Midterm on Friday

Covers 1.1 - 3.1 with a focus on 2.1-3.1

Expect ~4 pages: 1 page multiple choice,
1 page short-answer, 2 multi-part
questions

Practice Exam: up later today, solutions today
or tomorrow

HW 7: due wed of week 10: up later
today

Final quiz: Friday of week 9

Final: 10:15 on Friday June 11th

Comment: Trig sub handles integrals that contain:

① $\sqrt{a^2 - x^2}$

② $\sqrt{a^2 + x^2}$

③ $\sqrt{x^2 - a^2}$

where a is any number.

Ex: $\int \sqrt{9 - x^2}\, dx$

can't use u-sub b/c $u = 9 - x^2 \Rightarrow du = -2x\,dx$ but $-2x$ doesn't appear

$= \int \sqrt{3^2 - x^2}\, dx$

$x = 3\sin(\theta)$

$dx = 3\cos(\theta)\, d\theta$

$= \int \sqrt{3^2 - 3^2 \sin^2(\theta)} \cdot 3\cos(\theta)\, d\theta$

$$= \int \sqrt{3^2} \sqrt{1-\sin^2(\theta)} \cdot 3\cos(\theta)\, d\theta$$

$$= 9 \int \sqrt{1-\sin^2(\theta)} \, \cos(\theta)\, d\theta$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

$$\cos(\theta) = \sqrt{1-\sin^2(\theta)}$$

$$= 9 \int \cos(\theta) \cdot \cos(\theta)\, d\theta$$

$$= 9 \int \cos^2(\theta)\, d\theta$$

$$= 9 \int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= 9 \left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) \right] + C$$

$$= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C \quad \leftarrow$$

we're not done b/c we have to rewrite this in terms of x

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$= \frac{9}{2}\theta + \frac{9}{2}\sin(\theta)\cos(\theta) + C$

Need to rewrite θ, $\sin(\theta)$, and $\underbrace{\cos(\theta)}_{\text{tough}}$ in terms of x.

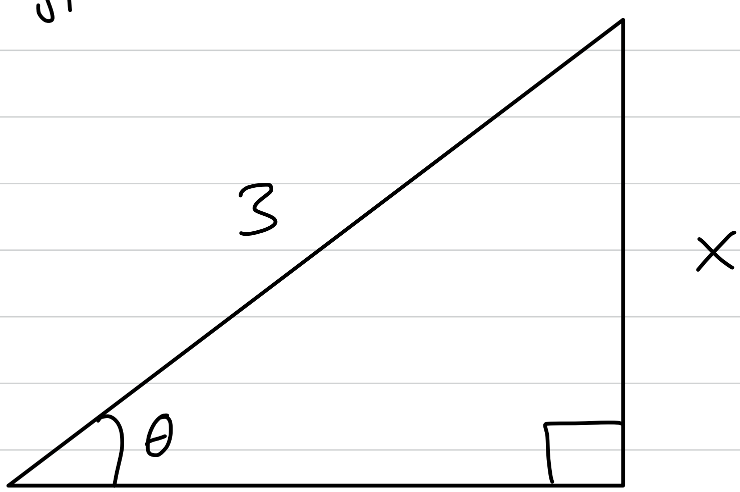
$x = 3\sin(\theta)$

$\sin(\theta) = x/3$

$\theta = \sin^{-1}(x/3)$

To solve for $\cos(\theta)$, draw a right triangle.

