Chapter I: Logic

Problem Solving

Ex: Solve $x^2 + 2x = -1$ for x.

$$x^2 + 2x + 1 = 0$$ ← specific situation

general fact

Use the quadratic formula:

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{-2 \pm \sqrt{4-4}}{2}$$

$$= \frac{-2 \pm 0}{2} = -1.$$

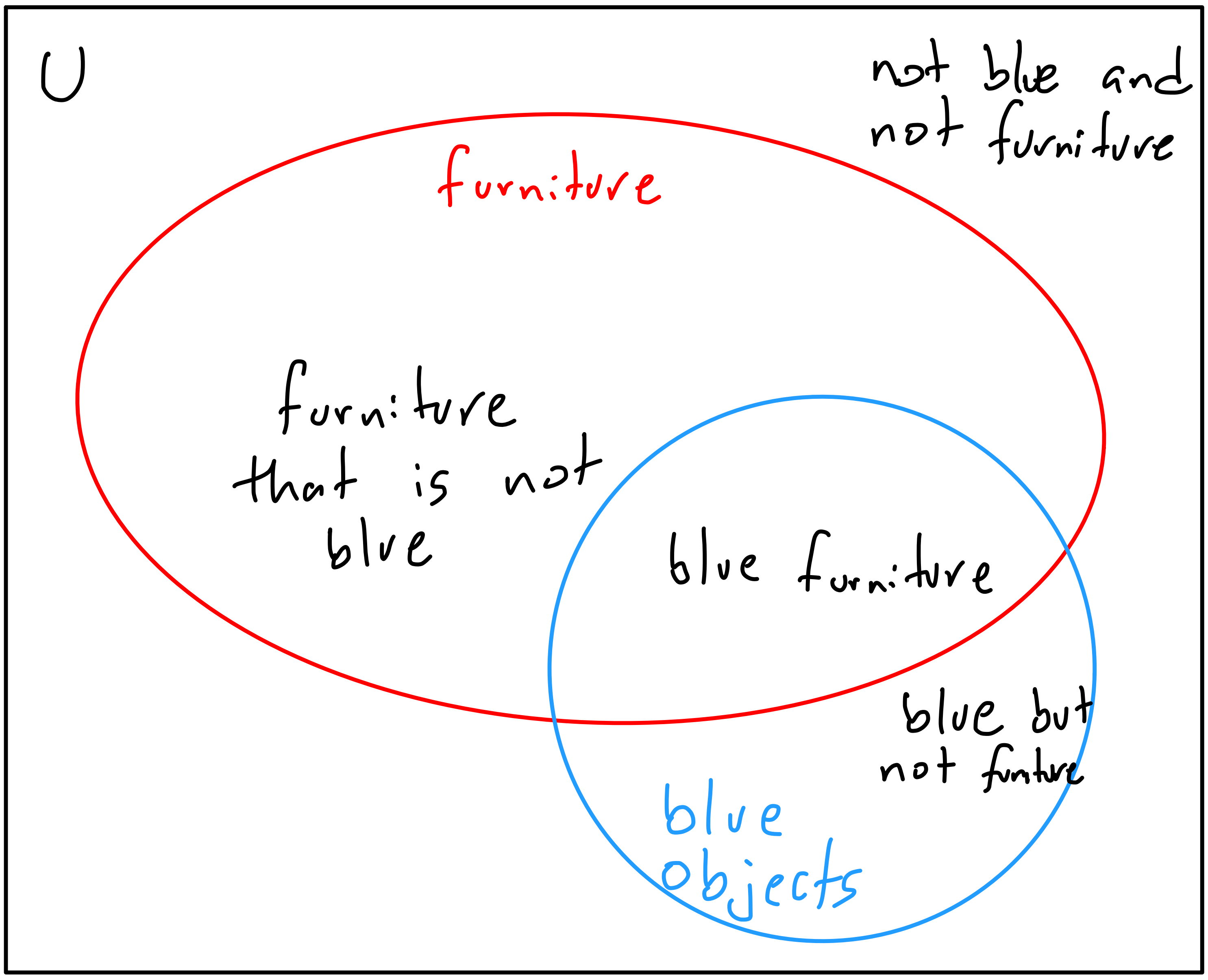
Comment: how did know how to solve this?

well, we know that we can use the quadratic formula whenever we have an equation of the form $ax^2+bx+c=0$. Here, $ax^2+bx+c=0$ is a general kind of problem, and $x^2+2x+1=0$ is a specific instance.

Def: Deductive reasoning is a method to solve problems by applying general knowledge to a specific situation.

Def: A Venn diagram is a set of overlapping figures that are contained within a universe U, typically drawn as a rectangle.

Ex:



Def: An argument is <u>valid</u> if the conclusion follows logically from the statements before it. It doesn't matter whether those statements or the conclusion are true.

Ex:

