Analysis Notes

Cruz Godar

Math 412, 413, and 414
Professor Hamilton
Cal Poly, Fall 2017–Spring 2018

I — A Construction of $\mathbb{R}$

Definition 1.1: A **Dedekind cut** is a set $A \subseteq \mathbb{Q}$ such that

1. $A \neq \varnothing$ and $A \neq \mathbb{Q}$.
2. If $r \in A$, then $q \in A$ for all $q \in \mathbb{Q}$ with $q < r$.
3. A does not have a maximum element — that is, if $r \in A$, then $r < s$ for some $s \in A$.

Definition 1.2: The **real numbers**, $\mathbb{R}$, are the set of all Dedekind cuts.

Definition 1.3: Let $A, B \in \mathbb{R}$. A is **less than** B, written $A \leq B$, if $A \subseteq B$.

Proposition 1.4: $\leq$ is a total order on $\mathbb{R}$.

Proof: Clearly, $\leq$ is reflexive, antisymmetric, and transitive, since $\subseteq$ is. Thus $\leq$ is a partial order on $\mathbb{R}$. To show that it is a total order, suppose $A \nleq B$. Then $A \nsubseteq B$, so there is an $a \in A$ with $a \notin B$. Let $b \in B$. Since $a \notin B$, $b \in B$, and B is a cut, $a > b$ (where $\leq$ here is the standard order on $\mathbb{Q}$), and since A is a cut, $b \in A$. Thus $B \subseteq A$, so $B \leq A$.

Definition 1.5: Let $A, B \in \mathbb{R}$. The **sum** of A and B is $A + B$ $\{a + b \mid a \in A, b \in B\}$.

Theorem 1.6: $\mathbb{R}$ is closed under addition.

Proof: Let $A, B \in \mathbb{R}$. To show $A + B \in \mathbb{R}$, we need to verify each of the three Dedekind cut axioms.

(1) Since $A \neq \varnothing$ and $B \neq \varnothing$, $A + B \neq \varnothing$. Since $A \neq \mathbb{Q}$ and $B \neq \mathbb{Q}$, there is an $s \in \mathbb{Q} \setminus A$ and a $t \in \mathbb{Q} \setminus B$, and since A and B are cuts, $a < s$ and $b < t$ for all $a \in A$ and $b \in B$. Thus $a + b < s + t$ for all $a \in A$ and $b \in B$, or equivalently, for all $a + b \in A + B$. Thus $s + t \notin A + B$, so $A + B \neq \mathbb{Q}$.

(2) Let $a + b \in A + B$ and let $s \in \mathbb{Q}$ such that $s < a + b$. Then $s - b < a$, so $s - b \in A$, since A is a cut. Thus $(s - b) + b = s \in A + B$.

(3) Let $a + b \in A + B$. Since A and B are cuts, there is an $s \in A$ and a $t \in B$ such that $a < s$ and $b < t$. Then $s + t \in A + B$ and $a + b < s + t$.

Proposition 1.7: Let $A, B, C \in \mathbb{R}$. Then $A + B = B + A$ and $(A + B) + C = A + (B + C)$.

Definition 1.8: The real numbers **zero** and **one** are defined as $\mathbf{0} = \{q \in \mathbb{Q} \mid q < 0\}$ and $\mathbf{1} = \{q \in \mathbb{Q} \mid q < 1\}$.

Proposition 1.9: For all $A \in \mathbb{R}$, $A + \mathbf{0} = A$.

Proof: $(\subseteq)$ Let $a + x \in A + \mathbf{0}$. Since $x < 0$, $a + x < a$, and since A is a cut, $a + x \in A$. Thus $A + \mathbf{0} \subseteq A$.

$(\supseteq)$ Let $a \in A$. Since A is a cut, there is an $s \in A$ such that $s > a$. Then $a - s < 0$, so $a - a \in \mathbf{0}$. Thus $a = s + (a - s) \in A + \mathbf{0}$, so $A \subset A + \mathbf{0}$.

Definition 1.10: Let $A \in \mathbb{R}$. The **additive inverse** of A is $-A = \{r \in \mathbb{Q} \mid r < -t \text{ for some } t \notin A\}$.

Proposition 1.11: Let $A \in \mathbb{R}$. Then $-A \in \mathbb{R}$.

Proposition 1.12: Let $A \in \mathbb{R}$. Then $A + (-A) = \mathbf{0}$.

