Def: Let $P(x)$ be a polynomial. The degree of P is the largest exponent on x.

Ex: $\deg\left(x^{\boxed{5}} + 2x^4 + x^{\boxed{5}} - 2\right) = 5$.

Ex: $\dfrac{x-1}{3x^2-2}$ will work with partial fractions since $\deg(x-1) = 1 < \deg(3x^2-2) = 2$.

Method (Partial Fractions, v1) Let $P(x)$ and $Q(x)$ be polynomials with $\deg P < \deg Q$ and such that $Q(x)$ splits into nonrepeating linear factors: $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ where $a_i \neq a_j$ for $i \neq j$. Then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

for some numbers $A_1, A_2, \ldots, A_n$.

Ex: $Q(x) = (x-1)(x+3)(x-2)$ works

$Q(x) = (x-1)^2 (x^2+1)$ doesn't

Ex: $$\int \frac{3x+2}{x^3 - x^2 - 2x}\, dx$$

$$= \frac{3x+2}{x(x^2-x-2)}$$

$$= \frac{3x+2}{x(x-2)(x+1)}$$

by Partial Fractions

$$= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

How do we find A, B, and C?

Multiply both sides of red by $Q(x) = x(x-2)(x+1)$.

$$3x + 2 = A(x-2)(x+1) + B(x)(x+1) + C(x)(x-2)$$

$$3x + 2 = A(x^2 - x - 2) + B(x^2 + x) + C(x^2 - 2x)$$

$$3x + 2 = (A + B + C)x^2 + (-A + B - 2C)x - 2A$$

Now set all the constant terms equal, all the coefficients on x equal, and so on.

$$2 = -2A \implies A = -1$$

$$3 = -A + B - 2C$$

$$0 = A + B + C$$

$$3 = 1 + B - 2C \Rightarrow B - 2C = 2$$

$$0 = -1 + B + C \Rightarrow B + C = 1$$

$$B = 2 + 2C$$

$$2 + 2C + C = 1$$

$$3C = -1$$

$$C = -1/3$$

$$B = 2 - 2/3 = 4/3$$

$$\int \frac{1}{x-2}\,dx = \ln(x-2) + C$$

$$\int \frac{-1}{x} + \frac{4/3}{x-2} + \frac{-1/3}{x+1}\,dx$$

$$= -\ln(x) + \frac{4}{3}\ln(x-2) - \frac{1}{3}\ln(x+1) + C.$$

<u>Method</u> (Partial Fractions, v2) Suppose P and Q are polynomials and $\deg P < \deg Q$. If Q factors as

$$Q(x) = (x-a_1)^{n_1}(x-a_2)^{n_2} \cdots (x-a_k)^{n_k},$$

then

$$\frac{P}{Q} = \left(\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_{n_1}}{(x-a_1)^{n_1}}\right)$$
$$+ \left(\frac{B_1}{(x-a_2)} + \frac{B_2}{(x-a_2)^2} + \cdots + \frac{B_{n_2}}{(x-a_2)^{n_2}}\right)$$
$$+ \cdots + \left(\frac{Z_1}{(x-a_k)} + \cdots + \frac{Z_{n_k}}{(x-a_k)^{n_k}}\right)$$

In short: if a linear factor is repeated n times in Q, add one term for each power of that factor that's $\leq n$.

$\underline{Ex}$: $\displaystyle\int \underbrace{\frac{x-2}{(2x-1)^2(x-1)}} dx$

$$= \frac{x-2}{\left(2\left(x-\frac{1}{2}\right)\right)^2(x-1)}$$

$$= \frac{1}{4} \cdot \underbrace{\frac{x-2}{\left(x-\frac{1}{2}\right)^2(x-1)}}$$

$$= \frac{A_1}{x-\frac{1}{2}} + \frac{A_2}{\left(x-\frac{1}{2}\right)^2} + \frac{B}{x-1}$$

