Prop: Let $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$ and $\vec{w} = w_1 \vec{i} + w_2 \vec{j}$.

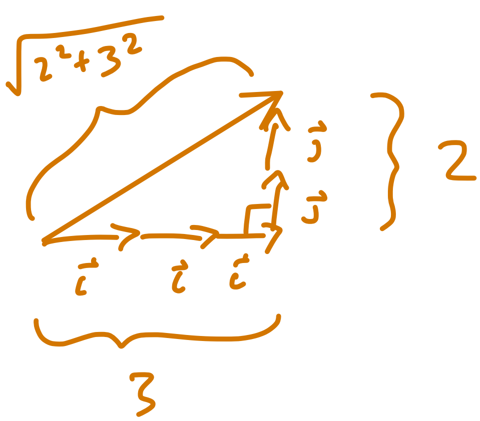
① $\vec{v} + \vec{w} = (v_1 + w_1)\vec{i} + (v_2 + w_2)\vec{j}$.

② $\vec{v} - \vec{w} = (v_1 - w_1)\vec{i} + (v_2 - w_2)\vec{j}$.

③ $c\vec{v} = (cv_1)\vec{i} + (cv_2)\vec{j}$

④ $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$.

Pythagorean Theorem



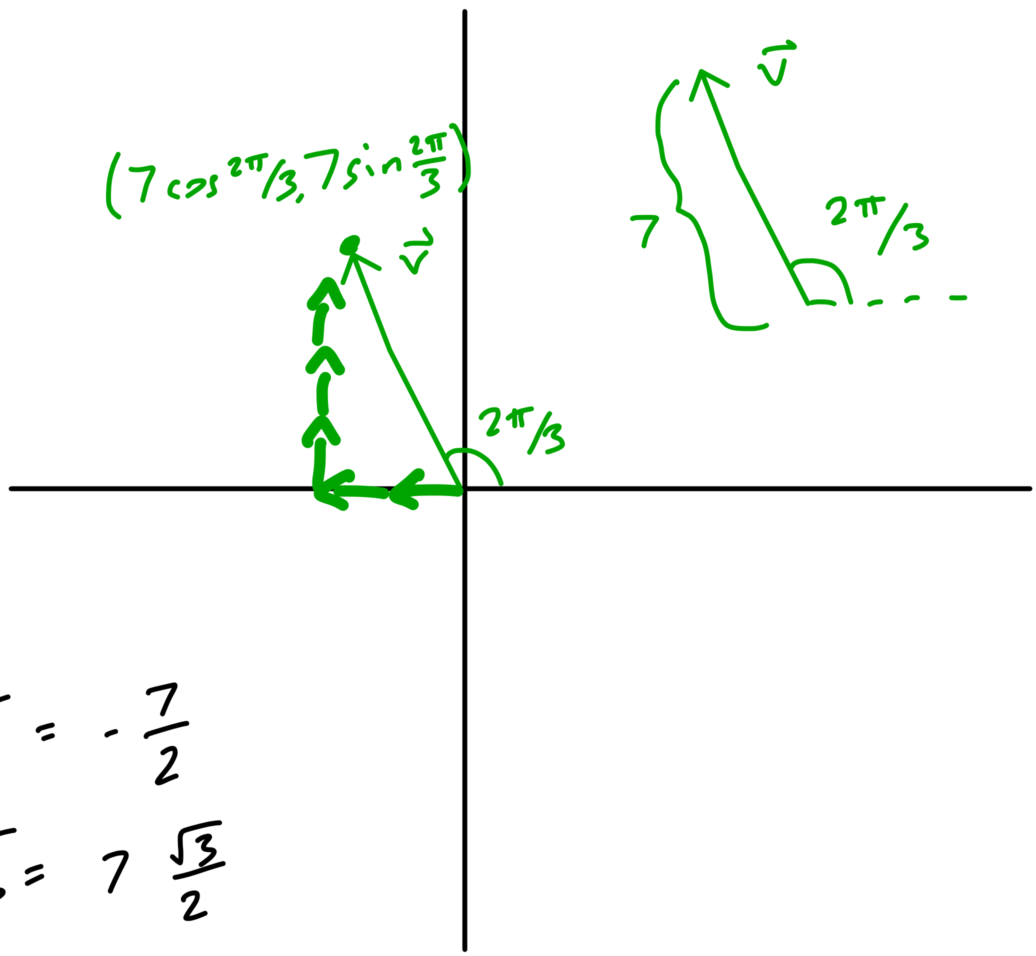
Ex: $\|3\vec{i} + 2\vec{j}\| = \sqrt{3^2 + 2^2} = \sqrt{13}$.

