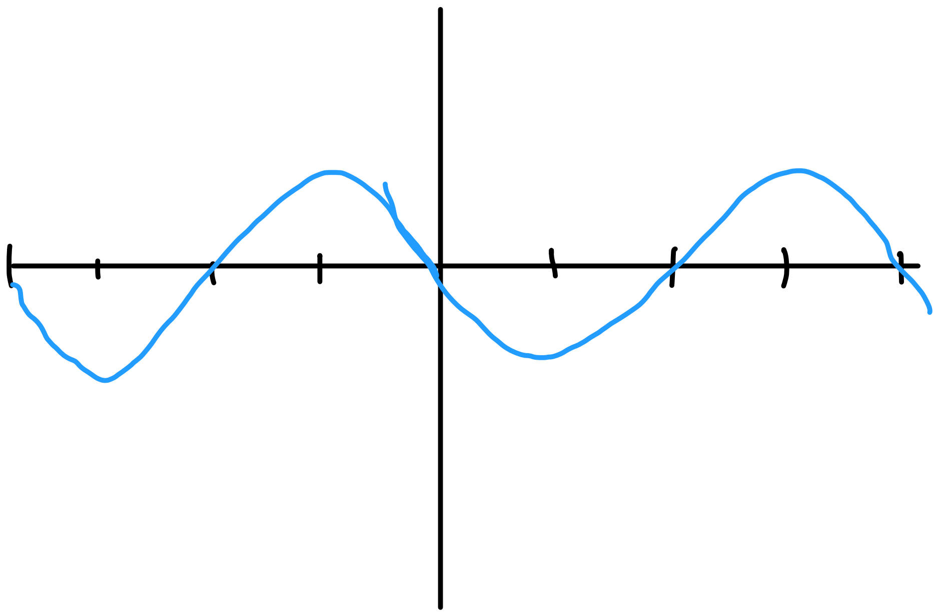
Sinusoidal Functions

Def: A function $f(x)$ is *sinusoidal* if $f(x) = A \sin(B(x-h)) + k$, where $A > 0$, $B > 0$, and h and k are real numbers. In other words, f is a transformation of $\sin x$.

Comment: We only allow $A > 0$ and $B > 0$ because $\sin x$ is periodic and symmetric about its midline, so a negative value of A or B is just a horizontal shift.

Ex: If $A=-1$, $B=1$, $h=0$, and $k=0$, then we have $-\sin x$.



Prop: Let $f(x) = A\sin(B(x-h))+k$.

1. Since the amplitude of $\sin x$ is 1 and f is a vertical stretch of $\sin x$ by a factor of A, the amplitude of f is A.

② The midline of f is k.

③ The period of f is $\frac{2\pi}{B}$.

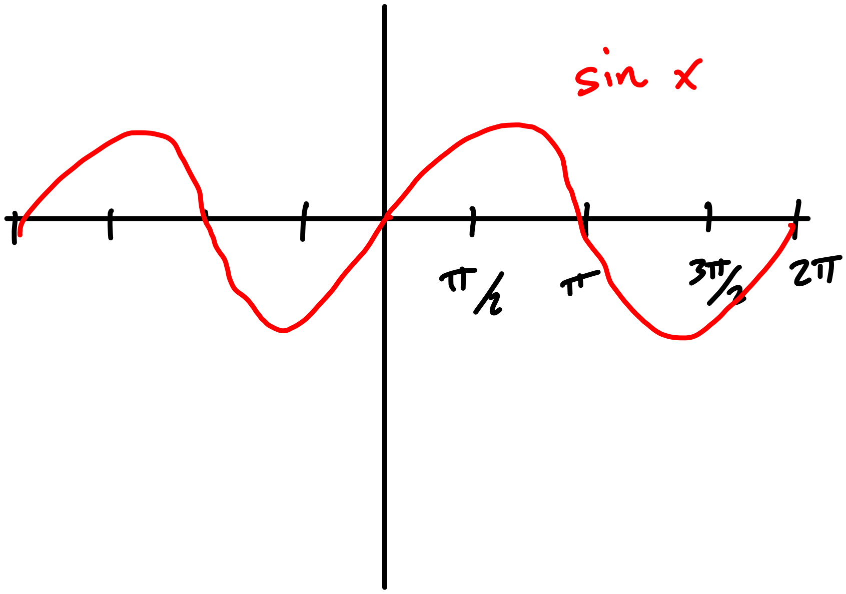
Ex: Graph $f(x) = 2\sin(2x-3)+1$.

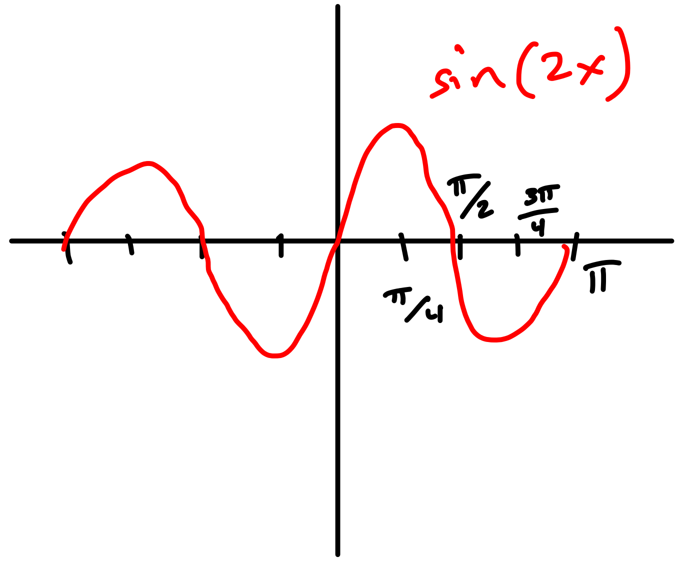
Get it in the right form:

