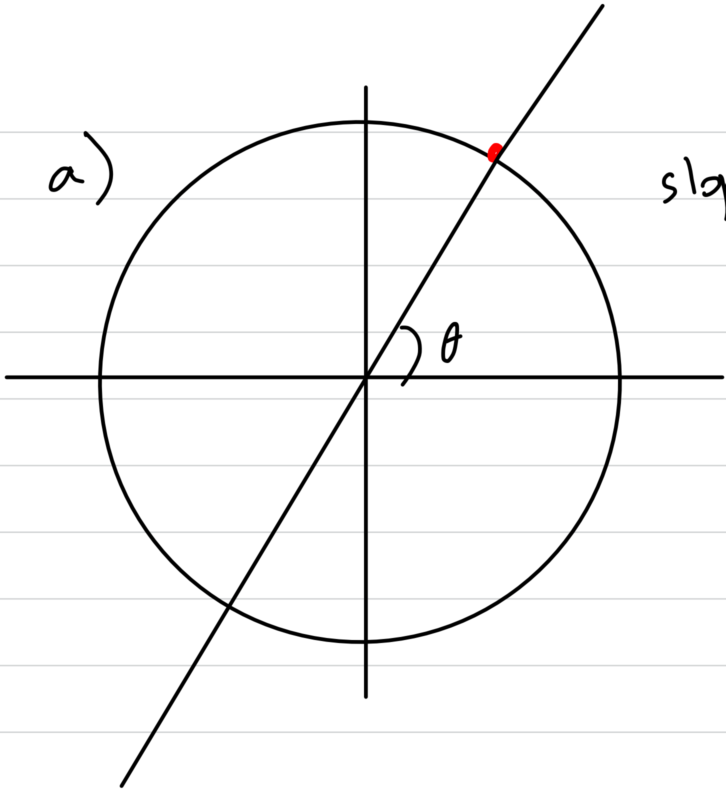
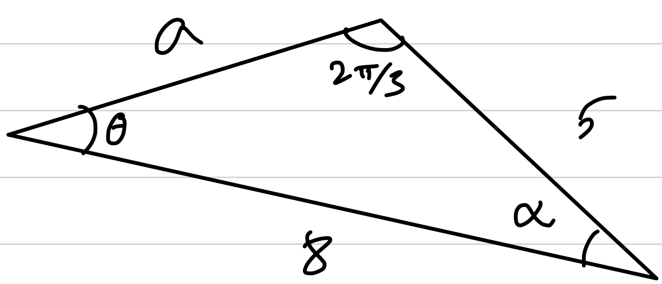
(1) a)



slope = $\tan(\theta)$

b) $\frac{\pi}{5}\left(\frac{180^\circ}{\pi}\right) = \frac{180^\circ}{5} = 36.$

c)



LoC: $5^2 = 8^2 + a^2 - 2 \cdot 8 \cdot a \cos(\theta)$

LoS: $\boxed{\frac{\sin(\theta)}{5} = \frac{\sin(2\pi/3)}{8}} = \frac{\sin(\alpha)}{a}$

d) $f(x) = 3\sin(2(x+2)) + 0$

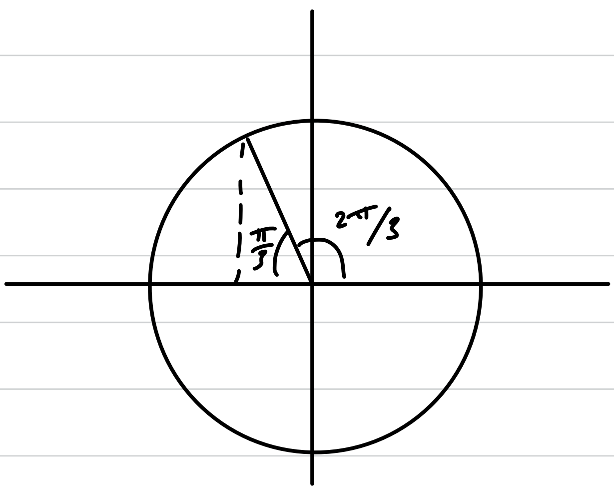
amplitude 3

midline 0

period $2\pi/2 = \pi$

② a) $\tan(2\pi/3) = -\tan(\pi/3) = -\sqrt{3}$

$$\left(\tan(2\pi/3) = \frac{2\tan(\pi/3)}{1-\tan(\pi/3)^2} \right)$$



or:

$$\tan(2\pi/3) = \frac{\sin(2\pi/3)}{\cos(2\pi/3)} = \frac{\sin(\pi/3)}{-\cos(\pi/3)} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}.$$

b) Exact value = no decimals.

$\sin(105°) = \sin(\frac{1}{2}(210°))$

$\sin(105°) = \sin(60° + 45°)$

$\sin(105°) = \sin(150° - 45°)$

$$\sin(105^\circ) = \sin(60^\circ + 45^\circ) = \sin(60^\circ)\cos(45^\circ) + \cos(60^\circ)\sin(45^\circ)$$

$$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\sin\left(\tfrac{1}{2}(210^\circ)\right) = \pm\sqrt{\frac{1-\cos(210^\circ)}{2}}$$

$$= \pm\sqrt{\frac{1-(-\sqrt{3}/2)}{2}}$$

$$= \pm\sqrt{\frac{1+\sqrt{3}/2}{2}}$$

$$= \pm\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$$

$$= \pm\sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \frac{\pm\sqrt{2+\sqrt{3}}}{2}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

positive since $\sin(105^\circ) > 0$

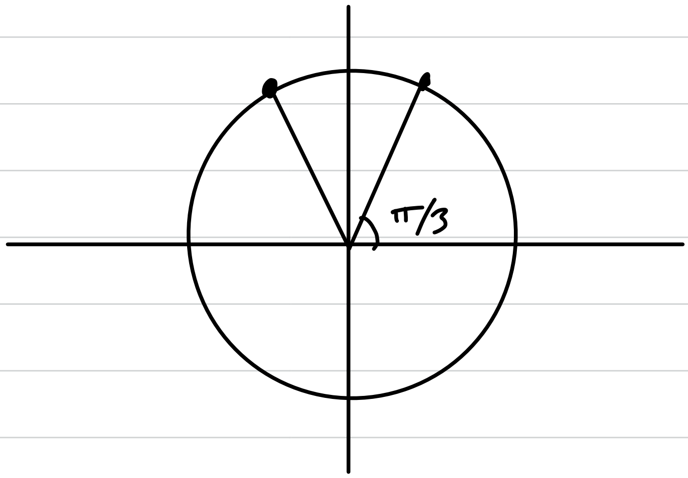
c) $f(x) = 2\sin(x - h) - \sqrt{3}$

$h \neq 0$ because then $f(0) = 2\sin(0 - 0) - \sqrt{3} = -\sqrt{3} \underset{0}{\overset{\pi}{}}$