$(3\vec{i} + 2\vec{j}) + (\vec{i} - \vec{j}) = 4\vec{i} + \vec{j}$.

Theorem: Let $\vec{v}$ be a vector with angle θ from the positive x-axis when it's placed at the origin. Then

$$\vec{v} = (\|\vec{v}\| \cos\theta)\,\vec{i} + (\|\vec{v}\| \sin\theta)\,\vec{j}.$$

Ex:



$$7\cos\frac{2\pi}{3} = -\frac{7}{2}$$

$$7\sin\frac{2\pi}{3} = 7\,\frac{\sqrt{3}}{2}$$

So $\vec{v} = -\frac{7}{2}\vec{i} + \frac{7\sqrt{3}}{2}\vec{j}$.

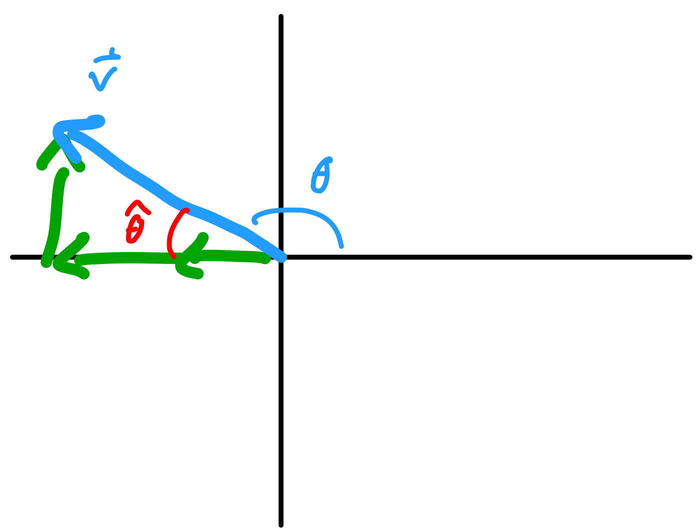
Comment:

$$\text{Magnitude} \xleftarrow{\text{Pythagorean theorem}} \vec{i} \text{ and } \vec{j}$$

$$\left.\begin{array}{c}\text{Magnitude} \\ \text{and} \\ \text{Direction}\end{array}\right\} \xrightarrow{\text{sin and cos}} \vec{i} \text{ and } \vec{j}$$

$$\text{Direction} \xleftarrow{\text{arc tan}} \vec{i} \text{ and } \vec{j}$$

Ex: Find the angle that $-2\vec{i}+\vec{j}$ makes with the positive x-axis.



