Midterm 1

Math 112

Spring 2021

You have 50 minutes to complete this exam. **Show all your work. You may use a scientific calculator, but not a graphing one.** When you're finished, first check your work if there is time remaining, then scan the exam and upload it to Canvas. If you have a question, don't hesitate to ask — I just may not be able to answer it.

1. (32 points) Multiple choice. You don't need to show your work.

a) (8 points) A function f is even if for all x in its domain,

A) $f(-x) = -f(x)$.

B) $f(-x) = f(x)$.

C) $-f(x) = f(x)$.

D) $f(x) = f(f(x))$.

b) (8 points) For which of the following values of θ is $\cos(\theta) = 0$ and $\sin(\theta) = -1$?

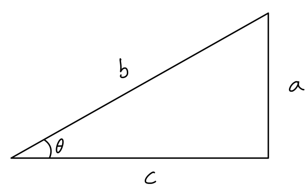
A) $\theta = 0°$.

B) $\theta = 90°$.

C) $\theta = 180°$.

D) $\theta = 270°$.

c) (8 points) In the following right triangle,



A) $\sin(\theta) = \frac{a}{b}$.

B) $\sin(\theta) = \frac{a}{c}$.

C) $\sin(\theta) = \frac{c}{a}$.

D) $\sin(\theta) = \frac{b}{c}$.

d) (8 points) The graph of $\cos(3x + 3)$ is the graph of $\cos(x)$, but

A) Horizontally stretched by a factor of 3 and shifted right 1.

B) Horizontally stretched by a factor of $\frac{1}{3}$ and shifted right 3.

C) Horizontally stretched by a factor of 3 and shifted left 3.

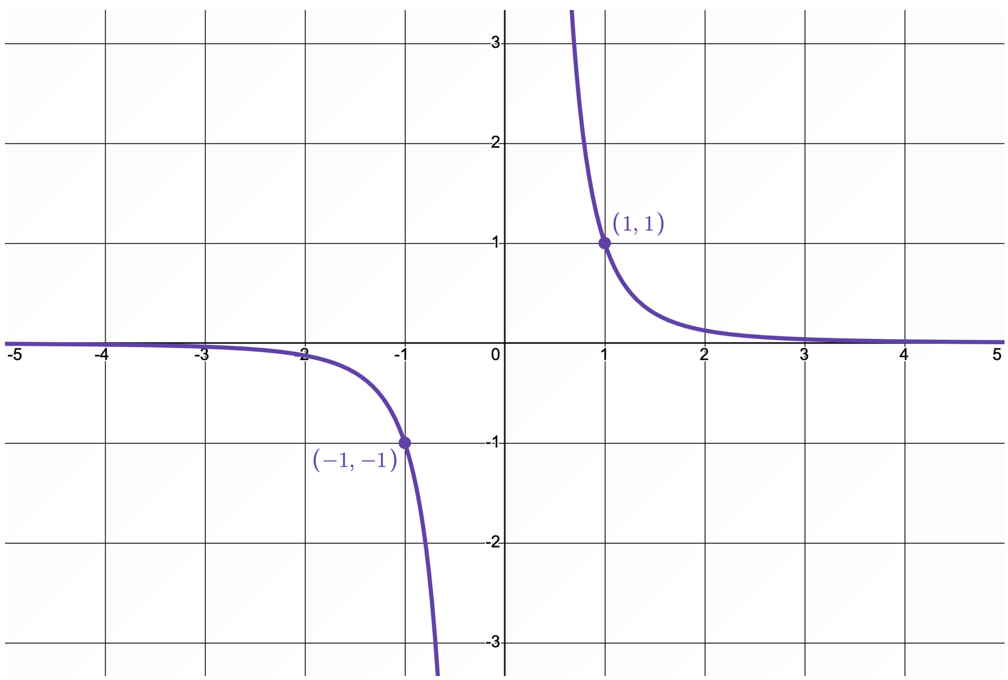
D) Horizontally stretched by a factor of $\frac{1}{3}$ and shifted left 1.

