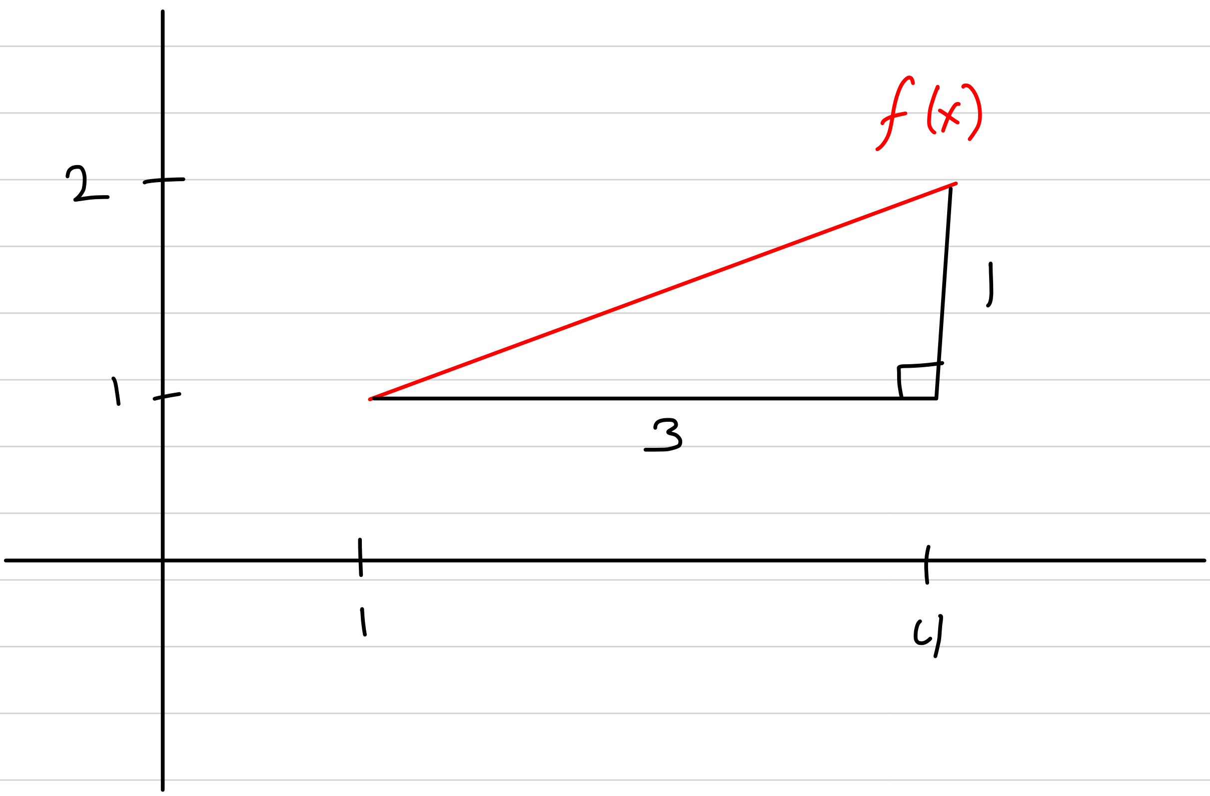
# Arc Length and Surface Area

**Def:** The arc length of $f(x)$ on $[a,b]$ is the length of the graph of $f$ between $x=a$ and $x=b$

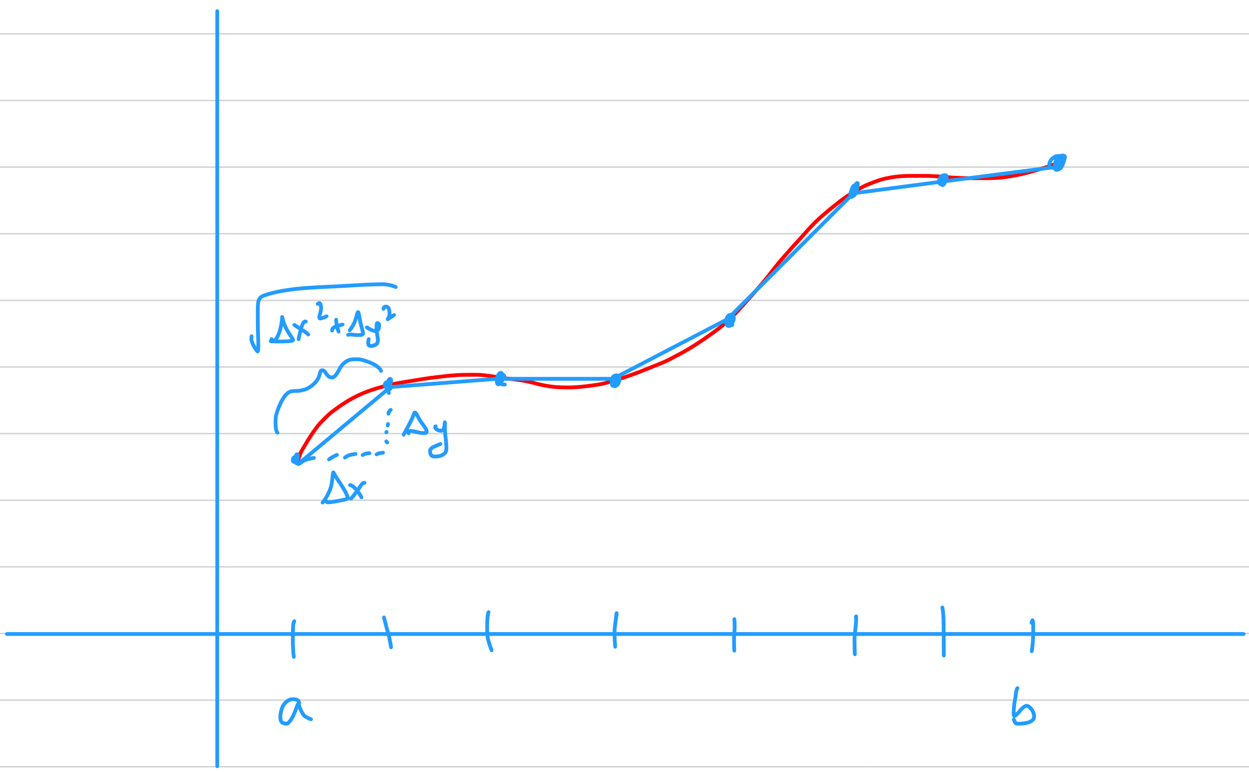
Ex




The arc length of $f$ on $[1,4]$ is

$\sqrt{1^2+3^2} = \sqrt{10}$.

**Comment:** We can approximate any curve by line segments:




$$\sqrt{\Delta x^2+\Delta y^2} = \sqrt{\Delta x^2\left(1+\frac{\Delta y^2}{\Delta x^2}\right)}$$

$$= \Delta x\sqrt{1+\frac{\Delta y^2}{\Delta x^2}}$$

As $\Delta x \to 0$, $\Delta x = dx$ and $\left(\frac{\Delta y}{\Delta x}\right)^2 = \left(\frac{dy}{dx}\right)^2$

$$= (f'(x))^2$$