$\tan\hat{\theta} = \frac{1}{2}$

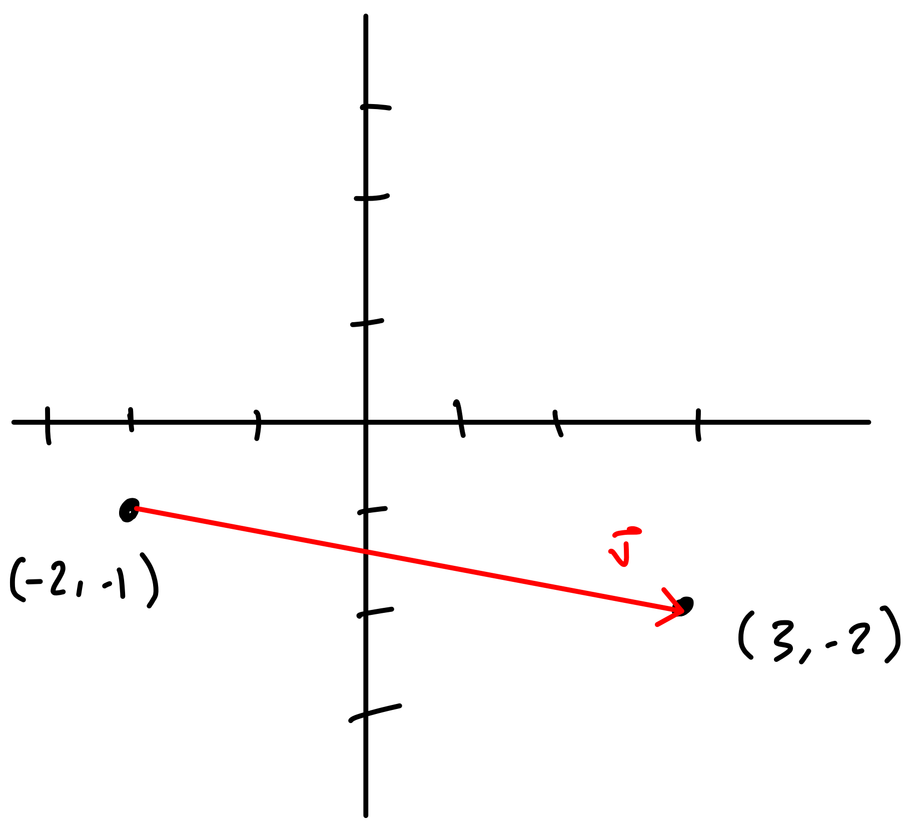
$\hat{\theta} = \arctan(1/2)$ since $\hat{\theta}$ is in $[-\pi/2, \pi/2]$

$\hat{\theta} = .464$, so $\theta = \pi - .464 = 2.678$.

Prop: Let (x_1, y_1) and (x_2, y_2) be two points in the plane. The vector that starts at (x_1, y_1) and ends at (x_2, y_2) is

$(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$.

Ex:



$$\vec{v} = (3 - (-2))\vec{i} + (-2 - (-1))\vec{j}$$

$$= 5\vec{i} - \vec{j}.$$

The Dot Product

Def: Let $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$ and $\vec{w} = w_1 \vec{i} + w_2 \vec{j}$.

The dot product of $\vec{v}$ and $\vec{w}$ is

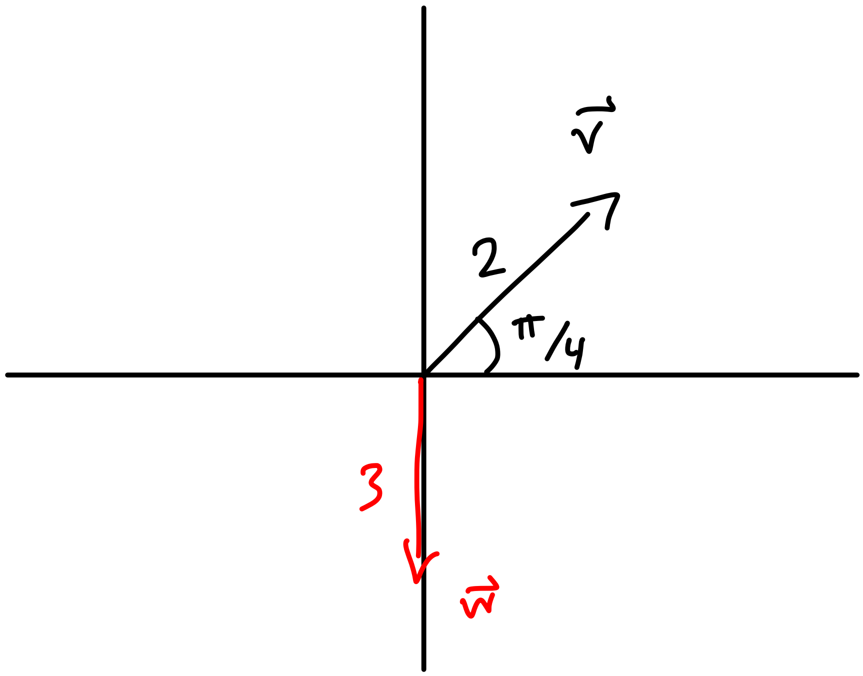
$$\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2.$$