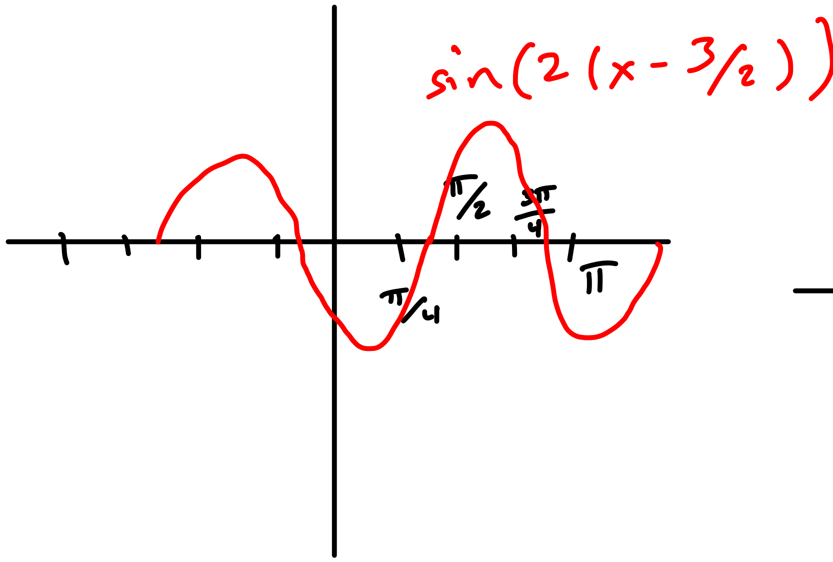
$$f(x) = 2\sin\left(2\left(x-\tfrac{3}{2}\right)\right)+1.$$

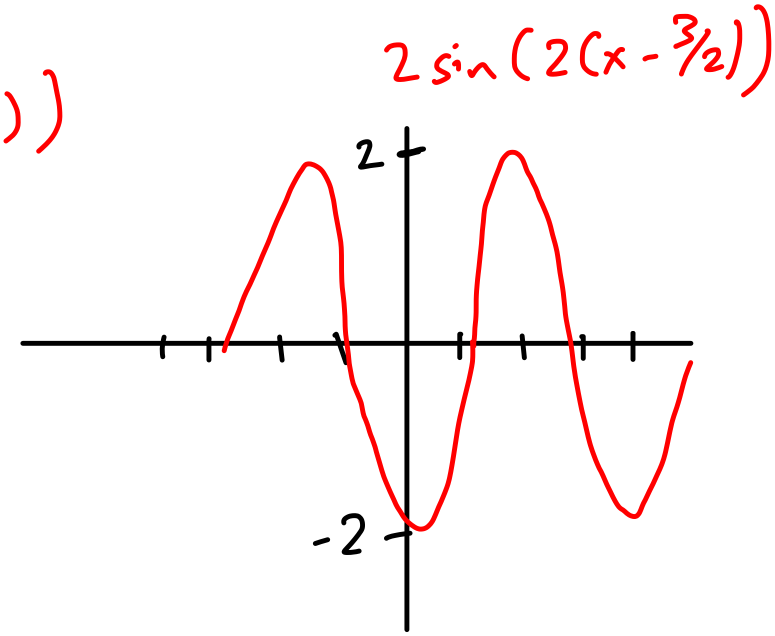
Now:

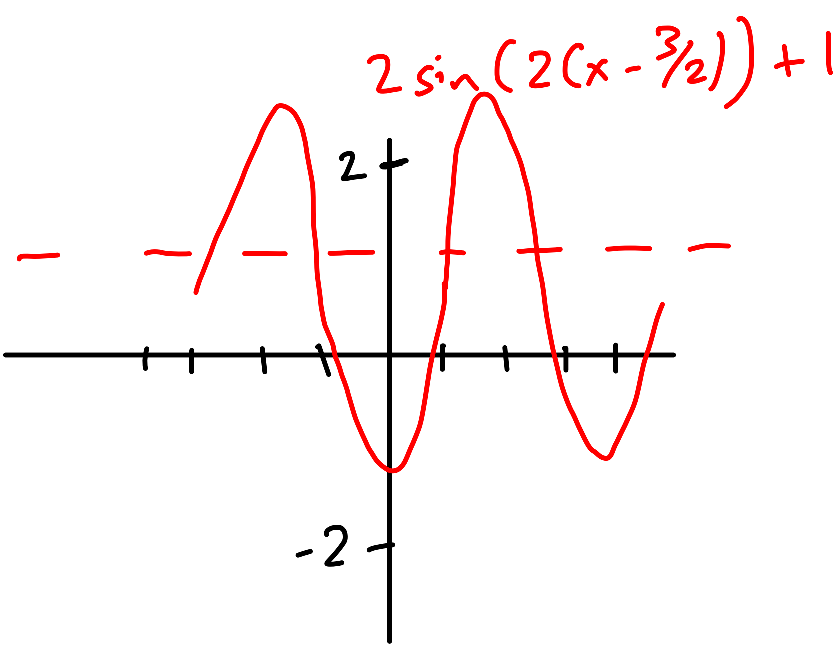
- horizontal stretch by a factor of $1/2$
- horizontal shift $3/2$ units to the right
- vertical stretch by a factor of 2
- vertical shift 1 unit up











Check: amplitude = 2,

midline = 1, period $= \frac{2\pi}{2}$

$= \pi$

✓

Ex: Find a formula for a sinusoidal function $f(x)$ with period $\boxed{4,}$ midline $\boxed{2,}$ and amplitude $\boxed{3,}$ such that the graph of f contains the point $(1,2)$ and it's increasing at that point.

$$A = 3$$

$$k = 2$$

$$\text{period} = \frac{2\pi}{B} = 4, \quad \text{so } B = \frac{2\pi}{4} = \pi/2.$$

