64
64
63
63
60
58

A

57
57
55
$\boxed{54}$
53

B
85 %

51
51
51
50
45

C

44
40

D

36
31

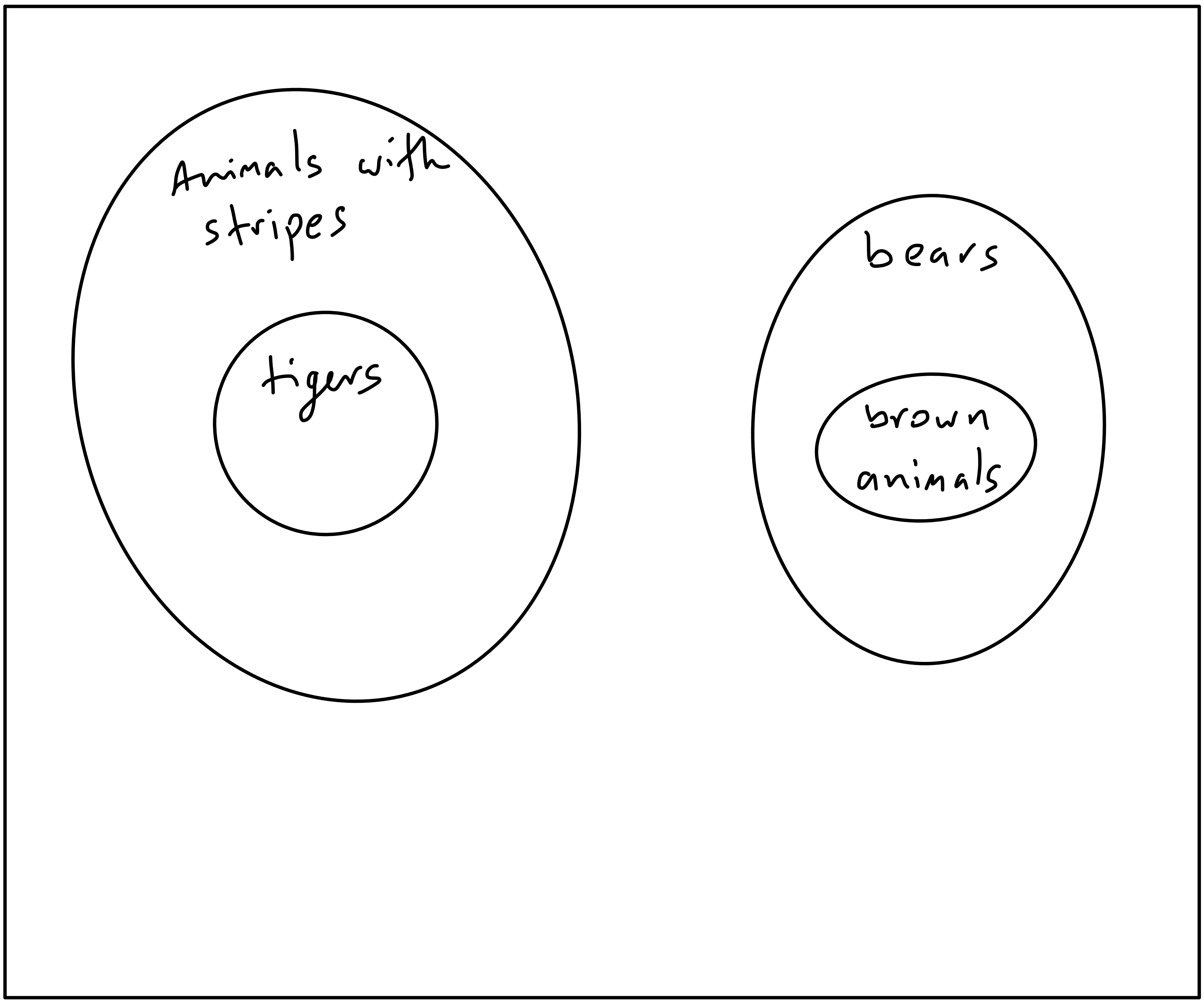
F

① $p \to (q \wedge r)$

p	q	r	$q \wedge r$	$p \to (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

② 1. All tigers have stripes.
2. Nothing with stripes is a bear.
3. All brown animals are bears.

No tigers are brown.



Therefore, the argument is valid.