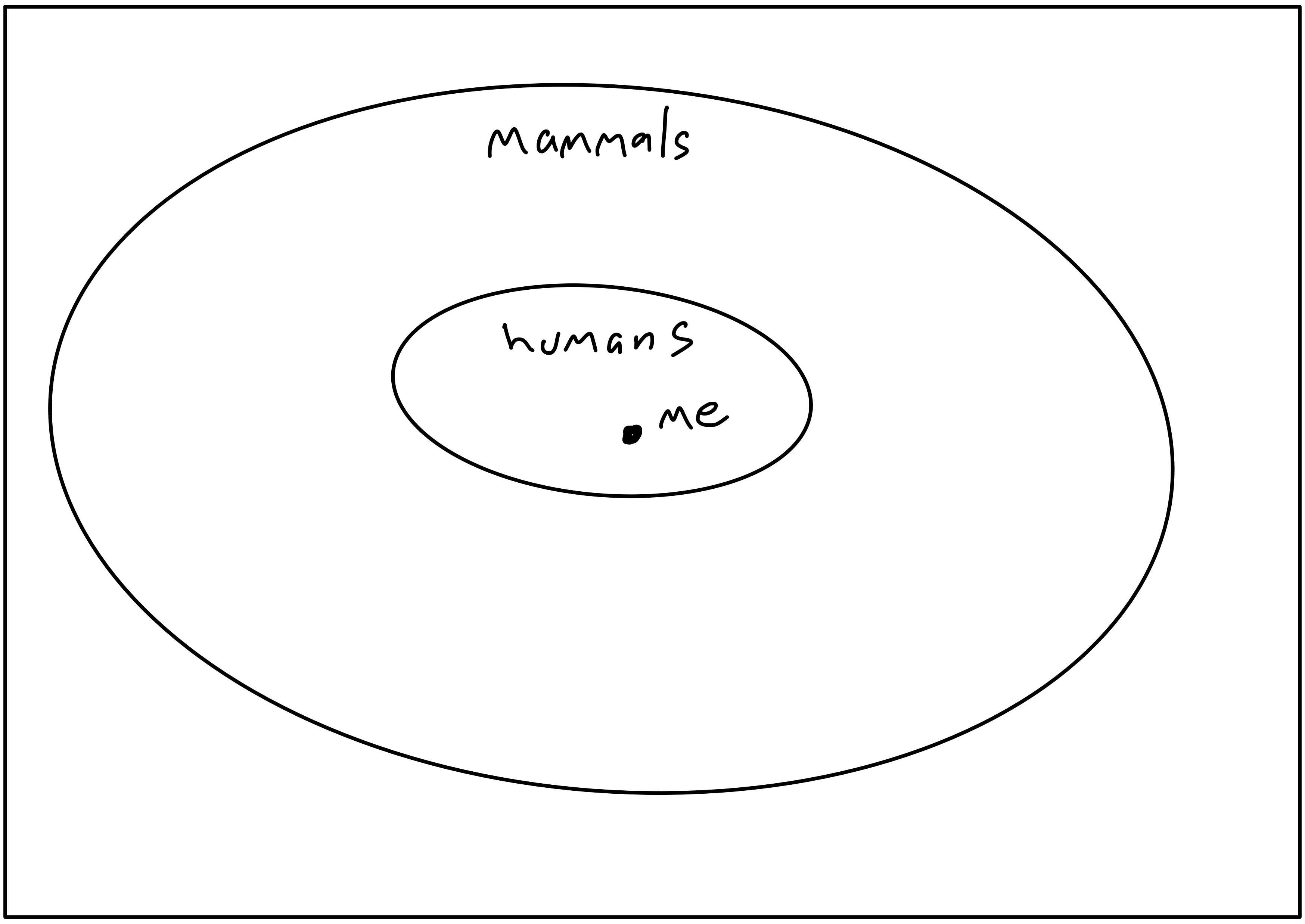
1. All humans are mammals.
2. I am a human.

I am a mammal.

Method (Showing an argument is valid):

Draw a Venn diagram that follows all the statements and assumes nothing else. Then demonstrate that the conclusion must be true

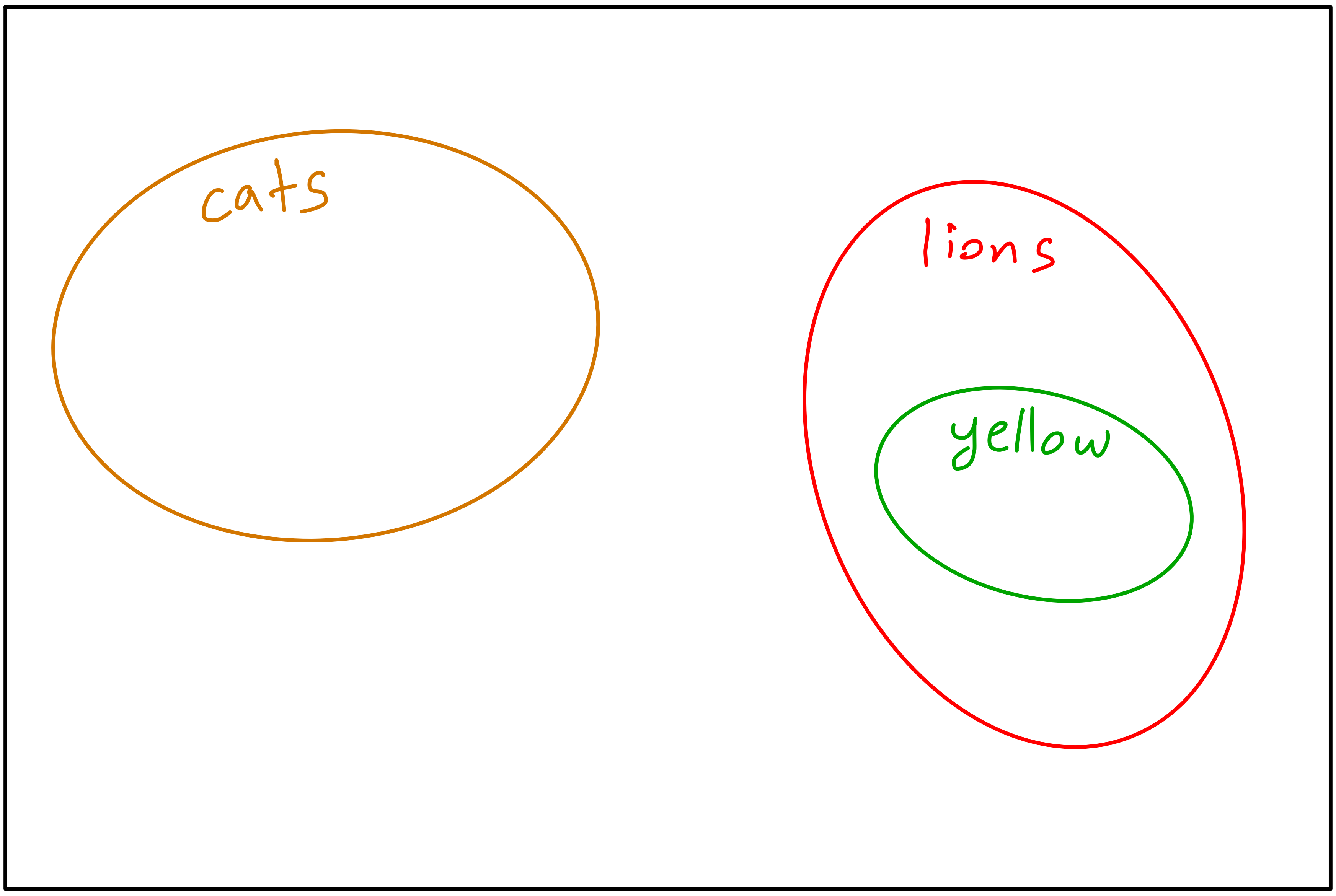
Ex: we want to draw a Venn diagram involving humans, mammals, and me.



