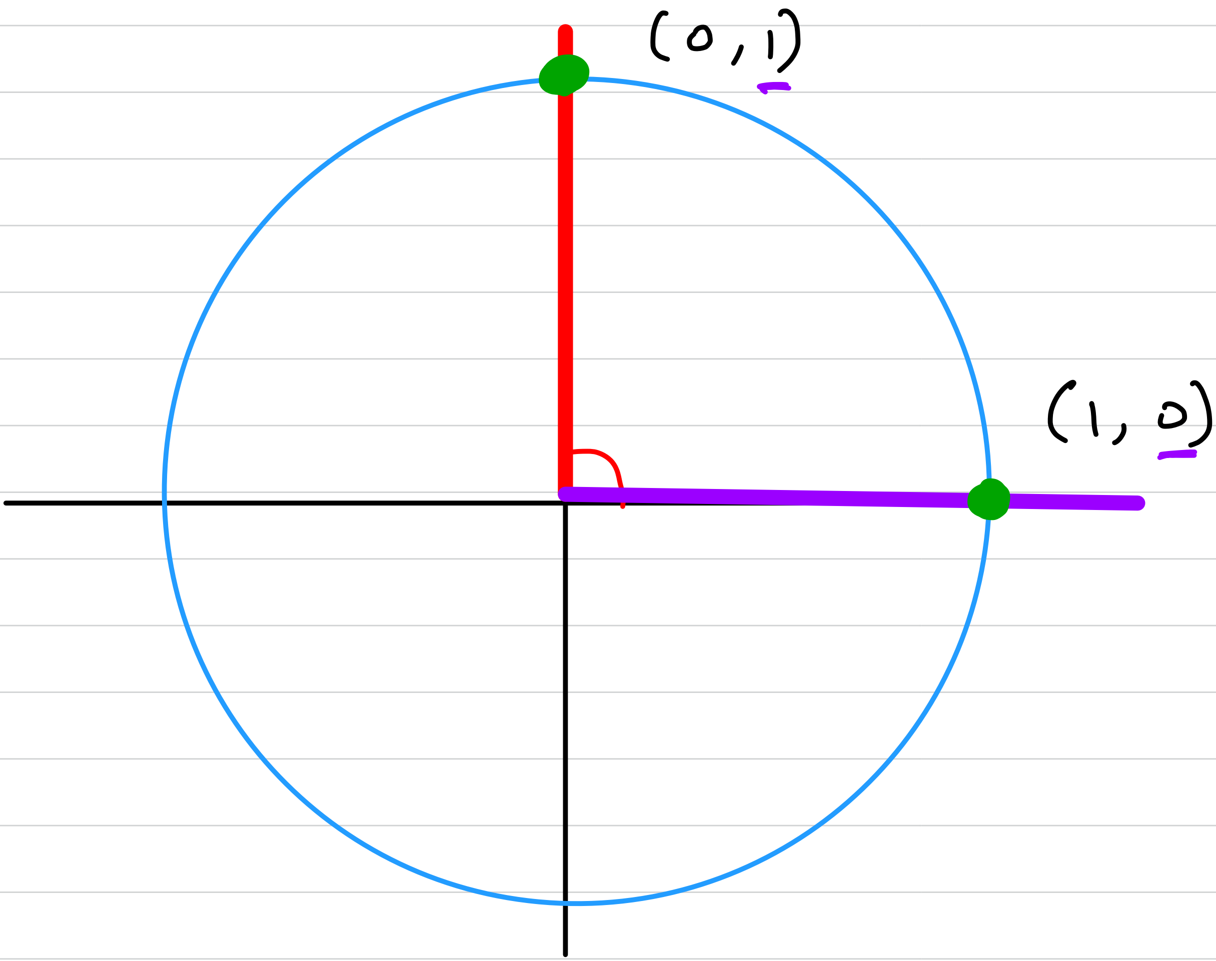
Sine and Cosine

Def: Let θ be an angle on the unit circle measured counter-clockwise from the positive x-axis. The sine and cosine of θ are the y and x coordinates of the point on the unit circle with angle θ, respectively. We write $\sin(\theta) = y$ and $\cos(\theta) = x$, where (x, y) is point on the unit circle with angle θ.

Comment: Both sin and cos take in angles and output distances.

Ex: Find $\sin(90°)$ and $\sin(0°)$.



$\sin(0°) = 0$ $\cos(0°) = 1$

$\sin(90°) = 1$ $\cos(90°) = 0$

Comment: your scientific calculator can find decimal values of sin and cos. But be careful —

most calculators have a degree mode and a "radian" mode.

For now, just make sure it's in degree mode whenever you want to calucate something with degrees

Prop: ① For any angle θ, $-1 \le \sin(\theta) \le 1$ and $-1 \le \cos(\theta) \le 1$.

② sin and cos are periodic functions with period $360°$, midline 0, and amplitude 1.

