Chapter I: Logic

Problem Solving

Ex: Solve $x^2+2x=-1$ for x.

$$x^2+2x+1=0$$ specific situation

general fact

Use the quadratic formula:

$$x=\frac{-2\pm\sqrt{2^2-4\cdot1\cdot1}}{2\cdot1}=\frac{-2\pm\sqrt{4-4}}{2}$$

$$=\frac{-2\pm0}{2}=-1.$$

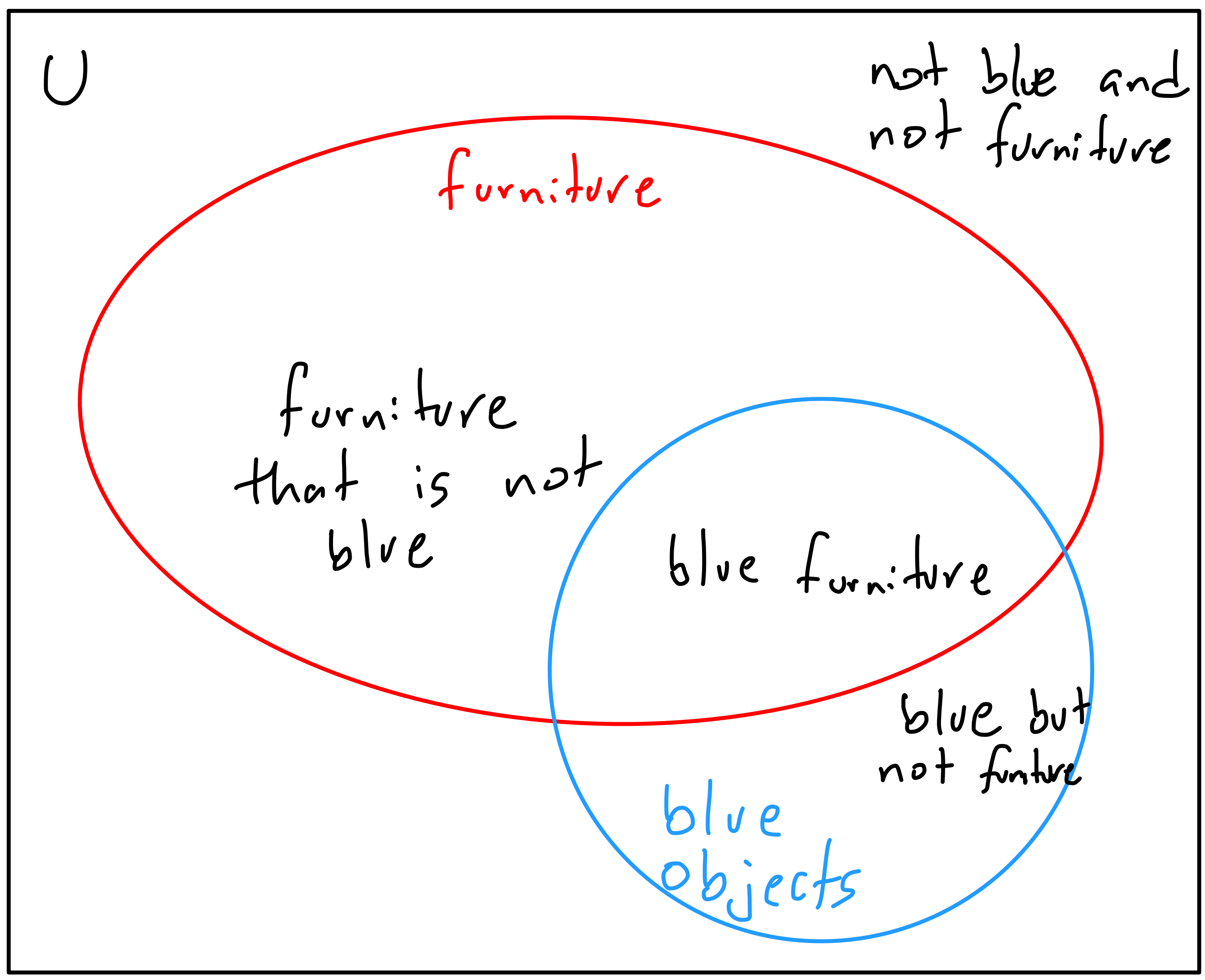
Comment: how did know how to solve this?

