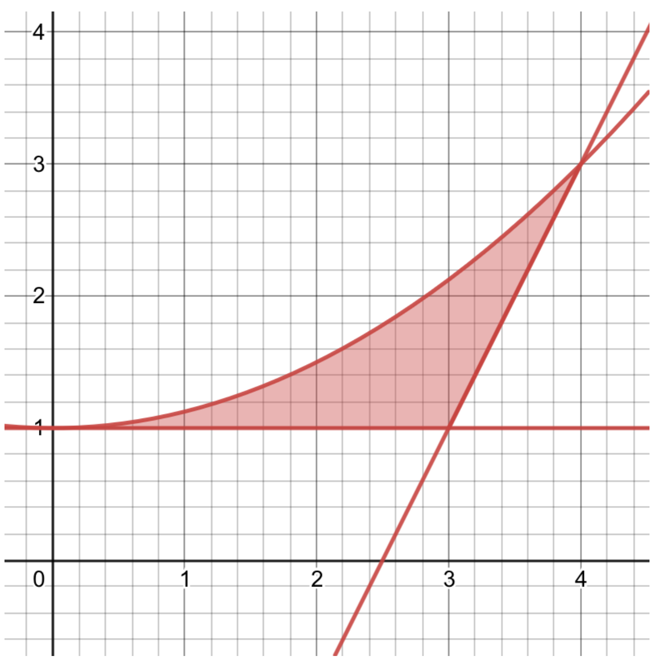
# Practice Midterm 2

Math 252

**Exercise 1:** Let $f(x)$ be a differentiable function with a continuous derivative. What is the arc length of $f$ between $x = 2$ and $x = 5$?

$\int_2^5 \sqrt{1 + (f'(x))^2}\, dx.$

**Exercise 2:** The shaded region below is bounded by the curves $y = 1$, $y = 2x - 5$, and $y = \frac{1}{8}x^2 + 1$. Find the shaded area.




Let's slice with respect to $y$, since then we don't have to split up the interval at $x = 3$. Then we need to solve for $y$: $x = \frac{y+5}{2}$ and $x = \sqrt{8y - 8}$ (we want the positive square root since $y > 0$ in the graph. Our limits are now $y = 1$ to $y = 3$, and since the linear function is the rightmost one, we have that the area is

$$\begin{aligned}\int_1^3 \left(\frac{y+5}{2} - \sqrt{8y-8}\right)\, dy &= \int_1^3 \left(\frac{y}{2} + \frac{5}{2} - (8y-8)^{1/2}\right)\, dy \\ &= \left[\frac{y^2}{4} + \frac{5}{2}y - \frac{(8y-8)^{3/2}}{3/2} \cdot \frac{1}{8}\right]\Bigg|_1^3 \\ &= \left(\frac{9}{4} + \frac{15}{2} - \frac{(16)^{3/2}}{3/2} \cdot \frac{1}{8}\right) - \left(\frac{1}{4} + \frac{5}{2} - \frac{(0)^{3/2}}{3/2} \cdot \frac{1}{8}\right) \\ &= \frac{5}{3}.\end{aligned}$$

**Exercise 3:** A tank in the shape of an inverted cone (like the problem from homework 5) has height 8 meters, radius 2 meters, and is filled up to 5 meters with water (weight density 9800 $\frac{N}{m^3}$). Find the work done by pumping it all out.

By similar triangles, the diameter of a slice at height $y$ is $\frac{4}{8}y = \frac{y}{2}$, so the radius is $\frac{y}{4}$. Then the area is $\frac{\pi}{16}y^2$, and the distance it needs to travel is $8-y$, so in total, we have

$$\int_0^5 9800\left(\frac{\pi}{16}y^2\right)(8-y)\, dy = \left[9800\frac{\pi}{16}\left(\frac{8}{3}y^3 - \frac{1}{4}y^4\right)\right]\Bigg|_0^5 \approx 340750.$$

**Exercise 4:** Let $R$ be the region bounded by $\sin(x)$ and $\dfrac{4}{\pi^2}x^2$ on $[0, \frac{\pi}{2}]$. Find the volume of the solid of revolution given by rotating $R$ about the $x$-axis (you may use any method you like).

Here we go. We'll use disks, since then the integral is with respect to $x$. Then the volume is

$$\pi\int_0^{\pi/2} \sin^2(x)\, dx - \pi\int_0^{\pi/2}\left(\frac{4}{\pi^2}x^2\right)^2\, dx = \pi\int_0^{\pi/2}\sin^2(x)\, dx - \pi\int_0^{\pi/2}\frac{16}{\pi^4}x^4\, dx.$$

Let's handle these terms one at a time. First,

$$\int_0^{\pi/2}\frac{16}{\pi^4}x^4\, dx = \left[\frac{16}{5\pi^4}x^5\right]\Bigg|_0^{\pi/2} = \frac{16}{5\pi^4}\cdot\frac{\pi^5}{32} = \frac{\pi}{10}.$$

Then we also have

$$\begin{aligned}\int_0^{\pi/2}\sin^2(x)\, dx &= \int_0^{\pi/2}\left(\frac{1}{2} - \frac{\cos(2x)}{2}\right)\, dx \\ &= \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]\Bigg|_0^{\pi/2} \\ &= \frac{\pi}{4}, \text{ since } \sin(0) = \sin(\pi) = 0.\end{aligned}$$

Putting it all together, we have that the area is

$$\pi \cdot \frac{\pi}{4} - \pi \cdot \frac{\pi}{10}.$$

**Exercise 5:** Set up, but do not solve, the integral for the surface area of the solid of revolution given by rotating $\ln(x)$ for $2 \leq x \leq 5$ about the $y$-axis.

Since we're revolving about the $y$-axis, we need to write this as a function of $y$ and have $y$-limits. We have $x = e^y$ and $y = \ln(2)$ to $y = \ln(5)$. Thus the integral is

$$\int_{\ln(2)}^{\ln(5)} 2\pi e^y \sqrt{1 + e^{2y}}\ dy.$$

**Exercise 6:** Find $\int t^2 \ln(t)\ dt$.

This is a product, so we should try integrating by parts. Letting $u = t^2$ and $dv = \ln(x)\ dx$ isn't a bad idea, but integrating $\ln(x)$ turns it into $x\ln(x) - x$, which is substantially more complicated, and differentiating it gives $\frac{1}{x}$, which is simpler. To that end, let $u = \ln(x)$ and $dv = x^2\ dx$. Then $du = \frac{1}{x}\ dx$ and $v = \frac{x^3}{3}$, so the integral becomes

$$\frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\ dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C.$$