Now let's find h. We know

$f(1) = 2$, and so $2 = 3\sin\left(\frac{\pi}{2}(1-h)\right)+2$

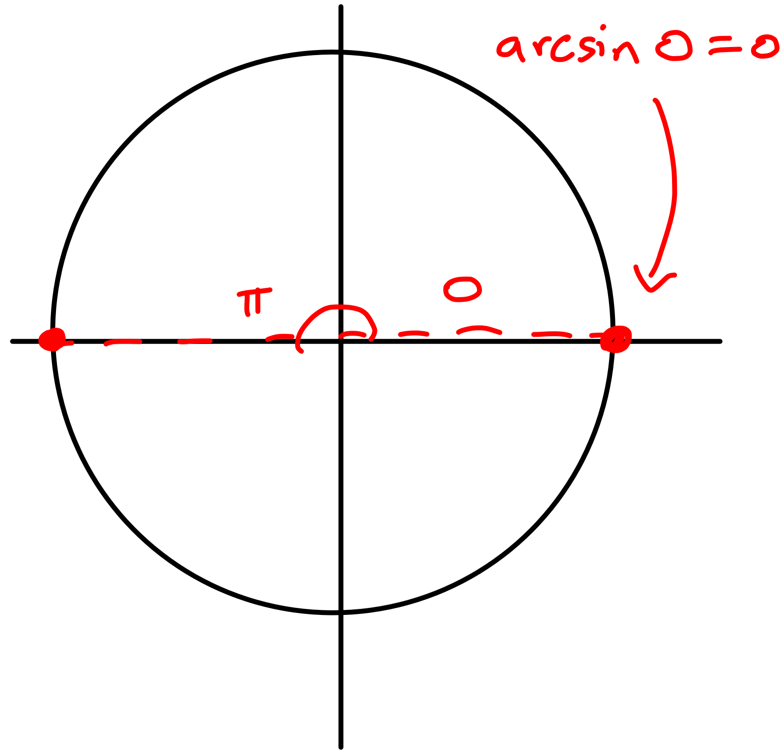
$$3\sin\left(\pi/2\,(1-h)\right) = 0$$

$$\sin\left(\pi/2\,(1-h)\right) = 0$$

$$\frac{\pi}{2}(1-h) = 0 + 2\pi n$$

or

$$\frac{\pi}{2}(1-h) = \pi + 2\pi n$$



$$1-h = 4n$$

or

$$1-h = 2+4n$$

$$h = 1-4n$$

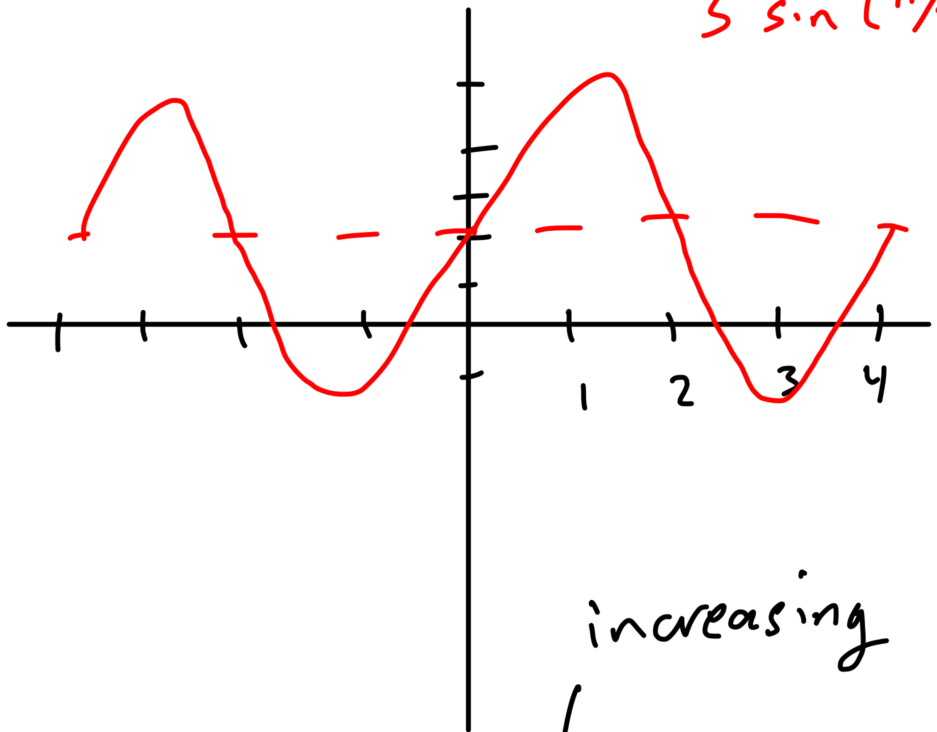
or

$$h = -1-4n$$

Try $h = 1-4n$ with $n=0$.

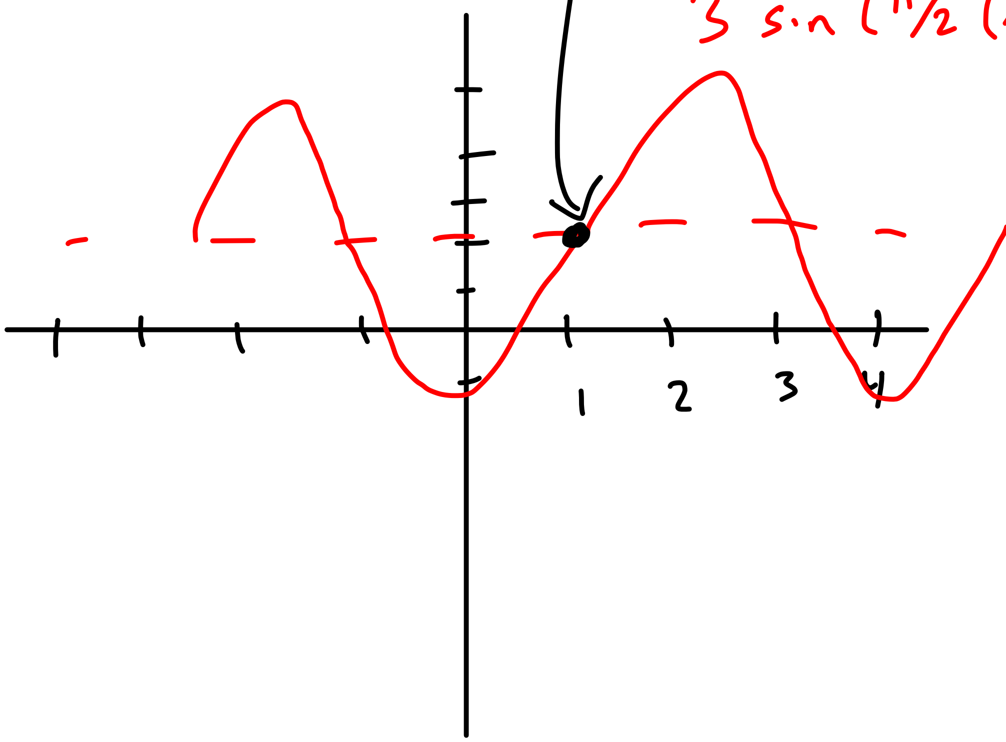
$h = 1$, so we have a horizontal shift left 1 unit.

