Quiz 4 Solutions:

① $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$E_1 = \{7\}$

$E_2 = \{2, 4, 6, 8\}$

$E_3 = \{3, 4, 5, 6\}$

② $p(E_1) = \frac{1}{8}$ $\quad o(E_1) = 1:7$

$p(E_2) = \frac{4}{8} = \frac{1}{2}$ $\quad o(E_2) = 4:4$

$p(E_3) = \frac{4}{8} = \frac{1}{2}$ $\quad o(E_3) = 4:4$

Let E be the event of rolling a 3 on the 8-sided die and F the event of rolling a 3 on the 4-sided one. Then we want

$$p(E \cup F) = p(E) + p(F) - p(E \cap F)$$

$n(E) = 4$ $\quad$ $p(E) = 4/32 = 1/8$

$n(F) = 8$ $\quad$ $p(F) = 8/32 = 1/4$

$n(E \cap F) = 1$ $\quad$ $p(E \cap F) = 1/32$

$n(S) = 32$

$$p(E \cup F) = 1/8 + 1/4 - 1/32 = 11/32.$$

3.7: Independence

Def: Events A and B are *independent* if $p(A|B) = p(A)$. What this means is that B taking place has no effect on the chance that A will take place.

Ex: if you toss two coins, the result of the second toss doesn't depend on the result of the first, so they're independent. In symbols, if A is the event of getting heads on the first toss and B is the event of getting heads on the second, then

$p(A) = 1/2$

$p(B) = 1/2$

$p(B \mid A) = 1/2$

Since $p(B \mid A) = p(B)$, A and B are independent.

<u>Ex</u>: If A is the event of drawing a heart off the top of a 52-card and B is the event of the card underneath it also being a heart, then

$$p(A) = 13/52 = 1/4$$
$$\neq$$
$$p(A \mid B) = 12/51$$

$\Longrightarrow$ A and B are <u>dependent</u>.

Comment: Independent vs Mutually exclusive

Independent means $p(A|B) = p(A)$.

Mutually exclusive means $p(A \cap B) = 0$.

Ex: Are the events A and B independent or mutually exclusive or neither, where A is having freckles and B is having red hair.

Since it's possible to have both freckles and red hair at the same time, A and B are mutually exclusive.

But having one makes you more likely to have the other, so A and B are dependent.

Theorem (Product rule for independent events)

If A and B are independent, then

$p(A \cap B) = p(A) \cdot p(B)$.

Ex: If A is the event of rolling a 3 on an 8-sided die and B is the event of rolling a 3 on a 4-sided die, then

$p(A \cap B) = \frac{1}{8} \cdot \frac{1}{4} = 1/32$ because

$p(A) = 1/8$

$p(B) = 1/4$

A and B are independent.

The Final

- 12 – 1:50 on Friday
- 1.5x midterm length (expect ~12 Qs)
- No outside resources (including a calculator)

- I'll post a list of topics

- Tue, Wed, Thu are open for questions, so come with questions ready

- office hours Wed + Fri as usual

If the first card is the 2 of spades, what is the probability that the second card is a spade?

Intuition: there are 12 spades and 51 cards left, so it should be $12/51$.

A: getting a spade, B: getting the 2 of spades

this is hard because $p(A)$ is hard.

If the first card is the 2 of spades, what is the probability that the second card is the ace of spades?

$$p(2 \cap ace) = \frac{1}{{}_{52}P_2} = \frac{1}{52 \cdot 51}$$

not asking for this.

$p(\text{ace} \mid 2) = 1/51$

12 burritos

5 spicy

7 not spicy

choose 3 at random

$$n(S) = {}_{12}C_3 = \frac{12!}{3!\,9!} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2}$$

$$= 2 \cdot 11 \cdot 10 = 220$$

(23) How many ways to choose 3 where all 3 are spicy?

$$ {}_5C_3 = \frac{5!}{3!\,2!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3 \cdot 2}}{\cancel{3 \cdot 2} \cdot \cancel{2}} = 10 $$

$\Rightarrow$ prob is $10/220 = 1/22$.

(24) How many ways to choose 3 where none is spicy?

$$ {}_7C_3 = \frac{7!}{3!\,4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 7 \cdot 5 = 35 $$

$35/220$

(25) Exactly one is spicy.

How many ways are there to choose 1 spicy burrito and 2 nonspicy ones?

$${}_5C_1 \cdot {}_7C_2 = \frac{5!}{1!\,4!} \; \frac{7!}{2!\;5!}$$

$$= 5 \cdot \frac{7 \cdot 6}{2} = 5 \cdot 7 \cdot 3$$

$$= 105$$

$105/220$

(26) Exactly two are spicy.

$$_5C_2 \cdot {}_7C_1 = \frac{5!}{2!\,3!} \cdot \frac{7!}{1!\,6!}$$

$$= \frac{5 \cdot 4}{2} \cdot 7$$

$$= 5 \cdot 2 \cdot 7 = 70$$

$70/220$

(27) At most one is spicy.
(so either none or one)

since those are mutually exclusive,

we get $35/220 + 105/220$

$$= 140/220$$

(28) At least one is spicy.