$\downarrow$ multiplying both sides by $\left(x-\frac{1}{2}\right)^2(x-1)$

$$x-2 = A_1\left(x-\frac{1}{2}\right)(x-1) + A_2(x-1) + B\left(x-\frac{1}{2}\right)^2$$

$$x-2 = A_1\left(x^2-\frac{3}{2}x+\frac{1}{2}\right) + A_2(x-1) + B\left(x^2-x+\frac{1}{4}\right)$$

$$0x^2+x-2 = (A_1+B)x^2 + \left(-\frac{3}{2}A_1+A_2-B\right)x + \left(\frac{1}{2}A_1-A_2+\frac{1}{4}B\right)$$

$$A_1 + B = 0 \quad (1)$$

$$-\frac{3}{2}A_1 + A_2 - B = 1 \quad (2)$$

$$\frac{1}{2}A_1 - A_2 + \frac{1}{4}B = -2 \quad (3)$$

$$A_1 = -B \quad (1)$$

$$\frac{3}{2}B + A_2 - B = 1 \quad (2)$$

$$-\frac{1}{2}B - A_2 + \frac{1}{4}B = -2 \quad (3)$$

$$\frac{1}{2}B + A_2 = 1 \quad (2)$$

$$A_2 = 1 - \frac{1}{2}B$$

$$-\frac{1}{4}B - A_2 = -2 \quad (3)$$

$$\frac{1}{4}B + A_2 = 2$$

$$\frac{1}{4}B + 1 - \frac{1}{2}B = 2$$

$$-\frac{1}{4}B + 1 = 2$$

$$-\frac{1}{4}B = 1$$

$B = -4$

$A_2 = 1 - \frac{1}{2}(-4) = 1 + 2 = 3$ 2

$A_1 = 4$ 1

$$\frac{1}{4} \cdot \frac{x-2}{(x-\frac{1}{2})^2(x-1)} = \frac{1}{4}\left(\frac{4}{x-\frac{1}{2}} + \frac{3}{(x-\frac{1}{2})^2} - \frac{4}{x-1}\right)$$

So $\displaystyle\int \frac{1}{4} \cdot \frac{x-2}{(x-\frac{1}{2})^2(x-1)}\,dx =$

$$\int \frac{1}{4}\left(\frac{4}{x-\frac{1}{2}} + \frac{3}{(x-\frac{1}{2})^2} - \frac{4}{x-1}\right) dx$$

$$= \int \frac{1}{x-\frac{1}{2}} + \frac{3/4}{(x-\frac{1}{2})^2} - \frac{1}{x-1}\,dx$$

$$= \ln\left(x-\tfrac{1}{2}\right) + \frac{3}{4} \cdot \frac{\left(x-\frac{1}{2}\right)^{-1}}{-1} - \ln(x-1) + C$$

Method (Partial Fractions, v3): Let P and Q be polynomials with $\deg P < \deg Q$ such that Q factors into linear factors and irreducible quadratic factors (e.g. x^2+1)

- For every linear factor $(x-a)^n$, add $\dfrac{A_1}{x-a} + \dfrac{A_2}{(x-a)^2} + \cdots + \dfrac{A_n}{(x-a)^n}$.

- For every irreducible quadratic factor $(ax^2+bx+c)^n$, add

$$\frac{A_1x+B_1}{ax^2+bx+C} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

Ex: $$\int \frac{2x-3}{x^3+x}\,dx$$

$$= \frac{2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Multiply both sides by $x(x^2+1)$

$$2x-3 = A(x^2+1) + (Bx+C)(x)$$

$$2x-3 = (A+B)(x^2) + Cx + A$$

$A + B = 0$

$C = 2$

$A = -3$

$B = 3$

$$\int \frac{-3}{x} + \frac{3x+2}{x^2+1}\, dx$$

$$= \int -\frac{3}{x} + \frac{3x}{x^2+1} + \frac{2}{x^2+1}\, dx$$

$$= -3\int \frac{1}{x}\,dx + 3\int \frac{x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx$$

