u-sub: use when you are integrating a composite function: one function *of* another function.

outer function: $\sin(x)$

Ex: $\sin(x^2)$

inner function: x^2

$u = x^2$
$du = (x^2)' dx$
$du = 2x\,dx$

outer: t^2

Ex: $(t^2+2t+3)^2$

inner: t^2+2t+3

$u = t^2+2t+3$
$du = (2t+2)\,dt$

outer: $\frac{1}{y}$

Ex: $\frac{1}{\cos(y)}$

inner: $\cos(y)$

$u = \cos(y)$
$du = -\sin(y)\,dy$

Once you've identified the outer and inner functions, let u = inner function.

Then $du = u'\,dx$

Then any x must be removed from the integral. If there are x that you can't turn into either u or du, then this u doesn't work.

Ex:

$$\int x \sin(x^2)\, dx \qquad u = x^2 \qquad du = 2x\, dx$$

$$\frac{1}{2} du = x\, dx$$

$$= \int \sin(u)\, \frac{1}{2} du$$

Ex:

$$\int \underbrace{(6t+6)}_{?} \underbrace{(t^2+2t+3)}_{u^2}{}^2\, dt \qquad u = t^2+2t+3$$

$$du = (2t+2)\,dt$$

$$3\,du = (6t+6)\,dt$$

$$= \int u^2 \cdot 3\, du$$

Ex:

$$\int \frac{\sin(y)}{\cos(y)}\, dy \qquad u = \cos(y)$$

$$du = -\sin(y)\, dy$$

$$-du = \sin(y)\,dy$$

$$= \int \frac{\sin(y)}{u} \, dy$$

$$= \int \frac{1}{u} \sin(y) \, dy$$

$$= \int -\frac{1}{u} \, du$$

Ex: integrating a piecewise function.

$$f(x) = \begin{cases} -1, & 0 \le x \le 2 \\ 2x, & 2 < x \le 5 \end{cases}$$

$g'(x) = f(x)$

Goal: find $g(x)$, given $g(0) = 3$.

We know that g is an antiderivative of f, so by FTC, $\int_0^a f(x)\,dx = g(a) - \underbrace{g(0)}$

$$g(a) - 3$$

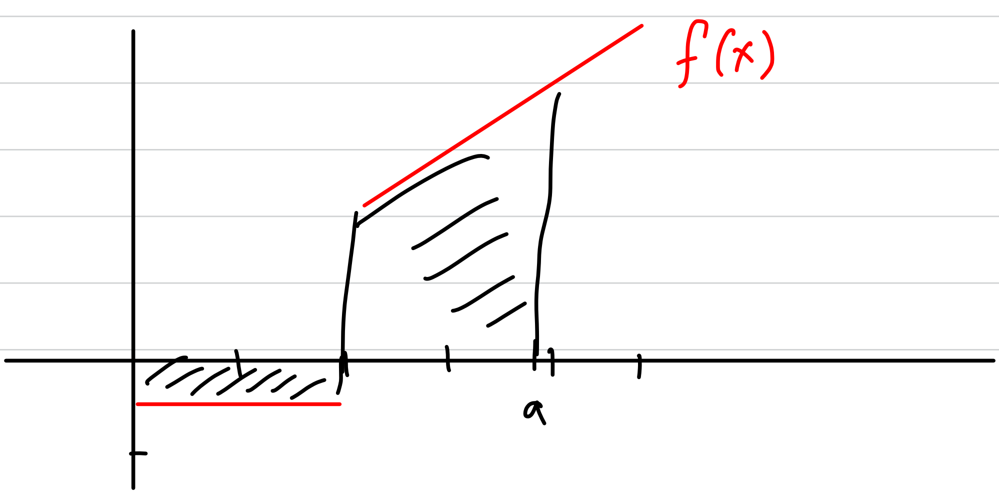
for any a. So we can write

$g(a) = 3 + \int_0^a f(x)\,dx$, and this will let us solve for g.

The hard part is that f is a piecewise function, so we have to handle $\int_0^a f(x)\,dx$ differently if $0 \le a \le 2$ or $2 < a \le 5$

If $0 \le a \le 2$, then $\int_0^a f(x)\,dx = \int_0^a -1\,dx$, b/c all of the x-values are between 0 and 2, and so $f(x) = -1$.