$0 = 2\sin(0 - h) - \sqrt{3}$

$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \pi/3$

$\frac{\sqrt{3}}{2} = \sin(-h)$



$-h = \pi/3 + 2\pi n$

or

$-h = 2\pi/3 + 2\pi n$

Let's take $-h = \pi/3 + 2\pi(0)$

$h = -\pi/3$

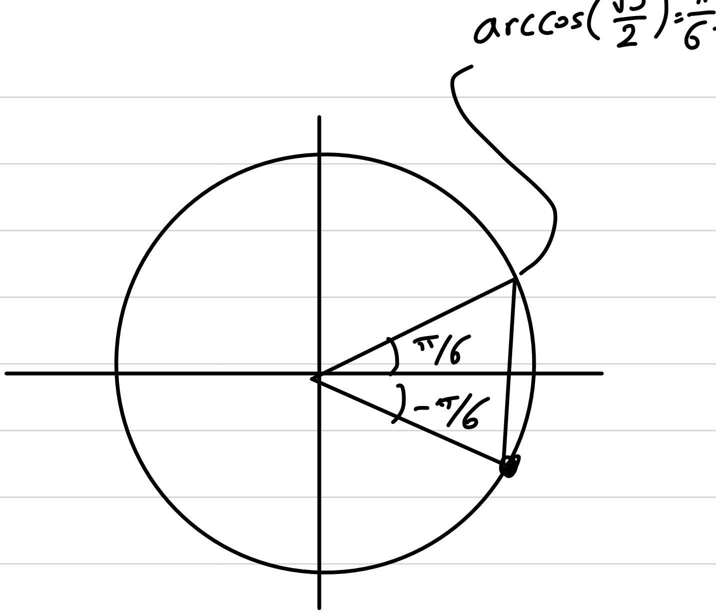
$f(x) = 2\sin(x + \pi/3) - \sqrt{3}$.

d) $\cos(\theta) = \frac{\sqrt{3}}{2}$

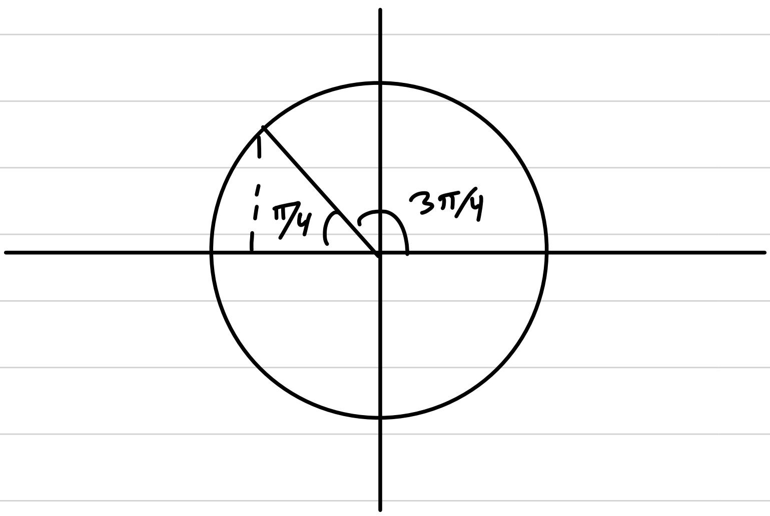
$\theta = \frac{\pi}{6} + 2\pi n$

or

$\theta = \frac{11\pi}{6} + 2\pi n$



③ a) $y(\pi/4) = 1 - \tan(3\pi/4) = 1 - (-1) = 2.$



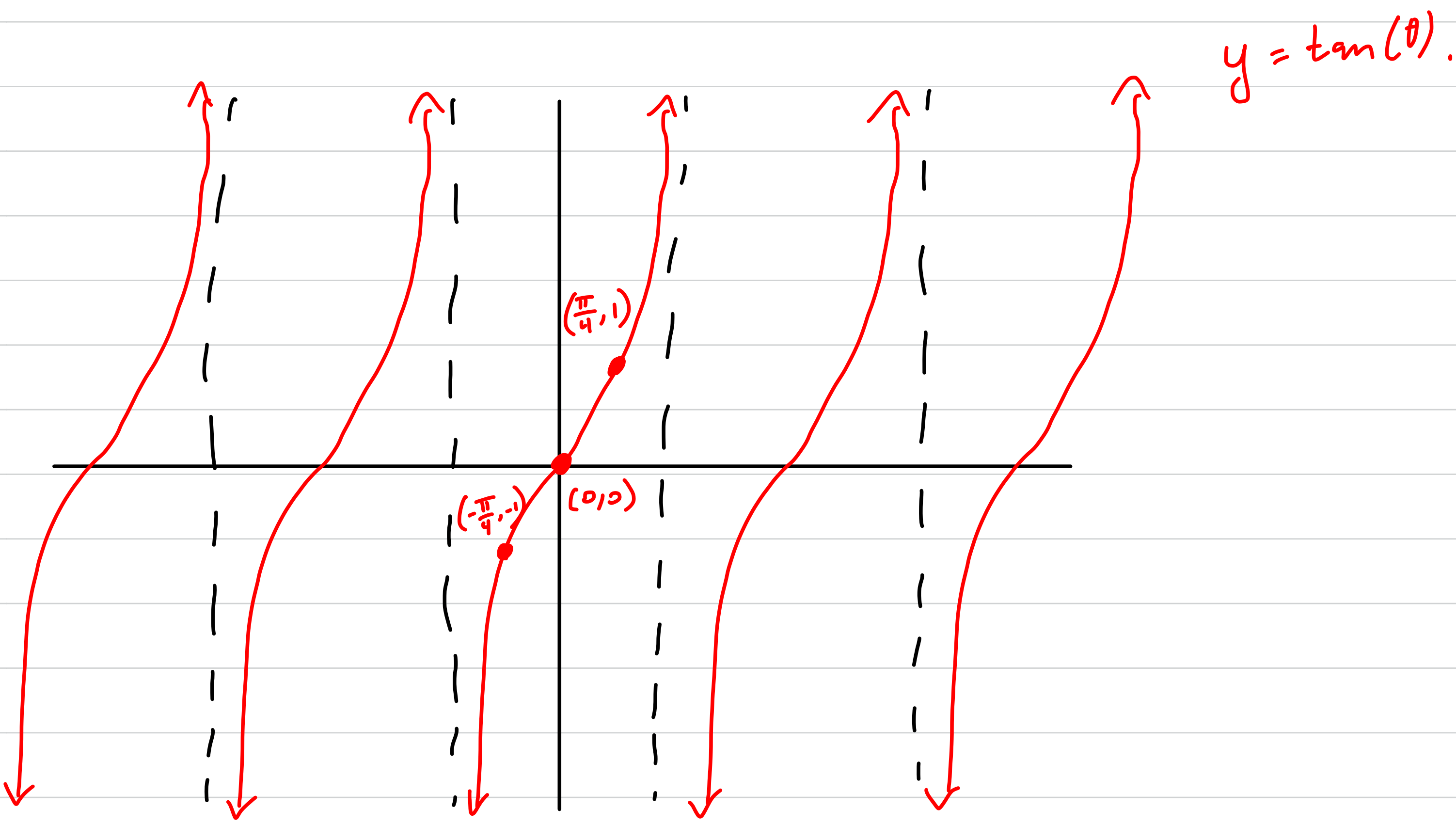
either: $\tan(\pi/4) = 1$ and slope < 0, so $\tan(3\pi/4) = -1$