Since that dot lives inside the set of mammals, it must be the case that I am a mammal.

Ex: 1. No cats are lions.
2. All yellow animals are lions.

No cat is yellow.



Since the set of cats and the set of yellow animals don't overlap, no cat is yellow.

<u>Comment</u>: Venn diagrams only work when the argument uses deductive reasoning.

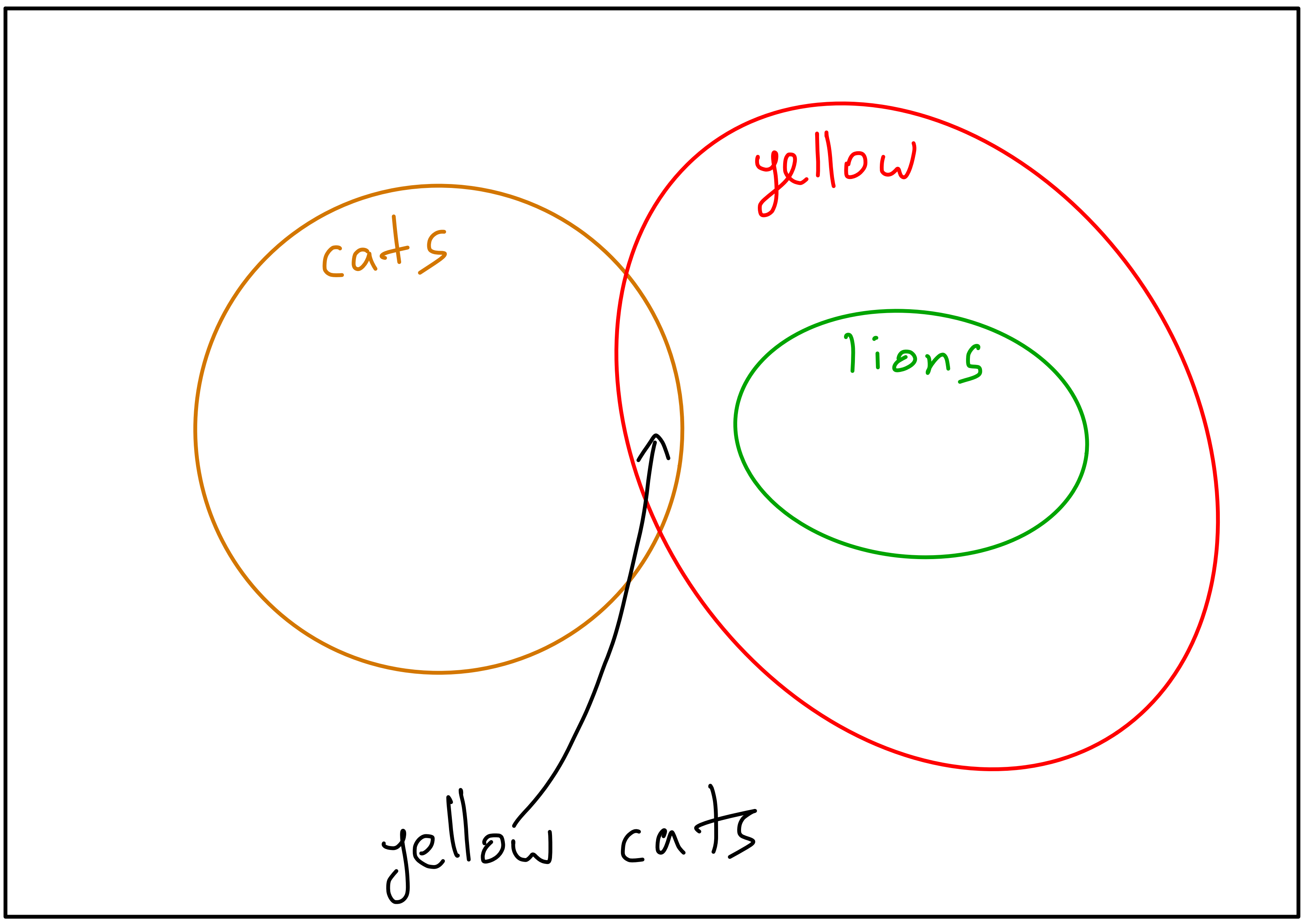
<u>Method</u> (Showing an argument is invalid): Construct a Venn diagram that satisfies the statements but not the conclusion.

<u>Ex</u>: 1. No cats are lions.
2. All lions are yellow.

No cat is yellow.

To show this is invalid, we would need to draw a situation where:

1. No cats are lions.
2. All lions are yellow.
3. Some cats are yellow.



Def: Inductive reasoning is a method to solve problems by finding a pattern in a few specific cases and conjecturing that the pattern holds in general.

Ex:

1. I got stung by a bee last month and it hurt.

2. I got stung by a bee today and it hurt.

Bee stings hurt.

Comment: We can't say for sure if an inductive argument is valid or not

$$\text{general} \xrightarrow{\text{deductive}} \text{specific}$$

$$\text{specific} \xrightarrow{\text{inductive}} \text{general}$$

1.2: Compound Statements

Def: A statement is a sentence that is either true or false.

Ex: UO is a college campus. ✓

It is raining. ✓

UO is the best university. ✗

Is it raining? ✗

This sentence is a lie. ✗

Def: Let p and q be statements.

① The negation of p, written $\sim p$ or $\neg p$, is the statement that is true when p is false and false when p is true. We read $\sim p$ as "not p"

② The conjunction of p and q, written $p \wedge q$, is the statement that is true when both p and q are true, and false if either p or q is false. We read $p \wedge q$ as "p and q".

(3) The disjunction of p and q, written $p \vee q$, is the statement that is true if at least one of p and q is true and false if both are false. We read $p \vee q$ as "p or q".

(4) The implication $p \rightarrow q$ is the statement that is true if whenever p is true, q is also true. We read $p \rightarrow q$ as "p implies q".

In summary:

$\neg p$	true when p is false
$p \wedge q$	true when p is true and q is true
$p \vee q$	true when p is true or q is true
$p \to q$	true when p being true means q is true

Ex: Let p be the statement "today is Monday". Then $\sim p$ is "today is not Monday".