(3) p : All tigers have stripes

q : Nothing with stripes is a bear

r : All brown animals are bears

s : No tigers are brown

$(p \wedge q \wedge r) \rightarrow s$

<u>OR</u>

p : you are a tiger

q : you have stripes

r : you are a bear

s : you are a brown animal

1. $p \rightarrow q$
2. $q \rightarrow \sim r$
3. $s \rightarrow r$

$\overline{\quad p \rightarrow \sim s \quad}$ (or $s \rightarrow \sim p$)

(4) 1. You eat only if you are hungry
2. If you go to a restaurant, then you eat

You are hungry if you go to a restaurant.

This is valid.

Let p be "you eat"
q be "you are hungry"
r be "you go to a restaurant".

Then $P_1 \equiv p \to q$
$P_2 \equiv r \to p$
$C \equiv r \to q$

p	q	r	P_1	P_2	C	$P_1 \wedge P_2$	$P_1 \wedge P_2 \rightarrow C$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	F	F	T
T	F	F	F	T	T	F	T
F	T	T	T	F	T	F	T
F	T	F	T	T	T	T	T
F	F	T	T	F	F	F	T
F	F	F	T	T	T	T	T

✓

⑤ You are hungry if you go to the restaurant.

≡ If you go to the restaurant, then you are hungry.

Converse: If you are hungry, then you go to the restaurant.

Inverse: If you don't go to the restaurant, then you are not hungry.

Contrapositive: If you are not hungry, then you do not go to the restaurant.

For <u>any</u> statement, the contrapositive is equivalent to it.

⑥ A : UO students who are currently taking 105.

B : UO students who have taken and passed 105.

$A \cup B$: UO students who have either taken 105 or are currently in 105.

$A \cap B$: UO students who are currently taking 105 but who have also taken and passed it before.

A': UO students who are not currently in 105.

B': UO students who have not passed 105.

(7) Which is/are true:

i. $A \cup B = \emptyset$

ii. $\boxed{A \cap B = \emptyset}$

iii. $A' = \emptyset$

iv. $B' = \emptyset$

(8) C has 15 elements

D has 10

$C \cup D$ has 17

$n(C \cap D)$?

$$n(C \cup D) = n(C) + n(D) - n(C \cap D)$$

$$17 = 15 + 10 - n(C \cap D)$$

$$n(C \cap D) = 8.$$

Ex: ${}_7P_3$ "7 permute 3"

this is the number of ways to choose 3 objects from a group of 7 and then arrange them
(permute)

$${}_7P_3 = {}_7C_3\,(3!) = \frac{7!}{\cancel{3!}(7-3)!}\,\cancel{(3!)}$$

$$= \frac{7!}{4!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}{\cancel{4\cdot 3\cdot 2\cdot 1}}$$

$$= 7\cdot 6\cdot 5$$

$$= 210.$$

$${}_7C_3 = \frac{7!}{3!\,(7-3)!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}{3\cdot 2\cdot 1\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}$$

$$= \frac{7\cdot \cancel{6}\cdot 5}{\cancel{3\cdot 2}} = 7\cdot 5 = 35$$

2.5: Infinite Sets

We know that the number of elements in a set A is $n(A)$.

<u>Def</u>: Two sets A and B are equivalent, written $A \sim B$, if we can pair every element of A with a unique element of B, and vice versa.

<u>Ex</u>: $\{1, 2, 3\} \sim \{a, b, x\}$ because we have the pairing

$$\begin{array}{ccc} 1 & 2 & 3 \\ | & | & | \\ a & b & x \end{array}$$

Theorem: If $A \sim B$, then $n(A) = n(B)$.

Ex: Let $\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$. Let $E = \{0, 2, 4, 6, 8, \cdots\}$ be the set of positive even numbers. We would think that $n(\mathbb{N}) > n(E)$, but this isn't true!

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ | & | & | & | & | & | & \\ 0 & 2 & 4 & 6 & 8 & 10 & \cdots \end{array}$$

We're forced to conclude that $n(\mathbb{N}) = n(E)$.