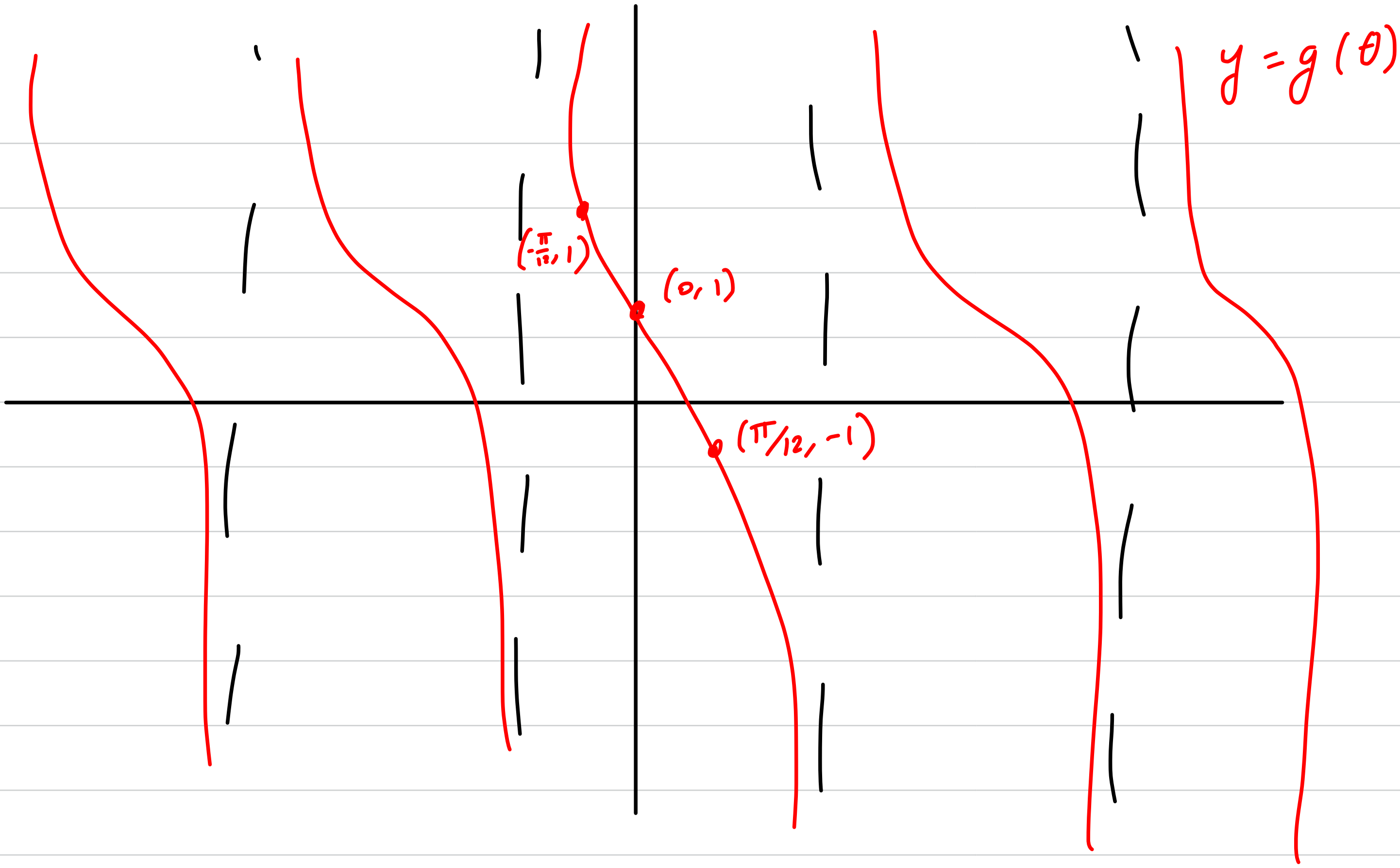
or

$\sin(3\pi/4) = \frac{\sqrt{2}}{2}$ $\cos(3\pi/4) = -\frac{\sqrt{2}}{2}$, so $\tan(3\pi/4) = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1.$

b) $g(\theta)$ is a transformation of $\tan(\theta)$.

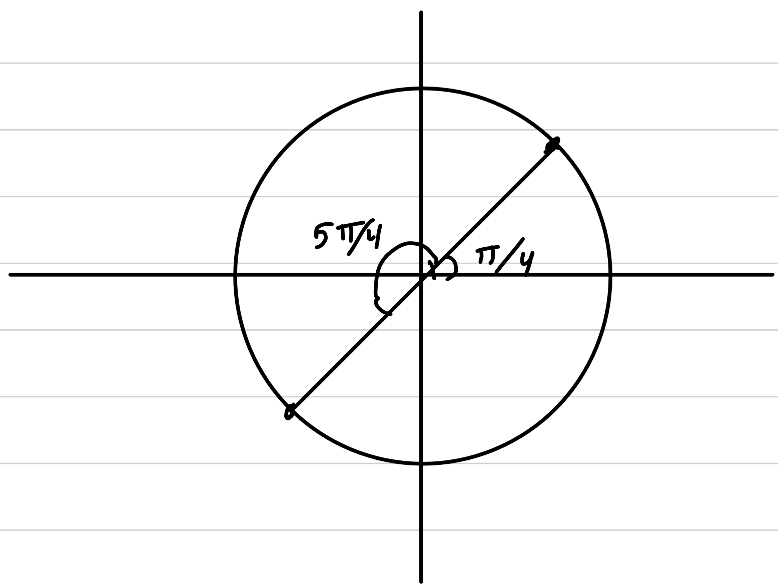
- Horizontal stretch by a factor of 1/3
- Vertical reflection
- Vertical shift up 1