2. (32 points) Let $f(x) = \dfrac{1}{x^3}$.

a) (8 points) Is f an even function? Is it an odd one? Explain your answer.

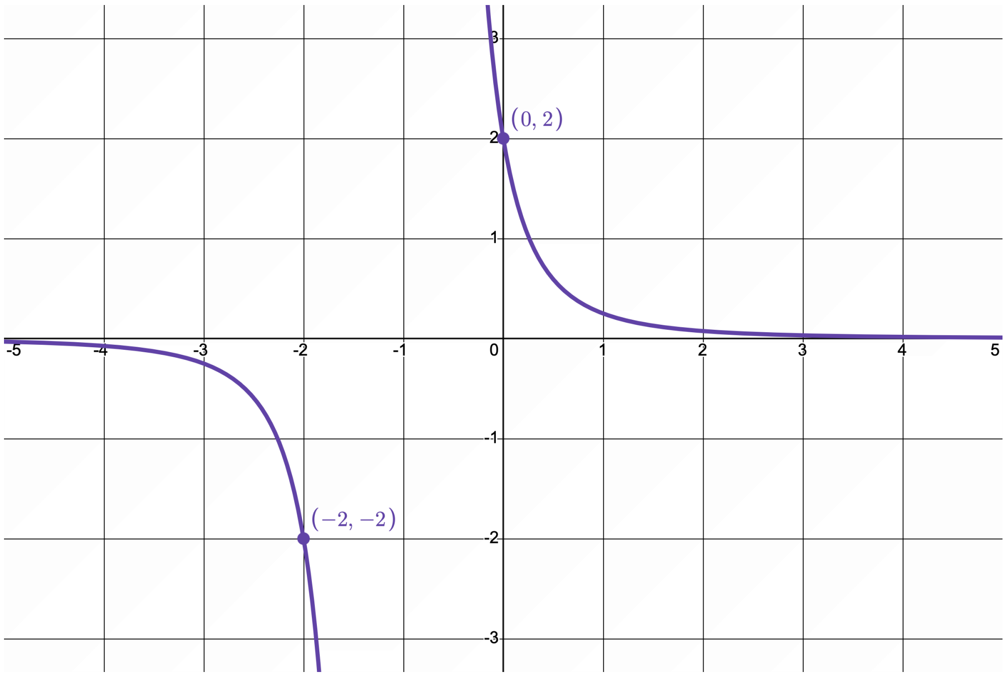
It's odd, since $\dfrac{1}{(-x)^3} = -\dfrac{1}{x^3}$.

b) (8 points) Sketch a graph of f, labeling at least two points.



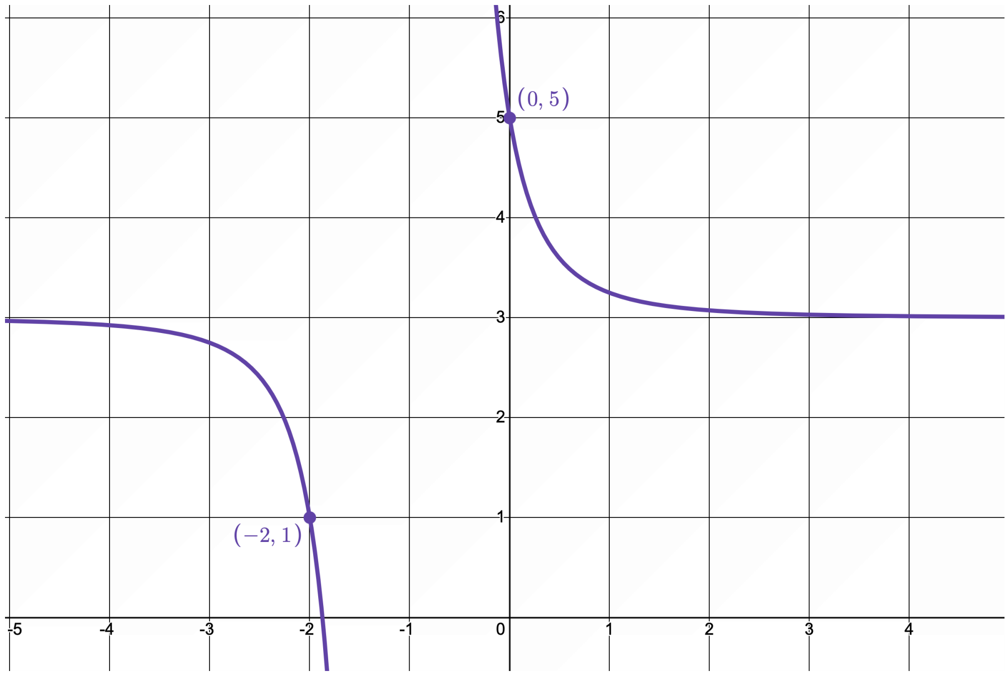
c) (8 points) Sketch a graph of $\frac{2}{(x+1)^3}$. To receive credit, you must list each transformation you apply and sketch a graph after each one, as in class. Continue to label the points you labeled in part b).