well, we know that we can use the quadratic formula whenever we have an equation of the form $ax^2+bx+c=0$. Here, $ax^2+bx+c=0$ is a general kind of problem, and $x^2+2x+1=0$ is a specific instance.

Def: Deductive reasoning is a method to solve problems by applying general knowledge to a specific situation.

Def: A Venn diagram is a set of overlapping figures that are contained within a universe U, typically drawn as a rectangle.

Ex:



Def: An argument is **valid** if the conclusion follows logically from the statements before it. It doesn't matter whether those statements or the conclusion are true.

Ex:

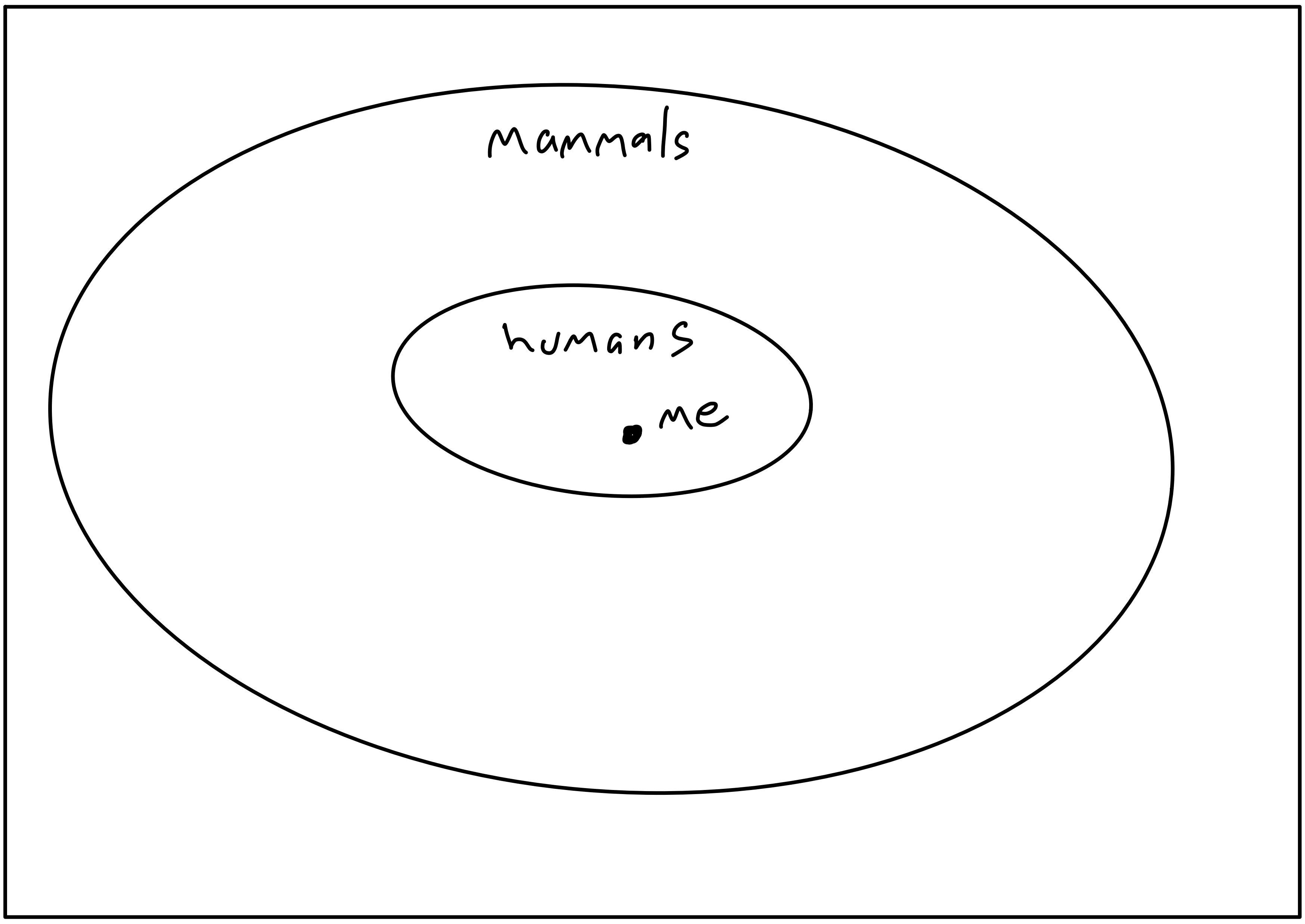
1. All humans are mammals.
2. I am a human.

