Def: Let $f(x)$ be continuous on $(c, b]$. Then $\int_c^b f(x)\,dx = \lim_{a \to c} \int_a^b f(x)\,dx$.

If f is continuous on $[a, c)$, then $\int_a^c f(x)\,dx = \lim_{b \to c} \int_a^b f(x)\,dx$

If f is continuous on $[a, b]$ except at a point c in $[a, b]$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Ex: $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$ Improper because $\frac{1}{\sqrt{4-x}}$ is undefined when $x = 4$, so it's continuous on $[0, 4)$.

So $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to 4} \int_0^b \frac{1}{\sqrt{4-x}}\,dx$

$$= \lim_{b\to 4} \int_0^b (4-x)^{-1/2}\,dx$$

$u = 4 - x$

$du = -dx \quad | \quad dx = -du$

$$= \lim_{b\to 4} \int_0^b -u^{-1/2}\,du$$

$$= \lim_{b\to 4} \left[-\frac{u^{1/2}}{1/2} \right]\Bigg|_0^b$$

$$= \lim_{b\to 4} \left[-2(4-x)^{1/2} \right]\Bigg|_0^b$$

$$= \lim_{b\to 4} \left(-2(4-b)^{1/2} + 2(4)^{1/2} \right)$$

$$= \lim_{b\to 4} \left(-2\overbrace{(\underbrace{4-b}_{\to 0})^{1/2}}^{\to 0} \right) + 2\cdot 2$$

$= 4.$

Ex: $\int_{-1}^{1} \frac{1}{x^3}\,dx$ Defined on $[-1,1]$ except at 0.

$$= \underbrace{\int_{-1}^{0} \frac{1}{x^3}\,dx} + \int_{0}^{1} \frac{1}{x^3}\,dx$$

$$\lim_{b\to 0} \int_{-1}^{b} \frac{1}{x^3}\,dx$$

$$= \lim_{b\to 0} \left[\frac{x^{-2}}{-2}\right]\Big|_{-1}^{b}$$

$$= \lim_{b\to 0} \left(\frac{1}{-2b^2} - \frac{1}{-2(-1)^2}\right)$$

$$= -\infty$$

So $\int_{-1}^{0} \frac{1}{x^3}\,dx$ diverges

So $\int_{-1}^{1} \frac{1}{x^3}\,dx$ diverges too.

Chapter IV: Differential Equations

Def: A differential equation (or a DE) is an equation involving a function $y = f(x)$ and its derivatives. They look a little bit like polynomials, except instead of increasing the power on x, we increase the number of derivatives on y.

Ex: $y' = 2x$ ← solution: $y = x^2 + C$ for any C.

Def: A solution to a DE is a function $y = f(x)$ that satisfies the equation.

Ex: $y' + 3y = 6x + 11$

Solution is $y = e^{-3x} + 2x + 3$

Why? We don't know (yet), but we can verify that it works:

$y' = -3e^{-3x} + 2$

$y' + 3y = -3e^{-3x} + 2 + 3e^{-3x} + 6x + 9$

$= 6x + 11$ ✓

Ex: You throw a baseball straight up. Initially, it's 3m off the ground and travelling at 10 m/s up. Given that acceleration due to gravity is $-9.81\ m/s^2$, find $v(t)$, the velocity of the ball at time t.

$v' = -9.81$ ← this is a DE!