Ex: $(2\vec{i} + \vec{j}) \bullet (3\vec{i} - 4\vec{j}) = 2 \cdot 3 + 1(-4)$

$= 6 - 4 = 2.$

$(3\vec{i} + \vec{j}) \bullet \vec{0} = (3\vec{i} + \vec{j}) \bullet (0\vec{i} + 0\vec{j})$

$= 3 \cdot 0 + 1 \cdot 0 = 0.$



$$\vec{v} \bullet \vec{w} = \left(\left(2 \cos \pi/4\right) \hat{i} + \left(2 \sin \pi/4 \ \vec{j}\right) \right) \bullet \left(-3\vec{j}\right)$$

$$= \left(2 \cos \pi/4\right)(0) + \left(2 \sin \pi/4\right)(-3)$$

$$= -6 \sin \pi/4$$

$$= -6\left(\frac{\sqrt{2}}{2}\right) = -3\sqrt{2}.$$

Comment: ① This is a scalar. We will not have a way in this class to multiply two vectors and get another vector.

② This is completely different from scalar multiplication.

$$(\text{number}) \cdot (\overrightarrow{\text{vector}}) = \overrightarrow{\text{vector}}$$

$$(\overrightarrow{\text{vector}}) \bullet (\overrightarrow{\text{vector}}) = \text{number}$$

③ Make it clear that you're using the dot product. Don't write $\vec{v}\vec{w}$, $\vec{v}\cdot\vec{w}$, or $\vec{v}\times\vec{w}$ — all of these mean other things in math. Your book uses $\vec{v}\cdot\vec{w}$ for whatever reason.

<u>Prop</u>: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors and c a scalar.

① $\vec{0} \bullet \vec{v} = 0$

zero vector

zero

② $\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$

③ $(c\vec{v}) \bullet \vec{w} = \vec{v} \bullet (c\vec{w}) = c(\vec{v} \bullet \vec{w})$

④ $\vec{u} \bullet (\vec{v} + \vec{w}) = (\vec{u} \bullet \vec{v}) + (\vec{u} \bullet \vec{w})$.

⑤ $\vec{v} \bullet \vec{v} = \|\vec{v}\|^2$.

If $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$,

$$\vec{v} \bullet \vec{v} = v_1^2 + v_2^2 = \|\vec{v}\|^2.$$

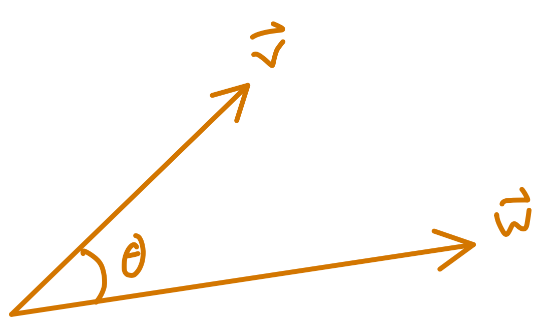
Comment: You might expect a property like $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$ — but this doesn't make sense! $\vec{v} \cdot \vec{w}$ is a number, so we cannot dot it with $\vec{u}$.

Theorem: Let $\vec{v}$ and $\vec{w}$ be vectors that form an angle of θ between them.

Then $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$.

<u>Comment</u>: Since $\cos 0 = 1$, $\cos \pi/2 = 0$, and $\cos \pi = -1$, this formula tells us that $\vec{v} \cdot \vec{w}$ measures the degree to which $\vec{v}$ and $\vec{w}$ are parallel.

<u>Prop</u>: Let $\vec{v}$ and $\vec{w}$ be vectors. If $\vec{v} \cdot \vec{w} > 0$, then the angle θ between $\vec{v}$ and $\vec{w}$ is acute. If $\vec{v} \cdot \vec{w} < 0$, θ is obteuse.



$\vec{v} \cdot \vec{w} > 0$

$\vec{u} \cdot \vec{x} < 0$

Theorem: If $\vec{v}$ and $\vec{w}$ are nonzero vectors and θ is the angle between them, then $\theta = \arccos\left(\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}\right)$.

Ex: Find the angle between $\vec{v} = 3\vec{i} + \vec{j}$ and $\vec{w} = 2\vec{i} - \vec{j}$.

