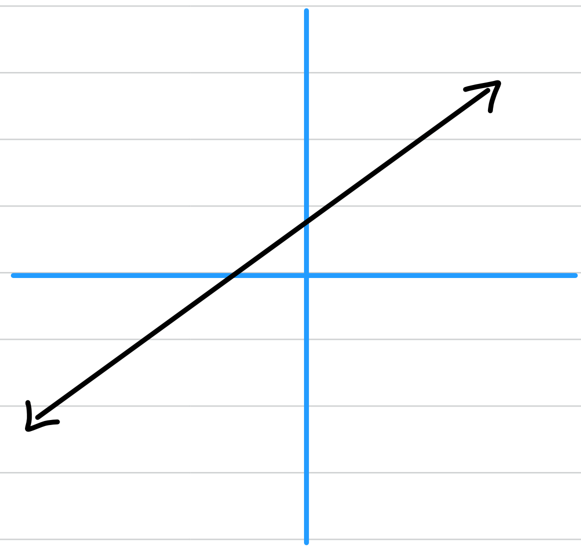
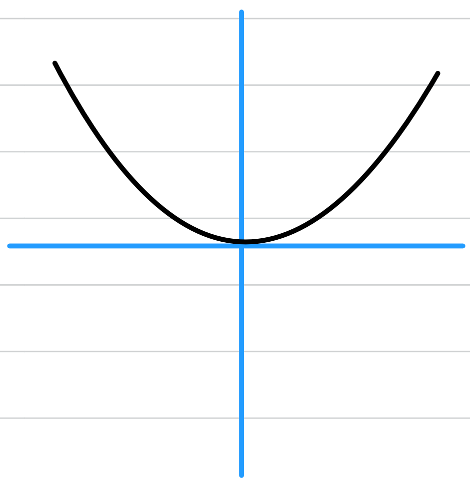
<u>Comment</u>: Recall from precalc or 111:

Lines: $f(x) = mx + b$

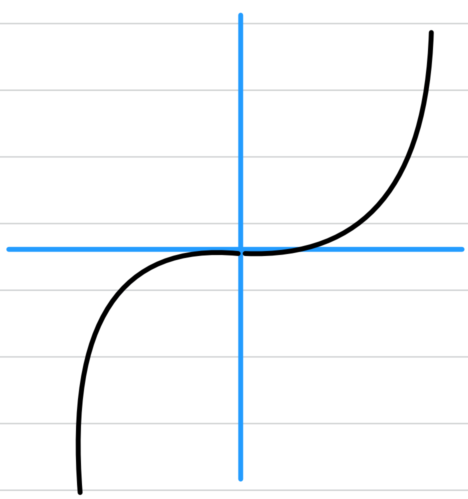


Power functions:

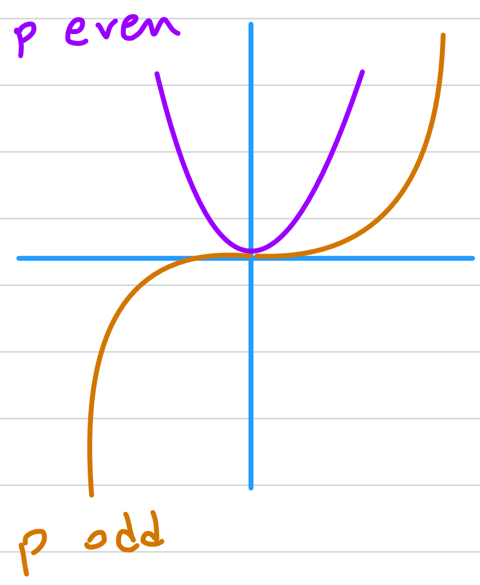
$f(x) = x^2$



$f(x) = x^3$



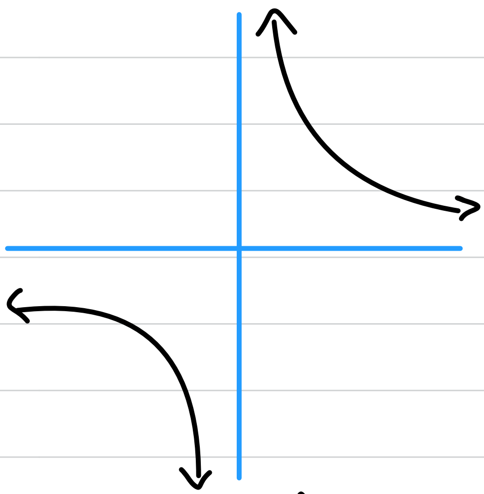
$f(x) = x^p$

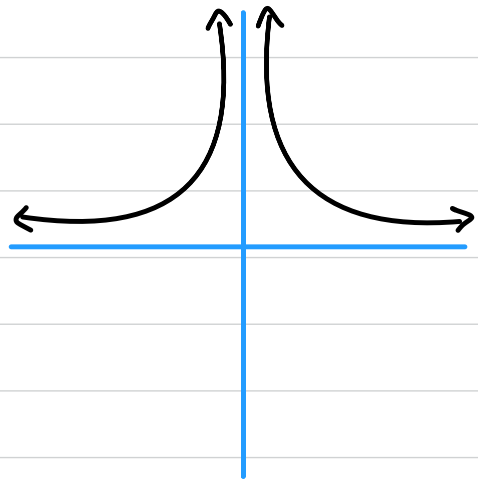


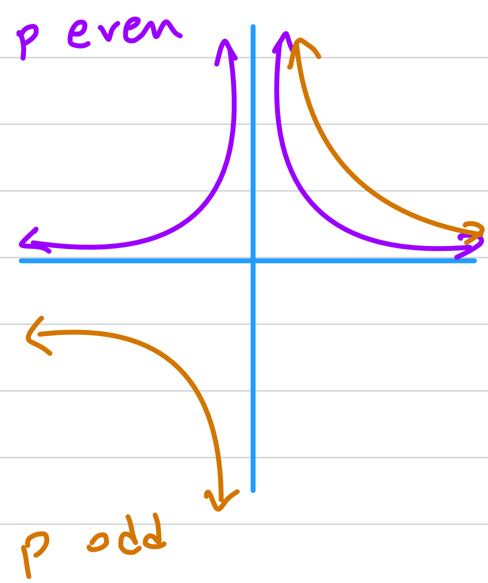
e.g. $x^3, x^5, x^7, x^9, \dots$

Reciprocal functions:

$f(x) = \frac{1}{x}$ $\qquad$ $f(x) = \frac{1}{x^2}$ $\qquad$ $f(x) = \frac{1}{x^p}$



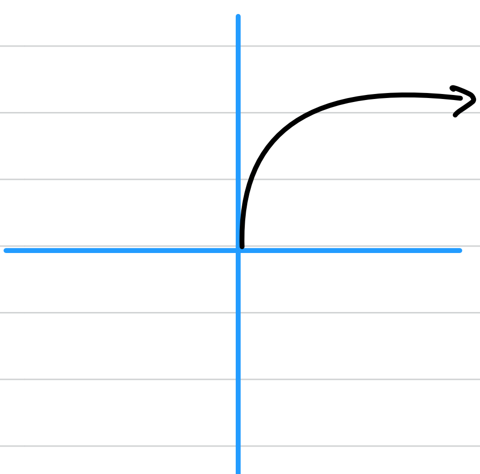


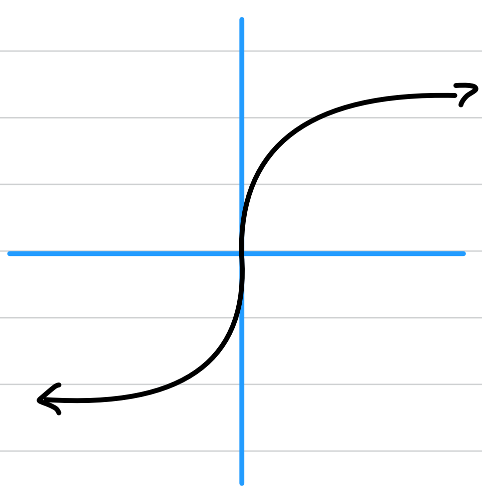


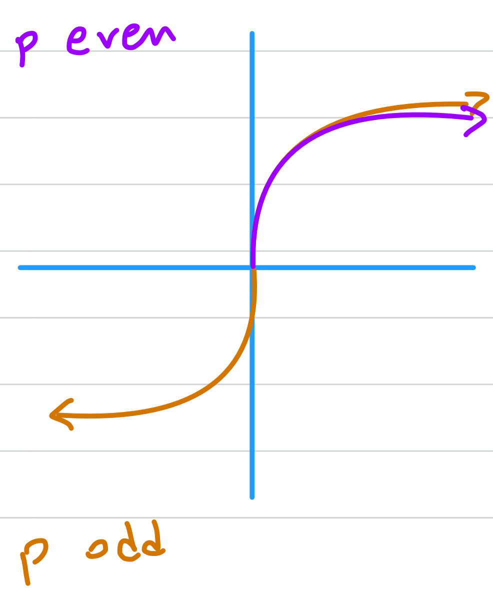
domain : $(-\infty, 0) \cup (0, \infty)$

Radical functions:

$f(x) = \sqrt{x}$ $\qquad$ $f(x) = \sqrt[3]{x}$ $\qquad$ $f(x) = \sqrt[p]{x}$



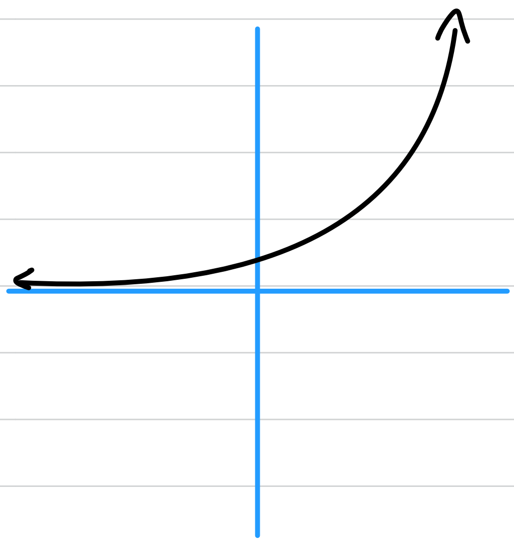


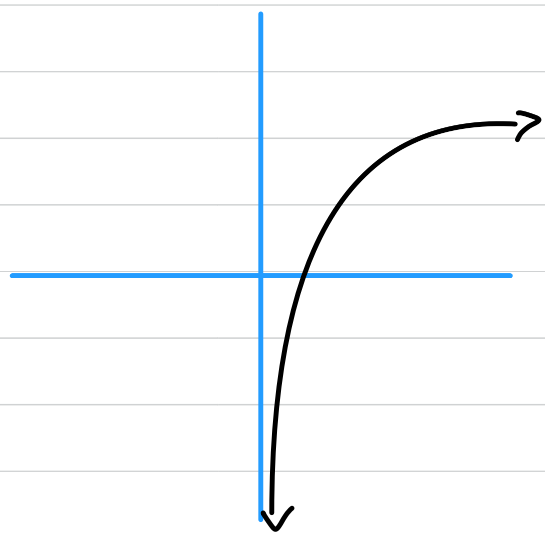


domain : $[0, \infty)$

Exponential and logarithmic functions:

$f(x) = e^x$ $\qquad$ $f(x) = \ln(x)$





<u>Comment</u>: These are called <u>elementary functions</u>. There are a lot more than them, but they are the "building blocks" of all functions.

<u>Def</u>: A function f is <u>even</u> if for all x in its domain, $f(-x) = f(x)$. The

function is <u>odd</u> if $f(-x) = -f(x)$. A function's evenness or oddness is called its <u>parity</u>. Most functions are neither even nor odd.

<u>Ex</u>: $f(x) = x$ is odd, since $f(-x) = -x = -f(x)$.

$f(3) = 3$
$f(-3) = -3$

<u>Ex</u>: Is $f(x) = x^4$ even, odd, or neither?

$$f(-x) = (-x)^4 = (-x)(-x)(-x)(-x) = x^4 = f(x)$$

$\Rightarrow f$ is even.

$f(3) = 3^4 = 81$

$f(-3) = (-3)^4 = 81$

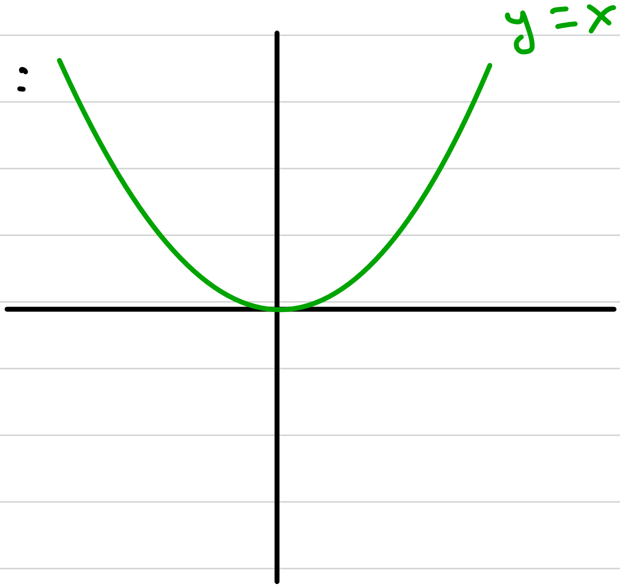
Prop: $f(x) = x^p$ is an even function if p is even, and it's an odd function if p is odd.

Ex: $f(x) = x^4 + x$ is neither even nor odd, since $f(-x) = (-x)^4 + (-x) = x^4 - x$,

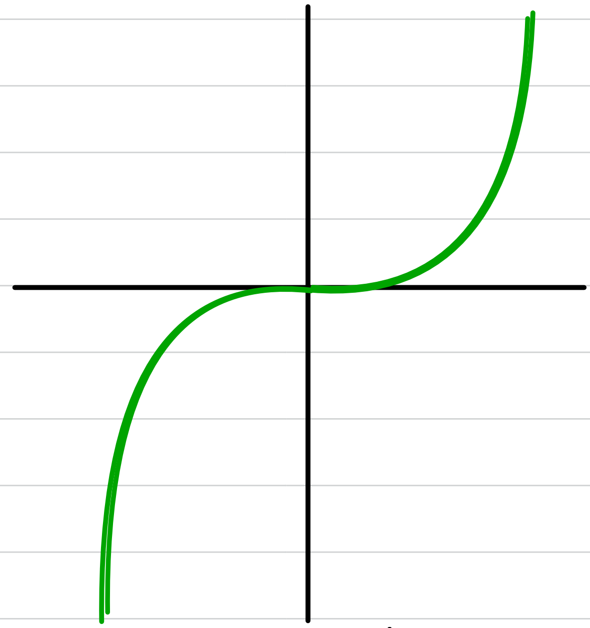
and $\underbrace{x^4 - x \neq f(x)}_{\text{not even}}$ and $\underbrace{x^4 - x \neq -f(x)}_{\text{not odd}}$, where $-f(x) = -x^4 - x$

Prop: Even functions have graphs that are symmetric about the y-axis, and odd functions have graphs that are symmetric about $180°$ rotations about the origin.