$v = \int v' = \int -9.81\, dt = -9.81t + C$

$v(0) = 10$, so $-9.81(0) + C = 10$

$C = 10$

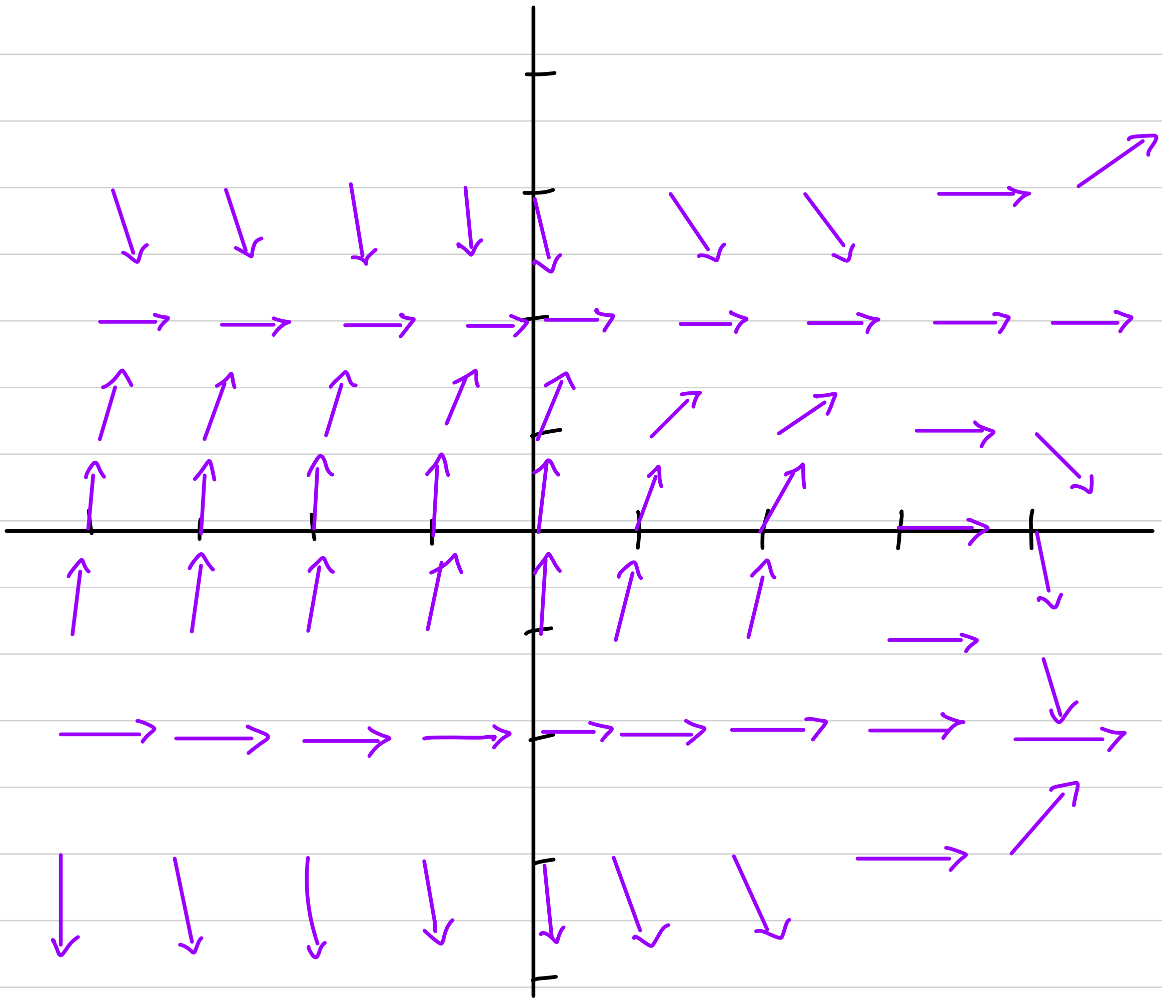
$v(t) = -9.81t + 10$

Direction Fields

Def: A direction field lets us visualize how a DE acts over time. To draw one, solve for y', then draw an arrow at every point (x,y) that corresponds to the slope given by y'.

Ex: $y' = (x-3)(y^2-4)$

e.g. at $(0,0)$, $y' = (0-3)(0-4) = 12$, so draw an arrow w/ slope 12 at $(0,0)$



Note: when $y=2$, $y'=0$, so $y=2$ is a solution to the DE.