③ For any angle θ,
$$(\sin(\theta))^2 + (\cos(\theta))^2 = 1.$$

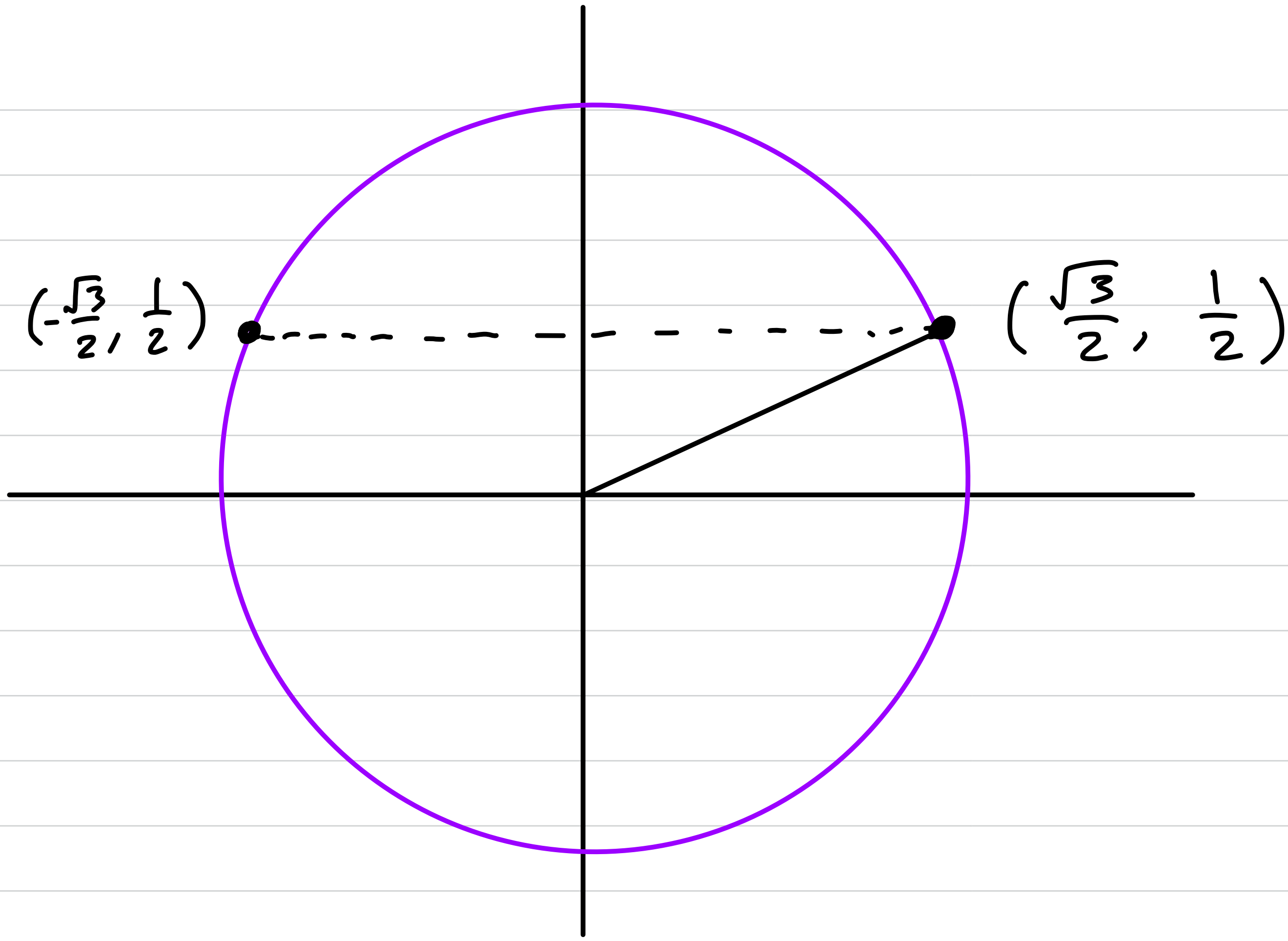
This is because $(\cos(\theta), \sin(\theta))$ is by definition a point on the unit circle.

<u>Prop</u>: Some values of sin and cos.

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

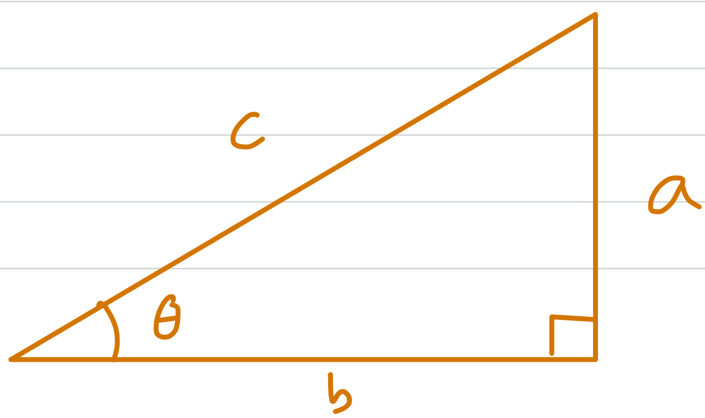
<u>Ex</u>: If $\sin\theta = 1/2$ and $0° \le \theta \le 360°$, what could $\cos\theta$ be?

Intuition says it must be $\sqrt{3}/2$. Instead, draw a circle.



