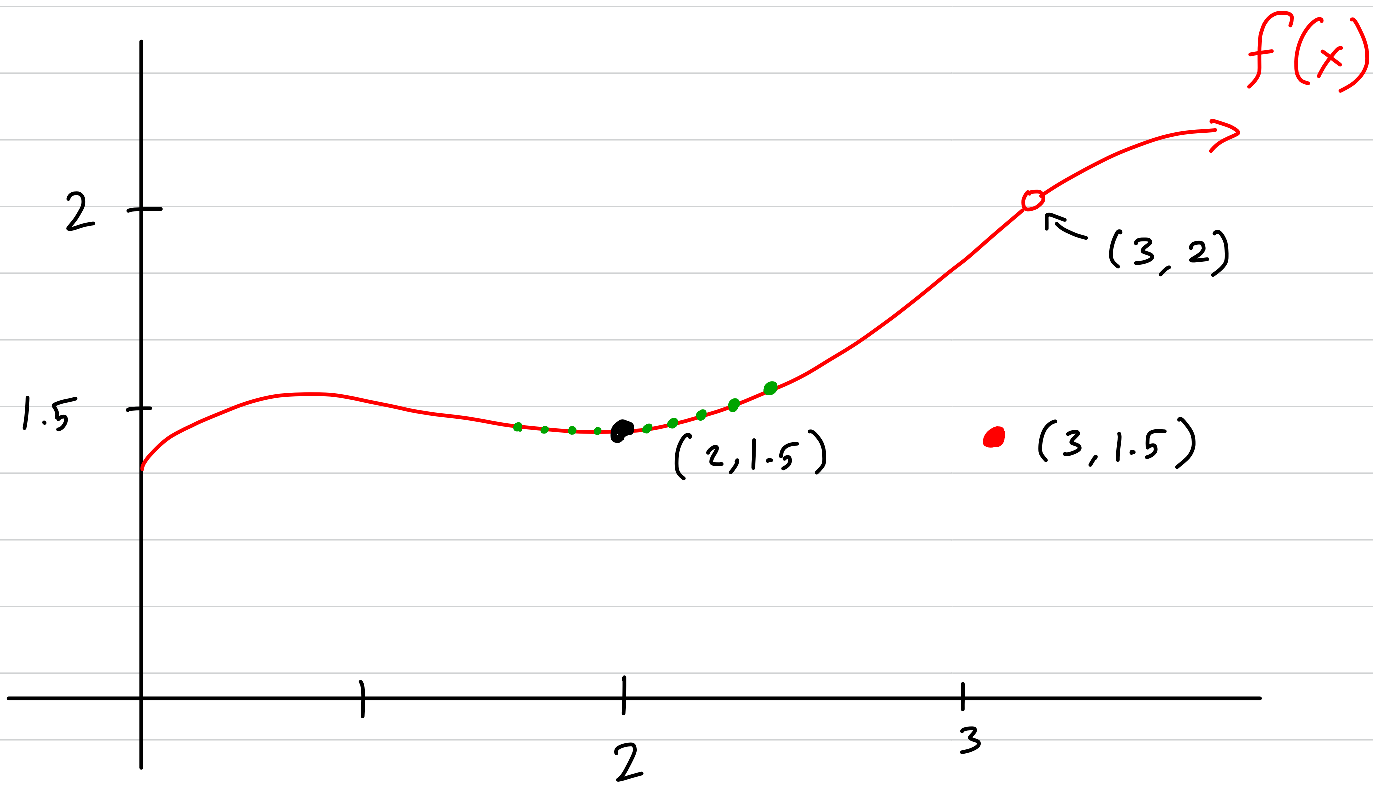
Review of Calc I.

Limits

- Estimate with table or graph
- The value a function "should" take at a point, regardless of what it actually does take.



$f(2) = 1.5$ $\qquad$ $f(3) = 1.5$

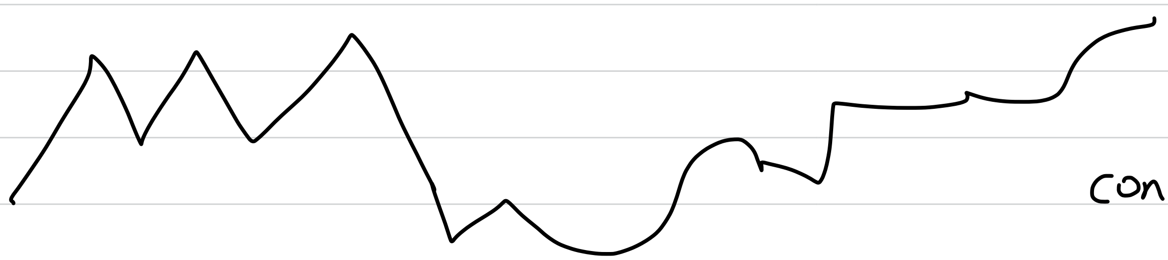
$\lim_{x \to 2} f(x) = 1.5$ $\qquad$ $\lim_{x \to 3} f(x) = 2$

f is continuous if $\lim_{x \to a} f(x) = f(a)$ for all a

The previous f is continuous at 2, but not at 3.

Polynomials, sin, and cos all are continuous.

Think of continuity as being able to draw the graph of a function without picking up your pen.