$$(x-3)(y^2-4) = (x-3)(4-4) = 0 = y'$$

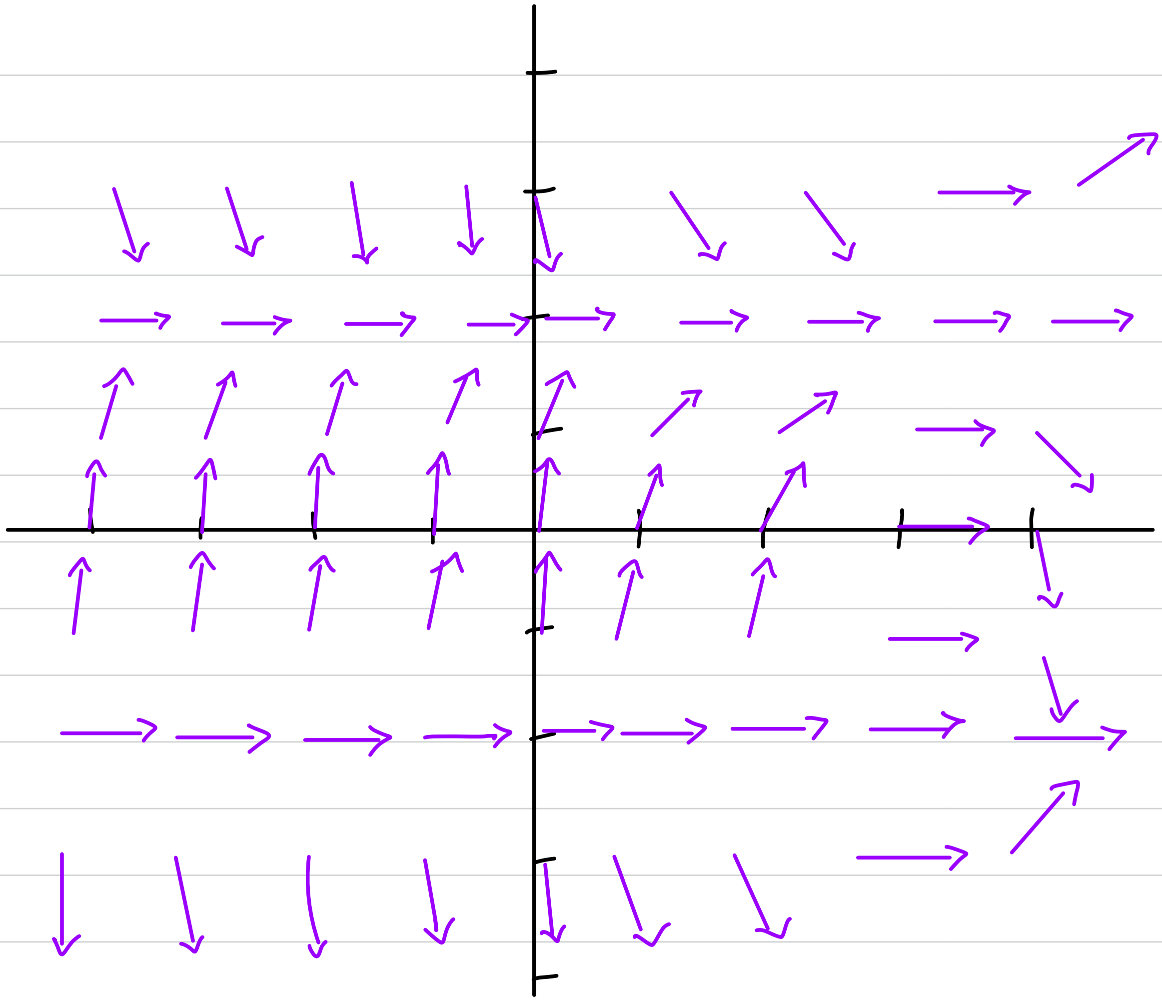
Def: An equilibrium solution to a DE

is a solution of the form $y=c$ for some constant c. They appear as horizontal stripes of arrows in direction fields.

End of term survey: live (check your email) If 50% of the class responds, everyone gets 2% EC on the final. Fill out by Sunday night

Practice final: posted by Friday night.

Office hours: today + Friday, also on Monday of finals week from 11:30 - 12:50 as usual.



