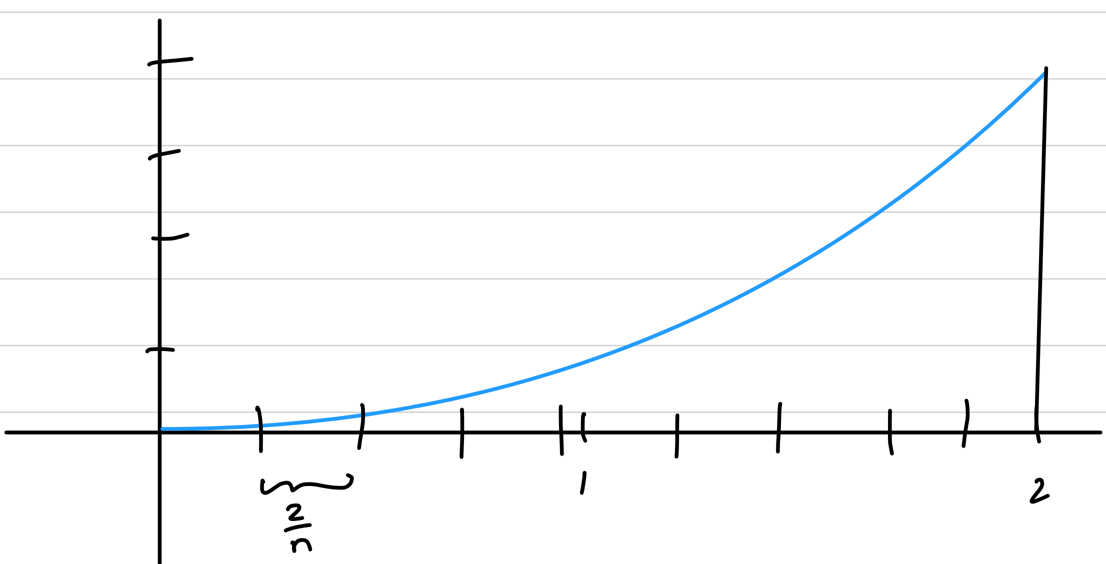
Ex: Evaluate $\int_0^2 t^2 \, dt$



First, get a regular partition of $[0,2]$ with n subintervals.

The width of each subinterval is $\frac{2}{n}$.

The ith subinterval is $\left[\frac{2}{n}(i-1), \frac{2}{n}i\right]$

Can pick x_i^* to be any point in that interval – let's pick the right endpoint $\frac{2}{n}i$.

So, $\displaystyle\int_0^2 t^2\,dt \overset{\text{DEF}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2}{n}i\right)\frac{2}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}i\right)^2 \frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n^2} i^2 \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right)$$

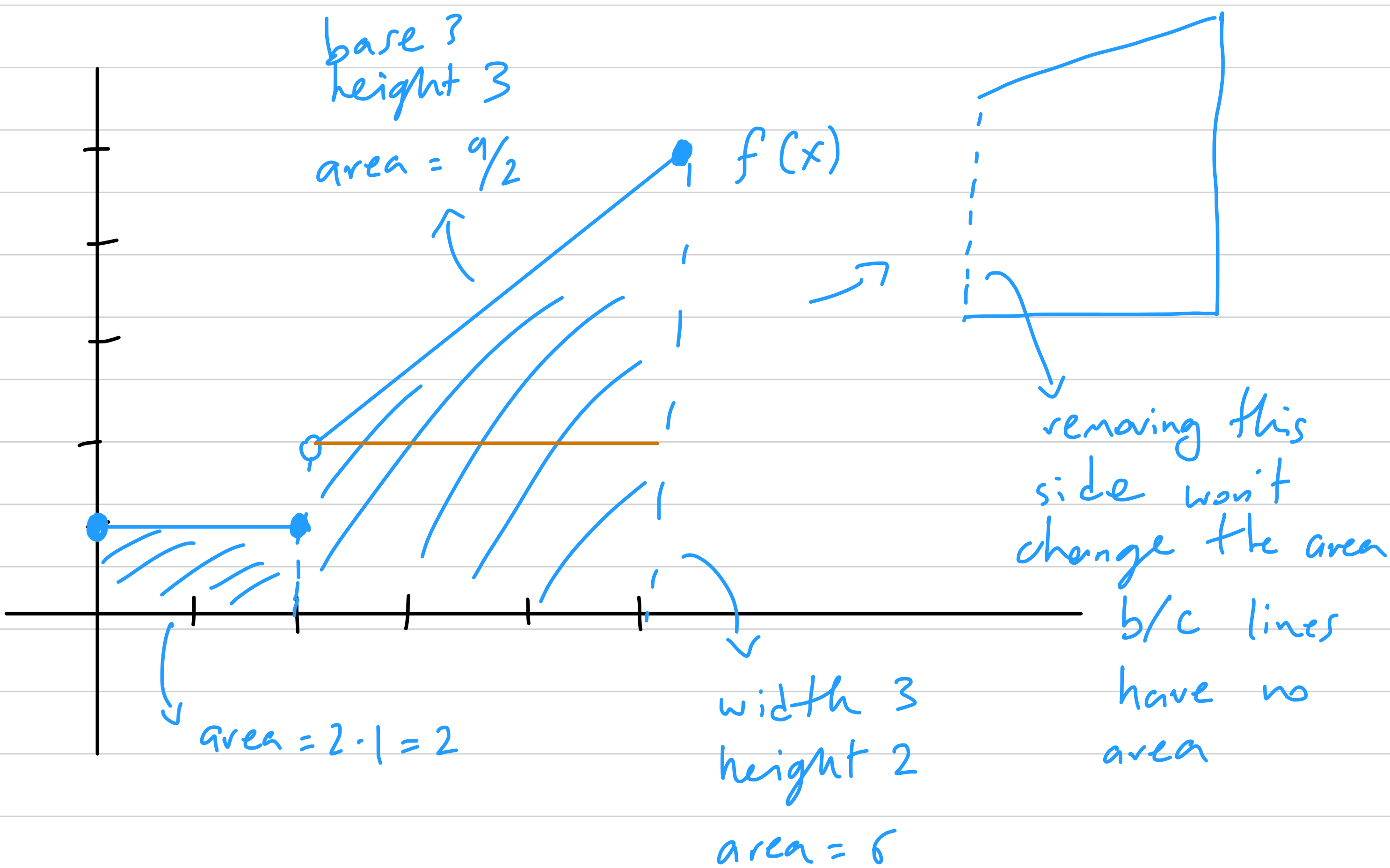
$$= \lim_{n\to\infty} \left(\frac{8}{n^3} \cdot \frac{2n^3 + n^2 + 2n^2 + n}{6} \right)$$

$$= \lim_{n\to\infty} \left(\frac{2\cdot 8}{6} + \frac{8\cdot 3}{6n} + \frac{8}{6n^2} \right)$$

$$= \frac{2\cdot 8}{6}$$

$$= 8/3.$$

Ex: Compute $\int_0^5 f(x)\,dx$, where $f(x) = \begin{cases} 1, & 0 \le x \le 2 \\ x, & 2 < x \le 5 \end{cases}$



In total, $\int_0^5 f(x)\,dx = 2 + 6 + \frac{9}{2} = \frac{25}{2}$.

