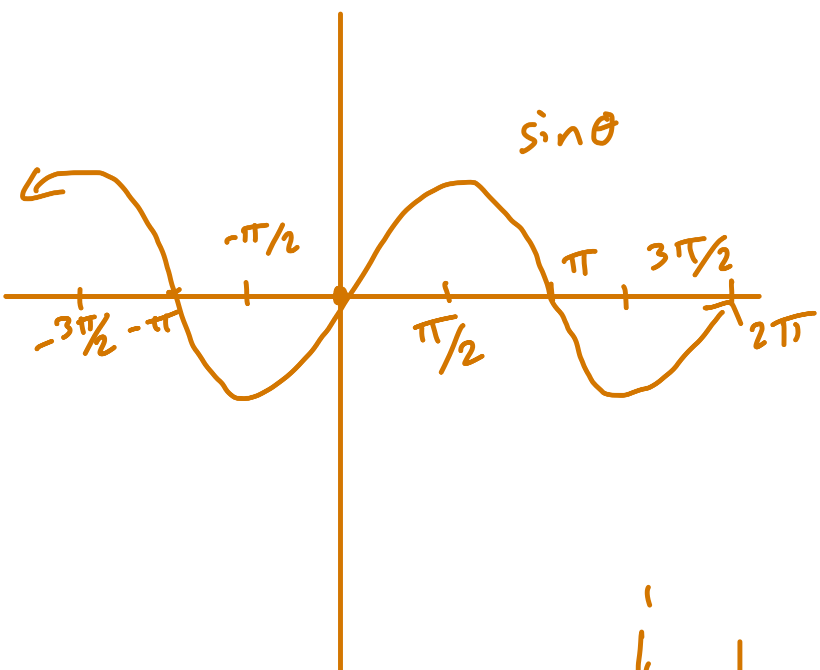
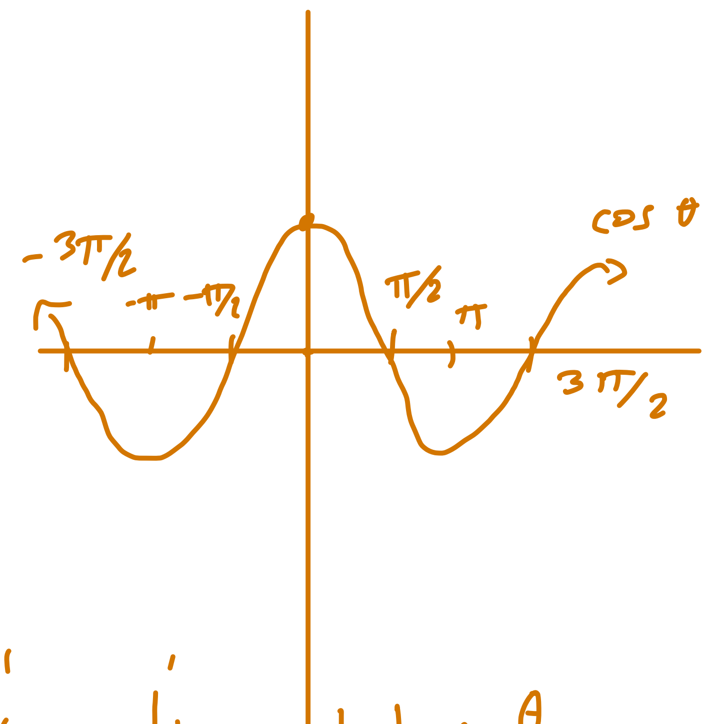
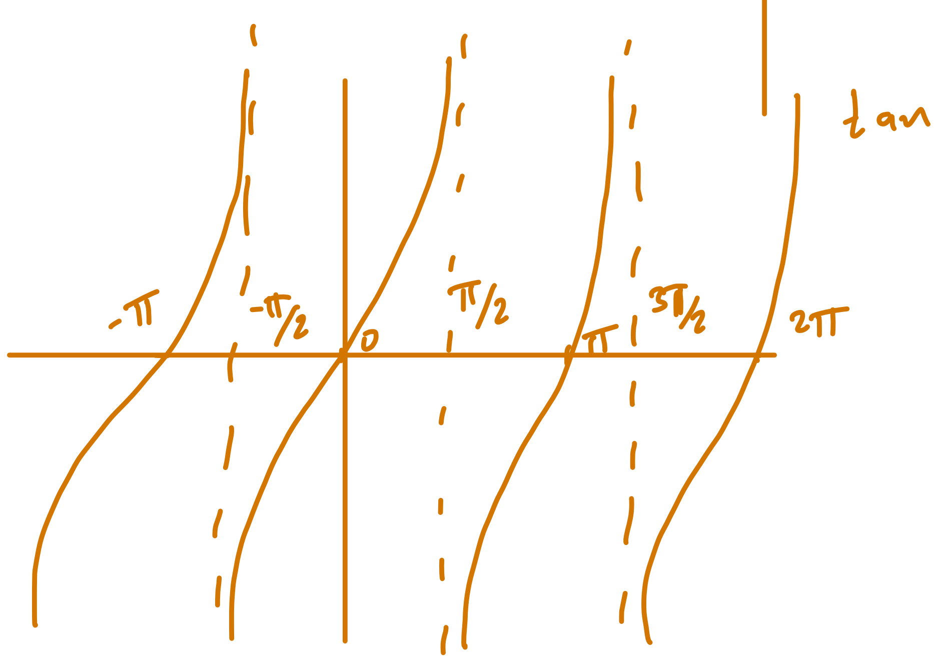
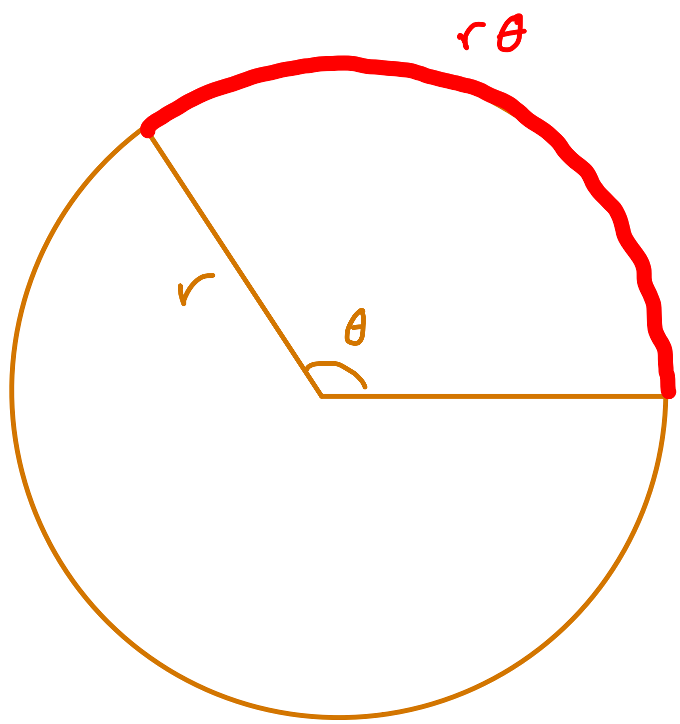
Theorem: The graphs of $\sin\theta$, $\cos\theta$, and $\tan\theta$, when θ is measured in radians, are the same as when it's measured in degrees, but horizontally stretched by a factor of $\frac{\pi}{180^\circ}$.