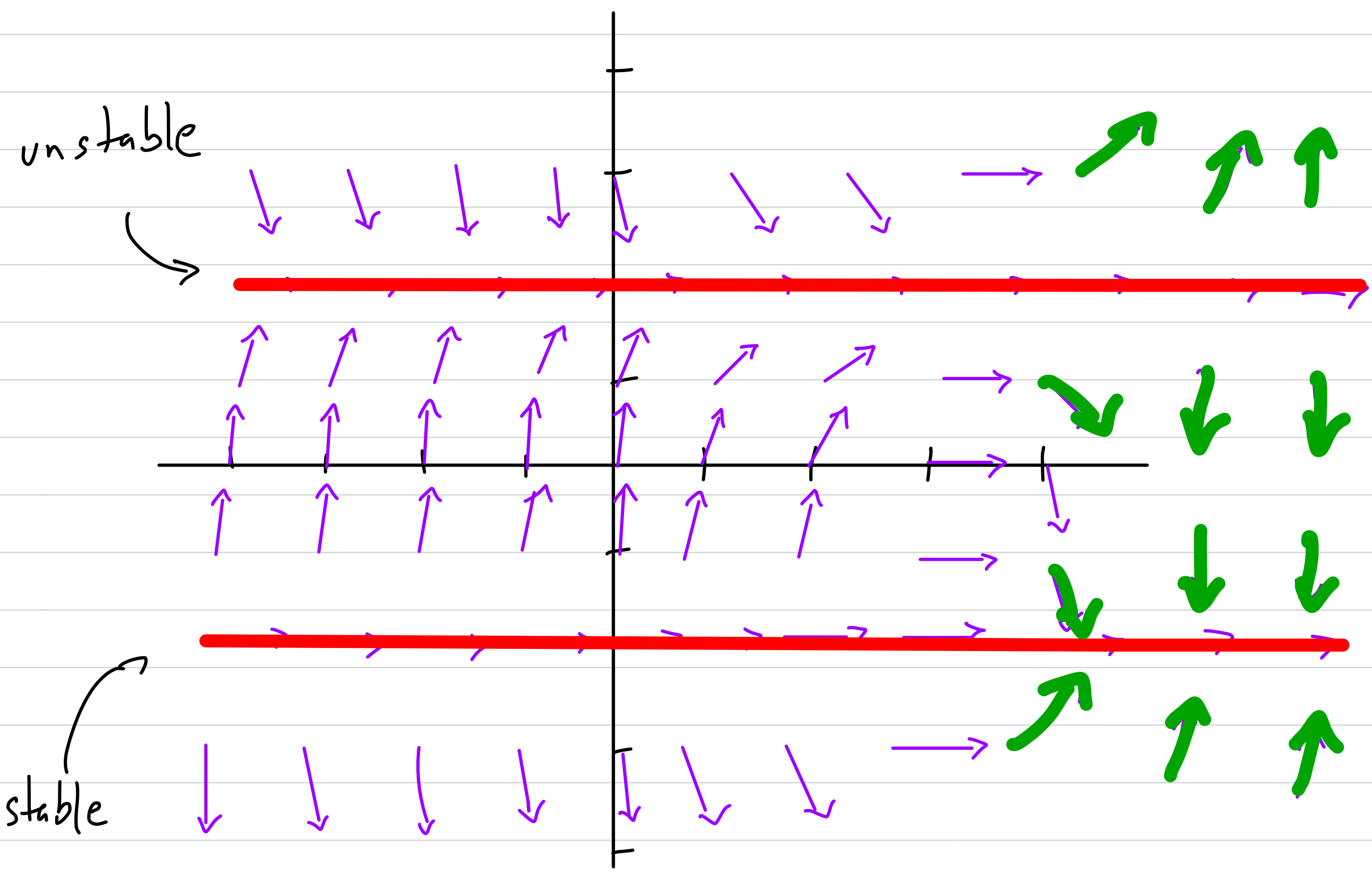
$$y' = (x-3)(y^2-4)$$

a solution to this DE is a function $y = f(x)$ whose derivative satisfies this equation — equivalently, a solution is a function whose graph's slope matches the arrows' slopes at every point. This looks like the function is "following the flow" of the direction field.

Def: An equilibrium solution to a DE is a solution of the form $y=c$ for some constant c. They appear as horizontal strips of arrows in direction fields.

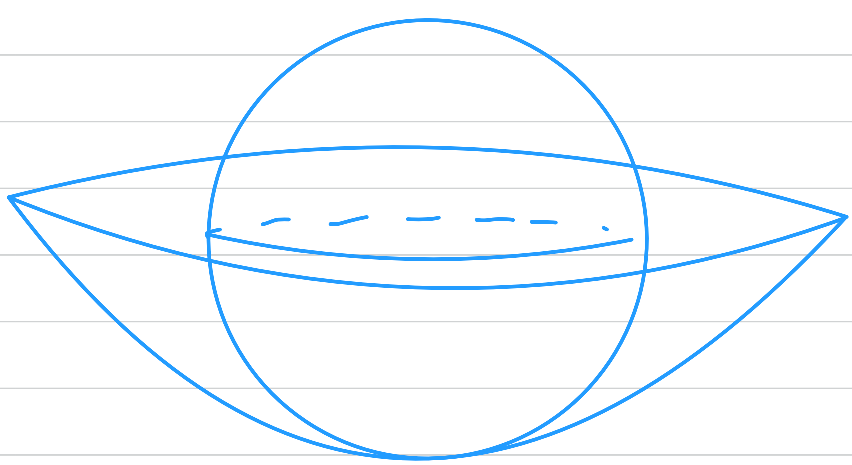
Ex: Find all equlibrium solutions of $y' = (x-3)(y^2-4)$. We don't an equation of the form $y = f(x)$, so we can't solve $y = c$. Instead, notice that if $y = c$, then $y' = 0$, so set $(x-3)(y^2-4) = 0$. Then $x = 3$ or $y^2 = 4$, so $x = 3$, $y = 2$, or $y = -2$. Because equilibrium solutions are only of the form $y = c$, we only want $y = \pm 2$.

Def: Suppose $y = c$ is an equilibrium solution to a DE $y' = f(x,y)$. We say that c is a stable equilibrium if as $x \to \infty$, a small band of solutions to the DE around $y = c$ approaches $y = c$ (i.e. the line $y = c$ "attracts" the surrounding direction field). We say c is an <u>unstable</u> equilibrium if a small band of solutions to DE around $y=c$ is moving away from $y=c$ (i.e. the line $y=c$ "repels" the surrounding direction field).