c) $g(\theta) = 0$

$\tan(3\theta) = 1$



$3\theta = \pi/4 + 2\pi n$

or

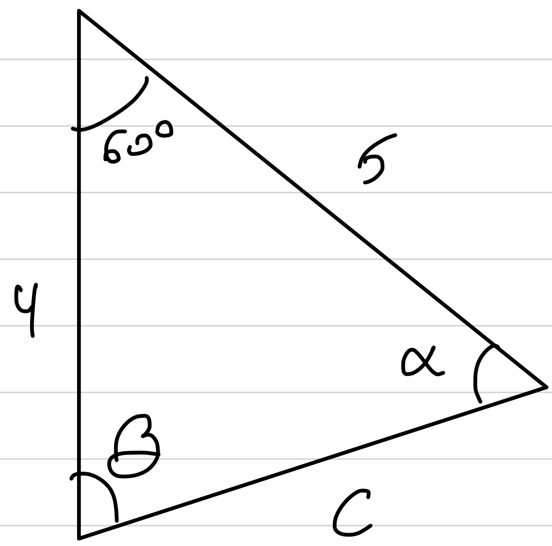
$3\theta = 5\pi/4 + 2\pi n$

$\theta = \pi/12 + \frac{2\pi n}{3}$

or

$\theta = 5\pi/12 + \frac{2\pi n}{3}$

④ a)



$$c^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cdot \cos(60°)$$

$$c^2 = 16 + 25 - 40\left(\frac{1}{2}\right)$$

$$c^2 = 21$$

$$c = \sqrt{21}.$$

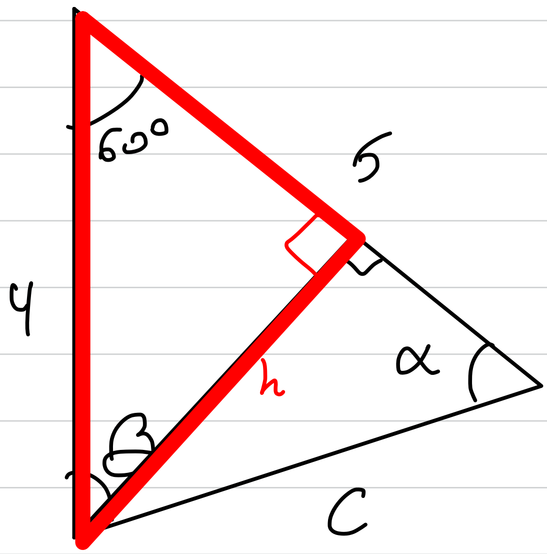
b) $$\frac{\sin(\alpha)}{4} = \frac{\sin(60°)}{c} = \frac{\sqrt{3}/2}{\sqrt{21}}$$

$$\alpha = \arcsin\left(4\left(\frac{\sqrt{3}/2}{\sqrt{21}}\right)\right)$$

c) $$\alpha + \beta + \gamma = 180°$$

$B = 180° - 60° - \alpha.$

d)



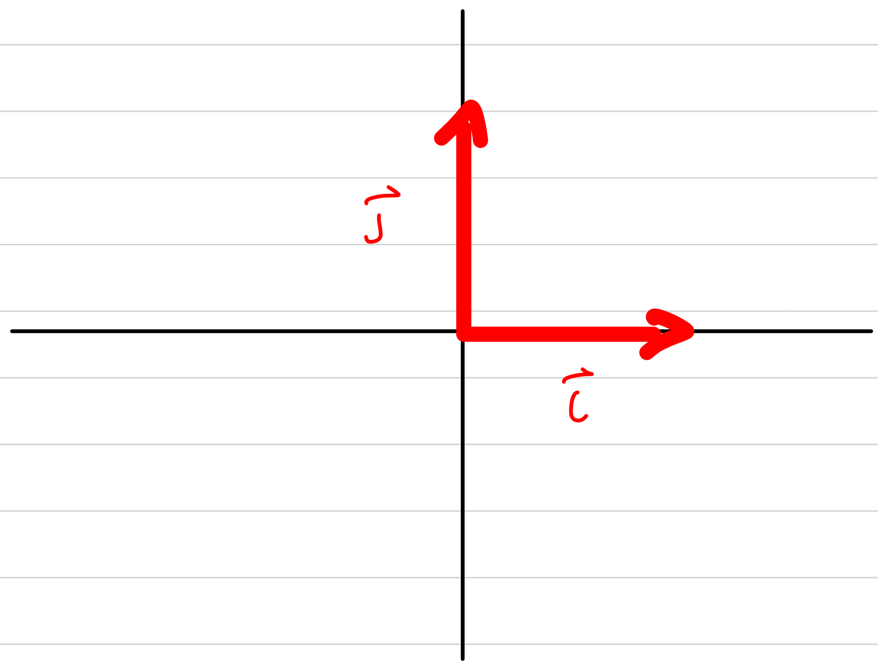
$$\sin(60°) = \frac{h}{4}$$

$$h = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$$A = \frac{1}{2}(5)(2\sqrt{3}) = 5\sqrt{3}.$$