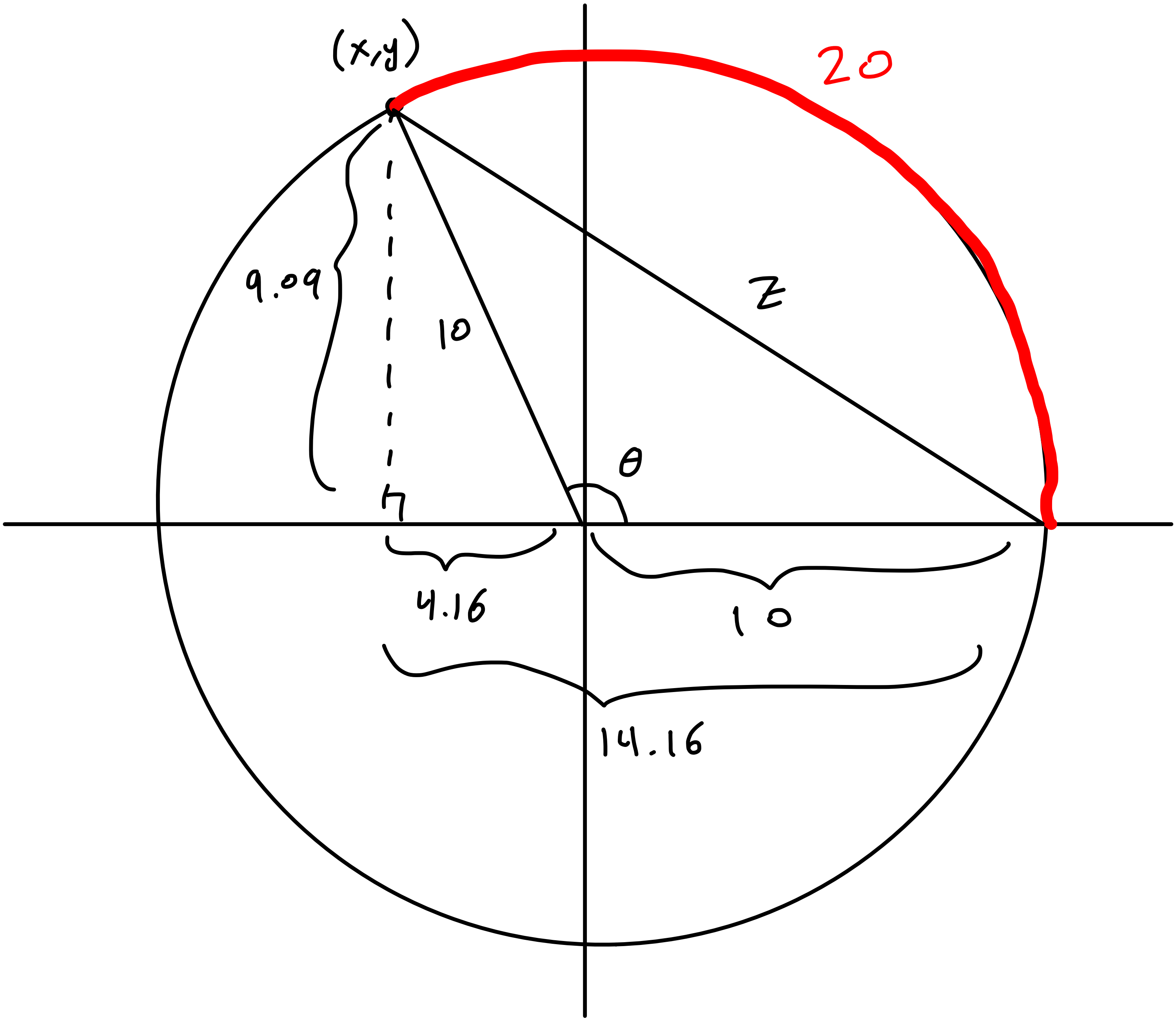




Theorem: The arc length of an arc with angle θ on a circle of radius r is $r\theta$ — but only if θ is measured in radians.



Ex: Find θ, x, y, and z.



Let's first look at the arc length formula: $20 = 10\theta$, so $\theta = 2$. Then $x = 10\cos 2$ and $y = 10\sin 2$, so $x = -4.16$ and $y = 9.09$. To find z, use the Pythagorean theorem:

$9.09^2 + 14.16^2 = z^2$. Then $z = 16.83$.

Trig in Non-right Triangles

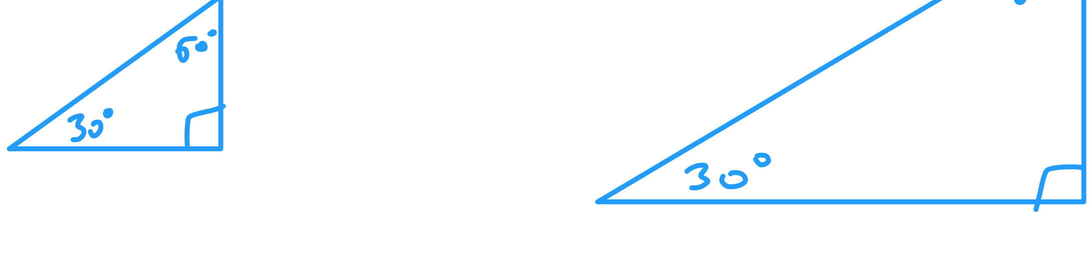