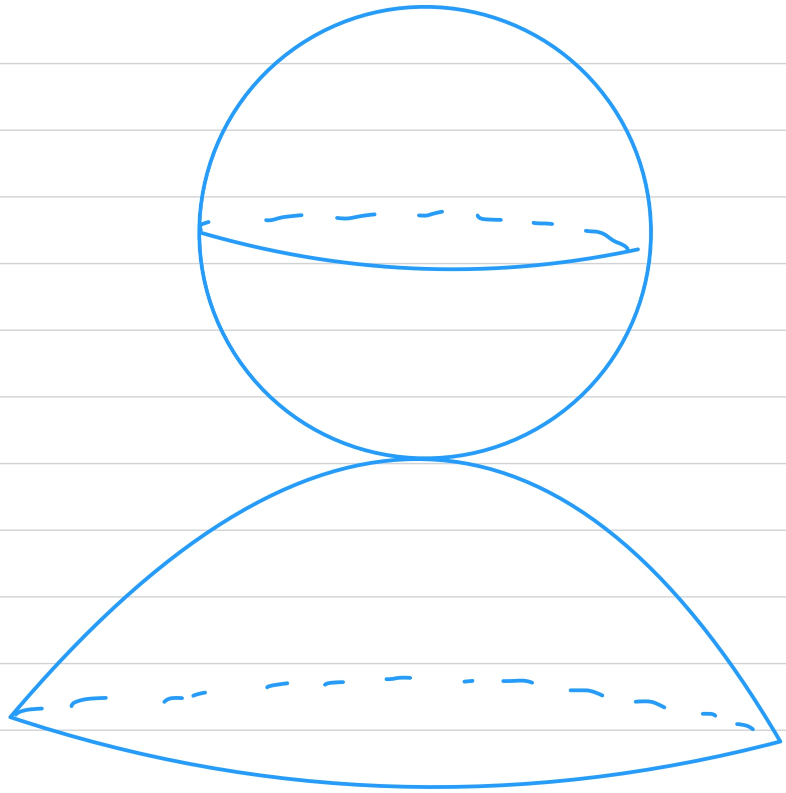


Note: when you're drawing a direction field, set $y' = 0$ to find the equilibria, but also to find the places where x-values cause the derivative to be zero (e.g. $x = 3$ in the above picture). Try to make the direction field large enough to cover all of these x-values.

Comment: think of stable vs unstable equilibria as a sphere balancing in/on a bowl



← if you move the sphere slightly in any direction, it will go back to the bottom (i.e. the bottom is a stable equilibrium)



if you move the sphere slightly in <u>any</u> direction, then it will roll away from the top (i.e. the top is an unstable equilibrium)

Separable DEs

Def: A separable DE is one that can be written in the form $y' = f(x)\,g(y)$ for some functions f and g.

Ex:

$$y' = \underbrace{(x-3)}_{f(x)}\underbrace{(y^2-4)}_{g(y)}$$

$$y' - xy + 2y - 3x + 6 = 0$$

$$y' = xy - 2y + 3x - 6$$

$$y' = (x-2)(y+3)$$

Method (Solving Separable DEs):

① Separate into $y' = f(x)\,g(y)$.

② Rewrite y' as $\frac{dy}{dx}$: $\frac{dy}{dx} = f(x)\,g(y)$.

③ Multiply both sides by dx and divide both sides by $g(y)$. $\frac{1}{g(y)}\,dy = f(x)\,dx$.

④ Integrate both sides.

⑤ Solve for y. If this is an initial value problem, also solve for C.

Ex: Solve $y' = (x-2)(y+3)$, $y(0) = 1$.

