Final : 10:15 AM Monday, March 15^{th}

Expect ~6 pages, otherwise same parameters as midterm

End of Term survey: now open on Duckweb
Responses go to me and a few math dept members
part of a teaching record.

If 50% of the class responds, everyone 2% EC on the final

4.3.8 B $\vec{v} = 4\vec{i} - 10\vec{j}$

$\vec{w} = 6\vec{j} - 20\vec{i}$

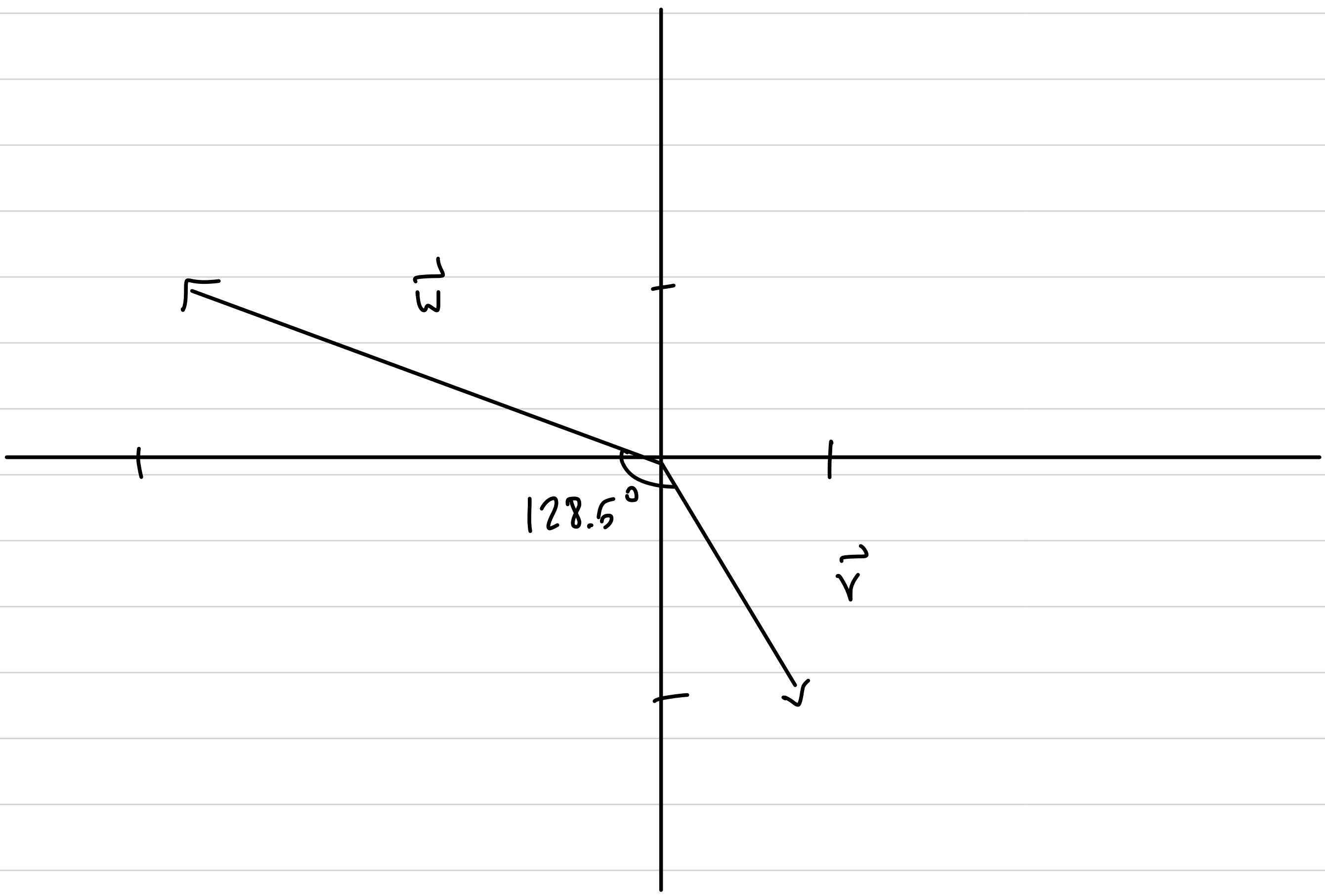
$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}\right)$$

$$\vec{v} \cdot \vec{w} = 4(-20) + (-10)(6) = -80 - 60 = -140$$

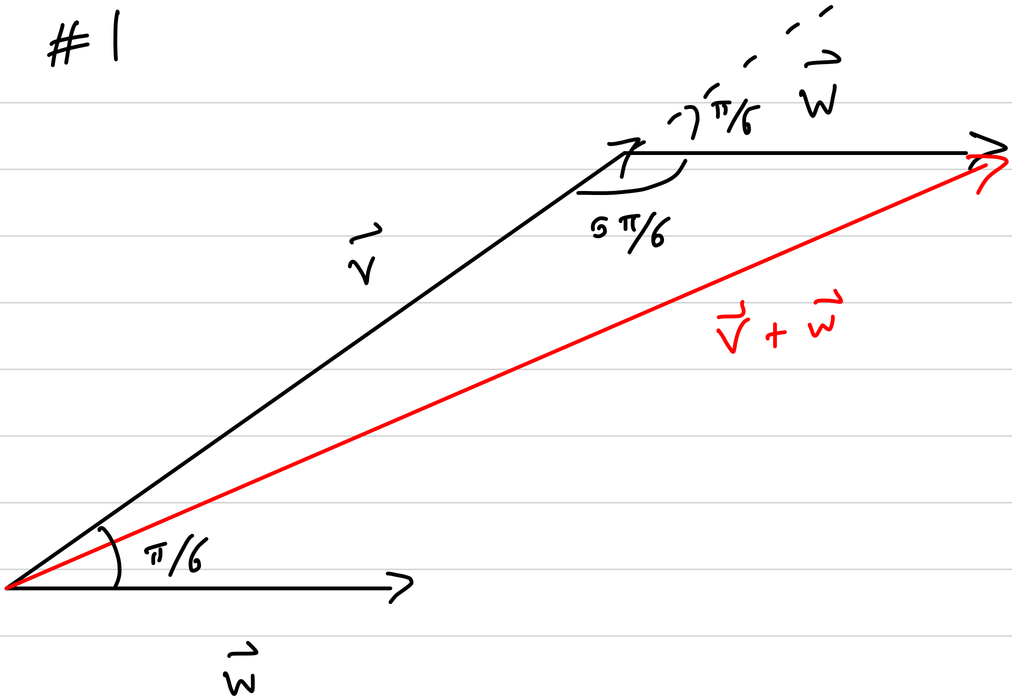
$$\|\vec{v}\| = \sqrt{4^2 + (-10)^2} = \sqrt{116}$$