(this is the same as not none being spicy)

so it's $1 - 35/220 = 185/220$.

(29) At least two are spicy.

2 or 3

$\Rightarrow$ not (0 or 1)

$\Rightarrow 1 - 140/220 = 80/220$

(30) At most two are spicy

0, 1, or 2

$\Rightarrow$ not 3

$\Rightarrow 1 - 10/220 = 210/220$

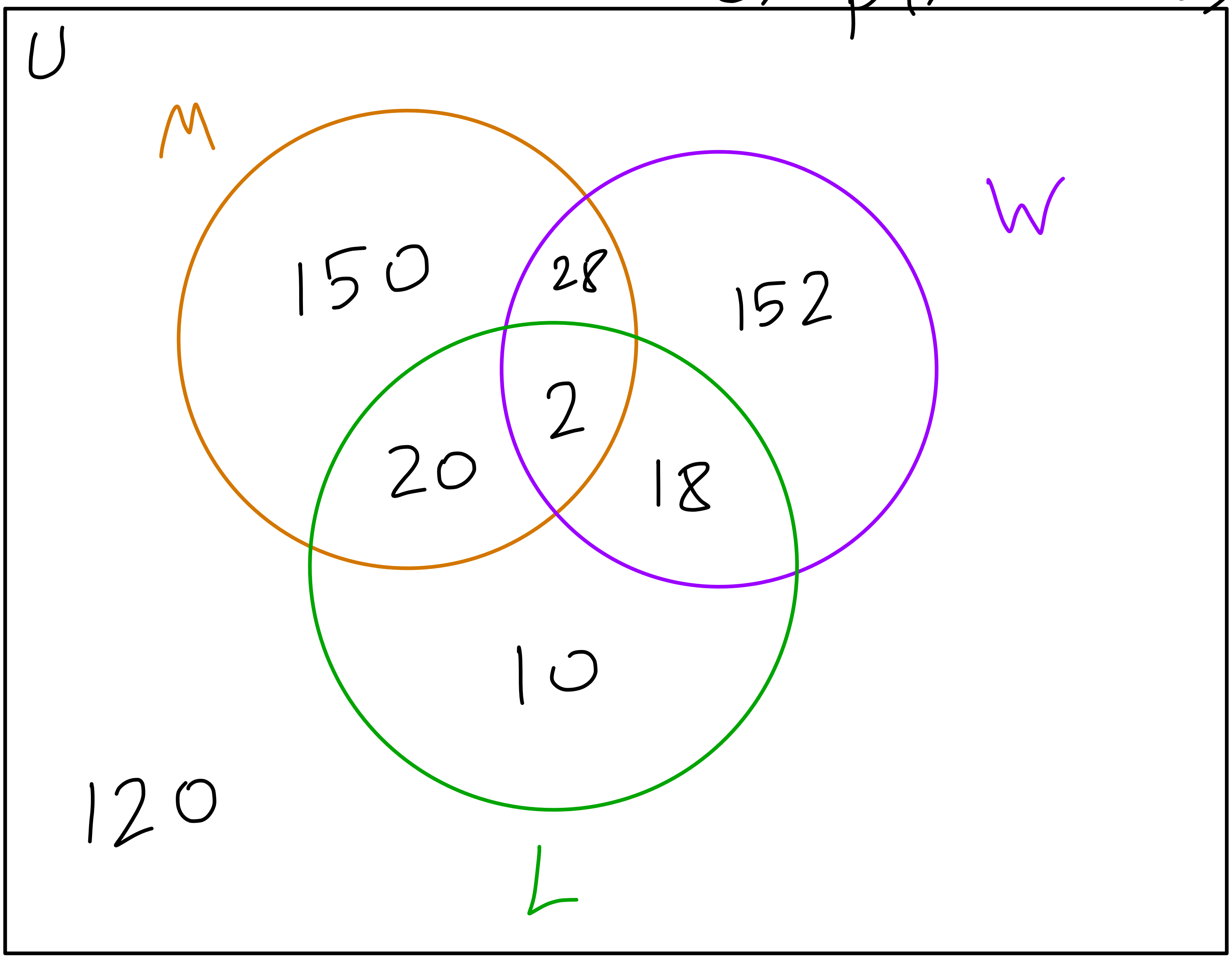
HW 3, 1

a) $p(M \cap W) = 30/500$

b) $p(L') = 450/500$

c) $p(M' \cap W' \cap L') = 120/500$

d) $p(M \cap W \cap L) = 2/500$



$n(U) = 500$

$n(M) = 200$

$n(W) = 200$

$n(L) = 50$

$n(M \cap W) = 30$

$n(W \cap L) = 20$

$n(M \cap W \cap L) = 2$

$n(M \cap W' \cap L') = 150$

a) $n(M \cap W \cap L') = 28$

b) $n(M \cap W' \cap L) = 20$

c) $n(M' \cap W' \cap L') = 120$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

A : the event of being dealt a 4-ace hand

B : the event of being dealt an ace off the top of the deck

$p(A \mid B)$ = the probability of being dealt a 4-ace hand when you're already been dealt one ace.