I am a mammal.

Method (Showing an argument is valid):

Draw a Venn diagram that follows all the statements and assumes nothing else. Then demonstrate that the conclusion must be true

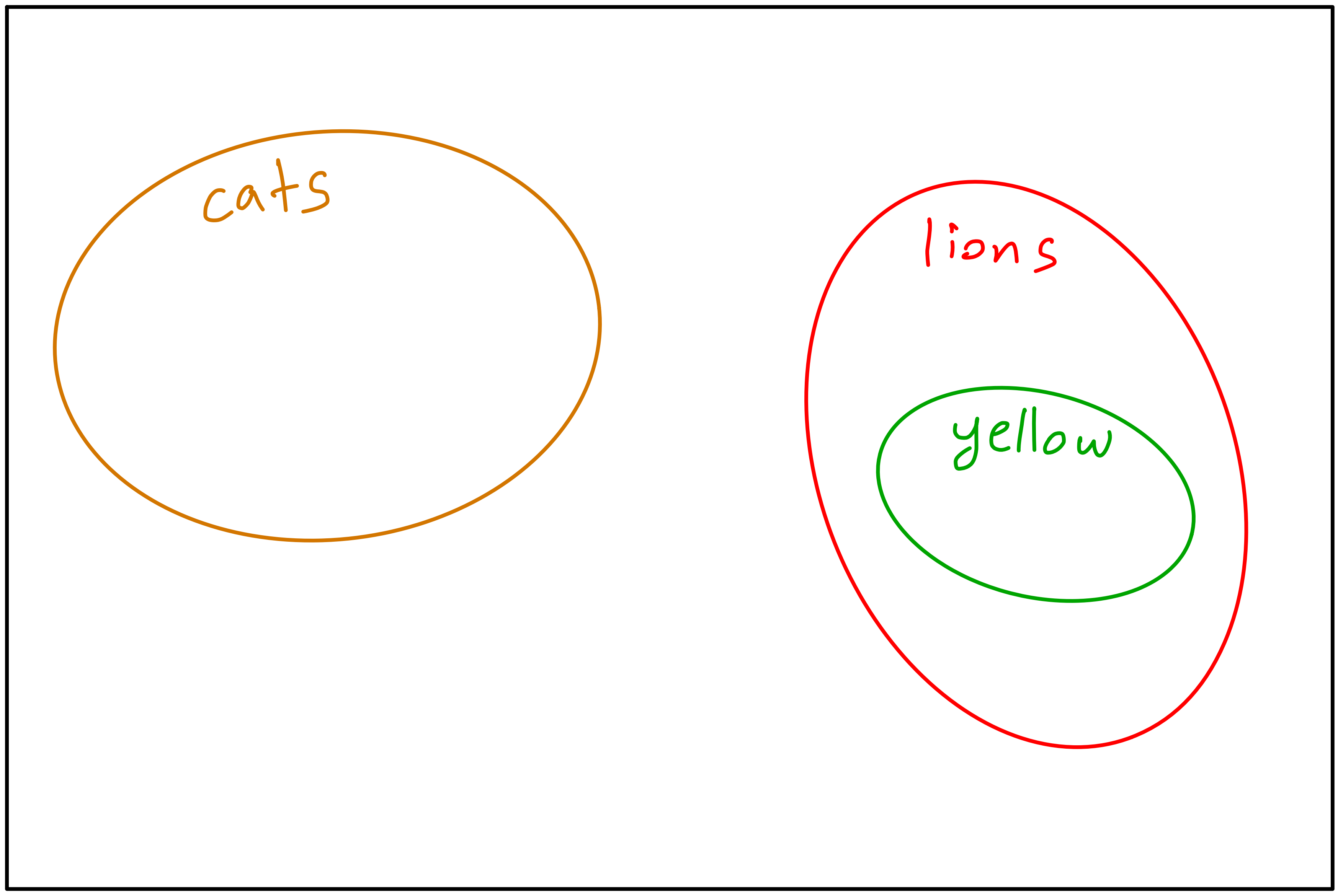
Ex: we want to draw a Venn diagram involving humans, mammals, and me.



