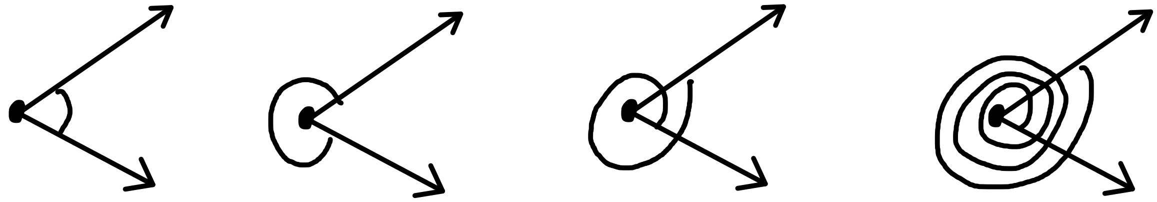
Chapter 2: Geometry and basic Trigonometry

Geometry Review

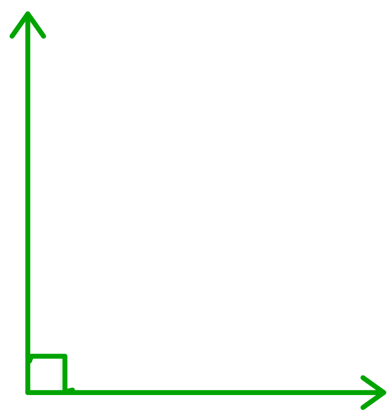
Def: A ray is a line with one endpoint.

Def: An angle is the object formed by two rays that share their endpoints. Always label the arc.

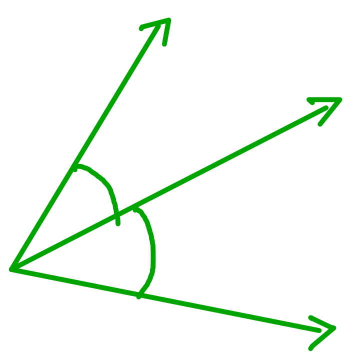
Ex:



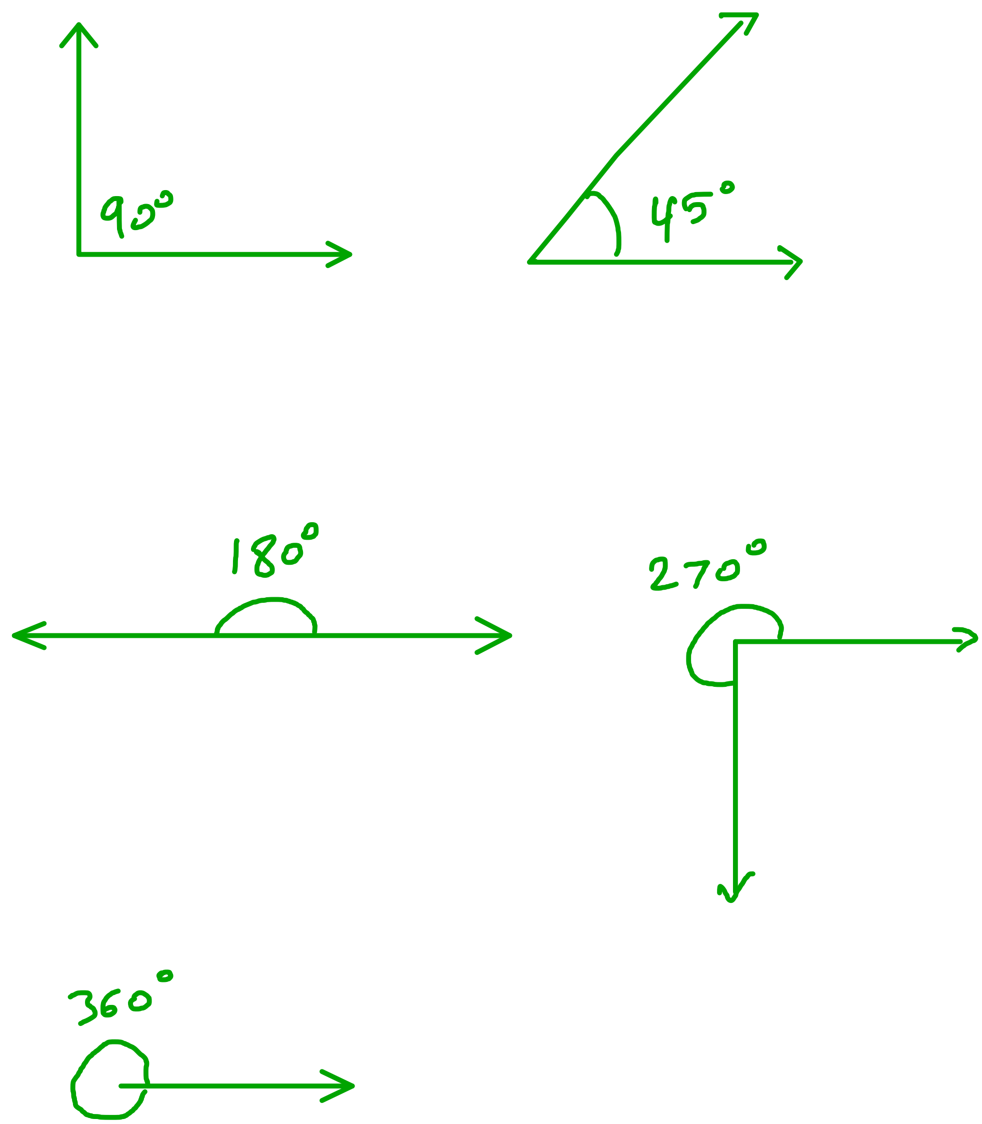
Def: A *right angle* is the angle formed by two perpendicular rays.



Def: Two angles are *adjacent* if they share a ray.



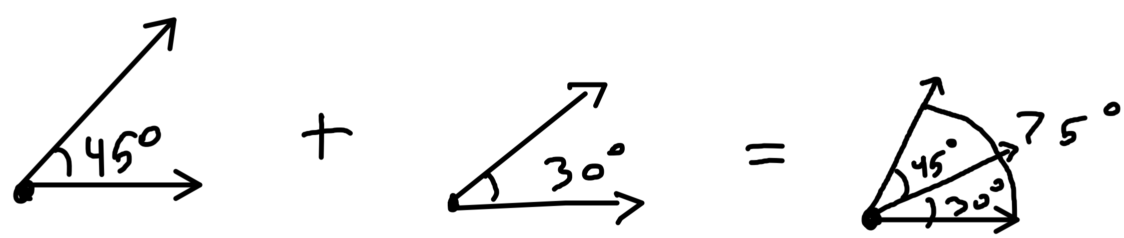
Def: A *degree* is $1/90^{th}$ of a right angle.



Def: An angle is *acute* if it is between $0°$ and $90°$. It's *obtuse* if it's between $90°$ and $180°$. It's *reflex* if it's between $180°$ and

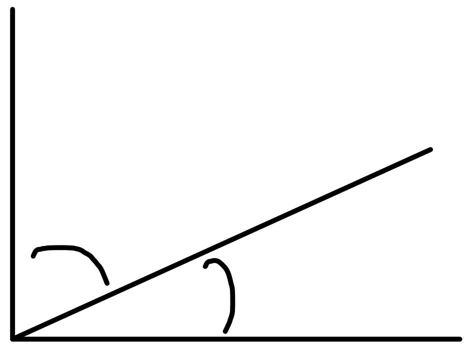
360°. All of these ranges are exclusive (e.g. A 90° angle is neither acute nor obtuse).

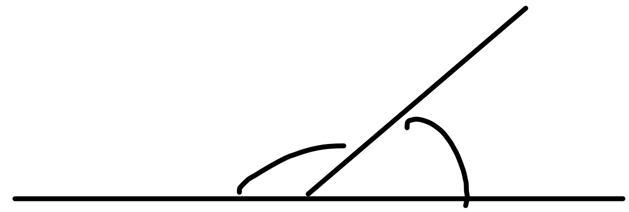
Def: The sum of two angles is the angle formed by making them adjacent.