Ex: $\displaystyle\int_3^6 \sqrt{9-(x-3)^2}\,dx$

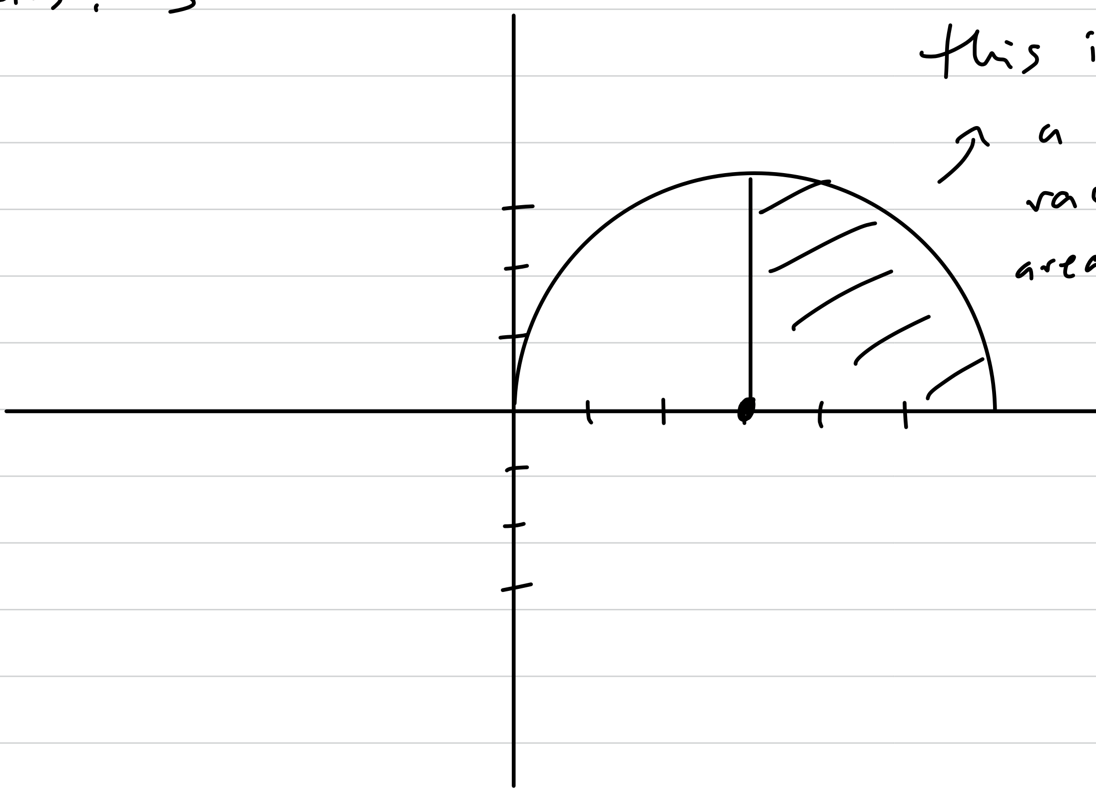
this is part of a circle

$y = \sqrt{9-(x-3)^2}$ ← this only gives + values, so it's only the top half of the circle.

$y^2 = 9-(x-3)^2$

$(x-3)^2 + y^2 = 3^2$

center : $(3,0)$
radius : 3

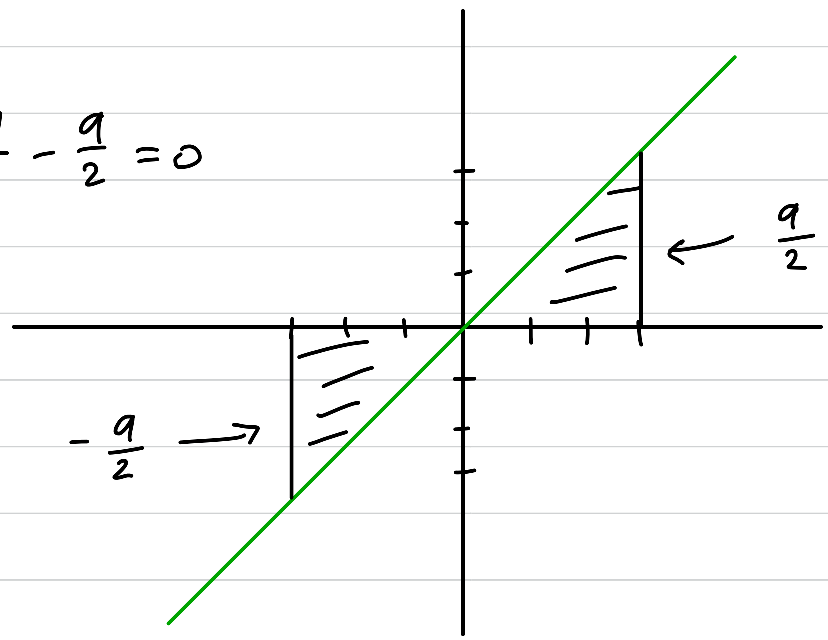


this is $\frac{1}{4}$ of a circle with radius 3, so area $= \frac{1}{4}(\pi \cdot 3^2)$
$= \frac{9}{4}\pi$.

$$\int_3^6 \sqrt{9-(x-3)^2}\, dx = \frac{9}{4}\pi.$$

Def: Integrals use signed area: area that's below the x-axis is counted as negative.

