

# Abstract Algebra Notes

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## I — Groups

**Definition 1.1:** A **group** is a set  $G$  equipped with a binary operation such that

1.  $ab \in G$  for all  $a, b \in G$ .
2.  $(ab)c = a(bc)$  for all  $a, b, c \in G$ .
3. There is an  $e \in G$  with  $ae = ea = a$  for all  $a \in G$ .
4. For all  $a \in G$ , there is an  $a^{-1} \in G$  with  $aa^{-1} = a^{-1}a = e$ .

**Example:** One important type of group is a *symmetry group*, formed by taking the collection  $S_X$  of symmetries of a set  $X$  — that is, structure-preserving bijections from  $X$  to itself. If  $X = \mathbb{R}$ , for instance, then  $S_X$  contains translational symmetries and stretching/shrinking ones.

**Definition 1.2:** Let  $n \in \mathbb{N}$ . The **integers modulo  $n$**  are the set  $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ .  $\mathbb{Z}_n$  is a group, with the operation given by addition mod  $n$ .

**Definition 1.3:** Let  $n \in \mathbb{N}$ . The group  **$U(n)$**  is defined as  $U(n) = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$ , with the operation given by multiplication mod  $n$ .

**Definition 1.4:** The **dihedral group** of degree  $n$  is the group  $D_n$  of symmetries of a regular  $n$ -gon, given by  $D_n = \{e, r, r^2, \dots, r^{n-1}, s, rs, r^2s, \dots, r^{n-1}s\}$ , where  $r$  is a rotation counter-clockwise by  $\frac{2\pi}{n}$ ,  $s$  is a reflection over the  $x$ -axis, and the operation is function composition. Note that  $sr = r^{-1}s$ .

**Definition 1.5:** The **symmetric group** of degree  $n$  is the group  $S_n$  of permutations on  $n$  elements, defined as  $\{\sigma : \{1, \dots, n\} \hookrightarrow \{1, \dots, n\}\}$ , where each  $\sigma$  is a bijection and the operation is given by function composition. We use **cycle notation** to denote elements of  $S_n$ : the element  $(124) \in S_4$ , for example, denotes the function  $\sigma$  with  $\sigma(1) = 2$ ,  $\sigma(2) = 4$ ,  $\sigma(4) = 1$ , and  $\sigma(3) = 3$ .

**Definition 1.6:** The **general linear group** of degree  $n$  over  $k$  is  $GL_n(k) = \{A \in M_n(k) \mid \det A \neq 0\}$ , with the operation given by matrix multiplication. Typically,  $k$  is  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{Z}$ .

**Definition 1.7:** The **special linear group** of degree  $n$  over  $k$  is  $SL_n(k) = \{A \in M_n(k) \mid \det A = 1\}$ , with the operation again given by matrix multiplication.

**Definition 1.8:** The **orthogonal group** of degree  $n$  is  $O_n = \{A \in M_n(\mathbb{R}) \mid AA^T = I\}$ , and the **special orthogonal group** of degree  $n$  is  $SO_n = \{A \in O_n \mid \det A = 1\}$ . The **unitary group**,  $U_n$ , and **special unitary group**,  $SU_n$ , are defined identically, except their matrices are taken from  $M_n(\mathbb{C})$  and require that  $A\overline{A}^T = I$ .

**Definition 1.9:** The **order** of a group  $G$  is  $|G|$ . A **finite group** is one that has finite order.

**Definition 1.10:** A group  $G$  is **Abelian** if  $ab = ba$  for all  $a, b \in G$ .

**Proposition 1.11:** Let  $G$  be a group. Then  $e \in G$  is unique.

**Proof:** Suppose there were  $e, e' \in G$  such that  $ae = ea = ae' = e'a = a$  for all  $a \in G$ . Then  $ee' = e$  and  $ee' = e'$ , so  $e = e'$ .

**Proposition 1.12:** Let  $G$  be a group. If  $ab = ac$  for  $a, b, c \in G$ , then  $b = c$ , and similarly, if  $ab = cb$ , then  $a = c$ .

**Proposition 1.13:** Let  $G$  be a group and  $a \in G$ . Then  $a^{-1}$  is unique.

**Proof:** If there were  $a^{-1}, (a^{-1})' \in G$  such that  $aa^{-1} = a^{-1}a = a(a^{-1})' = (a^{-1})'a = e$ , then  $aa^{-1} = a(a^{-1})'$ , so  $a^{-1}aa^{-1} = a^{-1}a(a^{-1})'$ , and so  $a^{-1} = (a^{-1})'$ .

**Definition 1.14:** Let  $G$  be a group. A set  $H \subseteq G$  is a **subgroup** of  $G$ , written  $H \leq G$ , if  $H$  is itself a group under  $G$ 's operation.

**Example:** For all groups,  $\{e\} \leq G$ , called the *trivial subgroup*, and  $G \leq G$ .

**Example:** If  $X$  is a regular, nonoriented  $n$ -gon and  $X'$  is a regular, oriented  $n$ -gon (so reflections count as symmetries of  $X$ , but not of  $X'$ ), then  $S'_X \leq S_X$ .

**Definition 1.15:**  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  is a group with its operation given by stretching and rotating the plane —  $z = re^{i\theta}$  represents the symmetry of stretching by  $r$  and rotating by  $\theta$  (or equivalently, moving 1 to  $z$ ).  $S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\} = \{z \in \mathbb{C}^* \mid |z| = 1\}$  is also a group under the same operation, so  $S^1 \leq \mathbb{C}^*$ .

**Definition 1.16:** The **quaternions** are the set

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = -1, ijk = -1\}.$$

$\mathbb{H}$  forms a group under addition, and  $\mathbb{H}^*$  a group under multiplication. Similarly to  $S^1$ , we define the **unit 3-sphere** as  $S^3 = \{z \in \mathbb{H}^* \mid |z| = 1\} \leq \mathbb{H}^*$ .

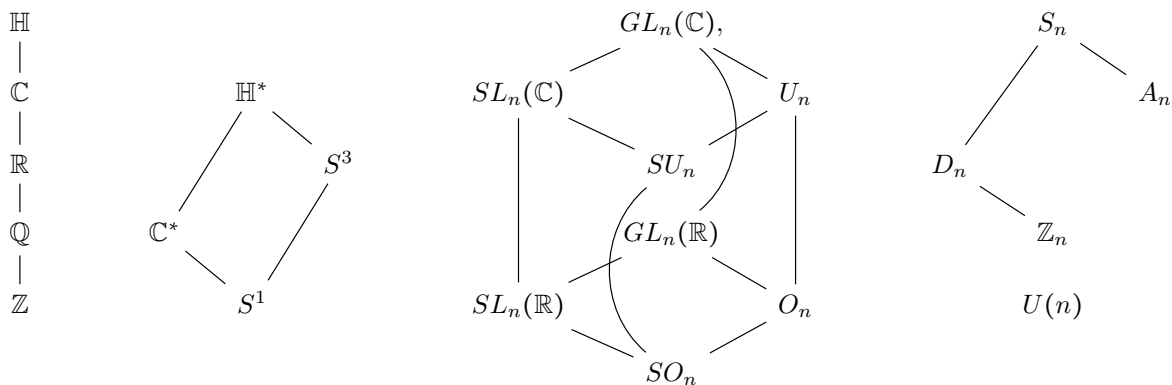
**Proposition 1.17:** Let  $G$  be a group. Then  $H \subseteq G$  is a subgroup of  $G$  if and only if

1.  $H \neq \emptyset$ .
2.  $ab \in H$  for all  $a, b \in H$ .
3. For all  $a \in H$ ,  $a^{-1} \in H$ .