Ex: 

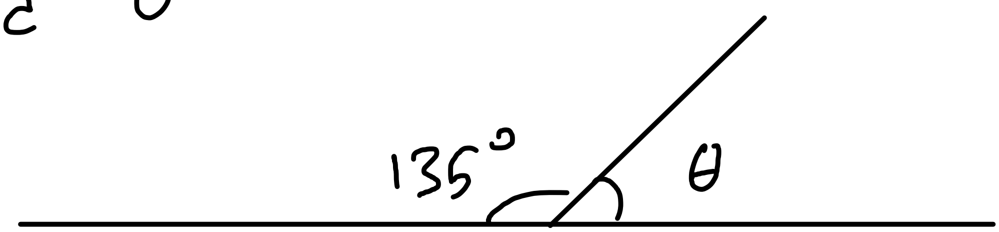
Def: Two angles are complementary if they sum to 90°. They are supplementary if they sum to 180°.

Ex:





Ex: find θ



Well, $135°$ and θ are supplementary

so $135° + \theta = 180°$. Thus $\theta = 45°$.

<u>Comment</u>: Angles are typically denoted with greek letters. Some common ones are:

θ — theta

ψ — psi

φ — phi

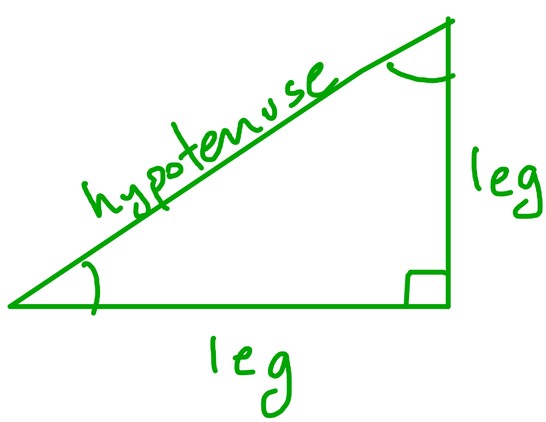
α — alpha

β — beta

γ — gamma

a $\quad \alpha$

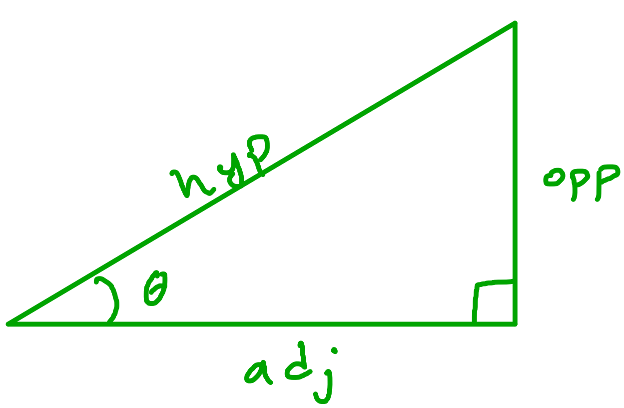
<u>Def</u>: A right triangle is a triangle with one right angle.

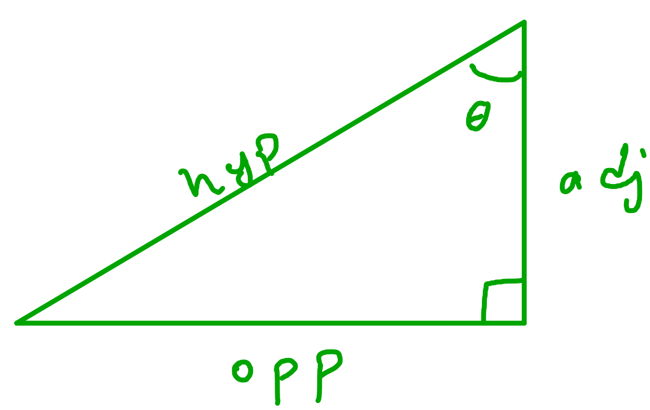


The side not touching the right angle is called the

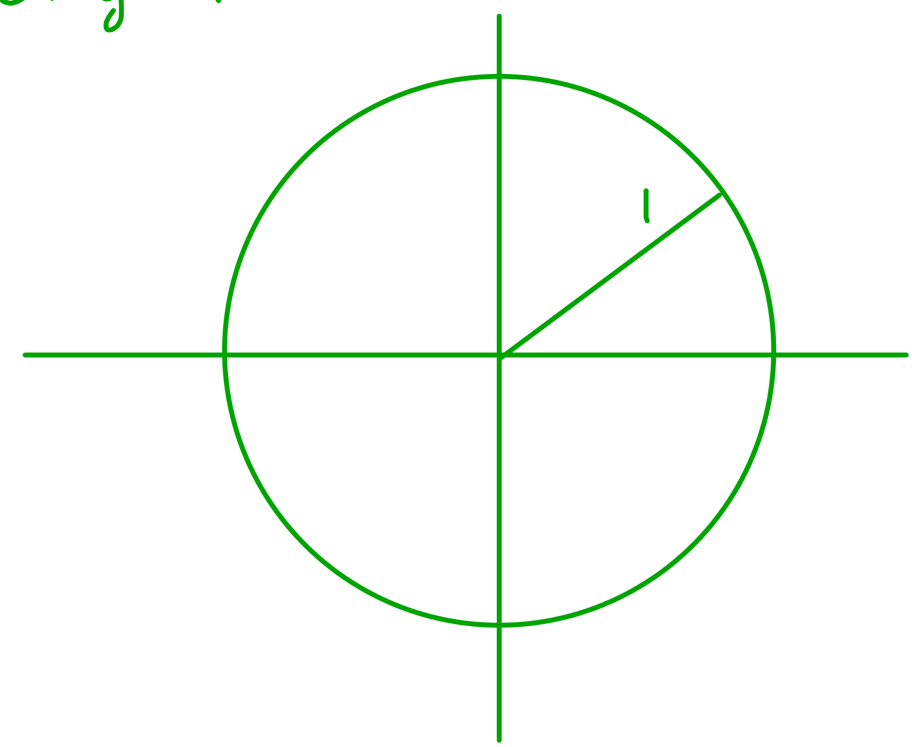
hypotenuse. The other two sides are called **legs**.

Fix an angle θ in a right triangle that is not the right angle. The **adjacent side** to θ is the leg of the triangle that touches θ. The **opposite side** to θ is the leg that does not touch θ.



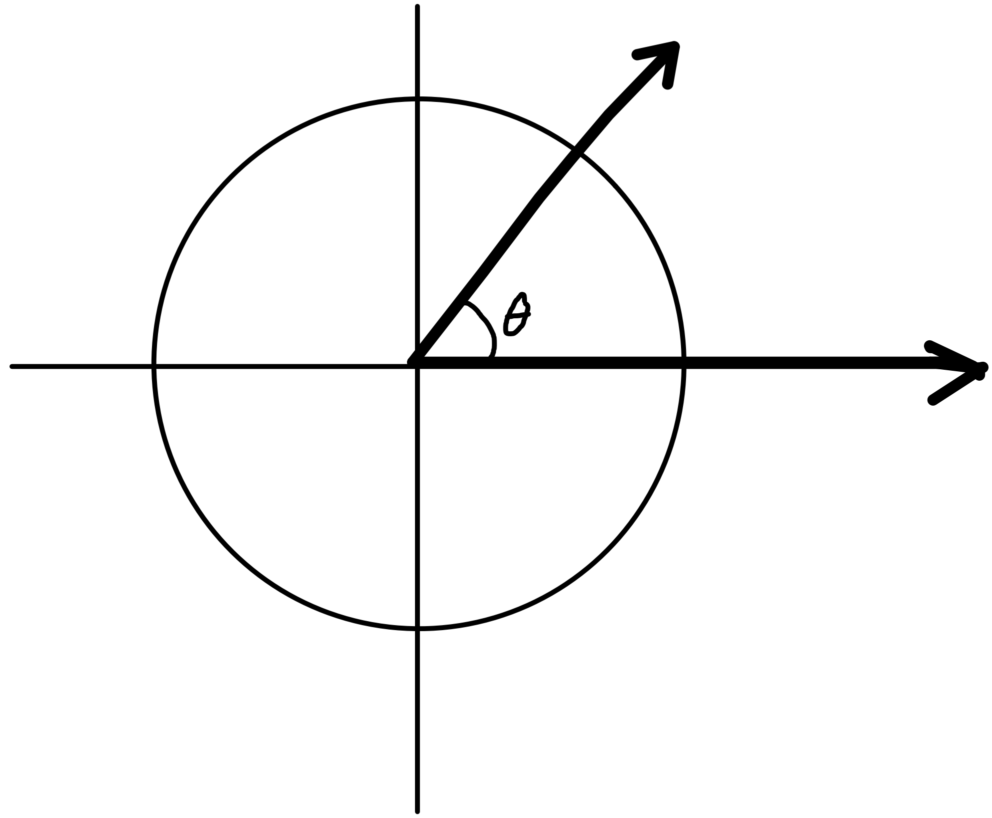


Def: The <u>unit circle</u> is the circle of radius 1 centered at the origin.