We want to add up all these segments, so we take $\int_a^b \sqrt{1+(f'(x))^2}\,dx$ to find the arc length.

Thm: Let $f$ be a function on $[a,b]$. The arc length of $f$ on $[a,b]$ is $\int_a^b \sqrt{1+(f'(x))^2}\,dx$. Note: this requires $f$ to be differentiable.

Ex: Find the arc length of $f(x) = 2x^{3/2}$ on $[0,1]$.

$f'(x) = 2 \cdot \frac{3}{2} x^{1/2} = 3x^{1/2}$

$(f'(x))^2 = 9x$

So arc length $= \int_0^1 \sqrt{1+9x}\,dx$

$u = 1 + 9x$

$du = 9\,dx$

$\frac{1}{9}\,du = dx$

$$= \int_0^1 \sqrt{u}\ \frac{1}{9}\,du$$

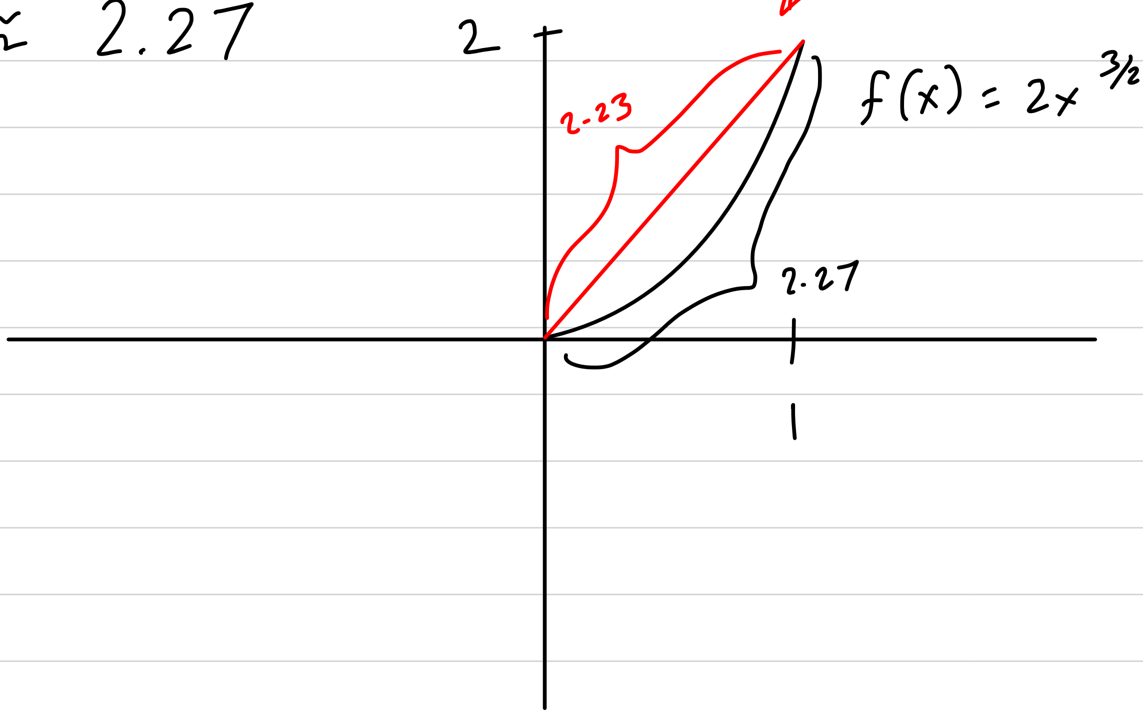
$$= \frac{1}{9}\left[\frac{u^{3/2}}{3/2}\right]\Big|_0^1$$