**Proof:** ( $\Rightarrow$ ) If  $H \leq G$ , then the only statement to prove is that the identity of  $H$  is the same as that of  $G$ , so that we can be sure that  $a^{-1} \in H$  is the same as  $a^{-1}$  in  $G$ . If the two identities are  $e_G$  and  $e_H$ , then  $e_H e_G = e_H$  in  $G$  and  $e_H e_H = e_H$  in  $H$ , so  $e_H e_G = e_H e_H$  in  $G$ , and therefore  $e_G = e_H$ . Thus the inverses are the same, and so all three conditions are met.

( $\Leftarrow$ ) Let  $a, b, c \in H$  (which is valid, since  $H$  is nonempty). Then  $a, b, c \in G$ , so  $(ab)c = a(bc)$ . Also, since  $a^{-1} \in H$  (and this is the inverse from  $G$ ),  $aa^{-1} = e_G \in H$ . We already know  $H$  is closed under  $G$ 's operation, so  $H \leq G$ .

**Example:** The major groups:



**Definition 1.18:** Let  $G$  be a group and  $a \in G$ . The **group generated by  $a$**  is  $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$ .

**Example:** In  $D_n$ ,  $\langle r \rangle = \{e, r, r^2, \dots, r^{n-1}\}$  and  $\langle s \rangle = \{e, s\}$ .

**Definition 1.19:** Let  $G$  be a group. The **order** of  $a \in G$  is  $|a| = |\langle a \rangle|$ , or equivalently, the smallest  $n \in \mathbb{N}$  such that  $a^n = e$  if it exists, or  $\infty$  if it does not.

**Definition 1.20:** The orders of the elements in  $U(9) = \{1, 2, 4, 5, 7, 8\}$  are

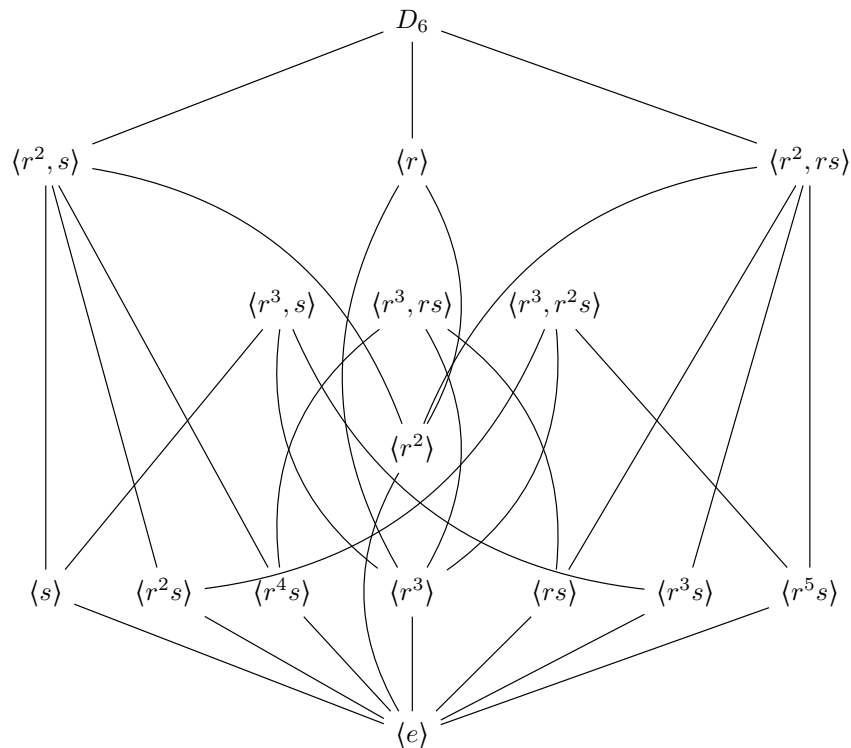
$$|1| = 1, \quad |2| = 6, \quad |4| = 3, \quad |5| = 6, \quad |7| = 3, \quad |8| = 2.$$

**Proposition 1.21:** Let  $G$  be a group and  $a \in G$ . Then  $|a| = |a^{-1}|$ .

**Proof:** Since  $a^k = e$  if and only if  $(a^k)^{-1} = e$ , if and only if  $(a^{-1})^k = e$ , the smallest  $n \in \mathbb{N}$  such that  $a^n = e$  will also be the smallest  $m \in \mathbb{N}$  such that  $(a^{-1})^m = e$ . Thus  $|a| = |a^{-1}|$ .

**Definition 1.22:** A group  $G$  is **cyclic** if  $G = \langle a \rangle$  for some  $a \in G$ , called a **generator** of  $G$ .

**Example:** The subgroup lattice of  $D_6$ :

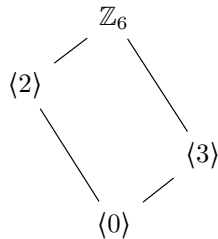


**Theorem 1.23:** A subgroup of a cyclic group is cyclic.

**Proof:** Let  $G = \langle a \rangle$  and  $H \leq G$ . Then for all  $h \in H$ ,  $h = a^m$  for some  $m \in \mathbb{Z}$ . Let  $S = \{m \in \mathbb{N} \mid a^m \in H\}$ , let  $m_0$  be the minimum element of  $S$ , and let  $h = a^m \in H$  be arbitrary. Then  $m = m_0q + r$  for some  $q, r \in \mathbb{Z}$  with  $0 \leq r < m_0$ , so  $a^m = a^{m_0q+r}$ , or equivalently,  $a^r = a^{m-m_0q}$ . Since  $a^m = h \in H$  and  $a^{m_0q} = (a^{m_0})^q \in H$ ,  $a^{m-m_0q} = a^r \in H$ . Thus if  $r \neq 0$ ,  $r \in S$  by definition, but  $r < m_0$ , the smallest element in  $S$ .  $\nmid$  Thus  $r = 0$ , and so  $h = (a^{m_0})^q$  for some  $q \in \mathbb{Z}$ . Since  $h$  was arbitrary,  $H = \langle a^{m_0} \rangle$ .

**Example:** Since  $\mathbb{Z}$  is cyclic, every subgroup of  $\mathbb{Z}$  is of the form  $\langle a \rangle$  for  $a \in \mathbb{Z}$ .

**Example:** The subgroups of  $\mathbb{Z}_6$  are  $\langle 0 \rangle$ ,  $\langle 3 \rangle$ ,  $\langle 2 \rangle$ , and  $\langle 1 \rangle = \mathbb{Z}_6$ .



**Theorem 1.24:** Let  $G = \langle a \rangle$  be cyclic with  $|G| = n$ . Then for any  $k \in \mathbb{N}$ ,  $|a^k| = \frac{n}{\gcd(n,k)}$ .