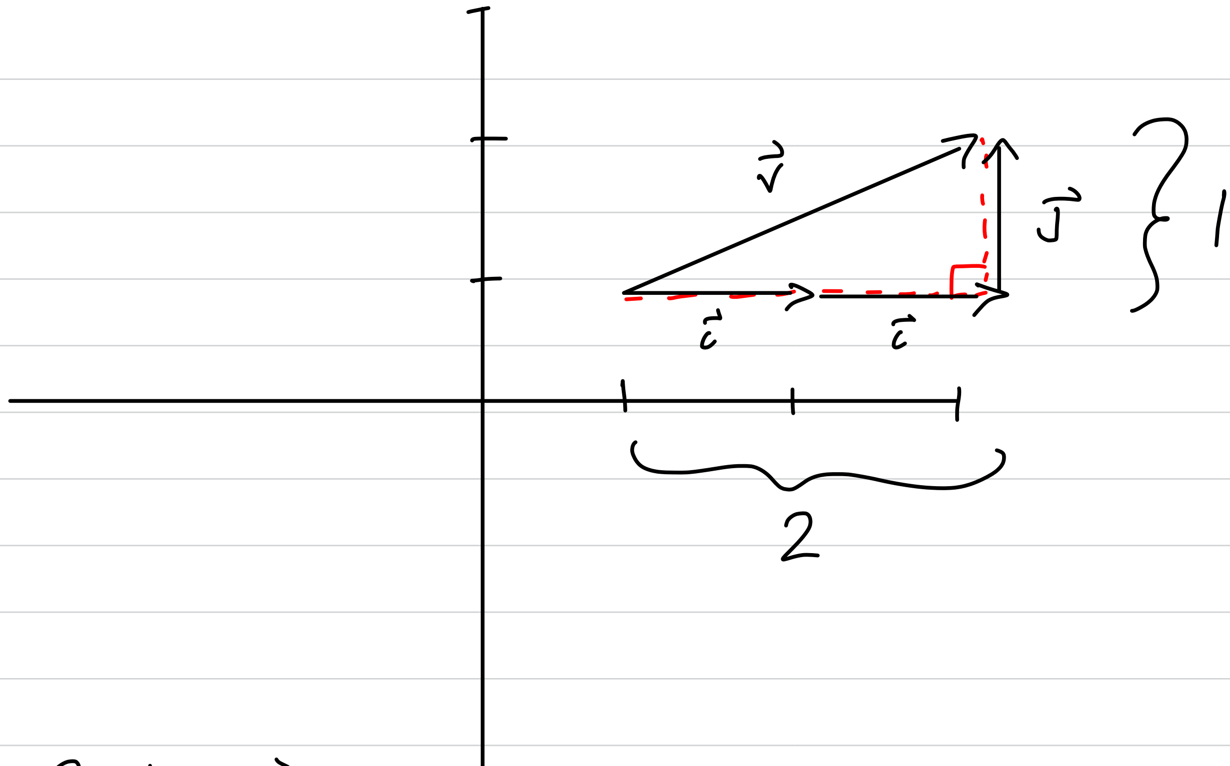
Def: A <u>unit vector</u> is a vector $\vec{v}$ with $\|\vec{v}\| = 1$.

Def: The two <u>standard unit vectors</u> in two dimensions are $\vec{i}$ and $\vec{j}$, where $\vec{i}$ points in the positive-x direction and $\vec{j}$ in the positive-y direction.



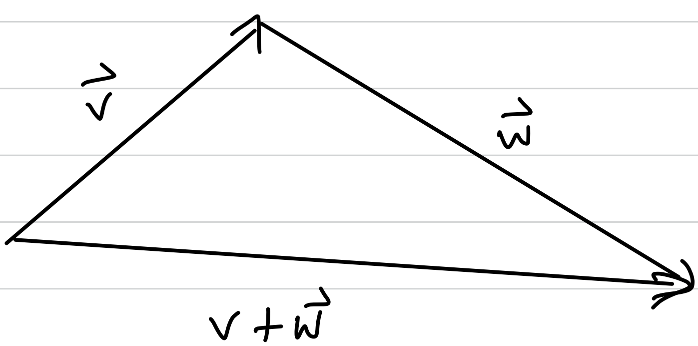
Theorem: Any vector $\vec{v}$ can be written as $\vec{v} = c\,\vec{i} + d\,\vec{j}$ for a unique pair of scalars c and d. This is called the unit vector decomposition of $\vec{v}$.

Ex:



$\vec{v} = 2\vec{i} + \vec{j}$

Ex:



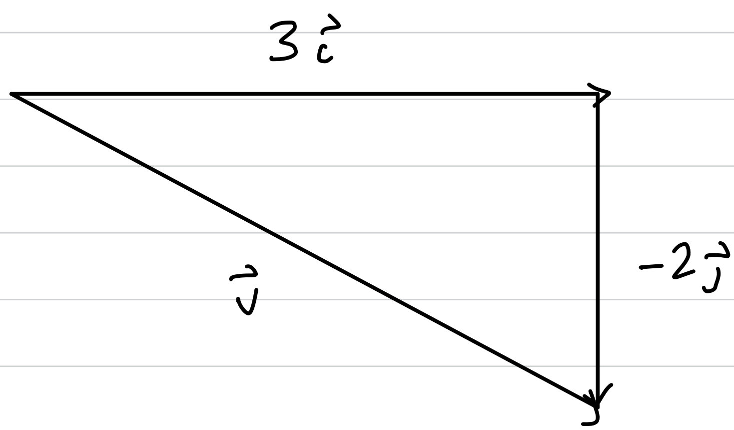
<u>Prop</u> Let $\vec{v} = c\vec{i} + d\vec{j}$ and $\vec{w} = e\vec{i} + f\vec{j}$.

1. $\vec{v} + \vec{w} = (c+e)\vec{i} + (d+f)\vec{j}$
2. $\vec{v} - \vec{w} = (c-e)\vec{i} + (d-f)\vec{j}$

③ $b\vec{v} = (bc)\vec{i} + (bd)\vec{j}$

④ $\|\vec{v}\| = \sqrt{c^2+d^2}$ (Pythagorean theorem)

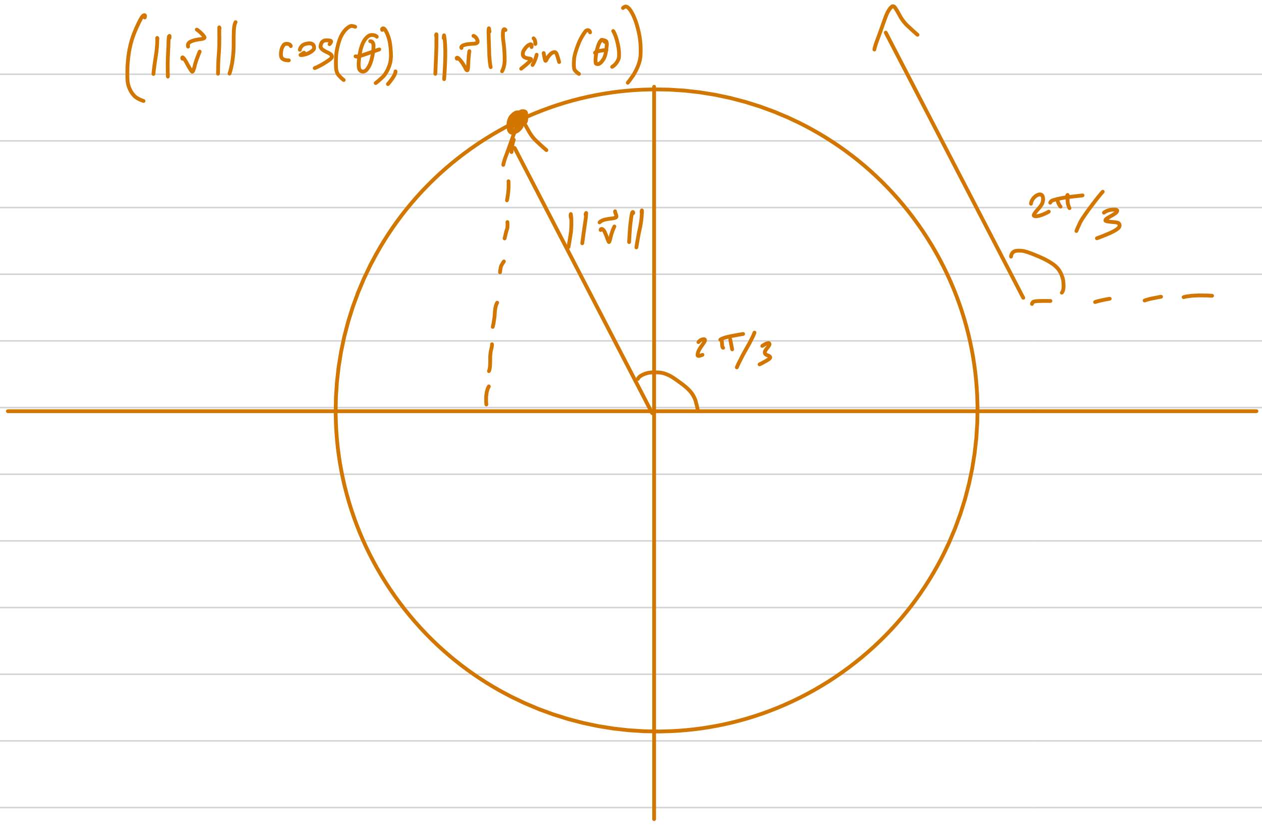
Ex: if $\vec{v} = 3\vec{i} - 2\vec{j}$, then $\|\vec{v}\| = \sqrt{3^2+(-2)^2} = \sqrt{13}$