$\sin(\theta) = \frac{opp}{hyp} = \frac{x}{3}$



$\cos(\theta) = \frac{adj}{hyp}$

Find adj w/ Pythagorean theorem

$$x^2 + adj^2 = 3^2$$

$$adj^2 = 3^2 - x^2$$

$$adj = \sqrt{9 - x^2}$$

$$\cos(\theta) = \frac{\sqrt{9-x^2}}{3}$$

$$\int = \frac{9}{2}\theta + \frac{9}{2}\sin(\theta)\cos(\theta) + C$$

$$= \boxed{\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C}$$

Comment: Trig formulas to know:

1. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
2. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

③ $\sin^2(\theta) = 1 - \cos^2(\theta)$

④ $\cos^2(\theta) = 1 - \sin^2(\theta)$

⑤ $1 + \tan^2(\theta) = \sec^2(\theta)$

⑥ $\sec^2(\theta) - 1 = \tan^2(\theta)$

⑦ $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$

⑧ $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$

Method (Trig Sub)

① Determine what trig function to use

$\sqrt{a^2 - x^2}$: $x = a\sin(\theta)$

$\sqrt{a^2 + x^2}$: $x = a\tan(\theta)$

$\sqrt{x^2 - a^2}$: $x = a\sec(\theta)$

② Find dx and substitute.

③ Simplify the integral with $\cos^2(\theta) = 1 - \sin^2(\theta)$,

$\tan^2(\theta) + 1 = \sec^2(\theta)$, or $\sec^2(\theta) - 1 = \tan^2(\theta)$.

Factor out an a from the square root to apply one of these.

④ Integrate using the methods of 3.2

⑤ Substitite back for x: you'll probably need to draw a right triangle.

Ex: $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$

of the form $\sqrt{a^2+x^2}$, so use $x = 2\tan(\theta)$

$dx = 2\sec^2(\theta)\,d\theta$.

$$= \int \frac{1}{2\sqrt{1+\tan^2(\theta)}} \cdot 2\sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{\sqrt{\sec^2(\theta)}} \cdot \sec^2(\theta)\, d\theta$$

$$= \int \frac{1}{\sec(\theta)} \cdot \sec^2(\theta)\, d\theta$$

$$= \int \sec(\theta)\, d\theta$$

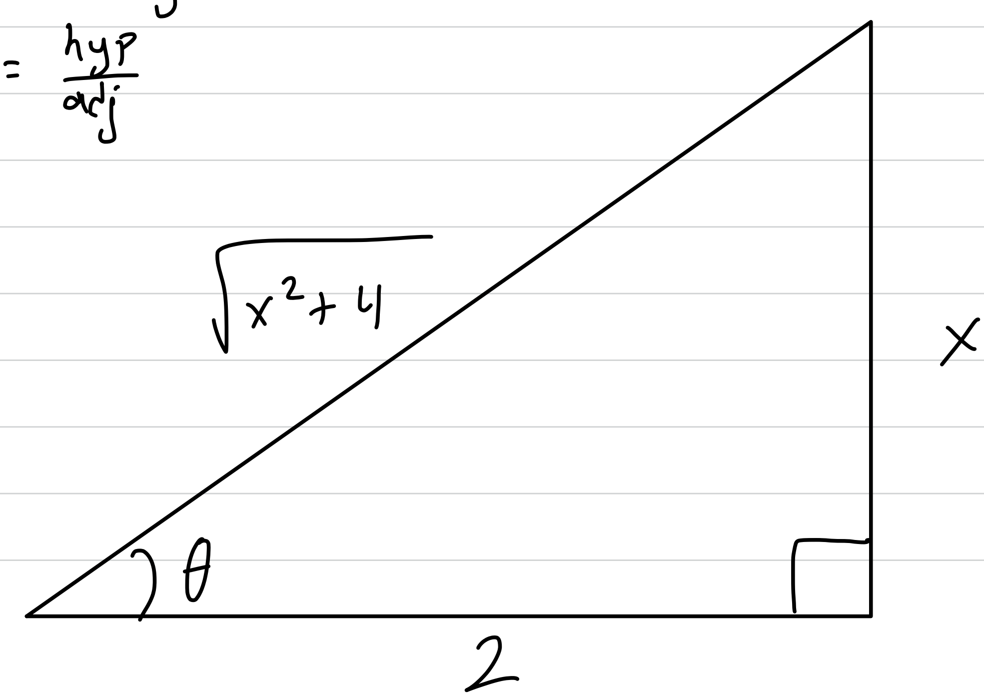
$$= \ln|\sec(\theta) + \tan(\theta)| + C$$ ← from 3.2