$$= \frac{1}{9}\left[\frac{(1+9x)^{3/2}}{3/2}\right]\Big|_0^1$$

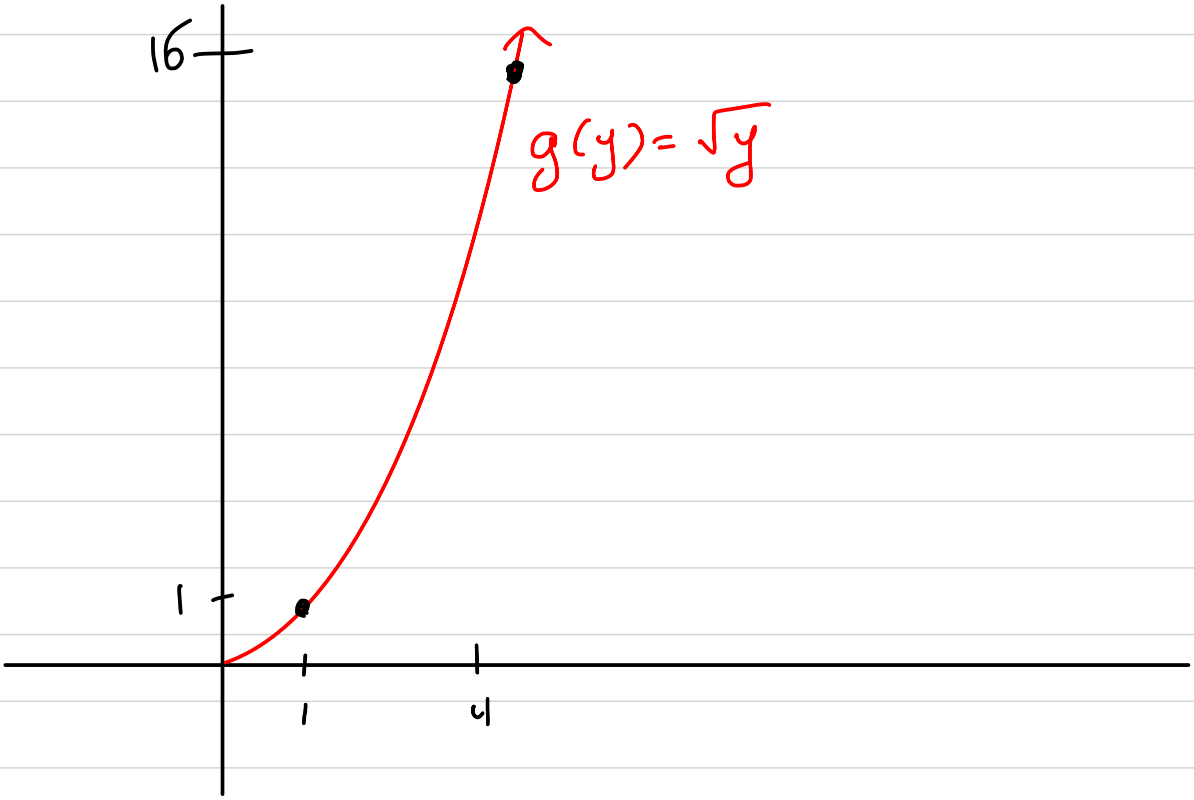
$$= \frac{1}{9}\left(\frac{10^{3/2}}{3/2} - \frac{1}{3/2}\right)$$

$\sqrt{1^2+2^2} = \sqrt{5}$

$\approx 2.27$



Ex: Let $g(y) = \sqrt{y}$. Find the arc length of $g$ between $x = 1$ and $x = 4$.

