Midterm 2

Math 252

Spring 2021

You have 50 minutes to complete this exam and upload it to Canvas. **You may use a scientific calculator, but no other resources.** When you're finished, first check your work if there is time remaining, then scan the exam and upload it. If you have a question, don't hesitate to ask — I just may not be able to answer it.

disk: same (i.e. x)
shell: opposite (i.e y)

1. (32 points) Multiple choice. You don't need to show any work.

a) (8 points) Suppose $y = f(x)$, and that the graph of f is rotated about the x-axis. Then

A) the shell method integrates with respect to y and the disk method with respect to x.

B) the shell method integrates with respect to x and the disk method also with respect to x.

C) the shell method integrates with respect to x and the disk method with respect to y.

D) the shell method integrates with respect to y and the disk method also with respect to y.

b) (8 points) The area between two functions can be negative...

$\int f(x) - g(x)\, dx$

$f(x) \geq g(x)$

A) when one function goes below the x-axis.

B) when both functions are to the left of the y-axis.

C) when the functions cross over each other.

D) never.

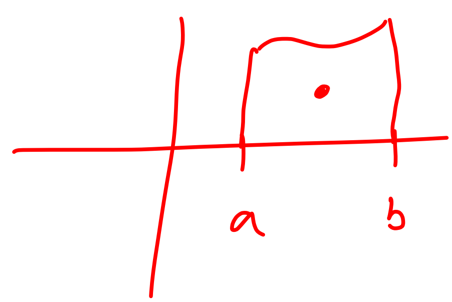
c) (8 points) If a lamina is given by $f(x)$ on $[a,b]$ with $f(x) \geq 0$, then it is always true that

A) $a \leq \overline{x} \leq b$.

B) $\overline{y} \geq 0$.

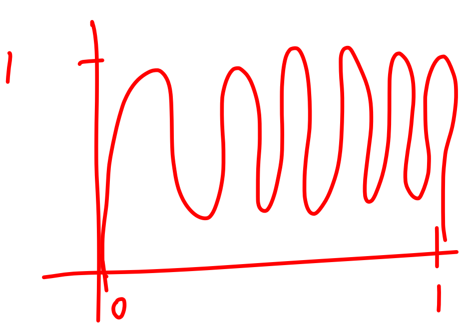
C) both.

D) neither.



d) (8 points) True or false: there is a number L such that for any continuous function f on $[0,1]$, the arc length of f is less than L.

e) (4 points extra credit) Justify your answer to part d): if you answered true, then find L. If you answered false, then given any number L, find a function on $[0,1]$ with arc length of at least L.



$\int \sqrt{1+(f'(x))^2}\, dx$

Can make f' as big as you want

2. (32 points) Short answer.

a) (8 points) Let f and g be continuous functions on $[1,6]$ such that $f(x) \geq g(x)$ on $[1,3]$ and $g(x) \geq f(x)$ on $[3,6]$. What is the area between f and g on $[1,6]$?

$$\int_1^3 f(x) - g(x)\, dx + \int_3^6 g(x) - f(x)\, dx$$

b) (8 points) Let $f(x) = 3x^2$. Set up the integrals to find the volume of the solid given by rotating the graph of f on $[0,3]$ about the x-axis, using **both** the disk and shell methods. Don't solve either of the integrals.

disk : $\int_0^3 \pi (3x^2)^2\, dx$

shell : $y = 3x^2 \quad x = \sqrt{\frac{y}{3}} \quad \left.\begin{array}{l} x = 0 \Rightarrow y = 0 \\ x = 3 \Rightarrow y = 27 \end{array}\right\} \int_0^{27} 2\pi y \sqrt{\frac{y}{3}}\, dy$

c) (8 points) Let L be a lamina bounded above by $f(x)$ on $[1,2]$. Write the three integrals necessary to calculate $\overline{x}$ and $\overline{y}$.

$$M_x = \int_1^2 \frac{1}{2} f(x)^2\, dx$$

$$M_y = \int_1^2 x f(x)\, dx$$

$$m = \int_1^2 f(x)\, dx$$

d) (8 points) Evaluate $\int xe^{2x}\, dx$.