Proof: $(\subseteq)$ Let $a + n \in A + (-A)$. Since $n \in -A$, there is a $t \notin A$ such that $n < -t$, and since $a \in A$ and $t \notin A$, $a < t < -n$, so $a + n < 0$. Thus $a + n \in \mathbf{0}$, so $A + (-A) \subseteq \mathbf{0}$.

$(\supseteq)$ Let $x \in \mathbf{0}$, let $\varepsilon = \frac{|x|}{2} = -\frac{x}{2}$, and let $t \in \mathbb{Q}$ such that $t \notin A$ but $t - \varepsilon \in A$. Since $t \notin A$, $-(t + \varepsilon) \in -A$, since $t < -(-(t + \varepsilon))$ and therefore $-(t + \varepsilon) < -t$. Then $x = -2\varepsilon = -(t + \varepsilon) + (t - \varepsilon) \in A + (-A)$, so $\mathbf{0} \subseteq A + (-A)$.

Definition 1.13: Let $A, B \in \mathbb{R}$. If $A \geq \mathbf{0}$ and $B \geq \mathbf{0}$, then the **product** of A and B is

$$AB = \{ab \mid a \in A, b \in B, a \geq 0, b \geq 0\} \cup \mathbf{0}.$$

If $A \geq \mathbf{0}$ and $B < \mathbf{0}$, then $AB = -(A(-B))$, if $A < \mathbf{0}$ and $B \geq \mathbf{0}$, then $AB = -((-A)B)$, and if $A < \mathbf{0}$ and $B < \mathbf{0}$, then $AB = (-A)(-B)$.

Theorem 1.14: Let $A, B, C \in \mathbb{R}$. Then $AB \in \mathbb{R}$, $AB = BA$, $(AB)C = A(BC)$, $\mathbf{1}A = A$, and if $A \neq \mathbf{0}$, then there is an $A^{-1} \in \mathbb{R}$ with $AA^{-1} = \mathbf{1}$.

Definition 1.15: A set $U \subseteq \mathbb{R}$ is **bounded above** if there is a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in U$. We call B an **upper bound** for U, and define **bounded below** and **lower bound** similarly.

Definition 1.16: Let $U \in \mathbb{R}$ such that $U \neq \varnothing$ and U is bounded above. We define $S(U) = \bigcup_{A \in U} A$.

Theorem 1.17: Let $U \subset \mathbb{R}$ be nonempty and bounded above. Then $S(U)$ is a cut.

Proof: (1) Since $U \neq \varnothing$ and $U \subseteq S(U)$, $S(U) \neq \varnothing$. Since U is bounded above, there is a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in U$. Then $A \subseteq B$ for all $A \in U$, so $S(U) = \bigcup A \subseteq B$. Since $B \neq \mathbb{Q}$, $S(U) \neq \mathbb{Q}$.

(2) Let $a \in S(U)$ and $q < a$. Then $a \in A$ for some $A \in U$, and since A is a cut and $q < a$, $q \in A \subseteq S(U)$.

(3) Let $a \in S(U)$. Then $a \in A$ for some $A \in U$, and since A is a cut, there is a $q \in A \subseteq S(U)$ with $a < q$.

Proposition 1.18: Let $U \subseteq \mathbb{R}$ be nonempty and bounded above. Then $S(U)$ is an upper bound for U.

Proof: For all $A \in U$, $A \subseteq \bigcup A = S(U)$, so $A \leq S(U)$.

Definition 1.19: A set $U \subseteq \mathbb{R}$ has a **supremum**, or least upper bound, if there is a $B \in \mathbb{R}$ such that B is an upper bound for U and $B \leq C$ for any upper bound C for U. We define the **infimum**, or greatest lower bound, similarly, and write $\sup U$ and $\inf U$ for the supremum and infimum.

Proposition 1.20: Let $U \subseteq \mathbb{R}$ be nonempty and bounded above. Then $S(U) = \sup U$.

Proof: Let C be an upper bound for U. Then $A \leq C$ for all $A \in U$, so $A \subseteq C$ for all $A \in U$. Then $S = \bigcup A \subset C$, so $S \leq C$.

Theorem 1.21: (The Completeness of the Reals) Every nonempty, bounded above subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R}$.

II — The Reals

Proposition 2.1: Let $A \subseteq \mathbb{R}$. If $\sup A \in A$, then $\sup A = \max A$.