Ex: Let p be "all lions are cats", q be "some lions are cats" and r be "no lions are cats". Then:

$\sim p$ is "some lions are not cats".

$\sim q$ is "no lions are cats".

$\sim r$ is "some lions are cats".

Ex If p is "today is Monday" and q is "it is raining", then $p \wedge q$ is "today is Monday and it is raining".

Ex: $p \vee q$ is "today is Monday or it is raining". Note this is what we call inclusive or: it's okay if it's Monday and it's raining at the same time.

Ex: $p \to q$ is "If today is Monday, then it is raining". Note that it is irrelevant if today is not Monday. The point is that $p \to q$ is true if p is false.

Ex: If p is "the Sun is blue" and q is "humans are mammals", then $p \to q$ is true.

$p \to q$

- p is true
  - q is true
    - $p \to q$ is true
  - q is false
    - $p \to q$ is false
- p is false
  - $p \to q$ is true

Def: In $p \Rightarrow q$, p is called the antecedent and q is called the consequent.

Ex: "If $\underbrace{\text{I exercise}}_{\text{antecedent}}$, then $\underbrace{\text{I am healthy}}_{\text{consequent}}$"

1.3: Truth Tables

<u>Def</u>: A truth table is a way to write down exactly when a compound statement is true.

<u>Ex</u>: The truth table for $p \wedge q$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex: The truth table for $p \to q$:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that p being true does not make q true. But it does make $p \to q$ true (sometimes we say $p \to q$ is "vacuously" true).

Ex: the truth tables for $\sim p$ and $p \vee q$.

p	$\sim p$
T	F
F	T

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex: find the truth table for

$$p \wedge \sim (q \vee r).$$

In general, if we have n variables, we need 2^n rows.

Here, $n=3$, so we need 8 rows.

p	q	r	$q \vee r$	$\sim(q \vee r)$	$p \wedge \sim(q \vee r)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

Ex: If you are a lion, then you are a cat.

Comment: In an implication $p \to q$, we say either:

- p is sufficient for q.
- q is necessary for p.

Ex: "Being a doctor is necessary for being a surgeon" is the same as "If you are a surgeon, then you are a doctor."

Def: Let p and q be statements. We say p and q are logically equivalent, written $p \equiv q$, if p is true exactly when q is true.

Ex: Show for any p and q,

$$p \to q \equiv (\sim p) \vee q.$$

p	q	$\sim p$	$p \to q$	$(\sim p) \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same

<u>Comment</u>: $p \equiv q$ does <u>not</u> mean that p and q are the same statement.

<u>Ex</u>: If p is "Air contains oxygen" and q is "An apple is fruit", then both p and q are <u>always</u> true, so $p \equiv q$.

<u>Theorem</u> (De Morgan's Laws):

① $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

② $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

Proof: Write the truth tables for $\sim(p \wedge q)$, $\sim p \vee \sim q$, $\sim(p \vee q)$, and $\sim p \wedge \sim q$.

p	q	$\sim p$	$\sim q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	F	F	F	F
T	F	F	T	T	T	F	F
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

same

same

Ex: Find the negation of "It is Friday and I receive a paycheck".

If p is "It is Friday" and q is "I receive a paycheck",

then $\sim(p \wedge q) \equiv \sim p \vee \sim q$, which is "It is not Friday or I don't receive a paycheck".

1.4: More About Conditionals

<u>Def</u>: Consider the conditional $p \to q$.

1. The converse of $p \to q$ is $q \to p$.

2. The inverse of $p \to q$ is $\sim p \to \sim q$.

3. The contrapositive of $p \to q$ is $\sim q \to \sim p$.

Ex: If you are a lion, then you are a cat.

Converse: If you are a cat, then you are a lion.

Inverse: If you are not a lion, then you are not a cat.

Contrapositive: If you are not a cat, then you are not a lion.

Ex: Find the contrapositive of

"Being a doctor is necessary for being a surgeon", and express it using either necessary or sufficient syntax.

Step 1: convert into an if-then.

Since the necessary refers to the consequent (the "then" part), we have "If you are a surgeon, then you are a doctor".

Step 2: Take the contrapositive.

"If you are not a doctor, then you are not a surgeon".

Step 3: convert to necessary/sufficient

Let's use sufficient: that refers to the antecedent, so we have

"Not being a doctor is sufficient for not being a surgeon".

Theorem: For any statements p and q,

(1) $p \to q \equiv \sim q \to \sim p$

(2) $q \to p \equiv \sim p \to \sim q$

So:

Original $\equiv$ Contrapositive.

Converse $\equiv$ Inverse.

Comment: A statement "p only if q" is equivalent to "If p, then q".

<u>Comment</u>: A statement "p if and only if q" is equivalent to "If p, then q, and if q, then p". This means $p \equiv q$.

Sometimes you'll see $p \leftrightarrow q$ or $p \Leftrightarrow q$ — they mean the same thing.

15: Analyzing Arguments

Ex: 1. All humans are mortal.
2. Socrates is a human.

Socrates is mortal.

If P_1 is "All humans are mortal" and P_2 is "Socrates is a human", and C is "Socrates is mortal", then the claim of this argument is $P_1 \wedge P_2 \to C$.

If $P_1 \wedge P_2 \to C$ is always true, then the argument is valid. Otherwise, it's invalid.

To do this, we need to write P_1, P_2, and C as compound statements. So, let p be "you are a human", q be "you are a mortal", and r be "you are Socrates". Then:

$$P_1 \equiv p \to q \qquad P_2 \equiv r \to p$$
$$C \equiv r \to q$$

p	q	r	P_1	P_2	C	$P_1 \wedge P_2$	$P_1 \wedge P_2 \to C$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	F	F	T
T	F	F	F	T	T	F	T
F	T	T	T	F	T	F	T
F	T	F	T	T	T	T	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

Since every entry in the final column is true, the argument is valid.

<u>Ex</u>: 1. If the defendent is innocent, they do not go to jail.

2. The defendent does not go to jail.

The defendent is innocent.

p: "the defendent is innocent"

q: "the defendent goes to jail"

$P_1 \equiv p \rightarrow \sim q$

$P_2 \equiv \sim q$

$C \equiv p$

p	q	$\sim q$	P_1	P_2	C	$P_1 \wedge P_2$	$P_1 \wedge P_2 \to C$
T	T	F	F	F	T	F	T
T	F	T	T	T	T	T	T
F	T	F	T	F	F	F	T
F	F	T	T	T	F	T	F

The F in the final column means the the argument is not valid. The situation that breaks the argument is the one in row 4: p false and q false, so the defendent was guilty and did not go to jail.

