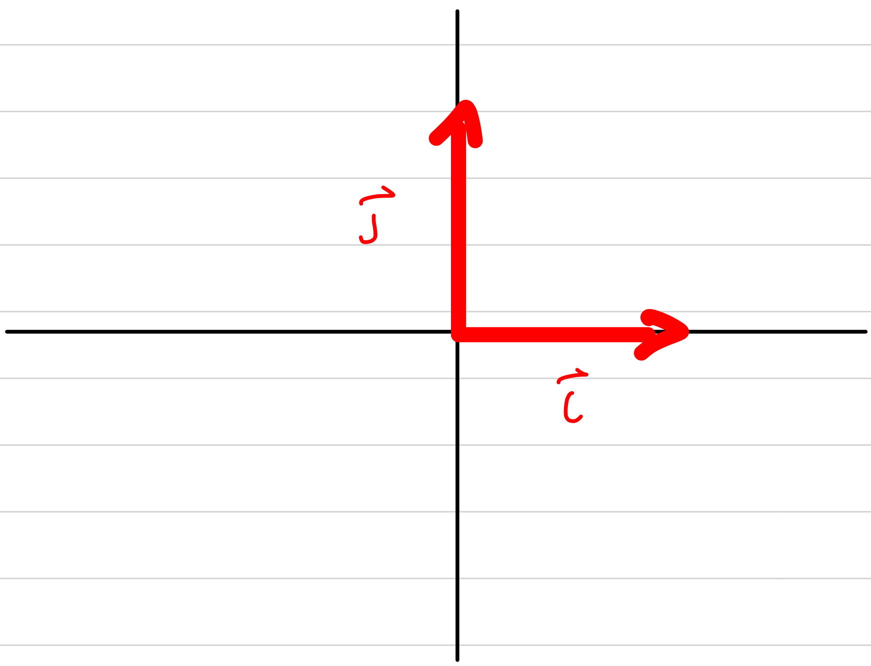
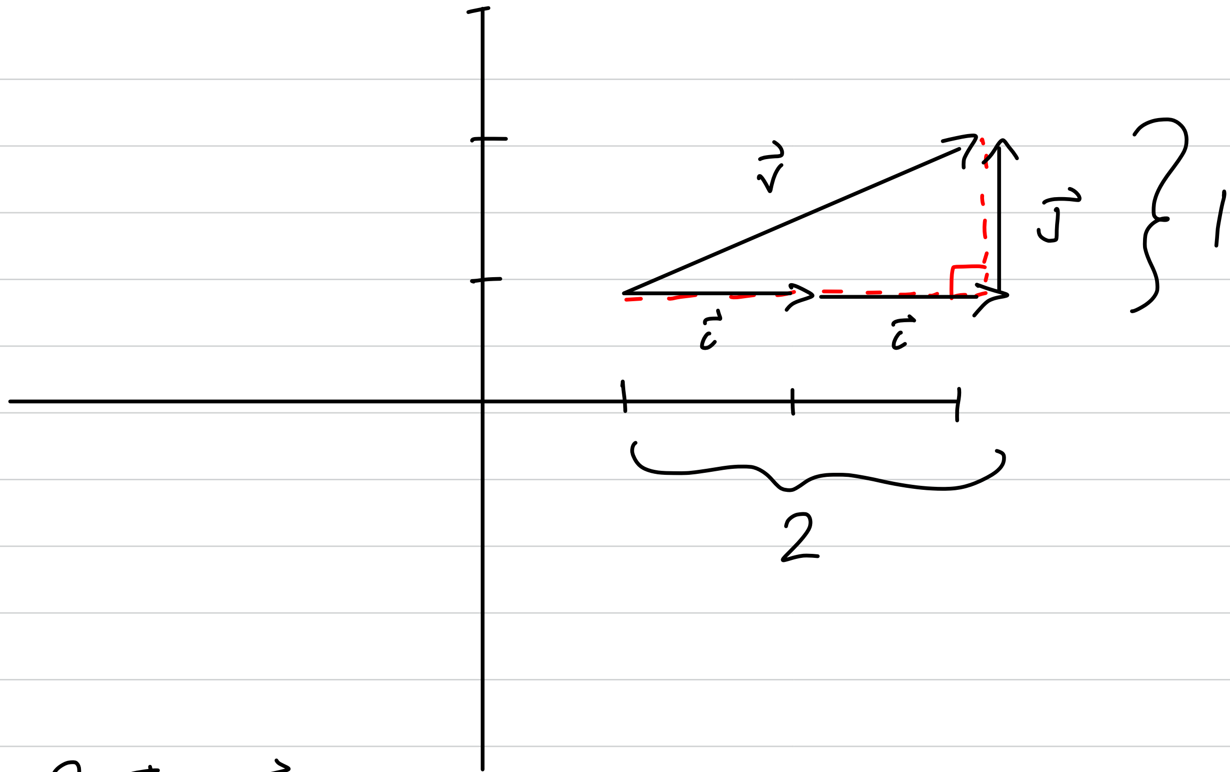
**Def:** A <u>unit vector</u> is a vector $\vec{v}$ with $\|\vec{v}\| = 1$.