If $2 < a \le 5$, then $\int_0^a f(x)\,dx = \int_0^2 -1\,dx + \int_2^a 2x\,dx$



$0 \leq a \leq 2$: $\int_0^a f(x)\,dx = \int_0^a -1\,dx = [-x]\Big|_0^a = -a - 0$

$= -a$

$2 < a \leq 5$: $\int_0^a f(x)\,dx = \underbrace{\int_0^2 -1\,dx}_{[-x]\big|_0^2} + \underbrace{\int_2^a 2x\,dx}_{[x^2]\big|_2^a}$

$= (-2 - 0) + (a^2 - 2^2)$

$= -2 + a^2 - 4$

$= a^2 - 6$

Since $g(a) = 3 + \int_0^a f(x)\,dx$,

$$g(a) = \begin{cases} 3 - a, & 0 \leq a \leq 2 \\ a^2 - 3, & 2 < a \leq 5 \end{cases} \quad \checkmark$$

This is how to approach exercise 1 on HW 3

Exam 1 : Friday 1.1 – 1.7

Review on Wednesday

Midterm course evals are open : access through duckweb

If 50% of the class responds, then everybody gets 2% EC on midterm (do before the midterm)

Midterm: roughly 4 pages, 1 page of multiple choice, 1 page of short-answer (quick) problem, 2 page-length questions. All four pages equally weighted