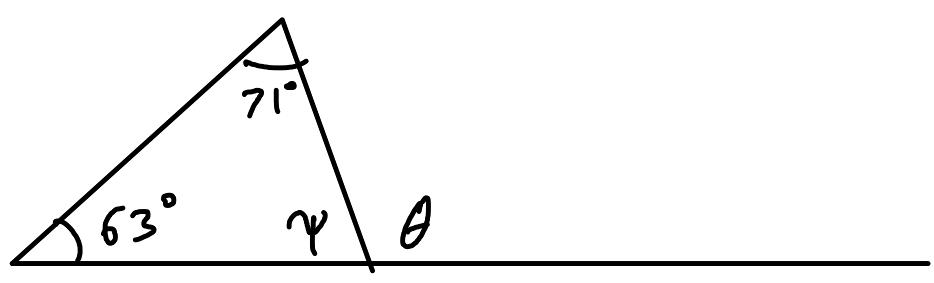
Def: An <u>angle on the unit circle</u> is an angle with one ray being the positive x-axis.

Ex:



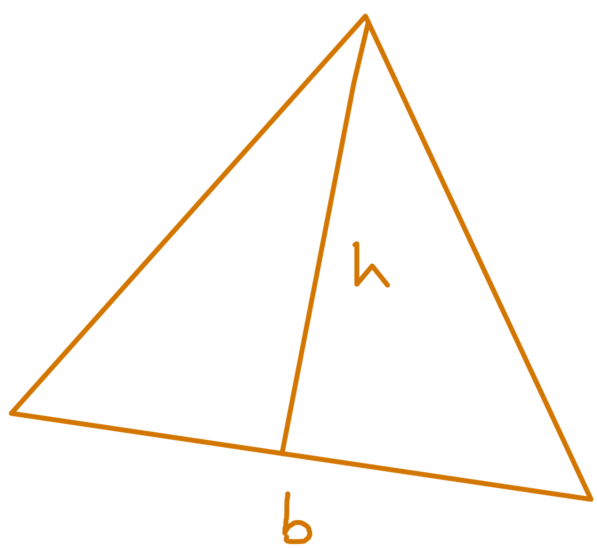
Prop: The angles of any triangle sum to $180°$.

Ex: Find θ:



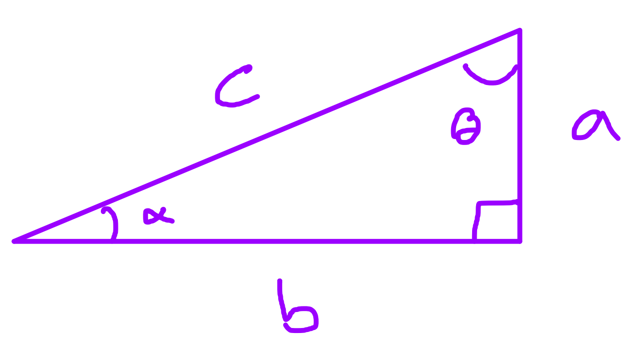
Well, the third angle in the triangle, ψ, has $63° + 71° + \psi = 180°$, so $\psi = 46°$.
But $\theta + \psi = 180°$, so $\theta = 134°$.

Prop: Let b be one side of a triangle and h the shortest distance from b to the vertex opposite b. Then the area of the triangle is $\frac{1}{2}bh$.

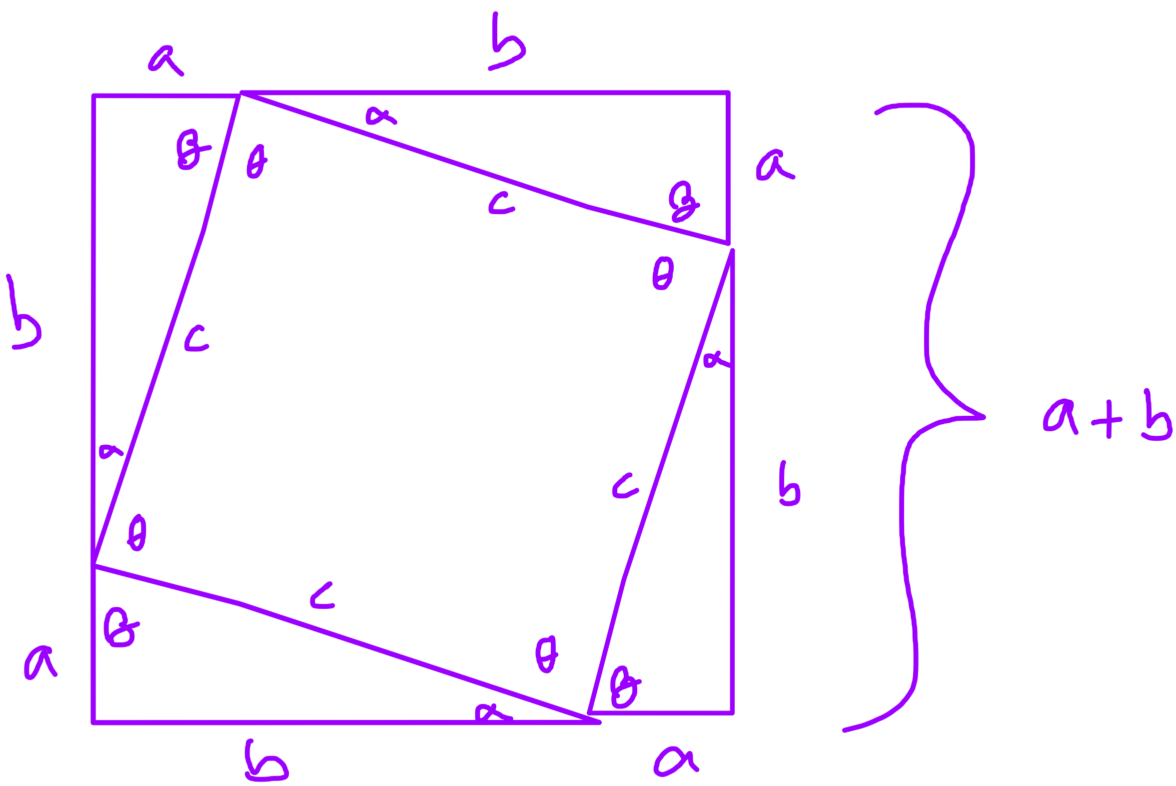


Theorem: (Pythagorean) In a right triangle with legs a and b and hypotenuse c, $a^2+b^2=c^2$.

Proof:



Arrange four copies of this triangle in a square.



Now $\alpha + \beta + \theta = 180°$, since they're supplementary, but $\alpha + \beta + 90° = 180°$, since those angles form a triangle. Thus $\theta = 90°$, s the inner shape is a square. Now compare the areas. The area of the bigger square is $(a+b)^2$.

It's also $4 \times$ (area of triangle) + (area of the smaller square) = $4(\frac{1}{2}ab) + c^2$. Therefore,

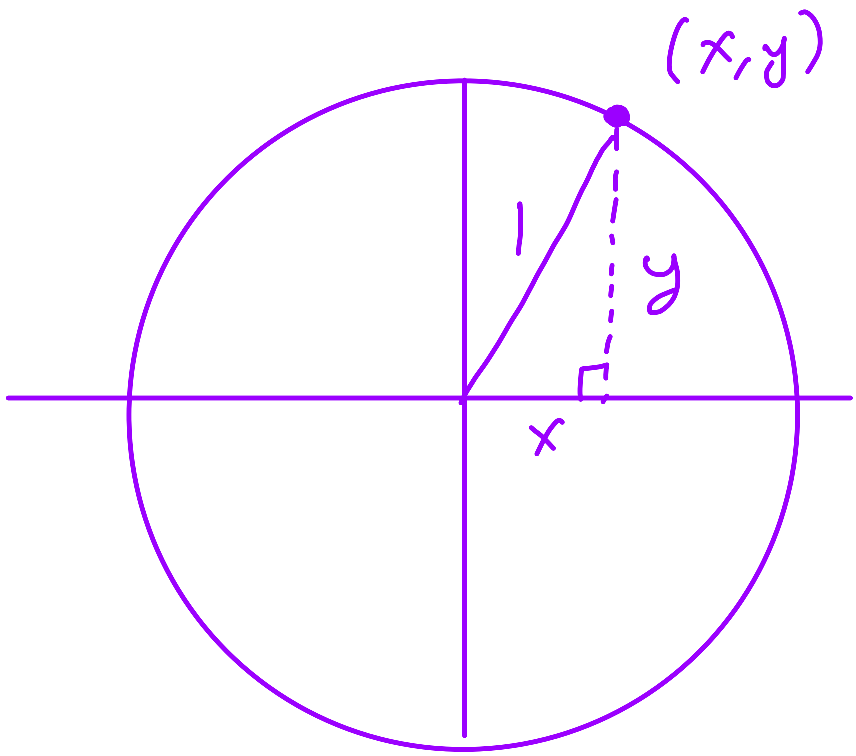
$$(a+b)^2 = 4\left(\frac{1}{2}ab\right) + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$a^2 + b^2 = c^2.$$

Corollary: A point (x,y) is on the unit circle if and only if $x^2+y^2=1$.