Ex: $$\int_{-3}^{3} x \, dx = \frac{9}{2} - \frac{9}{2} = 0$$



Prop: ① $\int_a^a f(x)\, dx = 0.$

② $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

③ $\int_a^b (f(x)+g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x) dx$

④ $\int_a^b c \cdot f(x)\, dx = c\int_a^b f(x)\, dx$

⑤ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

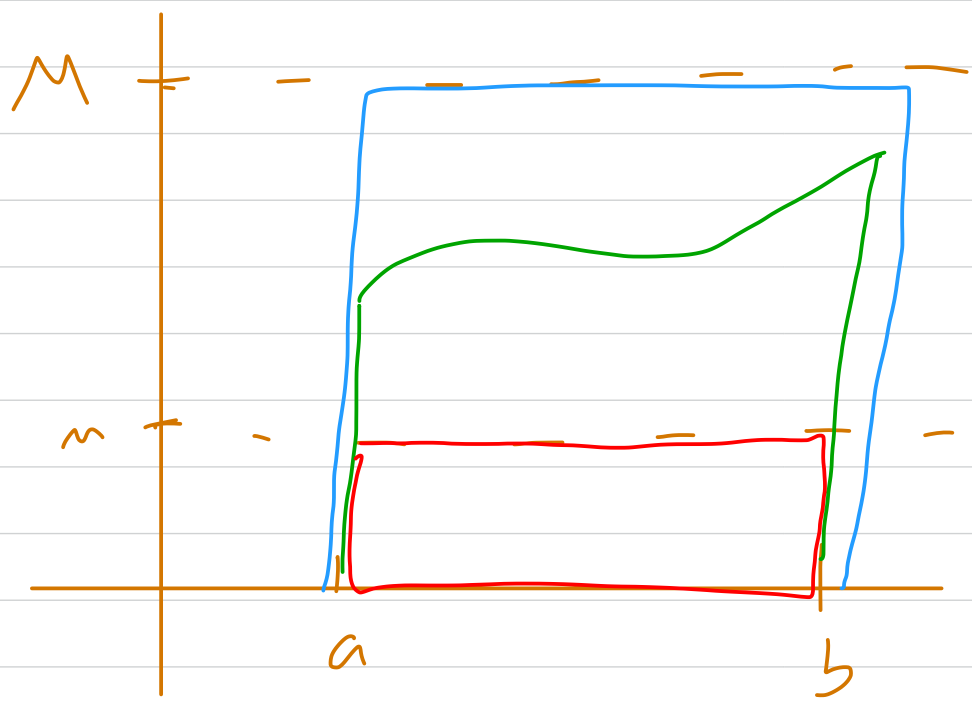
⑥ If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$

⑦ If $f(x) \geq g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

⑧ If $m \leq f(x) \leq M$ on $[a,b]$, then

$$\boxed{(b-a)m} \leq \boxed{\int_a^b f(x)\,dx} \leq \boxed{(b-a)M}.$$



Def: The average value of f on $[a,b]$ is
$\frac{1}{b-a}\int_a^b f(x)\,dx$.

think adding up a bunch of values and diving by the "number" of things you added.

Ex: The average value of x^2 on $[0,2]$ is
$\frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left(\frac{8}{3}\right) = \frac{8}{6}$

Antiderivatives

Def: Let f be a function. F is an antiderivative of f if $F' = f$.

Ex: $x^3/3$ is an antiderivative of x^2 because

$$\frac{d}{dx}\left[x^3/3\right] = \frac{1}{3}\frac{d}{dx}\left[x^3\right] = \frac{1}{3}(3x^2) = x^2.$$

$x^3/3 + 2$ is also one, since $\frac{d}{dx}[2] = 0$.

So $x^3/3 + C$ is an antiderivative for any number C.

Thm: For any function f, every antiderivative of f is of the form $F(x) + C$ for some C.

Ex: $1/x \Rightarrow \ln|x| + C$

$\sin(x) \Rightarrow -\cos(x) + C$

$e^x \Rightarrow e^x + C$

Def: The indefinite integral of $f(x)$ is $\int f(x)\,dx = F(x) + C$, where $F(x)$ is an antiderivative of f. (This looks like horrible notation, but we'll eventually see that it's not.)

Ex: $\int \sin(x)\,dx = -\cos(x) + C.$

Prop: $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$, since $\frac{d}{dx}\left[\frac{x^{p+1}}{p+1} + C\right] = x^p$

true if $p \neq -1$.

Prop: ① $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

② $\int c \cdot f(x)\,dx = c\int f(x)\,dx$

Def: An initial value problem is a problem of the form $\underbrace{\frac{dy}{dx} = f(x)}_{\text{determine } y \text{ by integrating}}$, $\underbrace{y(x_0) = y_0}_{\text{intial value}}$.

↳ solve for C

Ex Your velocity at time t is given by $v(t) = 3t + 2 \frac{ft}{s}$. After 1 second, you've moved 2 feet. Find a function $s(t)$ that gives the number of feet you've moved after t seconds.

(Remember: $s'(t) = v(t)$, so $s(t) = \int v(t)\,dt$)

$$s(t) = \int v(t)\,dt = \int (3t+2)\,dt = 3\int t\,dt + \int 2\,dt$$

$$= 3\left(\frac{t^2}{2} + C_1\right) + (2t + C_2)$$

In general, you can combine all the Cs into a single C.