**Proof:** Let  $d = \gcd(n, k)$ . We claim  $\langle a^k \rangle = \langle a^d \rangle$ .

( $\subseteq$ ) Since  $d|k$ ,  $k = dq$  for some  $q \in \mathbb{Z}$ . Then  $a^k = (a^d)^q \in \langle a^d \rangle$ , so  $\langle a^k \rangle \subseteq \langle a^d \rangle$ .

( $\supseteq$ ) By Bezout's identity,  $d = ks + nt$  for some  $s, t \in \mathbb{Z}$ , so  $a^d = a^{ks+nt} = (a^k)^s (a^n)^t = (a^k)^s (e)^t = (a^k)^s \in \langle a^k \rangle$ . Thus  $\langle a^d \rangle \subseteq \langle a^k \rangle$ .

Now  $\langle a^k \rangle = \langle a^d \rangle$ , so in particular,  $|a^k| = |a^d|$ . We know  $|a^d| = \frac{n}{d}$ , since otherwise, if  $|a^d| = m < \frac{n}{d}$ ,  $a^{dm} = e$  and  $dm < n$ , so  $|G| = |a| < n$ .  $\nmid$  Thus  $|a^k| = |a^d| = \frac{n}{d}$ .

**Corollary 1.24.1:** Let  $G = \langle a \rangle$  with  $|G| = n$ . Then the generators of  $G$  are  $a^k$  with  $\gcd(k, n) = 1$ .

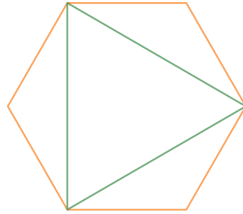
**Proposition 1.25:** Let  $G$  be a cyclic group and  $a \in G$  such that  $G \neq \langle a \rangle$ . Then no element of  $\langle a \rangle$  is a generator for  $G$ .

**Proof:** Suppose  $|G| = n$ . Since  $\langle a \rangle \neq G$ ,  $|a| < |G|$ . Now every element of  $\langle a \rangle$  is of the form  $a^k$  for some  $k \in \mathbb{Z}$ , and  $|a^k| = \frac{|a|}{\gcd(k, |a|)} \leq |a| < |G|$ , so no element of  $\langle a \rangle$  is a generator for  $G$ .

**Definition 1.26:** Two groups  $G$  and  $H$  are **isomorphic** if there is a bijection  $\varphi : G \rightarrow H$  such that  $\varphi(ab) = \varphi(a)\varphi(b)$  for all  $a, b \in G$ .  $\varphi$  is called an **isomorphism**, and we write  $G \simeq H$ .

**Example:**  $\mathbb{Z}_4 \simeq \langle i \rangle \leq \mathbb{C}^*$ , since the map  $\varphi : \mathbb{Z}_4 \rightarrow \langle i \rangle$  given by  $\varphi(a) = i^a$  is an isomorphism.

**Example:** Let  $H = \{x \in D_6 \mid x \text{ preserves an inscribed triangle}\} \leq D_6$ . Then  $H \simeq D_3$ .



**Proposition 1.27:** Isomorphisms preserve identities, inverses, subgroups, and order (both of elements and groups).

## II — The Symmetric Group

**Definition 2.1:** A  **$k$ -cycle** is an element of  $S_n$  of the form  $(a_1 \cdots a_k)$ .

**Definition 2.2:** A **transposition** is a 2-cycle.

**Proposition 2.3:** Every element of  $S_n$  can be expressed as a product of disjoint cycles.

**Example:** To remove duplicate numbers, start with 1 and pass it through the permutation, then pass the result through, and so on. Since every permutation is a bijection by definition, every number will be sent to a unique other. Once the permutation results in 1 again, close the cycle and continue with the next lowest unused number. For instance, if  $\sigma = (124)(314)$ ,  $\sigma(1) = 1$ , so we start again with 2:  $\sigma(2) = 4$ . Then  $\sigma(4) = 3$ , and  $\sigma(3) = 2$ , so the permutation is  $(243)$ .

**Proposition 2.4:** Disjoint cycles commute.

**Proposition 2.5:** Every element of  $S_n$  can be expressed as a product of transpositions.

**Proof:** For an arbitrary  $\sigma \in S_n$ , express  $\sigma$  as a product of disjoint cycles. Then for an individual cycle  $(a_1 \cdots a_k)$ ,  $(a_1 \cdots a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_3)(a_1 a_2)$ .

**Definition 2.6:** Let  $\pi \in S_n$ . The **permutation matrix** for  $\pi$  is  $P_\pi$ , defined by  $(P_\pi)_{ij} = 1$  if  $\pi(j) = i$  and 0 if not.

**Example:** For  $\pi = (243) \in S_4$ ,

$$P_\pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

**Proposition 2.7:** All permutation matrices are orthogonal (that is, the columns form an orthonormal basis for  $\mathbb{R}^n$ , so  $\det P_\pi = \pm 1$ ).

**Definition 2.8:** A permutation  $\pi \in S_n$  is **even** if  $\det P_\pi = 1$ , and **odd** if  $\det P_\pi = -1$ .

**Definition 2.9:** The **alternating group** of degree  $n$  is  $A_n = \{\pi \in S_n \mid \pi \text{ is even}\} \leq S_n$ .

**Proposition 2.10:** For  $n \geq 2$ ,  $|A_n| = \frac{n!}{2}$ .

**Proof:** Define  $f : A_n \longrightarrow S_n \setminus A_n$  by  $f(\pi) = (12)\pi$ . Then  $f^{-1}(\pi) = (12)\pi$ , so  $f$  is invertible. Thus  $|A_n| = |S_n \setminus A_n|$ , so  $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$ .

**Theorem 2.11:** Let  $\pi \in S_n$ . Then  $\pi \in A_n$  if and only if  $\pi$  can be expressed as the product of evenly many transpositions. Moreover, when  $\pi$  is expressed in such a way, there are always the same number of transpositions (mod 2).

**Proof:** Suppose  $\pi = \tau_1 \cdots \tau_k$ , where each  $\tau_i$  is a transposition. Then  $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$ . Since  $P_{\tau_i}$  has exactly 2 columns permuted from  $I_n$ ,  $\det P_{\tau_i} = -1$ , so  $\det P_\pi = (-1)^k$ . Thus  $\pi$  is even if and only if  $k$  is even, proving both claims.



**Theorem 2.12:** Let  $n \geq 3$ . Then  $A_n$  is generated by 3-cycles.

**Proof:** Let  $\pi \in A_n$ . Then  $\pi$  is the product of evenly many transpositions. Group them in pairs, and consider an arbitrary pair, say  $(ab)(cd)$ .

If  $(ab)$  and  $(cd)$  have no common entries, then  $(ab)(cd) = (ab)(bc)(bc)(cd) = (abc)(bcd)$  (which is valid, since  $b \neq c$  by assumption).

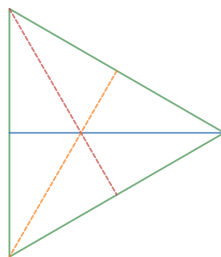
If  $(ab)$  and  $(cd)$  have one entry in common, then without loss of generality,  $(cd) = (bd)$ . Then  $(ab)(cd) = (ab)(bd) = (abd)$ .