<u>Comment</u>: There are certain relations that hold in any triangle involving the trig functions, where right triangles are just a special case.

Comment: Note that in a triangle, there are six pieces of information: three angles and three sides. In a right triangle, we know one: an angle. If we have two more, and one is a side, we can find all six.

one side + one angle : trig

two sides : Pythagorean theorem

(three angles doesn't work)



Comment: We'll always need three pieces of information to use trig on non-right triangles.

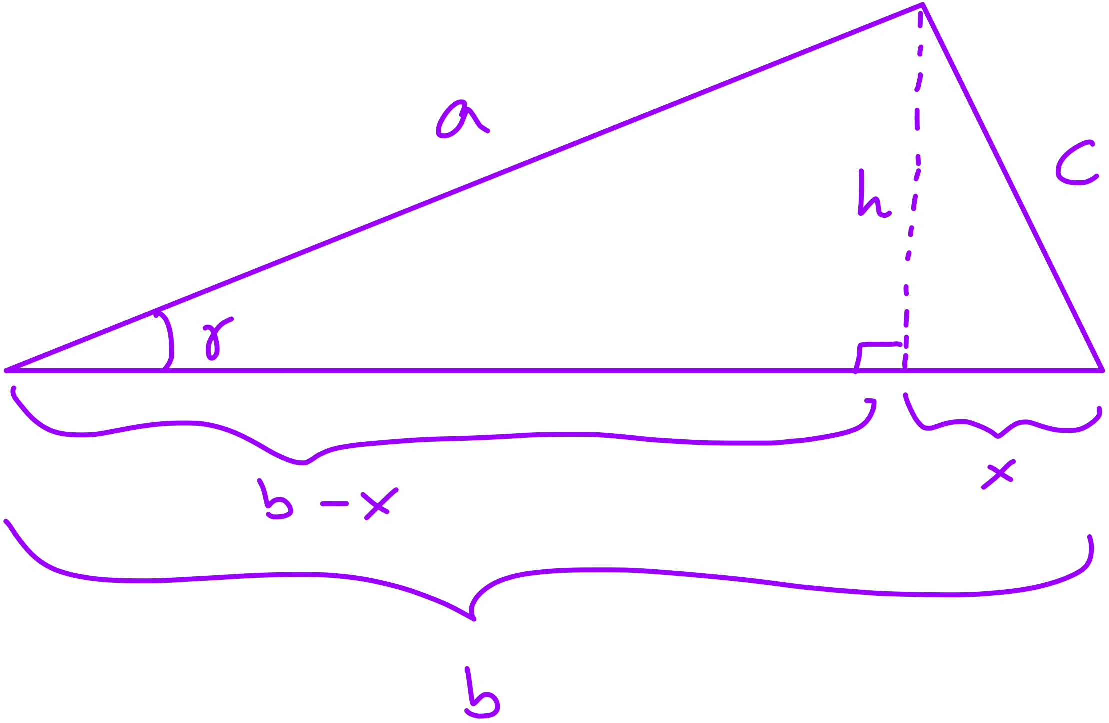
Theorem (The Law of Cosines):