$\|\vec{v}\| = \sqrt{3^2 + 1^2}$

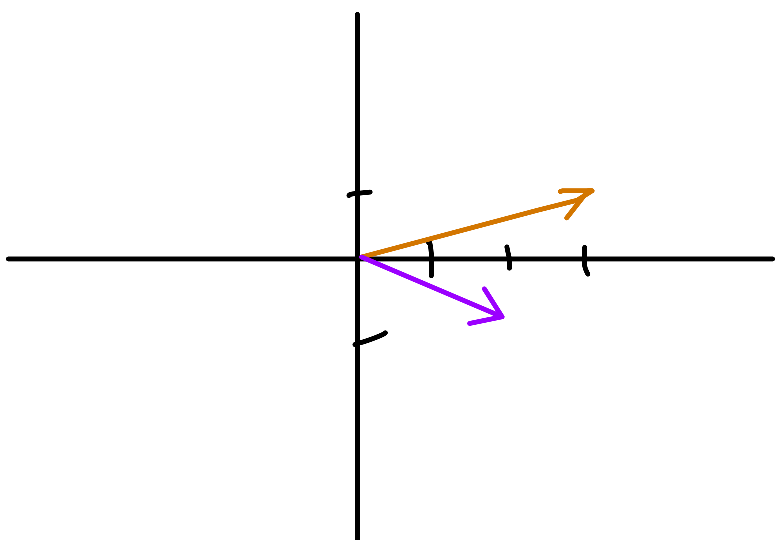
$\|\vec{w}\| = \sqrt{2^2 + (-1)^2}$

$\vec{v}\cdot\vec{w} = 3\cdot 2 + 1(-1) = 6 - 1 = 5.$

So $\theta = \arccos\left(\dfrac{5}{\|\vec{v}\|\,\|\vec{w}\|}\right) = \arccos\left(\dfrac{5}{\sqrt{10}\sqrt{5}}\right)$

$= \arccos\left(\dfrac{5}{\sqrt{50}}\right) = \arccos\left(\dfrac{5}{5\sqrt{2}}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right)$

$= \pi/4.$



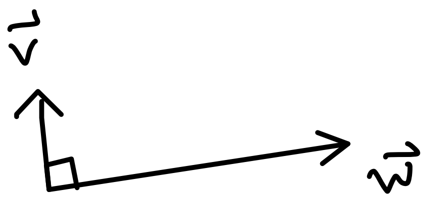
Comment: Note that we are not solving for the angles that these vectors make with the positive x-axis. We know how to do that with arctan. We're finding the angle they make with one another.

Def: Two vectors $\vec{v}$ and $\vec{w}$ are orthogonal (literally right-angled) if $\vec{v} \cdot \vec{w} = 0$.

Comment: Your book uses "perpendicular" — orthogonal vectors are perpendicular when neither of them is the zero vector.

When one or both is the zero vector, then they can be orthogonal without being perpendicular.

Ex:



$\vec{v} \cdot \vec{w} = 0$, since they're perpendicular, so $\vec{v}$ and $\vec{w}$ are orthogonal.

Ex: $2\vec{i} + 3\vec{j}$ and $-3\vec{i} + 2\vec{j}$ are orthogonal, since $(2\vec{i} + 3\vec{j}) \cdot (-3\vec{i} + 2\vec{j}) = 0$.

Ex: $\vec{0}$ is orthogonal to every vector, since $\vec{0} \cdot \vec{v} = 0$ for all $\vec{v}$.

$\underline{Ex}$: Find all vectors perpendicular to $-3\vec{i} + \vec{j}$.

If $\vec{w} = w_1\vec{i} + w_2\vec{j}$, then $(-3\vec{i} + \vec{j}) \bullet \vec{w} = 0$ gives $-3w_1 + w_2 = 0$. Then $3w_1 = w_2$.

Let $w_1 = t$, so that $w_2 = 3t$. Then $\vec{w} = t\vec{i} + 3t\vec{j}$ for any real number t, except for $t = 0$. When $t = 0$, $\vec{w} = \vec{0}$, so $\vec{w}$ and $-3\vec{i} + \vec{j}$ are <u>orthogonal</u>, but not <u>perpendicular</u>.