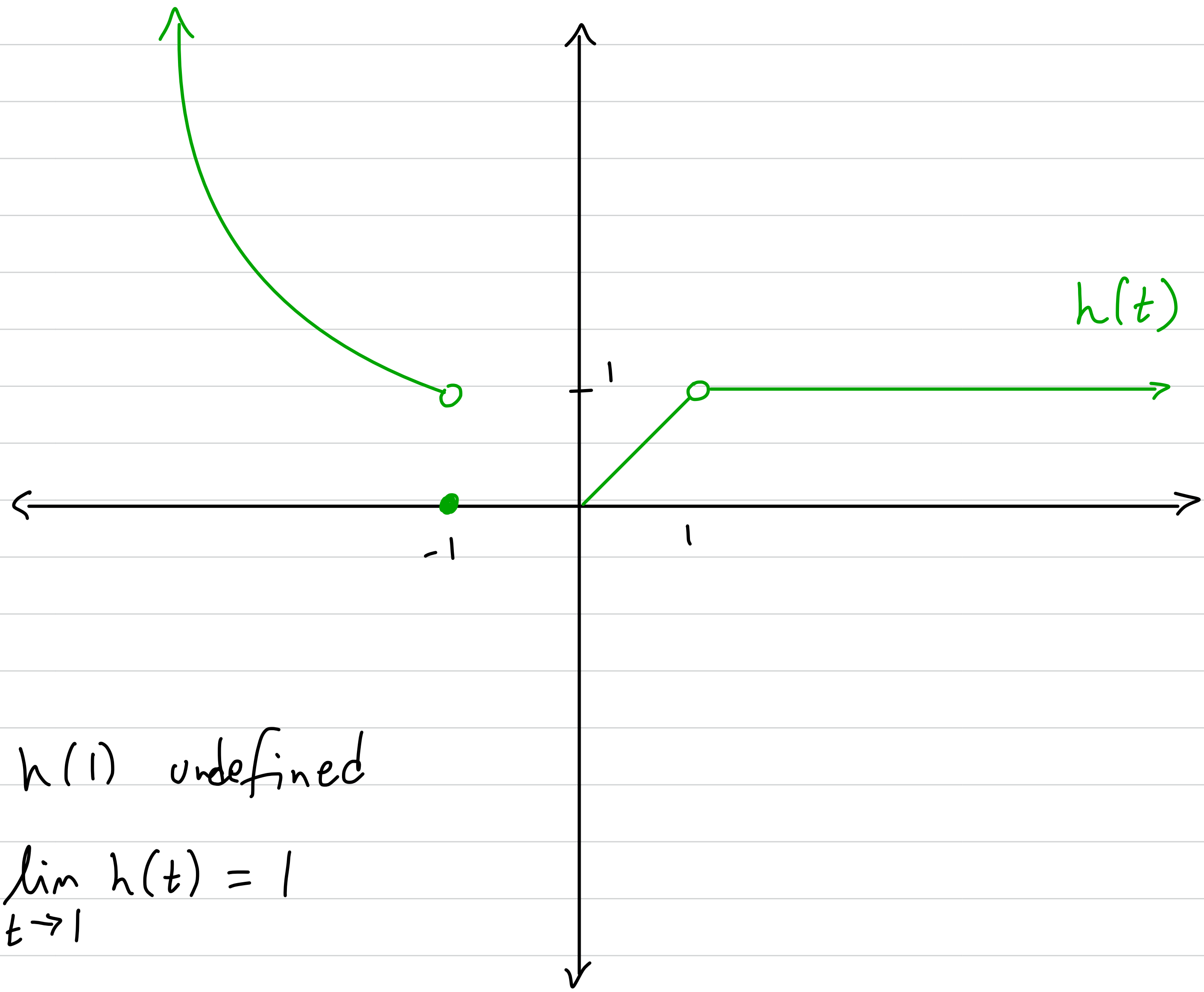


continuous

not continuous

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

$$\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right).$$



$h(1)$ undefined

$\lim_{t\to 1} h(t) = 1$

$h(-1) = 0$

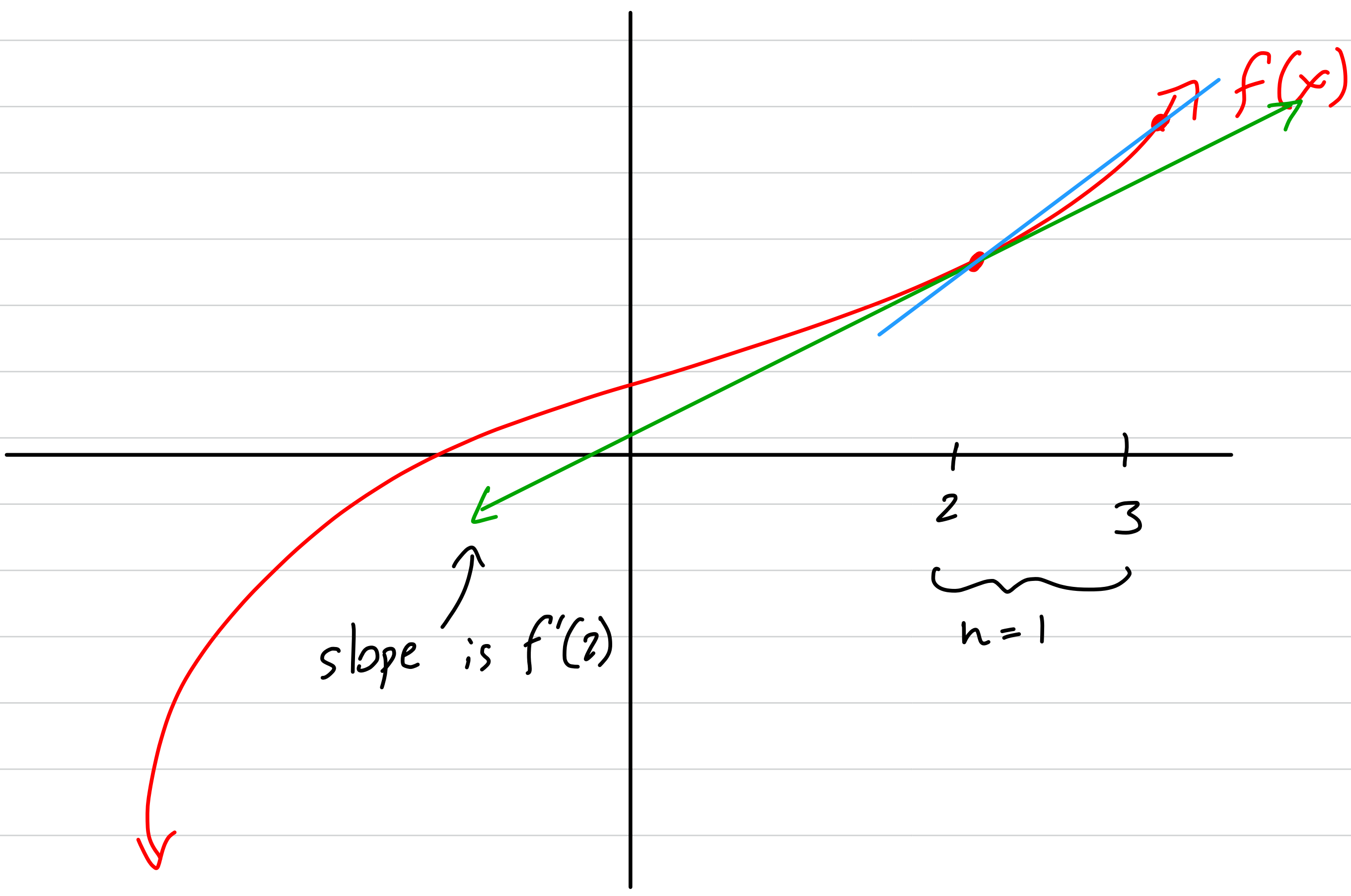
$\lim_{t\to -1} h(t)$ does not exist (because you can't approach from the right)

$\lim_{t\to -1^-} h(t) = 1$

↳ means only approach from the left

Derivatives

The derivative of a function f is a function f', where $f'(x)$ is the slope of the tangent line to the graph of f at x.