$$= \frac{3t^2}{2} + 2t + \underbrace{(3C_1 + C_2)}_{C}$$

$$= \frac{3t^2}{2} + 2t + C$$

Now $s(1) = 2$, so $2 = \frac{3}{2} + 2 + C$

$$C = -3/2$$

$$\Rightarrow s(t) = \frac{3t^2}{2} + 2t - 3/2 .$$

The Fundamental Theorem of Calculus

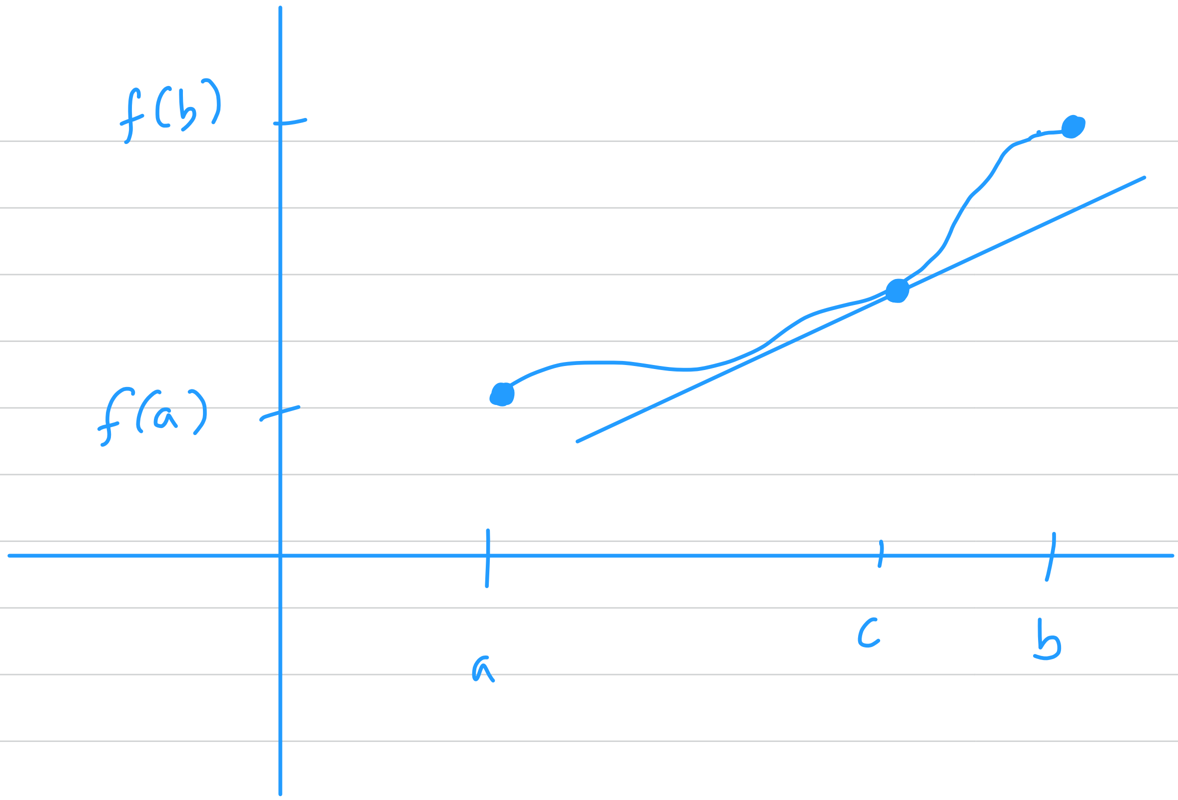
Recall: MVT says if f is differentiable on $[a,b]$, then there is a point $c \in [a,b]$ (i.e. $a \le c \le b$) such that

in

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

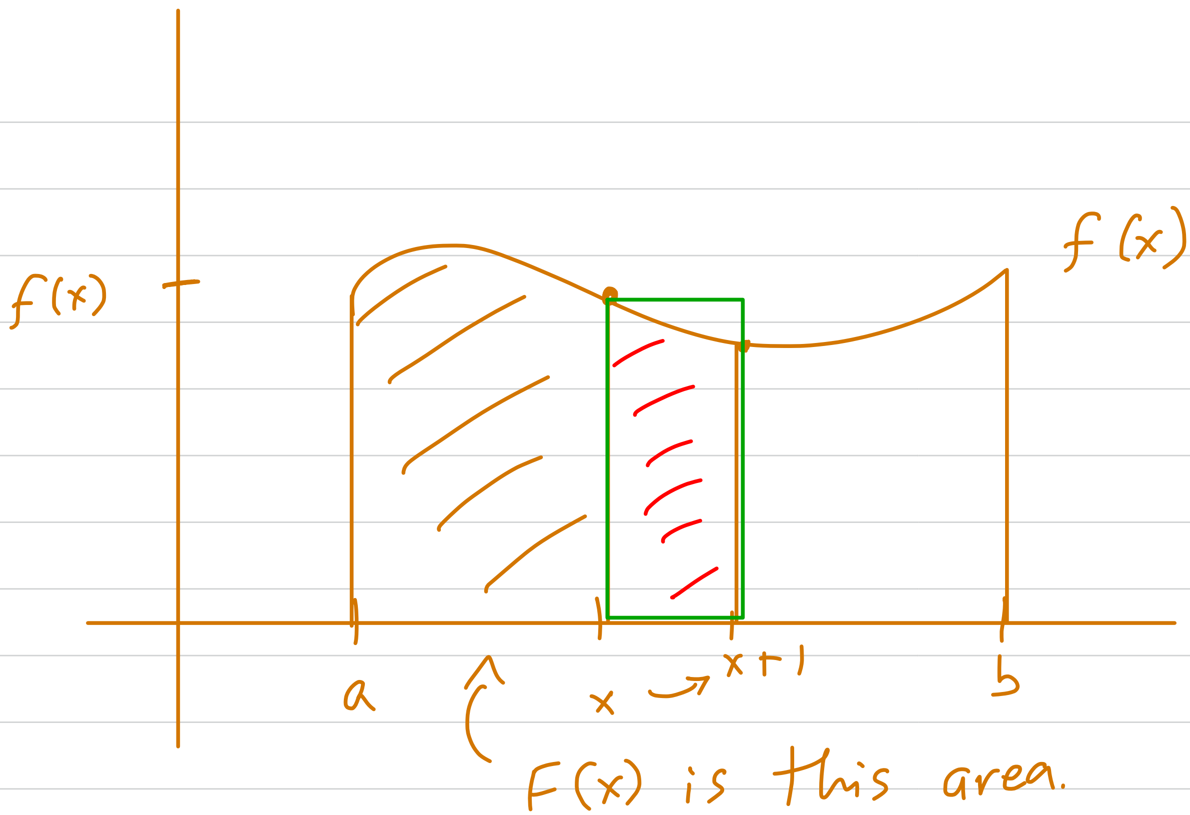
tangent line

secant line



Theorem (MVT for integrals): if f is continuous on $[a,b]$, then there is a $c \in [a,b]$ such that $f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{average value of } f}$.

Theorem (FTC, part I): Let f be a continuous function on $[a,b]$ and define a function F by $F(x) = \int_a^x f(t)\,dt$.



Then $F'(x) = f(x)$ (i.e. F is an antiderivative of f).

This should be somewhat intuitive: $F'(x)$ is approximately the amount F increases by going from x to $x+1$.

Comment: It's more important that there is an antiderivative than that it's equal to this expression — FTC I won't help us calculate antiderivatives

Ex: Find $\frac{d}{dx}\left[\int_1^x \frac{1}{t^3+t}\,dt\right]$

$$= \boxed{\frac{1}{x^3+x}}.$$

$$\frac{d}{dx}\left[\underbrace{\int_1^4 \frac{1}{t^3+t}\,dt}\right] = 0 \quad \text{b/c is constant}$$

$$\frac{d}{dx}\left[\int_1^x \frac{1}{t^3+t}\,dt\right]\Bigg|_{x=4} = \frac{1}{4^3+4} = \frac{1}{68}$$

Ex: Find $\frac{d}{ds}\left[\int_0^{\sqrt{s}} \sin(t)\,dt\right]$

Let $F(s) = \int_0^s \sin(t)\,dt$. Then

$F'(s) = \sin(s)$.