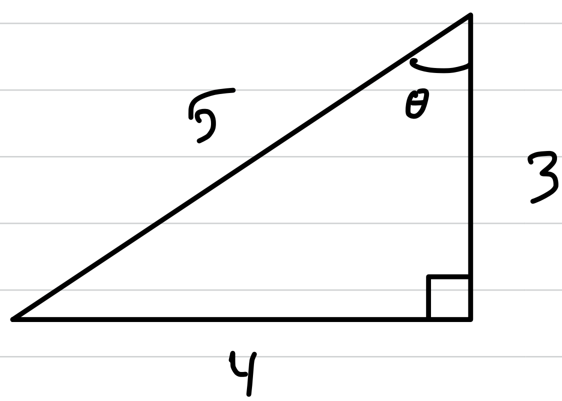
So $\cos(\theta) = \frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$.

Theorem: In the right triangle as shown, $\sin(\theta) = \frac{a}{c}$ and $\cos(\theta) = \frac{b}{c}$.



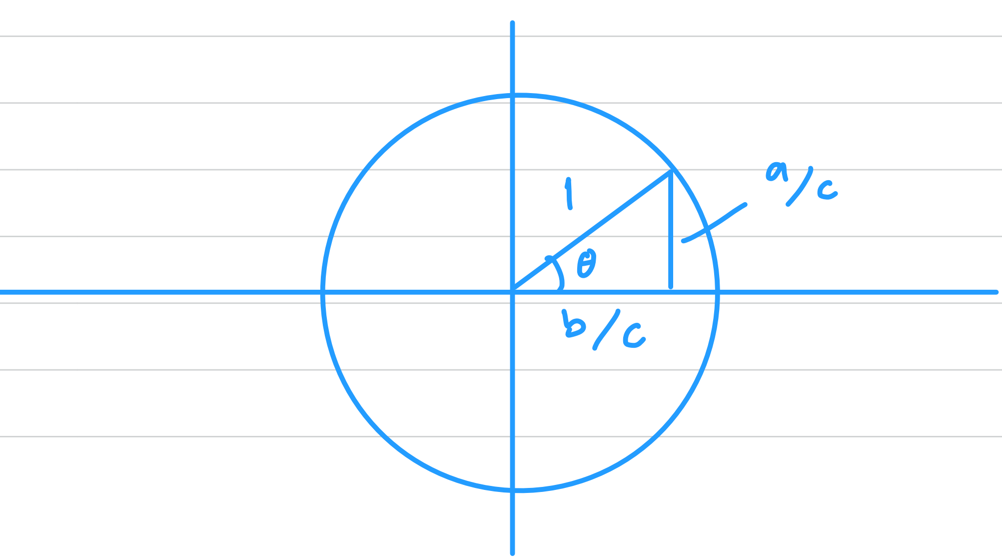
In general, $\sin(\theta) = \dfrac{\text{opp}}{\text{hyp}}$ and $\cos(\theta) = \dfrac{\text{adj}}{\text{hyp}}$ in any right triangle.

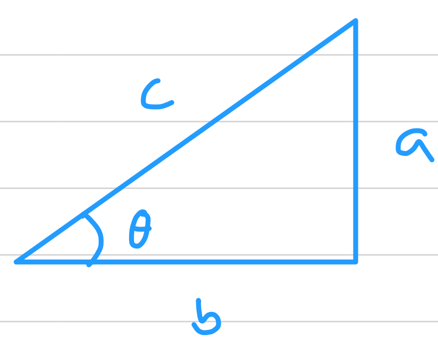
Ex:



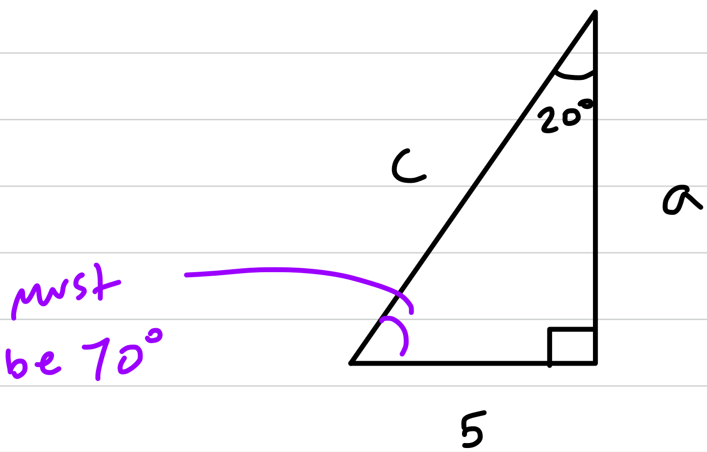
$$\sin\theta = \frac{4}{5} \qquad \cos(\theta) = \frac{3}{5}$$

Comment: This works by similar triangles





Ex: Find a and c.