$u = x^2+1$
$du = 2x\,dx$

$$= -3\ln(x) + 3\int \frac{1}{u}\cdot\frac{1}{2}\,du + 2\tan^{-1}(x) + C$$

$$= -3\ln(x) + \frac{3}{2}\ln(u) + 2\tan^{-1}(x) + C$$

$$= -3\ln(x) + \frac{3}{2}\ln(x^2+1) + 2\tan^{-1}(x) + C.$$

Ex: $\displaystyle\int \frac{3x-1}{x^2(x^2+1)^2}\,dx$

x : linear

x^2+1 : irreducible quadratic (i.e. can't be factored)

$$\frac{3x-1}{x^2(x^2+1)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2}$$

$$3x-1 = A_1 x (x^2+1)^2 + A_2 (x^2+1)^2 + (B_1x+C_1)(x^2)(x^2+1)$$
$$+ (B_2x+C_2)(x^2)$$

$$3x-1 = A_1 x(x^4+2x^2+1) + A_2(x^4+2x^2+1) +$$
$$(B_1x+C_1)(x^4+x^2) + (B_2x+C_2)x^2$$

$$3x-1 = (A_1+B_1)x^5 + (A_2+C_1)x^4 + (2A_1+B_1+B_2)x^3$$

$$(2A_2+C_1+C_2)x^2 + A_1x + A_2$$

$$\left.\begin{aligned} A_1 + B_1 &= 0 \\ A_2 + C_1 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} 3 + B_1 &= 0 \\ -1 + C_1 &= 0 \end{aligned}\right\} \quad \begin{aligned} B_1 &= -3 \\ C_1 &= 1 \end{aligned}$$

$$\left.\begin{aligned} 2A_1 + B_1 + B_2 &= 0 \\ 2A_2 + C_1 + C_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} 6 - 3 + B_2 &= 0 \\ -2 + 1 + C_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} B_2 &= -3 \\ C_2 &= 1 \end{aligned}$$

$A_1 = 3$

$A_2 = -1$

$$\int \frac{3}{x} - \frac{1}{x^2} + \frac{-3x+1}{x^2+1} + \frac{-3x+1}{(x^2+1)^2} \, dx$$

$$3\ln(x) + \frac{1}{x} - 3\int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$-3\int \frac{x}{(x^2+1)^2}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$

$$\int \frac{x}{x^2+1}\,dx \quad : \quad \begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \\ \tfrac{1}{2}\,du &= x\,dx \end{aligned}$$

$$= \int \frac{1}{u} \cdot \frac{1}{2}\,du$$

$$= \frac{1}{2}\ln(u) + C$$

$$= \frac{1}{2}\ln(x^2+1) + C.$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C.$$

$$\int \frac{x}{(x^2+1)^2}\,dx :$$

$$u = x^2+1$$
$$du = 2x\,dx$$
$$\frac{1}{2}\,du = x\,dx$$

$$= \int \frac{1}{u^2} \cdot \frac{1}{2}\,du$$

$$= -\frac{1}{2}\,u^{-1} + C$$

$$= -\frac{1}{2}\,(x^2+1)^{-1} + C$$

$$\int \frac{1}{(x^2+1)^2}\,dx$$

$$x = \tan(\theta)$$
$$dx = \sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{(\tan^2(\theta)+1)^2} \cdot \sec^2(\theta)\,d\theta$$

$$= \int \frac{1}{\sec^4(\theta)} \cdot \sec^2(\theta)\, d\theta$$

$$= \int \frac{1}{\sec^2(\theta)}\, d\theta$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$= \int \cos^2(\theta)\, d\theta$$