Friday 9:30 – 10:20 scan + submit to Canvas

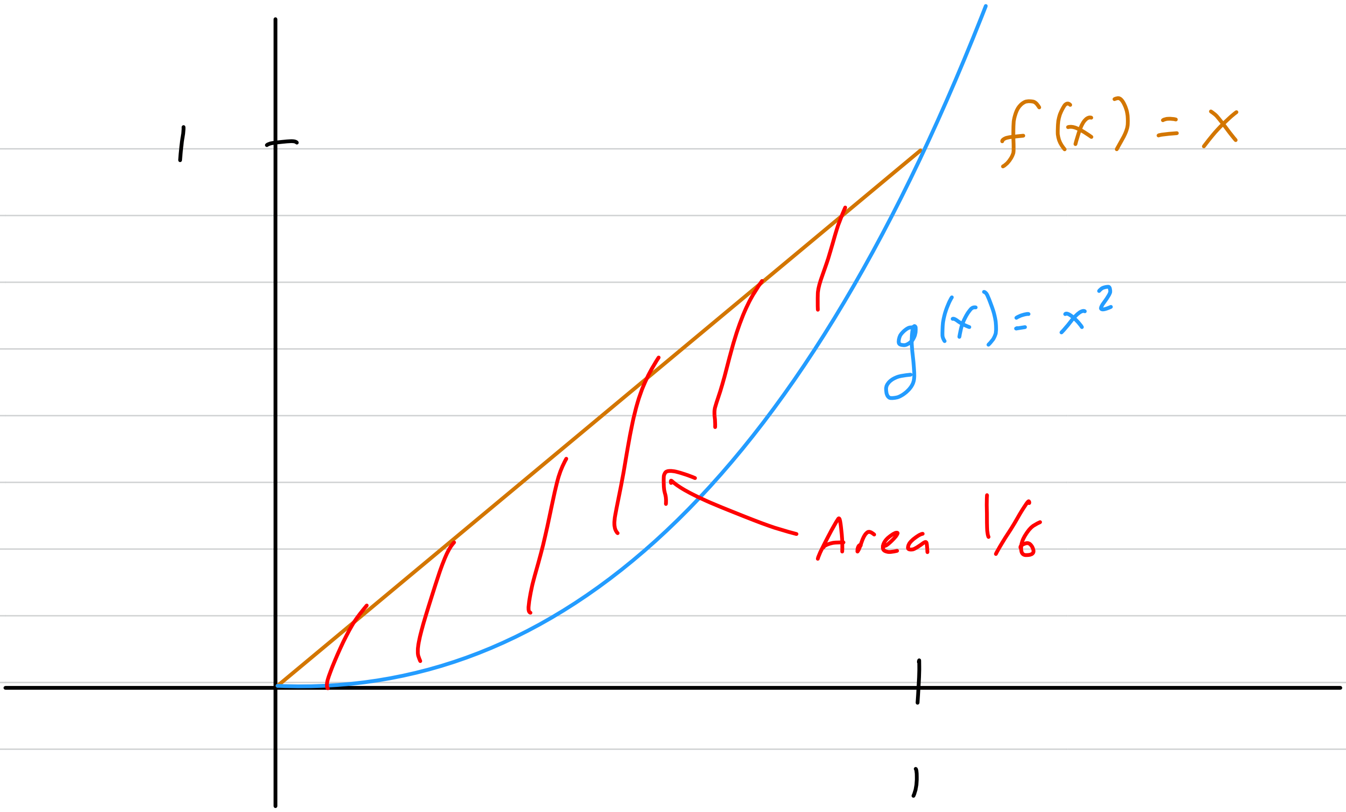
Chapter 2: Why we Care

The Area Between Curves

Prop: Let $f(x)$ and $g(x)$ be fnctions with $f(x) \geq g(x)$ The area between the graphs of f and g between $x=a$ and $x=b$ is

$$\int_a^b (f(x)-g(x))\,dx.$$

Ex: Find the area between $f(x)=x$ and $g(x)=x^2$ between $x=0$ and $x=1$.



$$\int_0^1 (x - x^2)\,dx$$

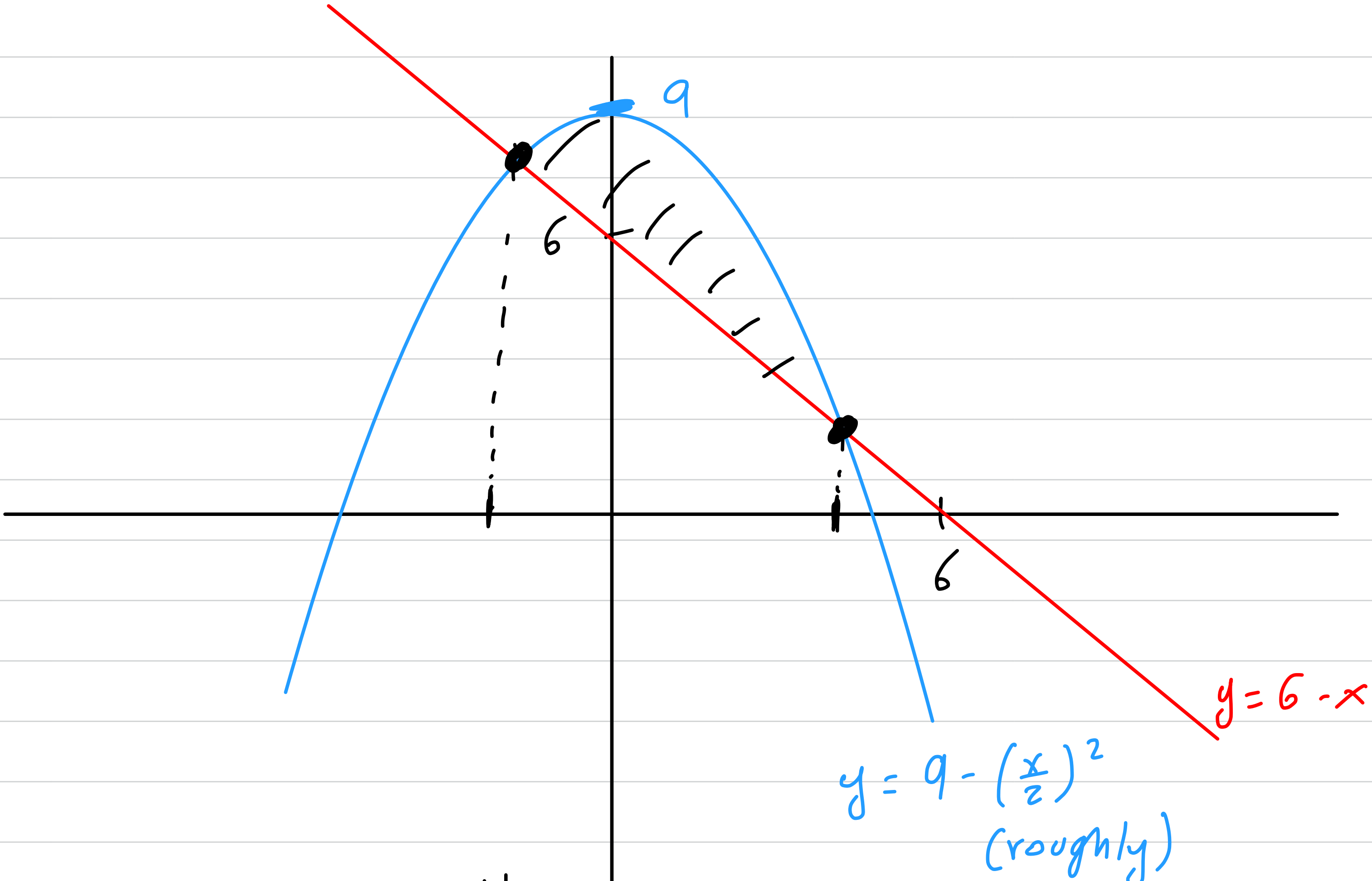
$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1$$