Ex:



even



odd

Ex: Show $g(x) = x e^{-x^2}$ is odd.

$$g(-x) = (-x)\, e^{-(-x)^2}$$

$$= (-x)\, e^{-(-x)(-x)}$$

$$= -x\, e^{-x^2}$$

$$= -g(x)$$

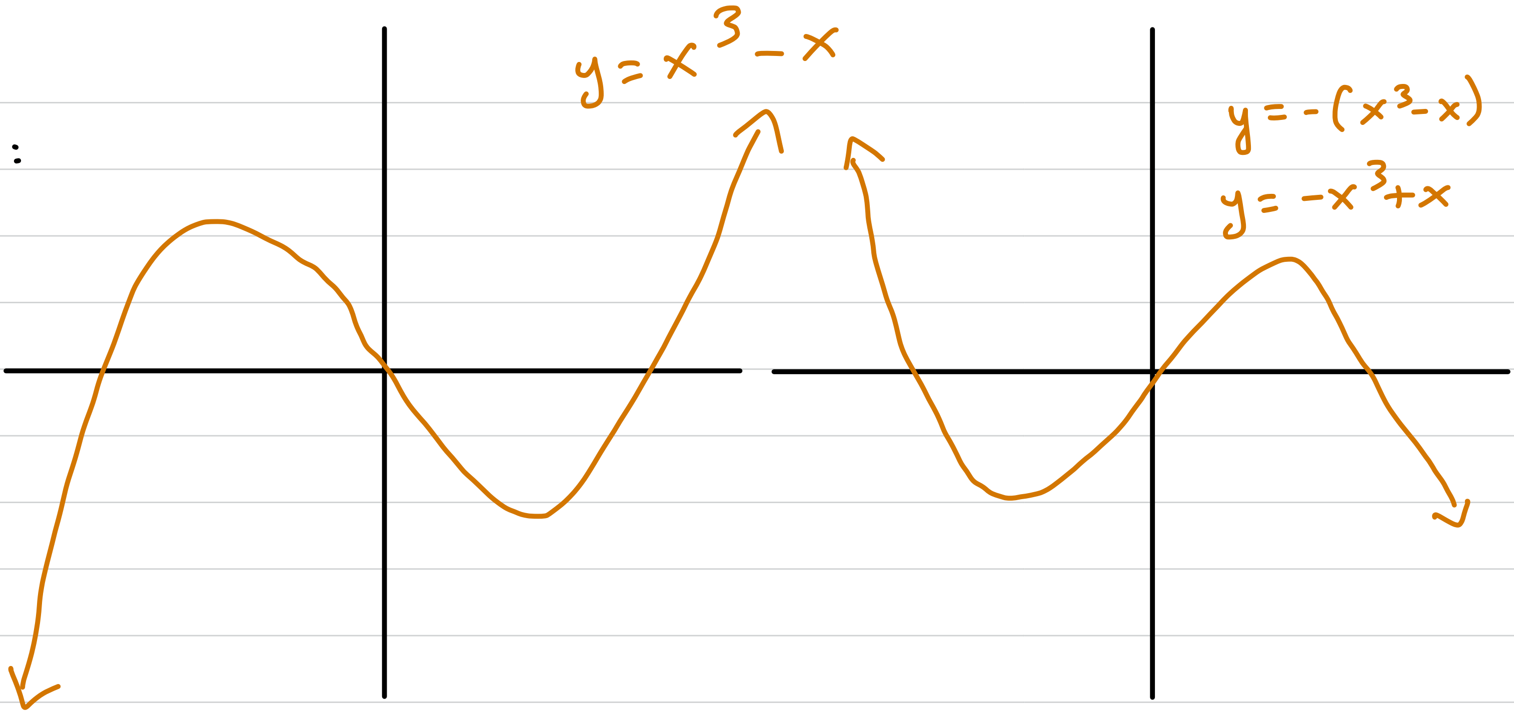
$\Rightarrow$ g is an odd function.

Vertical Transformations

Comment: If $h(t)$ gives your height in inches t years after you're born. How can we modify h so that it gives your height in centimeters?

Prop: Let f be a function. The graph of $y=-f(x)$ is the same as the graph of $y=f(x)$, but vertically reflected about the x-axis.

Ex:



$f(x) = x^3 - x$

<u>Def</u>: If we have a function $f(x)$ and we're graphing $-f(x)$, $f(x)$ is called the <u>parent function</u> and the negation is called a <u>transformation</u> of the parent function.