**Def:** The two <u>standard unit vectors</u> in two dimensions are $\vec{i}$ and $\vec{j}$, where $\vec{i}$ points in the positive-$x$ direction and $\vec{j}$ in the positive-$y$ direction.



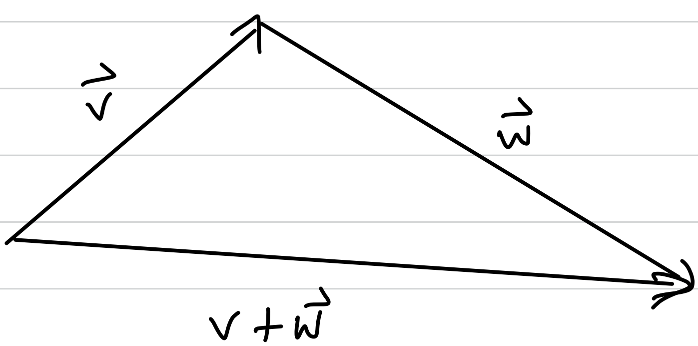

**Theorem:** Any vector $\vec{v}$ can be written as $\vec{v} = c\vec{i} + d\vec{j}$ for a unique pair of scalars $c$ and $d$. This is called the unit vector decomposition of $\vec{v}$.

Ex:




$\vec{v} = 2\vec{i} + \vec{j}$

Ex:



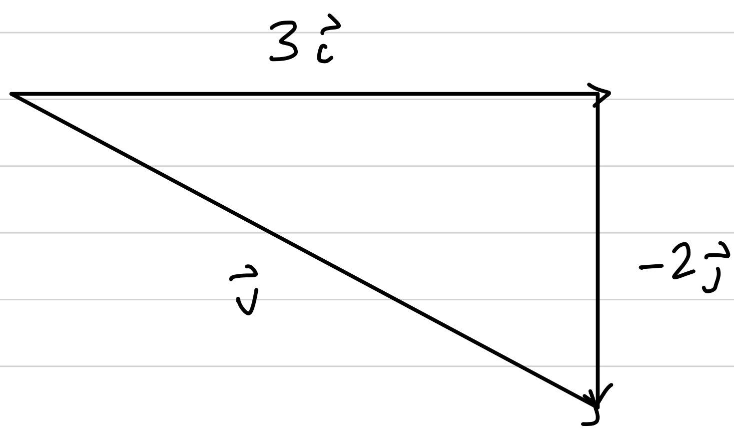

<u>Prop</u> Let $\vec{v} = c\vec{i} + d\vec{j}$ and $\vec{w} = e\vec{i} + f\vec{j}$.

1. $\vec{v} + \vec{w} = (c+e)\vec{i} + (d+f)\vec{j}$
2. $\vec{v} - \vec{w} = (c-e)\vec{i} + (d-f)\vec{j}$

③ $b\vec{v} = (bc)\vec{i} + (bd)\vec{j}$

④ $\|\vec{v}\| = \sqrt{c^2+d^2}$ (Pythagorean theorem)