Finally, if  $(ab)$  and  $(cd)$  share two entries, they are identical, so  $(ab)(cd) = e$ . Thus each pair is expressible as either one or two 3-cycles or the identity, and since we can omit the identities,  $\pi$  is the product of 3-cycles.

### III — Cosets and Lagrange's Theorem

**Definition 3.1:** Let  $G$  be a group,  $H \leq G$ , and  $a \in G$ . The **left coset** of  $H$  in  $G$  with representative  $a$  is the set  $aH = \{ah \mid h \in H\}$ , and right cosets are defined similarly.

**Example:** Let  $H = \langle r \rangle \leq D_3$ . Then  $eH = rH = r^2H = \{e, r, r^2\}$  and  $sH = rsH = r^2sH = \{s, rs, r^2s\}$ . If  $K = \langle s \rangle$ , then  $eK = sK = \{e, s\}$ ,  $rK = rsK = \{r, rs\}$ , and  $r^2sK = \{r^2, r^2s\}$ .



Here,  $K$  fixes the solid blue line,  $rK$  moves it to the red dashed line, and  $r^2K$  to the yellow dashed line. The red and yellow lines are called **conjugate substructures** to the blue line. Conjugate substructures can be found by applying every element of a group — or just those that generate distinct cosets — to the original substructure.

**Example:** The cosets of  $S^1 \leq \mathbb{C}^*$  are all the concentric circles around the origin.

**Comment:** In general, left cosets are more important than right ones, since function composition operates right-to-left.

**Proposition 3.2:** Let  $G$  be a group and  $H \leq G$ . Then the following are equivalent:

1.  $aH = bH$ .
2.  $b \in aH$ .
3.  $b^{-1}a \in H$ .

**Definition 3.3:** Let  $G$  be a group and  $H \leq G$ . The **index** of  $H$  in  $G$  is  $[G : H] = |\{aH \mid a \in G\}|$ .

**Theorem 3.4: (Lagrange)** Let  $G$  be a group and  $H \leq G$ . Then the set of left cosets  $\{aH \mid a \in G\}$  partitions  $G$  into subsets of equal cardinality.

**Proof:** For all  $b \in G$ ,  $b \in bH$ , so  $G \subseteq \bigcup aH$ . And clearly  $\bigcup aH \subseteq G$ , so  $G = \bigcup aH$ . Now we claim  $aH \cap bH$  is either  $aH$  or  $\emptyset$  for all  $a, b \in G$ . Suppose  $aH \cap bH \neq \emptyset$ . Then there is a  $c \in aH \cap bH$ , so by the previous result,  $cH = aH$  and  $cH = bH$ , and therefore  $aH = bH$ . Thus  $G$  is the disjoint union of the cosets of  $H$ , so they are a partition.

To show that all cosets have the same cardinality, define a function  $f : aH \rightarrow H$  by  $f(ah) = h$ . Then  $f$  is a bijection, so  $|aH| = |H|$ .

**Corollary 3.4.1:** Let  $G$  be a group and  $H \leq G$ . Then  $|G| = [G : H]|H|$ .

**Proof:** Since  $G = \bigsqcup aH$ ,  $|G| = \sum |aH| = \sum |H| = \sum_{k=1}^{[G:H]} |H| = [G : H]|H|$ .

**Corollary 3.4.2:** Let  $G$  be a finite group and let  $H \leq G$ . Then  $|H| \mid |G|$ .

**Corollary 3.4.3:** Let  $G$  be a finite group and let  $a \in G$ . Then  $|a| \mid |G|$ .

**Corollary 3.4.4:** Let  $G$  be a finite group and let  $a \in G$ . Then  $a^{|G|} = e$ .

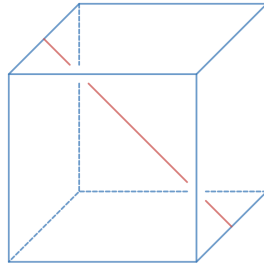
**Corollary 3.4.5:** Let  $a, n \in \mathbb{N}$  with  $\gcd(a, n) = 1$ . Then  $a^{\varphi(n)} \equiv 1 \pmod{n}$ .

**Proof:** With  $G = U(n)$ ,  $|G| = |\{a \in \mathbb{N} \mid a < n, \gcd(a, n) = 1\}| = \varphi(n)$ , so  $a^{|G|} = a^{\varphi(n)} = 1$ .

**Corollary 3.4.6:** Let  $a \in \mathbb{N}$  and  $p$  prime. If  $p \nmid a$ , then  $a^{p-1} \equiv 1 \pmod{p}$ .

**Example:** Find the order of the group of orientation-preserving symmetries of a regular cube.

Let  $G$  be this group and let  $H \leq G$  be the subgroup fixing the red line.



There are six elements of  $G$  that fix the red line (pick one of the two vertices on the line to face up — then there are three symmetries that permute the three vertices adjacent to it, making six in total). And there are four conjugate substructures of the line, so there are four distinct cosets of  $H$ . Thus  $[G : H] = 4$ , and therefore  $|G| = [G : H]|H| = 24$ .

**Theorem 3.5:** There is only one cyclic group of each finite order, and only one of any infinite order, up to isomorphism.

**Proof:** Let  $G$  be cyclic with  $|G| = n$ . Then  $G = \langle a \rangle$  for some  $a \in G$ , so the map  $\varphi : G \rightarrow \mathbb{Z}_n$  given by  $a^k \mapsto k \cdot 1$  is an isomorphism.

If  $G$  is infinite, then  $G$  must be countable, since it can be written as  $\{e, a, a^2, \dots\}$ . Then there is an isomorphism given by  $a^k \mapsto k \cdot 1$ . Not that it is *not* sufficient to claim that  $G \simeq \mathbb{Z}$  since  $G$  is countable — that definition requires only a bijection, which need not be multiplicative.

**Proposition 3.6:** Let  $\varphi : G \rightarrow G'$  be an isomorphism. Then

1.  $|G| = |G'|$ .
2.  $\varphi(e_G) = e_{G'}$ .
3.  $\varphi(a^{-1}) = \varphi(a)^{-1}$ .
4.  $|\varphi(a)| = |a|$ .

5. If  $H \leq G$ , then  $\varphi(H) \leq \varphi(G) = G'$ .

6. If  $G$  is Abelian, then so is  $G'$ .

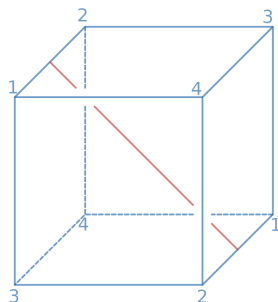
**Theorem 3.7: (Cayley)** Let  $G$  be a finite group of order  $n$ . Then  $G$  is isomorphic to a subgroup of  $S_n$ .

**Proof:** Let  $a \in G$  and define  $\lambda_a : G \rightarrow G$  by  $\lambda_a(b) = ab$ . Then if we number the elements of  $G$  from 1 to  $n$ ,  $\lambda_a \in S_n$ . Let  $G' = \{\lambda_a \in S_n \mid a \in G\}$ . We claim  $G \simeq G'$ , where the isomorphism is given by  $\lambda(a) = \lambda_a$ .