$\Rightarrow$ # ways to get dealt 3 more aces

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ways to get dealt 4 more cards

$= 48/249900$

$2/10000$

48

${}_{51}C_4 = 249900$

$$\frac{48}{{}_{50}C_3} = \frac{48}{19600} = 2/1000$$

3.4 #1

$$\frac{\text{\# ways to pick 30 birthdays where two are the same}}{\text{\# ways to pick 30 birthdays}}$$

If this is E, $n(E') = {}_{365}P_{30}$

365^{30}

$\Rightarrow p(E') = \dfrac{{}_{365}P_{30}}{365^{30}}$

So $p(E) = 1 - p(E') = .7$

If you have 20 objects and you want to order them, the number of ways is $20!$

If you have 20 objects and you want to choose 5 and order them, the number of ways is ${}_{20}P_5$

If you have 20 objects and you want to pick 5 and not order them, there are ${}_{20}C_5$ ways.

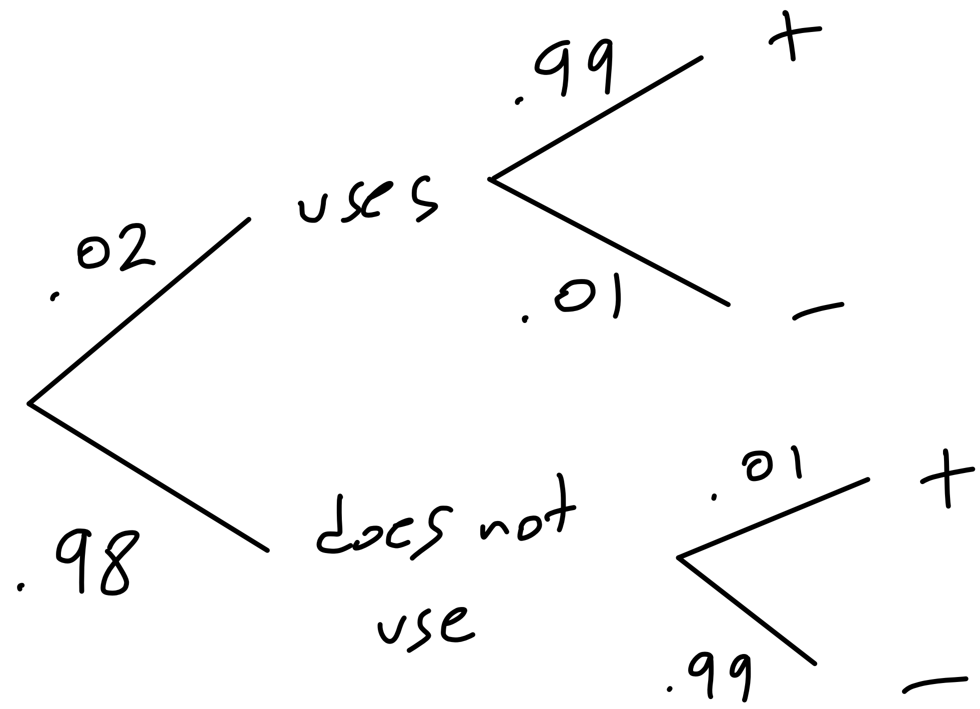
If you have 5 slots and you want to put one of the 20 objects in each slot, but you can reuse the objects and order matters, there are 20^5 ways to put them there.

$$p(A|B) = \frac{p(B|A)\,p(A)}{p(B)}$$

a) $p(A) = .02$.

$p(B|A) = .99$.

b) $p(B)$



$= .02 \cdot .99 + .98 \cdot .01$

$= .0296$

c) $p(A|B) = \dfrac{(.99)(.02)}{.0296} = .67$

d) $p(A'|B) = 1 - p(A|B) = 1 - .67 = .33$

A randomly selected person with a positive test has a $\sim 1/3$ chance to not actually be using this drug.

