Name: ____________________

Midterm 2

Math 253

Fall 2022

You have 50 minutes to complete this exam and turn it in. You may use a scientific calculator, but not a graphing one, and you may not consult the internet or other people. If you have a question, don't hesitate to ask — I just may not be able to answer it. **Enough work should be shown that there is no question about the mathematical process used to obtain your answers.**

1. (16 points) Multiple choice. You don't need to show your work.

a) (4 points) Suppose $a_n \geq 0$ for all n. Which of the following circumstances is **not** possible?

A) $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

B) $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges. This one — it would be a series that converges absolutely but not conditionally.

C) $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

D) $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges.

b) (4 points) Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$. What is its interval of convergence?

A) $(0, 3)$.

B) $(-1, 1)$.

C) $(-3, 3)$. This by the ratio test.

D) $(-\infty, \infty)$

c) (4 points) What are the first three terms of $\left(\sum_{n=0}^{\infty} n x^n\right)\left(\sum_{n=0}^{\infty} 2^n x^n\right)$?

A) $0 + x + 4x^2$.

B) $1 + x + 2x^2$.

C) $0 + 2x + 8x^2$. This: the product is $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 2^k k\right) x^n$

D) $1 + 4x + 4x^2$.

d) (4 points) The series $\sum_{n=1}^{\infty} \frac{1}{(-n)^n}$

A) converges absolutely. This is an alternating series that converges absolutely by the comparison test, since $\dfrac{1}{n^n} \le \dfrac{1}{n^2}$.

B) converges conditionally.

C) diverges.

2. (32 points) Short-answer. Explain your reasoning and/or show your work for each question.

a) (8 points) Does the series $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^{2/3}}$ converge or diverge?

The divergence test is inconclusive, so let's try something else. By the comparison test, $\frac{\ln(n)}{n^{2/3}} \geq \frac{1}{n^{2/3}} \geq \frac{1}{n}$, and since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so must this sum.

b) (8 points) Does the series $\sum_{k=0}^{\infty} \frac{2^{k+1}}{k!} k^2$ converge or diverge?

The factorials and exponentials indicate this is a great fit for the ratio test. We have

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{\frac{2^{k+2}}{(k+1)!}(k+1)^2}{\frac{2^{k+1}}{k!}k^2} \right| &= \lim_{k\to\infty} \left| \frac{2}{k+1} \cdot \frac{(k+1)^2}{k^2} \right| \\
&= \lim_{k\to\infty} \left| \frac{2(k+1)}{k^2} \right| \\
&= \lim_{k\to\infty} \left| \frac{2(k+1)}{k^2} \right| \\
&= 0.
\end{aligned}$$

Therefore, this series converges.

c) (8 points) Does $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ converge or diverge?

We've seen this one in a number of places before. The divergence test is inconclusive, the ratio and root tests are too, and it's not alternating. However, we can integrate it easily!

$$\begin{aligned}
\int_2^\infty \frac{1}{x\ln(x)}\,dx &= \int_2^\infty \frac{1}{u}\,du \quad \left(u = \ln(x),\ du = \frac{1}{x}\,dx\right) \\
&= [\ln(u)]\big|_2^\infty \\
&= [\ln(\ln(x))]\big|_2^\infty \\
&= \lim_{b\to\infty} (\ln(\ln(b)) - \ln(\ln(2))) \\
&= \infty.
\end{aligned}$$

Therefore, the series diverges.

d) (8 points) Let $f(x) = \sum_{n=0}^{\infty} x^n$ and $g(x) = \sum_{n=2}^{\infty} \left(x - \frac{1}{2}\right)^n$. What is the interval of convergence of $f + g$?

The interval of convergence of f is $(-1, 1)$ and the interval for g is $\left(-\frac{1}{2}, \frac{3}{2}\right)$ since they're both geometric series (the $n = 2$ in g doesn't change its convergence). The interval of convergence of the sum is the overlap of the two intervals, so $\left(-\frac{1}{2}, 1\right)$.

3. (32 points) Consider the series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$.

a) (8 points) Does $f(-1)$ converge absolutely, converge conditionally, or diverge?

$f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is the alternating Harmonic series, so conditionally (since the absolute value of this is the regular harmonic series, which diverges.)

b) (12 points) Estimate $f\left(-\frac{1}{2}\right)$ to within .02 of its actual value.

$f\left(-\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n}$, which is alternating. To estimate it, we know the Nth remainder R_N satisfies

$$R_N \leq \frac{1}{2^{N+1}(N+1)}.$$

To find when that is less than 0.02, we can try some values of N. The first one that works is $N = 3$, so our approximation is

$$\sum_{n=1}^{3} \frac{(-1)^n}{2^n n} = -\frac{1}{2} + \frac{1}{8} - \frac{1}{24} = -\frac{5}{12}.$$

c) (12 points) Using your answer to part a), determine the interval and radius of convergence of f.

Let's apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|x\frac{n}{n+1}\right|$$
$$= |x|$$

The series therefore converges on $(-1,1)$, and in fact on $[-1,1)$ by the answer to part a). Note: we could also get straight to this by noting that the series is centered at $x = 0$ and converges *conditionally* at $x = -1$, meaning that must be the boundary of the interval of convergence. That means the interval is at least $[-1,1)$, and we can rule out $x = 1$ since it's the harmonic series.

d) (2 points extra credit) Using your answer to part b), what do suspect the exact value of $f\left(-\frac{1}{2}\right)$ is? Hint: it involves ln.

This is $\ln(1.5) \approx 0.4055$! You might recall that this sum is equal to $\ln(1-x)$ from work we've done in class.