$$= \frac{1^2}{2} - \frac{1^3}{3}$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}.$$

Ex: Find the area of the region bounded by the graphs of $9-\left(\frac{x}{2}\right)^2$ and $6-x$.



we don't know the limits!

$$\int \left(9-\left(\frac{x}{2}\right)^2-(6-x)\right)dx$$

The limits are the x-coordinates of the intersection points of the two graphs.

So set $9-\left(\frac{x}{2}\right)^2 = 6-x$ and solve for x.

$-\frac{1}{4}x^2 + x + 3 = 0$

$x^2 - 4x - 12 = 0$

$(x-6)(x+2) = 0$

$x = 6$ or $x = -2$

$$\text{Area} = \int_{-2}^{6} \left(9 - \left(\frac{x}{2}\right)^2 - (6-x)\right) dx$$

$$= \int_{-2}^{6} \left(-\frac{1}{4}x^2 + x + 3\right) dx$$

$$= \left[-\frac{1}{12}x^3 + \frac{x^2}{2} + 3x\right]\Bigg|_{-2}^{6}$$

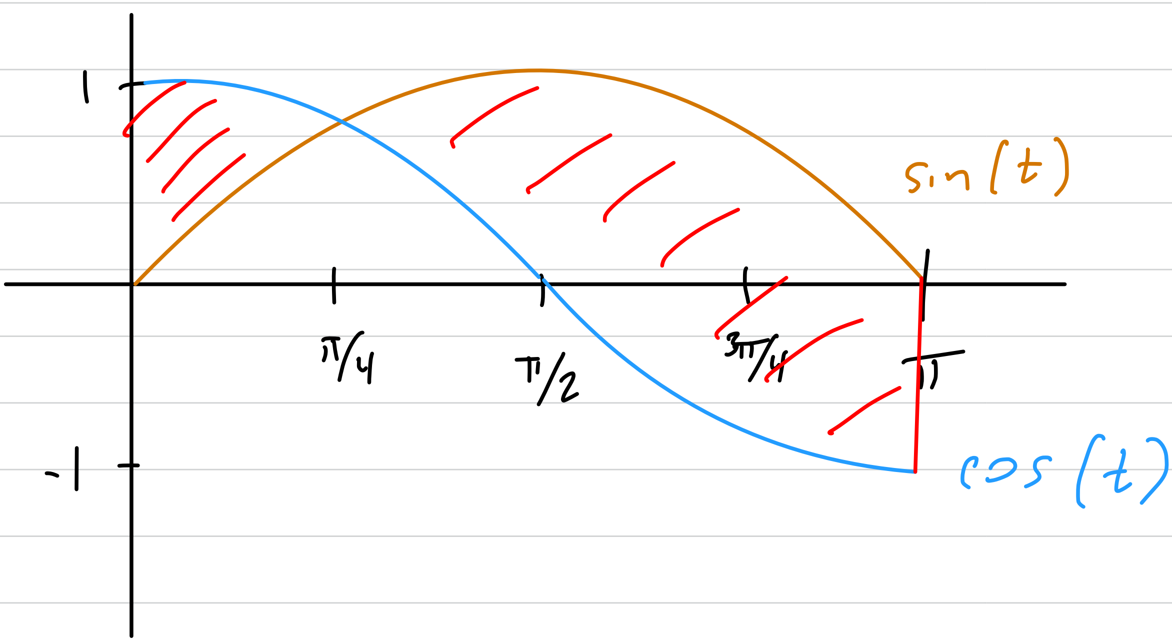
$$= \left(-\frac{1}{12}(216) + 18 + 18\right) - \left(-\frac{1}{12}(-8) + 2 + (-6)\right)$$

$$= (-18 + 18 + 18) - \left(\frac{2}{3} + 2 - 6\right)$$

$= 18 - \frac{10}{3}$

$= \frac{44}{3}$

Ex: Find the area between the graphs of $\sin(t)$ and $\cos(t)$ on $[0, \pi]$.