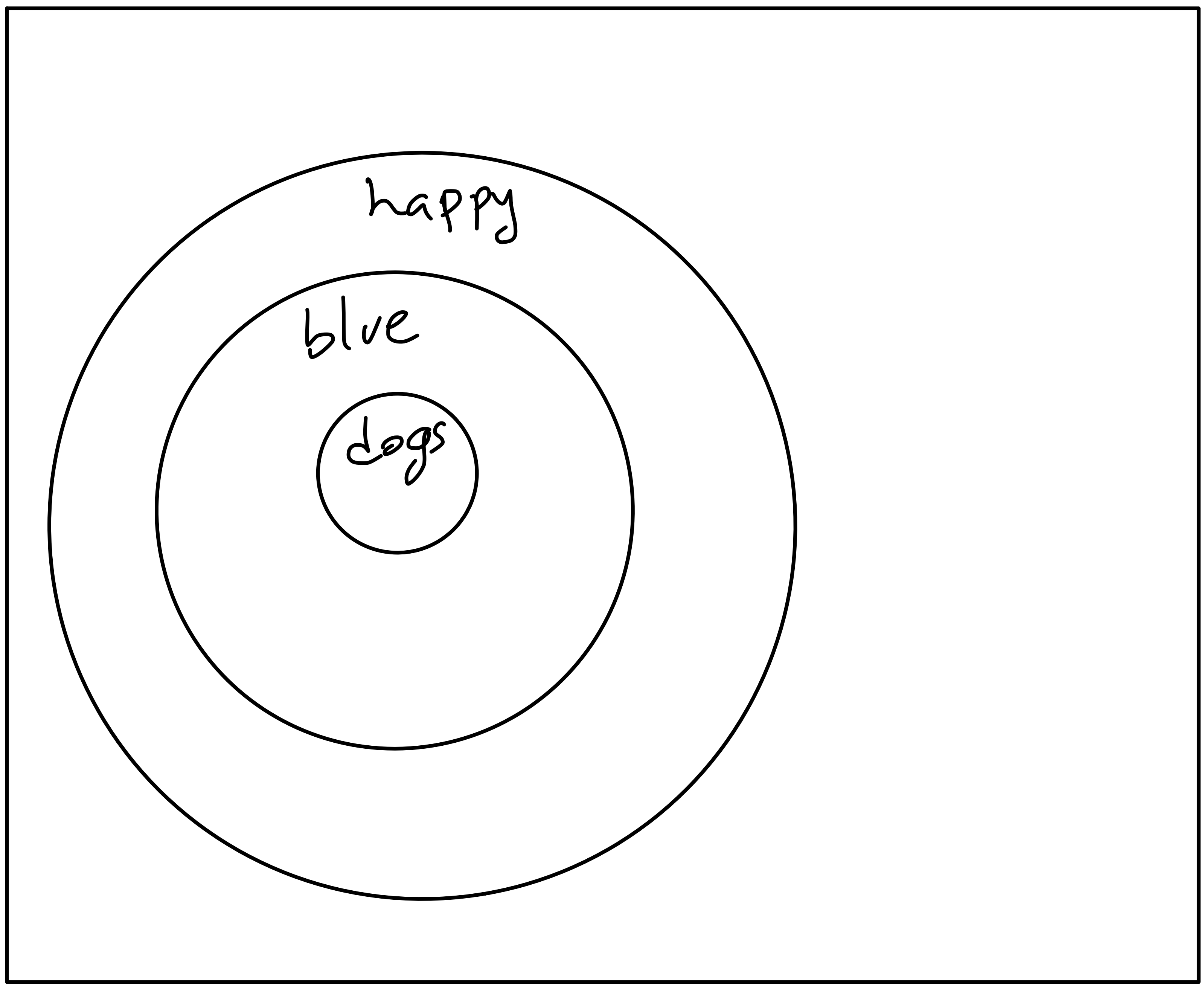
Ex: roll a 6-sided die. If you roll a 1, you lose \$10. If you roll a 2, 3, or 4, you gain \$1, and if you roll a 5 or 6, you gain \$3. What is the expected value of rolling the die?

-10	1	3
$1/6$	$3/6$	$2/6$
$-10/6$	$3/6$	$6/6$

$$-10/6 + 3/6 + 6/6 = \boxed{-1/6}$$

1. All dogs are blue
2. All blue things are happy

All dogs are happy ✓



1. All dogs are blue
2. All blue things are happy

All dogs are happy ✓

1. If you are a dog, then you are blue
2. If you are blue, then you are happy

1. $p \to q$
2. $q \to r$

If you are a dog, then you are happy

1. $p \to q$
2. $q \to r$

$p \to r$

p: you are a dog
q: you are blue
r: you are happy

1. $p \to q$

2. $q \to r$

$p \to r$

$P_1 : p \to q$

$P_2 : q \to r$

$C : p \to r$

p	q	r	P_1	P_2	C	$P_1 \land P_2$	$P_1 \land P_2 \to C$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

✓

Write the set of all real numbers that are less than 2 or bigger than 5.