$$\|\vec{w}\| = \sqrt{(-20)^2 + 6^2} = \sqrt{436}$$

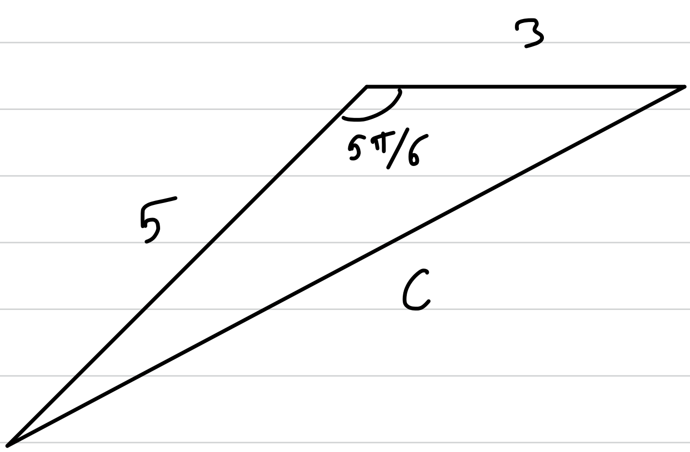
$$\theta = \arccos\left(\frac{-140}{\sqrt{116}\,\sqrt{436}}\right) = 128.5^\circ.$$



Quiz 7, #1



#2 $\quad \|\vec{v} + \vec{w}\| \neq 3+5 = 8$



By LoC, $\quad c^2 = 5^2 + 3^2 - 2 \cdot 3 \cdot 5 \cdot \overbrace{\cos(5\pi/6)}^{-\sqrt{3}/2}$

$$c^2 = 34 - 30\left(-\sqrt{3}/2\right)$$

$$= 34 + 15\sqrt{3}$$

$$c = \sqrt{34 + 15\sqrt{3}}$$

$$c = \|\vec{v} + \vec{w}\|$$

$\vec{v} = \vec{i} - \vec{j}$

$\vec{w}$: $\|\vec{w}\| = 2$, angle $\pi/4$ from horizontal

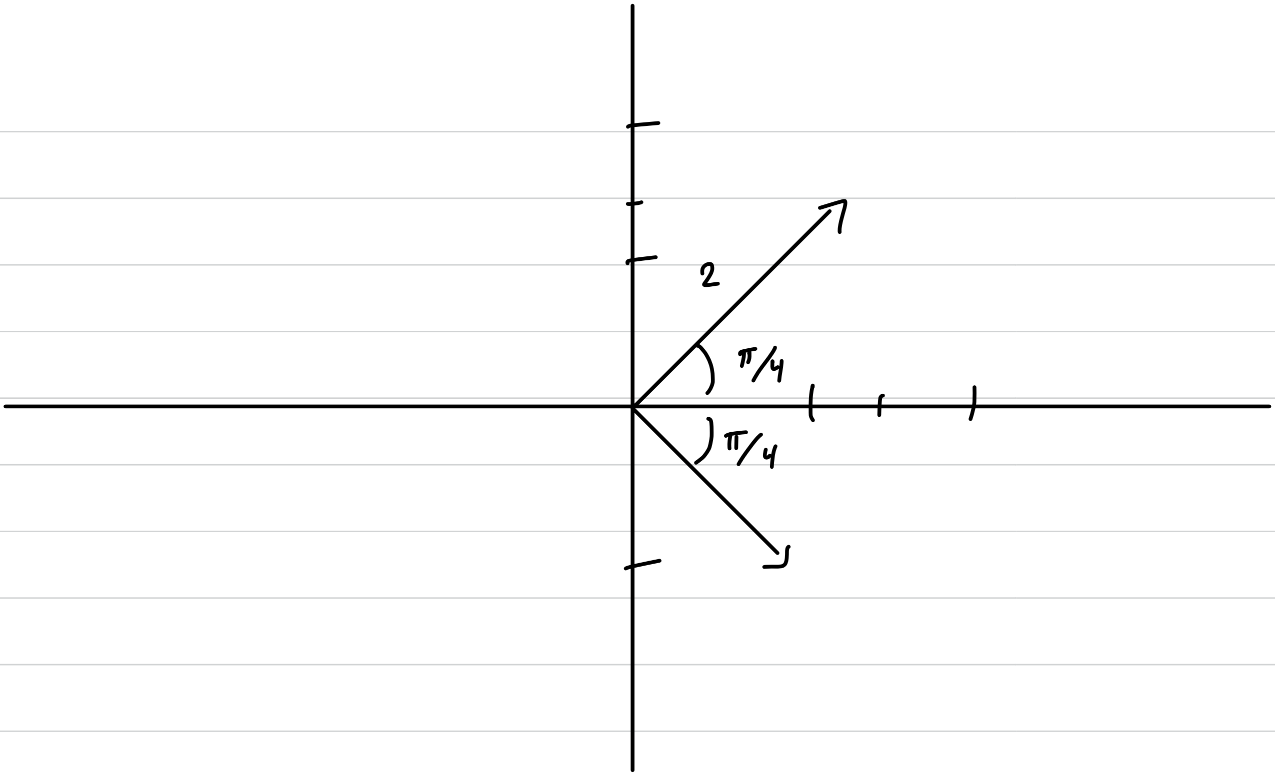
$$\vec{w} = 2\cos(\pi/4)\,\vec{i} + 2\sin(\pi/4)\,\vec{j}$$

$$= 2\left(\frac{\sqrt{2}}{2}\right)\vec{i} + 2\left(\frac{\sqrt{2}}{2}\right)\vec{j}$$

$$= \sqrt{2}\,\vec{i} + \sqrt{2}\,\vec{j}$$

$$\vec{v} \bullet \vec{w} = (1)(\sqrt{2}) + (-1)(\sqrt{2}) = \sqrt{2} - \sqrt{2} = 0$$