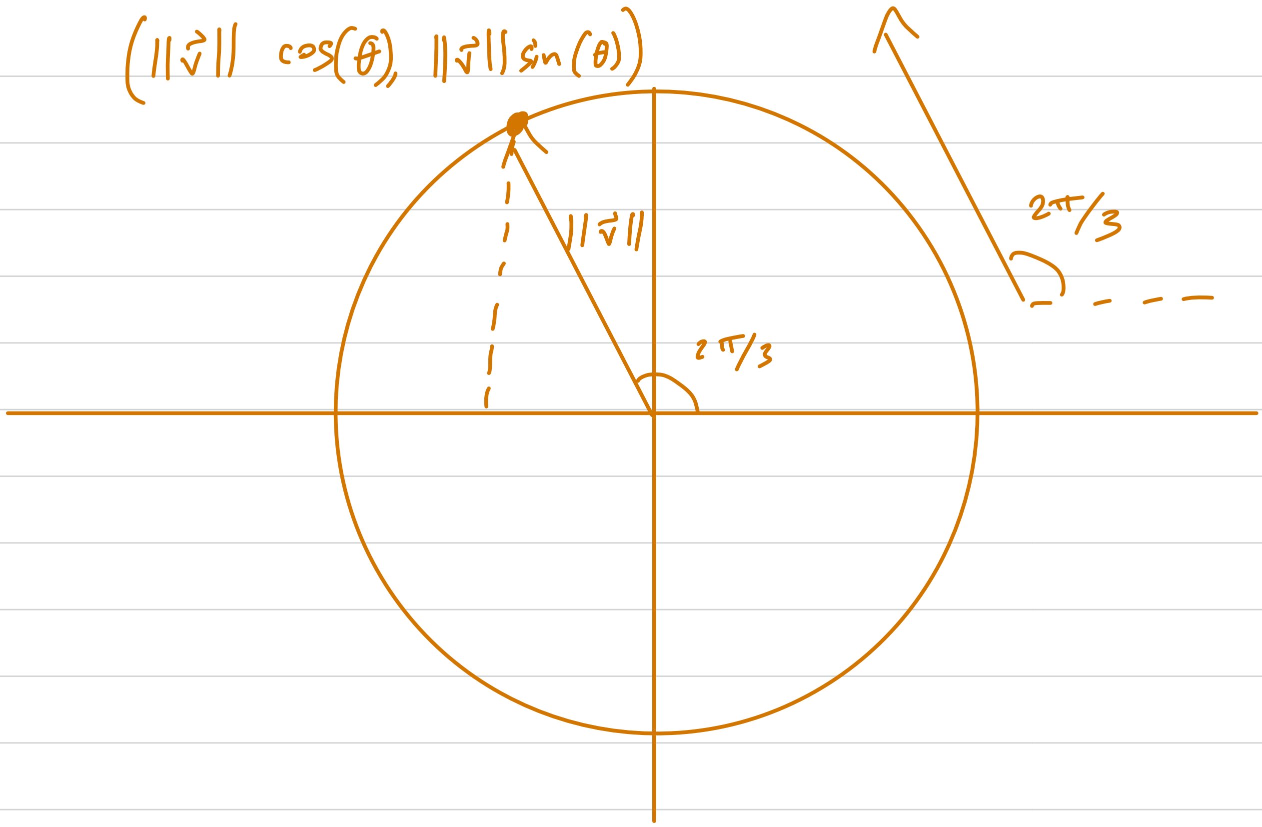
Ex: if $\vec{v} = 3\vec{i} - 2\vec{j}$, then $\|\vec{v}\| = \sqrt{3^2+(-2)^2} = \sqrt{13}$



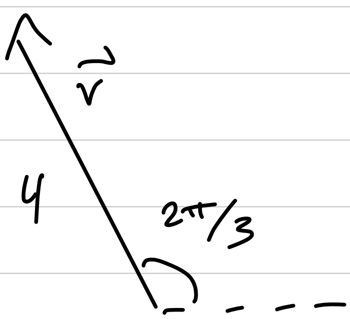

Comment ($\vec{i}$ and $\vec{j}$ components) $\longleftrightarrow$ ($\|\vec{v}\|$ and angle)

Prop: Let $\vec{v}$ be a vector with angle $\theta$ from the horizontal. Then

$$\vec{v} = (\|\vec{v}\|\cos(\theta))\,\vec{i} + (\|\vec{v}\|\sin(\theta))\,\vec{j}.$$




Ex:

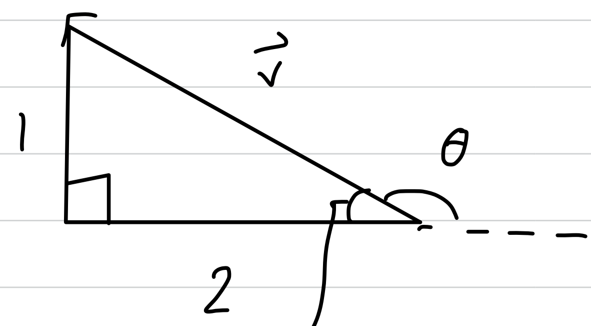



$$\vec{v} = 4\cos(2\pi/3)\,\vec{i} + 4\sin\left(\tfrac{2\pi}{3}\right)\vec{j}$$

$$= -2\,\vec{i} + 2\sqrt{3}\,\vec{j}$$

Comment: Given the unit vector decomposition of a vector $\vec{v}$, we can find its angle with the horizontal via arctan.