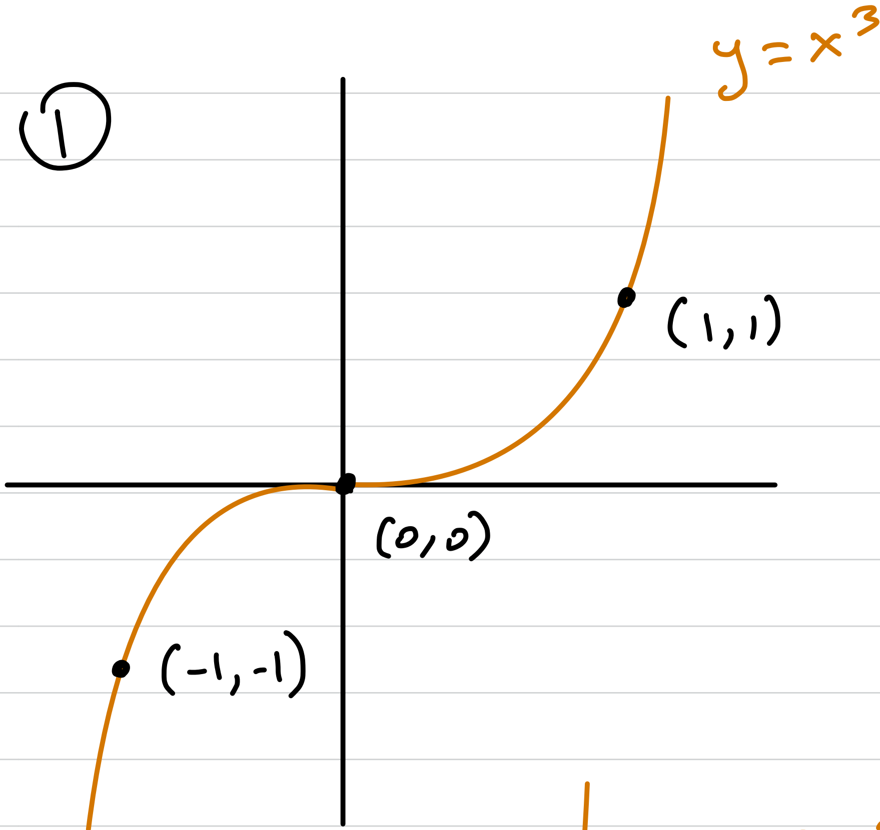
Prop: Let f be a function and c a number with $c > 0$. The graph of $y = cf(x)$ is the same as the graph of $y = f(x)$, but vertically stretched by a factor of c. What this means is that if (x, y) was a point on the graph of $y = f(x)$, then (x, cy) is a point on the graph of $y = cf(x)$.

Ex: Let $f(x) = x^3$. Graph $y = 2x^3$.

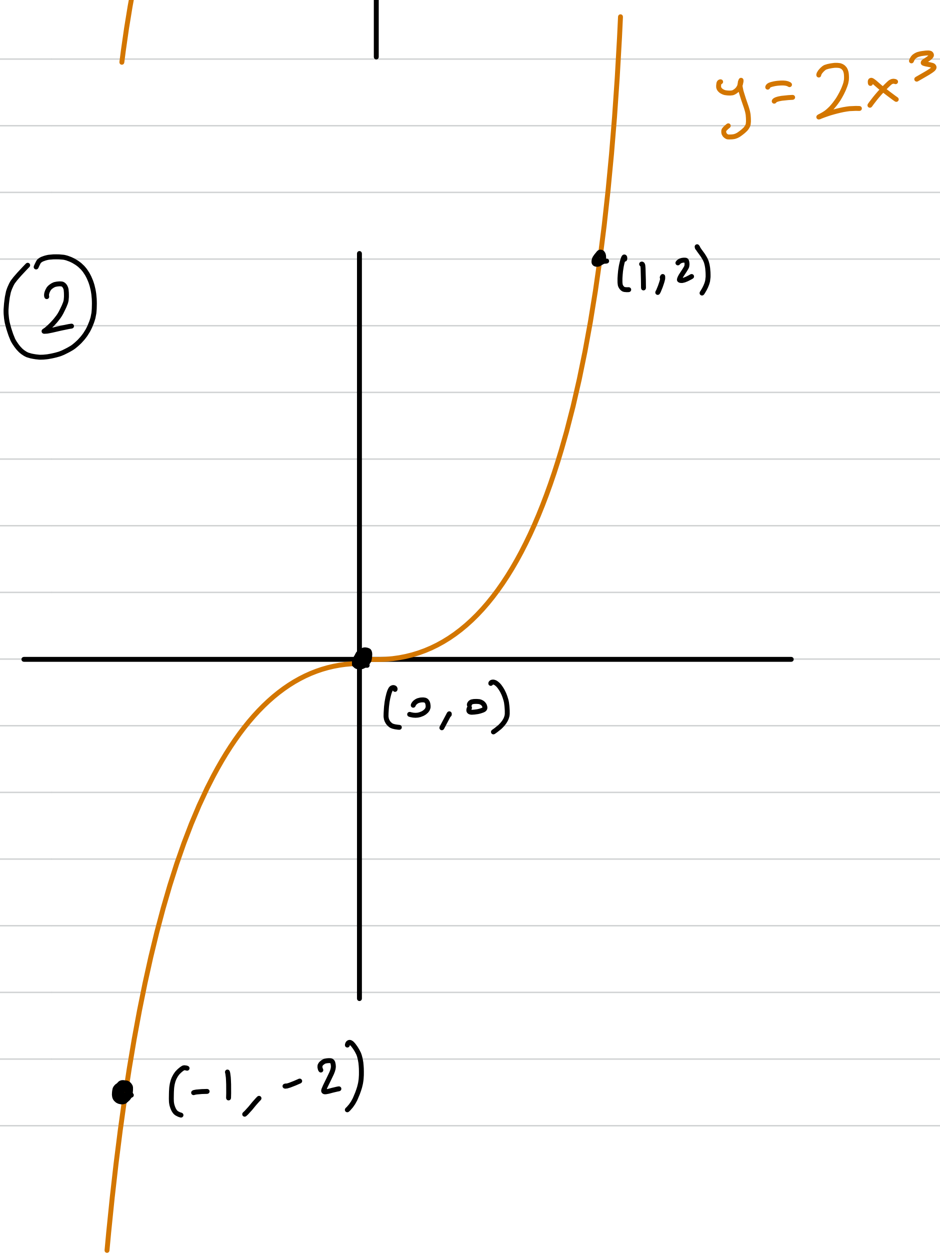
① Parent function is $y = x^3$.