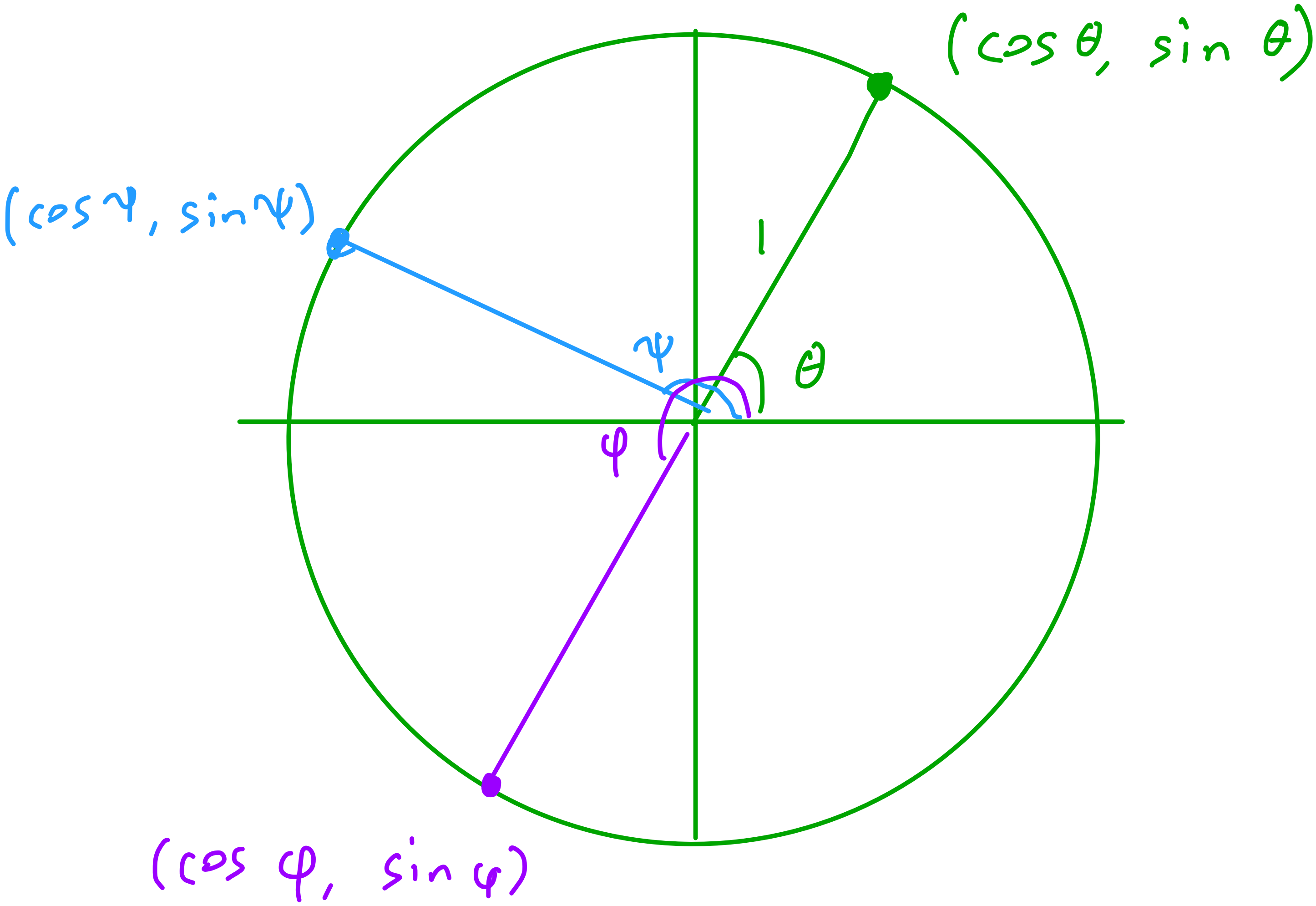
Proof:



Sine and Cosine

Def: Let θ be an angle on the unit circle (so one measured counter-clockwise from the positive x-axis).

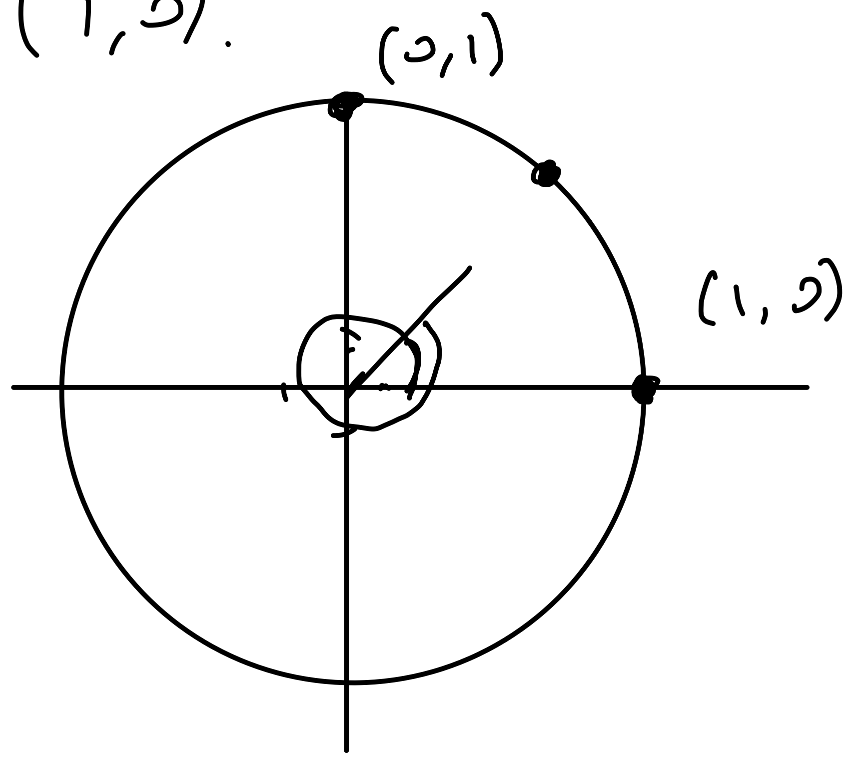
The cosine and sine functions are defined so that $\cos\theta = x$ and $\sin\theta = y$, where (x,y) is the point on the unit circle with angle θ.



Comment: Sine and cosine are probably the first functions you've seen that cannot be easily approximated.

Ex: $\cos 0^\circ = 1$ and $\sin 0^\circ = 0$,

since the point on the unit circle with angle 0° is $(1, 0)$.



$\cos 90^\circ = 0$ $\quad$ $\sin 90^\circ = 1$

$\cos 180^\circ = -1$ $\quad$ $\sin 180^\circ = 0$

$\cos 270^\circ = 0$ $\quad$ $\sin 270^\circ = -1$

$\cos 360^\circ = 1$ $\quad$ $\sin 360^\circ = 0$

Comment: calculators have two modes: degree mode and radian mode. Type in cos 90. If you get 0, you're in degree mode. If you don't, you're in radian mode. Know how to switch between the two.

Def: We write $\cos^k \theta$ to mean $(\cos\theta)^k$. We do this to avoid confusion with $\cos(\theta^k)$. Similarly for $\sin\theta$.