1.6 : Deductive Proof

Ex: If you are a lion, then you are a cat. If this is true, and you are a lion, then you must be a cat. In other words, we know that this argument is valid:

1. $p \to q$
2. p

$\overline{\qquad\qquad}$

q

we shouldn't need a truth table for this!

Def: The nine elementary rules of inference are:

① Modus Ponens (MP):

$$\begin{array}{l} 1.\ p \to q \\ 2.\ p \\ \hline \quad q \end{array}$$

② Modus Tollens (MT):

$$\begin{array}{l} 1.\ p \to q \\ 2.\ \sim q \\ \hline \quad \sim p \end{array}$$

③ Hypothetical Syllogism (HS):

$$\begin{array}{l} 1.\ p \to q \\ 2.\ q \to r \\ \hline \quad p \to r \end{array}$$

④ Disjunctive Syllogism (DS):

1. $p \vee q$

2. $\sim p$

---

q

⑤ Constructive Dilemma (CD):

1. $p \rightarrow q$

2. $r \rightarrow s$

3. $p \vee r$

$q \vee s$

⑥ Absorption (Abs):

1. $p \rightarrow q$

$p \rightarrow (p \wedge q)$

⑦ Simplification (Simp.):

$$\begin{array}{l} 1.\ p \wedge q \\ \hline p \end{array}$$

⑧ Conjunction (Conj.):

$$\begin{array}{l} 1.\ p \\ 2.\ q \\ \hline p \wedge q \end{array}$$

⑨ Addition (Add.):

$$\begin{array}{l} 1.\ p \\ \hline p \vee q \end{array}$$

Ex: Show this argument is valid:

1. $p \to q$
2. $q \to r$
3. $\sim r$
4. $p \vee s$

---

s

1. $p \to q$	
2. $q \to r$	
3. $\sim r$	
4. $p \vee s$.	
5. $p \to r$	1, 2, HS
6. $\sim p$	5, 3, MT
7. s	4, 6, DS

<u>Ex</u>: 1. $\sim(A \lor B)$

2. $\sim C \rightarrow (A \land \sim D)$

$C \land \sim A$

1. $\sim(A \lor B)$	
2. $\sim C \rightarrow (A \land \sim D)$	
3. $\sim A \land \sim B$	1, De Morgan
4. $\sim A$	3, Simp.
5. $\sim C \rightarrow A$	2, Simp.
6. $\sim(\sim C)$	5, 4, MT
7. C	6
8 $C \land \sim A$	7, 4, Conj.

Chapter 2: Sets and Counting

<u>Def</u>: A set is a collection of objects such that any object is either in the set or not.

<u>Ex</u>: All integers less than 10 and bigger than -5 form a set.

<u>Ex</u>: All good bands do not form a set, because it's not well-defined whether a given band is in the set.

<u>Def</u>: A set in set-builder notation is written in one of two ways:

(1) Listing every element in the set.

<u>Ex</u>: $A = \{1, 2, 3, 4\}$.

$B = \{1, 2, \ldots, 20\}$

(2) Listing a general element and the conditions it satisfies.

<u>Ex</u>: $C = \{\underline{x} \,\underline{|}\, \underline{x^2 = 4}\} = \{2, -2\}$

"The set of all $\underline{x}$ <u>such that</u> $\underline{x^2 = 4}$"

Def: If x is an element of a set A, we write $x \in A$ ("x is in A").
If not, we write $x \notin A$.

Comment:

1. Sets don't care about the order of their elements.

Ex: $\{1, 2, 3\} = \{2, 3, 1\}$.

2. Sets don't care about duplicate elements.

Ex: $\{1, 2, 3\} = \{1, 1, 1, 1, 2, 2, 3\}$.

3. Sets are equal if they have exactly the same elements.

Def: The cardinality of a set A is the number of distinct elements in A, written $n(A)$ (or more commonly, $|A|$).

Ex $n(\{a,b,c\}) = 3$.

$n(\{d, d, c, b, b\}) = 3$.

Def: A set A is a subset of a set B if every element of A is an element of B, written $A \subseteq B$. If $A \neq B$, we can write $A \subset B$ (similar to $\leq$ and $<$).

Ex:

$\{1\} \subseteq \{1,2\}$.

$\{a,b,c\} \subseteq \{a,b,\dots,y,z\}$.

$\{a,b,c\} \subseteq \{a,b,c\}$.

$\{1\} \subset \{1,2\}$ since $\{1\} \neq \{1,2\}$.

proper subset

Def: A set A is finite if $n(A)$ is a finite number.

Theorem: If A and B are finite and $A \subseteq B$, then $n(A) \leq n(B)$.

Def: The empty set is $\emptyset = \{\}$.