**Ex:** $\vec{v} = -2\vec{i} + \vec{j}$




$= \arctan(1/2) = .464$, so $\theta = \pi - .464$

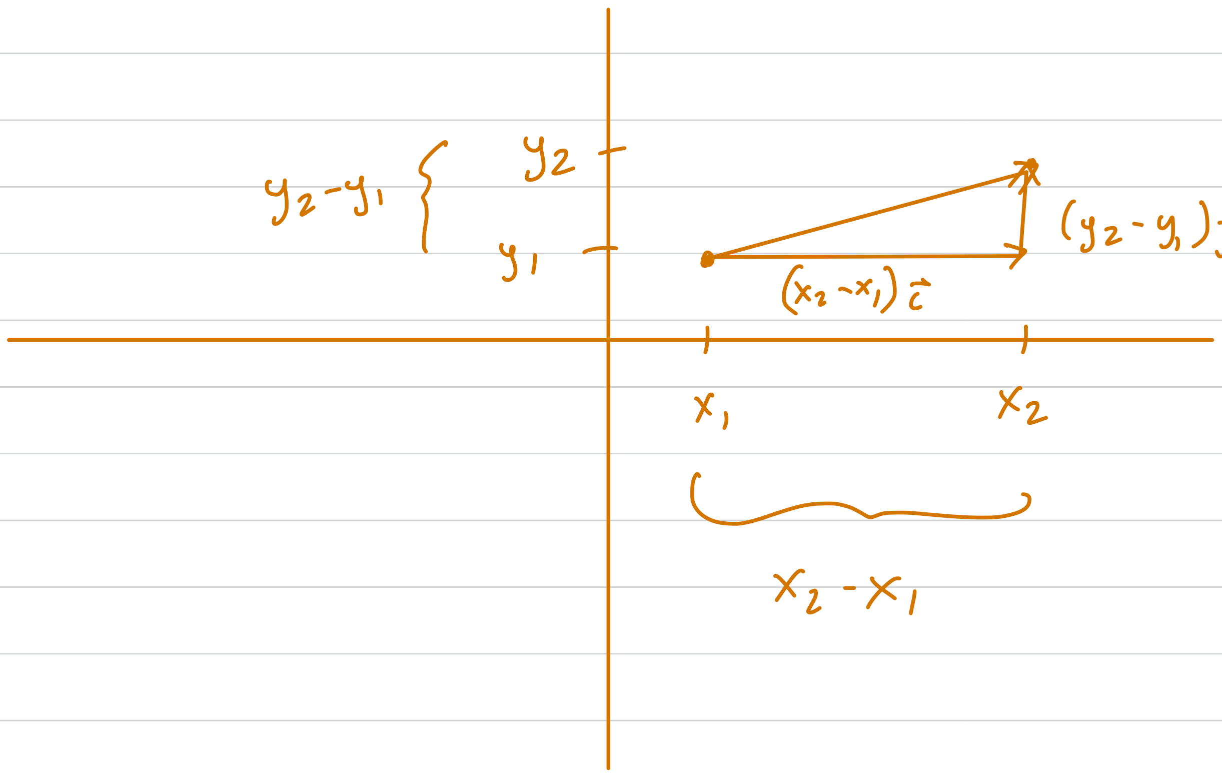
$= 2.678$.

Announcement: change to HW 7 (problem 2 easier)