Clearly,  $\lambda$  is onto, since  $G' = \lambda(G)$  by definition. To show that  $\lambda$  is injective, suppose  $\lambda(a) = \lambda(b)$ . Then  $\lambda_a = \lambda_b$ , so in particular,  $\lambda_a(e) = \lambda_b(e)$ . Thus  $a = b$ . Finally,  $\lambda$  is multiplicative, since  $\lambda(ab) = \lambda_{ab} = \lambda_a \lambda_b = \lambda(a)\lambda(b)$ . Thus  $\lambda$  is an isomorphism, so  $G \simeq G' \leq S_n$ .

**Example:** Determine the rotation group of the cube.

Let the group be  $G$ . By our previous work, we know  $|G| = 24$ , and by Cayley's Theorem, we know  $G \leq S_6$ . But  $|S_6| = 720$ , and there are far too many order-24 subgroups of  $S_6$  to determine the correct one (or even easily make a complete list). Instead, fix the four diagonals from the previous example and number the vertices 1–4 by which line they are contained in. Then each element of  $G$  must fix one of the four diagonals by the previous example, so  $G \leq S_4$ .



Now each rotation about one of the four diagonal axes fixes one number and permutes the other three, so it is a 3-cycle. Since we have all 8 possible 3-cycles from this method,  $G$  contains  $A_4$ . Draw a new line that connects the midpoints of the edges with endpoints 1 and 2 (shown in the above figure). A rotation about this axis fixes 3 and 4 and permutes 1 and 2, so  $(12) \in G$ . Thus  $|G| \geq |A_4| + 1 = 13$ , since  $(12) \notin A_4$ , and since  $|G| \mid |S_4| = 24$ ,  $|G| = 24$ . Thus  $G = S_4$ .

Since we can form a regular octahedron by replacing each face of the cube with a vertex and each vertex with a face (that is, the cube and octahedron are *dual*), the rotation group of the octahedron is also  $S_4$ .

**Definition 3.8:** Let  $G$  be a group. An **automorphism** of  $G$  is an isomorphism from  $G$  to itself. Notice that every automorphism is completely determined by where it sends the generators of  $G$ , since  $\varphi(a_1^{k_1} \dots a_n^{k_n}) = \varphi(a_1)^{k_1} \dots \varphi(a_n)^{k_n}$ .

**Definition 3.9:** Let  $G$  be a group. The **automorphism group** of  $G$  is  $\text{Aut } G$ , the group of automorphisms of  $G$ .

**Example:** The automorphism group of  $\mathbb{Z}$  is  $\{e, \varphi\}$ , where  $e : 1 \mapsto 1$  and  $\varphi : 1 \mapsto -1$ . Thus  $\text{Aut } \mathbb{Z} \simeq \mathbb{Z}_2$ .

**Example:** For any group  $G$ ,  $\gamma_a : G \rightarrow G$  defined by  $\gamma_a(b) = aba^{-1}$  is an automorphism.

**Definition 3.10:** Let  $G$  be a group. The **inner automorphism group** of  $G$  is  $\text{Inn } G = \{\gamma_a \mid a \in G\}$ .

**Definition 3.11:** Let  $G$  and  $G'$  be groups. The **direct product** of  $G$  and  $G'$  is  $G \times G' = \{(a, a') \mid a \in G, a' \in G'\}$ , where the group operation is defined as  $(a, a')(b, b') = (ab, a'b')$ .

**Proposition 3.12:** Let  $G$  and  $G'$  be groups. Then  $G \times G'$  is also a group, and  $|G \times G'| = |G||G'|$ .

**Proposition 3.13:** Let  $G$  and  $G'$  be groups and let  $(a, a') \in G \times G'$ . Then  $|(a, a')| = \text{lcm}(|a|, |a'|)$ .

**Definition 3.14:** An **internal direct product** is a group  $G$  such that there exist two subgroups  $H, K \leq G$  satisfying  $HK = \{hk \mid h \in H, k \in K\} = G$ ,  $H \cap K = \{e\}$ , and  $hk = kh$  for all  $h \in H$  and  $k \in K$ . In this case,  $G \simeq H \times K$ .

**Example:**  $D_6 = \langle r^2, s \rangle \langle r^3 \rangle$ .

**Proposition 3.15:**  $\simeq$  is an equivalence relation.

**Proposition 3.16:**  $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$  if and only if  $\gcd(m, n) = 1$ .

**Proposition 3.17:** Let  $H$  and  $K$  be groups. Then  $H \times K \simeq K \times H$ .

**Proposition 3.18:** If  $H \simeq H'$  and  $K \simeq K'$ , then  $H \times K \simeq H' \times K'$ .

**Definition 3.19:** Let  $G$  be a group. Two elements  $b, c \in G$  are **conjugate** if  $c = aba^{-1}$  for some  $a \in G$ .

**Definition 3.20:** Let  $G$  be a group. Two subgroups  $H, K \leq G$  are **conjugate** if  $K = aHa^{-1}$  for some  $a \in G$ .

**Proposition 3.21:** Conjugacy is an equivalence relation.

**Proof:** Let  $G$  be a group. For all  $b \in G$ ,  $b = ebe^{-1}$ . If  $c = aba^{-1}$  for some  $a, b, c \in G$ , then  $b = (a^{-1})c(a^{-1})^{-1}$ . Finally, if  $c = aba^{-1}$  and  $d = a'c(a')^{-1}$  for some  $a, a', b, c, d \in G$ , then  $d = (a'a)b(a'a)^{-1}$ .

**Theorem 3.22:** Let  $G$  be a group,  $a, b \in G$ , and  $H \leq G$ . Then  $|b| = |aba^{-1}|$ ,  $|H| = |aHa^{-1}|$ , and  $H \simeq aHa^{-1}$ .

**Example:** For  $D_6$ ,  $r^{-1} = srs = srs^{-1}$ , so  $r$  and  $r^{-1} = r^5$  are conjugate. Similarly,  $r^2$  and  $r^4$  are conjugate. Since  $r^3 \in Z(D_6)$ , it is only conjugate to itself.  $s, r^2s$ , and  $r^4s$  are all conjugate to one another, as are  $rs, r^3s$  and  $r^5s$ .

**Example:** In  $S_5$ ,  $(12345)$  and  $(13524)$  are conjugate by the amazing conjugation trick. Similarly, so are any two elements with the same cycle structure.

**Comment:** In a sense, conjugate elements and subgroups have the same “type”.

**Definition 3.23:** A subgroup  $H \leq G$  is **normal** if  $H = aHa^{-1}$  for all  $a \in G$ , or, equivalently,  $aH = Ha$  for all  $a \in G$ . We write  $H \triangleleft G$ .

**Definition 3.24:** A group is **simple** if it contains no nontrivial normal subgroups.

**Example:** All Abelian groups are normal, since  $aHa^{-1} = aa^{-1}H = H$ . In particular,  $Z(G) \triangleleft G$  for all groups  $G$ .