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative measures rate of change.
Classic example: if $s(t)$ is position at time t, then $s'(t)$ is velocity at time t, and $s''(t) = \left(s'(t)\right)'$ is acceleration at time t.

f differentiable $\Longrightarrow$ f is continous
You need to be continous to have a chance at being differentiable.

Also write $\frac{d}{dx}[f(x)]$ to mean $f'(x)$

$$\frac{d}{dx}[0] = 0$$

$$\frac{d}{dx}[x^p] = px^{p-1}$$

$$\frac{d}{dx}[cf] = c\,\frac{d}{dx}[f]$$ for any number c

$$(f+g)' = f' + g'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$

Ex: $$\frac{d}{dx}[\sin(x^3)] = \frac{d}{dx}[\sin(x)]\Big|_{x^3}\,\frac{d}{dx}[x^3]$$

$$= \cos(x^3)\,3x^2 = 3x^2\cos(x^3).$$

$$\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$$

$$\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$$

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f(x))}$$

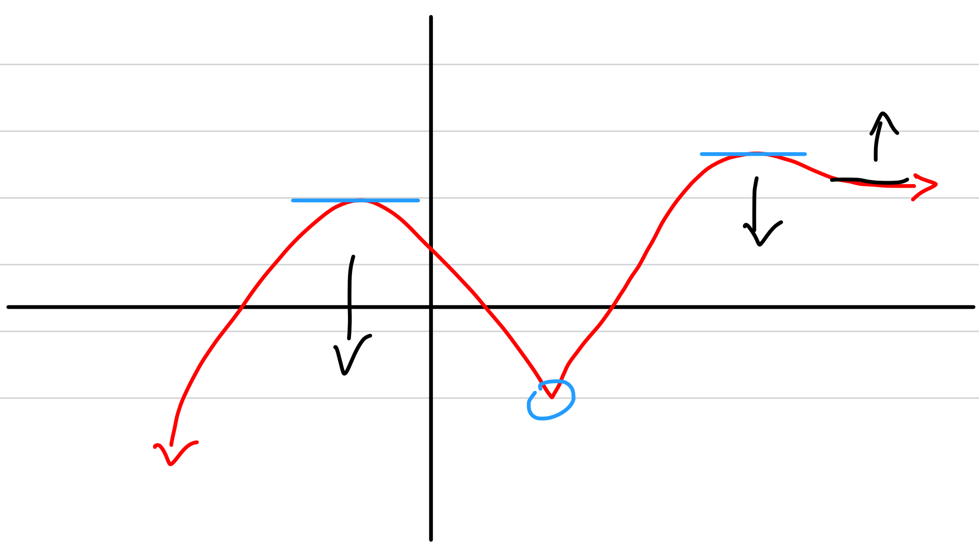
$$\frac{d}{dx}\left[b^x\right] = b^x \ln(b)$$

$$\frac{d}{dx}\left[e^x\right] = e^x \leftarrow \star$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}.$$

Optimizing functions (i.e. finding extrema)

A critical point $x = a$ is a point where $f'(a)$ is zero or undefined.

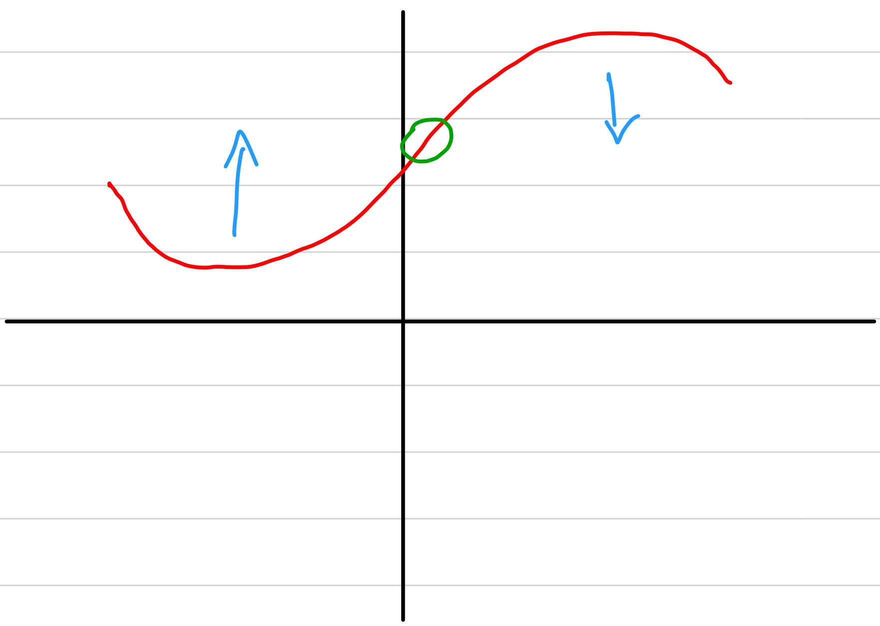


f'' measures concavity of f $\cup$ = concave up

$\cap$ concave down

Plug critical points into f'' to determine if they're maxima or minima: if $f''(a) < 0$, the function is concave down at a, and so a is a local maximum.