② Apply a vertical stretch by a factor of 2: $y = 2x^3$

1



2



Prop: Let f be a function and d a number. The graph of $y = f(x) + d$ is the graph of $y = f(x)$ shifted d units up (or down if d is negative). This means that every point (x, y) on the graph of $y = f(x)$ becomes $(x, y+d)$ on the graph of $y = f(x) + d$.

Ex: Graph $y = -2x^2 + 3$.

When working with multiple vertical transformations, apply them starting from the one closest to the parent function and working outward.

① Parent function: $y = x^2$.

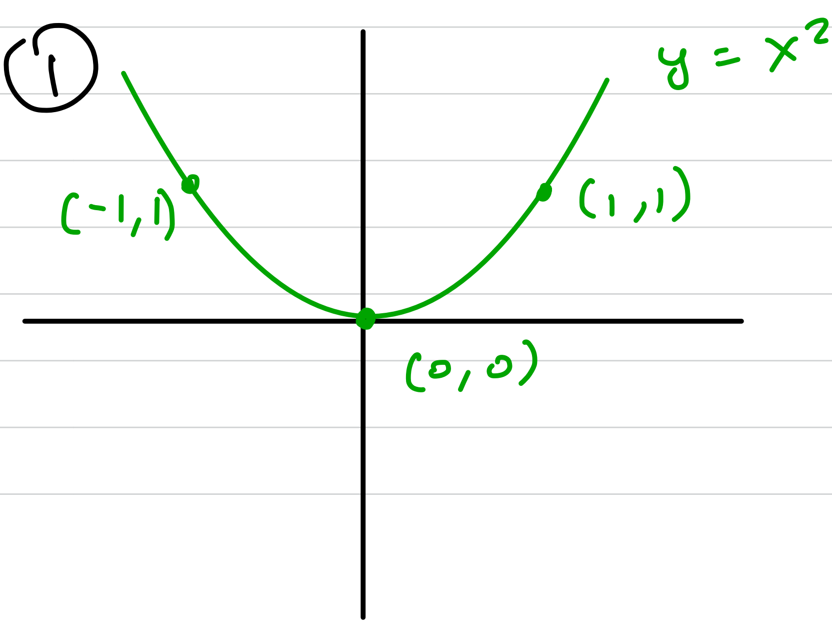
② Vertical stretch by a factor of 2: $y = 2x^2$.

③ Vertical reflection: $y = -2x^2$.

④ Vertical shift 3 units up: $y = -2x^2 + 3$.

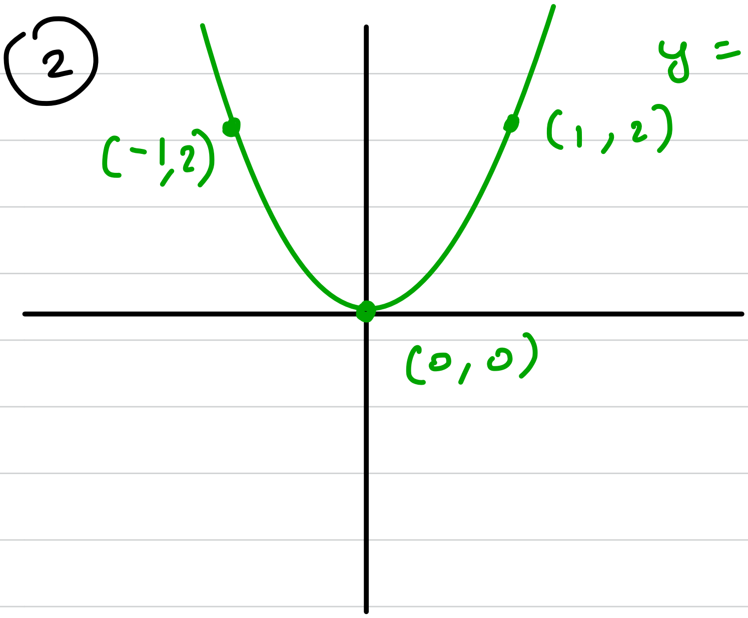
Note that this won't always be the order: $y = 2(-x^2 + 3)$ would have reflection first, then shift, then stretch.