This is a vertical stretch by a factor of 2 and a horizontal shift 1 unit to the left, in either order.



d) (8 points) Apply a vertical shift of 3 units upward to the function in part c). Write the new equation of the function and sketch a graph. Again, continue to label the points you've been labeling for the past two parts.

The new equation is $\frac{2}{(x+1)^3} + 3$.



6
5
4
3
2
1
-5
-4
-3
-2
-1
0
1
2
3
4
(0, 5)
(−2, 1)

3. (32 points) You go for a walk in a flat field. You first walk 3 miles east, then turn and walk another 2 miles south.

a) (8 points) You now walk home in a straight line. How many miles will you walk in that straight line before you get home? Leave your answer in exact form (i.e. no decimals).

$3^2 + 2^2 = 13$, so $\sqrt{13}$ miles.

b) (12 points) You left your house going due east and came home at an angle. What is the cosine of the angle between due east and the line you came home on? Again, leave your answer in exact form. (Hint: you don't need to find the angle itself.)

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{3}{\sqrt{13}}$.

c) (12 points) You want that final walk home to be 4 miles. If you still start by walking 3 miles east, but now walk x miles south, then walk home in a straight line, what must x be?

$9 + x^2 = 16$, so $x = \sqrt{7}$.

4. (32 points) Consider a circle of radius 1 centered at the origin.

a) (8 points) What are the coordinates of a point on this circle with angle θ counter-clockwise from the positive x-axis?

$(\cos(\theta), \sin(\theta))$.

b) (8 points) Now suppose the circle has radius 3. What are the coordinates of that point with angle θ now?

$(3\cos(\theta), 3\sin(\theta))$.

c) (8 points) A point on this circle with radius 3 has angle 150° from the positive x-axis. Sketch a picture of this point and find its coordinates, leaving them in exact form. You must explain how you calculate any trig functions to receive credit.

$(3\cos(150°), 3\sin(150°)) = \left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$, since the reference angle is 30°.

d) (8 points) If the circle were centered at $(-1, -2)$, what would the coordinates of that point be?

$\left(-\frac{3\sqrt{3}}{2} - 1, -\frac{1}{2}\right)$.

Midterm 2

Math 112

Fall 2021

You have 50 minutes to complete this exam. When you're finished, first check your work if there is time remaining, then turn it in. If you have a question, don't hesitate to ask — I just may not be able to answer it.

1. (32 points) Multiple choice. You don't need to show your work.

a) (8 points) The slope of a line that passes through the origin and a point on the unit circle with angle θ is

A) $\sin(\theta)$.

B) $\tan(\theta)$.

C) $\sec(\theta)$.

D) $\cos(\theta)$.

b) (8 points) The arc length of a $72°$ angle in a circle of radius 2 is

A) $\frac{2\pi}{72}$.

B) $\frac{4\pi}{5}$.

C) 2π.