Inflection point: $f''(a) = 0$

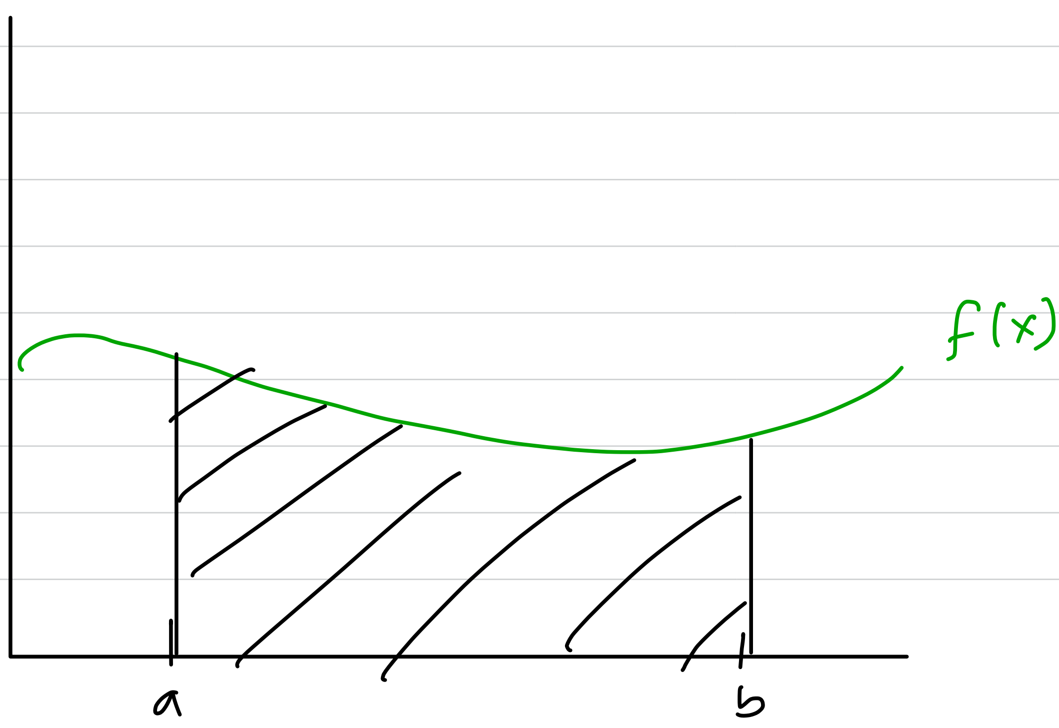


L'Hôpital: if $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$,

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$.

Ex: $\lim\limits_{x \to \infty} \frac{x}{x^2} = \frac{\infty}{\infty}$, so $\lim\limits_{x \to \infty} \frac{x}{x^2} = \lim\limits_{x \to \infty} \frac{1}{2x} = 0$

Comment: We have two goals: find areas under curves and run differentiation backward.



We'll find the area under this curve by taking better and better approximations and then taking a limit. This will involve adding a lot of small areas, so we need tools for dealing with sums.

Def: Sigma notation. We write $\sum_{i=1}^{n} a_i$ to mean $a_1 + a_2 + a_3 + \dots + a_n$, and we read it as "sum from $i=1$ to n of a_i".

i is called the index variable (you can use any variable, not just i)

Ex: $\sum_{i=2}^{5} 3 = \underset{i=2}{3} + \underset{i=3}{3} + \underset{i=4}{3} + \underset{i=5}{3} = 12$

$$\sum_{i=-2}^{1}(i-2) = (-2-2)+(-1-2)+(0-2)+(1-2)$$

$$= -4-3-2-1 = -10$$

Ex: We can work backward:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25} = \sum_{i=1}^{5}\frac{1}{i^2}.$$

Prop: ① $\sum_{i=1}^{n} c\,a_i = c\sum_{i=1}^{n} a_i$

② $\sum_{i=1}^{n} c = nc$

③ $\sum_{i=1}^{n}(a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

$\hookrightarrow (1+2)+(3+4)+(5+6) = (1+3+5)+(2+4+6)$

④ $\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i$

$\hookrightarrow a_1+a_2+a_3+\cdots+a_n = (a_1+a_2+\cdots+a_k)+(a_{k+1}+\cdots+a_n)$

Ex: $\sum_{i=-5}^{4} i = \underbrace{\sum_{i=-5}^{0} i + \sum_{i=1}^{4} i}_{k=0} = (-5-4-3-2-1+0) + (1+2+3+4) = -15+10 = -5.$

Prop: (1) $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

(2) $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

(3) $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

Ex: $1^2 + 2^2 + 3^2 + \cdots + 100^2 = \sum_{i=1}^{100} i^2$

$= \dfrac{100(101)(201)}{6} = 338350$

Ex: $10^3 + 11^3 + 12^3 + \cdots + 30^3$

We're not starting at 1, so we can't directly use our formula.

want

$$\sum_{i=1}^{30} i^3 = \sum_{i=1}^{9} i^3 + \sum_{i=10}^{30} i^3$$

$$\frac{30^2(31)^2}{4} = \frac{9^2(10^2)}{4} + \sum_{i=10}^{30} i^3$$

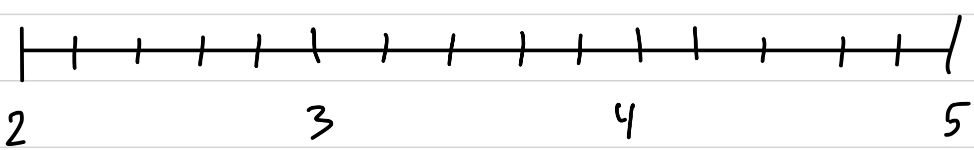
$$216225 = 2025 + \sum_{i=10}^{30} i^3$$

$$\sum_{i=10}^{30} i^3 = 216225 - 2025 = 214200.$$

Def: A <u>partition</u> of the interval $[a,b]$ is a set $\{x_0, x_1, \dots, x_n\}$ such that $x_0 < x_1 < x_2 < \cdots < x_n$ and $x_0 = a$ and $x_n = b$. The partition is <u>regular</u> if all the x_i are the same distance from one another.

Ex: A partition of $[2,5]$ is $\{2, 2.1, 2.9, 3, 4, 5\}$.

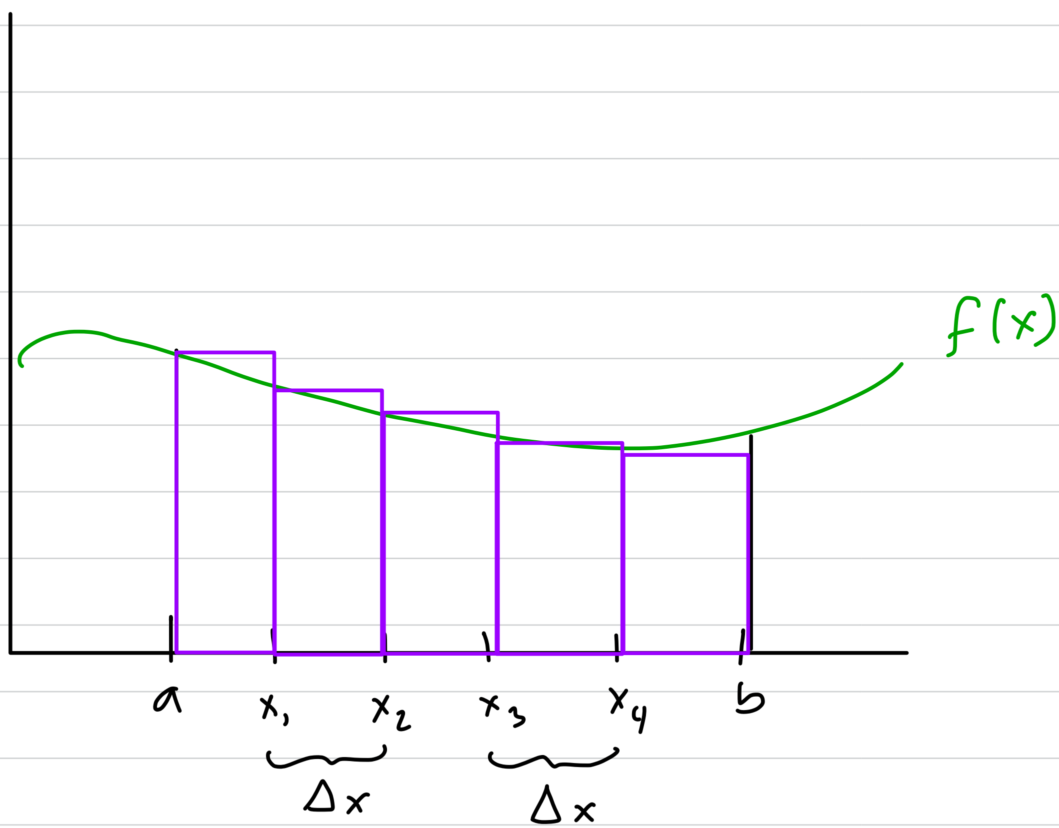
A regular partition of $[2,5]$ is $\{2, 2.2, 2.4, 2.6, \cdots, 4.8, 5\}$.