**Prop** Let $(x_1, y_1)$ and $(x_2, y_2)$ be two points in the plane. The vector that starts at

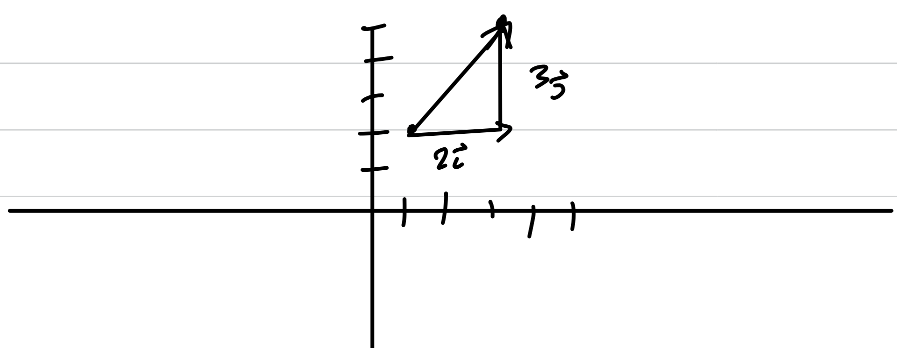
$(x_1, y_1)$ and ends at $(x_2, y_2)$ is

$(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$.




Ex: you walk from $(1,2)$ to $(3,5)$. What is the vector corresponding to your total travel?

$$(3-1)\vec{i} + (5-2)\vec{j} = 2\vec{i} + 3\vec{j}$$



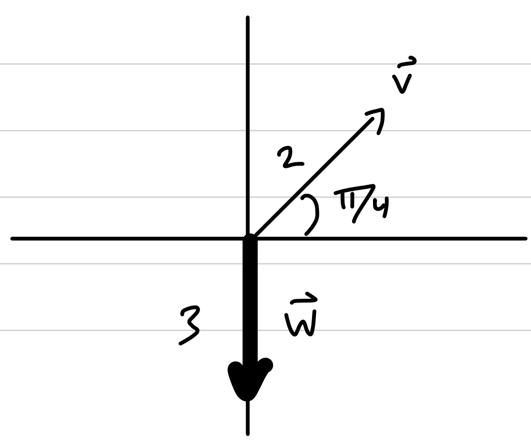
# The Dot Product

Comment: The dot product is a way to multiply two vectors, but it gives a scalar, not a vector.

Def: Let $\vec{v} = a\vec{i} + b\vec{j}$ and $\vec{w} = c\vec{i} + d\vec{j}$. The dot product of $\vec{v}$ and $\vec{w}$ is $\vec{v} \bullet \vec{w} = ac + bd$.

Ex $(2\vec{i} - \vec{j}) \bullet (3\vec{i} + 4\vec{j}) = 2 \cdot 3 + (-1) \cdot 4 = 2.$

Ex: Find $\vec{v} \bullet \vec{w}$:



We first have to find their unit vector decompositions. $\vec{v} = 2\cos(\pi/4)\,\vec{i} + 2\sin(\pi/4)\,\vec{j}$

$$= \sqrt{2}\,\vec{i} + \sqrt{2}\,\vec{j}$$

$$\vec{w} = -3\vec{j}$$

$$\vec{v} \bullet \vec{w} = (\sqrt{2})(0) + (\sqrt{2})(-3) = -3\sqrt{2}$$

Prop :

① $\vec{0} \bullet \vec{v} = \vec{v} \bullet \vec{0} = 0$

② $\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$

③ $c(\vec{v} \bullet \vec{w}) = (c\vec{v}) \bullet \vec{w} = \vec{v} \bullet (c\vec{w})$

④ $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

⑤ $\underbrace{\vec{v} \bullet \vec{v} = \|\vec{v}\|^2}$ (think of $x\ x = |x|^2$)