D) 72π.

c) (8 points) In which of the following cases can we not solve for another side of a triangle?

A) When we know three angles.

B) When we know two angles and a side.

C) When we know an angle and two sides.

d) (8 points) The function $f(x) = -5\sin(4(x+4))$ has

A) Amplitude 5, period 4, and midline 4.

B) Amplitude 5, period $\frac{\pi}{2}$, and midline 0.

C) Amplitude 5, period $\frac{\pi}{2}$, and midline 4.

D) Amplitude 5, period 4, and midline 0.

2. (32 points) Miscellaneous questions. The four parts here are unrelated to one another.

a) (8 points) Find the **exact value** of $\tan\left(\frac{2\pi}{3}\right)$. Show all your work — if you use a reference angle, you must draw a picture.

This angle is in quadrant II, and its reference angle is $\frac{\pi}{3}$. Tangent is negative there, and so $\tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$.

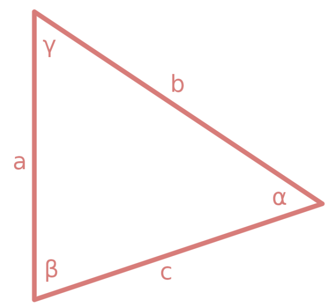
b) (8 points) Find the **exact value** of $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ in radians. Show all your work.

Since arccos outputs angles in $[0, \pi]$, we're looking for an angle θ with $\cos(\theta) = -\frac{\sqrt{2}}{2}$ and $0 \leq \theta \leq \pi$. That's $\theta = \frac{3\pi}{4}$.

c) (8 points) Find **all** solutions to the equation $\sin(\theta) = -\frac{\sqrt{3}}{2}$, where θ is in radians. Show all your work.

We start by taking arcsin of both sides, which gives us $\theta = -\frac{\pi}{3}$. The other angle on the unit circle with that y-coordinate is $\frac{4\pi}{3}$, and so the general solution is $\theta = -\frac{\pi}{3} + 2\pi k$ or $\theta = \frac{4\pi}{3} + 2\pi k$

d) Write the Law of Sines for the following triangle.



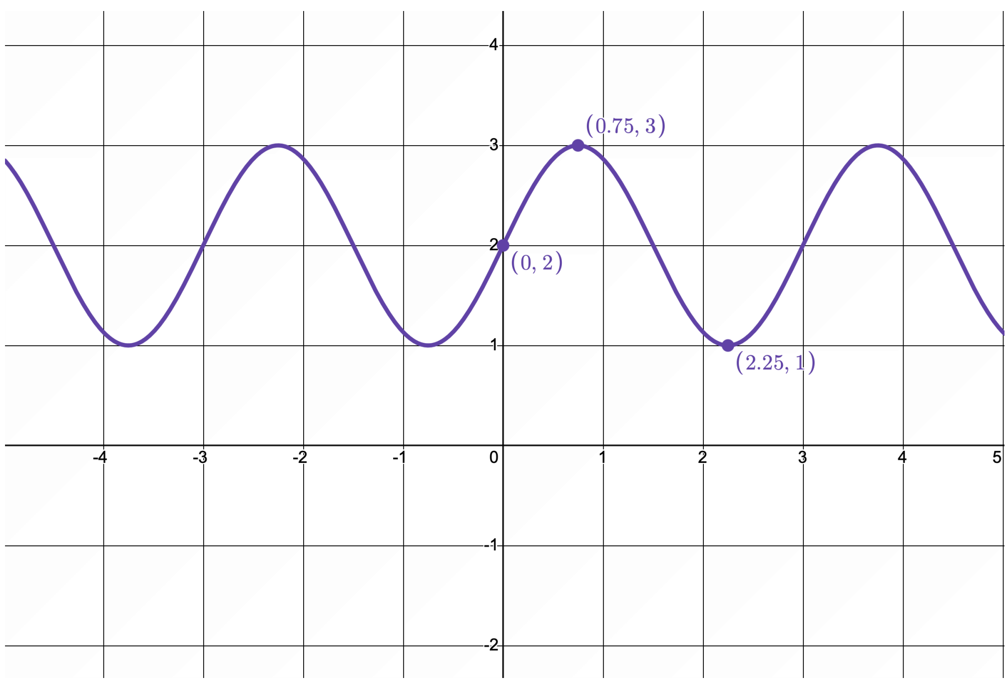
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

3. (32 points)

a) (12 points) Find the equation of a sinusoidal function $f(x)$ with amplitude 1, midline 2, and period 3, with no horizontal shift.