$$\{x \in \mathbb{R} \mid x < 2 \text{ or } x > 5\}$$

$$\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$$

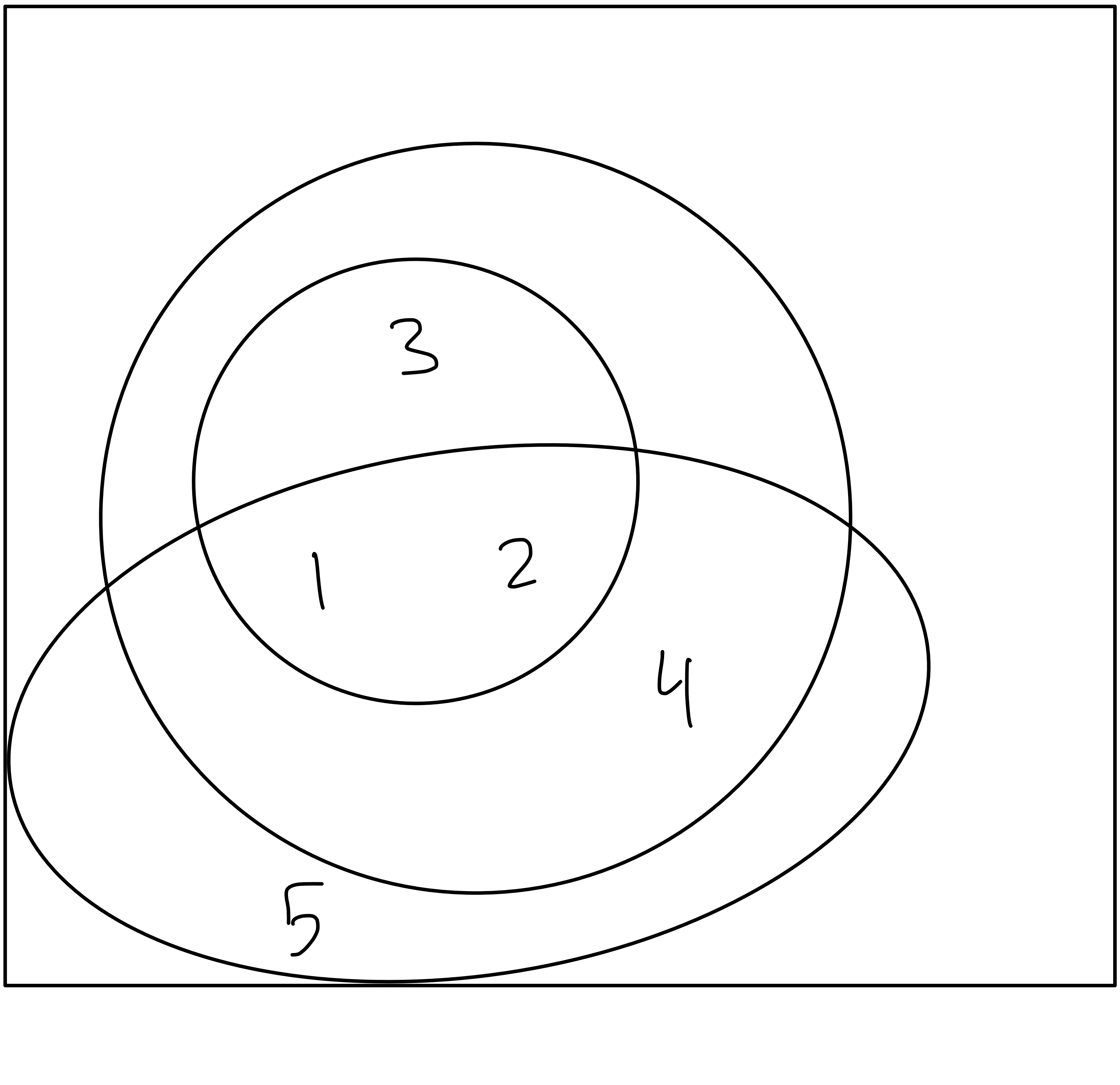
$$\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$$

If the universe $\mathcal{U} = \{1, 2, \dots, 10\}$, then $\{1, 2, 3\}' = \{4, 5, 6, 7, 8, 9, 10\}$.

$\cup$	$\vee$
$\cap$	$\wedge$
$'$	$\sim$

$\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$

$\{1, 2, 3\} \not\subseteq \{1, 2, 4, 5\}$



$$n(\{3,4,5,8\}) = 4$$

If $n(A)=4$, $n(B)=10$, and $n(A\cap B)=2$, then

$$n(A\cup B) = n(A) + n(B) - n(A\cap B)$$
$$= 4 + 10 - 2 = 12.$$

$$(A\cap B)' = A' \cup B'$$

$$(A\cup B) = A' \cap B'$$

How many ways can you shuffle a deck of cards?

Shuffling a deck of cards is equivalent to putting it in a specific order, so we need to find out how many orderings of 52 objects there are. This is $52!$.

How many 5-card hands are possible?

We are choosing an unordered group of 5 objects from 52, so it's ${}_{52}C_5 = \dfrac{52!}{5!\,47!} = \dfrac{52 \cdot 51 \cdot \overset{5}{\cancel{50}} \cdot 49 \cdot \overset{4}{\cancel{48}}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} =$

$52 \cdot 51 \cdot 5 \cdot 49 \cdot 4 = 2598960.$

How many ways can a 5-card hand be dealt in order?

$${}_{52}P_5 = {}_{52}C_5 \cdot 5! = 2598960 \cdot 120$$

$$= 311\,875\,200$$

If you can paint cards black, red, blue, green, yellow or white, how many ways can you paint 52 cards in order?

This is $6^{52} = 2.9 \cdot 10^{40}$

Roll a 12-sided die.