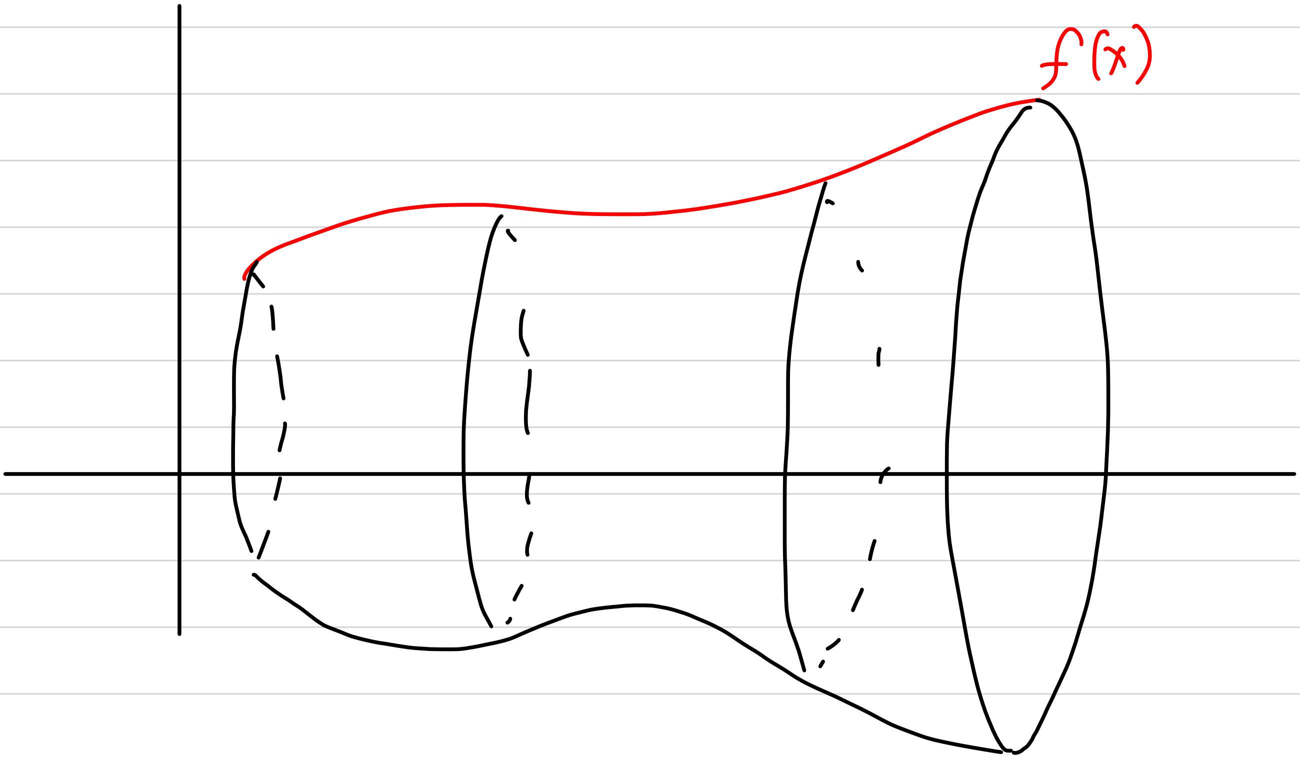



$$g'(y) = \frac{1}{2} y^{-1/2}$$

$$(g'(y))^2 = \frac{1}{4} y^{-1}$$

$$\int_1^{16} \sqrt{1 + \frac{1}{4} y^{-1}} \, dy$$

we can't integrate this (yet)




Thm: Let $f$ be a positive function on $[a,b]$. Then the surface area of the solid of revolution given by revolving $f$ about the $x$-axis is

$$\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Ex: Find the surface area of the solid of revolution generated by rotating the graph of $y=\sqrt{x}$ between $x=1$ and $x=9$ about the x-axis.

$$y' = \frac{1}{2}x^{-1/2}$$

$$\int_1^9 2\pi\sqrt{x}\sqrt{1+\frac{1}{4}x^{-1}}\,dx$$

$$= \int_1^9 2\pi\sqrt{x\left(1+\frac{1}{4}x^{-1}\right)}\,dx$$

$$= \int_1^9 2\pi\sqrt{x+\frac{1}{4}}\,dx$$

$$u = x + \frac{1}{4}$$

$$du = dx$$

$$= \int_1^9 2\pi\sqrt{u}\,du$$

$$= \left[2\pi\frac{u^{3/2}}{3/2}\right]\Big|_1^9$$

$$= \left[ 2\pi \frac{(x+\frac{1}{4})^{3/2}}{3/2} \right] \Big|_1^9$$

$$= 2\pi \left( \frac{(9+1/4)^{3/2}}{3/2} - \frac{(1+1/4)^{3/2}}{3/2} \right)$$

$$\approx 30.85$$

## Physical Applications

**Def:** Work is force $\cdot$ distance

**Ex:** You apply a $3\,\underset{\substack{\uparrow \\ \text{Newtons}}}{N}$ force to a cart to move it $9\,m$. Then you do $3 \cdot 9\ Nm = 27\ Nm$ of work.

Comment: we'll handle problems where the force you apply changes over time.

Ex: You apply a force on a cart of $x$ N when the cart has been pushed $x$ m, and you push it a total of 5 m. How much work do you do?

$\boxed{F(x) = x}$ ← Force applied at distance $x$.

$$\text{Work} = \int_0^5 F(x) \cdot dx = \int_0^5 x \, dx = \left[ \frac{x^2}{2} \right] \Big|_0^5$$

$$= \frac{25}{2} \text{ Nm.}$$

adding up very small dx times the force at that dx

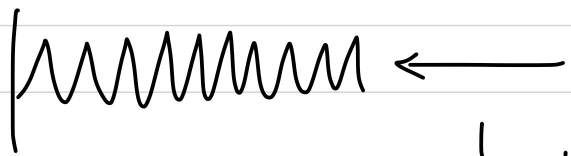
**Thm:** If you apply a force of $F(x)$ at distance $x$ and the distance ranges from $x=a$ to $x=b$, then the work done is $W = \int_a^b F(x)\,dx$.

**Thm** (Hooke's Law) The force to compress a spring (or to stretch it) is given by $F(x) = kx$ for some number $k$.

**Ex:** It takes 10 N to compress a spring .2 m. How much work is done if you stretch the spring .5 m?

$F(x) = kx$ for some $k$ by Hooke

Important: Hooke's law treats compression as a negative force



points in the negative direction

It also treats compression as moving in the negative direction.

$F(-.2) = -10$

$k(-.2) = -10$

$k = 50$

$F(x) = 50x$

$$W = \int_0^{.5} 50x \, dx = \left[ 50 \frac{x^2}{2} \right] \Big|_0^{.5} = 50 \frac{.25}{2} = 6.25 \text{ Nm}$$

**Prop:** Mass = density · volume

**Prop:** Let $\rho(x)$ be the density of a bar at distance $x$. If the bar exists from $x=a$ to $x=b$, then the mass of the bar is $\int_a^b \rho(x)\,dx$.