Def: (1) The natural numbers are

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}.$$

(2) The integers are

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

(3) The rational numbers are

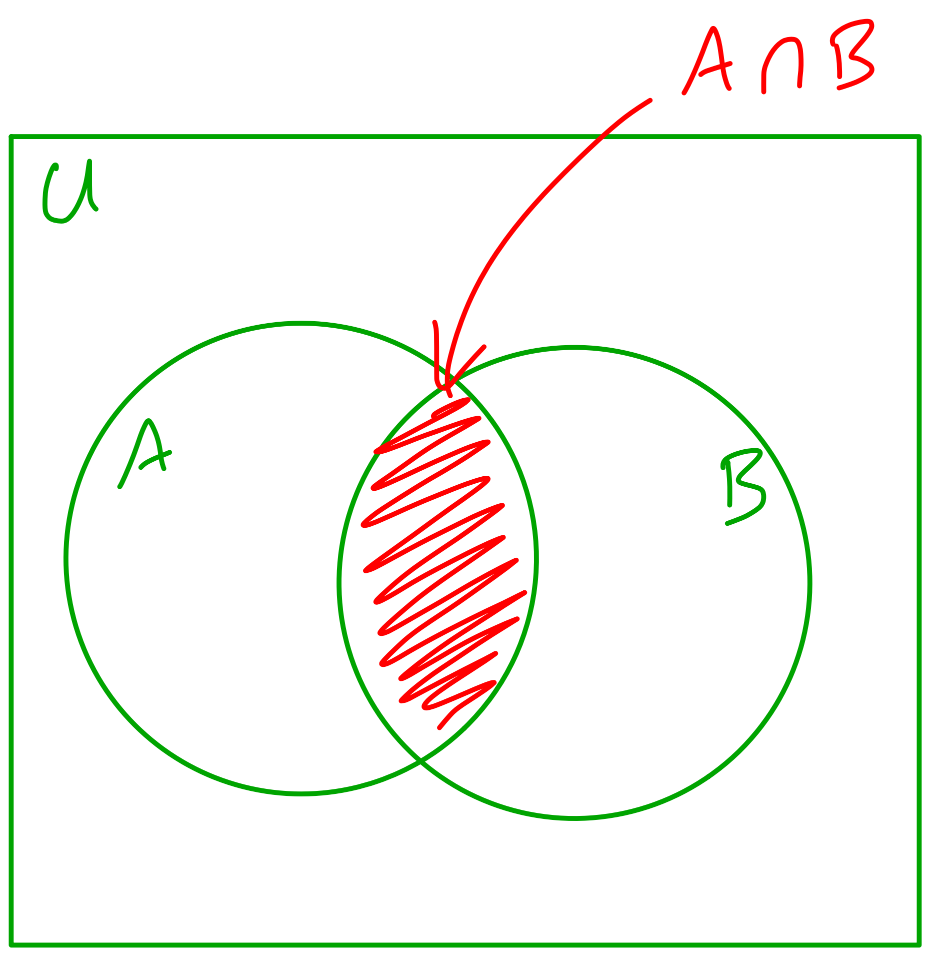
$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

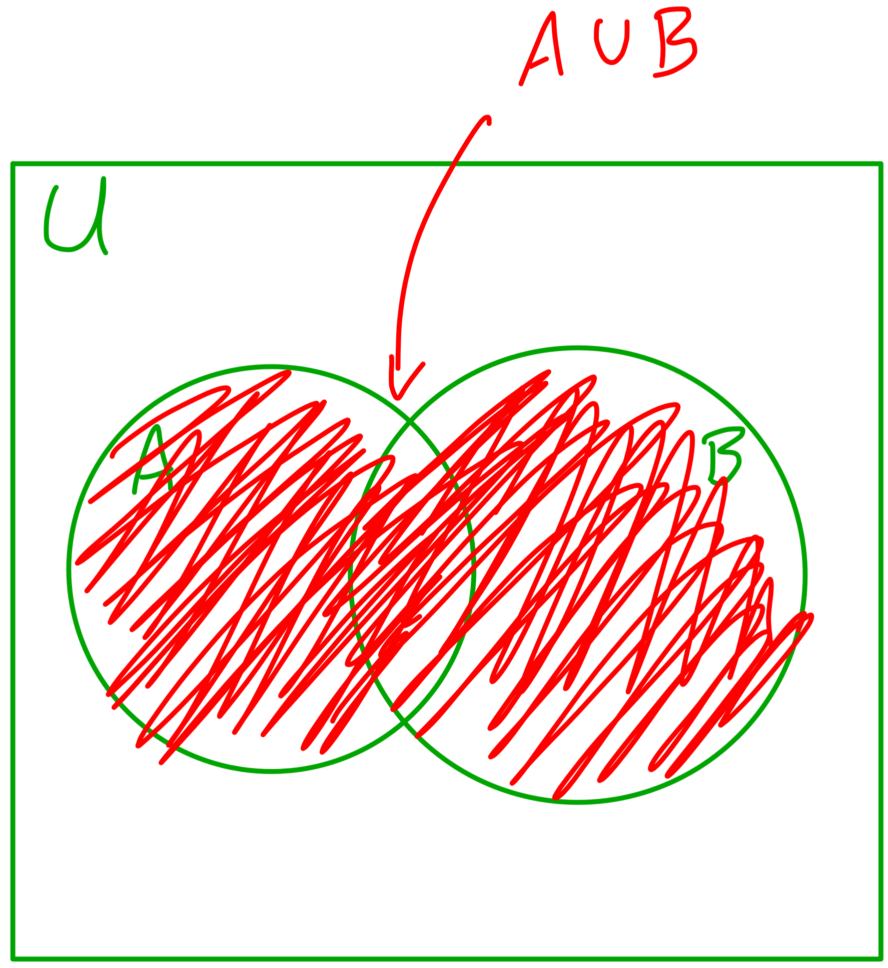
(4) The real numbers are the numbers on the number line, written $\mathbb{R}$.

(5) The complex numbers are $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$.