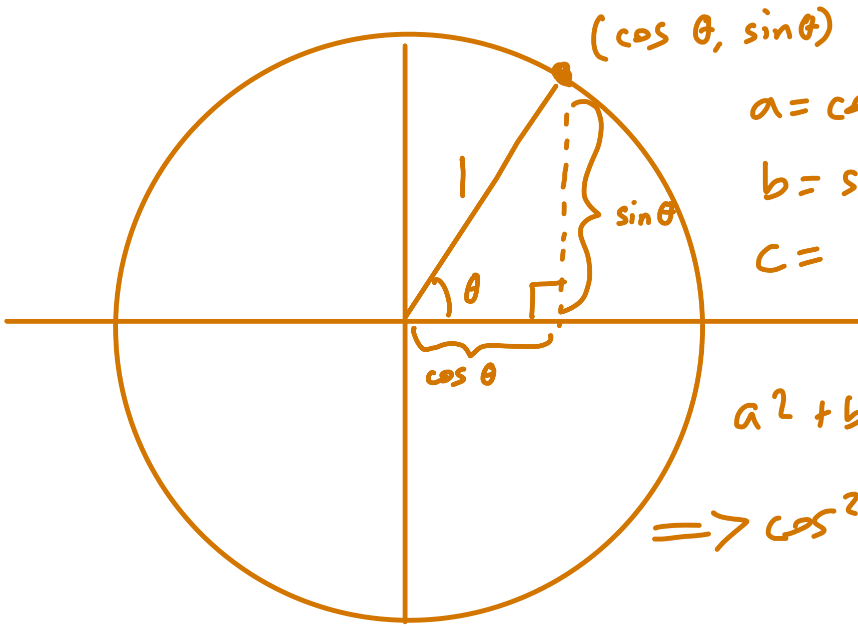
Prop: ① For all angles θ,

$-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$

② cos and sin are periodic with period $360°$, midline 0, and amplitude 1.

③ For any angle θ,

$\cos^2\theta + \sin^2\theta = 1.$



$a = \cos\theta$

$b = \sin\theta$

$c = 1$

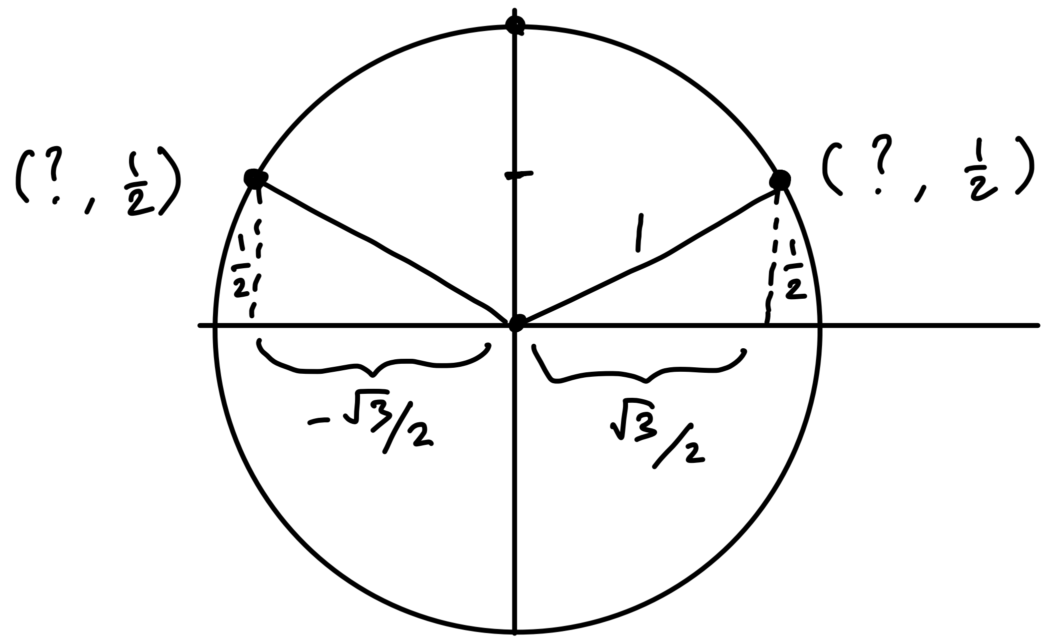
$a^2 + b^2 = c^2$

$\Rightarrow \cos^2\theta + \sin^2\theta = 1.$

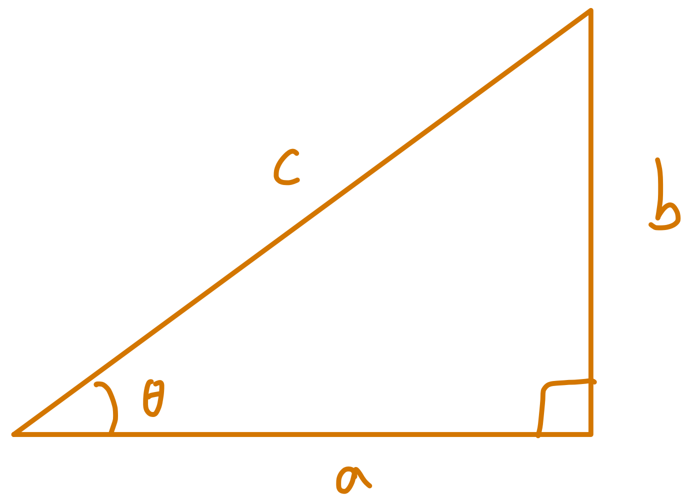
Ex: $\sin\theta = \frac{1}{2}$. What could $\cos\theta$ be?

We know $\cos^2\theta + \sin^2\theta = 1$, so $\cos^2\theta + \left(\frac{1}{2}\right)^2 = 1$. Thus $\cos^2\theta = 1 - \frac{1}{4} = \frac{3}{4}$, and so

$$\cos\theta = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}.$$

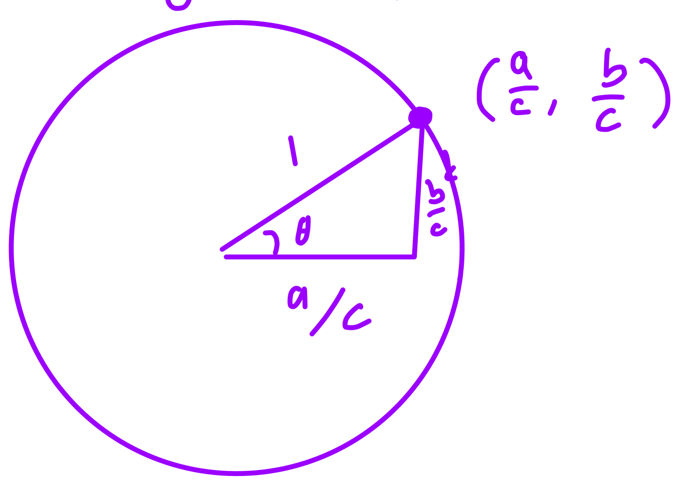


Theorem: Consider a right triangle as follows:



Then $\cos\theta = \frac{a}{c}$ and $\sin\theta = \frac{b}{c}$.

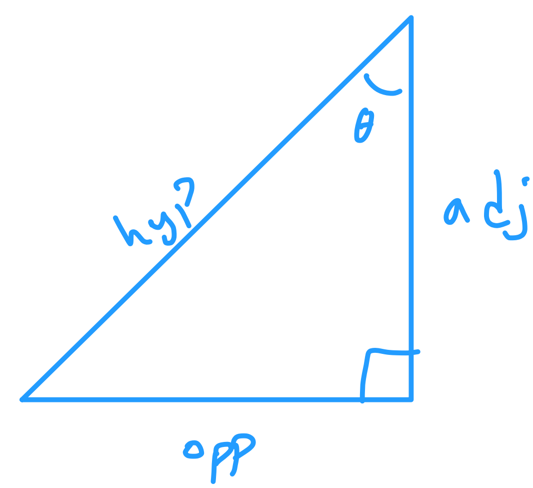
Proof: Similar triangles. Scale down by a factor of $1/c$:



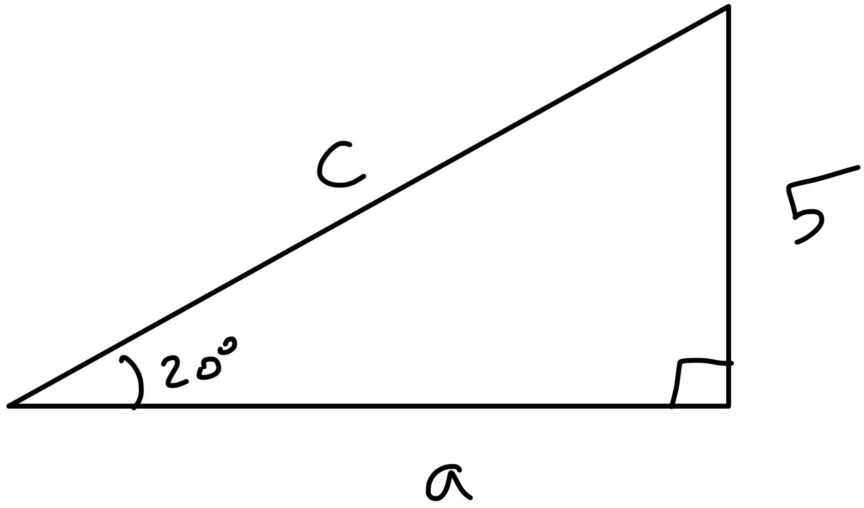
Now embed in a circle. $\square$

Comment: In a right triangle with a specified non-right angle θ, if opp is the side opposite θ and adj is the side adjacent θ, and hyp is the hypotenuse, then