$\vec{v} = a\vec{i} + b\vec{j}$

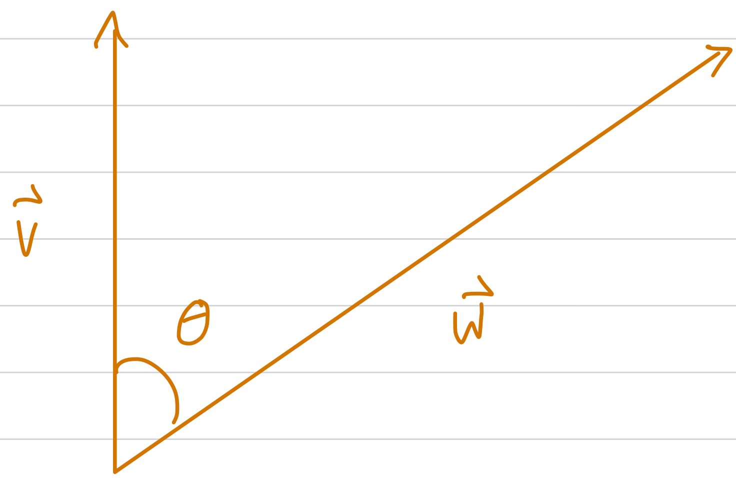
$\|\vec{v}\| = \sqrt{a^2+b^2}$

$\|\vec{v}\|^2 = a^2 + b^2$

$\vec{v} \bullet \vec{v} = a.a + b.b$

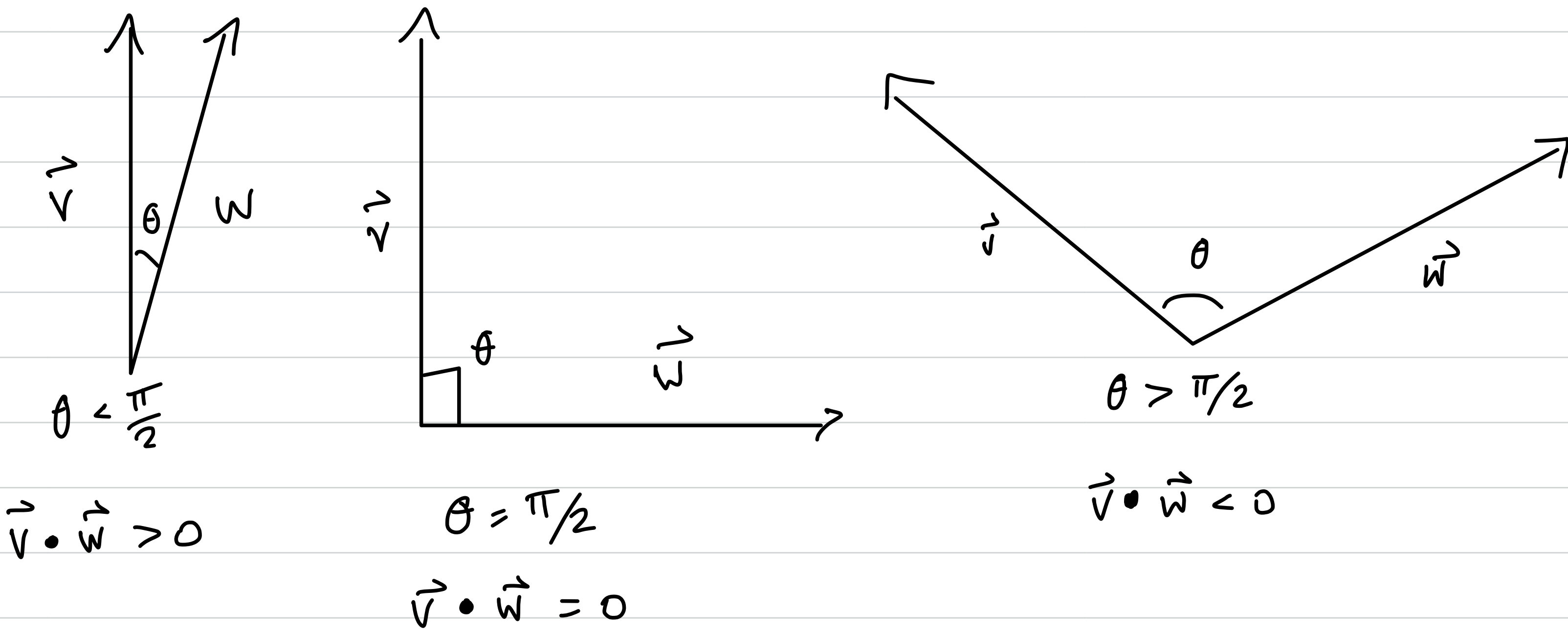
**Comment:** You might expect a property like $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$ – but this doesn't make sense! $\vec{u} \cdot \vec{v}$ is a scalar, and you can't dot scalars with vectors.

**Prop:** Let $\vec{v}$ and $\vec{w}$ be vectors that form an angle of $\theta$ when starting at the same point.




Then $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos(\theta)$.

**Comment:** In this sense, the dot product measures the degree to which $\vec{v}$ and $\vec{w}$ are parallel.




**Prop:** The angle between vectors $\vec{v}$ and $\vec{w}$ is

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}\right).$$

Ex: Find the angle between $\vec{v} = 3\vec{i} + \vec{j}$ and $\vec{w} = 2\vec{i} - \vec{j}$.