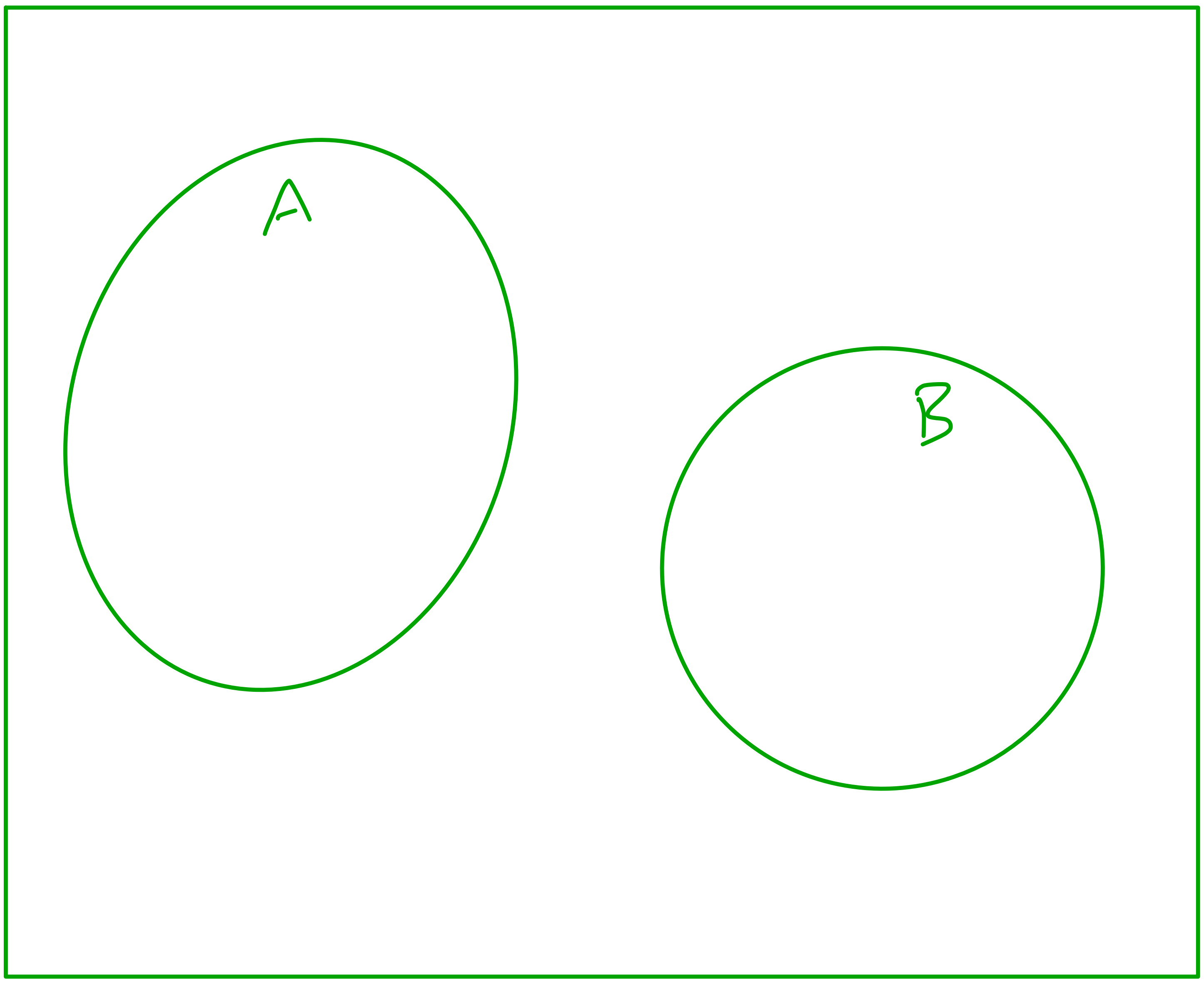
Comment: $\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Def: Let A and B be sets.
The intersection of A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
The union of A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. Note the similarity between $\cap$ and $\wedge$, and $\cup$ and $\vee$.