this means $\vec{v}$ and $\vec{w}$ are orthogonal

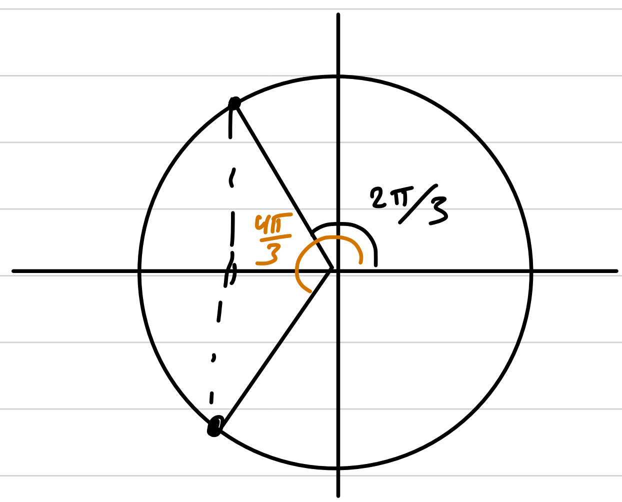


Find all solutions to $2\cos(3\pi x - 1) = -1$.

$$\cos(3\pi x - 1) = -\frac{1}{2}$$

① $\arccos\left(-\frac{1}{2}\right) = 2\pi/3$

② other angle $= 4\pi/3$



$$3\pi x - 1 = 2\pi/3 + 2\pi n$$

or

$$3\pi x - 1 = 4\pi/3 + 2\pi n$$

$$3\pi x = 2\pi/3 + 2\pi n + 1$$

or

$$3\pi x = 4\pi/3 + 2\pi n + 1$$

$$\boxed{\begin{gathered} x = \frac{2}{9} + \frac{2}{3}n + \frac{1}{3\pi} \\ \text{or} \\ x = \frac{4}{9} + \frac{2}{3}n + \frac{1}{3\pi} \end{gathered}}$$

If you're asked to find all solutions, you need to go this process. If you're only being asked to find one, the arc function of the other side is enough.

Final Topics

Chapter 1

- Parent functions and their graphs
  - Lines
  - x^2, x^3, x^p
  - $1/x$, $1/x^2$, $1/x^p$
  - $\sqrt{x}$, $\sqrt[3]{x}$, $x^{1/p}$
  - e^x, $\ln(x)$
  - trig functions

- Even and odd functions
  - Definition
  - Geometric interpretation (symmetry)

- Transformations
  - Vertical and horizontal stretches, reflections, and shifts
  - The order to apply them when there's

more than one

- Periodic functions
  - Definition
  - Period, amplitude, and midline
  - Graphing them given a small section

Chapter 2

- Basic geometry
  - Finding angles via complementary, supplementary, etc.
  - Reference angles
  - Area of a triangle
  - Pythagorean theorem
  - Opposite, adjacent, hypotenuse
  - The unit circle

- The three standard trig functions
  - Definition of sin, cos, tan
  - Special angles
  - $\sin(\theta)^2 + \cos(\theta)^2 = 1$
  - Reference angles with trig functions
  - Trig functions in right triangles
  - Graphs
  - Transformations of sin and cos interpreted as coordinates on a non-unit circle
  - The arc functions

Chapter 3

- Radians
  - Definition
  - Arc length

  $\left.\begin{array}{l}\text{Definition}\\ \text{Arc length}\end{array}\right\}$ basically the same
  - Trig functions of angles in radians

- Non-right triangles
  - Law of Cosines
  - Law of Sines

- Trig equations
  - Finding one solution with an arc function
  - Finding all the others

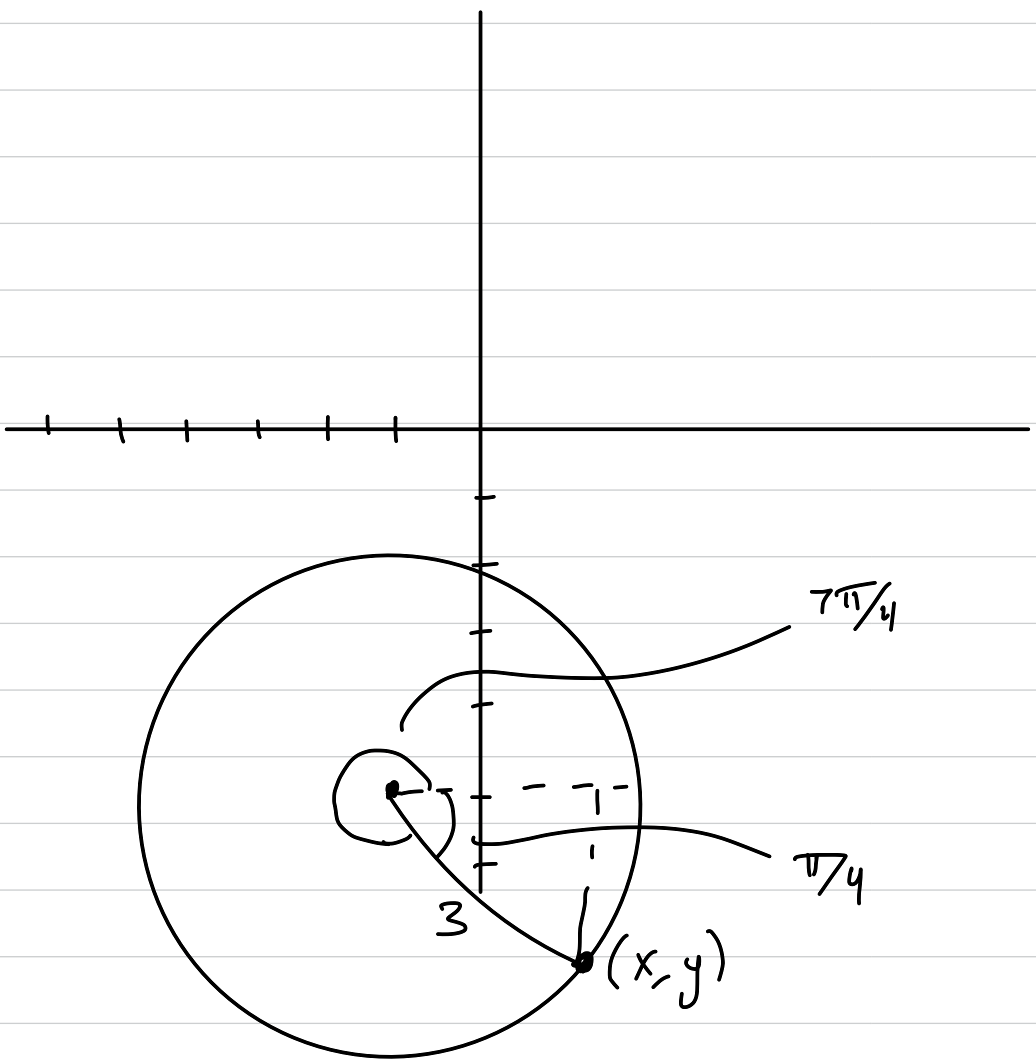
- Sinusoidal functions
  - Finding amplitude, midline, and period
  - Finding horizontal shift via a trig equation