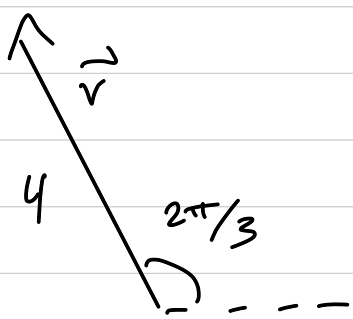
Comment $\left(\vec{i} \text{ and } \vec{j} \text{ components}\right) \longleftrightarrow \left(\|\vec{v}\| \text{ and angle}\right)$

Prop: Let $\vec{v}$ be a vector with angle θ from the horizontal. Then

$$\vec{v} = \left(\|\vec{v}\| \cos(\theta)\right)\vec{i} + \left(\|\vec{v}\| \sin(\theta)\right)\vec{j}.$$



Ex:

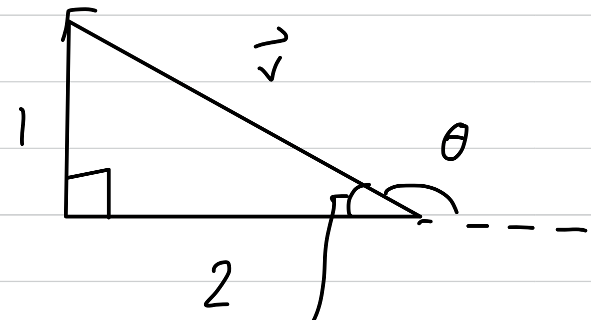


$$\vec{v} = 4\cos(2\pi/3)\,\vec{i} + 4\sin\left(\tfrac{2\pi}{3}\right)\vec{j}$$

$$= -2\,\vec{i} + 2\sqrt{3}\,\vec{j}$$

Comment: Given the unit vector decomposition of a vector $\vec{v}$, we can find its angle with the horizontal via arctan.

Ex: $\vec{v} = -2\vec{i} + \vec{j}$



$= \arctan(1/2) = .464$, so $\theta = \pi - .464$

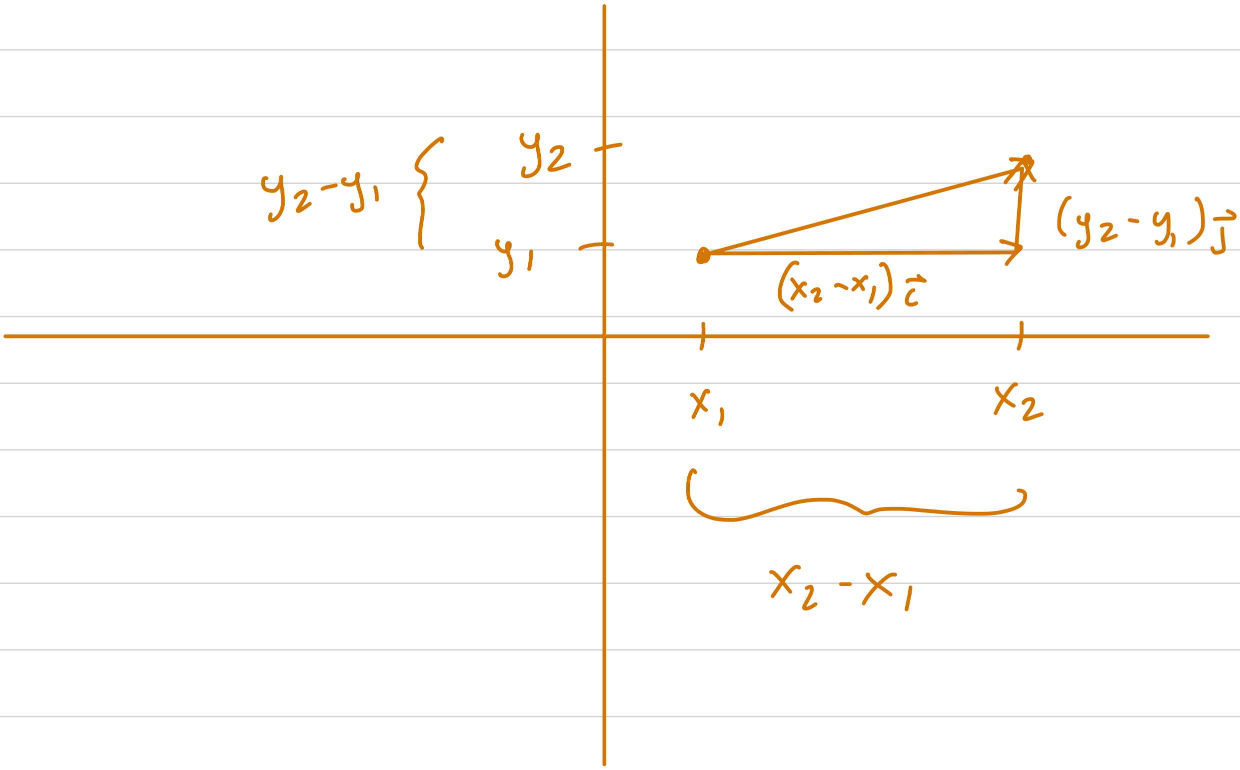
$= 2.678$.