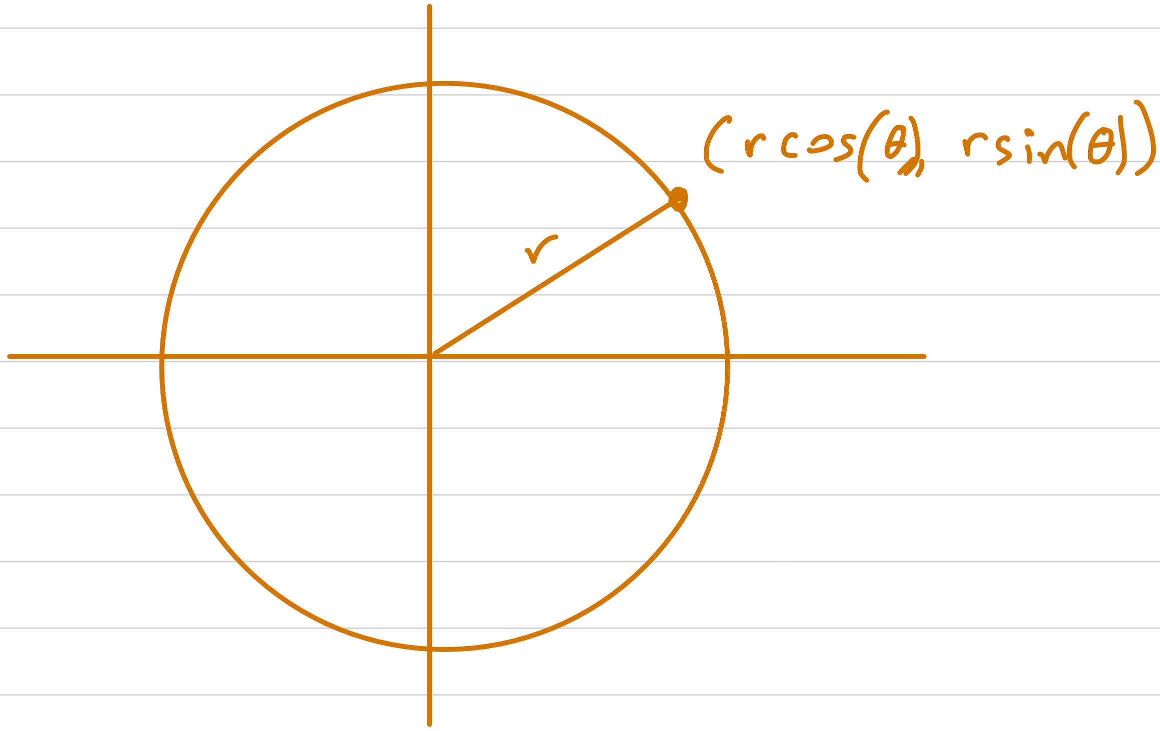
$$\sin(20°) = \frac{5}{c} \qquad\qquad \cos(20°) = \frac{a}{c}$$

calc

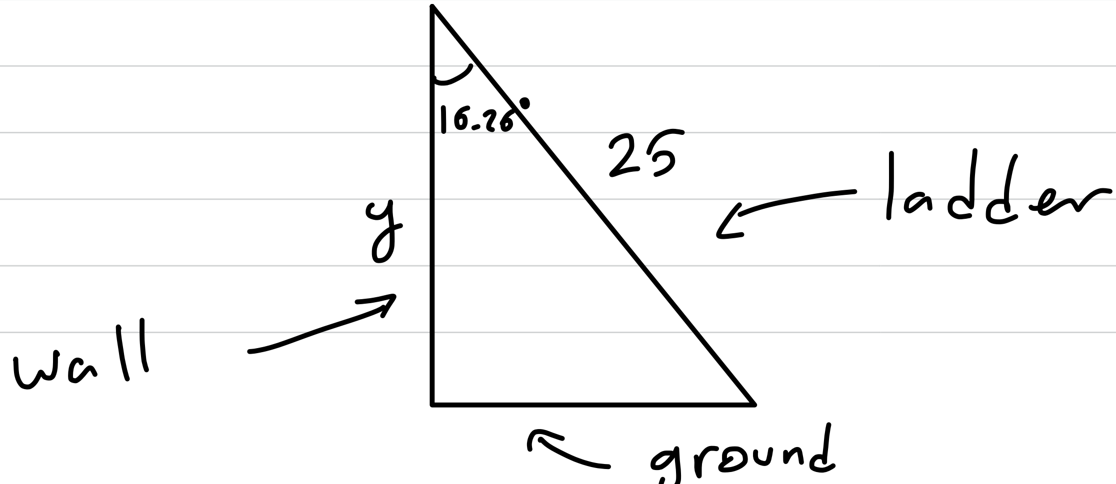
$$.342 = 5/c \qquad\qquad .94 = \frac{a}{14.62}$$

$$c = 14.62 \qquad\qquad a = 13.74$$

<u>Theorem</u>: In a circle of radius r, the coordinates of a point (x, y) on the circle with angle θ are $(r\cos(\theta), r\sin(\theta))$



Ex: You lean a ladder up against a wall. The ladder is 25 feet long, and it makes an angle of 16.26° with the wall. How far up does it reach?