In *any* triangle with sides a, b, and c, and angle γ opposite c,

gamma

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Proof:



In the right triangle on the left:

$\cos\gamma = \frac{b-x}{a}$, so $b - x = a\cos\gamma$,

and so $x = b - a\cos\gamma$. Also,

$\sin\gamma = \frac{h}{a}$, so $h = a\sin\gamma$.

By the Pythagorean theorem on the right triangle on the right, $x^2 + h^2 = c^2$,

so $(b - a\cos\gamma)^2 + (a\sin\gamma)^2 = c^2$.

$$\underbrace{b^2 - 2ab\cos\gamma + (a\cos\gamma)^2 + (a\sin\gamma)^2}_{=c^2}$$

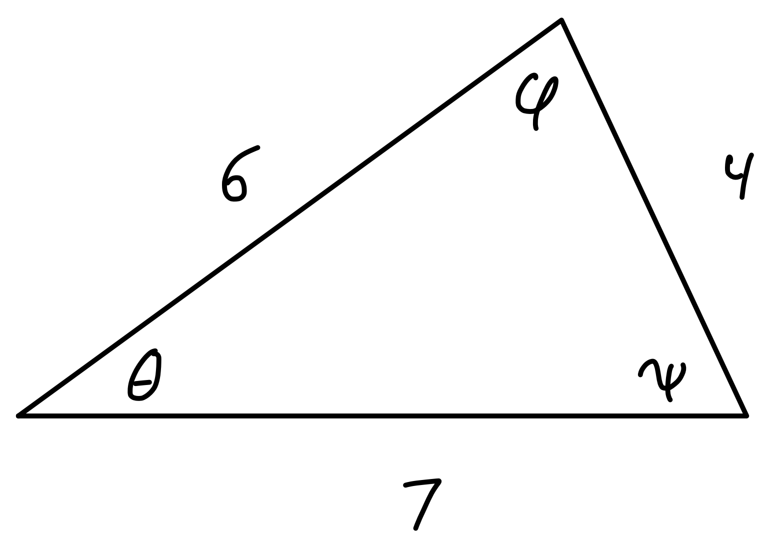
So $c^2 = a^2\left((\cos\gamma)^2 + (\sin\gamma)^2\right) + b^2 - 2ab\cos\gamma$

$c^2 = a^2 + b^2 - 2ab\cos\gamma$. $\square$

Comment: Use the Law of Cosines when:

- You know all three sides of a triangle and want to find an angle.
- You know two sides and an angle and want to find the other side.

Ex: find all the angles of this triangle:



The angle we solve for is the one opposite side c. So if we pick $a=4$, $b=6$, and $c=7$, the Law of Cosines will let us solve for φ.

Now $7^2 = 4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cos\varphi$, so

$49 = 16 + 36 - 48\cos\varphi$. Then

$-48\cos\varphi = 49 - 52 = -3$, so $\cos\varphi = \frac{3}{48} = \frac{1}{16}$.

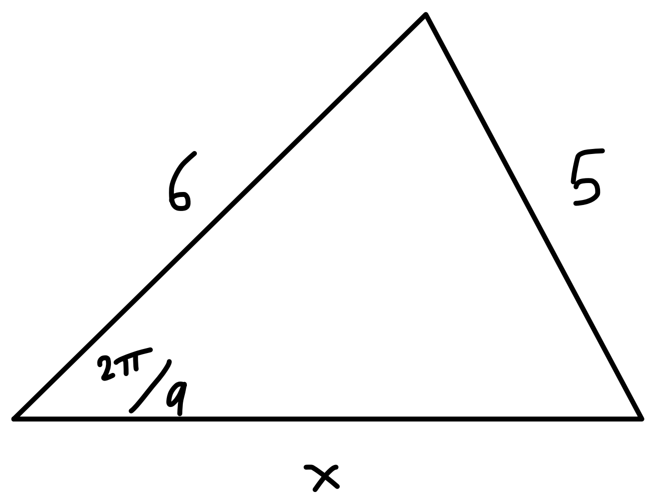
Then $\varphi = \arccos\left(\frac{1}{16}\right) = 1.51$.

Now with $a=6$, $b=7$, and $c=4$, we can solve for θ:

$4^2 = 6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cdot \cos\theta$. Then $\cos\theta = \frac{23}{28}$, so $\theta = \arccos\left(\frac{23}{28}\right) = .61$.