$f(x) = \sin\left(\frac{2\pi}{3}x\right) + 2.$

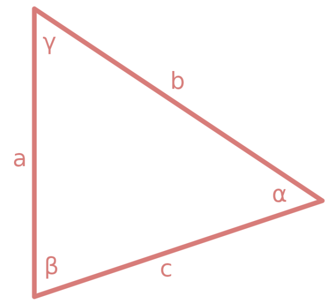
b) (8 points) Sketch a graph of f. Label at least three points.



c) (12 points) Find a horizontal shift of f so that its graph passes through the point $(1, 2.5)$ and it is increasing there.

If $\sin\left(\frac{2\pi}{3}(1-h)\right) + 2 = 2.5$, then $\sin\left(\frac{2\pi}{3}(1-h)\right) = \frac{1}{2}$. One possibility for h is therefore $\frac{2\pi}{3}(1-h) = \frac{\pi}{6}$, so $1 - h = \frac{1}{4}$ or equivalently $h = \frac{-3}{4}$. Looking at the graph, that moves the point $\left(\frac{1}{4}, 2.5\right)$ to $(1, 2.5)$, and since the graph was increasing at the first point, it is successfully increasing at the second point too.

4. (32 points) Consider the following triangle with sides a, b, and c, and angles α, β, and γ.



a) Given that $a = 4$, $b = 5$, and $\gamma = \dfrac{\pi}{3}$, find c. Leave your answer in **exact form**.

This is a Law of Cosines problem: $c^2 = 4^2 + 5^2 - 2(4)(5)\cos\left(\frac{\pi}{3}\right)$, so $c^2 = 16 + 25 - 20 = 21$. Therefore $c = \sqrt{21}$.

b) Use your answer to part a) to find α.

We can do this with the Law of Cosines again: $4^2 = 5^2 + 21 - 2 * 5\sqrt{21}\cos(\alpha)$, so $\cos(\alpha) \approx 0.6547$, and $\alpha \approx \arccos(0.6547) \approx 0.8571$.

c) Now find β.

This is just $\pi - \alpha - \gamma \approx 1.237$

d) (Bonus) Find the area of this triangle. (Hint: pick one side to be the base, then draw a line perpendicular to that base that reaches to the opposite vertex to split the triangle into two right triangles. Then use trig functions to find the length of that line.)

If we draw a line perpendicular to side a to the point with angle α and call its length h, then the right triangle formed with angle γ has $\sin(\gamma) = \frac{\sqrt{3}}{2} = \frac{h}{b} = \frac{h}{5}$. Therefore, $h = \frac{5\sqrt{3}}{2}$, so the area is $\frac{1}{2}ah = \frac{25\sqrt{3}}{4}$.

Final Exam

Math 112

Fall 2021

You have 2 hours to complete this exam. You may use a scientific calculator, but no other resources. When you're finished, first check your work if there is time remaining, then turn it in. If you have a question, don't hesitate to ask — I just may not be able to answer it.

Formulas

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1 + \cos(\theta)}$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

1. (64 points) Multiple choice. You don't need to show your work.

a) (8 points) Suppose θ is an angle in quadrant II. Which of the following is true?

A) $\arcsin(\sin(\theta)) = \theta$.

B) $\arccos(\cos(\theta)) = \theta$.

C) $\arctan(\tan(\theta)) = \theta$.

D) All of the above.

b) (8 points) Which of the following is **not** a transformation of $y = \sin(x)$?

A) $y = \sin(5x)$.