- When and how to use the double-angle, half-angle, and sum and difference formulas (but not exactly what they all are)

Chapter 4

- Vectors as quantities that measure a change in position
- Vector arithmetic
- Magnitude and direction
- Unit vector decompositions
- Changing between a unit vector decomposition and a magnitude-angle description
- The dot product
  - Unit vector definition
  - Magnitude-angle definition
  - Finding angle between vectors
  - Orthogonal vectors

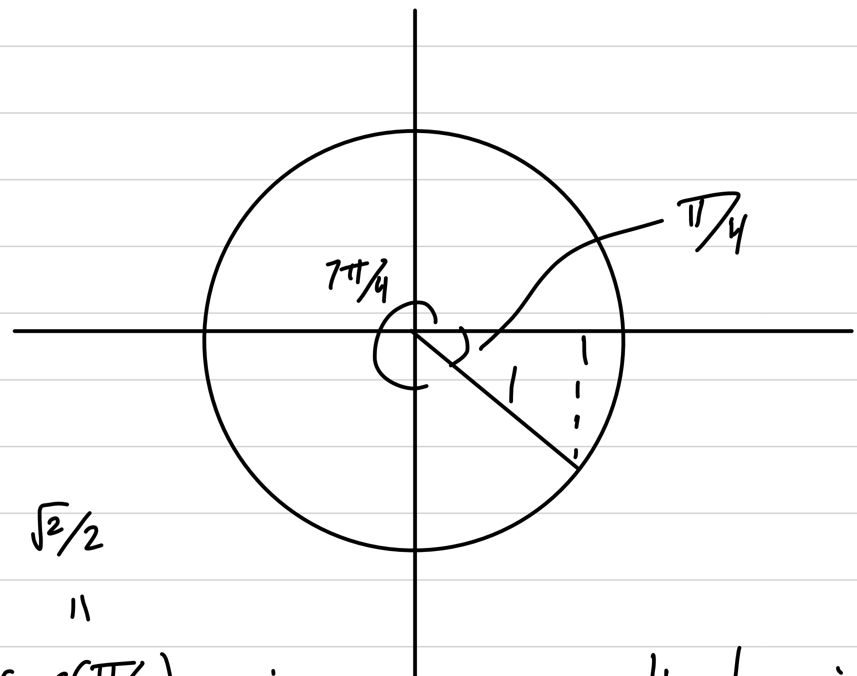
Find the coordinates of a point on a circle with radius 3 and center $(-1, -5)$, where the point has angle $7\pi/4$ from the horizontal.



$$x = 3\cos(\theta) + (-1) = 3\cos\left(\frac{7\pi}{4}\right) - 1 = 3\left(\frac{\sqrt{2}}{2}\right) - 1$$

$$y = 3\sin(\theta) + (-5) = 3\sin\left(\frac{7\pi}{4}\right) - 5 = 3\left(-\frac{\sqrt{2}}{2}\right) - 5$$

Now we need to find $\cos(7\pi/4)$ and $\sin(7\pi/4)$: so we use the unit circle



$$\underbrace{\cos(7\pi/4) = \cos(\pi/4)}_{\text{}}$$

$\sqrt{2}/2 = \cos(7\pi/4) = \cos(\pi/4)$ since x-coordinate is positive

$\sin(7\pi/4) = -\sin(\pi/4) = -\sqrt{2}/2$ since y-coordinate is negative

$$x = \frac{3\sqrt{2}}{2} - 1$$

$$y = -\frac{3\sqrt{2}}{2} - 5$$

Use the trig identity formulas when you're trying to find the exact value of a non-special angle

Ex: $\tan(15^\circ) = \tan\left({}^{30^\circ}\!/_{2}\right) \overset{\text{half-angle}}{=} \dfrac{\sin(30^\circ)}{1+\cos(30^\circ)} = \dfrac{1/2}{1+\sqrt{3}/2}$

$$\tan(15^\circ) = \tan(45^\circ - 30^\circ) \underset{\substack{\uparrow \\ \text{difference} \\ \text{formula}}}{=} \frac{\tan(45^\circ)-\tan(30^\circ)}{1+\tan(45^\circ)\tan(30^\circ)} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$$

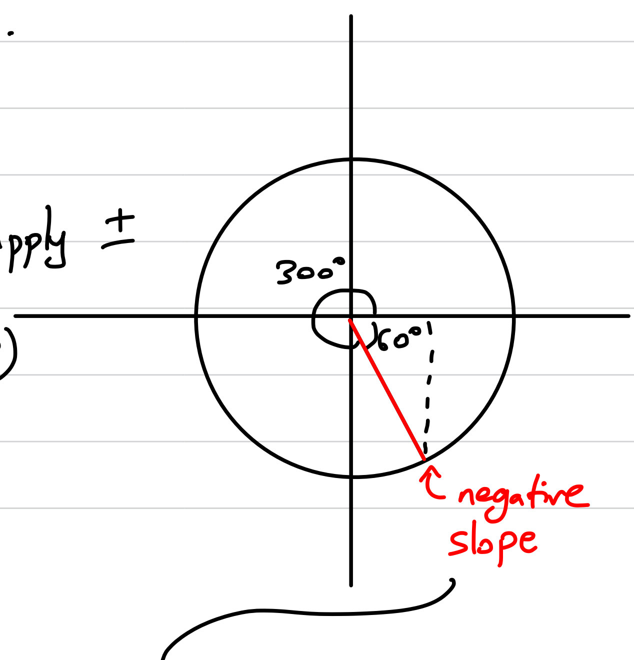
Quiz 4

Exact value of $\tan(300^\circ)$.