Issue: formula only works when one function is always bigger than the other. Solution: break up the integral

Know: $\sin(\pi/4) = \cos(\pi/4)$, so

$$\text{Area} = \int_0^{\pi/4} (\cos(t) - \sin(t))\,dt \; +$$

$$\int_{\pi/4}^{\pi} (\sin(t) - \cos(t))\,dt$$

$$= \Big[\sin(t) + \cos(t)\Big]\Big|_0^{\pi/4} \; +$$

$$\Big[-\cos(t) - \sin(t)\Big]\Big|_{\pi/4}^{\pi}$$

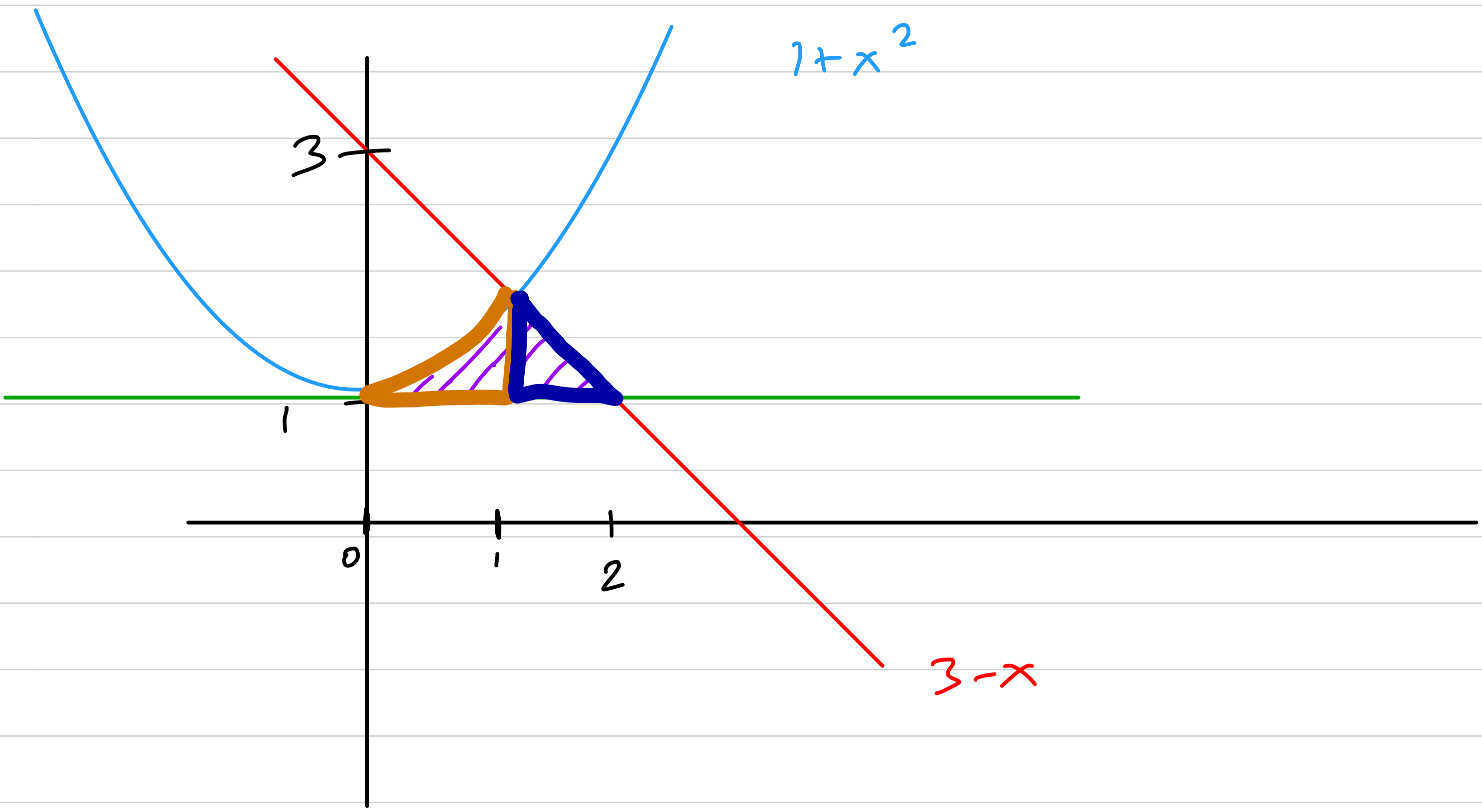
$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) \; +$$

$$(1 - 0) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2} - 1 + 1 + \sqrt{2}$$

$$= 2\sqrt{2}$$

Ex: Find the area of the region bounded by $1+x^2$, $3-x$, and 1.