If we know θ and φ, we also know ψ, since the angles of a triangle sum to $180° = \pi$. Thus $\psi = \pi - 1.51 - .61 = 1.02$.

Ex: Find x, where x is the longest side in the triangle.



We know two sides and an angle and want to solve for another side, so we'll use the Law of Cosines. Since we only know one angle, we want to make $c = 5$ since that's the side opposite the angle we know. b and a don't matter — either $b = 6$ or $a = 6$ work fine. Let's take $b = 6$ and $a = x$.

Now we have:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

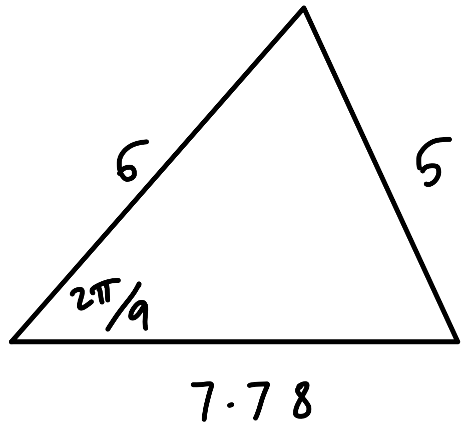
$$25 = x^2 + 36 - 2 \cdot x \cdot 6 \cos 2\pi/9$$

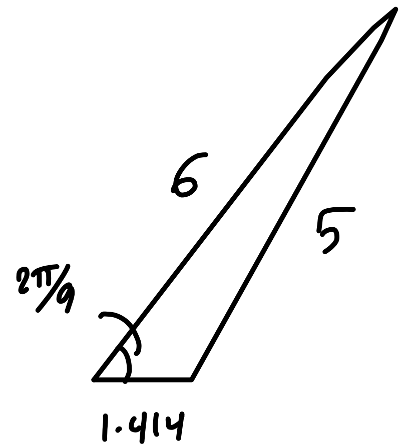
$$x^2 - \left(12 \cos 2\pi/9\right) x + 11 = 0$$

$$x = \frac{\left(12\cos 2\pi/9\right) \pm \sqrt{\left(-12\cos 2\pi/9\right)^2 - 4 \cdot 1 \cdot 11}}{2 \cdot 1}$$

$x = 1.414$ or $x = 7.78$. Remember that we want x to be the longest side in the triangle.

So we want $x = 7.78$.

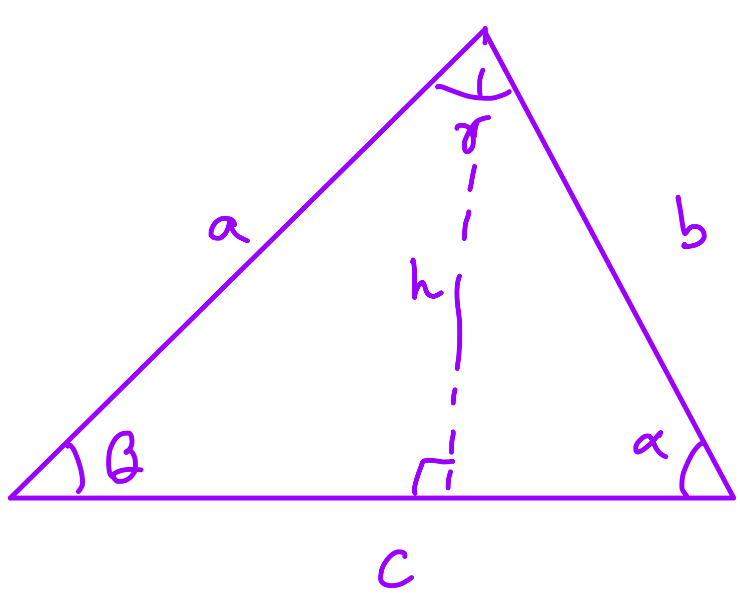




Theorem (Law of Sines): In *any* triangle with sides a, b, and c, and angles α, β, and γ opposite a, b, and c, respectively,

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$

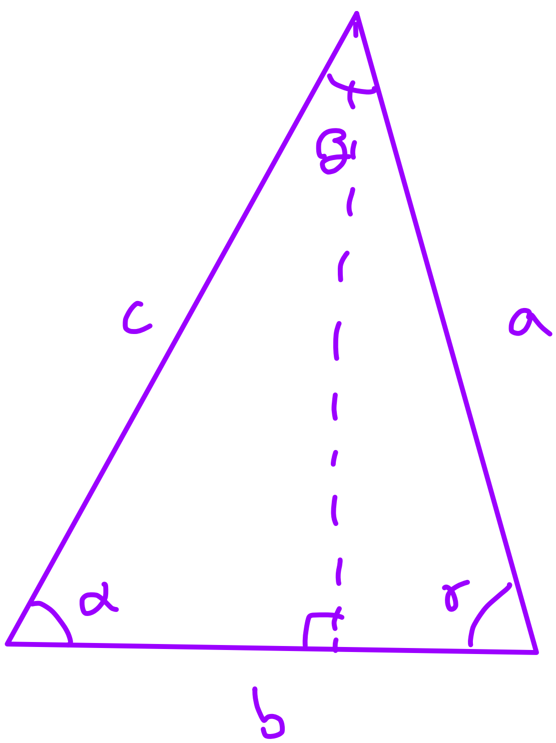
Proof:

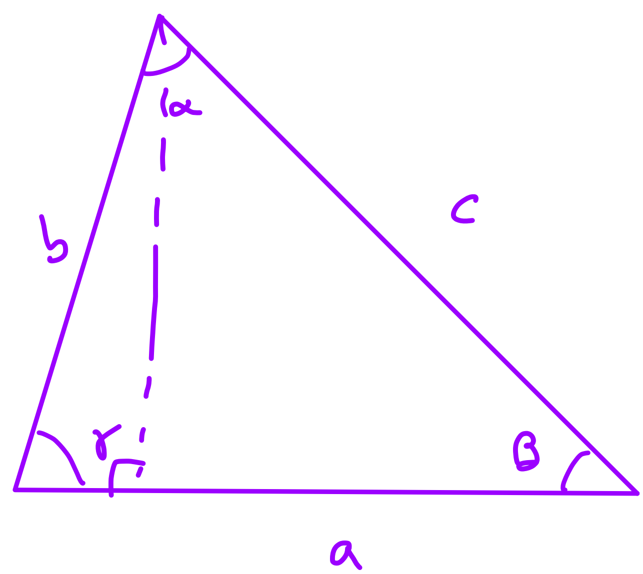


Now the area is $\frac{1}{2}ch$

$= \frac{1}{2}ca\sin\beta$

since $\sin\beta = \frac{h}{a}$





area is $\frac{1}{2}bc\sin\alpha$

area is $\frac{1}{2}ab\sin\gamma$

Now $\frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta = \frac{1}{2}ab\sin\gamma$

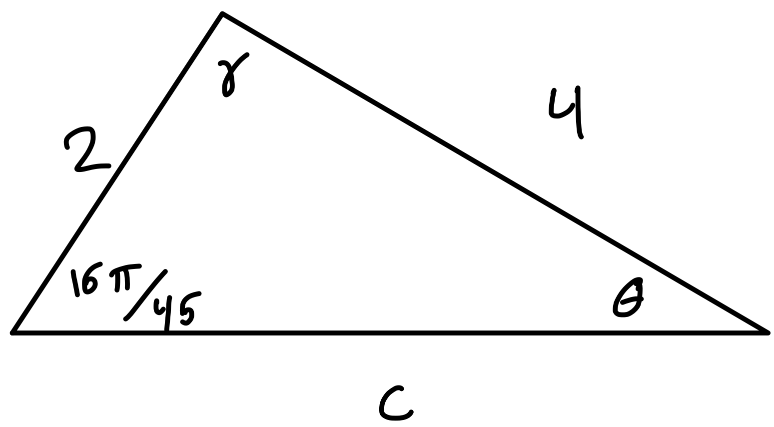
Multiply all sides by $\frac{2}{abc}$.

Then $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$. $\square$

Comment: Use the Law of Sines when:

- You know two sides and an angle and want to find a second angle.
- You know two angles and a side and want to find a second side.

Ex: Find θ.



By the Law of Sines,

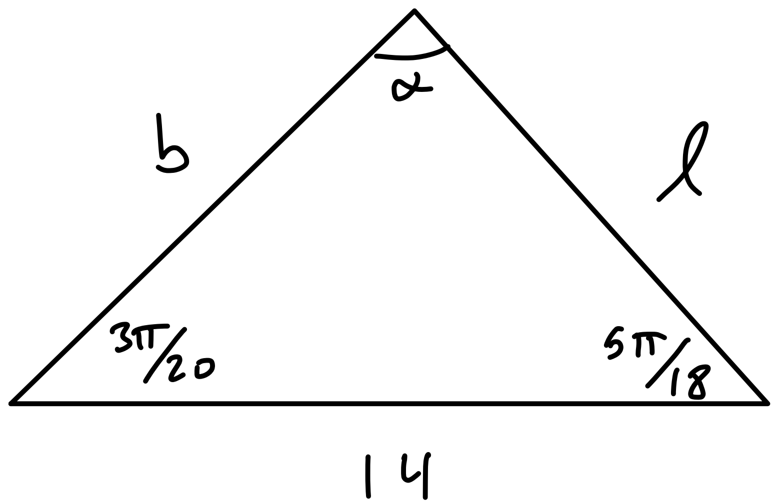
$$\boxed{\frac{\sin\theta}{2} = \frac{\sin 16\pi/45}{4}} = \frac{\sin\gamma}{c}$$

We'll always want to use two of the three sides in this equation.

Here, $\dfrac{\sin\theta}{2} = \dfrac{\sin 16\pi/45}{4}$, so

$\sin\theta = .45$, so $\theta = \arcsin(.45) = .47$.