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \text{and} \quad \sin\theta = \frac{\text{opp}}{\text{hyp}}.$$



Ex: Find a and c:



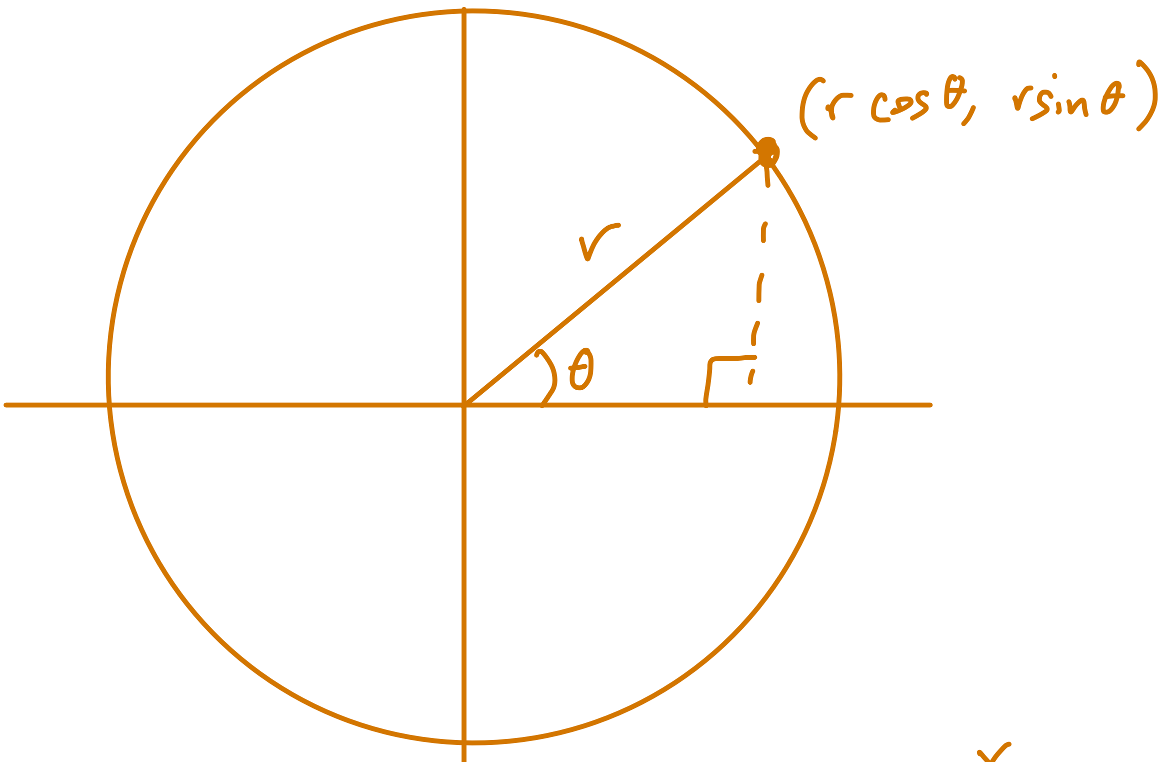
We know that $\cos 20^\circ = \frac{a}{c}$

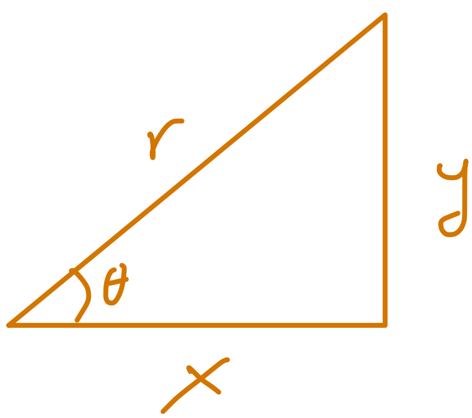
and $\sin 20^\circ = \frac{5}{c}$. But we can use a calculator to find $\sin 20^\circ$ and $\cos 20^\circ$: $\sin 20^\circ = .342$ and $\cos 20^\circ = .94$. Therefore,

$\frac{a}{c} = .94$ and $\frac{5}{c} = .342$ Then

$c = 5/.342 = 14.62$ and $a = .94 \cdot c = 13.74$

Theorem: In a circle with radius r, a point with angle θ has coordinates $(r\cos\theta,\ r\sin\theta)$.



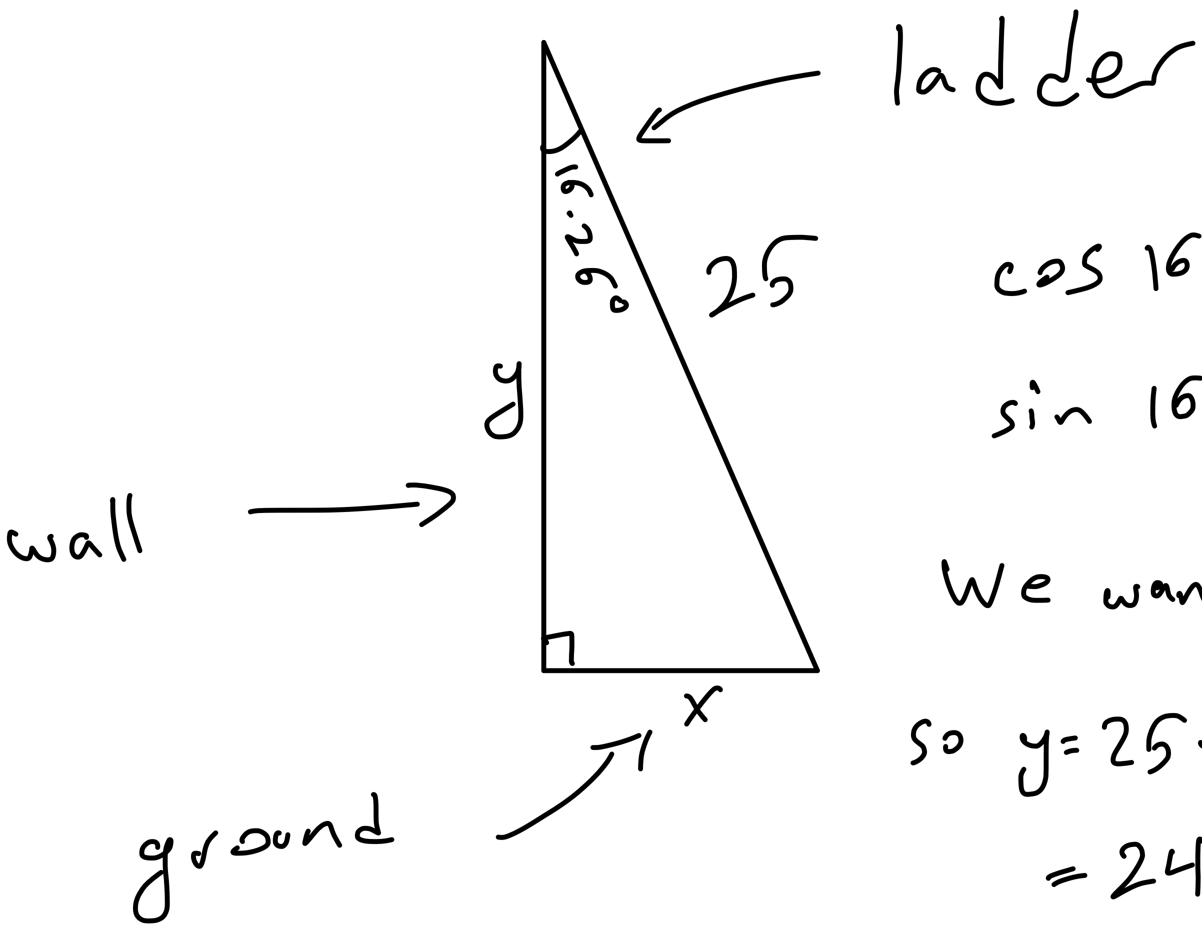


$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

$$\Rightarrow \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Ex: You lean a ladder up against a wall. The ladder is 25 feet long, and it makes an angle of 16.26° with the wall. How far up the wall did it reach?