$$\frac{dy}{dx} = (x-2)(y+3)$$

$$\frac{1}{y+3}\, dy = (x-2)\, dx$$

$$\int \frac{1}{y+3}\, dy = \int (x-2)\, dx$$

$$\ln(y+3) = \frac{x^2}{2} - 2x + C$$

$$y+3 = e^{\frac{x^2}{2} - 2x + C}$$

$$y = e^{x^2/2 - 2x + C} - 3$$

$$y(0) = 1 \implies 1 = e^{0^2/2 - 2\cdot 0 + C} - 3$$

$$1 = e^{C} - 3$$

$$4 = e^{C}$$

$$C = \ln(4)$$

$$\boxed{y = e^{\frac{x^2}{2} - 2x + \ln(4)} - 3}$$

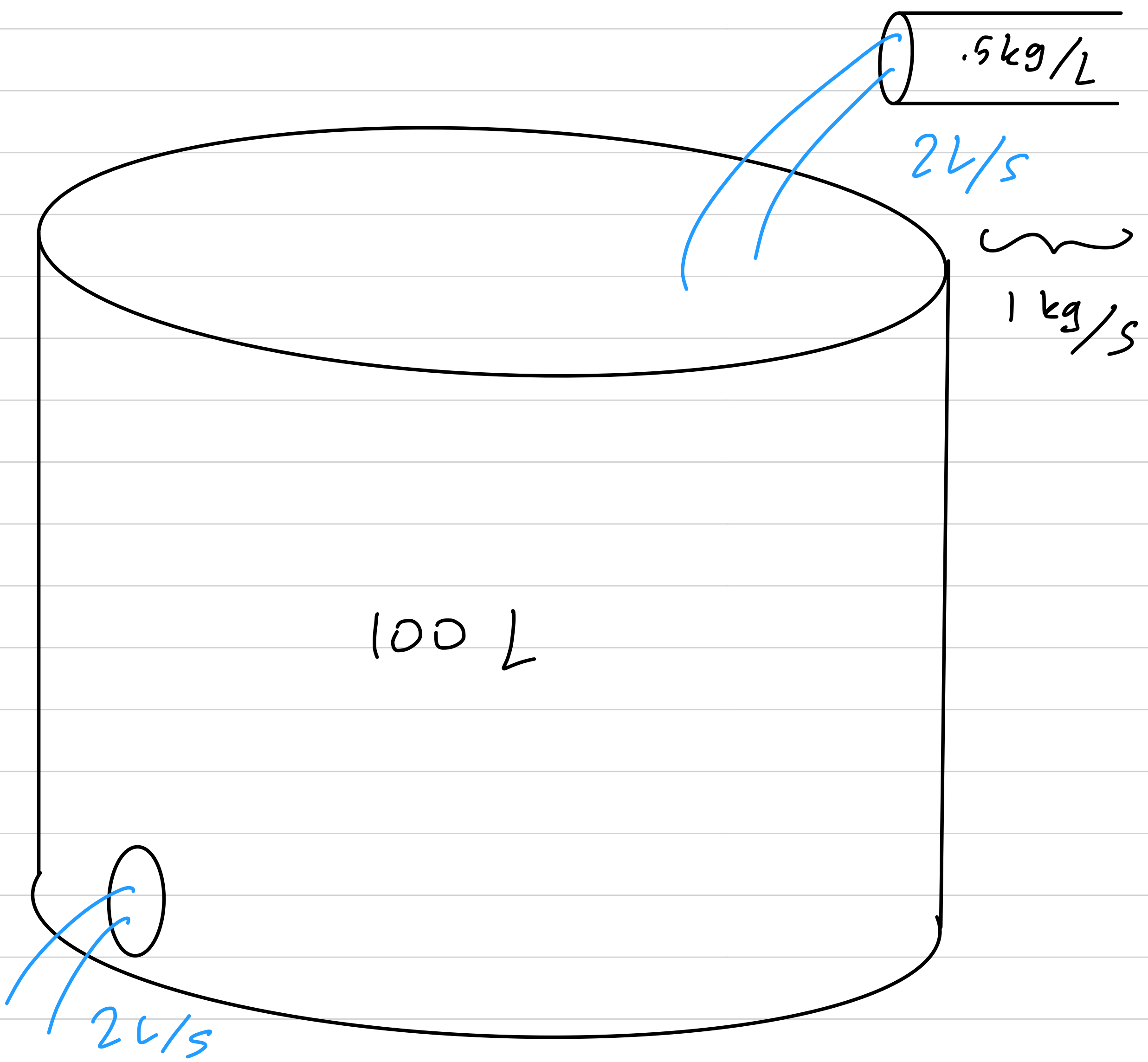
<u>Comment</u> With $y' = f(x)g(y)$, we can't divide by $g(y)$ when it's zero.

Actually, it's not an issue: when $g(y) = 0$, $y' = 0$, so we must have an equilibrium solution.

In practice, before trying to apply separation of variables, first set $g(y) = 0$ to find any equilibria.

<u>Ex</u>: Mixing in a tank. We have a 100L tank, and initially it contains 4 kg of salt (completely dissolved and well-mixed). Then we start draining the tank at

2 L/s. At the same time, we start pumping in a solution of .5 kg/L of saltwater at 2 L/s into the tank. Assume that the solution is always well-mixed. Find $m(t)$: the mass of salt in the tank at time t.



$m(0) = 4$

$$\underset{\substack{\uparrow \\ \text{rate of change of} \\ \text{the mass of salt}}}{m'} = \underbrace{(\text{salt inflow})}_{1} - (\text{salt outflow})$$