Since that dot lives inside the set of mammals, it must be the case that I am a mammal.

Ex: 1. No cats are lions.
2. All yellow animals are lions.

No cat is yellow.



Since the set of cats and the set of yellow animals don't overlap, no cat is yellow.

Comment: Venn diagrams only work when the argument uses deductive reasoning.

Method (Showing an argument is invalid): Construct a Venn diagram that satisfies the statements but not the conclusion.

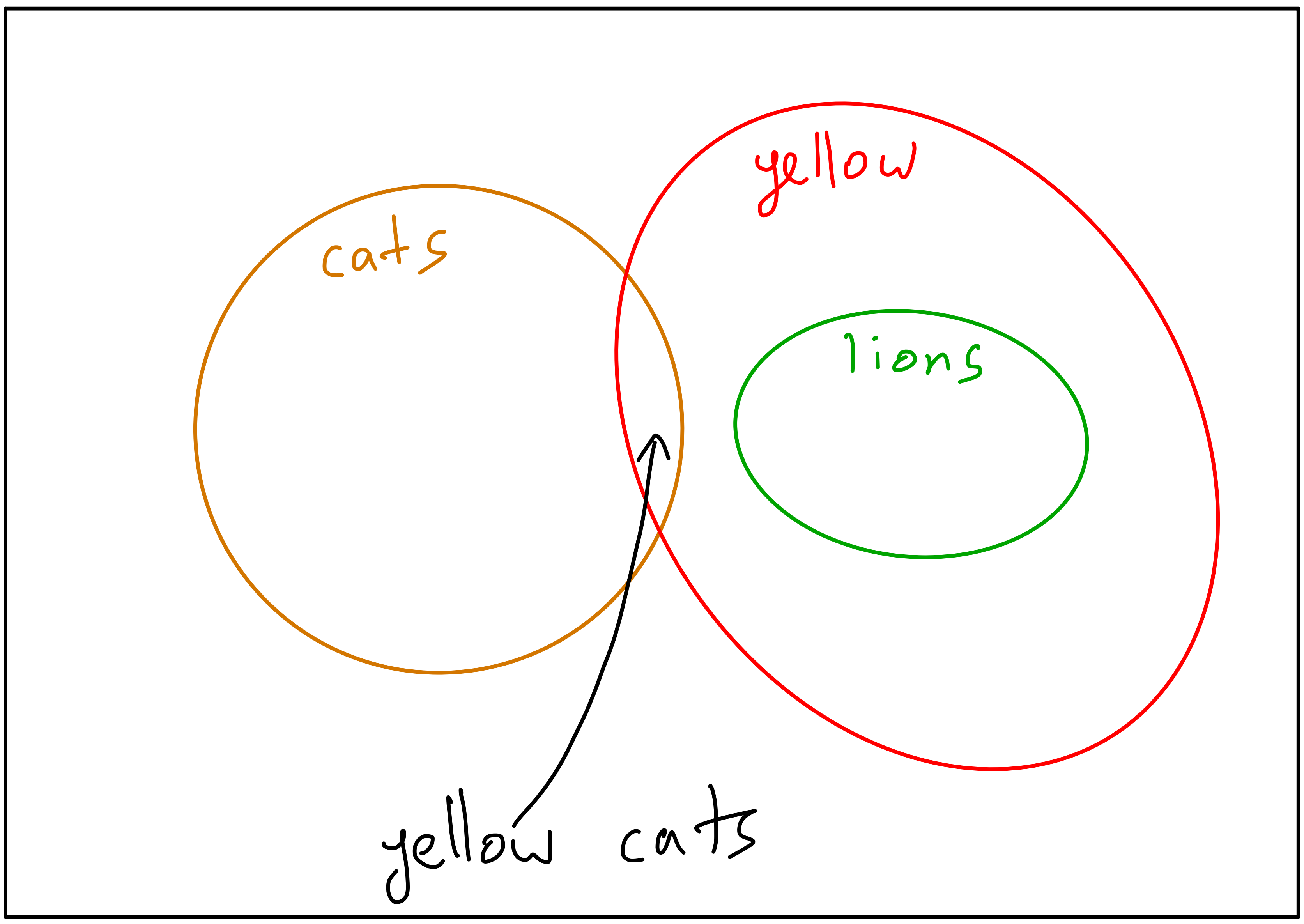
Ex:

1. No cats are lions.
2. All lions are yellow.

No cat is yellow.

To show this is invalid, we would need to draw a situation where:

1. No cats are lions.
2. All lions are yellow.
3. Some cats are yellow.



Def: Inductive reasoning is a method to solve problems by finding a pattern in a few specific cases and conjecturing that the pattern holds in general.

Ex:

1. I got stung by a bee last month and it hurt.

2. I got stung by a bee today and it hurt.

Bee stings hurt.

Comment: We can't say for sure if an inductive argument is valid or not

$$\text{general} \xrightarrow{\text{deductive}} \text{specific}$$

$$\text{specific} \xrightarrow{\text{inductive}} \text{general}$$

1.2: Compound Statements

Def: A <u>statement</u> is a sentence that is either true or false.

Ex: UO is a college campus. ✓

It is raining. ✓

UO is the best university. ✗

Is it raining? ✗

This sentence is a lie. ✗

Def: Let p and q be statements.

1. The negation of p, written $\sim p$ or $\neg p$, is the statement that is true when p is false and false when p is true. We read $\sim p$ as "not p"

2. The conjunction of p and q, written $p \wedge q$, is the statement that is true when both p and q are true, and false if either p or q is false. We read $p \wedge q$ as "p and q".

(3) The disjunction of p and q, written $p \vee q$, is the statement that is true if at least one of p and q is true and false if both are false. We read $p \vee q$ as "p or q".

(4) The implication $p \rightarrow q$ is the statement that is true if whenever p is true, q is also true. We read $p \rightarrow q$ as "p implies q".

In summary:

$\neg p$	true when p is false
$p \wedge q$	true when p is true and q is true
$p \vee q$	true when p is true or q is true
$p \to q$	true when p being true means q is true

Ex: Let p be the statement "today is Monday". Then $\sim p$ is "today is not Monday".

Ex: Let p be "all lions are cats", q be "some lions are cats" and r be "no lions are cats". Then:

$\sim p$ is "some lions are not cats".

$\sim q$ is "no lions are cats".

$\sim r$ is "some lions are cats".