①



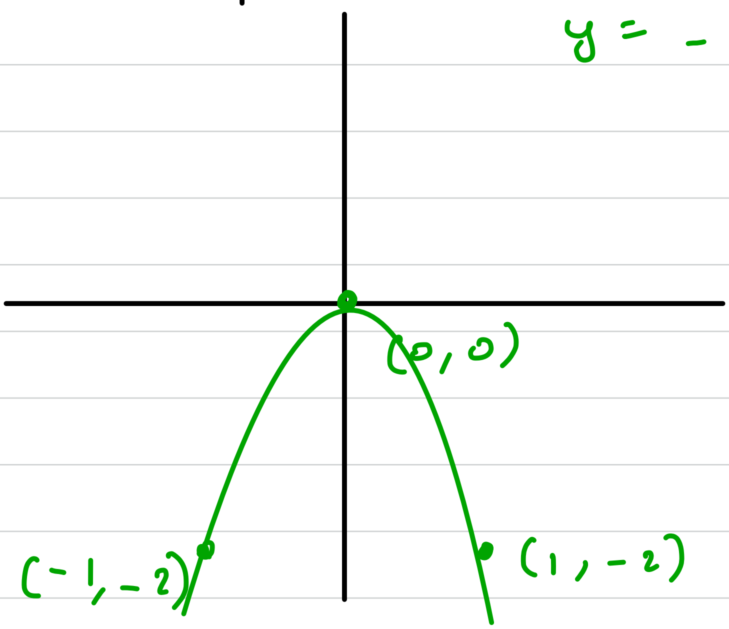
2

$y = 2x^2$



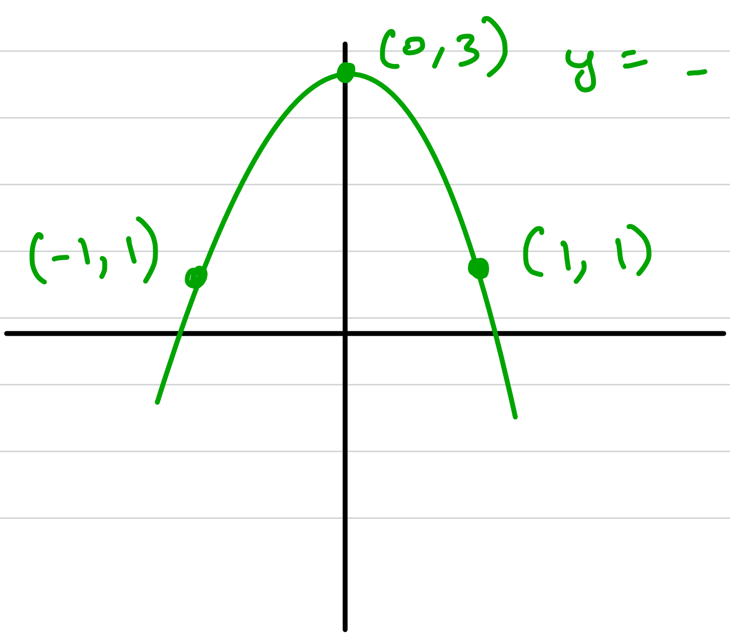
3

$y = -2x^2$



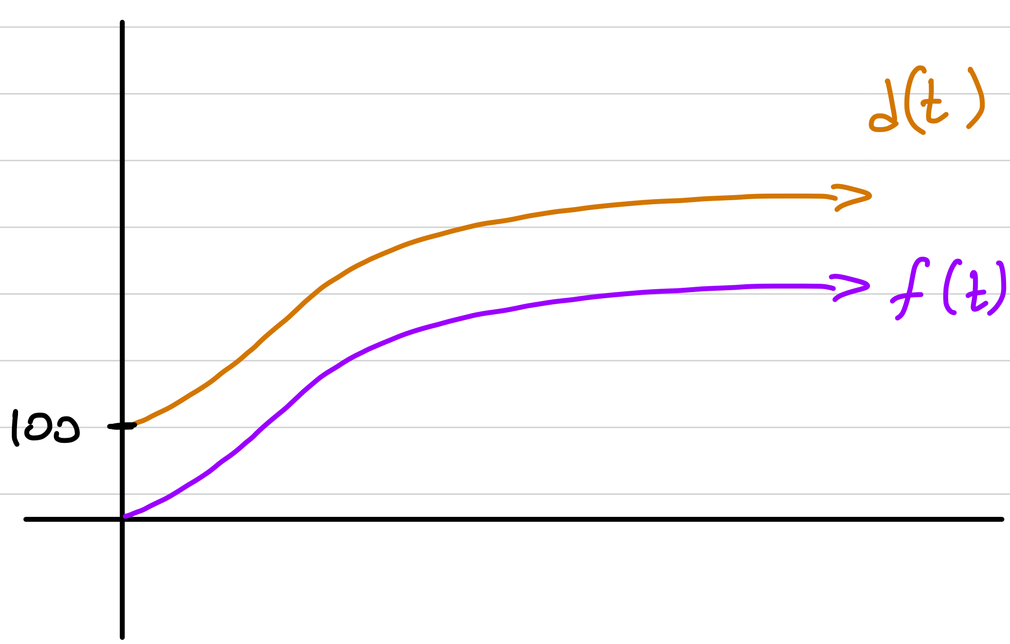
4

$y = -2x^2 + 3$



Comment: Vertical transformations are used to rescale the output of a function — e.g. $f(x)$ outputs inches, but you want feet.