$x = 2\tan(\theta)$

$\tan(\theta) = \frac{x}{2}$

$\tan(\theta) = \frac{opp}{adj}$

$\sec(\theta) = \frac{hyp}{adj}$



$$\sec(\theta) = \frac{\sqrt{x^2+4}}{2}$$

$$\int = \boxed{\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C}$$

Comment:

$$\left\{\begin{array}{l}\sin(\theta) = \frac{opp}{hyp} \\ \cos(\theta) = \frac{adj}{hyp} \\ \tan(\theta) = \frac{opp}{adj} \\ \sec(\theta) = \frac{hyp}{adj}\end{array}\right.$$

$$\csc(\theta) = \frac{hyp}{opp}$$

$$\cot(\theta) = \frac{adj}{opp}$$

Ex: $\int_0^1 x^3 \underbrace{\sqrt{1-x^2}} \, dx$

$\hookrightarrow x = a\sin(\theta) = \sin(\theta)$

$dx = \cos(\theta)\, d\theta$

$$= \int_0^1 \sin^3(\theta)\sqrt{1-\sin^2(\theta)} \cos(\theta)\, d\theta$$

$$= \int_0^1 \sin^3(\theta)\sqrt{\cos^2(\theta)} \cos(\theta)\, d\theta$$

$$= \int_0^1 \underbrace{\sin^3(\theta)}_{\text{odd exponent}} \cos^2(\theta)\, d\theta$$

$$= \int_0^1 \sin^2(\theta)\cos^2(\theta) \cdot \sin(\theta)\, d\theta$$

$$= \int_0^1 (1-\cos^2(\theta))\cos^2(\theta) \sin(\theta)\, d\theta$$

$$u = \cos(\theta)$$

$$du = -\sin(\theta)\,d\theta$$

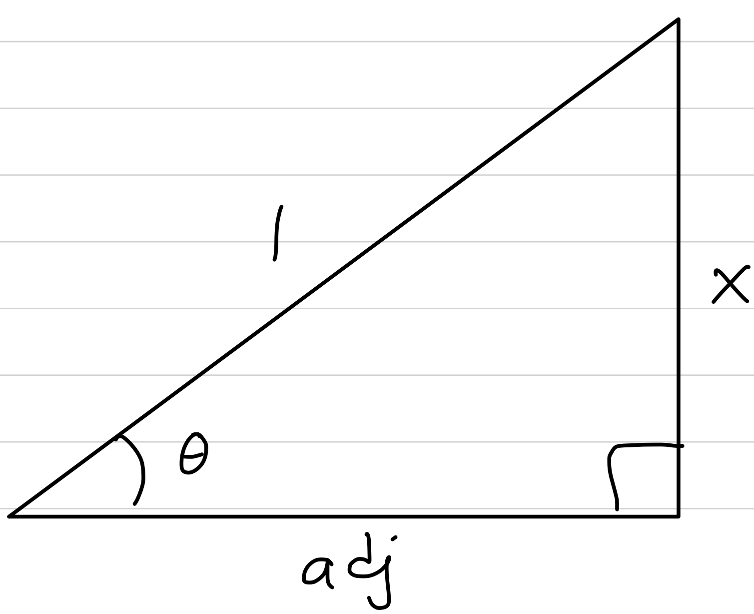
$$-du = \sin(\theta)\,d\theta$$

$$= -\int_0^1 (1-u^2)\,u^2\,du$$

$$= -\int_0^1 u^2 - u^4\,du$$

$$= -\left[\frac{u^3}{3} - \frac{u^5}{5}\right]\Bigg|_0^1$$

$$= -\left[\frac{\cos^3(\theta)}{3} - \frac{\cos^5(\theta)}{5}\right]\Bigg|_0^1$$