$3\sin\left(\frac{\pi}{2}(x)\right)+2$

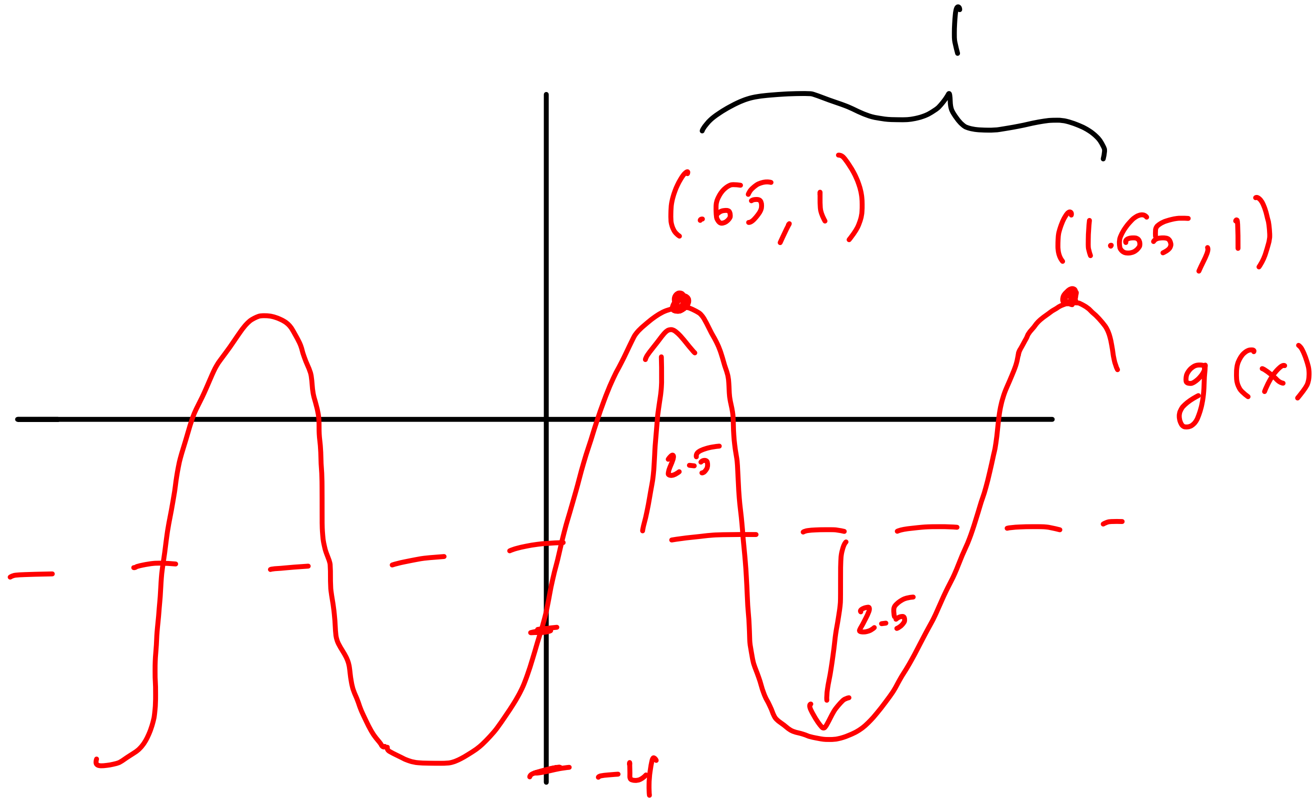


increasing

$3\sin\left(\frac{\pi}{2}(x-1)\right)+2.$



Ex: Find an equation for g, given that it's sinusoidal.



We know: midline is $\frac{M+m}{2} = \frac{1+(-4)}{2}$ $= -1.5$. The amplitude is 2.5.

The period is 1. So all we need is h. To find h, graph the function as if h were 0, and then find what horizontal shift we need.