**Example:**  $\{e, (12)(34), (13)(24), (14)(23)\} \triangleleft S_4$ .

**Proposition 3.25:** Let  $G$  be a groups and  $N \triangleleft G$ . Then the set product  $(aN)(bN) = abN$ .

**Definition 3.26:** Let  $G$  be a group and  $N \triangleleft G$ . The quotient group of  $G$  by  $N$  is  $G/N = \{aN \mid a \in G\}$  under the composition law  $(aN)(bN) = abN$ , with identity  $N$  and inverse  $(aN)^{-1} = a^{-1}N$ . Note that  $|G/N| = [G : N]$ .

**Example:**  $D_6/\langle r^2, s \rangle \simeq \mathbb{Z}_2$ .

$S_n/A_n = \{A_n, (12)A_n\} \simeq \mathbb{Z}_2$ .

For  $K = \{e, (12)(34), (13)(24), (14)(23)\}$ ,  $S_4/K \simeq D_3$ .

$\mathbb{Z}/5\mathbb{Z} \simeq \mathbb{Z}_5$ .

## IV — Morphisms

**Definition 4.1:** Let  $G$  and  $G'$  be groups. A **homomorphism** between  $G$  and  $G'$  is a function  $\varphi : G \longrightarrow G'$  such that  $\varphi(ab) = \varphi(a)\varphi(b)$  for all  $a, b \in G$ . Notice that every isomorphism is a homomorphism.

**Proposition 4.2:** Let  $G$  and  $G'$  be groups and  $\varphi : G \longrightarrow G'$  be a homomorphism. Then:

1.  $\varphi(e_G) = \varphi(e_{G'})$ .
2.  $\varphi(a^{-1}) = \varphi(a)^{-1}$ .
3.  $\varphi(G) \leq G'$ .

**Example:** Examples of homomorphisms:

Vector space linear transformations:  $T(v + w) = Tv + Tw$ .

The determinant,  $\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^*$ :  $\det AB = (\det A)(\det B)$ .

The complex modulus,  $|\cdot| : \mathbb{C}^* \longrightarrow \mathbb{R}^+$ :  $|ab| = |a||b|$ .

The exponential map,  $\exp : \mathbb{R} \longrightarrow S^1$ :  $\exp(a + b) = e^{2\pi i(a+b)} = e^{2\pi ia}e^{2\pi ib} = \exp(a)\exp(b)$ . (Remember that the group action in  $S^1$  is complex multiplication!)

For a group  $G$  with  $|G| = n$ , the map from Cayley's theorem,  $\lambda : G \longrightarrow S_n$ :  $\lambda(ab) = \lambda_a b = \lambda_a \lambda_b = \lambda(a)\lambda(b)$ .

**Theorem 4.3:** Let  $G$  and  $G'$  be groups,  $\varphi : G \longrightarrow G'$  be a homomorphism, and  $H' \leq G'$ . Then  $\varphi^{-1}(H') = \{h \in G \mid \varphi(h) \in H'\} \leq G$ . Moreover, if  $H' \triangleleft G'$ , then  $\varphi^{-1}(H') \triangleleft G$ .

**Proof:** Let  $H = \varphi^{-1}(H')$ . Since  $e_G \in G$  and  $\varphi(e_G) = e_{G'} \in H'$ ,  $e_G \in H$  by definition.

Let  $a, b \in H$ . Then  $\varphi(a), \varphi(b) \in H'$ , so  $\varphi(a)\varphi(b) = \varphi(ab) \in H'$ . Thus  $ab \in H$ .

Finally, let  $a \in H$ . Then  $\varphi(a) \in H'$ , so  $\varphi(a)^{-1} = \varphi(a^{-1}) \in H'$ . Thus  $a^{-1} \in H$ , and so  $H \leq G$ .

Now suppose  $H' \triangleleft G'$  and let  $a \in G$  be arbitrary. Then  $a(\varphi^{-1}(H'))a^{-1} = \varphi^{-1}(\varphi(a)H'\varphi(a^{-1})) = \varphi^{-1}(H')$ , since  $H' \triangleleft G'$ . Since  $H = \varphi^{-1}(H')$ ,  $aHa^{-1} = H$ . Thus  $H \triangleleft G$ .

**Definition 4.4:** Let  $G$  and  $G'$  be groups and  $\varphi : G \longrightarrow G'$  be a homomorphism. The **kernel** of  $\varphi$  is the set  $\ker \varphi = \{a \in G \mid \varphi(a) = e_{G'}\}$ .

**Proposition 4.5:** Let  $G$  and  $G'$  be groups and  $\varphi : G \longrightarrow G'$  be a homomorphism. Then  $\ker \varphi \triangleleft G$ .

**Proof:** Since  $\ker \varphi = \varphi^{-1}(\{e\})$  and  $\{e\} \triangleleft G'$ ,  $\ker \varphi \triangleleft G$ .

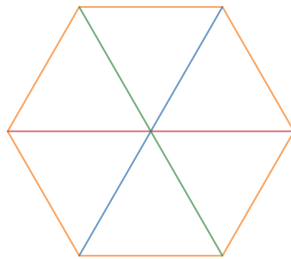
**Example:** With  $\varphi : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^*$  defined by  $\varphi(A) = \det A$ ,  $\ker \varphi = SL_n(\mathbb{R})$ , so  $SL_n(\mathbb{R}) \triangleleft GL_n(\mathbb{R})$ .

**Proposition 4.6:** Let  $\varphi : G \longrightarrow G'$  be a homomorphism and let  $a \in G$ . Then  $|\varphi(a)| \mid |a|$  and  $|\varphi(a)| \mid |G'|$ .

**Proposition 4.7:** Let  $G$  be a group such that each element  $a \in G$  permutes a set of conjugate substructures  $\{y_1, \dots, y_d\}$ . Then there is a homomorphism  $\varphi : G \longrightarrow S_d$ .



**Example:**



Here,  $\varphi : D_6 \longrightarrow S_3$ .

**Theorem 4.8: (The First Isomorphism Theorem)** Let  $G$  and  $G'$  be groups and let  $\varphi : G \longrightarrow G'$  be a homomorphism. Then  $G/(\ker \varphi) \simeq \varphi(G)$ .

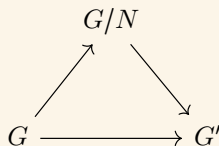
**Proof:** Let  $N = \ker \varphi$  and define  $\psi : G/N \longrightarrow G'$  by  $\psi(aN) = \varphi(a)$ .  $\psi$  is well-defined, since if  $aN = bN$ , then  $b^{-1}a \in N = \ker \varphi$ , so  $\varphi(b)^{-1}\varphi(a) = e$ , or equivalently,  $\psi(aN) = \psi(bN)$ . Moreover, since  $\psi(aNbN) = \psi(abN) = \varphi(ab) = \varphi(a)\varphi(b) = \psi(aN)\psi(bN)$ ,  $\psi$  is a homomorphism.

Now suppose  $\psi(aN) = \psi(bN)$ . Then  $\varphi(a) = \varphi(b)$ , so  $\varphi(b^{-1}a) = e$ . Then  $b^{-1}a \in \ker \varphi = N$ , so  $aN = bN$ . Thus  $\psi$  is injective, and certainly it is surjective onto its range, so  $\psi|_{\varphi(G)} : G/N \longrightarrow \varphi(G)$  is an isomorphism.