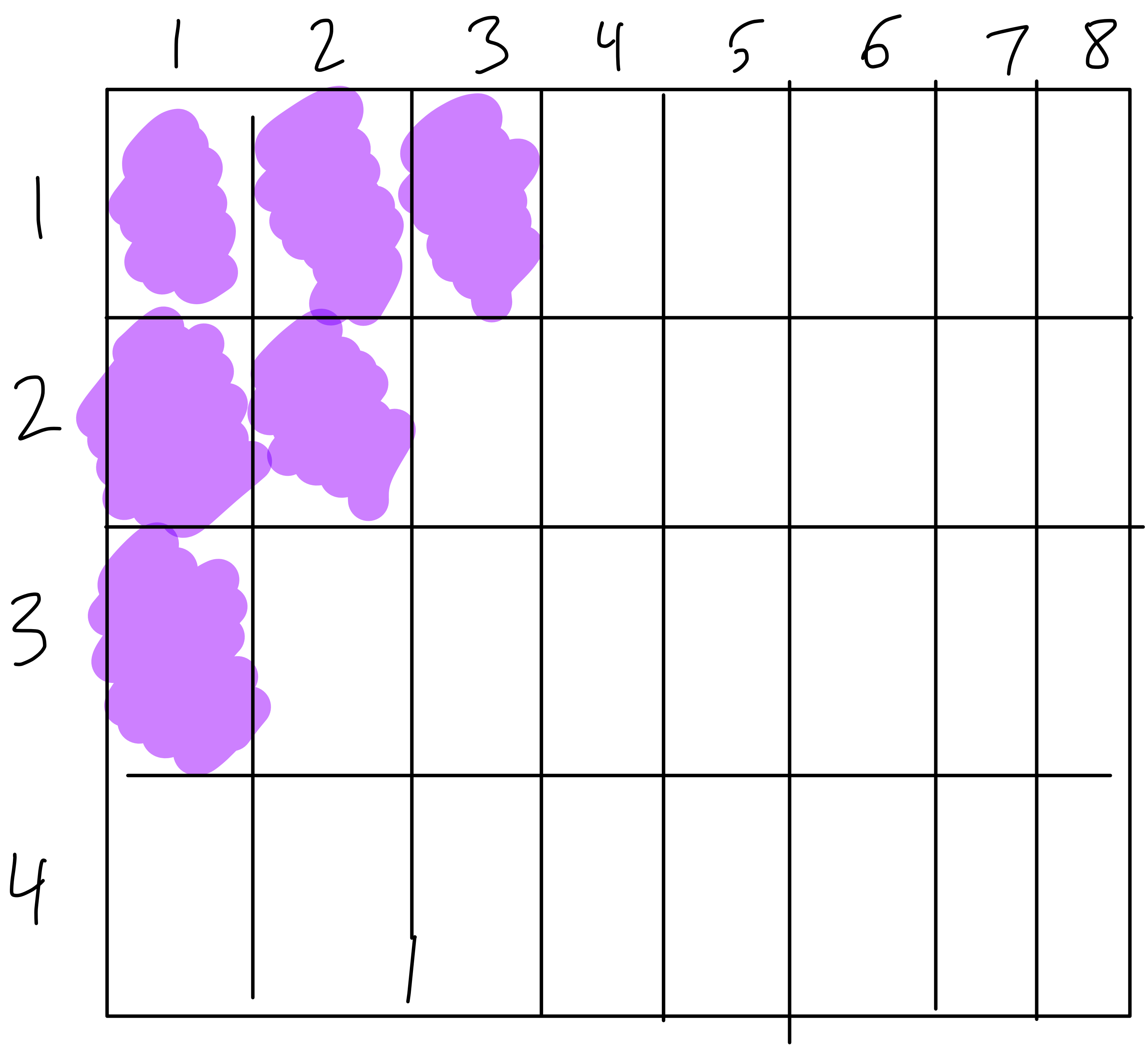
$$S = \{1, 2, 3, \ldots, 12\}$$

The event of rolling a 2 or 7 is $\{2, 7\}$.

The probability of this event is $2/12 = 1/6$.

The event of rolling a 12 and the event of rolling less than a 4 are mutually exclusive, since $\{12\} \cap \{1, 2, 3\} = \emptyset$.

If you roll an 8-sided die and a 4-sided die, what is the chance that the sum will be less than 5?



6 spaces that work out of 32 total, and evey space is equally likely, so the probability is 6/32

The probability of rolling at least one 4 is p(4 on 8-sided die) + p(4 on 4-sided die) - p(4 on both)

$$= 1/8 + 1/4 - 1/32 = 4/32 + 8/32 - 1/32$$

$$= 11/32.$$

The probability of <u>not</u> rolling at least one 4 is $1 - 11/32 = 21/32$

Two events A and B are independent

if $p(A \mid B) = p(A)$

Quiz 5 solutions

1. $\begin{cases} 6 \text{ objects} \\ \text{choose } 3 \\ \text{order matters} \end{cases}$

$$_6P_3 = ({}_6C_3)(3!)$$

$$= \frac{6!}{3!\,\cancel{3!}}\cancel{3!}$$

$$= \frac{6!}{3!}$$

$$= 6 \cdot 5 \cdot 4$$

$$= 120$$

$$\Rightarrow 1/120$$

(2)

1000	0
1/1000	999/1000
1	0

$1 + 0 = \boxed{1}$

\$1

On average, you'll win \$1 per time you play the raffle.