Announcement: change to HW 7 (problem 2 easier)

Prop Let (x_1, y_1) and (x_2, y_2) be two points in the plane. The vector that starts at

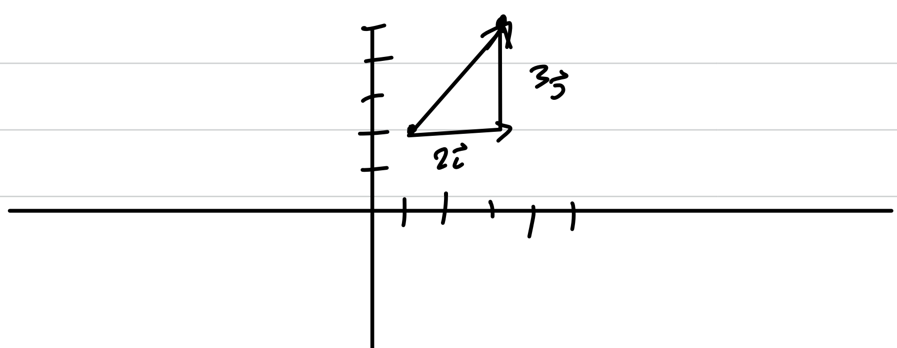
(x_1, y_1) and ends at (x_2, y_2) is

$$(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}.$$



Ex: you walk from $(1,2)$ to $(3,5)$. What is the vector corresponding to your total travel?

$$(3-1)\vec{i} + (5-2)\vec{j} = 2\vec{i} + 3\vec{j}$$



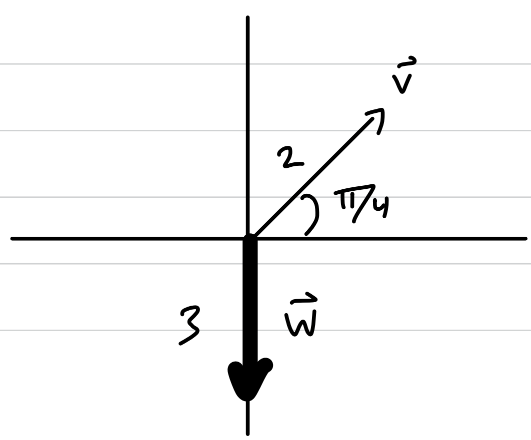
The Dot Product

Comment: The dot product is a way to multiply two vectors, but it gives a scalar, not a vector.

Def: Let $\vec{v} = a\vec{i} + b\vec{j}$ and $\vec{w} = c\vec{i} + d\vec{j}$. The dot product of $\vec{v}$ and $\vec{w}$ is $\vec{v} \bullet \vec{w} = ac + bd$.

Ex $(2\vec{i} - \vec{j}) \bullet (3\vec{i} + 4\vec{j}) = 2 \cdot 3 + (-1) \cdot 4 = 2$.

Ex: Find $\vec{v} \bullet \vec{w}$:



We first have to find their unit vector decompositions.

$$\vec{v} = 2\cos(\pi/4)\,\vec{i} + 2\sin(\pi/4)\,\vec{j}$$

$$= \sqrt{2}\,\vec{i} + \sqrt{2}\,\vec{j}$$

$$\vec{w} = -3\vec{j}$$

$$\vec{v} \bullet \vec{w} = (\sqrt{2})(0) + (\sqrt{2})(-3) = -3\sqrt{2}$$

Prop :

① $\vec{0} \bullet \vec{v} = \vec{v} \bullet \vec{0} = 0$

② $\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$

③ $c(\vec{v} \bullet \vec{w}) = (c\vec{v}) \bullet \vec{w} = \vec{v} \bullet (c\vec{w})$

④ $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

⑤ $\underbrace{\vec{v} \bullet \vec{v} = \|\vec{v}\|^2}$ (think of $x\,x = |x|^2$)

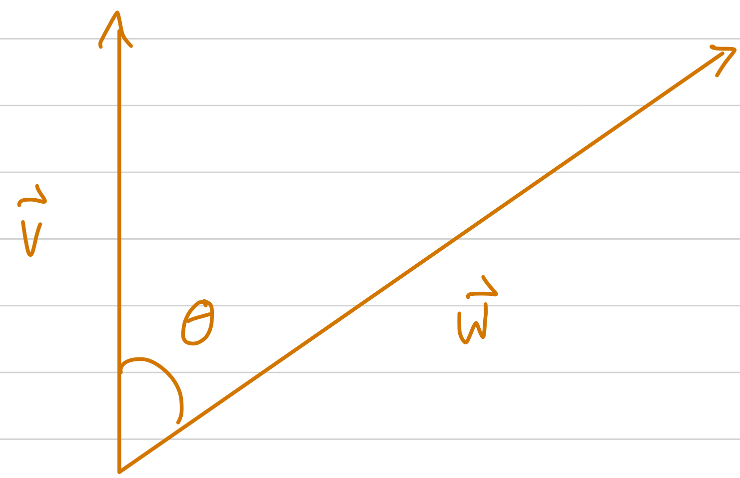
$\vec{v} = a\vec{i} + b\vec{j}$ $\quad \|\vec{v}\| = \sqrt{a^2+b^2}$

$\|\vec{v}\|^2 = a^2 + b^2$

$\vec{v} \bullet \vec{v} = a \cdot a + b \cdot b$

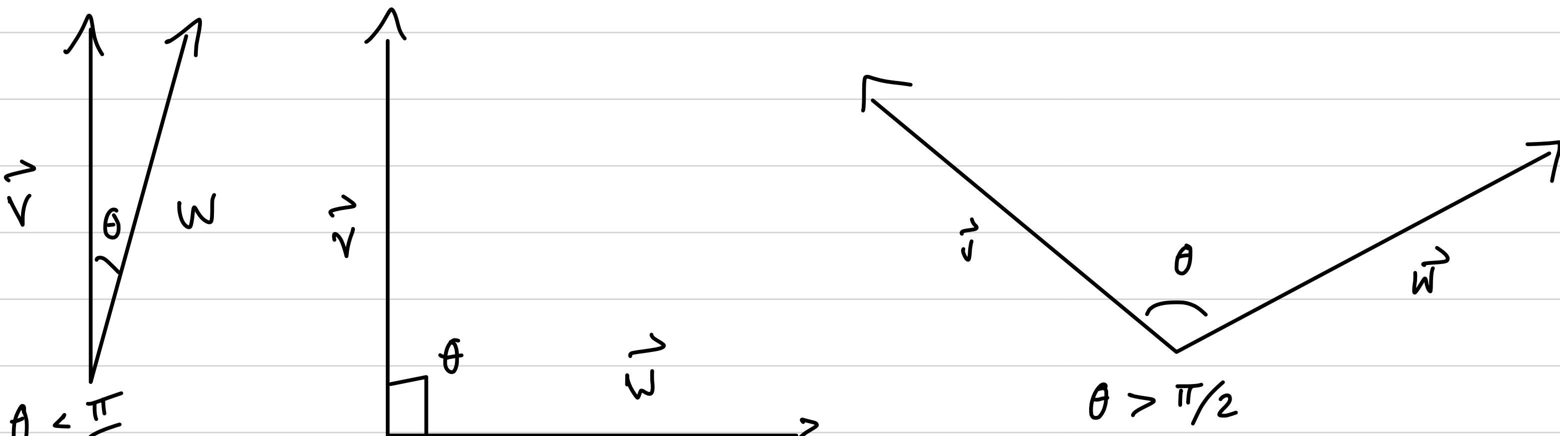
Comment: You might expect a property like $\vec{u} \bullet (\vec{v} \bullet \vec{w}) = (\vec{u} \bullet \vec{v}) \bullet \vec{w}$ — but this doesn't make sense! $\vec{u} \bullet \vec{v}$ is a scalar, and you can't dot scalars with vectors.