$u = x \qquad v = \frac{1}{2} e^{2x}$

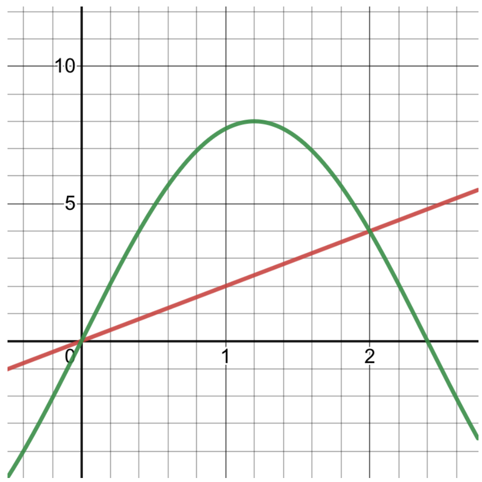
$du = dx \qquad dv = e^{2x}\, dx$

$$\frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x}\, dx$$

$$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

The rest of the problems require setting up and solving integrals. Half the credit is for the set-up and half for the solving.

3. (32 points) Consider the region given in the graph below, bounded by $f(x) = 8\sin\left(\frac{5\pi}{12}x\right)$ above and $g(x) = 2x$ below. These functions intersect at $(0,0)$ and $(2,4)$.



a) (16 points) Find the volume of the solid of revolution given by rotating the region about the y-axis.

Could use washers, but: we have to solve these functions for x, and there isn't a good way to have one on the right and one on the left

Instead, let's try shells: the functions are already in terms of x, and the green is always above the red.

$$\int_0^2 2\pi x \cdot 8\sin\left(\frac{5\pi}{12}x\right)dx - \int_0^2 2\pi x(2x)\,dx$$

$$16\pi\int_0^2 x\sin\left(\frac{5\pi}{12}x\right)dx - 4\pi\left[\frac{x^3}{3}\right]\Big|_0^2$$

$$u = x \qquad v = -\frac{12}{5\pi}\cos\left(\frac{5\pi}{12}x\right)$$

$$\downarrow \qquad\qquad \uparrow$$

$$du = dx \qquad dv = \sin\left(\frac{5\pi}{12}x\right)dx$$

$$= 16\pi\left[-\frac{12}{5\pi}x\cos\left(\frac{5\pi}{12}x\right) + \int\frac{12}{5\pi}\cos\left(\frac{5\pi}{12}x\right)dx\right]\Bigg|_0^2$$

$$-4\pi\left(\frac{8}{3}\right)$$

b) (16 points) Suppose the region is a lamina with density $\rho = 1$. Find the center of mass. ← harder than expected

$$M_x = \int_0^2 4 \sin^2\left(\frac{5\pi}{12} x\right) dx - \int_0^2 \frac{1}{2} 4x^2 \, dx$$

$$M_y = \int_0^2 8x \sin\left(\frac{5\pi}{12} x\right) dx - \int_0^2 2x^2 \, dx$$

$$m = \int_0^2 8 \sin\left(\frac{5\pi}{12} x\right) dx - \int_0^2 2x \, dx$$

Everything but $\int_0^2 4\sin^2\left(\frac{5\pi}{12}x\right)dx$ was done in part a)

$$4\int_0^2 \sin^2\left(\frac{5\pi}{12}x\right)dx = 4\int_0^2 \frac{1-\cos\left(\frac{10\pi}{12}x\right)}{2}dx$$

$$= 2\int_0^2 1-\cos\left(\frac{10\pi}{12}x\right)dx$$

$$= 2\left[x - \frac{12}{10\pi}\sin\left(\frac{10\pi}{12}x\right)\right]\Bigg|_0^2$$

$$= 2\left(2 - \frac{6}{5\pi}\sin\left(\frac{20\pi}{12}\right)\right).$$

4. (16 points) Evaluate $\int_1^2 x^2 \ln(x)\, dx$.

$u = \ln(x) \qquad v = \dfrac{x^3}{3}$

$\downarrow \qquad\qquad \uparrow$

$du = \dfrac{1}{x}dx \qquad dv = x^2 dx$

$$\left[\frac{x^3}{3}\ln(x) - \int \frac{x^3}{3} \cdot \frac{1}{x}\,dx \right]\Bigg|_1^2$$

$$= \left[\frac{x^3}{3}\ln(x) - \frac{x^3}{9} \right]\Bigg|_1^2$$

$$= \left(\frac{8}{3}\ln(2) - \frac{8}{9} \right) - \left(\frac{1}{3}\cancel{\ln(1)}^{\,0} - \frac{1}{9} \right)$$

$$= \boxed{\frac{8}{3}\ln(2) - \frac{7}{9}}$$