$$\sin(\theta) = x = \frac{x}{1}$$

$$x^2 + adj^2 = 1^2$$

$$adj = \sqrt{1-x^2}$$

$$\cos(\theta) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$= -\left[\frac{\cos^3(\theta)}{3} - \frac{\cos^5(\theta)}{5}\right]\Bigg|_0^1$$

$$= -\left[\frac{\sqrt{1-x^2}^{\,3}}{3} - \frac{\sqrt{1-x^2}^{\,5}}{5}\right]\Bigg|_0^1$$

$$= -\left(0 - \left(\frac{1}{3} - \frac{1}{5}\right)\right)$$

$$= \frac{1}{3} - \frac{1}{5}$$

$$= \frac{2}{15}$$

Note: we could have solved $\int_0^1 x^3\sqrt{1-x^2}\,dx$

with u-sub: $u = 1 - x^2 \quad | \quad x^2 = 1 - u$

$du = -2x\,dx \quad | \quad -\frac{1}{2}\,du = x\,dx$

$$= \int_0^1 x^2 \sqrt{1 - x^2} \cdot x\,dx$$

$$= -\frac{1}{2} \int_0^1 (1-u)\sqrt{u}\,du$$

Ex: Find arc length of $f(x) = x^2$ on $[0, \frac{1}{2}]$.

$f'(x) = 2x$

$(f'(x))^2 = 4x^2$

$$\int_0^{1/2} \sqrt{1 + 4x^2}\,dx$$

factor out

$$= \int_0^{1/2} 2\sqrt{\frac{1}{4} + x^2}\,dx$$

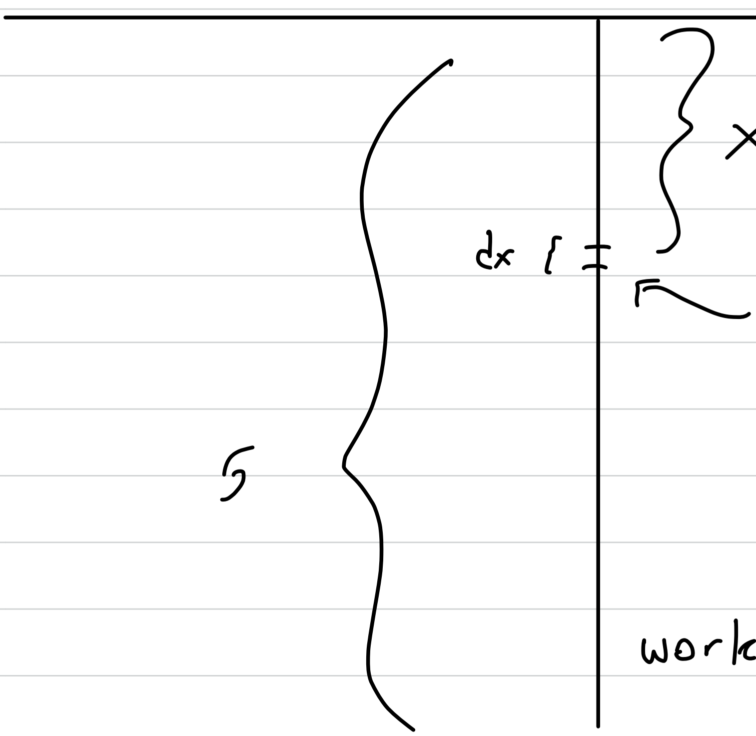
$$= 2\int_0^{1/2} \sqrt{\left(\tfrac{1}{2}\right)^2 + x^2}\ dx$$

$$x = \tfrac{1}{2}\tan(\theta)$$

$$dx = \tfrac{1}{2}\sec^2(\theta)\, d\theta$$

$$= 2\int_0^{1/2} \underbrace{\sqrt{\tfrac{1}{2}^2 + \tfrac{1}{2}^2\tan^2(\theta)}}_{\frac{1}{2}\sec(\theta)} \cdot \tfrac{1}{2}\sec^2(\theta)\, d\theta$$