What we want is $\frac{d}{ds}\left[F(\sqrt{s})\right]$, so we need to use the chain rule: $\frac{d}{ds}\left[F(\sqrt{s})\right] =$

$F'(\sqrt{s}) \cdot \frac{d}{ds}\left[\sqrt{s}\right] = \sin(\sqrt{s}) \cdot \frac{1}{2} s^{-\frac{1}{2}}$

$\left(\sqrt{s} = s^{1/2}, \text{ so } \frac{d}{ds}\left[\sqrt{s}\right] = \frac{1}{2} s^{-1/2} \right)$

(FTC, part II)

Thm: Let f be an integrable function. By FTC I, there is an antiderivative F for f. Then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Comment: This is strange! The area under the graph of f depends *only* on the value of F at 2 points.

Def: $$\left[F(x)\right]\Big|_a^b = F(b) - F(a)$$

Comment: We can restate FTC as
$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]\Big|_a^b.$$

Ex: $$\int_0^2 x^2\,dx = \left[x^3/3\right]\Big|_0^2 = 2^3/3 - 0^3/3 = 8/3.$$

where is the $+C$?

It's true that $\int x^2\,dx = x^3/3 + C$. But,
$$\left[x^3/3 + C\right]\Big|_0^2 = 2^3/3 + \cancel{C} - 0^3/3 - \cancel{C},$$ so

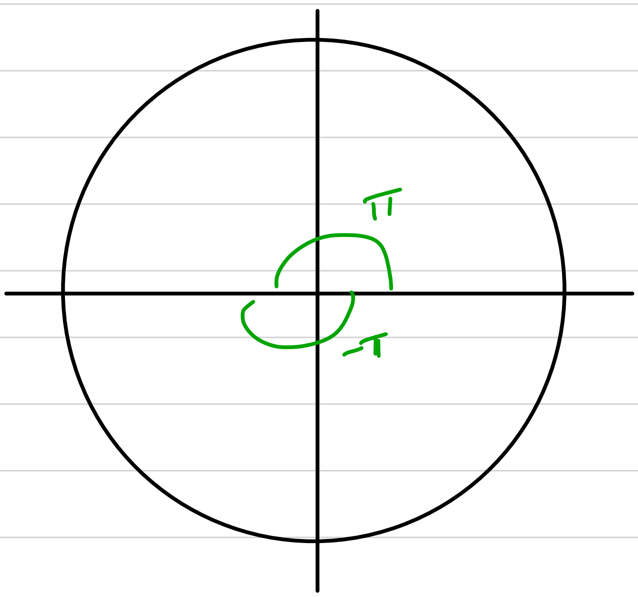
we can alway ignore C in definite integrals.

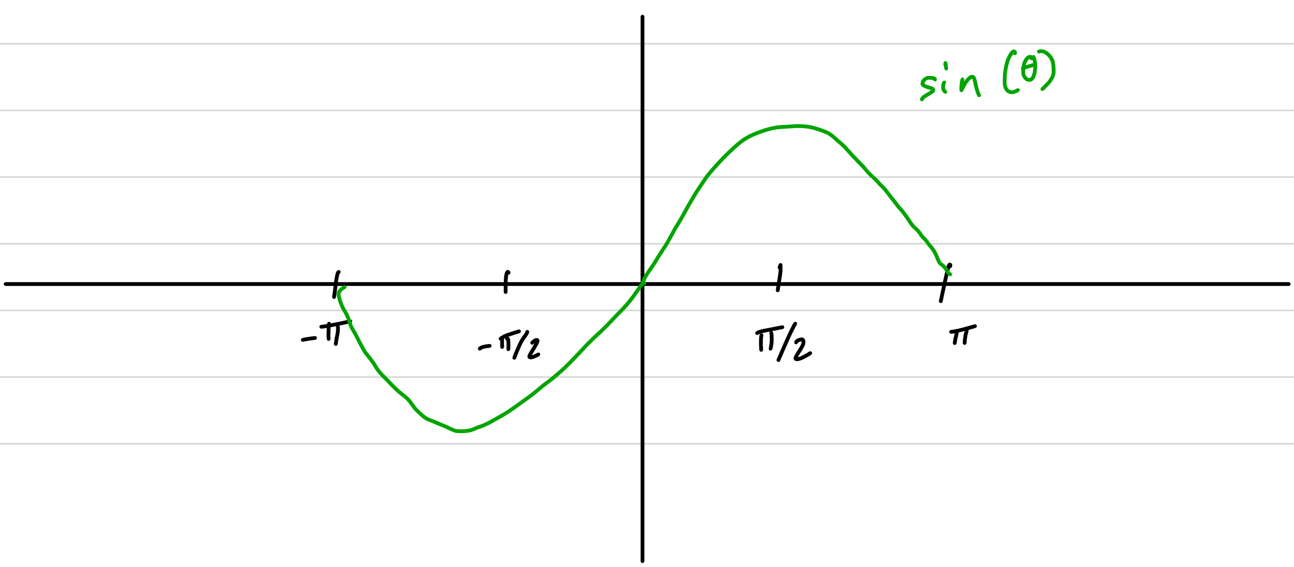
Ex: $\int_{-\pi}^{\pi} \sin(\theta)\,d\theta = \left[-\cos(\theta)\right]\Big|_{-\pi}^{\pi} = -\cos(\pi) + \cos(-\pi)$

$= -(-1) + (-1)$

$= 1 - 1 = 0$

$\int \sin(\theta)\,d\theta = -\cos(\theta) + C$ b/c $\frac{d}{d\theta}\left[\cos(\theta)\right] = -\sin(\theta)$





$\underline{Ex}$:

$$\int_1^9 \frac{x-1}{\sqrt{x}}\,dx = \int_1^9 \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right) dx$$

$$= \int_1^9 \left(\frac{x^1}{x^{1/2}} - \frac{1}{x^{1/2}}\right) dx$$

$$= \int_1^9 \left(x^{(1-1/2)} - x^{-1/2}\right) dx$$

$$= \int_1^9 \left(x^{1/2} - x^{-1/2}\right) dx$$

$$= \int_1^9 x^{1/2}\,dx - \int_1^9 x^{-1/2}\,dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]\Bigg|_1^9 - \left[\frac{x^{1/2}}{1/2}\right]\Bigg|_1^9$$

$$= \left(\frac{9^{3/2}}{3/2} - \frac{1^{3/2}}{3/2}\right) - \left(\frac{9^{1/2}}{1/2} - \frac{1^{1/2}}{1/2}\right)$$

$$= \left(\frac{\sqrt{9}^{\,3}}{3/2} - \frac{1}{3/2}\right) - \left(\frac{\sqrt{9}}{1/2} - \frac{1}{1/2}\right)$$

$$= \left(27 \cdot \frac{2}{3} - \frac{2}{3}\right) - \left(3 \cdot \frac{2}{1} - \frac{2}{1}\right)$$