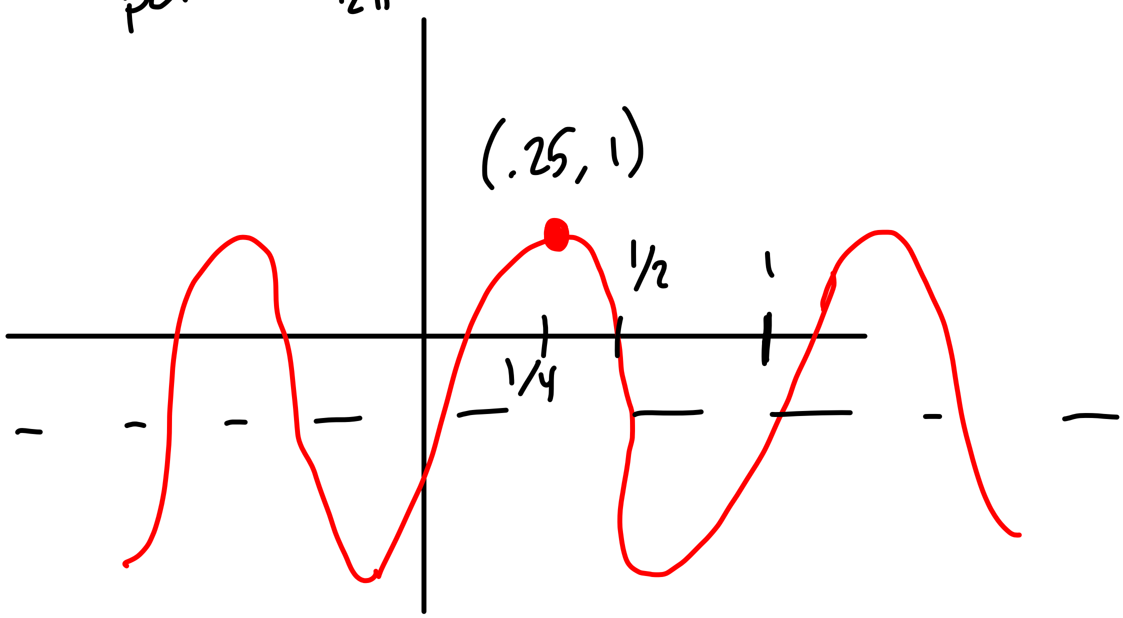
$A = 2.5$

$k = -1.5$

$\frac{2\pi}{B} = 1$, so $B = 2\pi$

Graph $2.5 \sin(2\pi(x)) - 1.5$

period $= \frac{2\pi}{2\pi} = 1$



therefore, we want $h = .4$.

In total, we have

$$2.5 \sin\left(2\pi\left(x - .4\right)\right) - 1.5$$

<u>Ex</u>: Find a sinusoidal function g with maximum 14, minimum 2, period 2, $g(11/6) = 11$, and such that g is decreasing at $x = 11/6$.

Since g is sinusoidal, $g = A \sin(B(x-h)) + k$

A = amplitude

$2\pi/B$ is the period

k is the midline

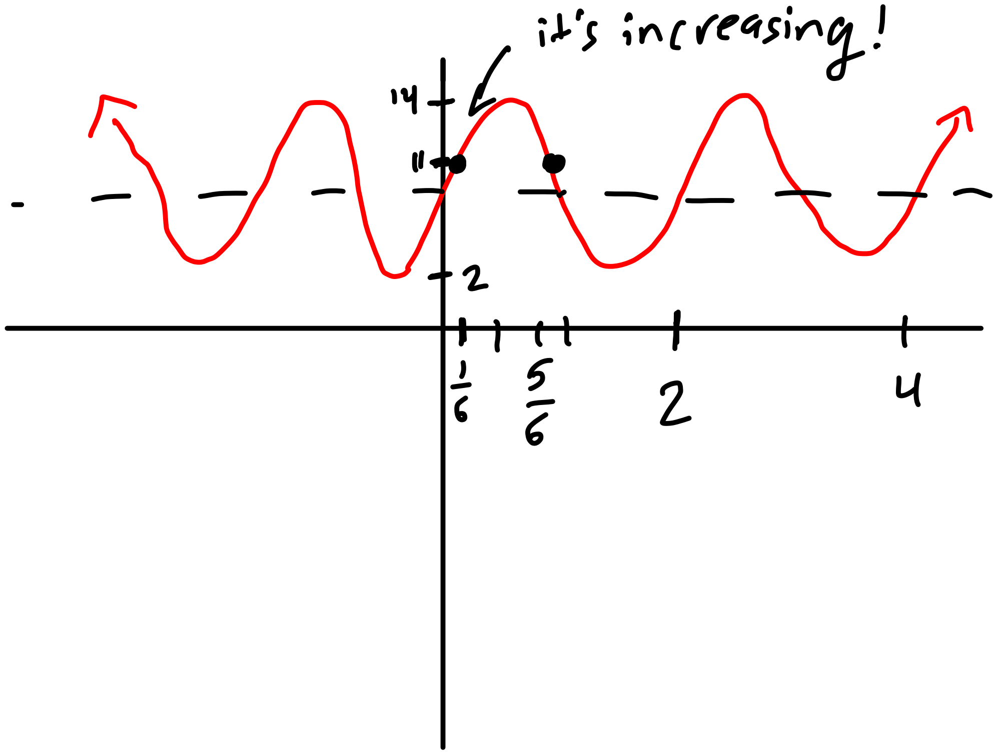
Recall that the midline is $\frac{\max + \min}{2}$.

Here, that's $\frac{14+2}{2} = \frac{16}{2} = 8$. The amplitude is how far the function gets away from its midline. That's $14-8=6$.

Finally, $\frac{2\pi}{B} = 2$, so $2B = 2\pi$, and so $B = \pi$.

To find h, assume it's 0, graph the function, and figure out what it should be.

If $h=0$, $g(x) = 6\sin(\pi x) + 8$.



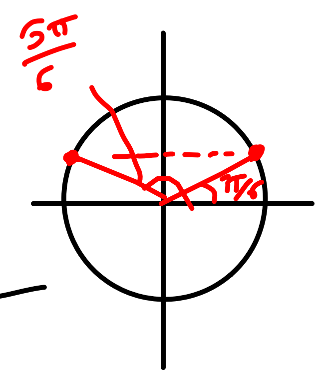
Find a point where $g(x) = 11$ and is decreasing, and then shift that point to $x = 11/6$.

$$6\sin(\pi x) + 8 = 11$$

$$6\sin(\pi x) = 3$$

$$\sin(\pi x) = \tfrac{1}{2}$$

$\arcsin(1/2) = \frac{\pi}{6}$



$$\pi x = \frac{\pi}{6} + 2\pi n$$

$$\pi x \overset{\text{or}}{=} \frac{5\pi}{6} + 2\pi n$$

So $x = \frac{1}{6} + 2n$ or $x = \frac{5}{6} + 2n$.

We need to find one of these x where the function is decreasing.

Let's try $1/6$. But g is increasing at $1/6$, so that won't work.