**Definition 4.9:** Let  $G$  be a group and  $N \triangleleft G$ . The **canonical projection** is  $\gamma : G \longrightarrow G/N$ , defined by  $g \mapsto gN$ .

**Definition 4.10:** A **commutative diagram** is a collection of sets and maps between them such that for any two sets  $A$  and  $B$  in the diagram, all paths of compositions from  $A$  to  $B$  are equal as functions.

**Theorem 4.11:** Let  $G$  and  $G'$  be groups and  $\varphi : G \longrightarrow G'$  be a homomorphism. Then the following diagram commutes, where  $\gamma$  is the canonical projection of  $G$  onto  $G/N$  and  $\psi$  is as before, defined by  $\psi(aN) = \varphi(a)$ .



**Example:** Let  $\varphi : V \rightarrow V'$  be a linear transformation between vector spaces and let  $W = \ker \varphi$ . Then  $V/W \simeq \varphi(V)$ .

With  $\varphi = \det$ ,  $GL_n/SL_n \rightarrow \mathbb{R}^*$ .

With  $\varphi(a + bi) = \sqrt{a^2 + b^2}$ ,  $\mathbb{C}^*/S^1 \simeq \mathbb{R}^+$ .

With  $\varphi(a) = e^{2\pi ia}$ ,  $\mathbb{R}/\mathbb{Z} \simeq S^1$ .

**Theorem 4.12: (The Second Isomorphism Theorem)** Let  $G$  be a group,  $H \leq G$ , and  $N \triangleleft G$ . Then  $HN$  is a group,  $H \cap N \triangleleft H$ , and  $H/(H \cap N) \simeq (HN)/N$ , where the isomorphism is given by  $\varphi(h(H \cap N)) = hN$ .

**Theorem 4.13: (The Third Isomorphism Theorem)** Let  $G$  be a group and  $H, N \triangleleft G$  such that  $H \subseteq N$ . Then  $N/H \triangleleft G/H$  and  $\frac{G/H}{N/H} \simeq G/N$ .

**Theorem 4.14: (The Correspondence Theorem)** Let  $G$  and  $G'$  be groups and  $\varphi : G \rightarrow G'$  a surjective homomorphism. Then  $\varphi$  is a bijection between  $\{H \leq G \mid \ker \varphi \leq H\}$  and  $\{H' \leq G'\}$ . Moreover, if  $N' \triangleleft G'$ , then  $\varphi^{-1}(N') \triangleleft G$ .

**Proof:** Since  $\varphi$  is surjective, for all  $h' \in H'$ , there is a  $g \in G$  such that  $\varphi(g) = h'$ . Thus  $\varphi(\varphi^{-1}(H')) = H'$ . Now let  $h \in H$ . Since  $\varphi^{-1}(\varphi(H)) = H$ . Thus  $\varphi$  is a bijection,

**Definition 4.15:** Let  $G$  be a group and  $a \in G$ . The **conjugacy class** of  $a$  is  $[a] = \{gag^{-1} \mid g \in G\}$ .

**Proposition 4.16:** Let  $G$  be a group. Then  $G = \bigsqcup_{a \in G} [a]$ .

**Proposition 4.17:** Let  $G$  be a group,  $a \in G$ , and  $G_a = \{g \in G \mid gag^{-1} = a\}$ . Then  $[G : G_a] = |[a]|$ .

**Theorem 4.18: (The Class Equation)** Let  $G$  be a group. Then  $|G| = \sum [G : G_a]$ , where the sum is over the  $a \in G$  that produce distinct conjugacy classes.

## V — Group Actions

**Definition 5.1:** Let  $G$  be a group,  $X$  a set, and  $S_X$  the group of permutations of  $X$ . An **action** of  $G$  on  $X$  is a homomorphism  $\varphi: G \rightarrow S_X$ . We say that  $X$  is a  $G$ -set, and write  $a(x)$  instead of  $(\varphi(a))(x)$  for  $x \in X$ . In this sense, every element of  $G$  is a function on  $X$ .

**Definition 5.2:** Let  $G$  be a group acting on a set  $X$ . The **stabilizer** of  $x \in X$  is the set  $G_x = \{a \in G \mid a(x) = x\} \leq G$ .

**Definition 5.3:** Let  $G$  be a group acting on a set  $X$ , with the action given by  $\varphi$ . The **kernel** of the action is  $\ker \varphi$ .

**Definition 5.4:** Let  $G$  be a group acting on a set  $X$ . The **orbit** of  $x \in X$  is  $\text{orb } x = \{a(x) \mid a \in G\}$ .

**Definition 5.5:** Let  $G$  be a group acting on a set  $X$ . The **fixed point set** corresponding to the action is  $X^G = \{x \in X \mid a(x) = x \text{ for all } a \in G\}$ .

**Example:**  $D_6$  acts on the regular hexagon,  $\mathbb{C}^*$  acts on the plane, and  $SO(3)$  acts on  $\mathbb{R}^3$ .

**Example:** If  $G$  is the group of symmetries of a structured set  $X$  and  $Y$  is a substructure of  $X$ , then  $G$  acts on the set of conjugate substructures of  $Y$ .

**Example:** A group  $G$  acts on itself by left multiplication, giving the isomorphism from  $G$  to a subgroup of  $S_{|G|}$ .

**Example:** A group  $G$  acts on the set of left cosets of a subgroup  $H \leq G$  by left multiplication, which gives the homomorphism from  $G$  to  $S_{|G/H|}$ .

**Example:** A group  $G$  acts on itself by conjugation, giving the homomorphism from  $G$  to  $\text{Aut } G$  with a range of  $\text{Inn } G$ . This is called the *adjoint action*. For  $a \in G$ ,  $G_a$  is called the *centralizer* of  $a$  and is denoted  $C(a)$ . The kernel of the action is the center of the group,  $Z(G)$ . The orbit of  $a \in G$  is  $a$ 's conjugacy class,  $[a]$ .

**Theorem 5.6:** There is a one-to-one correspondence between the sets  $\text{orb } x$  and  $G/G_x$ , and between the  $a(x)$  and  $aG_x$ . In particular, the number of distinct orbits of  $x$  is  $[G : G_x]$ , and therefore,  $[G : G_x] = |[x]|$ .

**Proposition 5.7:** Let  $G$  be a group acting on a set  $X$ ,  $a \in G$ , and  $x \in X$ . Then  $G_{a(x)} = aG_xa^{-1}$ .

**Definition 5.8:** Let  $G$  be a group acting on a set  $X$ .  $G$  acts **transitively** on  $X$  if for all  $x, y \in X$ , there is an  $a \in G$  such that  $a(x) = y$ .

**Theorem 5.9:** Let  $G$  be a group acting transitively on a set  $X$ . Let  $x_0 \in X$  and  $G_0 = G_{x_0}$ , and for all  $x \in X$ , let  $a_x \in G$  be such that  $a_x(x_0) = x$ . Then the action of  $G$  on  $X$  is equivalent to the action of  $G$  on  $G/G_0$ , in that there is a bijection from  $X$  to  $G/G_0$  given by  $x \mapsto a_xG_0$ , and  $a(x) = y$  if and only if  $a(a_xG_0) = a_yG_0$  for all  $a \in G$ .