$= \left(18 - \frac{2}{3}\right) - (6 - 2)$

$= 14 - 2/3$

Comment: The rest of the class is about

1. Finding antiderivatives
2. Applying integration to other problems

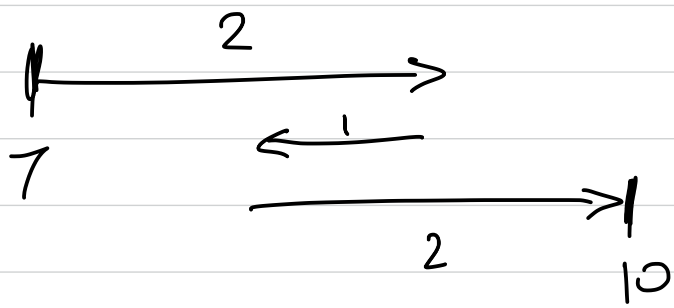
$$\int x^p \, dx = \frac{x^{p+1}}{p+1} + C \quad , \; p \neq -1$$

$$\int x^{-1} \, dx = \ln|x| + C$$

Ex: Find the net displacement of a particle traveling at $v(t) = 3t - 5$ m/s from time $t=0$ to $t=3$.

Net displacement : signed distance

Total distance : unsigned distance



Net displacement: 3
Total distance: 5

We want $s(3) - s(0)$, where s is the position of the particle.

$$s(3) - s(0) = \int_0^3 v(t)\,dt \quad b/c \quad \int v(t)\,dt = s(t) + C$$

$$b/c \quad s'(t) = v(t)$$

$$\int_0^3 (3t-5)\,dt = \left[3\,t^2/2 - 5t\right]\Big|_0^3 = \left(3\,3^2/2 - 5\cdot 3\right) - 0$$

$$= 27/2 - 15$$

$$= -3/2 .$$

So the particle moved $3/2$ m to the left from $t=0$ to $t=3$.

Ex: You move at rate $v(t) = t^2$ from $t = 0$ to $t = 2$. Then you move at rate $-t^3$ from $t = 2$ to $t = 3$. Find net displacement and total distance traveled.

Net displacement: $s(3) - s(0)$, where $s(t)$ is position at time t. So $s(3) - s(0) = \int_0^3 v(t)\,dt$

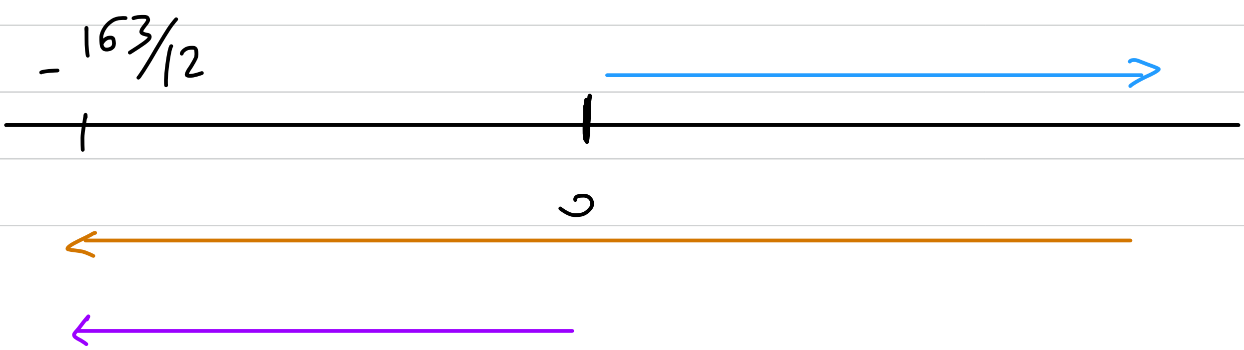
$$= \int_0^2 t^2\,dt + \int_2^3 -t^3\,dt = \left[t^3/3\right]\Big|_0^2 + \left[-t^4/4\right]\Big|_2^3 =$$

$$\left(2^3/3 - 0^3/3\right) + \left(-3^4/4 - \left(-2^4/4\right)\right) = \frac{8}{3} - \frac{81}{4} + \frac{16}{4} = -\frac{163}{12}.$$

Total distance traveled: $\int_0^2 |t^2|\,dt + \int_2^3 |-t^3|\,dt$

$$= \int_0^2 t^2\,dt + \int_2^3 t^3\,dt = \left[t^3/3\right]\Big|_0^2 + \left[t^4/4\right]\Big|_2^3$$

$$= 2^3/3 + 81/4 - 16/4 = 227/12.$$



Net displacement: $\longleftarrow$

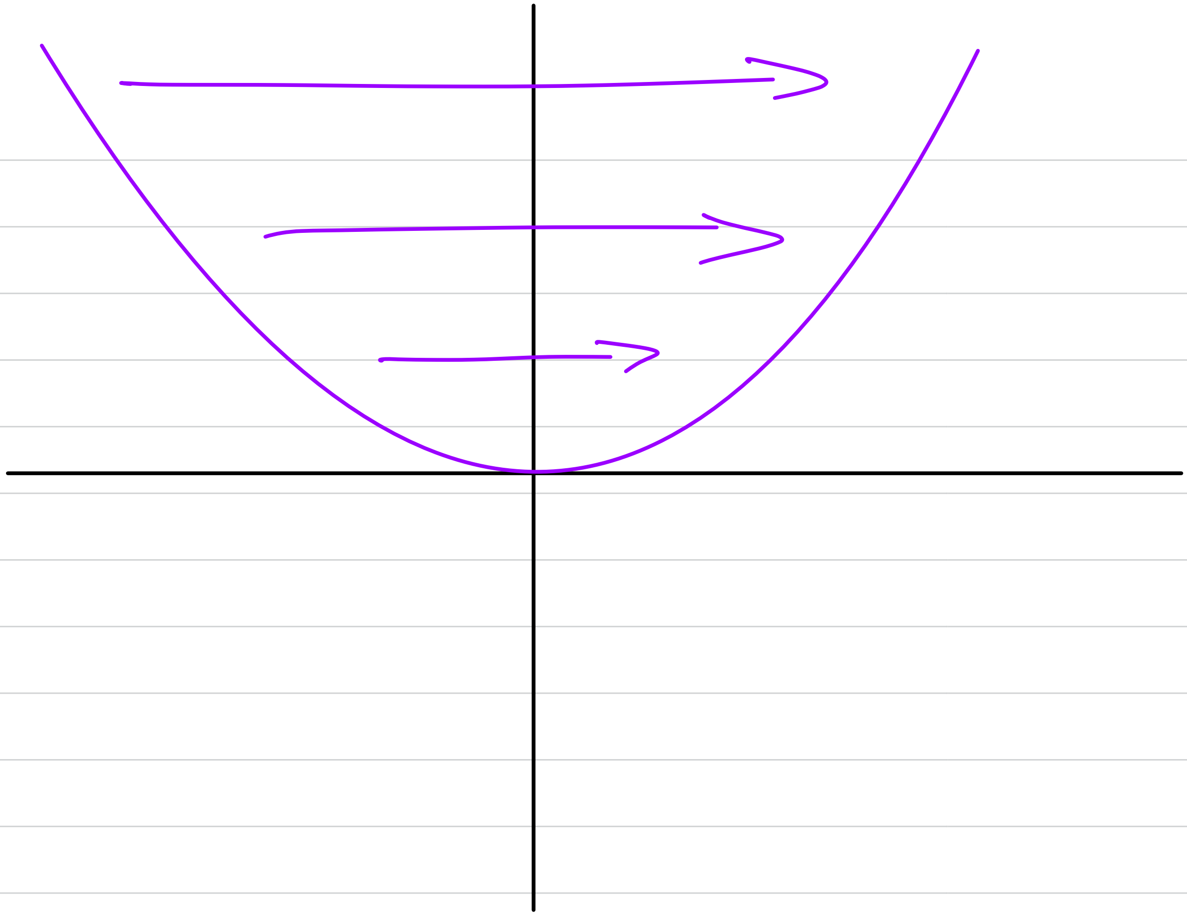
Total distance: $\longrightarrow \longrightarrow$