Either: know $\tan(60^\circ)$ and how to apply $\pm$

or

know $\sin(300^\circ)$ and $\cos(300^\circ)$



$\tan(60°) = \sqrt{3}$

$\tan(300°) = -\sqrt{3}$

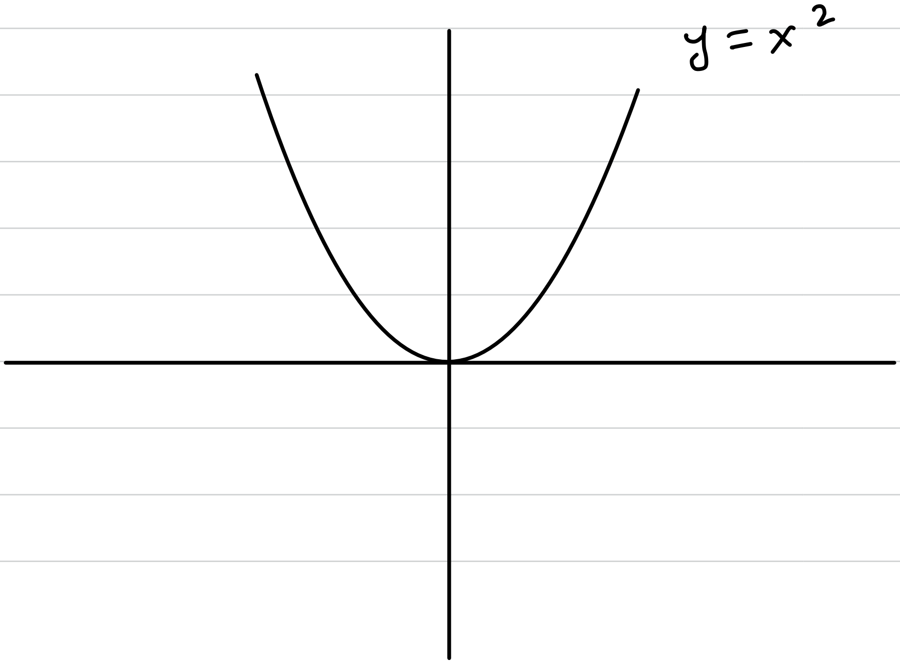
Or:

$$\tan(300°) = \frac{\overset{-\sin(60°)}{\overset{\parallel}{\sin(300°)}}}{\underset{\cos(60°)}{\underset{\parallel}{\cos(300°)}}} = \frac{-\sqrt{3}/2}{1/2} = -\frac{\sqrt{3}}{2} \cdot \frac{2}{1} = -\sqrt{3}$$

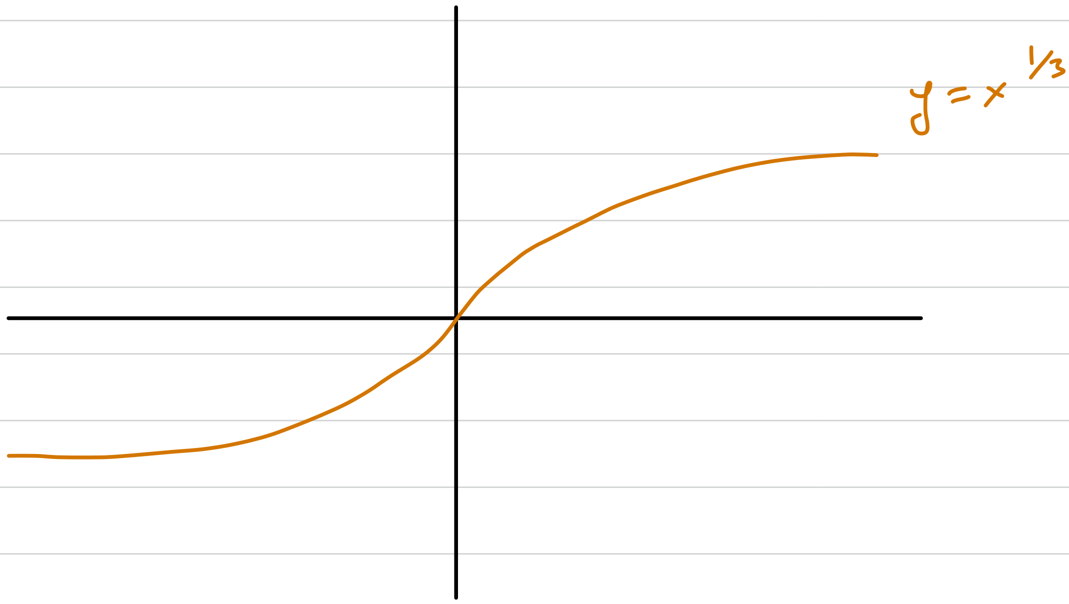
$y = x^{1/3}$ one-to-one ?

This means every y-value comes from only one x-value.

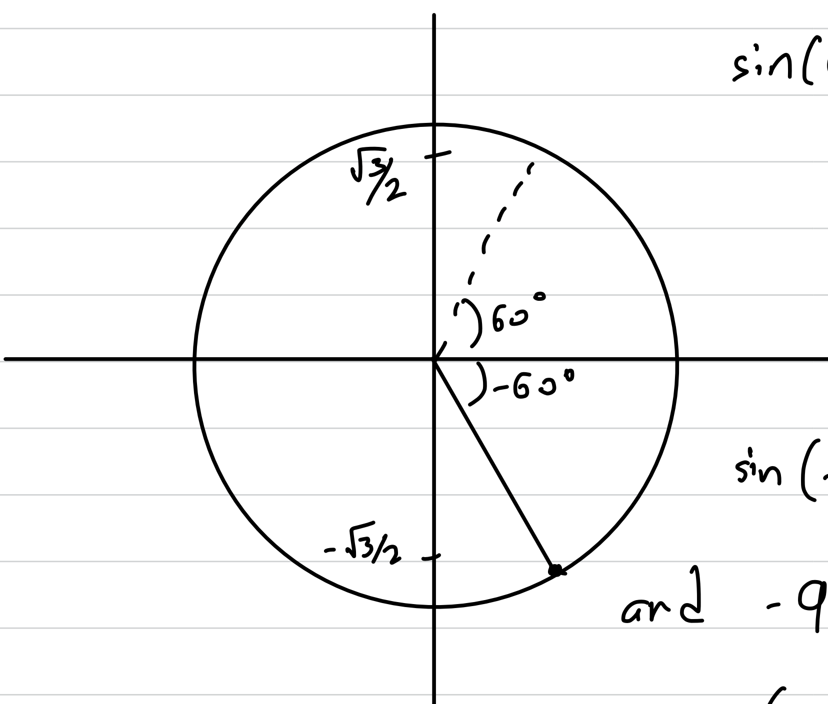
$y = x^2$



$y = x^{1/3}$ is one-to-one because it passes the HLT.