**Ex:** The density of a bar on $[\pi/2, \pi]$ is $\rho(x) = \sin(x)$. Find the mass.

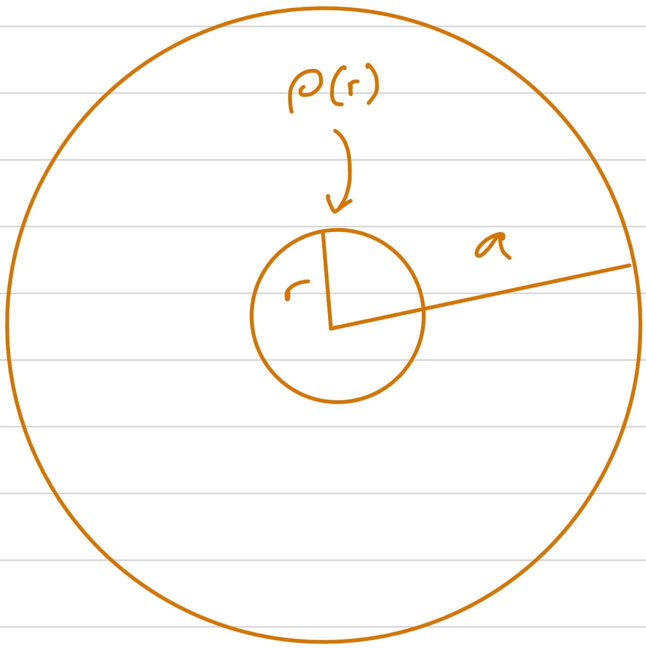
$$\int_{\pi/2}^{\pi} \sin(x)\,dx = \Big[-\cos(x)\Big]\Big|_{\pi/2}^{\pi}$$

$$= -\cos(\pi) + \cos(\pi/2)$$

$$= 1.$$

**Prop:** Let $\rho(r)$ be the density of a disc at radius $r$. If the disc has radius $a$, then its mass is

$$\int_0^a 2\pi r \rho(r)\, dr.$$




Ex: A disc of radius 4 has radial density $\rho(r) = \sqrt{r}$. Find its mass.

$$\int_0^4 2\pi r \sqrt{r}\, dr = 2\pi \int_0^4 r^{3/2}\, dr$$

$$= 2\pi \left[ \frac{r^{5/2}}{5/2} \right]\Big|_0^4$$

$$= 2\pi \quad \frac{4^{5/2}}{5/2}$$

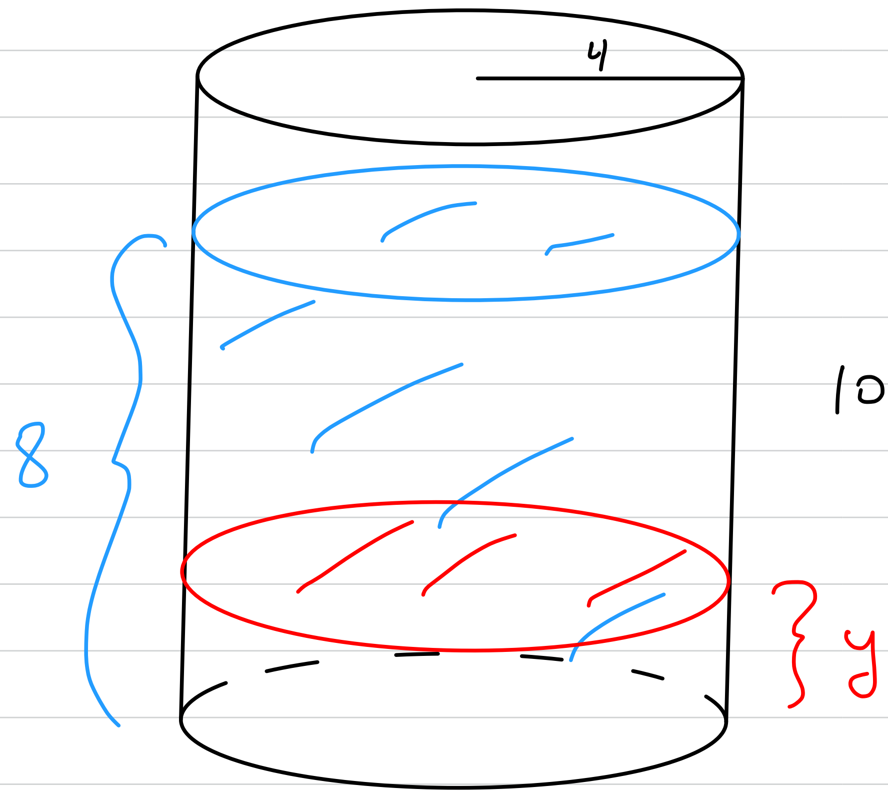
$$= 2\pi \cdot \frac{2}{5} \left(4^{1/2}\right)^5$$

$$= \frac{4\pi}{5} \cdot 2^5$$

$$= \frac{128\pi}{5}.$$

Prop: To find the work done by moving a fluid out of a container, first slice the tank into cross-sections perpendicular to the vertical direction (i.e. straight up). Find the area of a slice and the work required to move it, and then integrate the product of the two.

Ex: A cylindrical tank of height 10m and radius 4m is filled with water to a height of 8m. Find the work done by pumping all of the water out over the top of the tank.