<u>Comment</u>: Recall that a function f is <u>even</u> if $f(x) = f(-x)$ for all x, and it's <u>odd</u> if $-f(x) = f(-x)$ for all x.

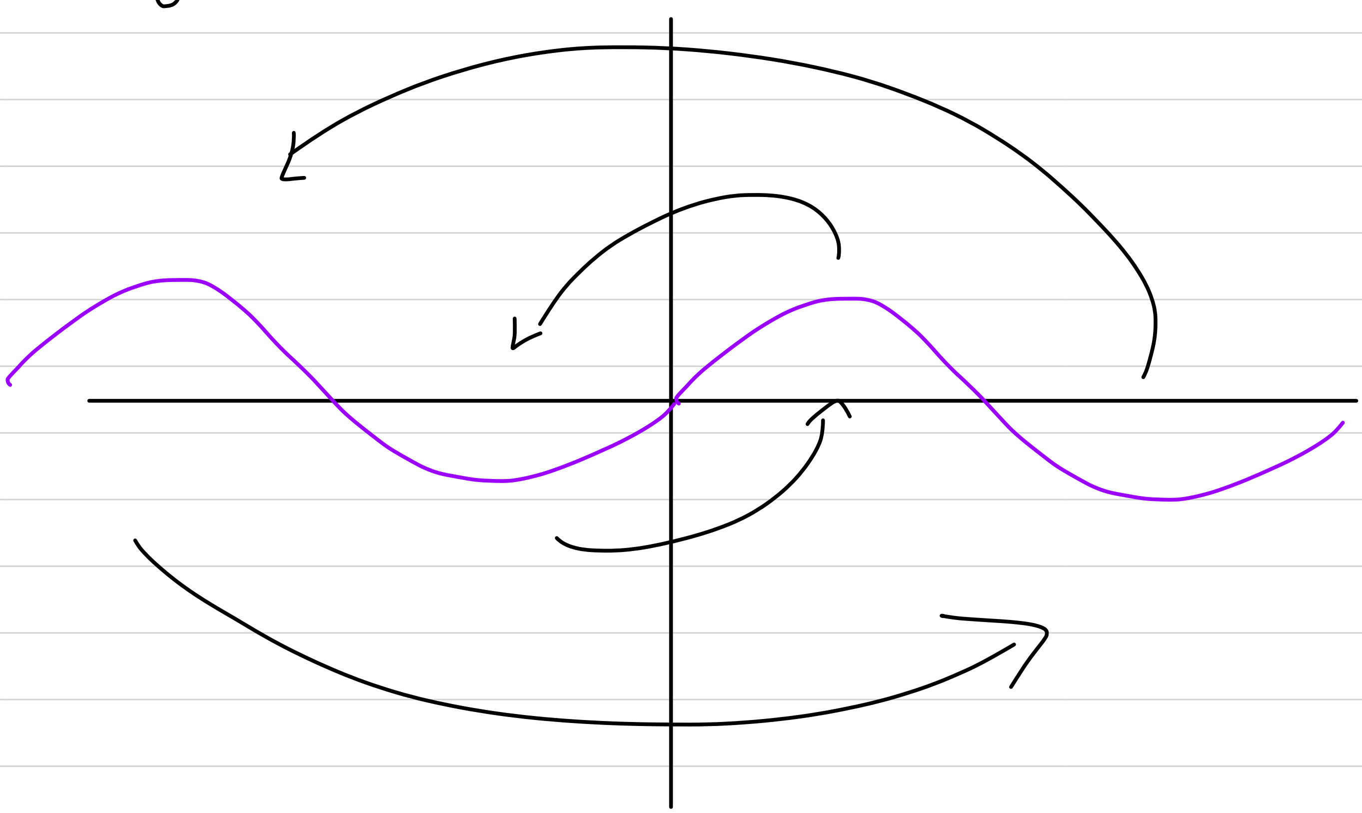
Even: symmetry about y-axis

Odd: rotational symmetry about the origin ($180°$)