$3-x=1$

$3-1=x$

$2=x$

$3-x=1+x^2$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2$ or $\boxed{x=1}$

$$\text{Area} = \boxed{\int_0^1 \left((1+x^2) - 1\right) dx} + \boxed{\int_1^2 \left((3-x) - 1\right) dx}$$

$$= \left[\frac{x^3}{3}\right]\Bigg|_0^1 + \left[2x - \frac{x^2}{2}\right]\Bigg|_1^2$$

$$= \frac{1}{3} + \left(2 \cdot 2 - \frac{2^2}{2}\right) - \left(2 \cdot 1 - \frac{1^2}{2}\right)$$

$$= \frac{1}{3} + 2 - \frac{3}{2}$$

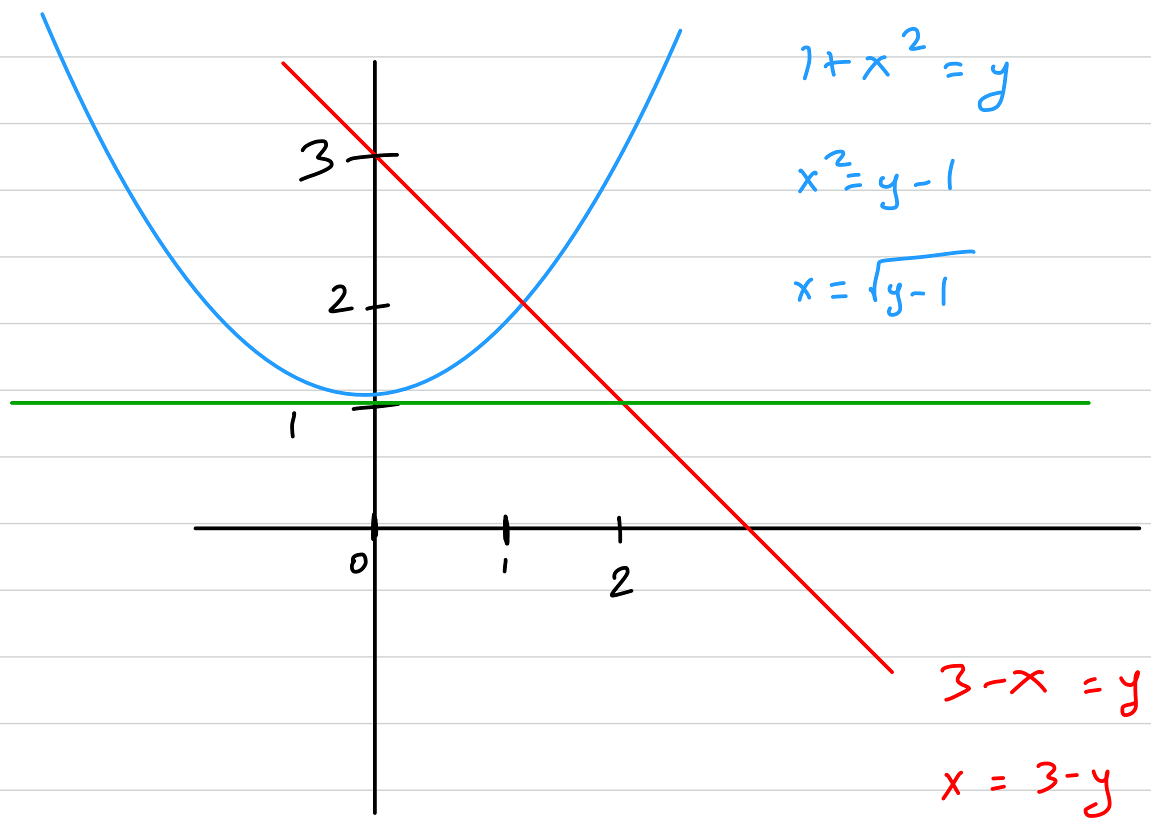
$$= \frac{1}{3} + \frac{1}{2}$$

$$= 5/6$$

Prop: Let $x = f(y)$ and $x = g(y)$ be two functions of y. Then area between f and g on $[a,b]$, where $f(y) \geq g(y)$, is

$$\int_a^b \left(f(y) - g(y)\right) dy.$$

Ex: Repeat the previous example using integration with respect to y.