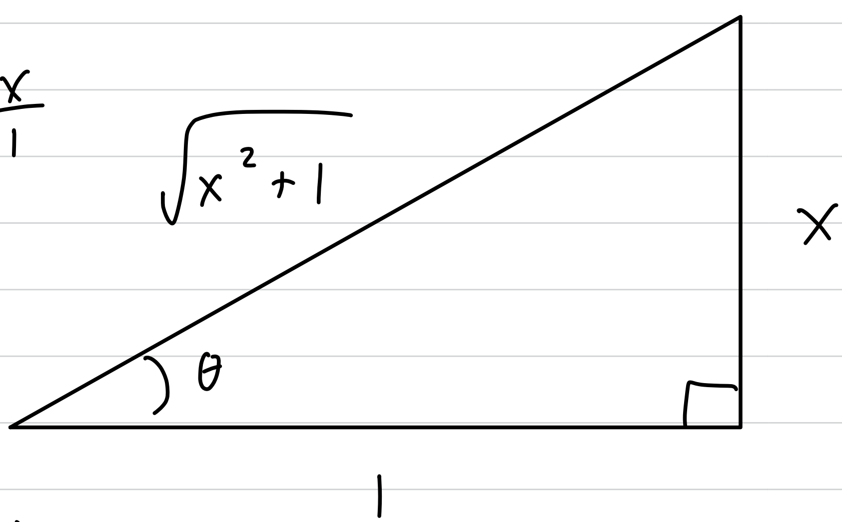
$$= \int \frac{1 + \cos(2\theta)}{2}\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta) + C\right)$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2} \cdot 2 \cdot \sin(\theta) \cdot \cos(\theta) + C\right)$$

$$\tan(\theta) = \frac{x}{1}$$

$$\theta = \tan^{-1}(x)$$



$$\sin(\theta) = \frac{x}{\sqrt{x^2+1}}$$

$$\cos(\theta) = \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{1}{2}\left(\tan^{-1}(x) + \frac{x}{\sqrt{x^2+1}} \cdot \frac{1}{\sqrt{x^2+1}}\right) + C$$

$$= \frac{1}{2}\left(\tan^{-1}(x) + \frac{x}{x^2+1}\right) + C$$

$$= 3\ln(x) + \frac{1}{x} - 3\left(\frac{1}{2}\ln(x^2+1)\right) + \tan^{-1}(x)$$

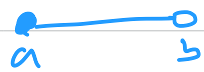
$$-3\left(-\frac{1}{2}(x^2+1)^{-1}\right) + \frac{1}{2}\left(\tan^{-1}(x) + \frac{x}{x^2+1}\right) + C.$$

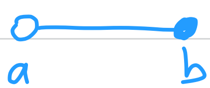
Improper Integrals
(§3.7)

Comment: An Improper integral is one where either:

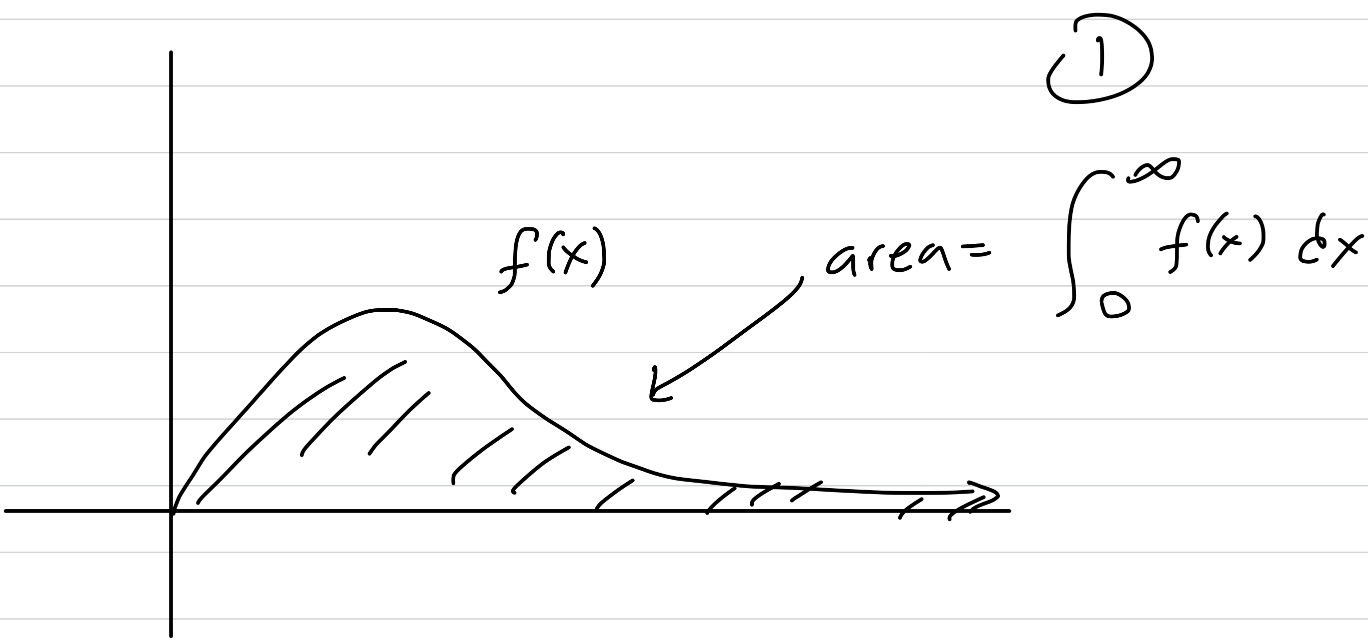
① In $\int_a^b f(x)\,dx$, $a = -\infty$ or $b = \infty$ (or both)