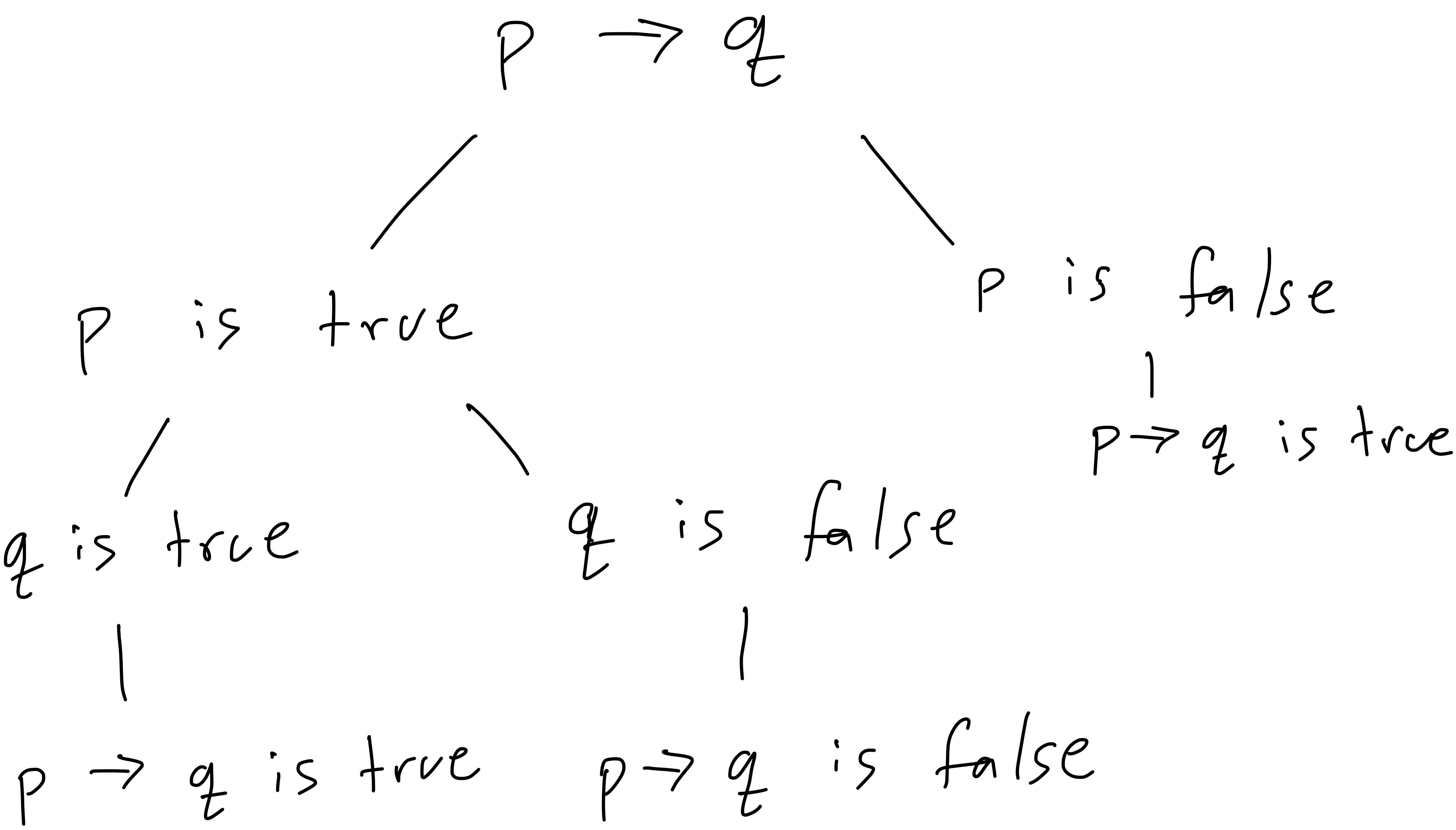
Ex If p is "today is Monday" and q is "it is raining", then $p \wedge q$ is "today is Monday and it is raining".

Ex: $p \vee q$ is "today is Monday or it is raining". Note this is what we call inclusive or: it's okay if it's Monday and it's raining at the same time.

Ex: $p \to q$ is "If today is Monday, then it is raining". Note that it is irrelevant if today is not Monday.

The point is that $p \to q$ is true if p is false.