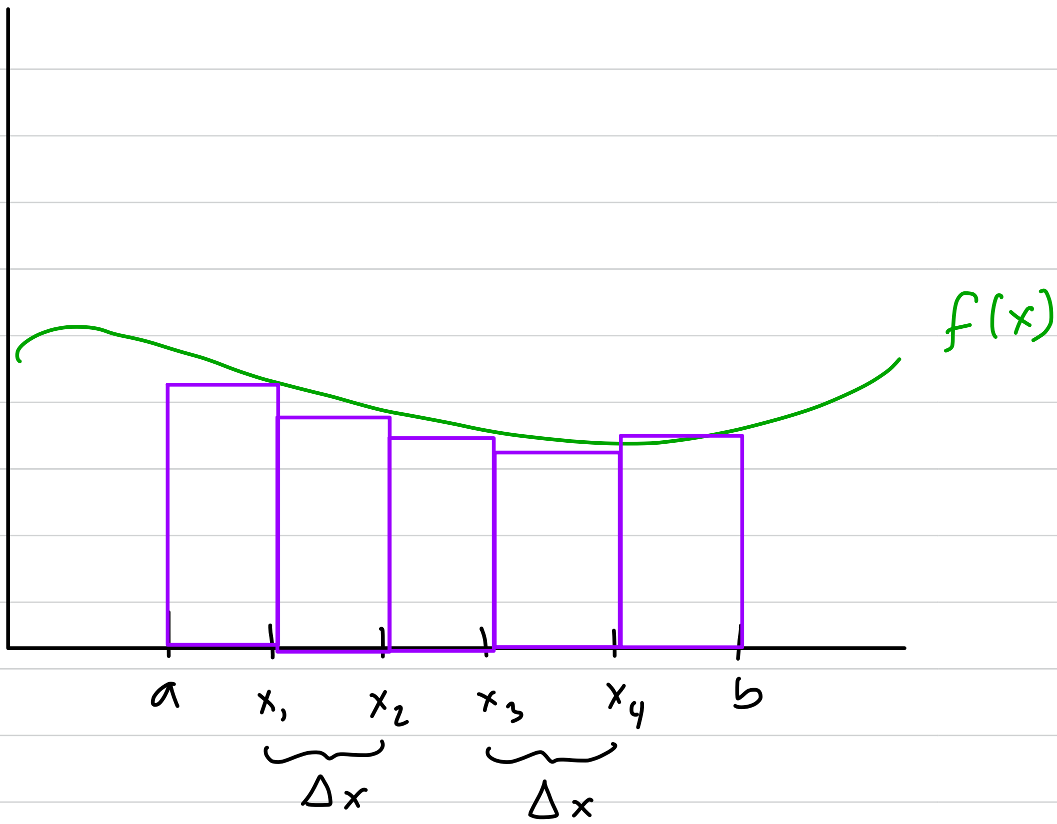
Def: Let f be a nonnegative function on $[a,b]$ and $\{x_0, x_1, \cdots, x_n\}$ a regular partition of $[a,b]$. Let A be the area under the graph of f on $[a,b]$. The left-endpoint

approximation of A is $L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$,
where Δx is the distance between the x_i.
Note: this is a sum of rectangles. The width is Δx and the height is $f(x_{i-1})$.
Similarly, the right-endpoint approximation of A is $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$,

Ex:



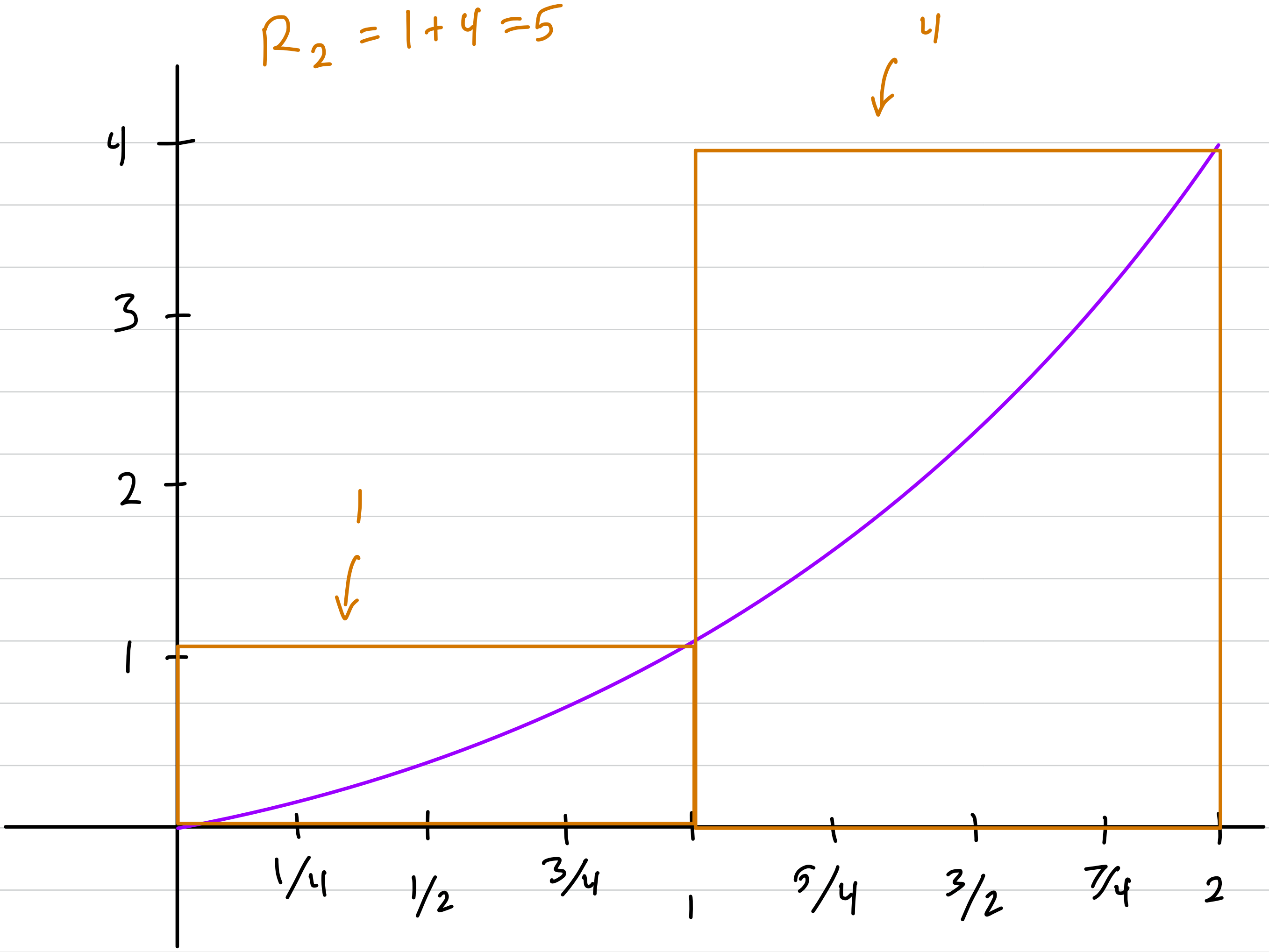
L_5 is the sum of the areas of the purple rectangles.