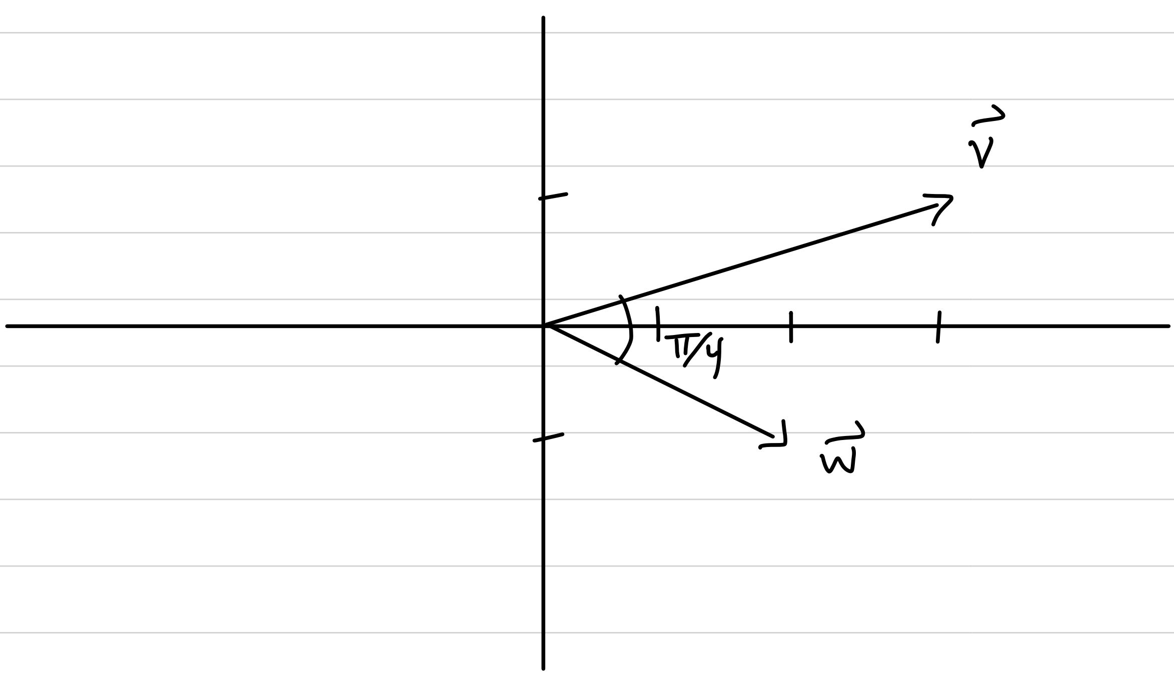
$\vec{v} \cdot \vec{w} = 3 \cdot 2 + (-1)(1) = 6 - 1 = 5$

$\|\vec{v}\| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$\|\vec{w}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$$\theta = \arccos\left(\frac{5}{\sqrt{10}\,\sqrt{5}}\right) = \arccos\left(\frac{5}{\sqrt{2}\,\underbrace{\sqrt{5}\,\sqrt{5}}_{5}}\right)$$

$$= \arccos\left(\frac{1}{\sqrt{2}}\right) = \pi/4 .$$



Comment: If you want the angle that vector makes with the horizontal, use arctan. If you want the angle that two vectors make with one another, use this.

Def: Vectors $\vec{v}$ and $\vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.

Comment: If neither $\vec{v}$ nor $\vec{w}$ is the zero vector, then orthogonal means perpendicular. Your book uses perpendicular, but we'll use orthogonal.

Ex: $\vec{v} = 2\vec{i} + 3\vec{j}$ and $\vec{w} = -3\vec{i} + 2\vec{j}$ are orthogonal, because $\vec{v} \cdot \vec{w} = 2(-3) + 3(2)$
$= -6 + 6 = 0$.

Ex: Find all vectors orthogonal to $-3\vec{i} + 2\vec{j}$.

Let $\vec{v} = a\vec{i} + b\vec{j}$ and solve $\vec{v} \cdot (-3\vec{i} + 2\vec{j}) = 0$

$-3a + 2b = 0$

First solve for a. $-3a = -2b$

$a = \frac{2}{3}b$

Set $b = t$ for a variable $t$.

$b = t$

$a = \frac{2}{3}t$

$\vec{v} = \frac{2}{3}t\,\vec{i} + t\,\vec{j}$ for any $t$.

(Note: $t = 3$ gives the previous example)

What's happening geometrically?

$$\vec{v} = t\left(\frac{2}{3}\vec{i} + \vec{j}\right)$$