Instead, let's try $5/6$. Now this one works! We want this to be at $11/6$, so we need to shift to the right by $6/6 = 1$. In total,

$$g(x) = 6 \sin(\pi(x-1)) + 8.$$

Relationships Between Trig Functions

Def: The <u>secant</u> function is

$$\sec\theta = \frac{1}{\cos\theta}.$$

The cosecant function is

$$\csc\theta = \frac{1}{\sin\theta}.$$

The cotangent function is

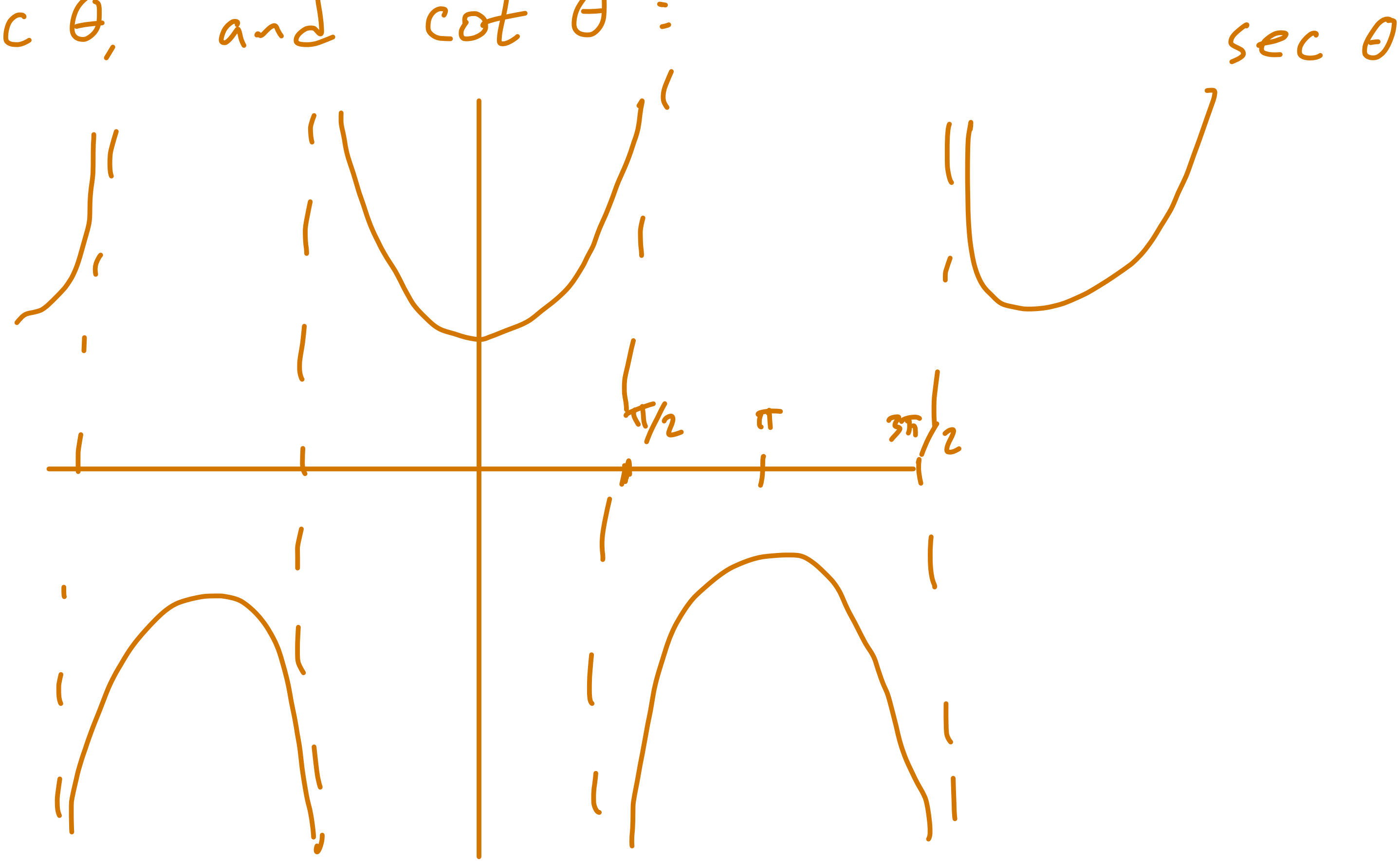
$$\cot\theta = \frac{\cos\theta}{\sin\theta}.$$

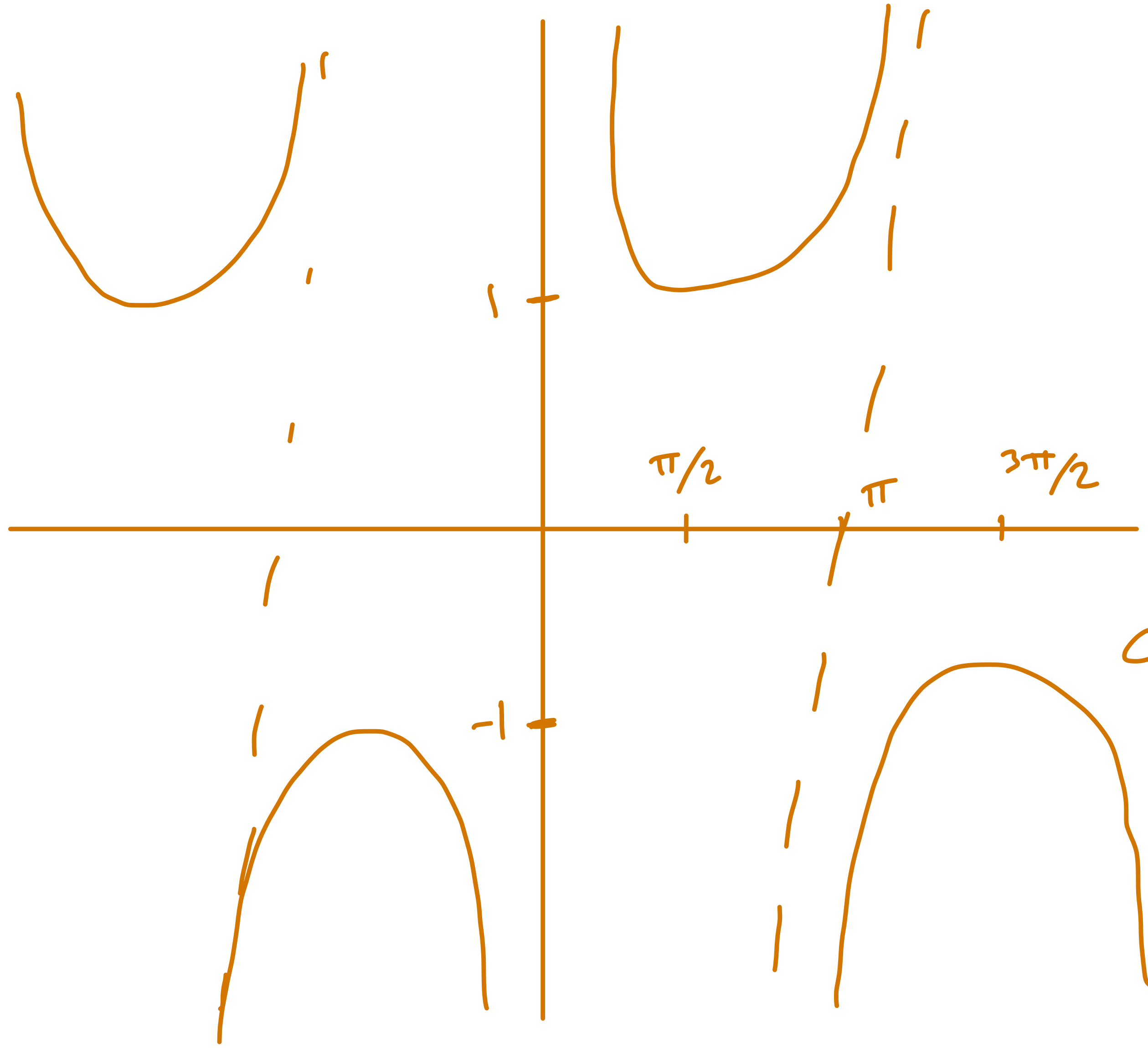
Ex $\csc(\pi/4) = \dfrac{1}{\sin(\pi/4)} = \dfrac{1}{\sqrt{2}/2}$

