (g(x))^2\,dx$

f top function
g bottom function

Shells: $\int_c^d 2\pi y\, h(y)\,dy$



Arc length: $\int_a^b \sqrt{1+(f'(x))^2}\,dx$

Surface area of revolution about x-axis:

$$\int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\,dx$$

Work: $\int_a^b F(x)\,dx$ F is force

Mass: $\int_a^b \rho(x)\,dx$ $\rho(x)$ = density at x

for a linear object

Mass of a circle: $\int_a^b 2\pi r\,\rho(r)\,dr$ $\rho(r)$ = radial density at r

Pumping: slice vertically and integrate the work done on each slice, which is (weight-density) (area) (distance)

Moments: $m = \rho \int_a^b f(x)\,dx$ for a lamina given by f

$$M_x = \rho \int_a^b \frac{1}{2}\left(f(x)\right)^2\,dx$$

$$M_y = \rho \int_a^b x\,f(x)\,dx$$

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}.$$

Chapter III: Techniques of Integration

Integration by Parts

Comment: u-sub worked on compositions:

$$\int f(g(x))\, g'(x)\, dx$$

$u = g(x)$
$du = g'(x)\, dx$

$$\int f(u)\, du$$

Integration by Parts will work on products of functions: $\int f(x)\, g(x)\, dx$

Thm (Integration by Parts)

$$\int u\,dv = uv - \int v\,du$$

In practice, when you have $\int f(x)g(x)\,dx$, let $u = f(x)$ and $dv = g(x)\,dx$. For this to work, you need to be able to differentiate f and integrate g. Fill out a table:

$$\begin{array}{ccc} u = f(x) & & v = G(x) \\ \text{differentiate} \downarrow & & \uparrow \text{integrate} \\ du = f'(x)\,dx & & dv = g(x)\,dx \end{array}$$

Now: $\int f(x)\,g(x)\,dx = f(x)\,G(x) - \int G(x)\,f'(x)\,dx.$

Ex. $\int x \sin(x)\,dx$

Choosing $u = \sin(x)$ and $dv = x\,dx$,

we get

$u = \sin(x)$ $\quad\quad$ $v = \frac{x^2}{2}$

$\downarrow$ $\quad\quad$ $\uparrow$

$du = \cos(x)\,dx$ $\quad\quad$ $dv = x\,dx$

$$= \frac{x^2}{2}\sin(x) - \underbrace{\int \frac{x^2}{2}\cos(x)\,dx}_{\text{worse than the start}}$$

Instead, try:

$u = x$ $\quad\quad$ $v = -\cos(x)$

$\downarrow$ $\quad\quad$ $\uparrow$

$du = dx$ $\quad\quad$ $dv = \sin(x)\,dx$

$$= -x\cos(x) - \int -\cos(x)\,dx$$

we can solve this

$= -x\cos(x) + \sin(x) + C.$

Ex: Find $\int \frac{\ln(x)}{x^3}\,dx$

Examples of choices for u and dv:

$u = \ln(x) \qquad dv = \frac{1}{x^3}\,dx$

$u = \frac{\ln(x)}{x} \qquad dv = \frac{1}{x^2}\,dx$

$u = \frac{1}{x^3} \qquad dv = \ln(x)\,dx$

Integrating $\ln(x)$ gives $x\ln(x) - x + C$, which is complicated, so we shouldn't have $\ln(x)$ in the dv. On the other hand, $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$, so we should try $u = \ln(x)$. Then $dv = \frac{1}{x^3}\,dx$

$u = \ln(x) \qquad v = \frac{x^{-2}}{-2} = -\frac{1}{2x^2}$

$du = \frac{1}{x}\,dx \qquad dv = \frac{1}{x^3}\,dx$

$$= -\frac{\ln(x)}{2x^2} - \int -\frac{1}{2x^2} \cdot \frac{1}{x}\, dx$$

$$= -\frac{\ln(x)}{2x^2} + \frac{1}{2}\int \frac{1}{x^3}\, dx$$

$$= -\frac{\ln(x)}{2x^2} + \frac{1}{2} \cdot \frac{-1}{2x^2} + C$$

$$= \frac{-\ln(x)}{2x^2} - \frac{1}{4x^2} + C$$

verify: $$\frac{d}{dx}\left[-\frac{\ln(x)}{2x^2} - \frac{1}{4x^2} + C \right]$$

$$= -\frac{\frac{1}{x} \cdot 2x^2 - \ln(x) \cdot 4x}{4x^4} + \frac{2}{4}x^{-3}$$

$$= -\frac{2x - 4x\ln(x)}{4x^4} + \frac{1}{2x^3}$$

$$= -\frac{1 - 2\ln(x)}{2x^3} + \frac{1}{2x^3}$$

$$= \frac{-1 + 2\ln(x)}{2x^3} + \frac{1}{2x^3}$$

$$= \frac{\cancel{-1} + 2\ln(x) + \cancel{1}}{2x^3}$$

$$= \frac{\ln(x)}{x^3} \checkmark$$

Comment: Integration by Parts is just the product rule for derivatives in reverse:

$$(uv)' = u'v + uv'$$

$$\int (uv)' = \int u'v + \int uv'$$

$$uv = \int v\,du + \int u\,dv$$

$$\int u\,dv = uv - \int v\,du$$

Ex: Sometimes, we have to apply Integration by Parts more than once.

$$\int x^2 e^{3x}\,dx$$

$u = x^2$ $\quad$ $v = \frac{1}{3}e^{3x}$

$du = 2x\,dx$ $\quad$ $dv = e^{3x}\,dx$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x}\,dx$$ ← do by Parts again!