B) $y = \cos(5x)$.

C) $y = \csc(5x)$.

D) $y = \sin(5x)\cos(5x)$.

c) (8 points) Let $\vec{v}$ be a 2-dimensional vector. Which of the following can $\vec{v}$ **not** have?

A) Negative magnitude.

B) Negative angle.

C) Negative $\vec{i}$ component.

D) Negative $\vec{j}$ component.

d) (8 points) What is the definition of the tangent function?

A) $\tan(\theta) = \dfrac{1}{\sin(\theta)}$.

B) $\tan(\theta) = \cos(\pi - \theta)$.

C) $\tan(\theta) = \sin(\theta)^2 + \cos(\theta)^2$

D) $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.

e) (8 points) The measure of $\dfrac{13\pi}{12}$ in degrees is

A) $3.403°$.

B) $195°$.

C) $62°$.

D) $.0594°$.

f) (8 points) If the dot product of two vectors is negative, then the angle θ between the two vectors satisfies

A) $\theta = 0°$.

B) $0° < \theta < 90°$.

C) $\theta = 90°$.

D) $90° < \theta \leq 180°$.

g) (8 points) How many solutions to $\sin(x) = \cos(x)$ are there such that $0 \leq x < 2\pi$?

A) 0.

B) 1.

C) 2.

D) 4.

h) (8 points) The function $y = 3(x+1)^5$ is a transformation of $y = x^5$ by

A) A vertical stretch and horizontal stretch.

B) A vertical stretch and horizontal shift.

C) A vertical shift and horizontal stretch.

D) A vertical shift and horizontal shift.

2. (32 points) Short-answer. Show all your work!

a) (8 points) What are the coordinates of a point on a circle of radius r with angle θ counter-clockwise from the positive x-axis?

$(r\cos(\theta), r\sin(\theta))$.

b) (8 points) Evaluate the **exact value** of $\sin(135°)$. You must draw a picture, identify the reference angle, and use it correctly.

This angle is in quadrant II and its reference angle is 45°, so $\sin(135°) = \sin(45°) = \frac{\sqrt{3}}{2}$.

c) (8 points) Why is $\arccos(4)$ undefined? Your answer should be one or two complete sentences.

Because there is no angle θ with $\cos(\theta) = 4$, since there is no point on the unit circle with x-coordinate 4.

d) (8 points) Find the angle between $\vec{v} = 2\vec{i}$ and $\vec{w} = -2\vec{i} + \vec{j}$.

The magnitudes of these vectors are $||\vec{v}|| = 2$ and $||\vec{w}|| = \sqrt{2^2 + 1} = \sqrt{5}$. The dot product is $\vec{v} \bullet \vec{w} = -4$, and so $-4 = (2)(\sqrt{5})\cos(\theta)$, so $\theta = \arccos\left(\frac{-4}{2\sqrt{5}}\right) \approx 2.678$.

3. A 2-dimensional vector $\vec{v}$ has magnitude 2 and angle $\frac{7\pi}{6}$ counter-clockwise from the positive x-axis.

a) (8 points) Find the unit vector decomposition for $\vec{v}$, showing all your work.

$\vec{v} = 2\cos\left(\frac{7\pi}{6}\right)\vec{i} + 2\sin\left(\frac{7\pi}{6}\right)\vec{j} = -\sqrt{3}\vec{i} - \vec{j}$.

b) (8 points) Vector $\vec{w}$ has unit vector decomposition $\vec{w} = 2\vec{i} - 3\vec{j}$. Find $||\vec{w}||$ and the angle $\vec{w}$ makes with the positive x-axis.

$||\vec{w}|| = \sqrt{2^2 + 3^2} = \sqrt{13}$, and the angle is $\arctan\left(\frac{-3}{2}\right) = -.983$.

c) Find $\vec{v} \bullet \vec{w}$.

This is $(-\sqrt{3})(2) + (-1)(-3) = -2\sqrt{3} + 3$.

d) (8 points) What is the angle between $\vec{v}$ and $\vec{w}$?