3.6 #21

Roll a pair of dice.

a) What is the probability that the sum is 4?

$E = \{(1,3), (2,2), (3,1)\}$

$n(E) = 3$

6 possible rolls per die

rolling something on the first die doesn't change the number of possible rolls on the second,

the FPoC says there $6 \cdot 6 = 36$ possible rolls.

$\Rightarrow 3/36 = 1/12$

b) What is the probability of getting a sum of 4 given that the sum is less than 6?

"Reducing the sample space"

What is the new S?

How many rolls of two dice have sums less than 6?

2	(1,1)
3	(1,2), (2,1)
4	(1,3), (2,2), (3,1)
5	(1,4), (2,3), (3,2), (4,1)

10 ways to do this

$n(S)$ is now 10

prob is therefore $3/10$.

A is the event that the sum is 4

B is the event that the sum is less than 6

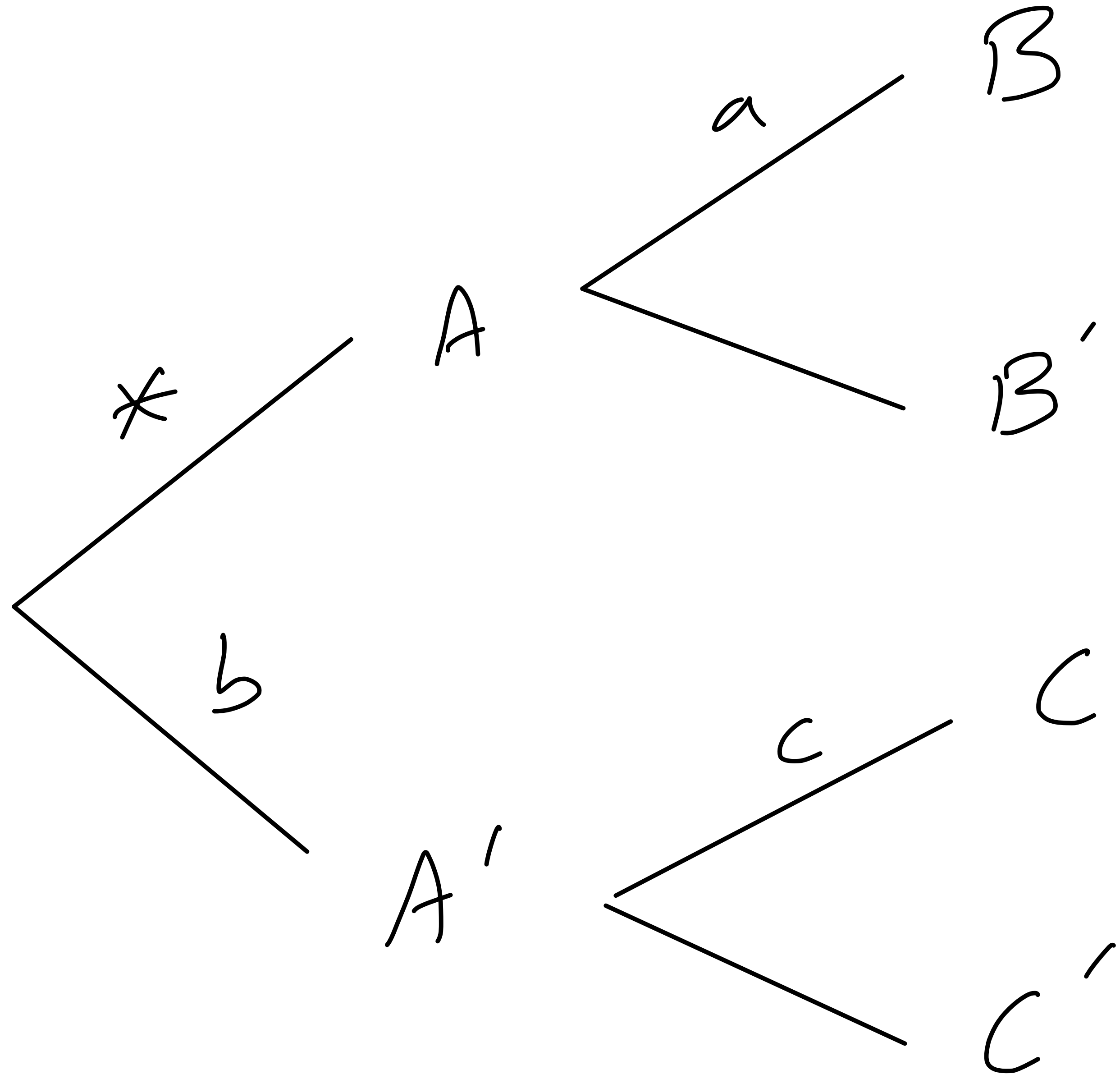
$p(A \mid B)$

c) $p(B \mid A)$

what is the probability that the sum is less than 6, given that the sum is 4?