$\arcsin\left(-\frac{\sqrt{3}}{2}\right)$: this is the angle whose sin is $-\frac{\sqrt{3}}{2}$



$\sin(60°) = \sqrt{3}/2$

$\sin(-60°) = -\sqrt{3}/2$

and $-90° \le -60° \le 90°$, so

$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -60°$.

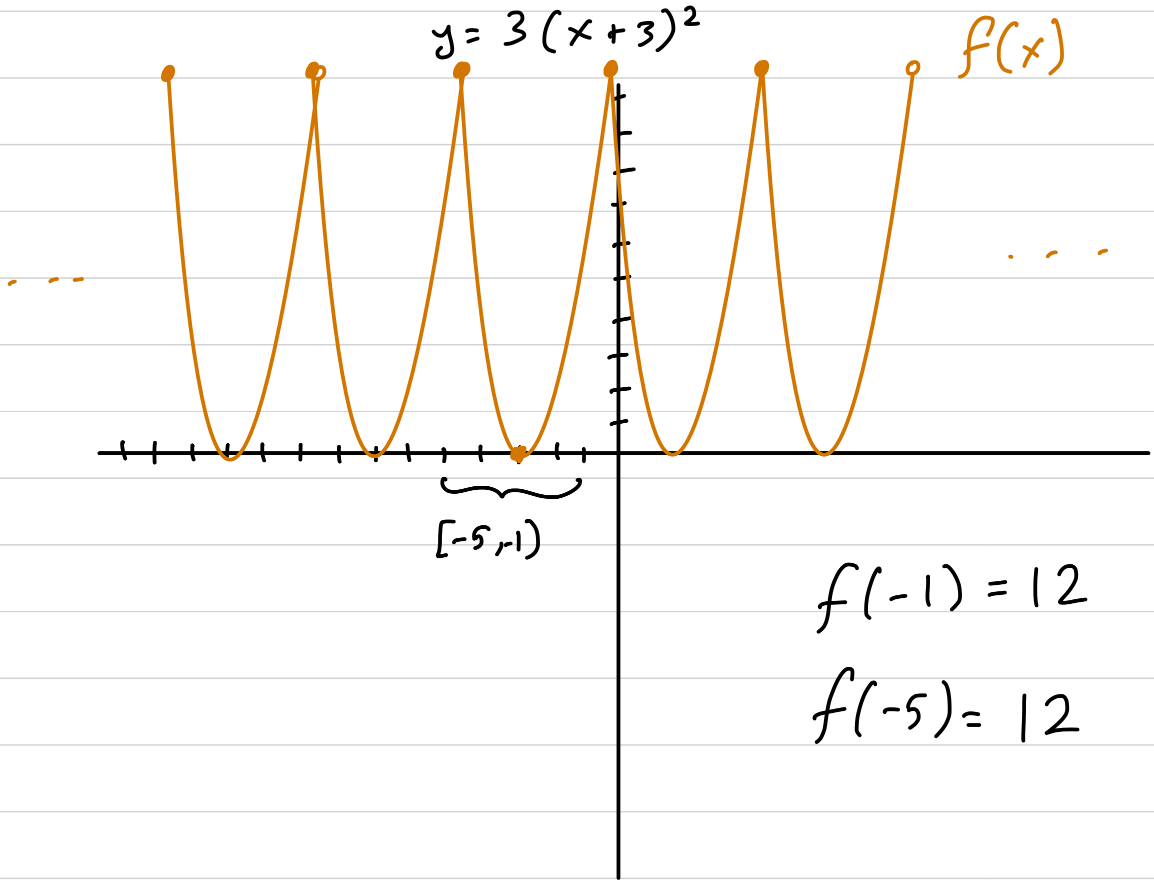
Let f be a periodic function with period 4 such that on the interval $[-5,-1)$, $f(x)=3(x+3)^2$.

Graph f

Parent function: x^2

Horizontal shift 3 left

Vertical stretch by a factor of 3.



$f(-1)=12$

$f(-5)=12$

$\vec{u} = 3\vec{i} - 2\vec{j}$

$\vec{v} = (t-1)\vec{i} + (t+1)\vec{j}$

angle between $\vec{u}$ and $\vec{v}$ is $2\pi/3$ } dot product

$\|\vec{v}\| \geq 4$

$\vec{u} \bullet \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(2\pi/3)$

$\|\vec{u}\| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\vec{u} \bullet \vec{v} = (\sqrt{13}) \|\vec{v}\| (-1/2)$

$\vec{u} \bullet \vec{v} = 3(t-1) - 2(t+1) = 3t - 3 - 2t - 2 = t - 5$

$t - 5 = (\sqrt{13}) \|\vec{v}\| (-1/2)$

$\|\vec{v}\| = \sqrt{(t-1)^2 + (t+1)^2} = \sqrt{t^2 - 2t + 1 + t^2 + 2t + 1}$

$= \sqrt{2t^2 + 2}$

$t - 5 = (\sqrt{13})(\sqrt{2t^2+2})(-1/2)$

$$\sqrt{2t^2+2} = \frac{-2t+10}{\sqrt{13}}$$

$$2t^2+2 = \left(\frac{-2t+10}{\sqrt{13}}\right)^2 = \frac{(-2t+10)^2}{13} = \frac{4t^2-40t+100}{13}$$