$$\cos(16.26^\circ) = \frac{y}{25}$$

$$\frac{24}{25} = \frac{y}{25}$$

$$y = 24$$

Reference Angles

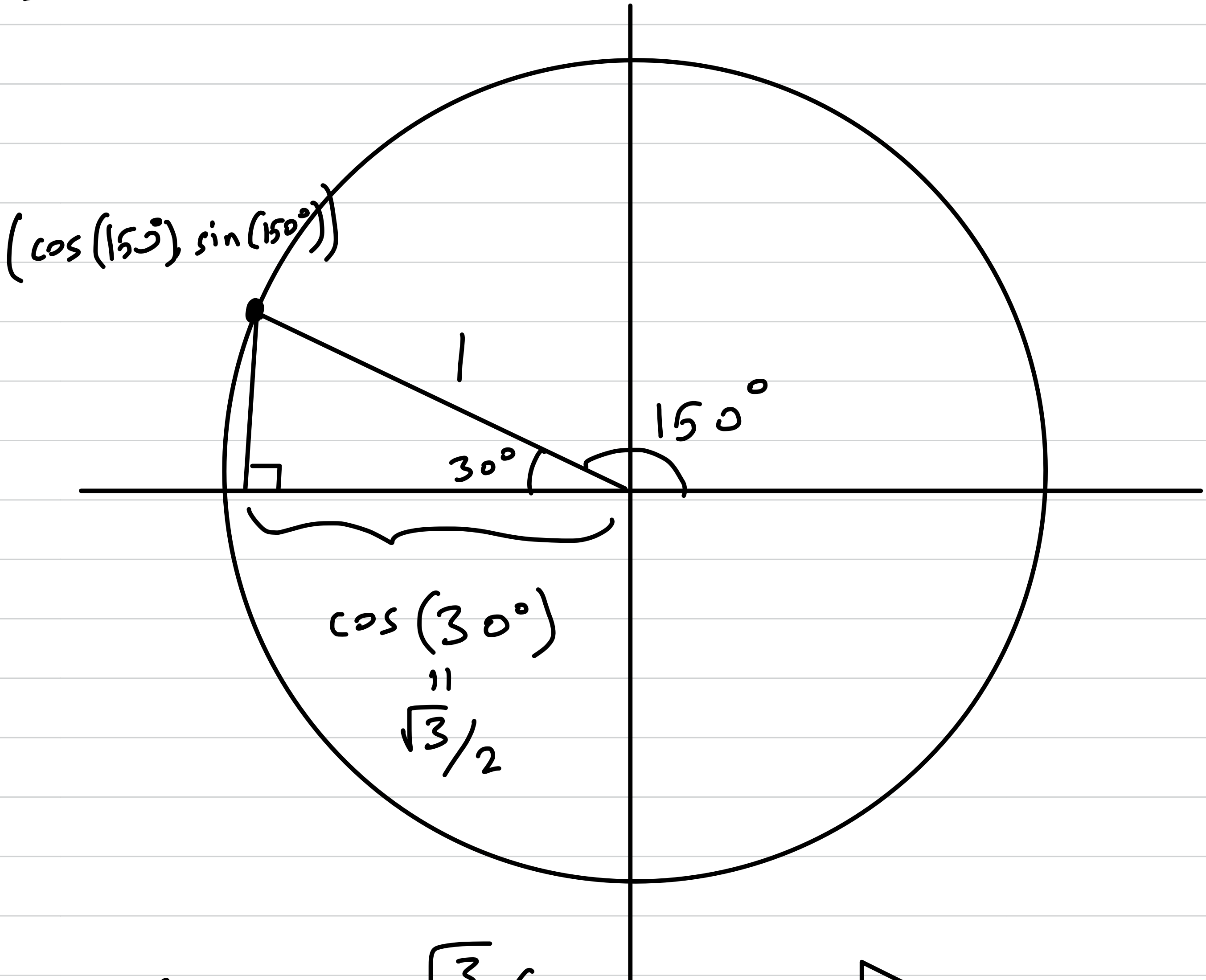
<u>Comment</u>: We know sin and cos for a few values of θ, but we'd like more.

θ	0°	30°	45°	60°	90°
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

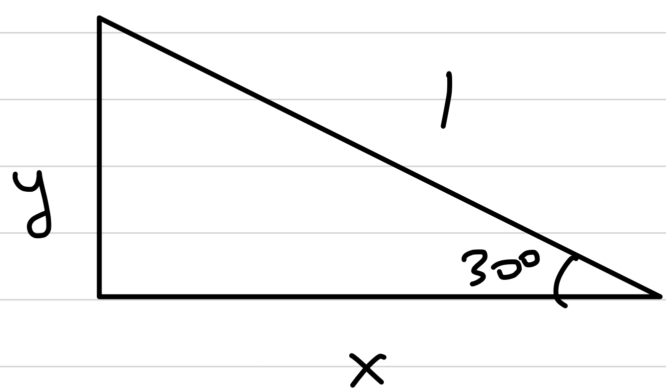
__Method__ (Finding exact values of trig functions using reference angles):

Given an angle θ, draw the point (x, y) on the unit circle with angle θ. Then draw a vertical line from (x, y) to the x-axis. This forms a triangle, called the __reference triangle__, and the angle made with the x-axis in this triangle is called the __reference angle__. Use that angle and our table of values to find $\sin\theta$ and $\cos\theta$, adding minus signs as needed