Area of a slice at height $y$ is $4^2 \cdot \pi = 16\pi$

Work done to move water at height $y$ out of the tank is force · distance

distance $= 10 - y$

force depends on gravity: for water, it's

$9800 \ N/m^3$

$$\text{Work: } \underbrace{9800 \cdot 16\pi}_{\text{force}} \underbrace{(10-y)}_{\text{distanc}}$$

$$\text{Total work required: } \int_0^8 9800(10-y) \cdot 16\pi \, dy$$

$$= 9800(16\pi) \int_0^8 10-y \, dy$$

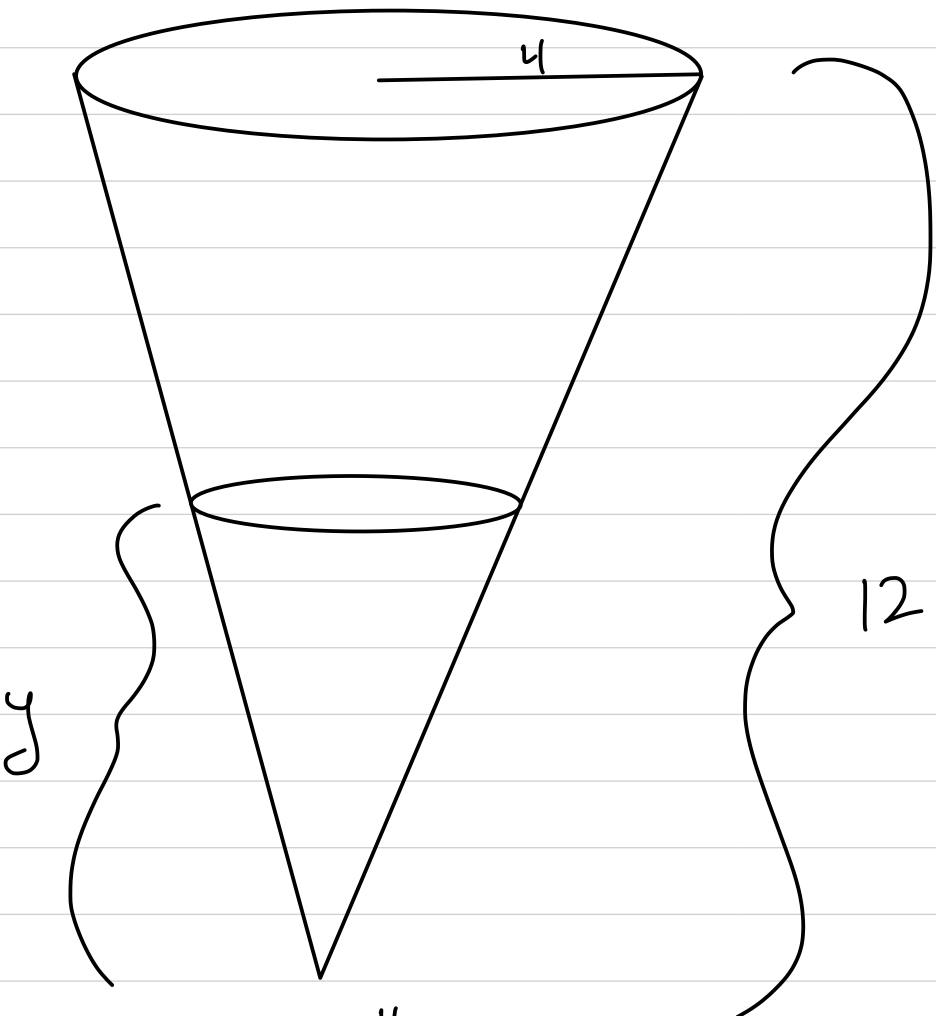
$$= 9800(16\pi)\left[10y - y^2/2\right]\Big|_0^8$$

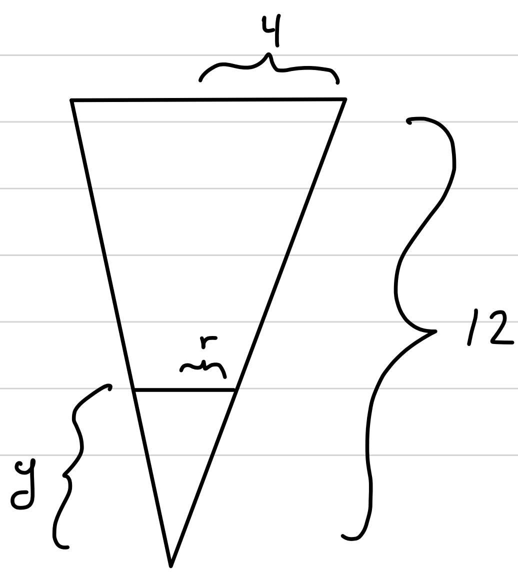
$$= 9800(16\pi)(80 - 32)$$

$$= 9800(16\pi)(48).$$

Ex: A 12m tall cone with radius 4m is filled with a fluid whose weight-density is $20000 \ N/m^3$. The fluid is pumped out over the top of the cone until a height of 4m of fluid is left.

Find the work done.




side view

Similar triangles: $\frac{r}{4} = \frac{y}{12}$, so $r = \frac{y}{3}$.

At height $y$, a slice has area $= \pi\left(\frac{y}{3}\right)^2$.

So the force required is $\pi\left(\frac{y}{3}\right)^2 \cdot 20000$

So the work required is $\pi\left(\frac{y}{3}\right)^2 \cdot 20000 \cdot (12-y)$.

So in total, we want $\int_4^{12} \pi\left(\frac{y}{3}\right)^2 \cdot 20000 \cdot (12-y)\,dy$.