**Proof:** The bijection exists immediately by the one-to-one correspondence between  $\text{orb } x_0 = X$  (since the action is transitive) and  $G/G_0$ . For the second statement,  $a(x) = y$  if and only if  $aa_xG_0(x_0) = a((a_xG_0)(x_0)) = y$ , since  $a_xG_0$  is the set of elements that send  $x_0$  to  $x$ . But this happens if and only if  $a(a_xG_0) = a_yG_0$ , since  $a_yG_0$  is the set of elements that send  $x_0$  to  $y$ .

**Comment:** By this theorem, we can treat any transitive action as an action on the left cosets of some subgroup. Therefore, all actions on geometric objects can be transformed into pure algebra by fixing a base point and considering the left cosets of its stabilizer subgroup.

**Definition 5.10:** Let  $G$  be a group acting on a set  $X$ .  $G$  acts **freely** on  $X$  if for all  $x \in X$ ,  $G_x = \{e\}$ .

**Definition 5.11:** Let  $G$  be a group acting on a set  $X$ .  $X$  is a **principal homogeneous set** for  $G$  if  $G$  acts freely and transitively on  $X$ .

**Example:** The punctured line  $\mathbb{E} \setminus \{0\}$  is a principal homogeneous set for  $\mathbb{R}^*$ .

The punctured plane  $\mathbb{E}^2 \setminus \{0\}$  is a principal homogeneous set for  $\mathbb{C}^*$ .

The punctured 3-space  $\mathbb{E}^3 \setminus \{0\}$  is *not* a principal homogeneous set for the group of symmetries of  $\mathbb{E}^3$  that preserve the orientation, scale, and the origin, since it is impossible to rotate the 2-sphere without fixing a point, and therefore the symmetry group cannot act freely.

The punctured 4-space  $\mathbb{E}^4 \setminus \{0\}$  is a principal homogeneous set for  $\mathbb{H}^*$ .

**Theorem 5.12: (The Class Equation)** Let  $G$  be a group. Then

$$|G| = |Z(G)| + \sum_{[a], a \notin Z(G)} [G : G_a].$$

**Proof:**  $[G : G_a] = 1$  if and only if  $[a] = \{a\}$ , if and only if  $a \in Z(G)$ .

**Example:** Does  $S_4$  have a normal subgroup of order 8? The class equation for  $S_4$  is  $24 = 1 + 6 + 8 + 6 + 4$ , counting the center, 2-cycles, 3-cycles, 4-cycles, and  $(2, 2)$ -cycles, respectively. A normal subgroup is the union of conjugacy classes. But every subgroup contains the identity, and there is no way to union conjugacy classes to result in a subgroup of order 8 that includes  $e$ .

**Definition 5.13:** A  **$p$ -group** is a group  $G$  with order  $|G| = p^n$  for some prime  $p$  and some  $n \in \mathbb{N}$ .

**Proposition 5.14:** Every nontrivial  $p$ -group has a nontrivial center.

**Proof:** Let  $G$  be a nontrivial  $p$ -group. Then  $|G| = p^n = |Z(G)| + \sum_{[a], a \notin Z(G)} [G : G_a]$ . Since  $|G_a| \mid p^n$  for all  $a \in G$ ,  $p \mid |G_a|$  if  $|G_a| \neq 1$ , so  $p \mid |G_a|$  if  $a \notin Z(G)$ . Thus  $p \mid \sum_{[a], a \notin Z(G)} [G : G_a]$ , and since  $p \mid p^n$  ( $n \neq 0$ ),  $p \mid |Z(G)|$ . Since  $p > 1$ ,  $|Z(G)| \neq 1$ .

**Definition 5.15:** Let  $G$  be a group. The **commutator subgroup** of  $G$  is  $G' = \langle [a, b] \mid a, b \in G \rangle$ , where  $[a, b] = aba^{-1}b^{-1}$ .

**Proposition 5.16:** Let  $G$  be a group and  $G'$  the commutator subgroup. Then  $G' \triangleleft G$ ,  $G/G'$  is abelian, and if  $N \triangleleft G$  and  $G/N$  is abelian, then  $G' \leq N$ .

**Definition 5.17:** Let  $G$  be a group. The  **$i$ th derived subgroup** of  $G$  is  $G^{(i)} = (G^{(i-1)})'$ , where  $G^{(1)} = G'$ .

**Definition 5.18:** A group  $G$  is **solvable** if  $G^{(n)} = \{e\}$  for some  $n \in \mathbb{N}$ .

**Example:** Every abelian group  $G$  is solvable, since  $G' = \{e\}$ .

**Comment:** In some sense, solvable groups are built up from abelian groups, but they are far more common than abelian groups while still sharing many nice properties.

**Definition 5.19:** A group  $G$  is **perfect** if  $G' = G$ .

**Theorem 5.20:** All  $p$ -groups are solvable.

**Proof:** Suppose not. Then there is a group  $G$  that is the smallest-order nonsolvable  $p$ -group. Since  $G' \leq G$ ,  $|G'| \leq |G|$ , so  $G' = G$ , since otherwise  $G'$  would be a smaller-order nonsolvable  $p$ -group. Now let  $\overline{G} = G/Z(G)$ , which is a group, since  $Z(G) \triangleleft G$ . Since  $G$  is a nontrivial  $p$ -group,  $Z(G) \neq \{e\}$ , so  $|\overline{G}| < |G|$ . Since  $G = G'$ , every element of  $G$  is of the form  $([a_1, b_1]) \cdots ([a_i, b_i])$ , and since  $|G| > |\overline{G}|$ , there is an onto homomorphism from  $G$  to  $\overline{G}$  given by  $[a, b] \mapsto [a, b]Z(G)$ , so every element of  $\overline{G}$  is of the form  $[a, b]Z(G)$ . Thus  $\overline{G} = \overline{G}'$ , so  $\overline{G}^{(n)} = \overline{G}$  for all  $n \in \mathbb{N}$ . Since  $G$  is a  $p$ -group,  $|G| = p^k$  for some  $k \in \mathbb{N}$ , and since  $Z(G) \leq G$ ,  $|Z(G)| = p^l$  for some  $l \in \mathbb{N}$ . Thus  $|\overline{G}| = |G/Z(G)| = p^{k-l}$ , and so  $\overline{G}$  is a  $p$ -group. But  $G$  is the smallest-order nonsolvable  $p$ -group, so  $\overline{G}$  must be solvable. But  $\overline{G}^{(n)} = \overline{G}$  for all  $n \in \mathbb{N}$ , so  $\overline{G} = \{e\}$ . But  $\overline{G} = G/Z(G)$ , so  $G = Z(G)$ , and so  $G$  is abelian, and therefore solvable.  $\nexists$

**Theorem 5.21:** All odd-ordered groups are solvable.

**Theorem 5.22:** Let  $G$  be a finite group and let  $p$  be a prime such that  $p \mid |G|$ . Then  $G$  contains an element of order  $p$ .