$\cos 16.26° = \frac{y}{25}$

$\sin 16.26° = \frac{x}{25}$

We want y,

so $y = 25 \cdot \cos 16.26°$

$= 24.$

Special Angles

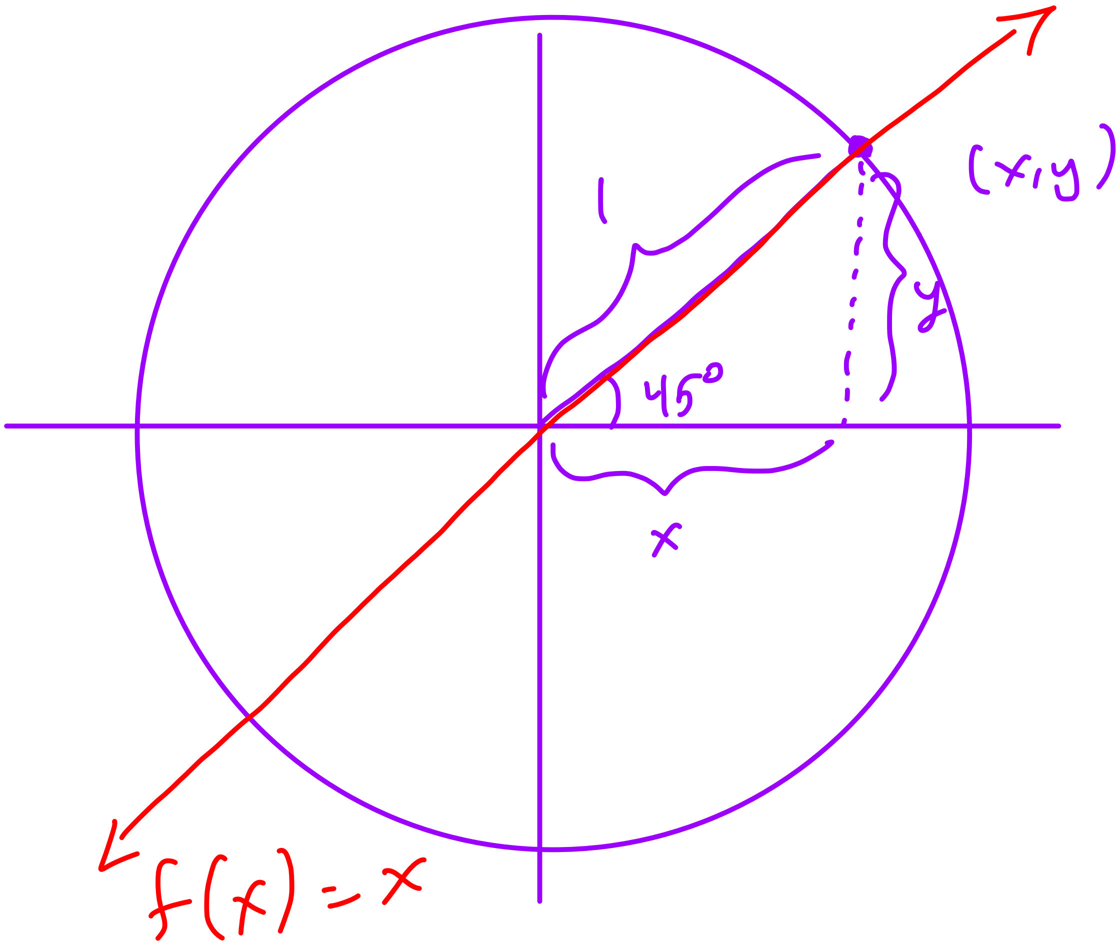
Comment:

We know values of $\sin\theta$ and $\cos\theta$ for $\theta = 0°, 90°, 180°, 270°, \ldots$.

This isn't enough! We would like more angles with exact values of sin and cos.

Theorem: $\cos 45° = \sin 45° = \frac{\sqrt{2}}{2}$.

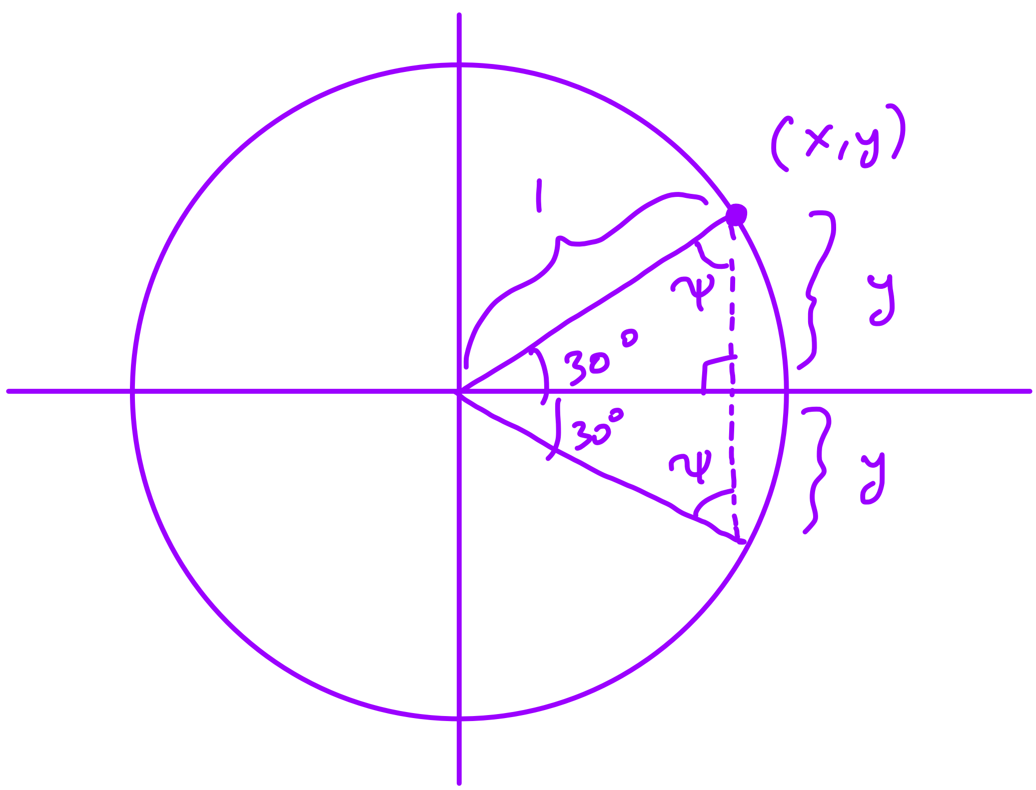
Proof: If $\cos 45° = x$ and $\sin 45° = y$, then we have:



This line is part of the line $f(x)=x$. Therefore, $\sin 45° = \cos 45°$, so $y=x$. Now $x^2+y^2=1$, so $x^2+x^2=1$, and so $2x^2=1$. Thus $x^2=\frac{1}{2}$, so $x=\pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\left(\frac{\sqrt{2}}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \pm\frac{\sqrt{2}}{2}$. $x \geq 0$ by inspection, so $x = \frac{\sqrt{2}}{2} = y$.

Theorem: $\cos 30^\circ = \frac{\sqrt{3}}{2}$ and $\sin 30^\circ = 1/2$.

Proof:

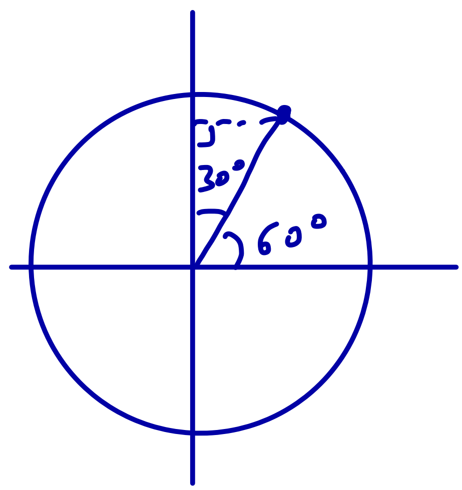


Notice: this triangle has angles 60°, ψ, and ψ. So $60^\circ + \psi + \psi = 180^\circ$, so $2\psi = 120^\circ$, and therefore $\psi = 60^\circ$.