rate of change of
the mass of salt

salt flow rate: $(L/s)(kg/L)$

inflow: $2\,L/s \cdot .5\,kg/L = 1\,kg/s$

outflow: $2\,L/s \cdot m\,kg/100\,L = \frac{m}{50}\,\frac{kg}{s}$

$$m' = 1 - \frac{m}{50}$$

$$\frac{dm}{dt} = 1 - \frac{m}{50}$$

$$\frac{dm}{1 - m/50} = dt$$

$$\int \frac{1}{1 - m/50}\, dm = \int 1\, dt$$

$$u = 1 - m/50$$

$$du = -\frac{1}{50}\, dm$$

$$-50 \int \frac{1}{u}\, du = t + C$$

$$-50 \ln(1 - m/50) = t + C$$

$$\ln(1 - m/50) = -\frac{t + C}{50}$$

$$1 - m/50 = e^{-\frac{t+C}{50}}$$

$$M/50 = 1 - e^{-\frac{t+C}{50}}$$

$$M = 50 - 50e^{-\frac{t+C}{50}}.$$

Since $M(0) = 4$,

$$4 = 50 - 50e^{\frac{-0+C}{50}}$$

$$-46 = -50e^{C/50}$$

$$46/50 = e^{C/50}$$

$$\frac{C}{50} = \ln(46/50)$$

$$C = 50\ln(46/50).$$

$$M = 50 - 50e^{\frac{-t + 50\ln(46/50)}{50}}$$

Wait! We didn't solve for the equilibria.

$$1 - \frac{M}{50} = 0$$

$M = 50$ kg ← not a valid solution,

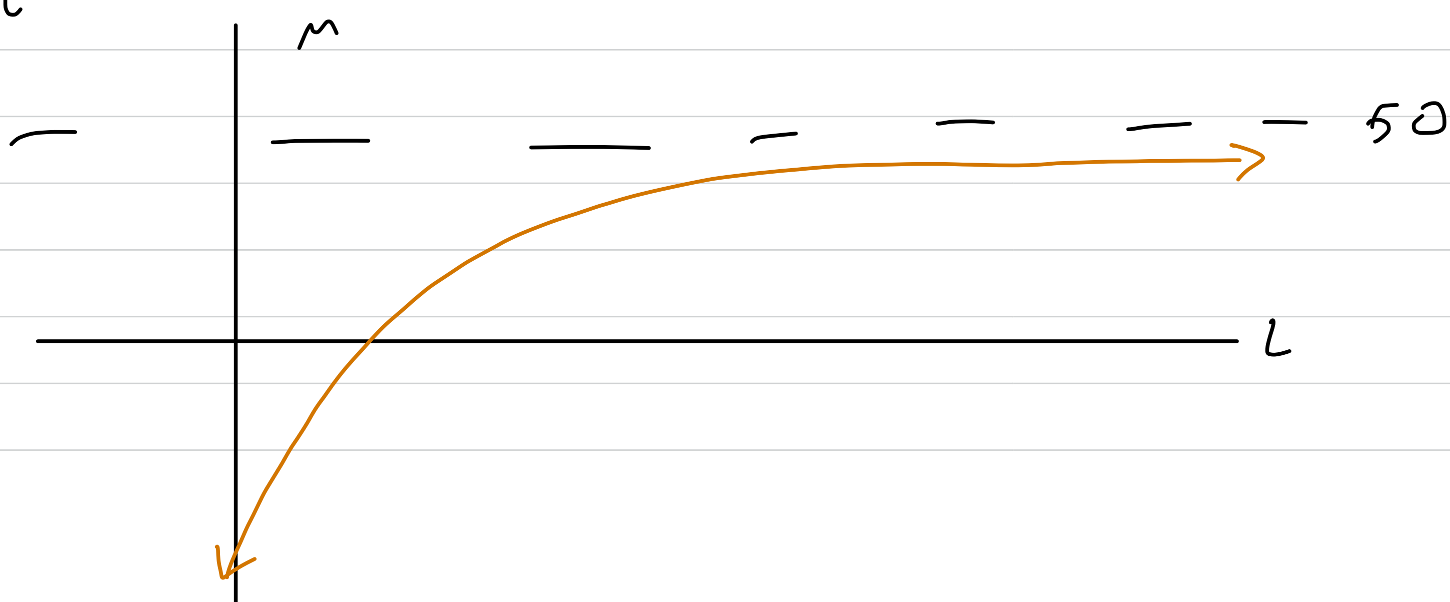
because we know $M(0) = 4 \neq 50$

Is 50 stable, unstable, or neither?

Make a guess based on the solution we have:

$$\lim_{t \to \infty} \left(50 - 50e^{\frac{-t + 50\ln(46/50)}{50}} \right) = 50$$

It's a good guess that 50 is a stable equilibrium.



Thm (Newton's Law of Cooling): Let $T(t)$ be the temperature of an object at time t and let T_s be the temperature of the surrounding material. Then $T' = k(T - T_s)$ for some number k.

Ex: A pizza is baked at $350°$ F. The temperature of the kitchen is $75°$F. After 5 minutes, the pizza is $340°$F. How much longer until it is $300°$F?

