

Name: \_\_\_\_\_

Homework 1 | Math 342 | Cruz Godar

*Due Wednesday of Week 2 at the start of class*

Complete the following problems and submit them as a pdf to Canvas. 8 points are awarded for thoroughly attempting every problem, and I'll select three problems to grade on correctness for 4 points each. Enough work should be shown that there is no question about the mathematical process used to obtain your answers.

## Section 0

In problems 1–3, do the following:

- a) Write the system of equations in the form  $A\vec{x} = \vec{b}$  for a matrix  $A$  and a vector  $\vec{b}$  of constants, and a vector  $\vec{x}$  of variables.
- b) Augment  $A$  with  $\vec{b}$  and row reduce the system to solve for  $\vec{x}$ .
- c) If  $A$  is square, find  $\det A$ .
- d) Let  $T$  be the linear transformation corresponding to the matrix  $A$ . Write down the domain and codomain for  $T$ .
- e) Find a basis for  $\ker T$  and use it to find a basis for  $\text{image } T$  using the fundamental theorem of linear algebra.
- f) Determine if  $T$  is one-to-one, onto, both, or neither. If it's both one-to-one and onto, find a formula for  $T^{-1}$ .

1.

$$x + y = 1$$

$$x - y = 3$$

2.

$$x + 2y - z = -1$$

$$2x + 3y - w = -2$$

3.

$$x + y + 2z = 4$$

$$2x + y - z = 2$$

$$-3x - y + 4z = 0$$

In problems 4–7, do the following:

- a) The given sets  $V$  and  $W$  are vector spaces. Determine whether the subset  $X$  of  $V$  is a subspace. If it is, show it satisfies all three subspace properties, and if not, give a specific example showing one of the properties fails.
- b) Determine whether the function  $T : V \rightarrow W$  is a linear transformation. If it is, show it satisfies the two properties, and if not, give a specific example showing one of them fails.
- c) If  $T$  is a linear transformation, find the matrix for  $T$  with respect to the standard bases for  $V$  and  $W$ .
- d) If  $T$  is a linear transformation, find a basis for  $\ker T$  and use it to find bases for  $V$  and  $\text{image } T$  using the fundamental theorem of linear algebra. Then extend the basis for  $\text{image } T$  to a basis for  $W$ , and find the matrix for  $T$  with respect to the bases for  $V$  and  $W$  you found.

4.  $V = \mathbb{R}^4$ ,  $W = M_{2 \times 2}(\mathbb{R})$ ,  $X$  is the set of vectors  $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$  such that  $x + 2y = w$ , and  $T : V \rightarrow W$  is

defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 0 & x \\ y+z & 2w \end{bmatrix}.$$

5.  $V = M_{2 \times 2}(\mathbb{R})$ ,  $W = M_{2 \times 2}(\mathbb{R})$ ,  $X$  is the set of matrices  $A$  such that  $\det A = 0$ , and  $T : V \rightarrow W$  is defined by

$$T(A) = A^T,$$

where  $A^T$  is the **transpose** of  $A$ , defined by  $(A^T)_{ij} = A_{ji}$  (it effectively flips  $A$  across its diagonal).

6.  $V$  is the space of polynomials with degree at most 3,  $W = \mathbb{R}^2$ ,  $X$  is the set of polynomials  $p(x)$  such that  $p'(x) = 1$ , and  $T : V \rightarrow W$  is defined by

$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b \\ c + d \end{bmatrix}.$$

7.  $V = \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ ,  $W = M_{2 \times 2}(\mathbb{R})$ ,  $X$  is the set of linear transformations  $S : \mathbb{R} \rightarrow \mathbb{R}^2$  such that  $S(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , and  $T : V \rightarrow W$  is defined by

$$T(S) = \begin{bmatrix} | & | \\ S(1) & S(-1) \\ | & | \end{bmatrix},$$

i.e. the outputs of  $S$  are placed as columns in a  $2 \times 2$  matrix.

## Section 1

In problems 8–10, find the eigenvalues and eigenvectors of the matrix.

8.  $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$

9.  $B = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}.$

10.  $C = \begin{bmatrix} 2 & 2 & -2 \\ -3 & 7 & 3 \\ -5 & 5 & 5 \end{bmatrix}.$

11. Let  $A = [a_{ij}]$  be an  $n \times n$  matrix that is upper triangular — that is,  $a_{ij} = 0$  whenever  $i > j$ . What are the eigenvalues of  $A$  in terms of the entries  $a_{ij}$ ? Justify your answer.

12. Let  $A$  be an  $n \times n$  matrix with eigenvectors  $\vec{v}_1, \dots, \vec{v}_n$  and corresponding eigenvalues  $\lambda_1, \dots, \lambda_n$ . If  $A$  is invertible, what are the eigenvectors and eigenvalues of  $A^{-1}$ ? Justify your answer.

13. Let  $A$  be an  $n \times n$  matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$ . What are the eigenvalues of  $A^T$ ? Justify your answer.