This triangle must therefore be equilateral, so $1 = 2y$. Thus $y = \sin 30° = 1/2$.

Also, $x^2 + y^2 = 1$, so $x^2 + \left(\frac{1}{2}\right)^2 = 1$, and so $x^2 = \frac{3}{4}$. Thus $x = \cos 30° = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$, and again, $x \geq 0$, so $\cos 30° = \frac{\sqrt{3}}{2}$.

<u>Corollary:</u> $\cos 60° = 1/2$ and $\sin 60° = \sqrt{3}/2$.



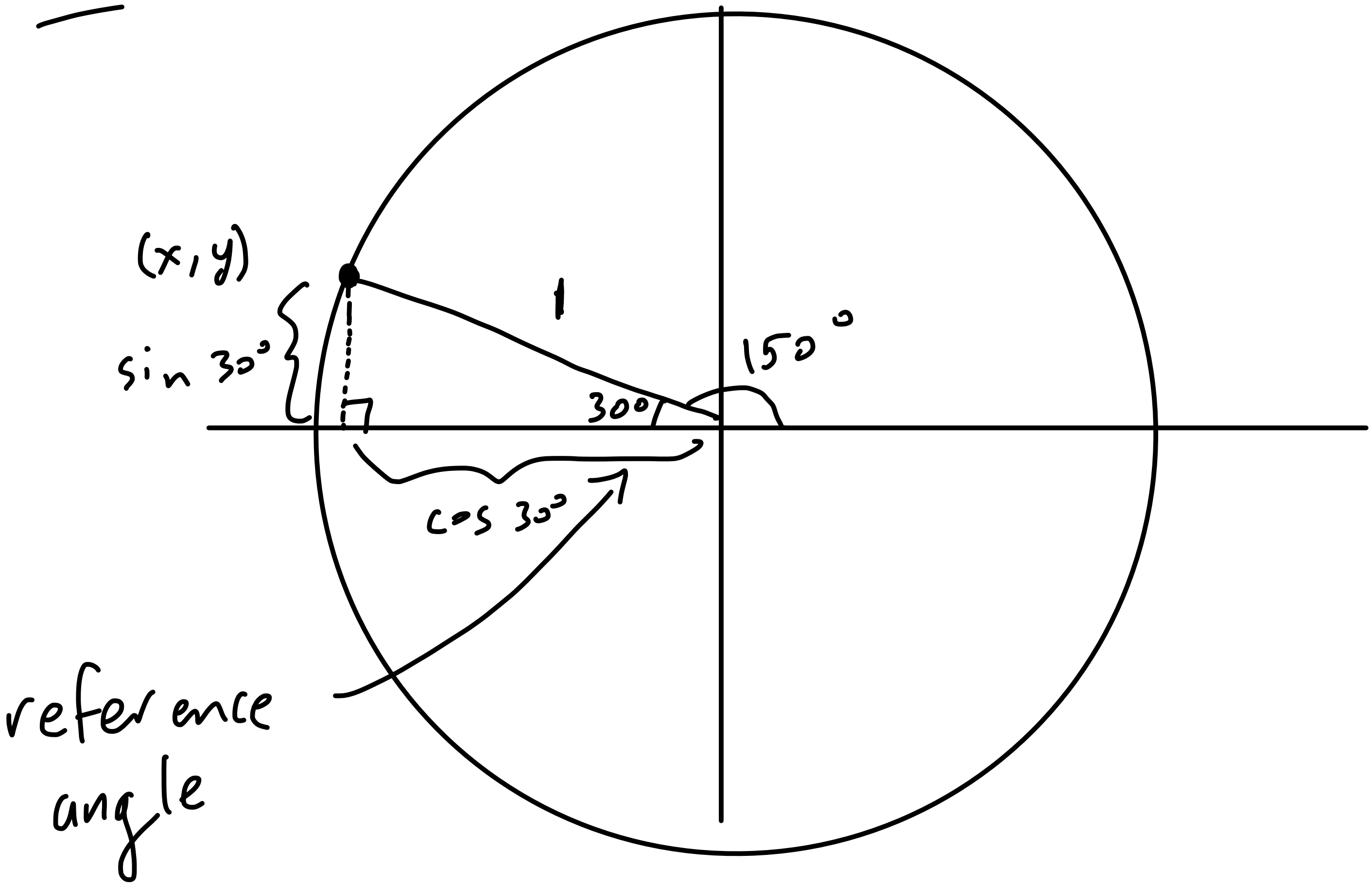
Comment: We have

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

This is something to memorize.

Method: (Finding trig functions for "nice" angles) Given a point (x,y) on the unit circle with angle θ, drop a perpendicular to the x-axis. This forms a right triangle, called the reference triangle. The angle on the x-axis is called the reference angle. Use sin and cos on the reference triangle to determine its side lengths.

<u>Ex</u>: Find $\cos 150^\circ$.

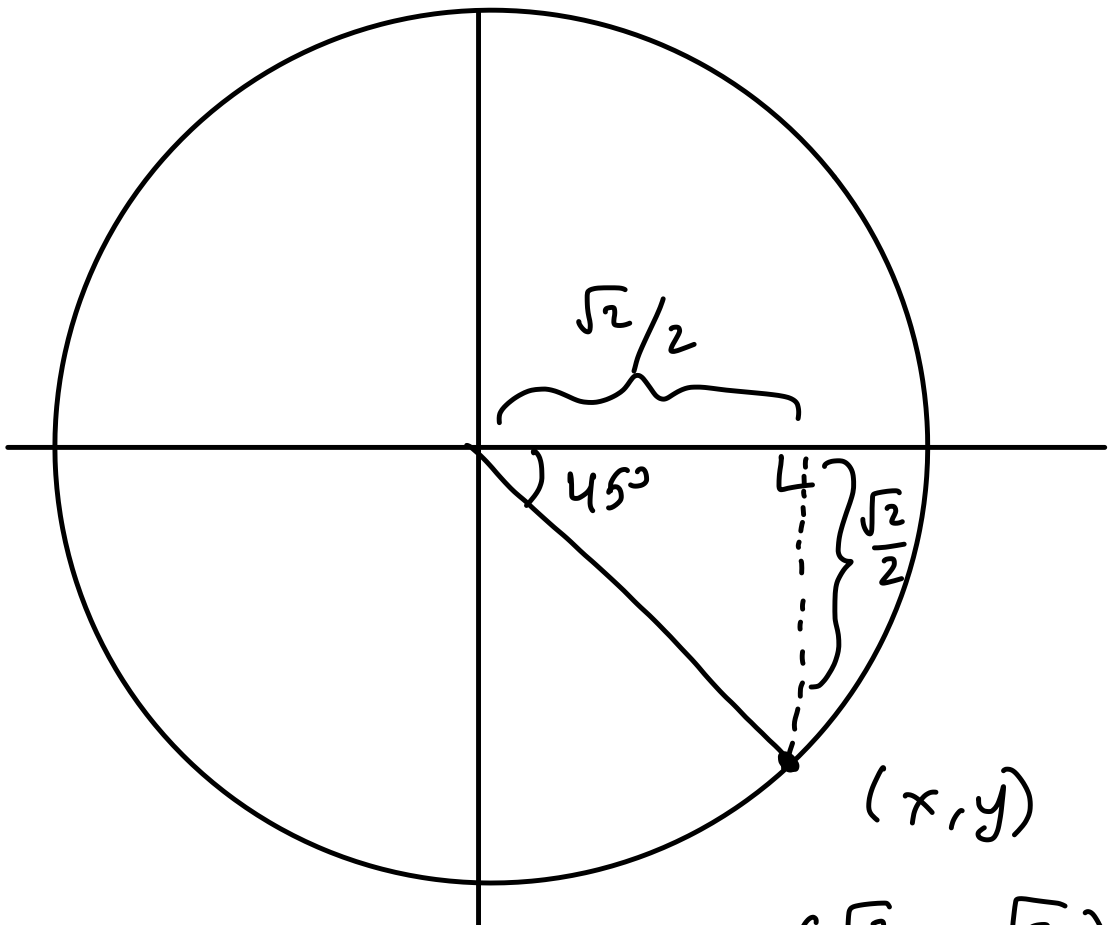


Now $\cos 30^\circ = \frac{\sqrt{3}}{2}$, so the bottom side of the triangle has length $\sqrt{3}/2$. But <u>x is negative</u>.

So we moved $\sqrt{3}/2$ in the negative direction. Therefore

$x = \cos 150^\circ = -\sqrt{3}/2$.

Ex: Find sin 315°.



$= \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$