$T(t)$ = temp of pizza at time t

$T(0) = 350$

$T(5) = 340$

$$T_s = 75$$

$$\frac{dT}{dt} = k(T - 75)$$

$$\frac{1}{T-75}\, dT = k\, dt$$

$$\int \frac{1}{T-75}\, dT = \int k\, dt$$

$$\ln(T - 75) = kt + C$$

$$T - 75 = e^{kt+C}$$

$$T = e^{kt+C} + 75$$

$$350 = e^{C} + 75 \qquad (T(0) = 350)$$

$$275 = e^{C}$$

$$C = \ln(275)$$

$$340 = e^{5k + \ln(275)} + 75 \qquad (T(5) = 340)$$

$$265 = e^{5k + \ln(275)}$$

$$\ln(265) = 5k + \ln(275)$$

$$5k = \ln(265) - \ln(275)$$

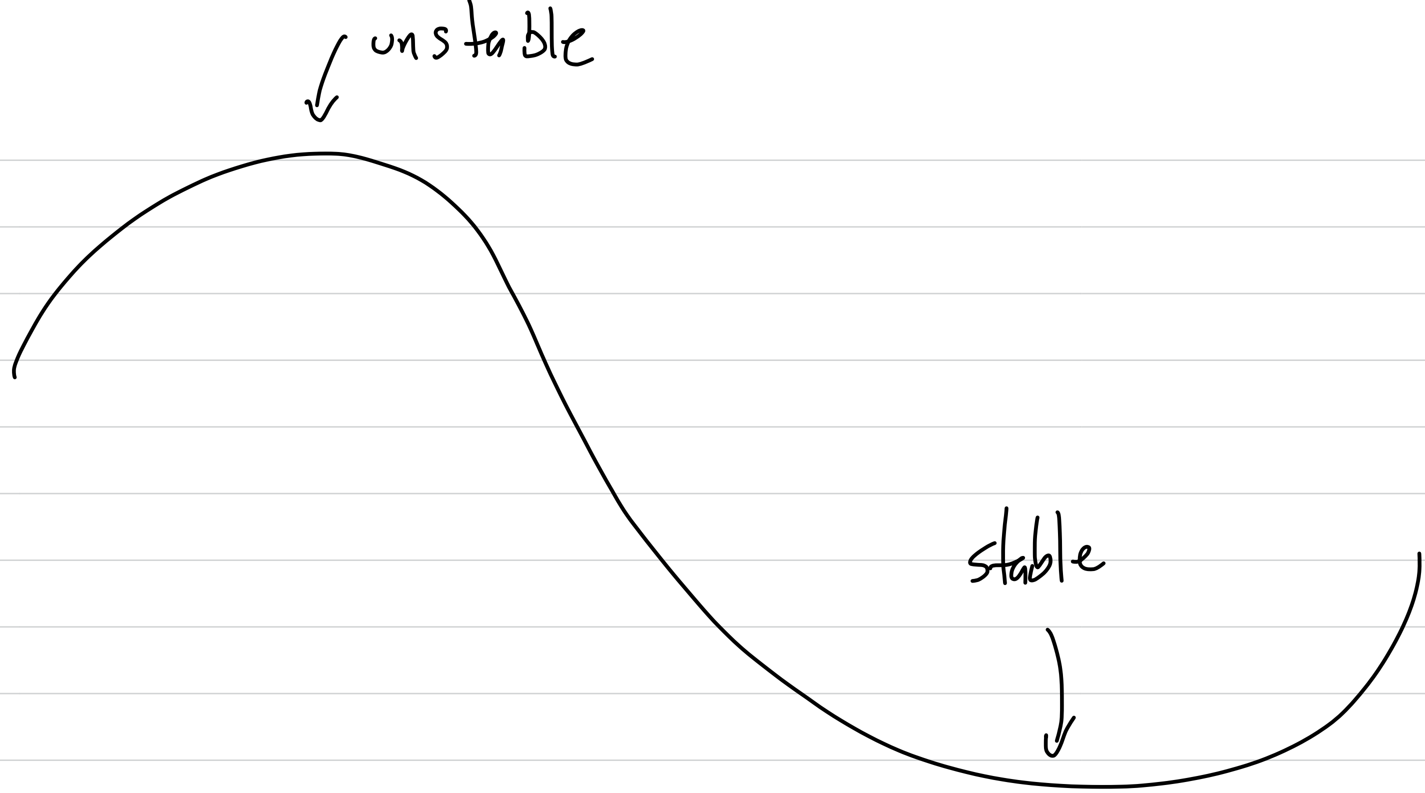
$$k = \frac{1}{5}\left(\ln(265) - \ln(275)\right)$$

$$T = e^{\frac{t}{5}\left(\ln(265) - \ln(275)\right) + \ln(275)} + 75.$$

equilibrium: $k(T - 75) = 0$

$$T = 75$$

$\lim_{t \to \infty} T(t) = 75$, so 75 is a stable equilibrium



unstable
stable