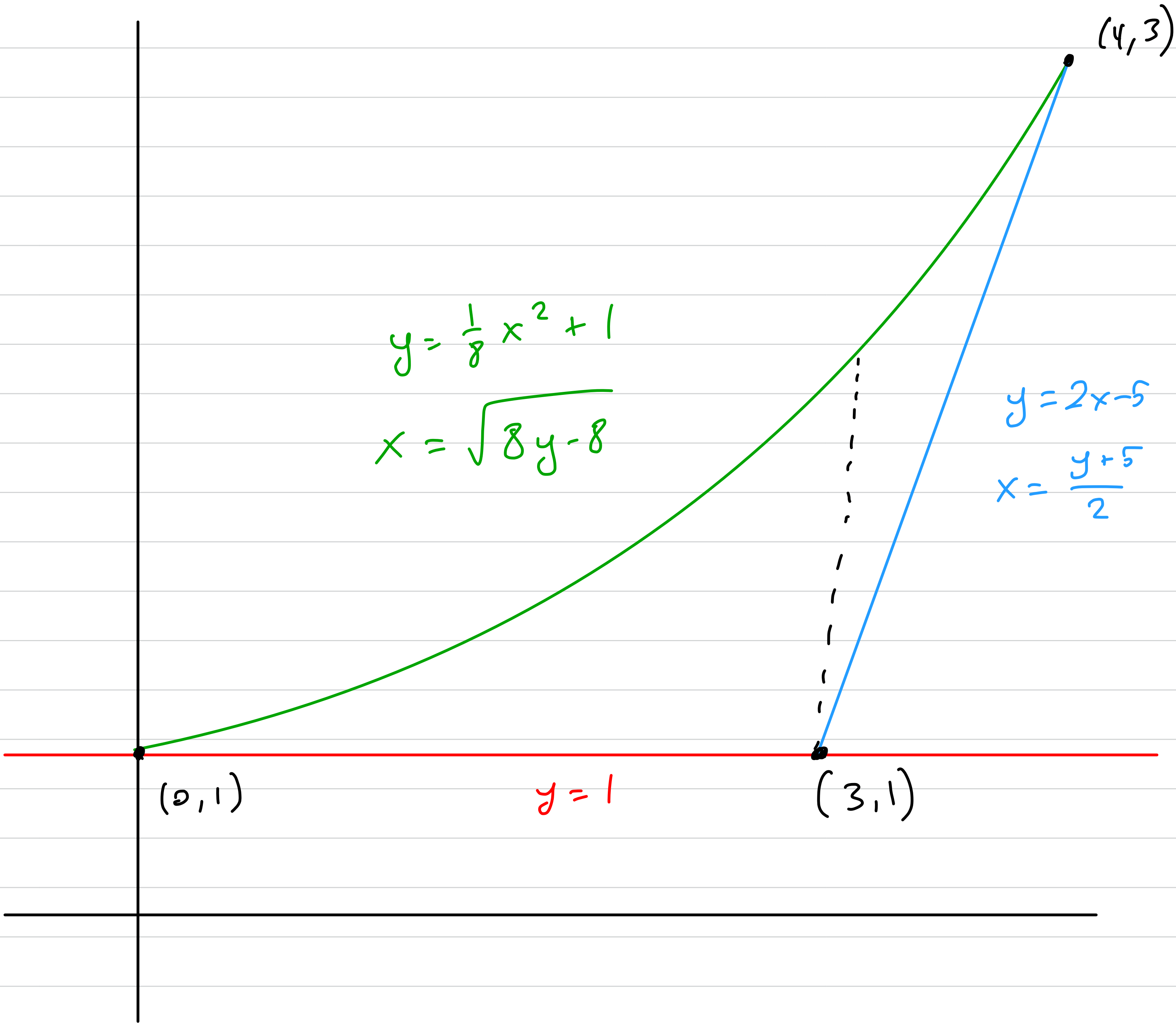
$$= 2\int_0^{1/2} \tfrac{1}{4}\cdot \sec^3(\theta)\ d\theta$$