Ex: Find $\cos(150°)$

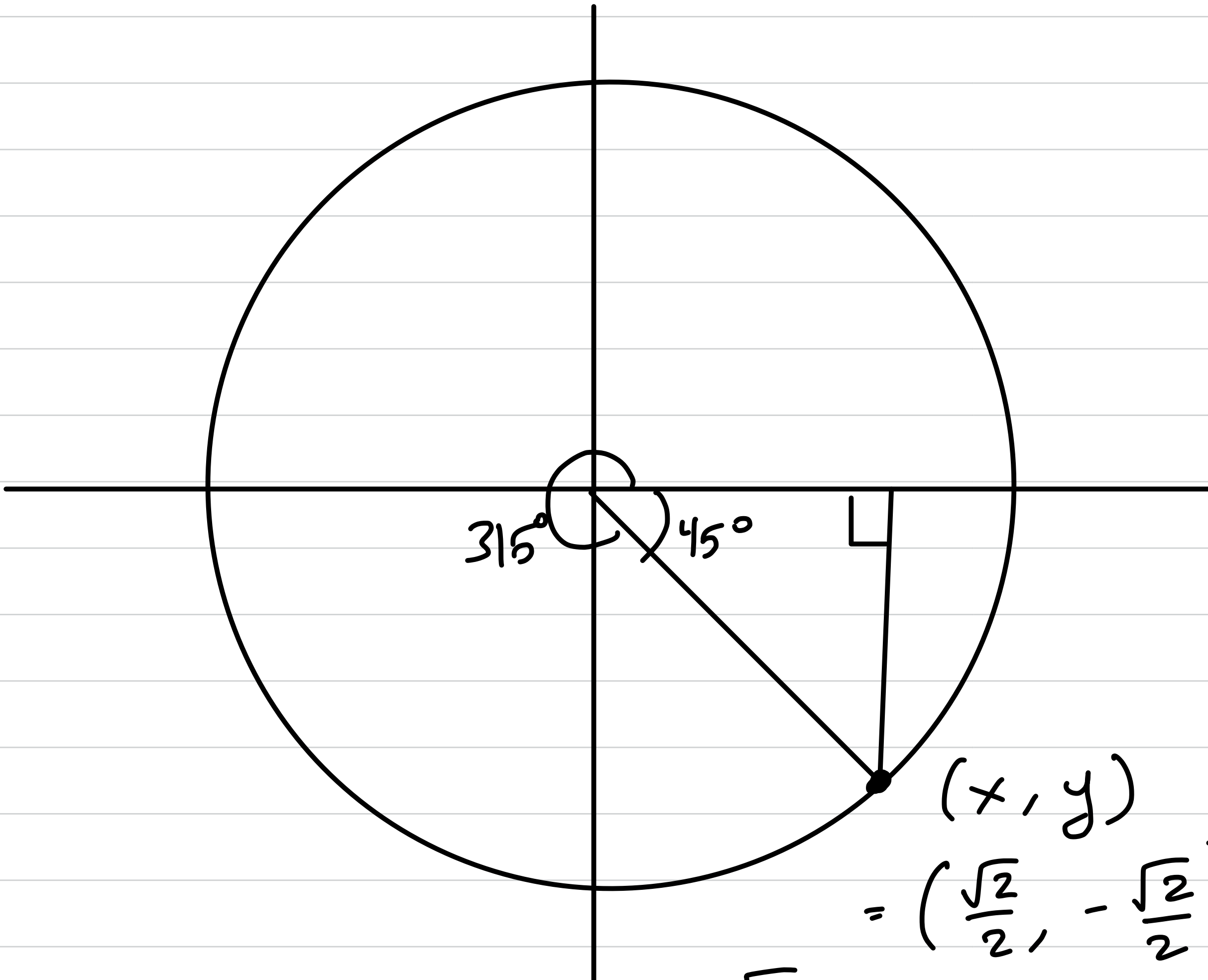


$\cos(150°) = -\sqrt{3}/2$.



$$\cos(30°) = \frac{x}{1} = x$$

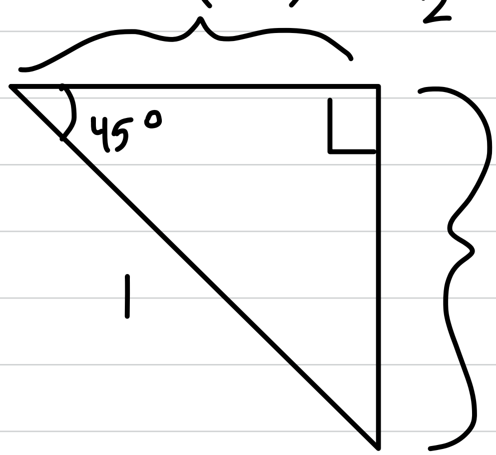
Ex: Find $\sin(315^\circ) = -\frac{\sqrt{2}}{2}$.



(x, y)

$= \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$\cos(45^\circ) = \frac{\sqrt{2}}{2}$