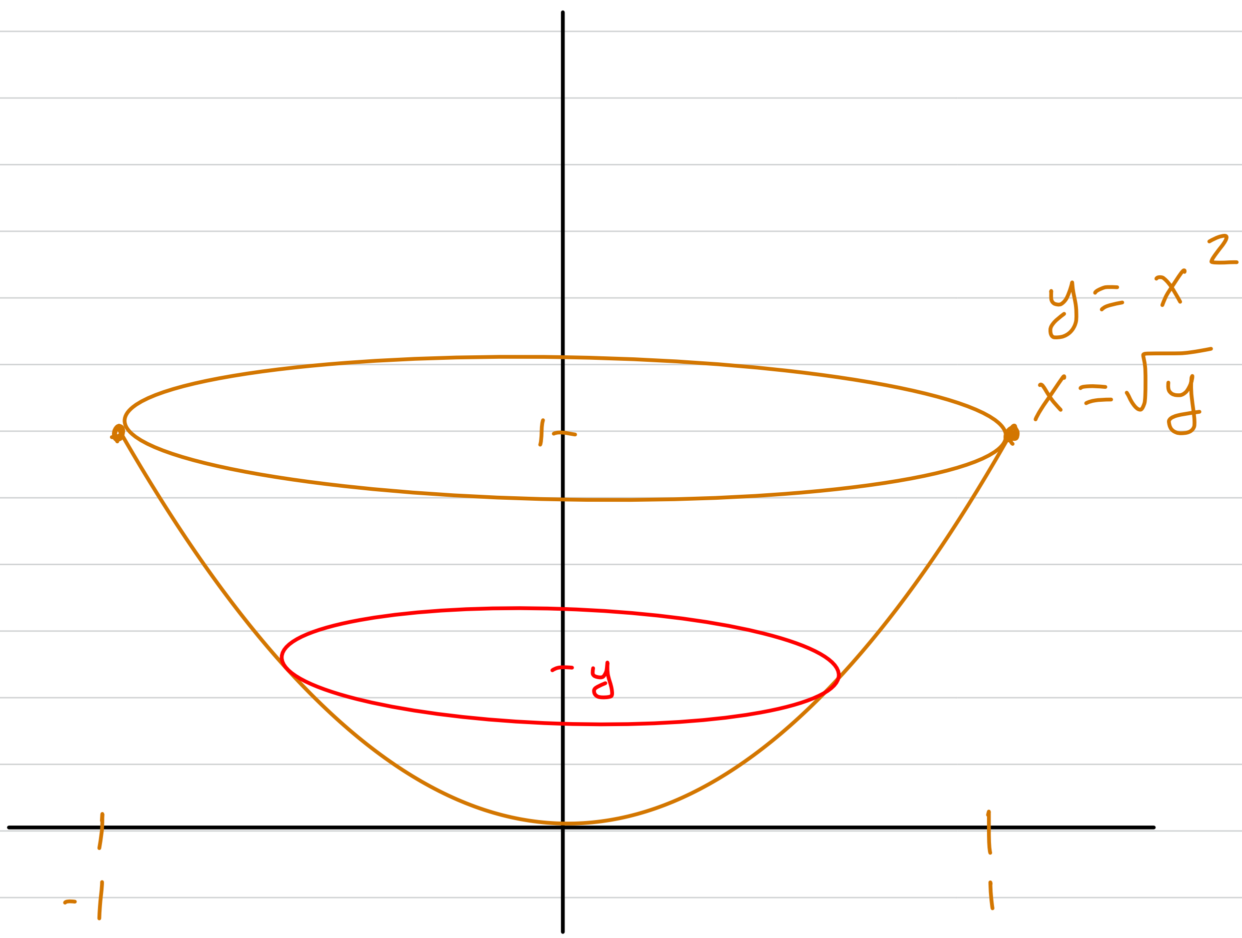
$$= \int_4^{12} \pi \cdot \frac{y^2}{9} \cdot 20000 \cdot (12-y)\,dy$$

$$= \frac{\pi}{9}(20000) \int_4^{12} 12y^2 - y^3\,dy$$

$$= \frac{\pi}{9}(20000)\left[4y^3 - y^4/4\right]\Big|_4^{12}$$

$$= \frac{\pi}{9}(20000)\left(\left(4\cdot 12^3 - 12^4/4\right) - \left(4\cdot 4^3 - 4^4/4\right)\right).$$

Ex: Let $f(x) = x^2$ on $[0,1]$. Revolve the graph of $f$ about the $y$-axis to produce a trough, and fill it with water ($9800\ N/m^3$). Find the work done by pumping it all out.




radius of a slice at height $y$ is $\sqrt{y}$.

So the area is $\pi \sqrt{y}^{2} = \pi y$

So the force applied is $9800 \pi y$

So the work done is $9800 \pi y (1-y)$

The total work is $\int_0^1 9800 \pi y (1-y) \, dy$.

$$= 9800 \pi \int_0^1 y - y^2 \, dy$$

$$= 9800 \pi \left[ \frac{y^2}{2} - \frac{y^3}{3} \right]\Big|_0^1$$

$$= 9800 \pi \left( \frac{1}{2} - \frac{1}{3} \right)$$

$$= 9800 \pi \left( \frac{1}{6} \right).$$

Ex: Find the arc length of $y = \frac{x^4}{4} + \frac{1}{8x^2}$ on $[1, 2]$

$$\int_1^2 \sqrt{1 + (f'(x))^2} \, dx$$

$$f'(x) = x^3 - \frac{1}{4}x^{-3}$$

$$(f'(x))^2 = x^6 - \frac{1}{2} + \frac{1}{16}x^{-6}$$

$$(f'(x))^2 + 1 = x^6 + \frac{1}{2} + \frac{1}{16}x^{-6}$$

$$= \left(x^3 + \frac{1}{4}x^{-3}\right)^2$$

$$\int_1^2 \left(x^3 + \frac{1}{4}x^{-3}\right) dx$$

## Centers of Mass

Def: The center of mass of a group of masses arranged in a straight line is the weighted average of their positions.