<u>Ex</u>: $y = x^2$ is even. $(-x)^2 = x^2$

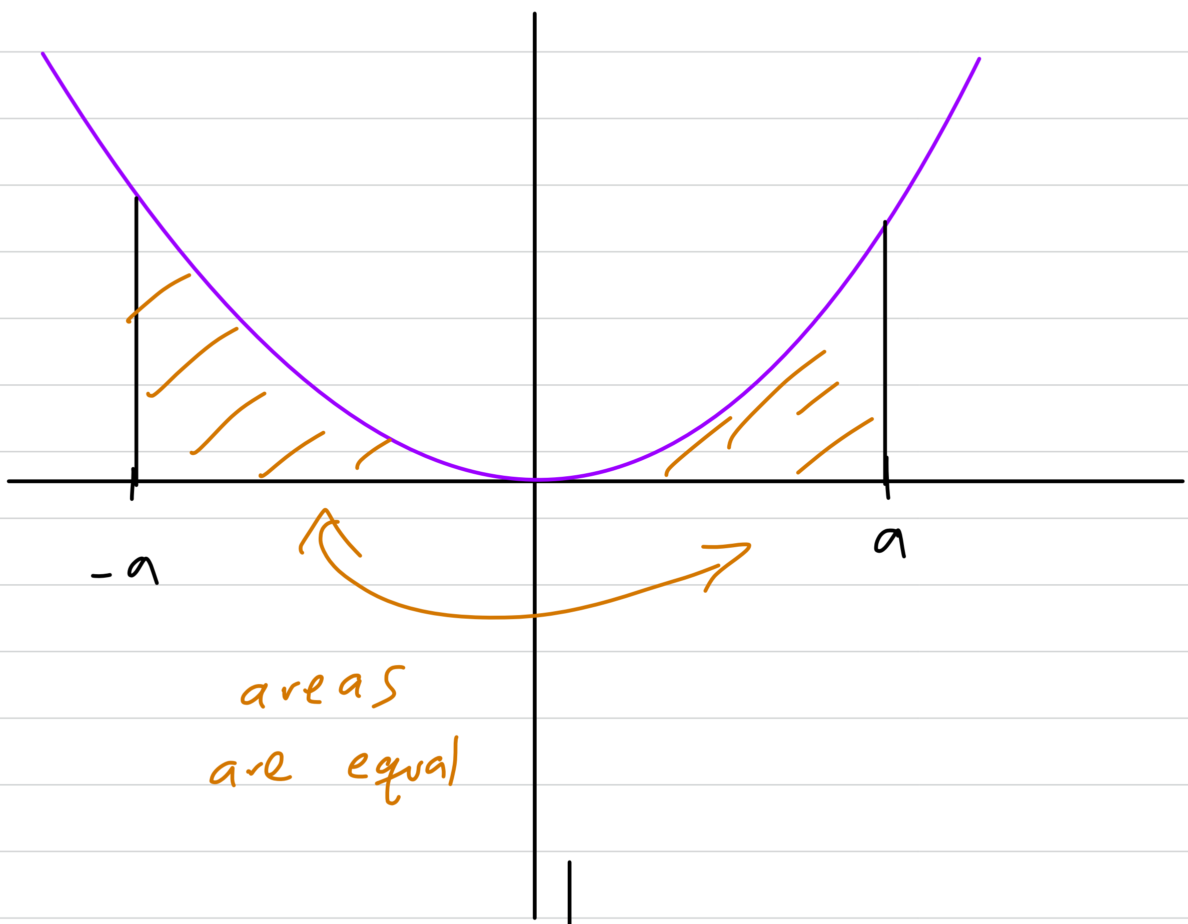


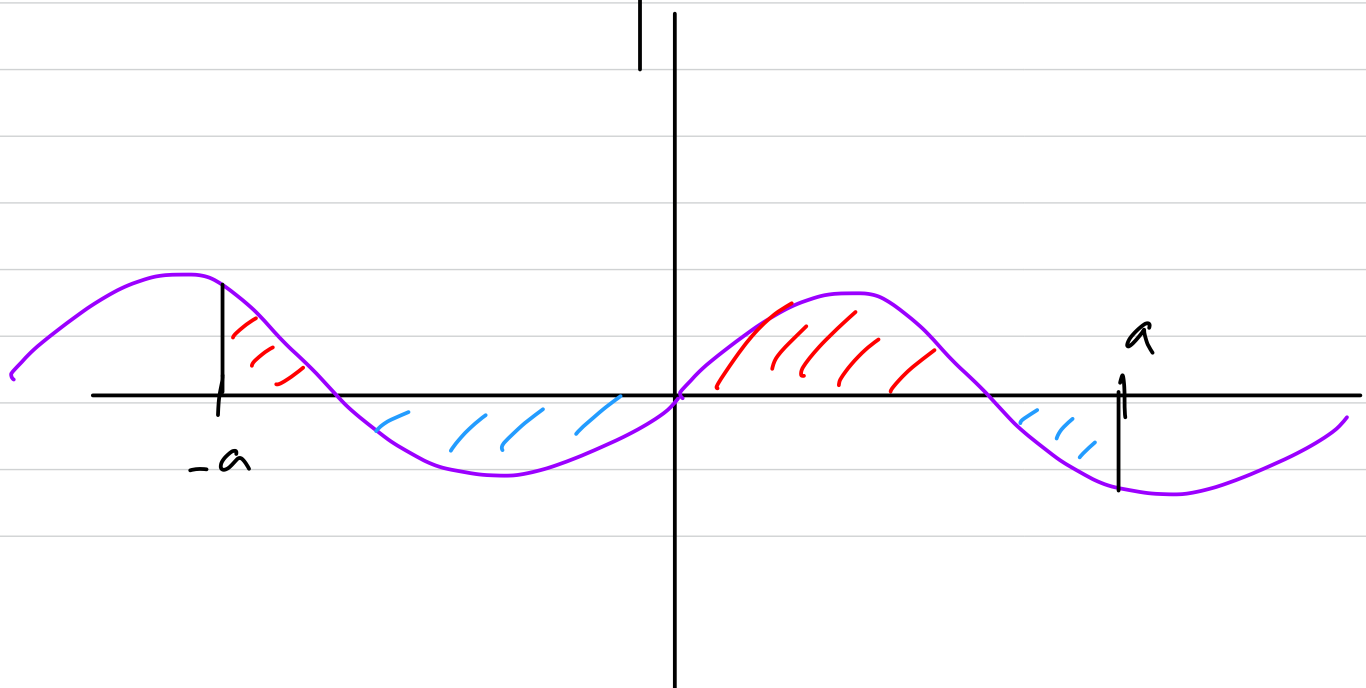
Ex: $y = \sin(x)$ is odd. $\sin(-x) = -\sin(x)$



Prop: If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.





u - substitution

Comment: Right now, the only good way we have to take definite integrals is with FTC, which requires antiderivatives. The only way we know to find those is to recognize them on sight. We'll develop four techniques to expand the kinds of functions we can integrate. u-sub is the first of these.

Thm (u-substitution) $\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C.$

This is just the chain rule backward.

Here's a better version:

If you have $\int f(g(x))\, g'(x)\, dx$, then

1. Set $u = g(x)$.

2. Write $\frac{du}{dx} = g'(x)$ as $du = g'(x)\, dx$

   looks weird, but it's fine, I promise

3. Rewrite the integral as $\int f(u)\, du$ and integrate to get $F(u) + C$

4. Substitute $u = g(x)$ to get $F(g(x)) + C$

$$\Rightarrow \int f(g(x))\, g'(x)\, dx = F(g(x)) + C$$

Comment: Use u-sub when you have a composition of functions, like when you'd use the chain rule.