R_5 is the sum of the areas of the purple rectangles.

Ex: Compute R_2, R_4, and R_8 for $f(x)=x^2$ on $[0,2]$.

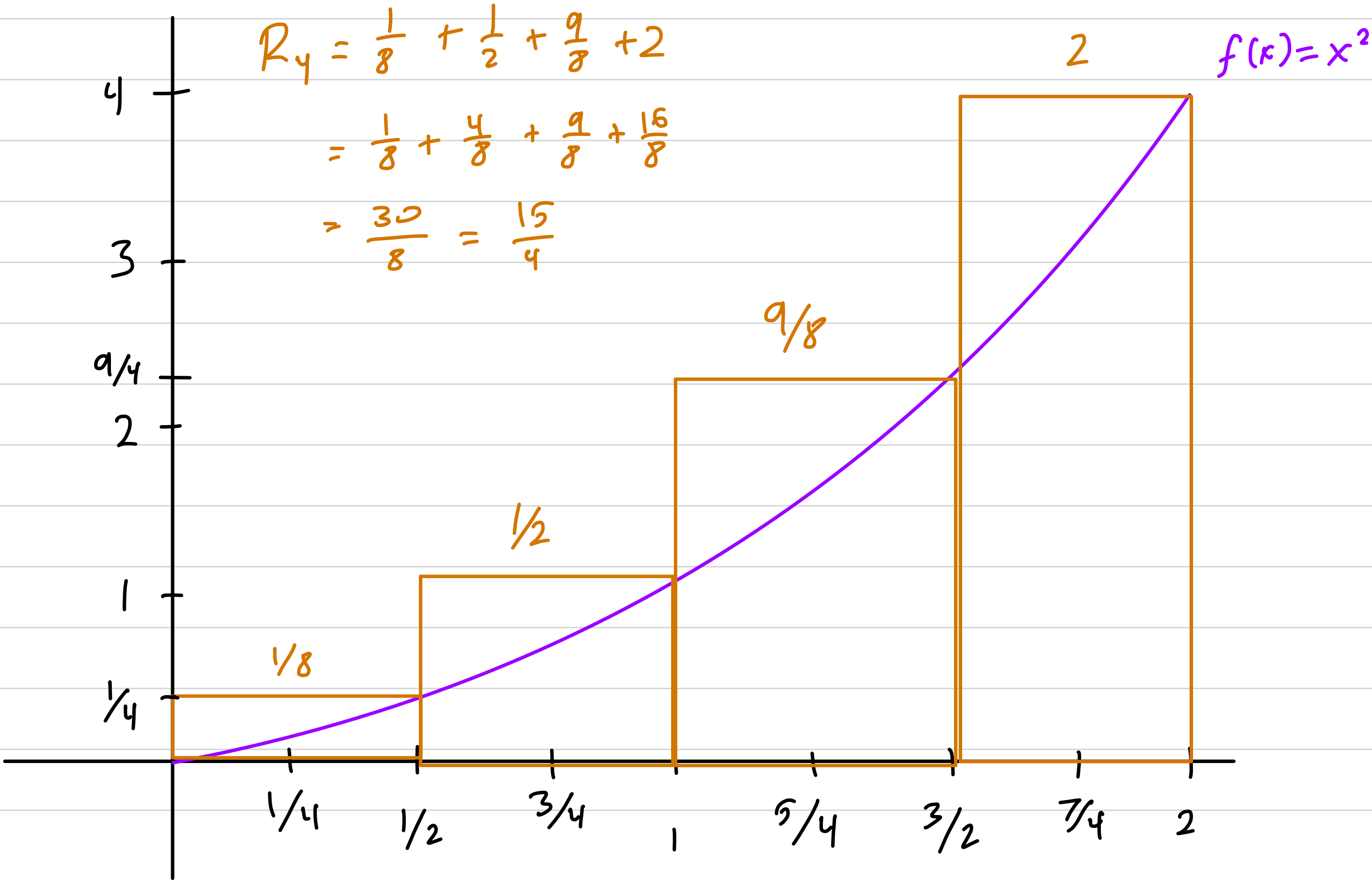
$R_2 = 1 + 4 = 5$

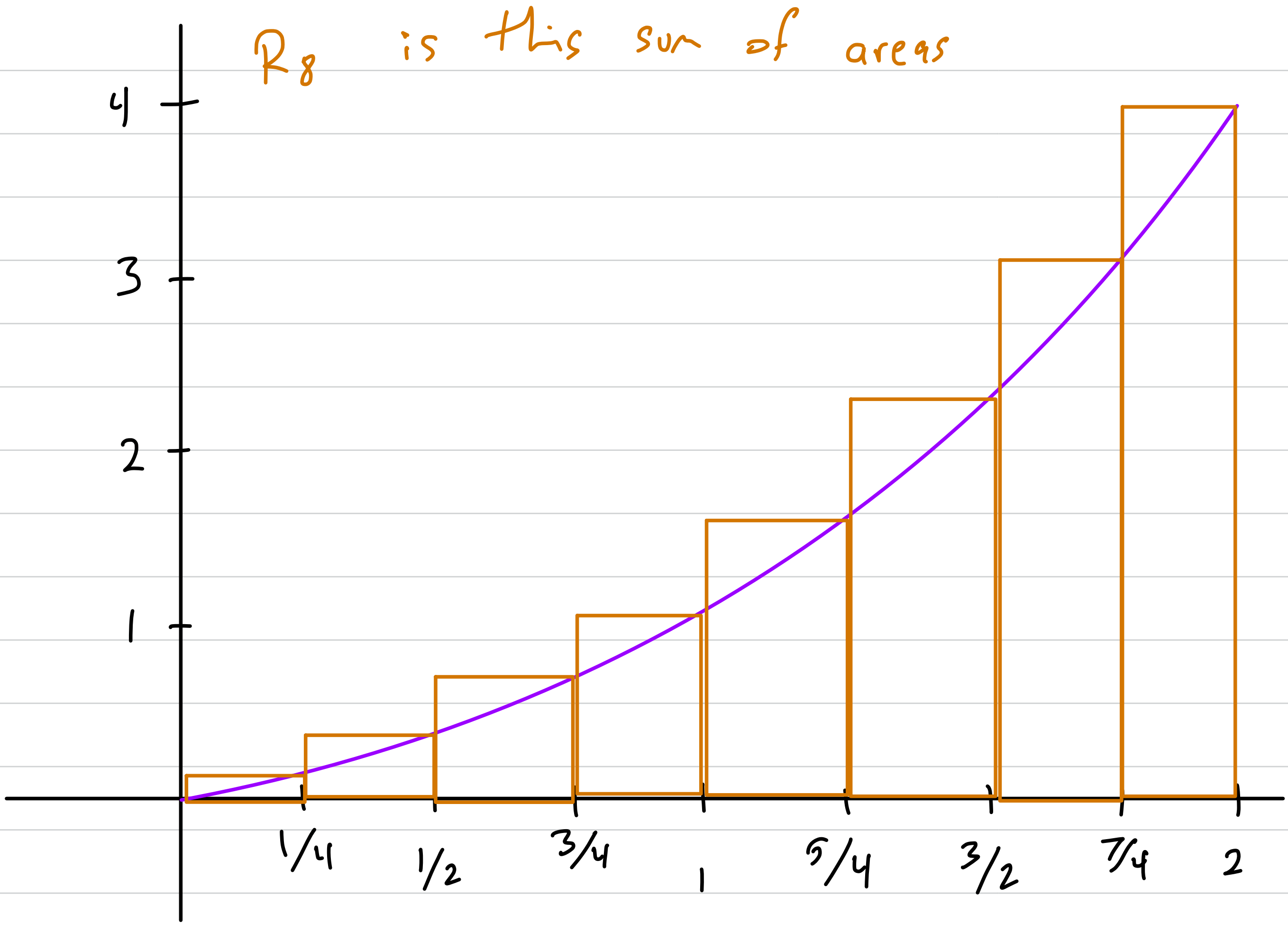


$R_4 = \frac{1}{8} + \frac{1}{2} + \frac{9}{8} + 2$

$= \frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \frac{16}{8}$

$= \frac{30}{8} = \frac{15}{4}$

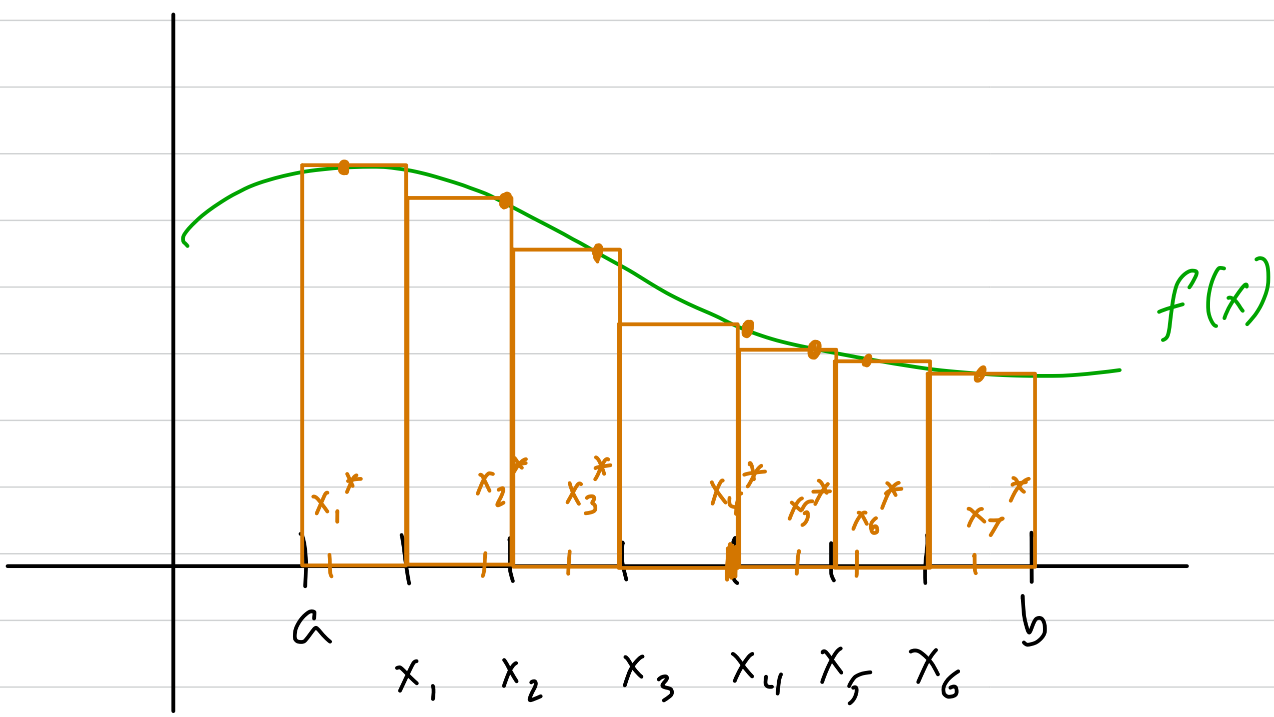




Comment: As n gets bigger, this approximation gets better. To make this work, we need something slightly more general than L_n or R_n.

Def: Let f be a nonnegative function on $[a,b]$. Let $\{x_0, x_1, \cdots, x_n\}$ be a regular partition of $[a,b]$. A Riemann sum of f on $[a,b]$ with this partition is $\sum_{i=1}^{n} f(x_i^*)\Delta x$, where x_i^* is <u>any</u> x between x_{i-1} and x_i. (The idea is that x_i^* can be any point in the subinterval, not just the left or right endpoint.)