Ex: A scientist measures the population of deer in a park. If the population is 100 at some point in time and t years later, the population is given by the function $d(t)$:



What does the function $f(t) = d(t) - 100$ represent?

$f(t)$ is a vertical shift of $d(t)$, down 100 units, so $f(t)$ is some sort of rescaling of the output of $d(t)$. Here, that rescaling is the number of new deer after the original 100, t years after we start measuring.

Horizontal Transformations

Comment: We can apply stretches, shifts, and reflections to the inside of a function, not just the outside

Prop: Let $f(x)$ be a function. The graph of $f(-x)$ is the graph of $f(x)$ reflected about the y-axis. The graph of $f(c \cdot x)$ for a number $c > 0$ is the graph of $f(x)$ horizontally stretched by a factor of $\boxed{\frac{1}{c}}$. The graph of $f(x+d)$ for a number d is the graph of $f(x)$ shifted to the right by $\boxed{-d}$ units.

Comment: it's not quite as simple to combine multiple horizontal transformations as it is to combine multiple vertical ones.

Ex: Graph $y = e^{-x} - 2$

① $y = e^{x}$: parent function

② $y = e^{-x}$: horizontal reflection

(Note: $-e^{x}$ would be a vertical reflection.

③ $y = e^{-x} - 2$: vertical shift 2 down

(Note: $y = e^{-(x-2)}$ would be a horizontal shift 2 right.

Horizontal transformations

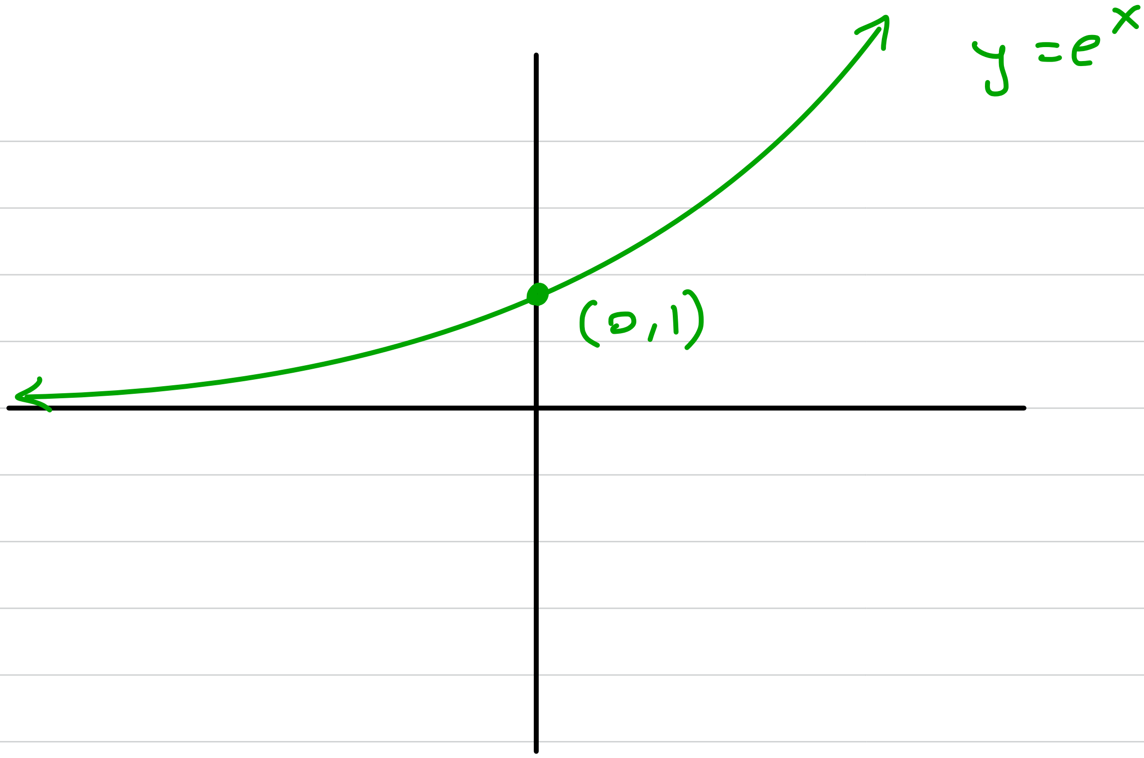
$\downarrow$

parent function

$\downarrow$

Vertical transformations

①



②

$y = e^{-x} - 2$

③