$$26t^2+26 = 4t^2-40t+100$$

$$22t^2+40t-74=0$$

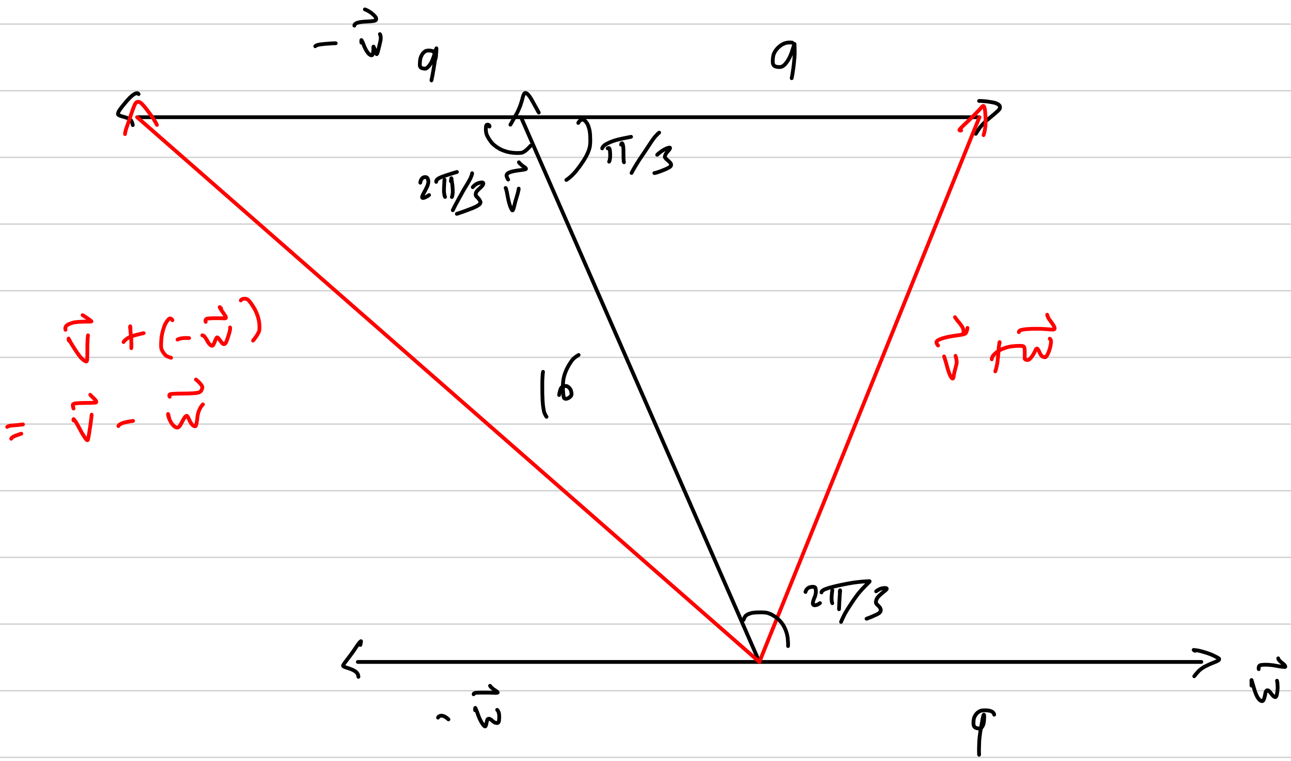
$$t = \frac{-40 \pm \sqrt{40^2-4(22)(-74)}}{2(22)} = -2.956 \text{ or } 1.138$$

$$t = \boxed{-2.956}: \quad \|\vec{v}\| = \sqrt{2t^2+2} = \sqrt{2(-2.956)^2+2} = 4.413 \ \checkmark$$

$$t = 1.138: \quad \cancel{\|\vec{v}\| = 2.142}$$

$\|\vec{v}\| = 16$

$\|\vec{w}\| = 9$



Let $c = \|\vec{v} - \vec{w}\|$

By LoC, $c^2 = 9^2 + 16^2 - 2 \cdot 9 \cdot 16 \cdot \cos(2\pi/3)$

$c^2 = 81 + 256 + 144$

$c = 21.93$

Find a sinusoidal function $f(x)$ such that:

① f has amplitude π, midline $-\pi$, period 3

② the graph of f passes through the point $(1, -2)$ and it is decreasing there.

$$f(x) = A \sin(B(x-h)) + k$$

$A = \pi$

$k = -\pi$

$2\pi/B = 3 \quad \Rightarrow \quad B = 2\pi/3$

$$f(1) = -2$$

$$-2 = \pi \sin\left(\frac{2\pi}{3}(1-h)\right) - \pi$$

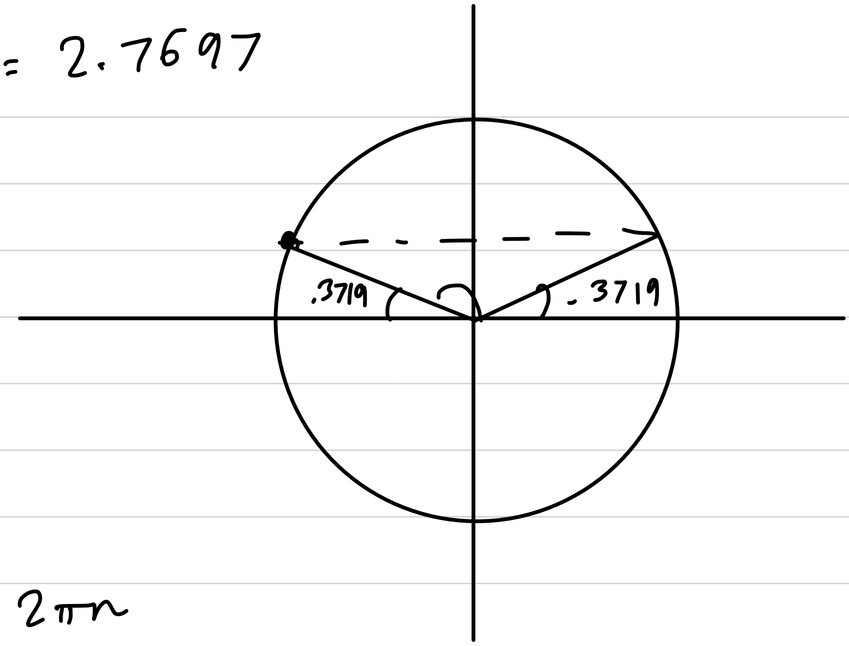
$$\frac{\pi - 2}{\pi} = \sin\left(\frac{2\pi}{3}(1-h)\right)$$

$$\arcsin\left(\frac{\pi-2}{\pi}\right)$$

(want to write $= \frac{2\pi}{3}(1-h)$, but that misses solutions)