Proposition 2.2: If $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$, then $\sup A \leq \sup B$.

Proof: Since $A \subseteq B$, $a \in B$ for all $a \in A$, and so since $\sup B \geq b$ for all $b \in B$, $\sup B \geq a$ for all $a \in A$. Then $\sup B$ is an upper bound for A, so $\sup A \leq \sup B$.

Theorem 2.3: Let s be an upper bound for $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for all $\varepsilon > 0$, there is an $a \in A$ with $s - \varepsilon < a$.

Proof: ($\Rightarrow$) Assume $s = \sup A$ and let $\varepsilon > 0$. Since $s - \varepsilon < s = \sup A$, $s - \varepsilon$ cannot be an upper bound for A. Thus there must be an $a \in A$ with $a > s - \varepsilon$.

Assume s is an upper bound for A and that for every $\varepsilon > 0$, there is an $a \in A$ such that $a > s - \varepsilon$. Let b be an upper bound for A and suppose $b < s$. Let $\varepsilon = \frac{s-b}{2}$. Since $a < b$ for all $a \in A$, there is no $a \in A$ such that $a > s - \varepsilon$, since $s - \varepsilon$ is the midpoint of s and b, and is therefore greater than b. $\lightning$

Theorem 2.4: (The Nested Interval Theorem) For each $n \in \mathbb{N}$, let $I_n = [a_n, b_n]$ be an interval such that $I_n \subseteq I_{n-1}$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof: Let $A = \{a_n \mid n \in \mathbb{N}\}$. Since A is nonempty and bounded above (by b_1, for instance), A has a least upper bound. In fact, each b_i is an upper bound for A, since otherwise the intervals would not be nested.

Let $s = \sup A$ and let $n \in \mathbb{N}$. Since s is an upper bound for A, $s \geq a_n$, and since b_n is an upper bound for A, $s \leq b_n$. Thus $s \in I_n$ for all $n \in \mathbb{N}$, so $s \in \bigcap I_n$.

Theorem 2.5: (The Well-Ordering Principle) Every nonempty subset of $\mathbb{N}$ has a minimum element.

Proposition 2.6: (The Archimedean Property) Let $x \in \mathbb{R}$. Then there is a $y \in \mathbb{N}$ with $y > x$.

Corollary 2.6.1: Let $x \in \mathbb{R}^+$. Then there is a $y \in \mathbb{N}$ with $\frac{1}{y} < x$.

Theorem 2.7: (The Density of $\mathbb{Q}$ in $\mathbb{R}$) Let $a, b \in \mathbb{R}$ with $a < b$. Then there is a $q \in \mathbb{Q}$ with $a < q < b$.

Proof: First, suppose $a \geq 0$. By the Archimedean property, let $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. Let m be the smallest natural greater than na. Then $m - 1 \leq na < m$, so $m \leq na + 1 < m + 1$. Since $na < m$, $a < \frac{m}{n}$, and since $m \leq na + 1$ and $\frac{1}{n} < b - a$, $m < n\left(b - \frac{1}{n}\right) + 1 = nb$. Thus $\frac{m}{n} < b$, and so $a < \frac{m}{n} < b$.

If $a < 0$ and $b > 0$, then $a < \frac{0}{1} < b$, and if $a < 0$ and $b \leq 0$, then since $-b < -a$ (and $-b, -a > 0$), there is a $q \in \mathbb{Q}$ with $-b < q < -a$, so $a < -q < b$.

Theorem 2.8: There is an $\alpha \in \mathbb{R}$ with $\alpha^2 = 2$.

Proof: Let $T = \{t \in \mathbb{R} \mid t^2 < 2\}$, which is is nonempty and bounded above, and let $\alpha = \sup T$. Suppose $\alpha < 2$. By the Archimedean principle, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{2-\alpha^2}{2\alpha+1}$, or equivalently, $\frac{2\alpha+1}{n} < 2 - \alpha^2$. Then

$$\begin{aligned}
\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\
&< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\
&= \alpha^2 + \frac{2\alpha + 1}{n} \\
&< \alpha^2 + (2 - \alpha^2) \\
&= 2,
\end{aligned}$$

so $\alpha + \frac{1}{n} \in T$, but $\alpha + \frac{1}{n} > \alpha = \sup T$. ↯ Similarly, $a > 2$ gives a contradiction.