$= \dfrac{2}{\sqrt{2}}.$

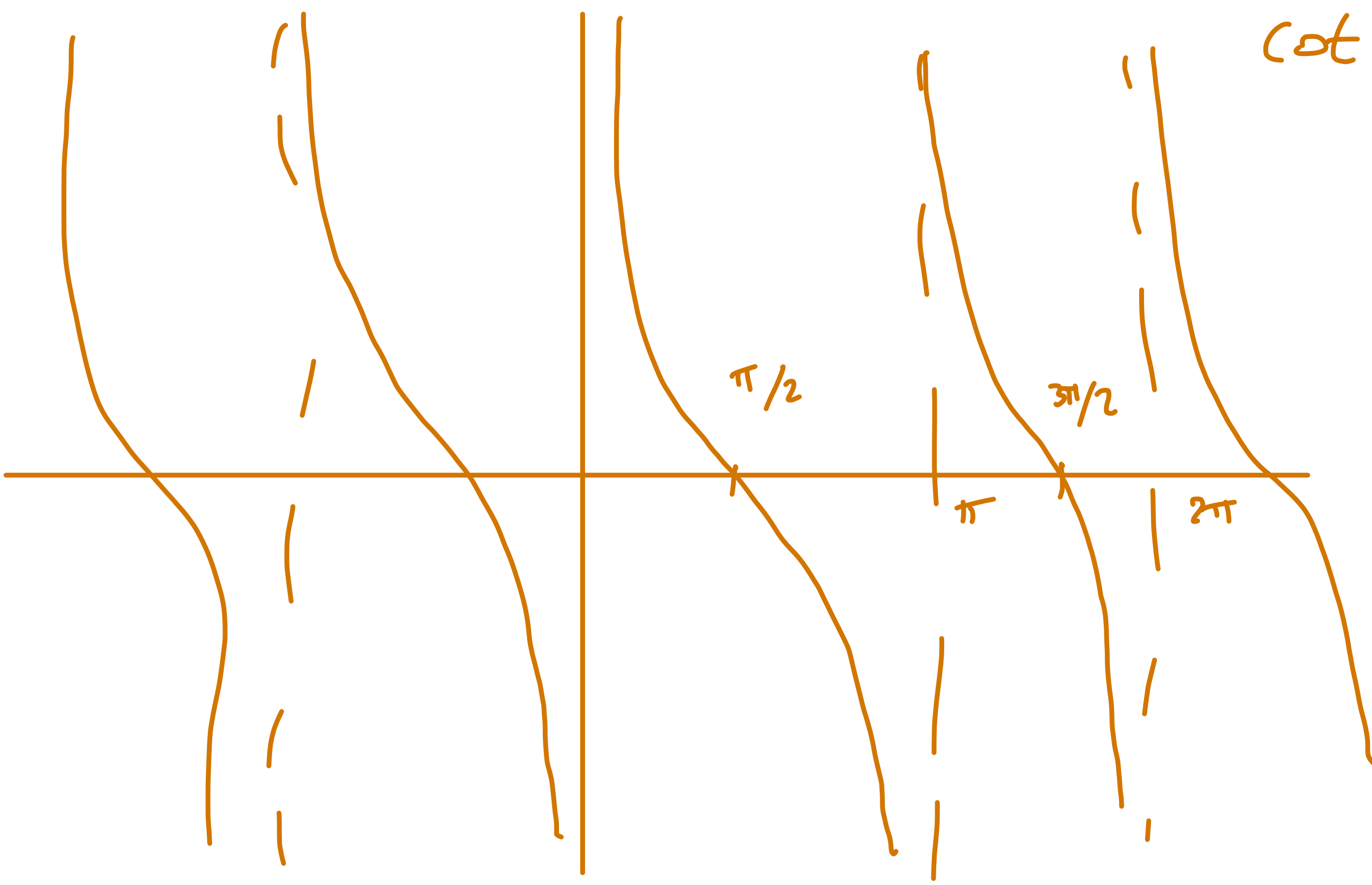
Ex $\cot(\pi/6) = \dfrac{\cos(\pi/6)}{\sin(\pi/6)} = \dfrac{\sqrt{3}/2}{1/2} = \sqrt{3}.$

Theorem: The graphs of $\sec\theta$, $\csc\theta$, and $\cot\theta$:





$\csc\theta$



$\cot\theta$

Theorem (The Basic Relationships)

1. $\sin^2\theta + \cos^2\theta = 1.$
2. $\cos(-\theta) = \cos\theta.$
3. $\sin(-\theta) = -\sin\theta.$
4. $\tan(-\theta) = -\tan\theta.$

Comment: You'll be provided the rest of theorems in this section on the midterm and final.