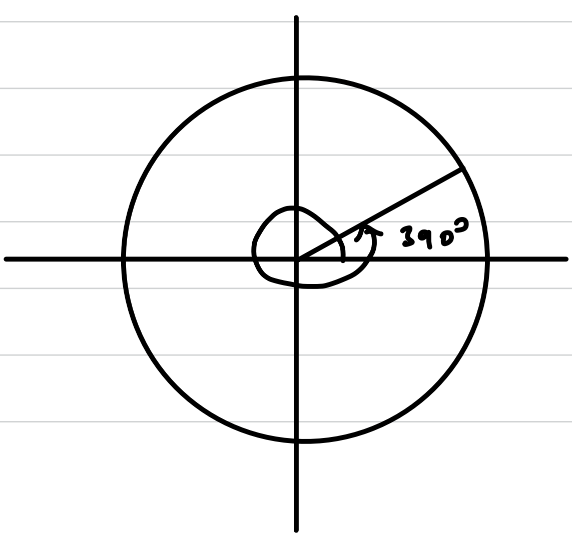


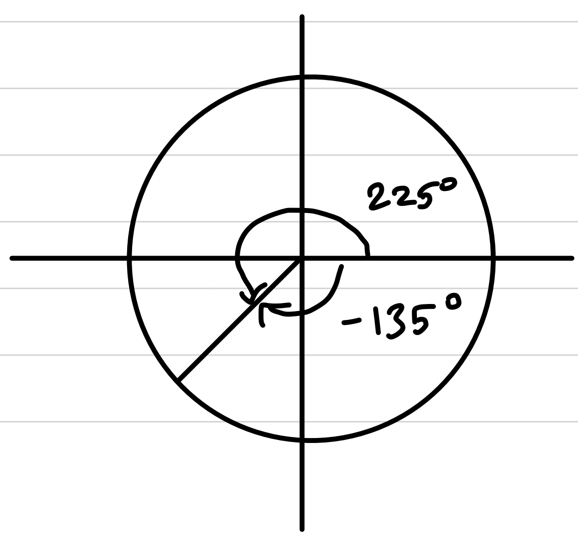
$\sin(45^\circ) = \frac{\sqrt{2}}{2}$

Comment: We've defined $\sin\theta$ and $\cos\theta$ for angles $0^\circ \le \theta \le 360^\circ$, but we can define them for any real number θ.

Def: An angle larger than 360° corresponds to wrapping around the circle more than once, and a negative angle corresponds to a clockwise measure on the circle.

Ex:





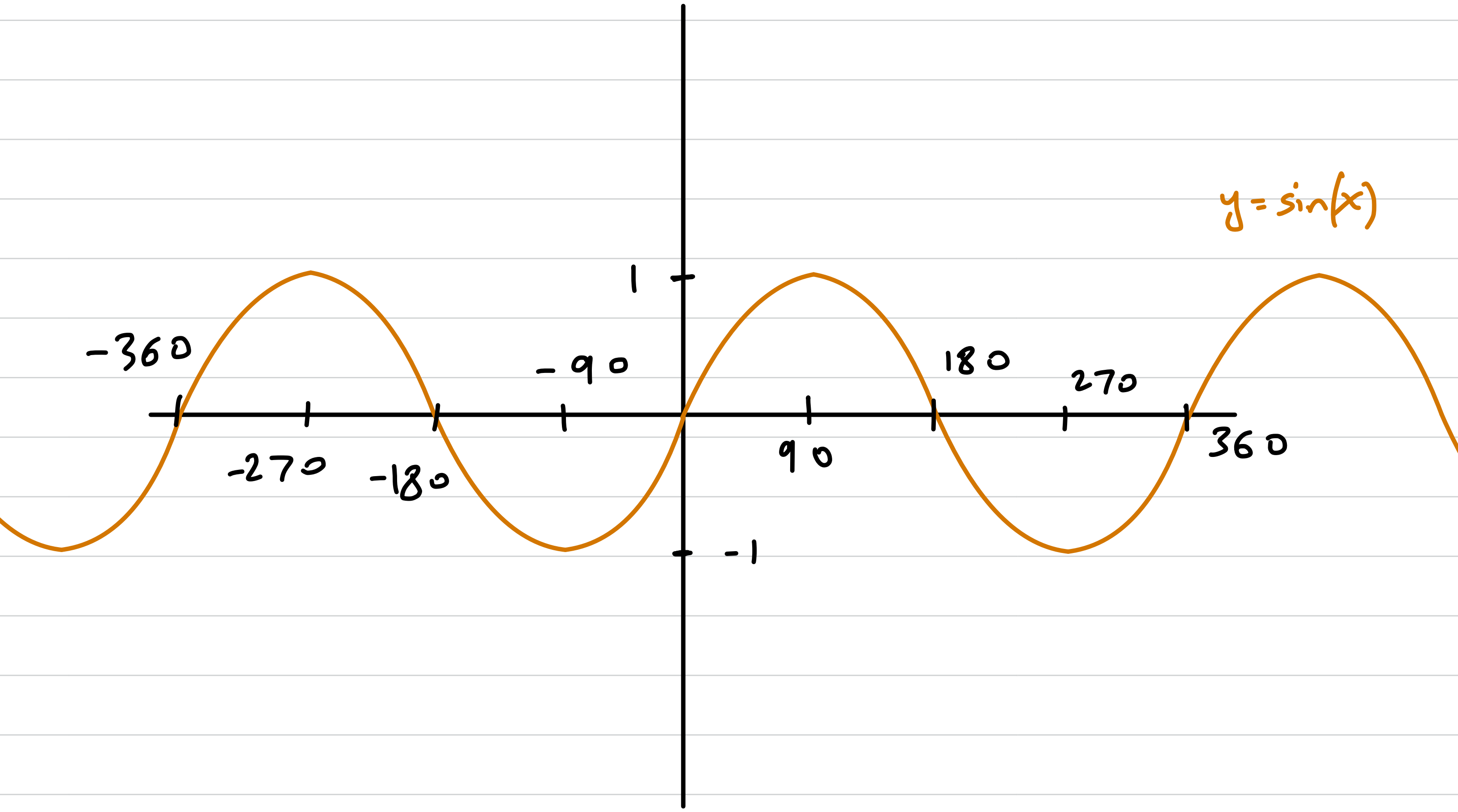
Because we define angles like this, sin and cos are periodic functions with period $360°$. So for example,