$$\| \\ .3719$$

other angle $= \pi - .3719 = 2.7697$



$$2\pi/3\,(1-h) = .3719 + 2\pi n$$

or

$$2\pi/3\,(1-h) = 2.7697 + 2\pi n$$

$$(1-h) = (.3719 + 2\pi n)\left(\frac{3}{2\pi}\right) = .1776 + 3n$$

or

$$(1-h) = (2.7697 + 2\pi n)\left(\frac{3}{2\pi}\right) = 1.3224 + 3n$$

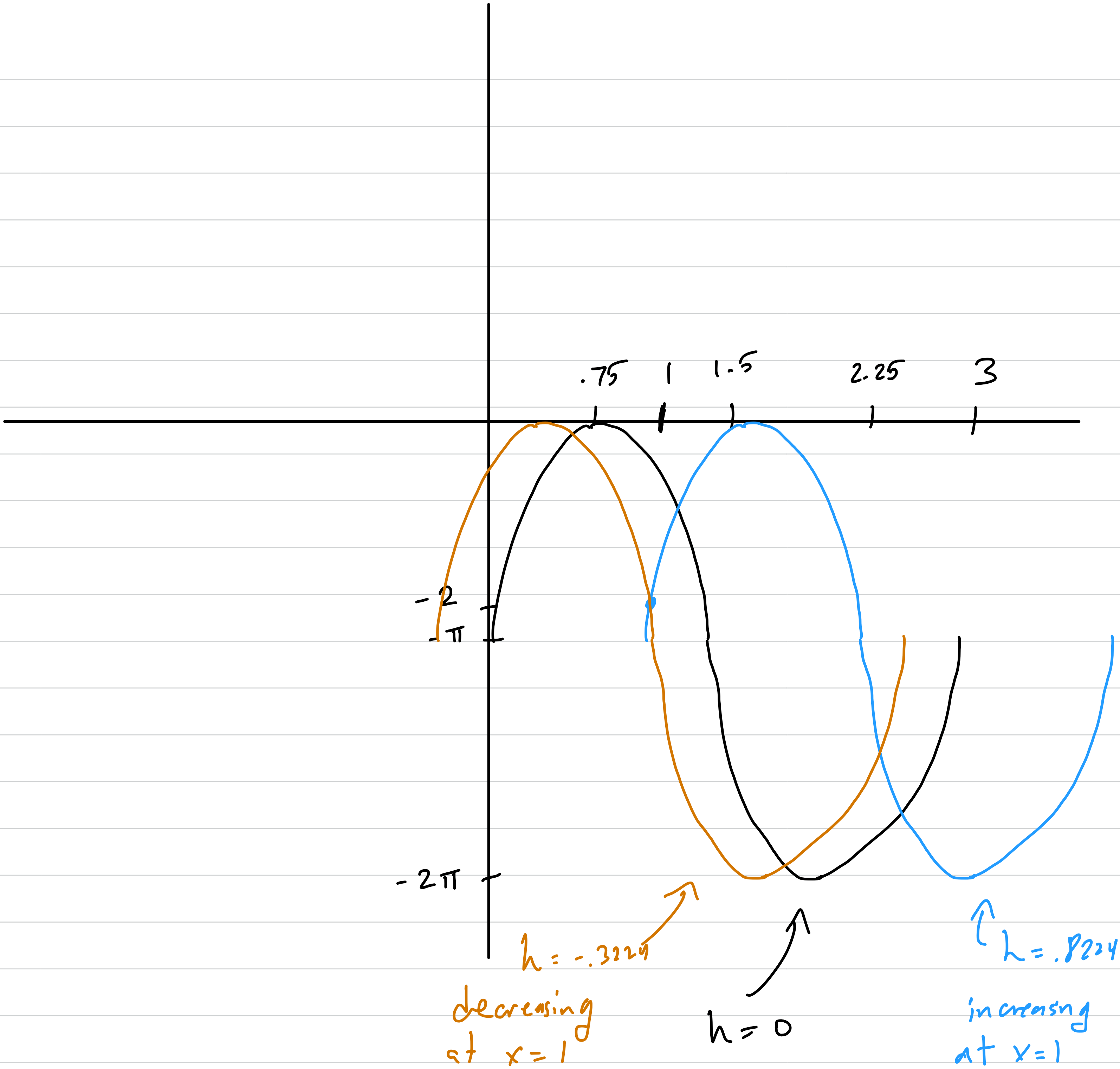
$$h = .8224 - 3n$$

or

$$h = -.3224 - 3n$$

Try $h = .8224$. $f(x) = \pi \sin\left(\frac{2\pi}{3}(x - .8224)\right) - \pi$

Is this decreasing at $(1,-2)$? We need to graph it.



h = -.3224 is what we want.

$$f(x) = \pi \sin\left(\frac{2\pi}{3}(x + .3224)\right) - \pi.$$

$\vec{v} = 2\vec{i} - 3.5\vec{j}$

$\vec{w}$ magnitude 3 and angle $2\pi/5$ clockwise from the horizontal

Find the angle between them

Need $\vec{v} \cdot \vec{w}$. Two formulas:

1. $\vec{i}$ and $\vec{j}$
2. magnitude and angle between ← trying to find this

$\vec{w} = 3\cos(-2\pi/5)\,\vec{i} + 3\sin(-2\pi/5)\,\vec{j}$

$.9271\,\vec{i} - 2.8532\,\vec{j}$

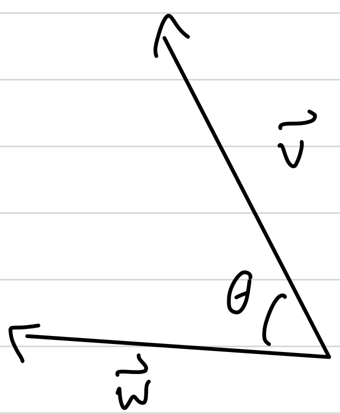
$\vec{v} \cdot \vec{w} = 2(.9271) + (-3.5)(-2.8532) = 11.8404.$

$11.8404 = \|\vec{v}\|\,\|\vec{w}\|\cos(\theta) = \left(\sqrt{2^2 + (-3.5)^2}\right)(3)\cos(\theta)$

$\cos(\theta) = .5765$

$\theta = \arccos(.5765) = .9551$

this is okay because arccos outputs angles between 0 and π, and two vectors have the angle between them always in that range.



$0 \le \theta \le \pi$

Reminder: fill out course eval!