Ex: On the x-axis, there is:

A 30 kg mass at $x=-2$

A 5 kg mass at $x=3$

A 10 kg mass at $x=6$

A 15 kg mass at $x=-3$

Where is the center of mass?

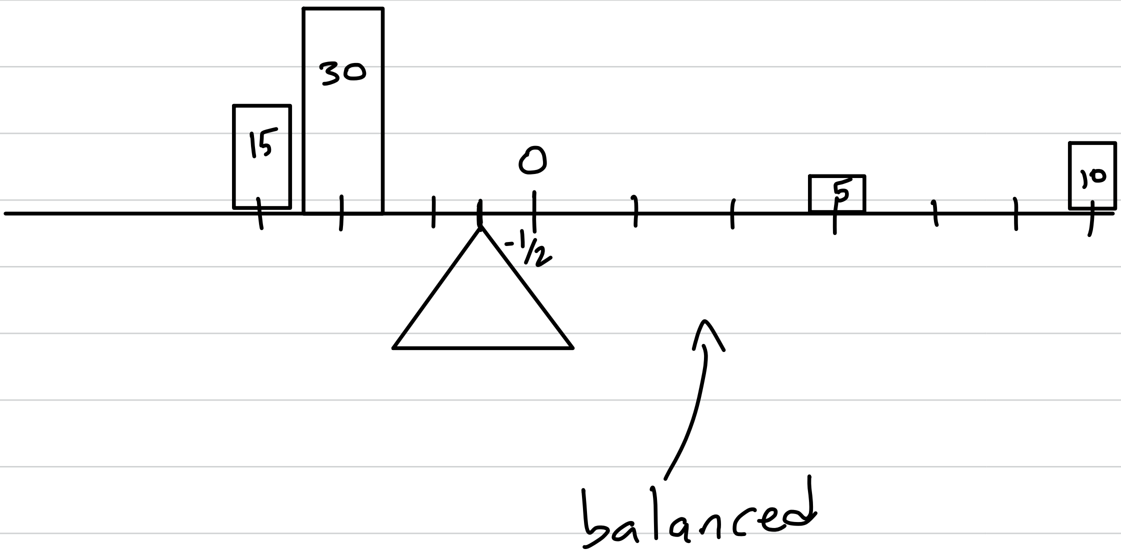
$$\frac{(-2)(30)+(3)(5)+(6)(10)+(-3)(15)}{30+5+10+15}$$

$$= \frac{-60+15-60-45}{60}$$

$$= \frac{-30}{60}$$

$$= -1/2$$

Comment: The center of mass of an object is the point on which it balances




Def: The centroid of an object is its center of mass, denoted $\bar{x}$ for a 1-dim object and $(\bar{x}, \bar{y})$ for a 2-dim object.

Def: For a 1-dimensional object, the moment is $M = \sum_{i=1}^{n} m_i x_i$, where $m_i$ is the ith mass and $x_i$ is its position.

**Prop**: For a 1-dim object, if $m$ is the total mass, then $\bar{x} = \frac{M}{m}$.

$m = \sum_{i=1}^{n} m_i$

$M$ ← moment

$m$ ← mass

**Ex**: In the first example, $m_1 = 30$, $m_2 = 5$, $m_3 = 10$, and $m_4 = 15$, $x_1 = -2$, $x_2 = 3$, $x_3 = 6$, and $x_4 = -3$.

$$m = 30 + 5 + 10 + 15 = 60$$

$$M = 30(-2) + 5(3) + 10(6) + 15(-3) = -30$$

$$\bar{x} = \frac{M}{m} = \frac{-30}{60} = -1/2.$$

**Def**: Let $m_i$ be the mass of the ith object and $(x_i, y_i)$ its location. Then there are two moments:

$$M_x = \sum_{i=1}^{n} m_i y_i$$

$$M_y = \sum_{i=1}^{n} m_i x_i$$

Prop: For a 2-dim object, $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$.

Ex: There is a 2 kg mass at $(-1, 3)$, a 6 kg mass at $(1, 1)$, and a 4 kg mass at $(2, -2)$. Find the center of mass.

$m_1 = 2$ $\quad x_1 = -1$ $\quad y_1 = 3$

$m_2 = 6$ $\quad x_2 = 1$ $\quad y_2 = 1$

$m_3 = 4$ $\quad x_3 = 2$ $\quad y_3 = -2$

$$M_x = 2 \cdot 3 + 6 \cdot 1 + 4(-2) = 4$$

$$M_y = 2(-1) + 6 \cdot 1 + 4 \cdot 2 = 12$$

$$m = 2 + 6 + 4 = 12$$

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1$$

$$\bar{y} = \frac{M_x}{m} = \frac{4}{12} = 1/3$$

centroid: $(1, \frac{1}{3})$.