$\sin(390°) = \sin(30°) = \frac{1}{2}$ and

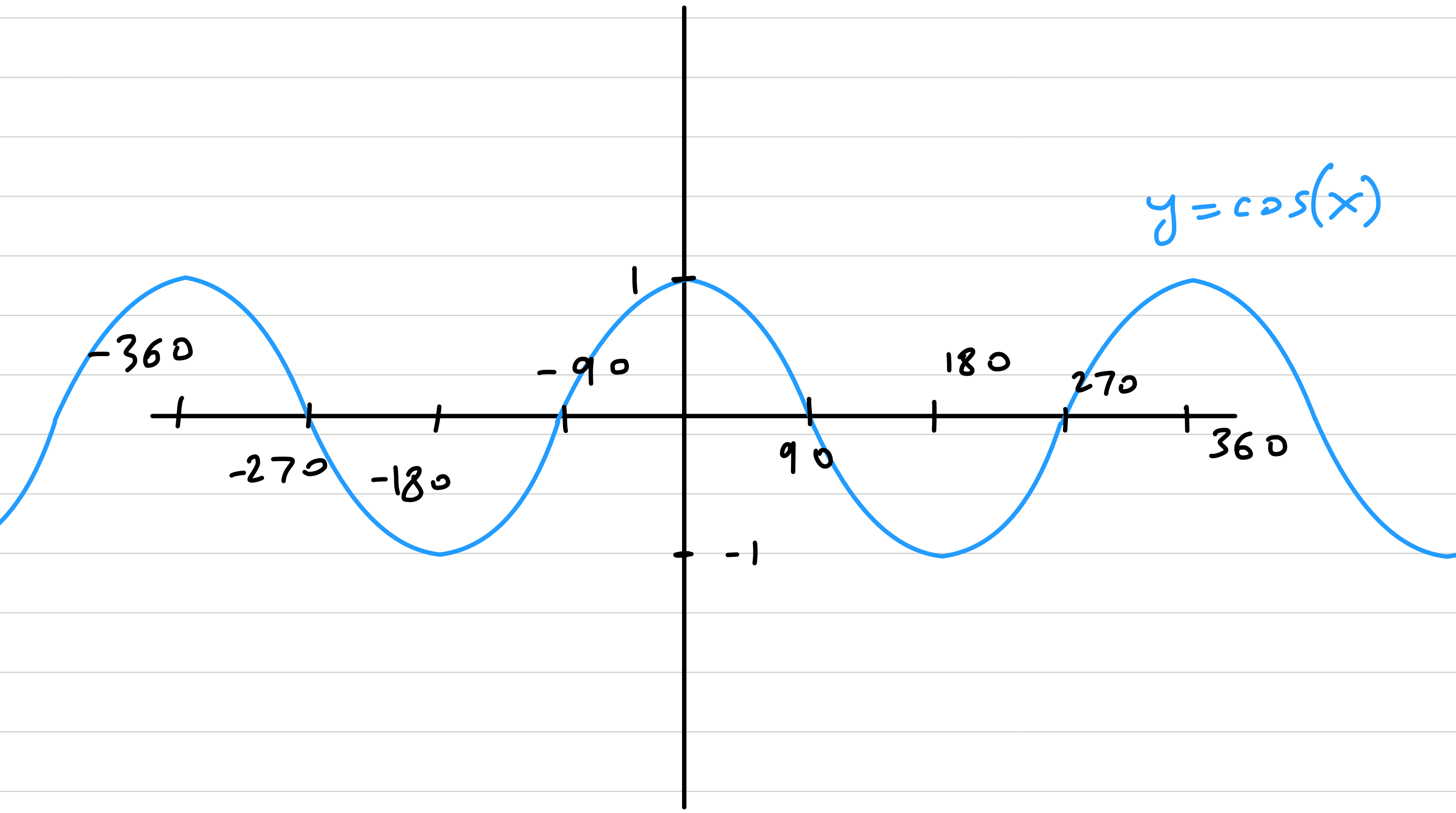
$\cos(-135°) = \cos(225°) = -\frac{\sqrt{2}}{2}$.

The graphs of sin and cos

Theorem ① The graphs of sin and cos are:



y = sin(x)
1
-1
-360
-270
-180
-90
90
180
270
360



y = cos(x)
1
-1
-360
-270
-180
-90
90
180
270
360

Comment: $\sin(x)$ and $\cos(x)$ are now parent functions for us.

(2) The domain of $\sin(x)$ and $\cos(x)$ is $(-\infty, \infty)$.

(3) $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$ for all x.

(4) The roots of $\sin(x)$ are $x = 180^\circ n$ for any integer n, and the roots of $\cos(x)$ are $x = 180^\circ n + 90^\circ$.

(5) $\sin(x)$ is odd and $\cos(x)$ is even.

(6) The midline of $\sin(x)$ and $\cos(x)$ is 0, and the amplitude is 1.