Remember that $\mathbb{Q}$ is the set of rational numbers. We can list all of them like this:

	0	1	-1	2	-2	3	-3
1	$0/1$	$1/1$	$-1/1$	$2/1$	$-2/1$	$3/1$	$-3/1$
2	$0/2$	$1/2$	$-1/2$	$2/2$	$-2/2$	$3/2$	$-3/2$
3	$0/3$	$1/3$	$-1/3$	$2/3$	$-2/3$	$3/3$	$-3/3$
4	$0/4$	$1/4$	$-1/4$	$2/4$	$-2/4$	$3/4$	$-3/4$
5	$0/5$	$1/5$	$-1/5$	$2/5$	$-2/5$	$3/5$	$-3/5$
6	$0/6$	$1/6$	$-1/6$	$2/6$	$-2/6$	$3/6$	$-3/6$

0	1	2	3	4	5	6	7	8	9	...
0	1	$1/2$	-1	$1/3$	$-1/2$	2	$1/4$	$-1/3$	-2	...

Therefore, $n(\mathbb{N}) = n(\mathbb{Q})$

Def: The first infinite cardinal is written $\aleph_0$ ("aleph-naught").

$n(\mathbb{N}) = \aleph_0$

$n(\mathbb{Q}) = \aleph_0$

Theorem: $n(\mathbb{R}) > \aleph_0$.

Proof: Suppose $n(\mathbb{R}) = n(\mathbb{N})$. Then we would have a pairing

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \\ | & | & | & | & | & \\ r_0 & r_1 & r_2 & r_3 & r_4 & \cdots \end{array}$$

real numbers

Now every real number r has a decimal expansion that we can write $r_0 = .a_0 a_1 a_2 a_3 a_4 \ldots$

For example, 32.1567912 has decimal $.1567912$.

So we have

$$
\begin{aligned}
r_0 &= .a_0 a_1 a_2 a_3 a_4 \ldots \\
r_1 &= .b_0 b_1 b_2 b_3 b_4 \ldots \\
r_2 &= .c_0 c_1 c_2 c_3 c_4 \ldots \\
&\vdots
\end{aligned}
$$

Let $R = .a_0 b_1 c_2 d_3 e_4 \ldots$

Let S be the same, but with every digit shifted up by one.

So if $R = .321718,$

$S = .432829.$

Now S can not appear in the list. So we didn't list all the real numbers!

Question: is there a set A for which $n(\mathbb{N}) < n(A) < n(\mathbb{R})$?

It's impossible to say.

3.2: Basic Probability

Def: An experiment is a process by which an outcome is obtained. The sample space is the set of all possible outcomes, and an event is a subset of the sample space.

Ex: rolling a die. The experiment is rolling the die. The sample space is $\{1, 2, 3, 4, 5, 6\}$.

Some examples of events:

$\{1\}$ (you roll a 1)

$\{1, 2, 3\}$ (you roll a 1, 2, or 3)

$\{1, 2, 3, 4, 5, 6\}$ (you roll anything)

<u>Def</u>: An event is <u>certain</u> or <u>guaranteed</u> if it always occurs, and <u>impossible</u> if it never does.

<u>Ex</u>: $\{1, 2, 3, 4, 5, 6\}$ is certain

$\{7\}$ is impossible.

Def: The probability of an event E with a sample space S is

$$P(E) = \frac{n(E)}{n(S)}$$

if all outcomes in the sample space are equally likely. It's also written

$$P(E), \quad P_r(E), \quad \text{and} \quad \mathbb{P}(E).$$

Def: The odds of an event E occuring is $o(E) = n(E) : n(E')$.

Ex: We flip a coin. The sample space is $\{H, T\}$. If $E = \{H\}$, then

$$p(E) = \frac{n(E)}{n(S)} = \frac{1}{2} = .5$$

Note that we could only do this because H and T are equally likely. The odds are $n(E) : n(E') = n(\{H\}) : n(\{T\})$

$= 1:1$.

Similarly, if $F = \{H, T\}$, then

$p(F) = \frac{2}{2} = 1$, and $o(F) = 2:0$.

Also, $p(\emptyset) = 0$ and $o(\emptyset) = 0:2$.

Ex: in real life, if you flip a coin 10 times, you might get 3 heads.

Def: The relative frequency of an event is the number of times in an experiment that an event occurs divided by the number of attempts.

Ex: if you flip a coin 10 times and get 3 heads, then the relative frequency of heads is $3/10 = .3$.

Theorem (The Law of Large Numbers):

If an experiment is repeated a large number of times, the relative frequency of an event is approximately equal to the probability of the event.