Ex: If p is "the Sun is blue" and q is "humans are mammals", then $p \to q$ is true.



Def: In $p \Rightarrow q$, p is called the antecedent and q is called the consequent.

Ex: $\underbrace{\text{"If I exercise,}}_{\text{antecedent}}$ then $\underbrace{\text{I am healthy"}}_{\text{consequent}}$

1.3: Truth Tables

<u>Def</u>: A truth table is a way to write down exactly when a compound statement is true.

<u>Ex</u>: The truth table for $p \wedge q$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex: The truth table for $p \to q$:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note that p being true does not make q true. But it does make $p \to q$ true (sometimes we say $p \to q$ is "vacuously" true).

Ex: the truth tables for $\sim p$ and $p \vee q$.

p	$\sim p$
T	F
F	T

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex: find the truth table for

$$p \wedge \sim (q \vee r).$$

In general, if we have n variables, we need 2^n rows.

Here, $n=3$, so we need 8 rows.

p	q	r	$q \vee r$	$\sim(q \vee r)$	$p \wedge \sim(q \vee r)$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

Ex: If you are a lion, then you are a cat.

Comment: In an implication $p \to q$, we say either:

- p is sufficient for q.
- q is necessary for p.

Ex: "Being a doctor is necessary for being a surgeon" is the same as "If you are a surgeon, then you are a doctor."

<u>Def</u>: Let p and q be statements. We say p and q are <u>logically equivalent</u>, written $p \equiv q$, if p is true exactly when q is true.

<u>Ex</u>: Show for any p and q,

$$p \to q \equiv (\sim p) \vee q.$$

p	q	$\sim p$	$p \to q$	$(\sim p) \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same

Comment: $p \equiv q$ does not mean that p and q are the same statement.

Ex: If p is "Air contains oxygen" and q is "An apple is fruit", then both p and q are always true, so $p \equiv q$.

Theorem (De Morgan's Laws):

① $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

② $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

Proof: Write the truth tables for $\sim(p \wedge q)$, $\sim p \vee \sim q$, $\sim(p \vee q)$, and $\sim p \wedge \sim q$.

p	q	$\sim p$	$\sim q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	F	F	F	F
T	F	F	T	T	T	F	F
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

same

same

Ex: Find the negation of "It is Friday and I receive a paycheck".

If p is "It is Friday" and

q is "I receive a paycheck",

then $\sim(p \wedge q) \equiv \sim p \vee \sim q$, which

is "It is not Friday or I don't receive a paycheck".

1.4: More About Conditionals

Def: Consider the conditional $p \rightarrow q$.

① The converse of $p \rightarrow q$ is $q \rightarrow p$.

② The inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

③ The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

Ex: If you are a lion, then you are a cat.

Converse: If you are a cat, then you are a lion.

Inverse: If you are not a lion, then you are not a cat.

Contrapositive: If you are not a cat, then you are not a lion.

Ex: Find the contrapositive of

"Being a doctor is necessary for being a surgeon", and express it using either necessary or sufficient syntax.

Step 1: convert into an if-then.

Since the necessary refers to the consequent (the "then" part), we have "If you are a surgeon, then you are a doctor".

Step 2: Take the contrapositive.

"If you are not a doctor, then you are not a surgeon".

Step 3: convert to necessary/sufficient

Let's use sufficient: that refers to the antecedent, so we have

"Not being a doctor is sufficient for not being a surgeon".

Theorem: For any statements p and q,

① $p \to q \equiv \sim q \to \sim p$

② $q \to p \equiv \sim p \to \sim q$

So:

Original $\equiv$ Contrapositive.

Converse $\equiv$ Inverse.

Comment: A statement "p only if q" is equivalent to "If p, then q".

Comment: A statement "p if and only if q" is equivalent to "If p, then q, and if q, then p". This means $p \equiv q$.

Sometimes you'll see $p \longleftrightarrow q$ or $p \Longleftrightarrow q$ — they mean the same thing.