This is $\arccos\left(\frac{-2\sqrt{3}}{2\sqrt{13}}\right) = 2.07.$

4. (32 points) Let $f(x)$ be a sinusoidal function with amplitude 2, midline 1, and period 3.

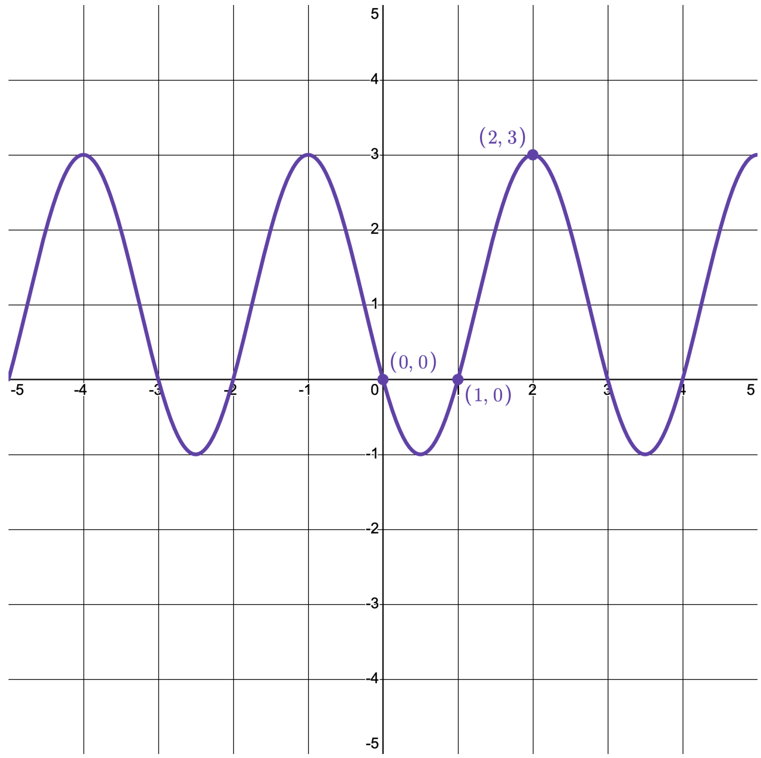
a) (12 points) Assuming there is no horizontal shift, find a formula for f.

$f(x) = 2\sin\left(\frac{2\pi}{3}x\right) + 1.$

b) (12 points) If $f(1) = 0$ and f is increasing there, find a formula for f.

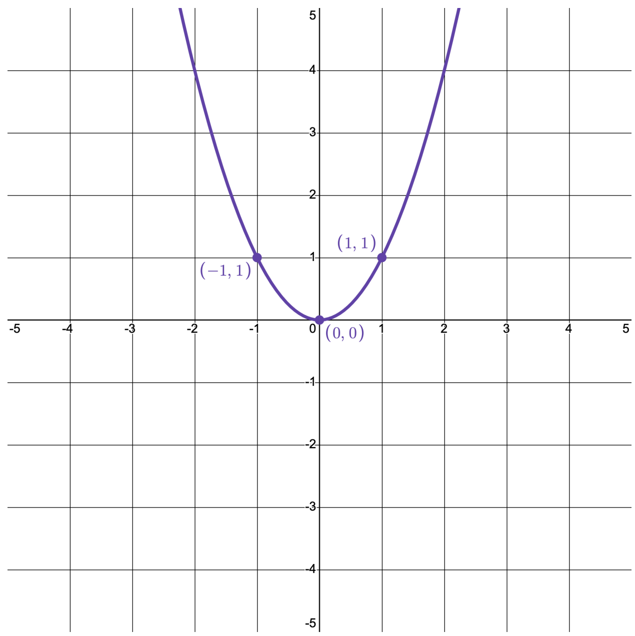
$2\sin(2\pi/3 - h) + 1 = 0$, so $h = \frac{5\pi}{6} + 2\pi n$ or $h = \frac{3\pi}{2} + 2\pi n$. The first one makes the function increasing, so $h = \frac{5\pi}{6}$.

c) (8 points) Sketch a graph of f, labeling at least three points.