② In $\int_a^b f(x)\,dx$, f is only continuous on $[a, b)$ or $(a, b]$

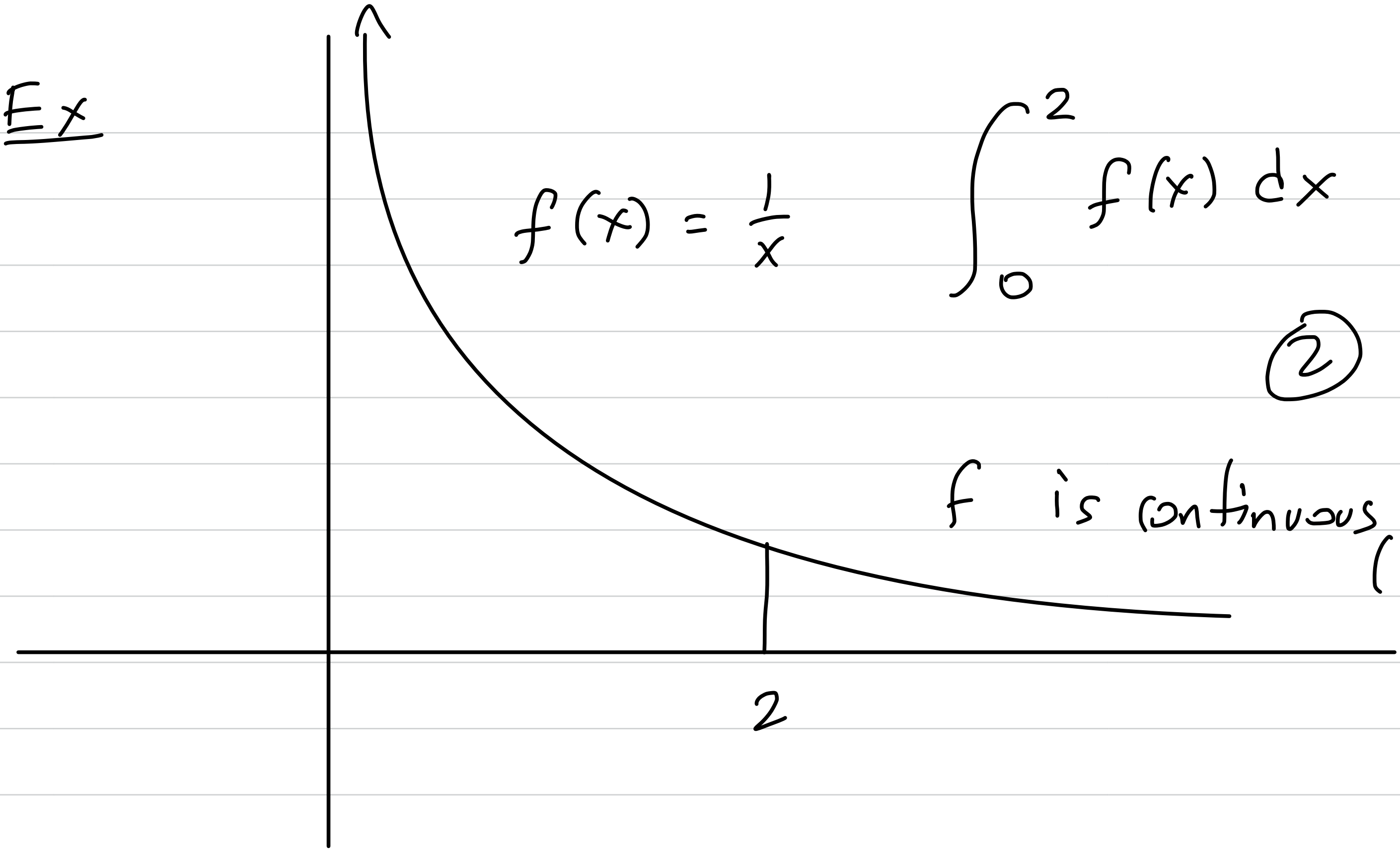




Ex:



Ex



$$\int_0^2 f(x)\,dx$$

2

f is continous on $(0, 2]$

Def: Let $f(x)$ be a function continous on $[a, \infty)$. Then $\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$.

If f is continous on $(-\infty, b]$, then $\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$.

If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

For any of these, if the limit is infinite, then we say the improper integral <u>diverges</u>.

Ex:

$$\int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x}\,dx$$

$$= \lim_{b \to \infty} \left[\ln(x) \right] \Big|_{1}^{b}$$

$$= \lim_{b \to \infty} \left(\ln(b) \right)$$

$$= \infty$$

So $\int_{1}^{\infty} \frac{1}{x}\,dx$ diverges.