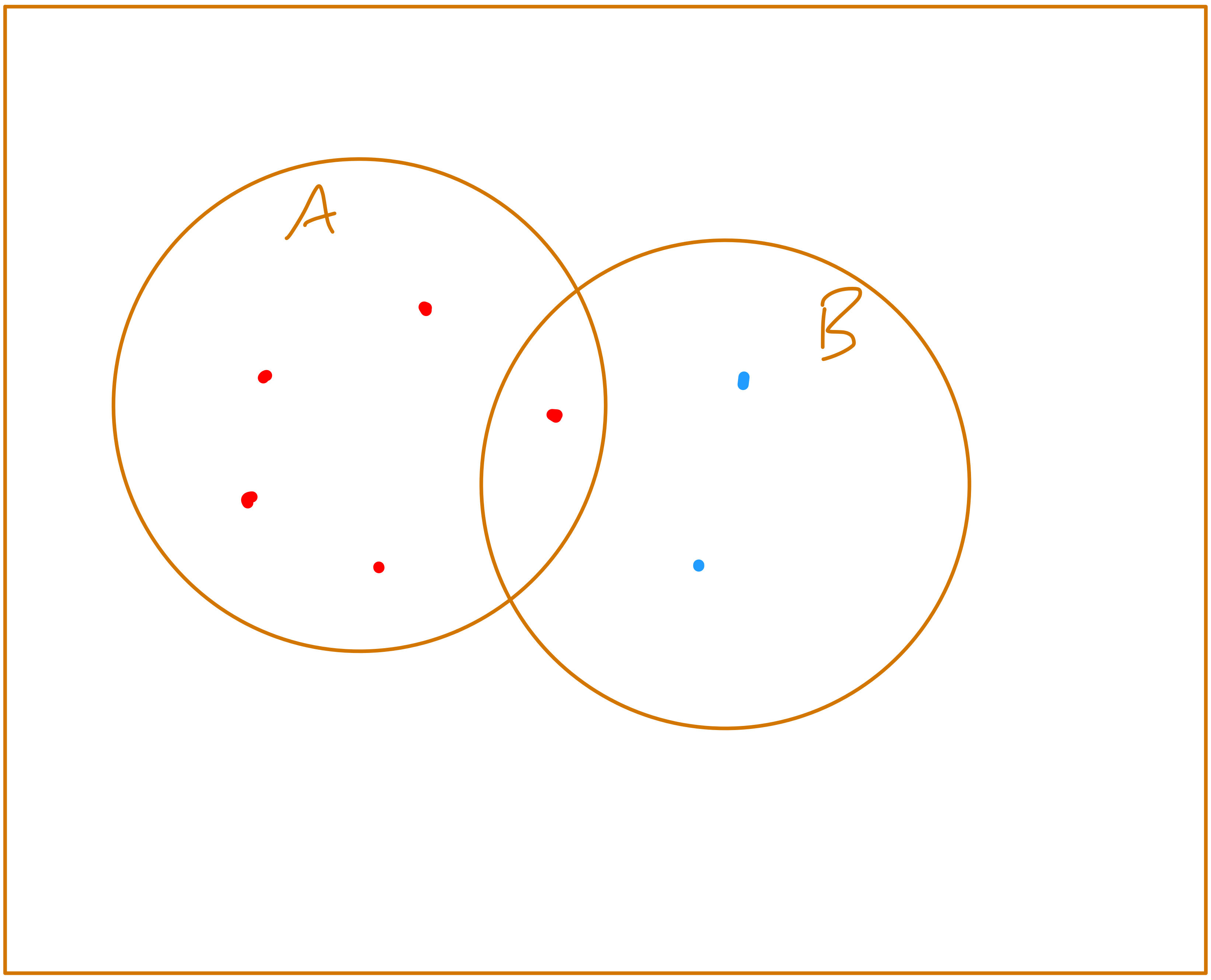


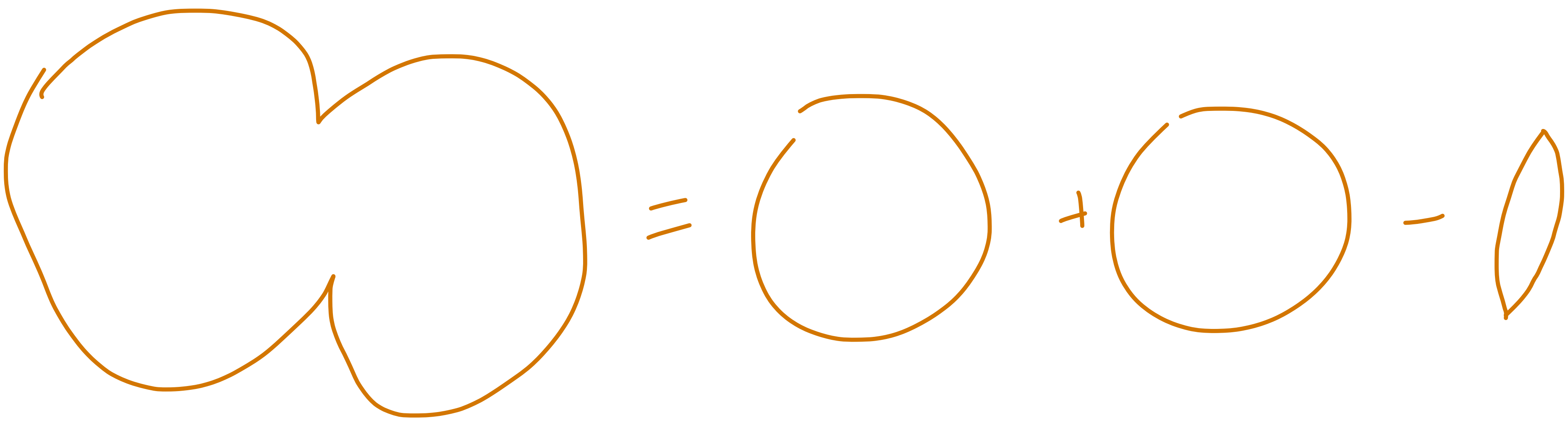


Def: Two sets A and B are disjoint if $A \cap B = \emptyset$.

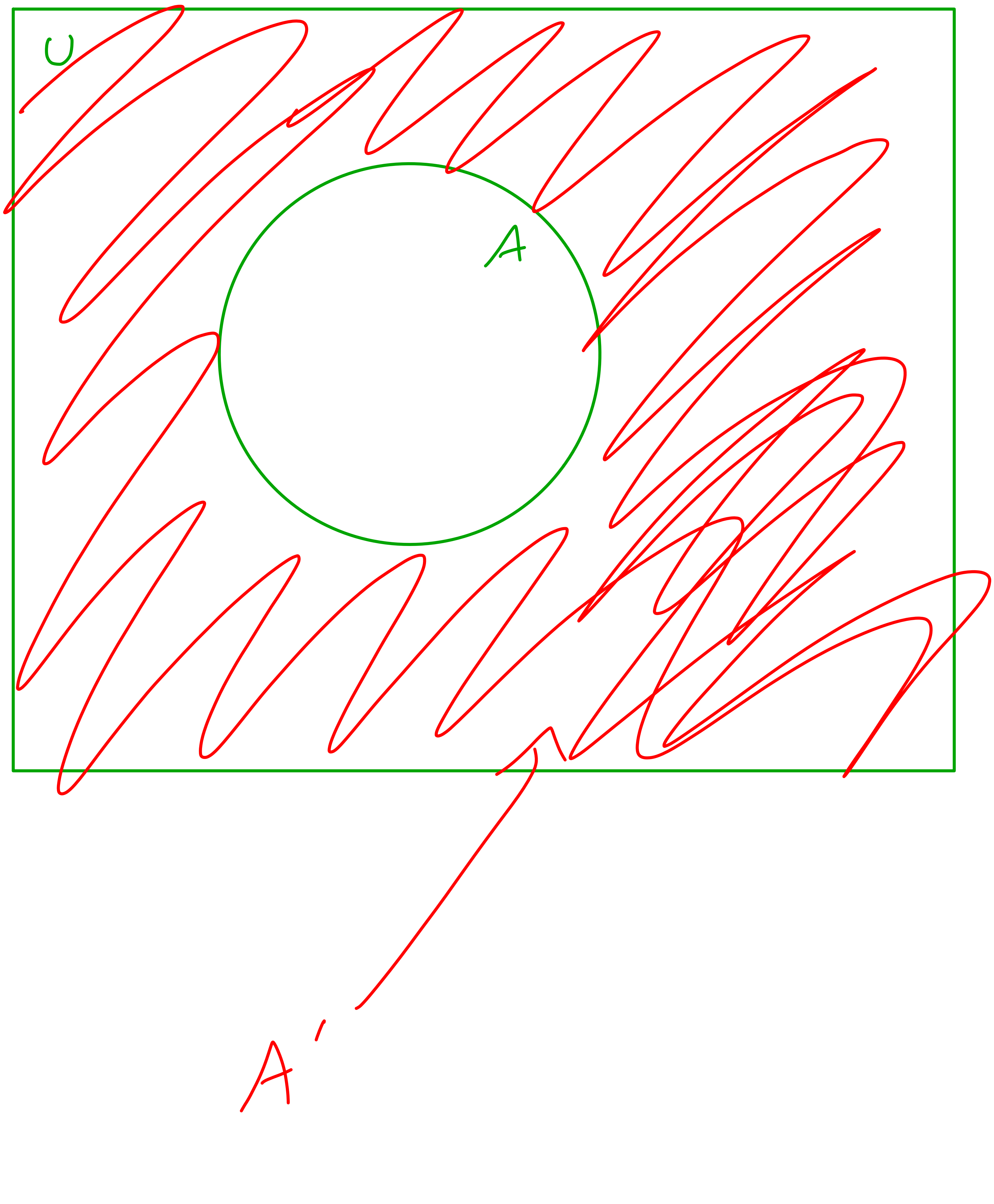


Theorem: Let A and B be finite sets. Then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.


A
B



Def: The complement of a set A is the set $A' = \{x \in U \mid x \notin A\}$.



Comment:

Logic	Set theory
$\wedge$	$\cap$
$\vee$	$\cup$
$\sim$	$'$
$\rightarrow$	$\subseteq$