Now, find the y-limits of the region.

$$\int_{1}^{2}\left((3-y)-\left(\sqrt{y-1}\right)\right)dy$$

$y=2$

$y=1$

$$= \int_1^2 \left(3 - y - \sqrt{y-1}\right) dy$$

$$= \left[3y - \frac{y^2}{2} - \frac{(y-1)^{3/2}}{3/2} \right]\Bigg|_1^2$$

$$= \left(3 \cdot 2 - \frac{2^2}{2} - \frac{(2-1)^{3/2}}{3/2}\right) - \left(3 \cdot 1 - \frac{1^2}{2} - \frac{(1-1)^{3/2}}{3/2}\right)$$

$$= \left(6 - 2 - \frac{2}{3}\right) - \left(3 - \frac{1}{2} - 0\right)$$

$$= \frac{10}{3} - \frac{5}{2}$$

$$= \frac{20}{6} - \frac{15}{6}$$

$$= 5/6 \quad \checkmark$$

Practice Midterm

Compute $\displaystyle \frac{d}{dx} \int_4^{e^{3x}} \frac{t^4 - 2}{t^2 + 2t + 1} \, dt$

$$\int_0^x \frac{d}{dt}\left[f(t)\right]dt \overset{\text{FTC II}}{=} f(x)-f(0)$$

$$\frac{d}{dx}\int_0^x f(t)\,dt \overset{\text{FTC I}}{=} f(x)$$

$$\frac{d}{dx}\int_4^x \frac{t^4-2}{t^2+2t+1}\,dt = \frac{x^4-2}{x^2+2x+1}$$

First, set $F(x) = \int_4^x \frac{t^4-2}{t^2+2t+1}\,dt$.

Know: $\frac{d}{dx}\left[F(x)\right] = \frac{x^4-2}{x^2+2x+1}$ by FTC I

Want: $\frac{d}{dx}\left[F(e^{3x})\right]$ ← use the chain rule

$= F'(e^{3x}) \cdot (e^{3x})'$

$= \dfrac{(e^{3x})^4-2}{(e^{3x})^2+2e^{3x}+1} \cdot 3e^{3x}.$

Typically, $F(x)$ is antiderivative of $f(x)$

so $F'(x) = f(x)$. Because indefinite integrals are

just antiderivatives, $\int f(x)\,dx \overset{DEF}{=} F(x) + C$.

Also: $\int_a^b f(x)\,dx \underset{\text{FTC II}}{=} F(b) - F(a)$

Why don't we use the 4?

$$\int_4^x \frac{t^4 - 2}{t^2 + 2t + 1}\,dt = F(x) - F(4), \text{ \underline{but}}$$

$$\frac{d}{dx}\left[F(x) - F(4)\right] = \frac{d}{dx}\left[F(x)\right]$$

6 $\quad v(t) = \dfrac{\ln(t + e)}{t + e}$

Find $s(t)$ assuming $s(0) = 3$.

$s'(t) = v(t)$, so $\int_0^x v(t)\,dt = s(x) - s(0)$

$= s(x) - 3$

$$\int_0^x v(t)\,dt = \int_0^x \frac{\ln(t+e)}{t+e}\,dt$$

$u = \ln(t+e)$

$$du = \frac{1}{t+e} \cdot \underbrace{(t+e)'}_{1}\,dt = \frac{1}{t+e}\,dt$$

$$= \int_0^x u\,du$$

$$= \left[\frac{u^2}{2}\right]\Big|_0^x$$

$$= \left[\frac{(\ln(t+e))^2}{2}\right]\Big|_0^x$$

$$= \frac{(\ln(x+e))^2}{2} - \frac{(\ln(e))^2}{2}$$

$$= \frac{(\ln(x+e))^2}{2} - \frac{1}{2}$$

So $$s(x) - 3 = \frac{(\ln(x+e))^2}{2} - \frac{1}{2}$$

$$s(x) = \frac{(\ln(x+e))^2}{2} + 5/2 .$$