(2, 3)
(0, 0)
(1, 0)
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
1
2
3
4
5

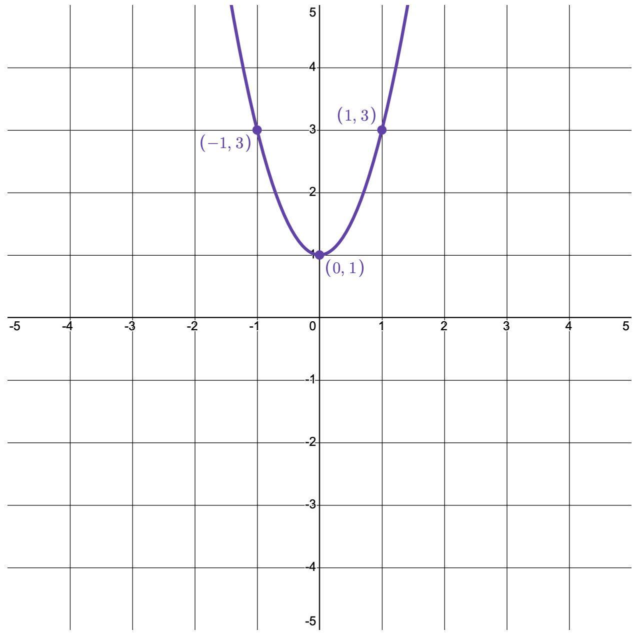
5. (32 points) Let $f(x) = x^2$.

a) (8 points) Sketch a graph of f, labeling at least three points.



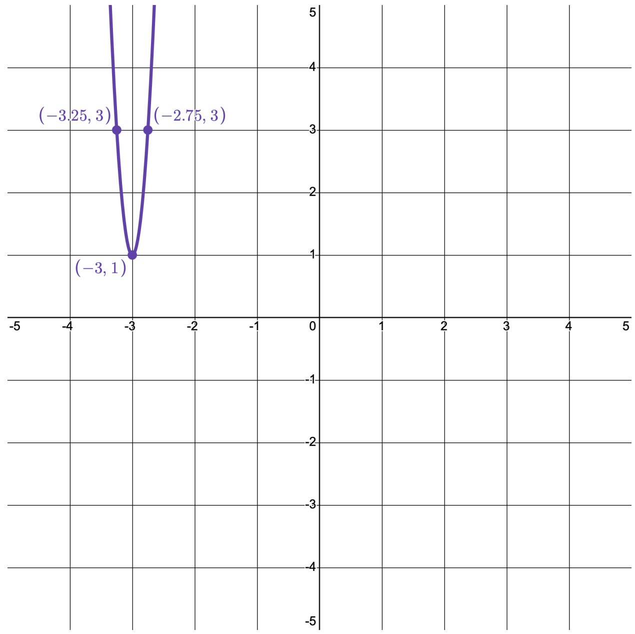
b) (8 points) Let $g(x) = 2x^2 + 1$. g is a transformation of f — list the transformation(s) you'd need to apply to f to get g, and then sketch a graph of g, labeling the points that correspond to the ones from part a).

g is a vertical stretch by a factor of 2, followed by a vertical shift up one.



c) (8 points) Let $h(x) = 2(4(x+3))^2 + 1$. h is a transformation of g — again, list the transformation(s) applied to g to get h, then sketch a graph of h, labeling the points that correspond to the ones from part a).

This one is a horizontal stretch by a factor of $\frac{1}{4}$, followed by a horizontal shift 3 to the left.



d) (8 points) If we want to apply a vertical stretch to h to make a new function k such that $k(1) = 1$, what must $k(x)$ be?

We know $k(x) = a \cdot h(x)$, so $k(1) = a \cdot h(1) = a(2(256) + 1)$. Thus $513a = 1$, so $a = \dfrac{1}{513}$.