Ex: $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

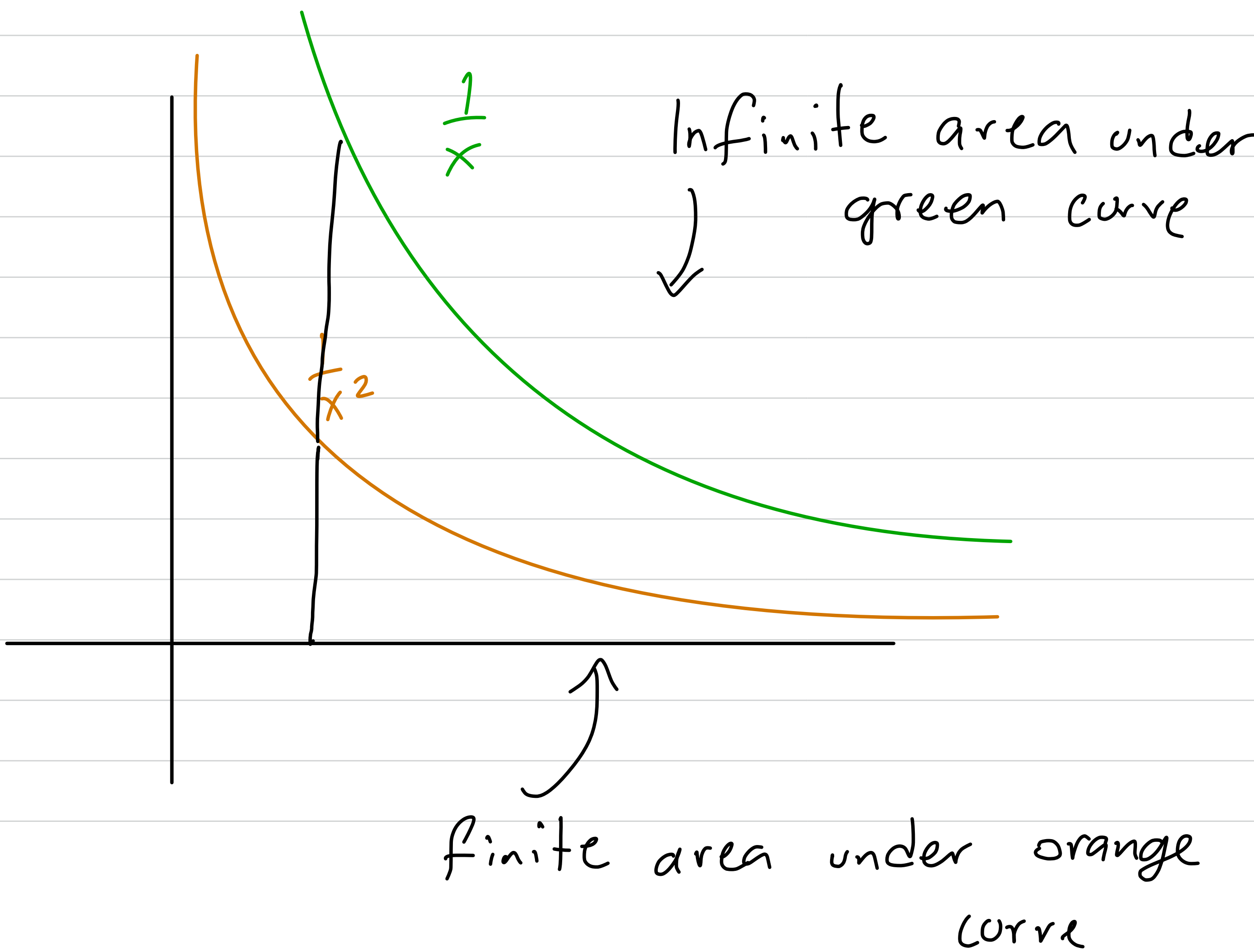
$$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]\Big|_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right)$$

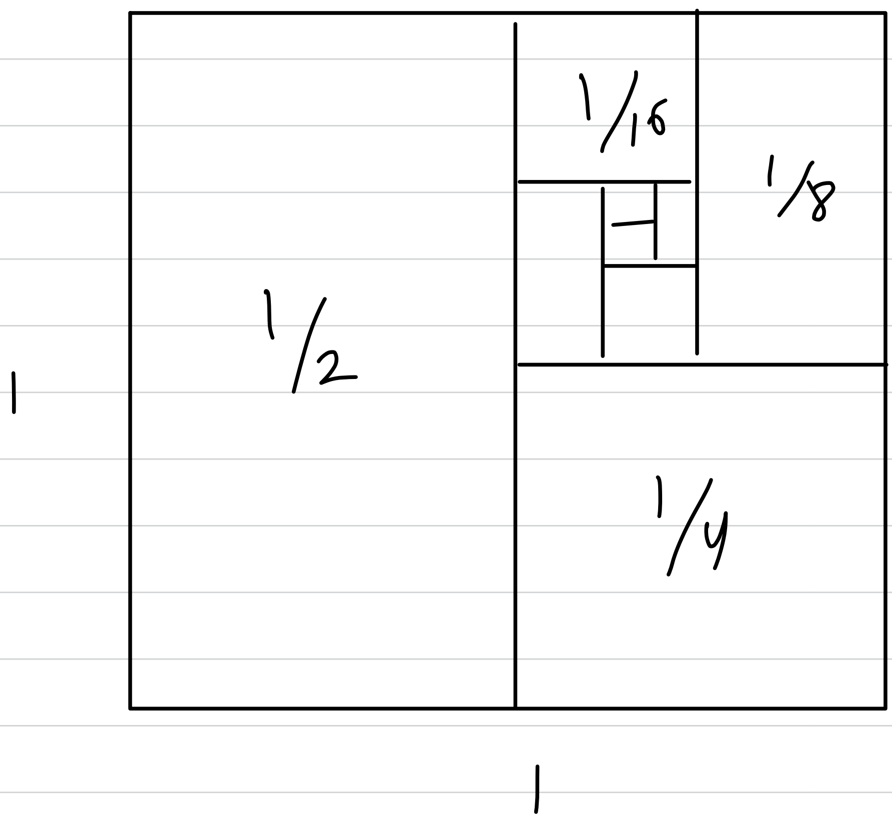
$$= 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