Prop: Let $\vec{v}$ and $\vec{w}$ be vectors that form an angle of θ when starting at the same point.



Then $\vec{v} \bullet \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos(\theta)$.

Comment: In this sense, the dot product measures the degree to which $\vec{v}$ and $\vec{w}$ are parallel.



$\theta < \frac{\pi}{2}$ $\qquad$ $\theta = \pi/2$ $\qquad$ $\theta > \pi/2$

$\vec{v} \cdot \vec{w} > 0$ $\qquad$ $\vec{v} \cdot \vec{w} = 0$ $\qquad$ $\vec{v} \cdot \vec{w} < 0$

Prop: The angle between vectors $\vec{v}$ and $\vec{w}$ is

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}\right).$$

Ex: Find the angle between $\vec{v} = 3\vec{i} + \vec{j}$ and $\vec{w} = 2\vec{i} - \vec{j}$.

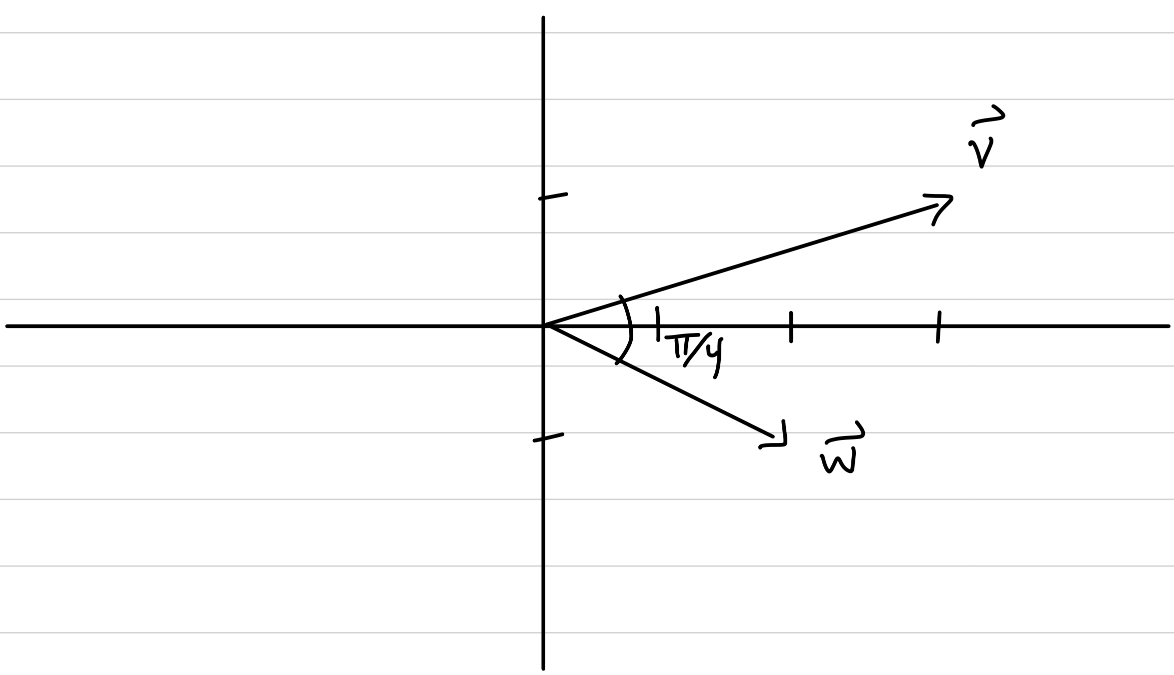
$$\vec{v} \cdot \vec{w} = 3 \cdot 2 + (-1)(1) = 6 - 1 = 5$$

$$\|\vec{v}\| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$\|\vec{w}\| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\theta = \arccos\left(\frac{5}{\sqrt{10}\,\sqrt{5}}\right) = \arccos\left(\frac{5}{\sqrt{2}\,\underbrace{\sqrt{5}\,\sqrt{5}}_{5}}\right)$$

$$= \arccos\left(\frac{1}{\sqrt{2}}\right) = \pi/4.$$



Comment: If you want the angle that vector makes with the horizontal, use arctan. If you want the angle that two vectors make with one another, use this.

Def: Vectors $\vec{v}$ and $\vec{w}$ are orthogonal if $\vec{v} \bullet \vec{w} = 0$.

Comment: If neither $\vec{v}$ nor $\vec{w}$ is the zero vector, then orthogonal means perpendicular. Your book uses perpendicular, but we'll use orthogonal.

Ex: $\vec{v} = 2\vec{i} + 3\vec{j}$ and $\vec{w} = -3\vec{i} + 2\vec{j}$ are orthogonal, because $\vec{v} \bullet \vec{w} = 2(-3) + 3(2)$
$= -6 + 6 = 0.$

Ex: Find all vectors orthogonal to $-3\vec{i}+2\vec{j}$.

Let $\vec{v}=a\vec{i}+b\vec{j}$ and solve $\vec{v}\cdot(-3\vec{i}+2\vec{j})=0$

$-3a+2b=0$

First solve for a. $-3a=-2b$

$a=\frac{2}{3}b$

Set $b=t$ for a variable t.

$b=t$

$a=\frac{2}{3}t$

$\vec{v}=\frac{2}{3}t\,\vec{i}+t\,\vec{j}$ for any t.

(Note: $t=3$ gives the previous example)

What's happening geometrically?

$$\vec{v} = t\left(\frac{2}{3}\vec{i} + \vec{j}\right)$$