Theorem (The Half-Angle Formulas):

1. $$\sin(\theta/2) = \pm\sqrt{\frac{1-\cos\theta}{2}}.$$

2. $$\cos(\theta/2) = \pm\sqrt{\frac{1+\cos\theta}{2}}.$$

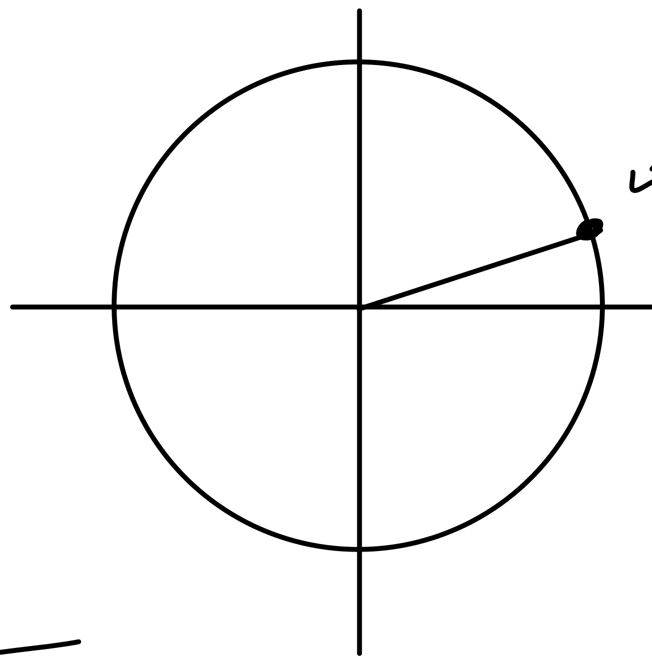
3. $$\tan(\theta/2) = \frac{\sin\theta}{1+\cos\theta}.$$

Comment: when there is a $\pm$, use the def of sin/cos (a point on the unit circle) to figure out if it's + or −. It can't be both!

Ex: Find $\sin(15^\circ)$ exactly.

$$\sin(15^\circ) = \sin\left(30^\circ/2\right) = \pm\sqrt{\frac{1-\cos 30^\circ}{2}}$$

$$= \pm\sqrt{\frac{1-\sqrt{3}/2}{2}} = \pm\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{4}}$$



$\sin 15^\circ$ is the y-value of this point by definition, so it must be positive.

$$= \sqrt{\frac{1}{2}-\frac{\sqrt{3}}{4}}.$$

Ex: $\tan\left(\pi/8\right) = \tan\left(\frac{\pi/4}{2}\right)$

$$= \frac{\sin \pi/4}{1+\cos \pi/4} = \frac{\sqrt{2}/2}{1+\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}/2}{\frac{2}{2}+\frac{\sqrt{2}}{2}}$$

$$= \frac{\sqrt{2}/2}{\frac{2+\sqrt{2}}{2}} = \frac{\sqrt{2}}{2}\cdot\frac{2}{2+\sqrt{2}} = \frac{\sqrt{2}}{2+\sqrt{2}}$$

Theorem (The Double-Angle Formulas):

1. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.
2. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.
3. $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

Ex: given that $\sin(25°) = .423$,

find $\sin(50°)$.

We know $\sin(50°) = 2\sin(25°)\cos(25°)$.

Now we need to solve for $\cos 25°$,

but $\sin^2(25°) + \cos^2(25°) = 1$, so

$\cos(25°) = \sqrt{1 - .423^2} = \sqrt{.821} = .906$.

Now $\sin(50°) = 2(.423)(.906) = .767$.

Theorem (The Sum and Difference Formulas):

1. $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
2. $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
3. $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
4. $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.
5. $\tan(\alpha+\beta) = \dfrac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$.

Ex: Find the exact value of $\cos(75°)$.

We could write $75° = 150°/2$ and use a half-angle formula.

Instead, let's write $75° = 30° + 45°$.

Now $\cos(75^\circ) = \cos(30^\circ + 45^\circ)$

$$= \cos(30^\circ)\cos(45^\circ) - \sin(30^\circ)\sin(45^\circ)$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Ex: Find an exact value for

$\tan\left(\frac{7\pi}{12}\right)$.

$$\frac{7\pi}{12} = \frac{4\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$$

Now $\tan\left(\frac{7\pi}{12}\right) = \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

$$= \frac{\tan(\pi/3) + \tan(\pi/4)}{1 - \tan(\pi/3)\tan(\pi/4)} = \frac{\sqrt{3} + 1}{1 - (\sqrt{3})(1)} = \frac{1+\sqrt{3}}{1-\sqrt{3}}.$$

Ex: Show that for any α and β,

$$\frac{\sin(\alpha+\beta)}{\cos(\alpha)\cos(\beta)} = \tan(\alpha) + \tan(\beta).$$

In general, start with the more complicated side and try to simplify it.

$$\frac{\sin(\alpha+\beta)}{\cos(\alpha)\cos(\beta)} = \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}$$

$$= \frac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}} + \frac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}$$

$$= \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} = \tan\alpha + \tan\beta.$$

Ex: Show that for all angles θ,

$$\frac{1}{2}(\cot\theta + \tan\theta) = \csc(2\theta)$$

$$\frac{1}{2}(\cot\theta + \tan\theta) = \frac{1}{2}\left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right)$$

$$= \frac{1}{2}\left(\frac{(\cos\theta)(\cos\theta) + (\sin\theta)(\sin\theta)}{(\sin\theta)(\cos\theta)}\right)$$

$$= \frac{1}{2}\left(\frac{\cos^2\theta + \sin^2\theta}{(\sin\theta)(\cos\theta)}\right)$$

$$= \frac{1}{2\sin(\theta)\cos(\theta)} = \frac{1}{\sin(2\theta)} = \csc(2\theta).$$