So



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

$$\sum_{i=1}^{\infty} \frac{1}{i} = \infty$$

$$\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx = \lim_{a\to-\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) \right]\Bigg|_a^0$$

$$= \underbrace{\frac{1}{2}\tan^{-1}(0)}_{0} - \lim_{a\to-\infty} \underbrace{\frac{1}{2}\tan^{-1}\left(\frac{a}{2}\right)}_{?}$$

Question: when does tan go to $-\infty$?

$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \to -\infty$ when sin is negative

and $\cos(\theta) \to 0$ — so at $\theta = -\pi/2$.

Therefore, $\lim_{a\to-\infty} \frac{1}{2}\tan^{-1}\left(\frac{a}{2}\right) = \frac{1}{2}\left(-\frac{\pi}{2}\right)$

$$= -\pi/4$$

$$\int_{-\infty}^{0} \frac{1}{x^2+4}\,dx = \pi/4.$$

$$\lim_{x\to\infty} \tan^{-1}(x) = \pi/2$$

$$\lim_{x \to -\infty} \tan^{-1}(x) = -\pi/2$$

<u>Ex</u>: $\int_{-\infty}^{\infty} x e^{x} \, dx$

$$= \int_{-\infty}^{0} x e^{x} \, dx + \int_{0}^{\infty} x e^{x} \, dx$$

$$u = x \qquad v = e^{x}$$

$$\downarrow \qquad \uparrow$$

$$du = dx \qquad dv = e^{x} \, dx$$

$$\int x e^{x} \, dx = x e^{x} - \int e^{x} \, dx$$

$$= x e^{x} - e^{x}$$

$$\int_{-\infty}^{\infty} x e^{x} \, dx = \lim_{a \to -\infty} \left[x e^{x} - e^{x} \right] \Big|_{a}^{0} +$$

$$\lim_{b \to \infty} \left[x e^{x} - e^{x} \right] \Big|_{0}^{b}$$

$$\lim_{a \to -\infty} \left[xe^{x} - e^{x} \right] \Big|_{a}^{0} =$$

$$-e^{0} - \lim_{a \to -\infty} \left(ae^{a} - e^{a} \right)$$

$$= -1 - \lim_{a \to -\infty} \underbrace{\left(e^{a} \right)}_{\downarrow \, 0} \underbrace{\left(a - 1 \right)}_{\downarrow \, -\infty}$$

$$e^{a} = \frac{1}{e^{-a}}$$

$$= -1 - \lim_{a \to -\infty} \frac{a-1}{e^{-a}}$$

$$= -1 - \lim_{a \to -\infty} \frac{\frac{d}{da}\left[a-1\right]}{\frac{d}{da}\left[e^{-a}\right]}$$ by L'Hôpital

$$= -1 - \lim_{a \to -\infty} \frac{1}{-e^{-a}} = -1 - 0 = -1$$

$$\lim_{b\to\infty} \left[xe^x - e^x\right]\Big|_0^b = \lim_{b\to\infty} \left(\left(b-1\right)e^b\right) + 1$$

$$= \infty$$

$$\int_{-\infty}^{\infty} xe^x \, dx = \infty - 1 = \infty$$

So the integral diverges.