$= 1.$

$a = p(B|A)$



a) Should * and a be added or multiplied?

Multiplied, and gives $p(A \cap B)$.

b) What about b and c?

Multiplied, and gives $p(A' \cap C)$.

c) What about $p(A \cap B)$ and $p(A' \cap C)$?

Added, and gives $p((A \cap B) \cup (A' \cap C))$

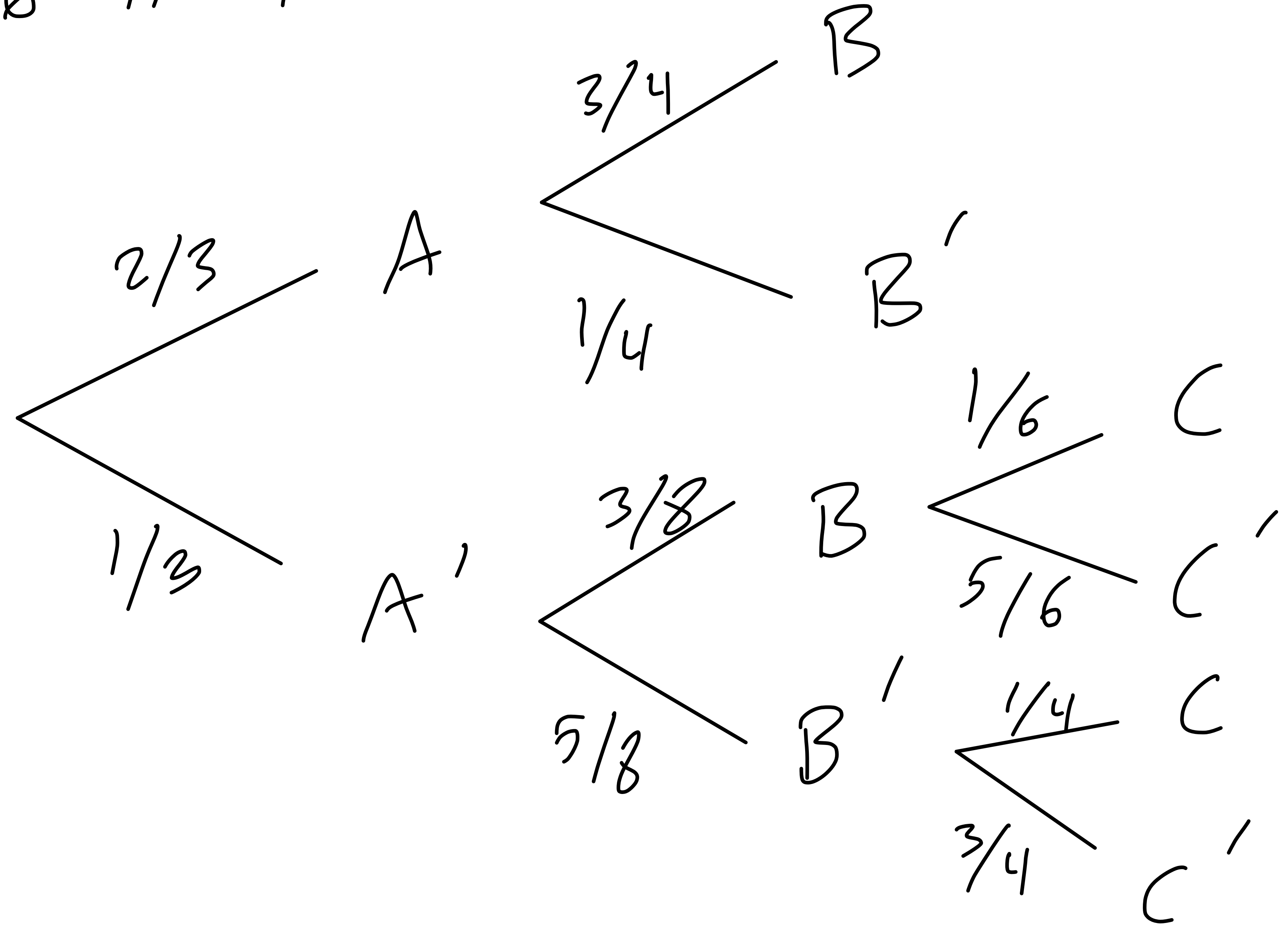
The reason the multiplication gave us an intersection in a) and b) is because tree diagrams are written with conditional probabilities.

$$p(A \cap B) = p(A)\, p(B|A)$$

The reason the addition worked in part c) is because $A \cap B$ and $A' \cap C$ are mutually exclusive.

Since if E and F are mutually exclusive, then $p(E) + p(F) = p(E \cup F)$.

3.6 # 7



a) $p(C \mid A') = p(B \cap C \mid A') + p(B' \cap C \mid A')$
$= 3/8 \cdot 1/6 + 5/8 \cdot 1/4$

b) $p(B \cap A') = p(A')\, p(B \mid A')$
$= 1/3 \cdot 3/8 = 1/8$.

$$p((B \cap C) \cup (B' \cap C) \mid A')$$