$$F = (100 - \sqrt{x})\, dx$$

distance = x

$$\text{work} = \int_0^5 (100 - \sqrt{x})(x)\ dx$$



$$\text{Area} = \int_0^3 \left(\frac{1}{8}x^2+1\right) - (1)\,dx + \int_3^4 \left(\frac{1}{8}x^2+1\right) - (2x-5)\,dx$$

$$\text{Area} = \int_1^3 \left(\frac{y+5}{2}\right) - \left(\sqrt{8y-8}\right)\,dy$$

Comment: Given an integral $\int f(x)\,dx$,

① If you know an antiderivative, you're done.

② If there's a composition, try u-sub, typically with u equal to the inside function, but in general just look for a u that has du present in the integral.

$$\int \sin(\underbrace{\cos(\ln(x))}_{u}) \cdot \underbrace{\left(-\sin(\ln(x)) \cdot \frac{1}{x}\right) dx}_{du}$$

$$\int \sin(\cos(\underbrace{\ln(x)}_{u})) \cdot \underbrace{\frac{1}{x}\,dx}_{du}$$

③ If there's a product and u-sub didn't work, try integrating by parts. In general, let u be the part of $f(x)$ that simplifies the most when differentiated while still keeping dv integrable.

4 If nothing is working and there is an expression of the form $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$, then try trig sub.

5 If all else fails and you have a function of the form $\frac{P(x)}{Q(x)}$, where P and Q are polynomials and $\deg P < \deg Q$, run partial fractions on it and go back to 1.

Def: Let $P(x)$ be a polynomial. The degree of P is the largest exponent on x.

Ex: $\deg\left(x^{\boxed{5}} + 2x^4 + x^{\boxed{5}} - 2 \right) = 5.$

Ex: $\dfrac{x-1}{3x^2-2}$ will work with partial fractions since $\deg(x-1) = 1 < \deg(3x^2-2) = 2.$

Method (Partial Fractions, v1) Let $P(x)$ and $Q(x)$ be polynomials with $\deg P < \deg Q$ and such that $Q(x)$ splits into nonrepeating linear factors: $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ where $a_i \neq a_j$ for $i \neq j$. Then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

for some numbers $A_1, A_2, \ldots, A_n$.

Ex: $Q(x) = (x-1)(x+3)(x-2)$ works

$Q(x) = (x-1)^2(x^2+1)$ doesn't

Ex: $\displaystyle\int \underbrace{\frac{3x+2}{x^3-x^2-2x}} \, dx$

$$= \frac{3x+2}{x(x^2-x-2)}$$

$$= \boxed{\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}}$$

by Partial Fractions

How do we find A, B, and C?

Multiply both sides of red by $Q(x) = x(x-2)(x+1)$.

$$3x+2 = A(x-2)(x+1) + B(x)(x+1) + C(x)(x-2)$$

$$3x+2 = A(x^2 - x - 2) + B(x^2 + x) + C(x^2 - 2x)$$

$$3x+2 = (A+B+C)x^2 + (-A+B-2C)x - 2A$$

Now set all the constant terms equal, all the coefficients on x equal, and so on.

$$2 = -2A \implies A = -1$$

$$3 = -A + B - 2C$$

$$0 = A + B + C$$

$3 = 1 + B - 2C \Rightarrow B - 2C = 2$

$0 = -1 + B + C \Rightarrow B + C = 1$

$B = 2 + 2C$

$2 + 2C + C = 1$

$3C = -1$

$C = -1/3$

$B = 2 - 2/3 = 4/3$

$\int \frac{1}{x-2}\,dx = \ln(x-2) + C$

$$\int \frac{-1}{x} + \frac{4/3}{x-2} + \frac{-1/3}{x+1}\,dx$$

$$= -\ln(x) + \frac{4}{3}\ln(x-2) - \frac{1}{3}\ln(x+1) + C.$$