$u = x$ $\quad$ $v = \frac{1}{3}e^{3x}$

$du = dx$ $\quad$ $dv = e^{3x}\,dx$

$$\int e^{3x}\,dx$$

$u = 3x$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$$\frac{1}{3}\int e^u\,du$$

$$\frac{1}{3}e^u + C$$

$$= \frac{1}{3} e^{3x} + C$$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3} x e^{3x} - \frac{1}{3}\int e^{3x}\, dx \right)$$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3} x e^{3x} - \frac{1}{3}\left(\frac{1}{3} e^{3x} + C\right)\right)$$

$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C.$$

<u>Ex:</u> $\int t^3 e^{t^2}\, dt$

Try $u = t^3$ and $dv = e^{t^2}\, dt$?

Problem: we can't integrate e^{t^2}.

Why? $\frac{d}{dt}\left[e^{t^2}\right] = e^{t^2} \cdot 2t$

Instead, notice that we can integrate te^{t^2} by u-sub, so we want to let $dv = te^{t^2}\,dt$, and therefore $u = t^2$.

$u = t^2$

$du = 2t\,dt$

$v = \int te^{t^2} = \frac{1}{2}e^{t^2}$

$dv = te^{t^2}\,dt$

$w = t^2$

$dw = 2t\,dt$

$\frac{1}{2}dw = t\,dt$

$v = \frac{1}{2}\int e^w\,dw = \frac{1}{2}e^w = \frac{1}{2}e^{t^2}$

$$= \frac{1}{2}t^2 e^{t^2} - \frac{1}{2}\int 2t e^{t^2}\, dt$$

$$= \frac{1}{2}t^2 e^{t^2} - \int t e^{t^2}\, dt$$ we just did this!

$$= \frac{1}{2}t^2 e^{t^2} - \frac{1}{2}e^{t^2} + C$$

Comment: To evaluate a definite integral by Parts, just evaluate the entire expression:

$$\int_a^b u\, dv = \left[uv - \int v\, du\right]\Big|_a^b$$

Ex: Find the volume of the solid given by rotating the graph of $y = e^{-x}$ from $x = 0$ to $x = 1$ about the y-axis.

$$= -\int u^4 - u^6 \, du$$

$$= -\left(\frac{u^5}{5} - \frac{u^7}{7} + C \right)$$

$$= -\frac{\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C.$$

Ex: $\int \cos^5(x) \sin^3(x) \, dx$

Apply method 1: we can pick either sin or cos to split b/c both powers are odd. Let's pick cos.

$$= \int \overbrace{\cos^4(x)}^{(\cos^2(x))^2} \sin^3(x) \cos(x) \, dx$$

$$= \int (1 - \sin^2(x))^2 \sin^3(x) \cos(x) \, dx$$

$u = \sin(x)$

$du = \cos(x) \, dx$

$$= \int (1-u^2)^2 u^3 \, du$$

$$= \int (1 - 2u^2 + u^4) u^3 \, du$$

$$= \int u^3 - 2u^5 + u^7 \, du$$

$$= \frac{u^4}{4} - \frac{2u^6}{6} + \frac{u^8}{8} + C$$

$$= \frac{\sin^4(x)}{4} - \frac{2\sin^6(x)}{6} + \frac{\sin^8(x)}{8} + C.$$

<u>Ex</u>: $\int \sin^2(x) \cos^4(x) \, dx$

Now both powers are even, so we apply method 2: rewrite all the sines as $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$ and the cosines as $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1-\cos(2x))(1+\cos(2x))^2 \, dx$$

$$= \frac{1}{8}\int (1-\cos(2x))\left(1+2\cos(2x)+\cos^2(2x)\right) dx$$

$$= \frac{1}{8}\int 1 + 2\cos(2x) + \cos^2(2x) - \cos(2x) - 2\cos^2(2x) - \cos^3(2x) \, dx$$

$$= \frac{1}{8}\int 1 + \cos(2x) - \cos^2(2x) - \cos^3(2x) \, dx$$

$$= \frac{1}{8}\left(x + \frac{1}{2}\sin(2x) - \int \cos^2(2x)\,dx - \int \cos^3(2x)\,dx\right)$$

Apply method ②: $\displaystyle\int \frac{1+\cos(4x)}{2}\,dx$

$$= \int \frac{1}{2} + \frac{1}{2}\cos(4x)\,dx$$

$$= \frac{1}{2}x + \frac{1}{8}\sin(4x)$$

Apply method ①: $\int \cos^2(2x)\cos(2x)\,dx$

$$= \int (1-\sin^2(2x))\cos(2x)\,dx$$

$u = \sin(2x)$

$du = 2\cos(2x)$

$\frac{1}{2}du = \cos(2x)\,dx$

$$= \int (1-u^2)\frac{1}{2}\,du$$

$$= \frac{1}{2}\left(u - \frac{u^3}{3}\right)$$

$$= \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x)$$

$$\frac{1}{8}\left(x + \frac{1}{2}\sin(2x) - \frac{1}{2}x - \frac{1}{8}\sin(4x) - \frac{1}{2}\sin(2x)\right.$$

$$\left. + \frac{1}{6}\sin^3(2x)\right) + C$$

Prop:

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

Why?

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\ln|\sec(x)|] = \frac{1}{\sec(x)} \cdot \sec(x)\tan(x)$$

$$= \tan(x)$$

$$\frac{d}{dx}[\ln|\sec(x) + \tan(x)|]$$

$$= \frac{1}{\sec(x)+\tan(x)} \cdot (\sec(x)\tan(x) + \sec^2(x))$$

$$= \sec(x)$$

<u>Comment</u>: There are formulas/methods for

$$\int \sec^j(x)\tan^k(x)\,dx,$$ but we don't care.