Ex: Find ℓ.



$$\frac{\sin \alpha}{14} = \frac{\sin 5\pi/18}{b} = \frac{\sin 3\pi/20}{\ell}$$

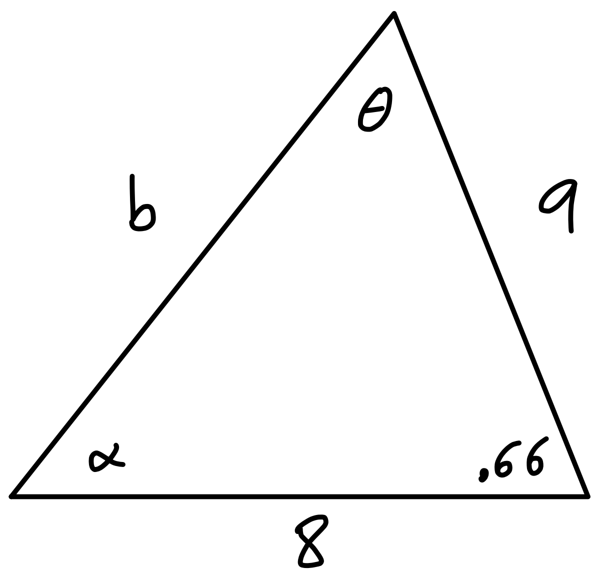
??

Wait! The angles of a triangle always sum to $180^\circ = \pi$. So

$\alpha + \frac{3\pi}{20} + \frac{5\pi}{18} = \pi$, so $\alpha = \pi - \frac{5\pi}{18} - \frac{3\pi}{20}$

$= \frac{103\pi}{180}$. Then $\frac{\sin(103\pi/180)}{14} = \frac{\sin\left(\frac{3\pi}{20}\right)}{\ell}$, so $\ell = \dfrac{14 \sin 3\pi/20}{\sin\left(\frac{103\pi}{180}\right)} = 6.52.$

Ex: Find angle θ.



We're given two sides and angle and are trying to find a second angle, so it makes sense to use the Law of Sines.

$$\frac{\sin \alpha}{9} = \frac{\sin(.66)}{b} = \frac{\sin \theta}{8}.$$

We're stuck again, and we can't solve for any more angles.

Solution: use the Law of Cosines!

First,

$$b^2 = 8^2 + 9^2 - 2 \cdot 8 \cdot 9 \cdot \cos(.66).$$

$b = 5.59$, so

$$\frac{\sin \alpha}{9} = \frac{\sin(.66)}{5.59} = \frac{\sin \theta}{8}$$

Now $\dfrac{\sin(.66)}{5.59} = \dfrac{\sin \theta}{8}$, so

$$\sin \theta = \frac{8 \sin(.66)}{5.59} = .877$$

Since θ is acute (it's between 0 and $\pi/2$), $\theta = \arcsin(.877) = 1.41$.

Trig Equations

<u>Comment</u>: The arc functions let us solve equations like $\sin\theta = \frac{1}{2}$: $\theta = \arcsin(\frac{1}{2}) = \pi/6$. But the arc functions only work if θ is in the right range (here, $[-\pi/2, \pi/2]$).

What if we want <u>all</u> possible θ with $\sin\theta = \frac{1}{2}$?

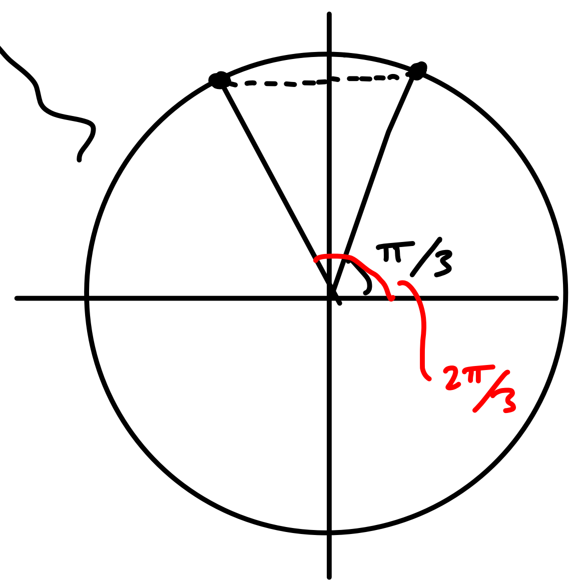
Ex: Find all values of θ such that

$$2\sin(3\theta+1)+4\sqrt{3}=5\sqrt{3}.$$

$$2\sin(3\theta+1)=\sqrt{3}$$

$$\sin(3\theta+1)=\frac{\sqrt{3}}{2}$$

$$\arcsin\left(\frac{\sqrt{3}}{2}\right)=\pi/3$$



$$3\theta+1=\frac{\pi}{3}+2\pi n$$

or

$$3\theta+1=\frac{2\pi}{3}+2\pi n$$

for any integer n

$$\theta=\frac{\pi}{9}-\frac{1}{3}+\frac{2\pi n}{3}$$

or

$$\theta=\frac{2\pi}{9}-\frac{1}{3}+\frac{2\pi n}{3}$$

for any integer n

Ex: Find all values of x such that

$$\frac{8\cos\left(\frac{\pi}{4}(x-5)\right)+10}{3} = 2 \quad \text{and}$$

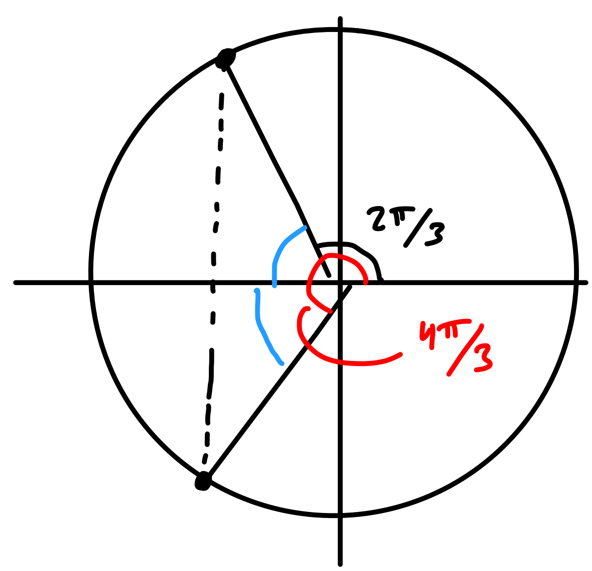
$$-8 \le x \le 2.$$

$$8\cos\left(\frac{\pi}{4}(x-5)\right)+10 = 6$$

$$\cos\left(\frac{\pi}{4}(x-5)\right) = -\frac{1}{2}$$

$$\arccos\left(-\frac{1}{2}\right) = 2\pi/3$$

$$\begin{cases} \frac{\pi}{4}(x-5) = \frac{2\pi}{3} + 2\pi n \\ \text{or} \\ \frac{\pi}{4}(x-5) = \frac{4\pi}{3} + 2\pi n \end{cases}$$



$$\begin{cases} x-5 = \frac{8}{3} + 8n \\ \text{or} \\ x-5 = \frac{16}{3} + 8n \end{cases}$$

$$x = \frac{23}{3} + 8n$$

or for any integer n.

$$x = \frac{31}{3} + 8n$$

We want only the x in $[-8, 2]$:

so write a table.

$\boxed{\begin{array}{c} x = -6.3 \\ \text{or} \\ x = -.33 \end{array}}$

n	$\frac{23}{3} + 8n$	$\frac{31}{3} + 8n$	
0	$23/3 \approx 7.67$	$31/3 \approx 10.3$	too big!
-2	too small $7.67 - 16 \approx -8.33$	$10.3 - 16 = \boxed{-6.3}$	
-1	$7.67 - 8 = \boxed{-.33}$	too big $10.3 - 8 = 2.3$	
-3	$7.67 - 24 = -16.33$	$10.3 - 24 = -14.3$	too small!

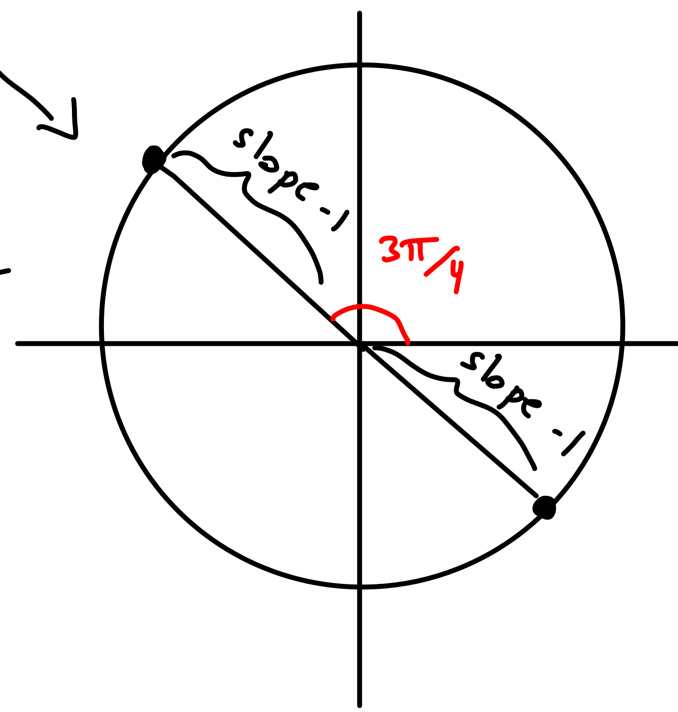
Ex: Find all solutions to

$$2\tan(y+1)+2=0.$$

$$2\tan(y+1)=-2$$

$$\tan(y+1)=-1$$

$\arctan(-1)=-\frac{\pi}{4}$



$$y+1=-\frac{\pi}{4}+2\pi n$$

or

$$y+1=\frac{3\pi}{4}+2\pi n$$

$$y=-1-\frac{\pi}{4}+2\pi n$$

or

$$y=-1+\frac{3\pi}{4}+2\pi n$$

for any integer n.