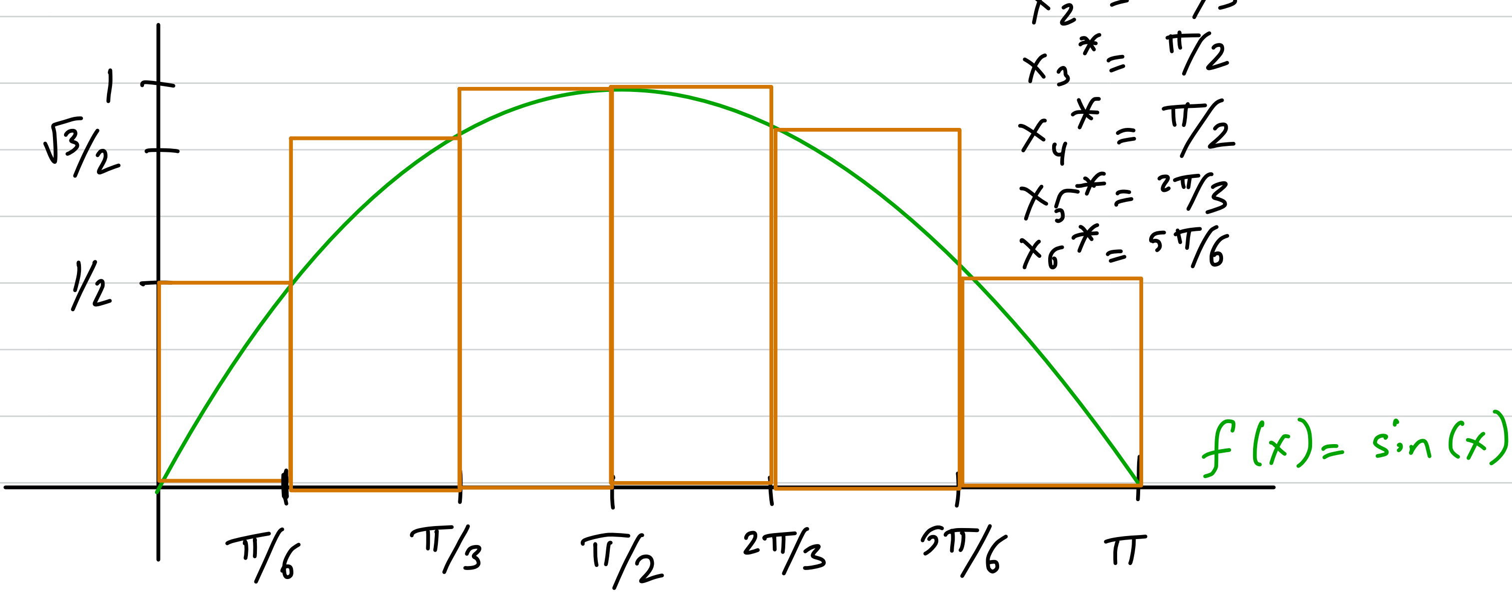
Ex:



Def: Let f be as before. The <u>upper sum</u> of f is the Riemann sum where every x_i^* is chosen to maximize f on its subinterval. The <u>lower sum</u> is the same, but where the x_i^* are chosen to minimize f.

Ex: Find the upper and lower sums with 6 subintervals of the function $f(x) = \sin(x)$ on $[0, \pi]$.

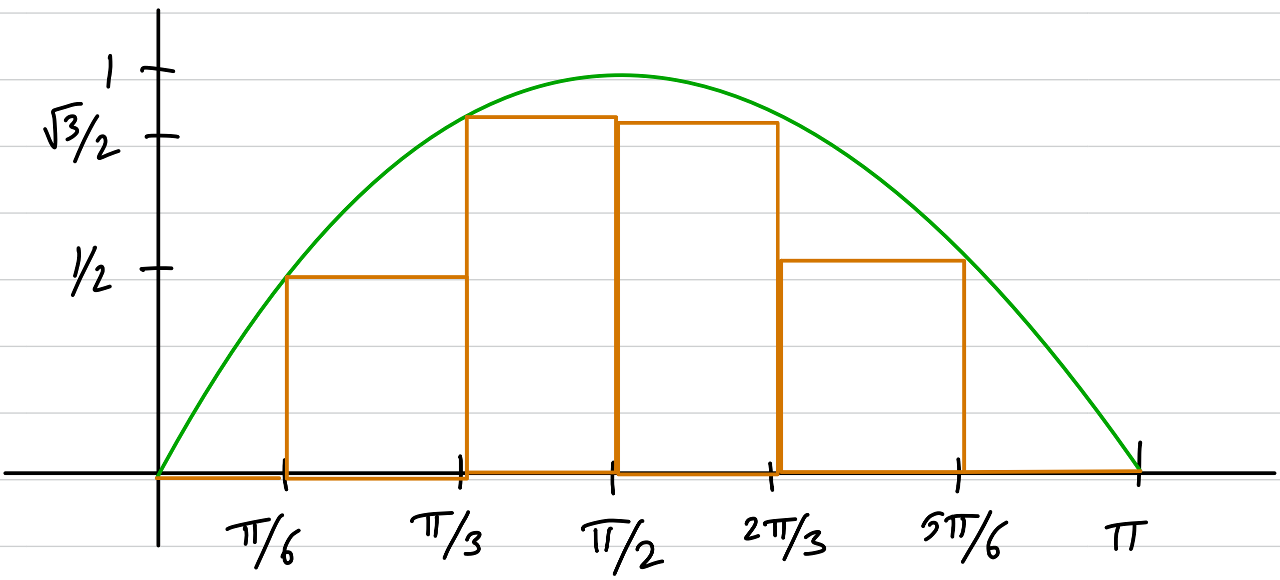
$x_1^* = \pi/6$
$x_2^* = \pi/3$
$x_3^* = \pi/2$
$x_4^* = \pi/2$
$x_5^* = 2\pi/3$
$x_6^* = 5\pi/6$



upper sum: $\left(\frac{\pi}{6}\cdot\frac{1}{2}\right)+\left(\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2}\right)+\left(\frac{\pi}{6}\cdot 1\right)+\left(\frac{\pi}{6}\cdot 1\right)+\left(\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2}\right)+\left(\frac{\pi}{6}\cdot\frac{1}{2}\right)$

$$= \frac{\pi}{6}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}+1+1+\frac{\sqrt{3}}{2}+\frac{1}{2}\right)$$

$$= \frac{\pi}{6}\left(3+\sqrt{3}\right)$$



lower sum $= \frac{\pi}{6}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{1}{2}\right)$

$$= \frac{\pi}{6}\left(1+\sqrt{3}\right)$$

So, the area A under the graph of $\sin(x)$ on $[0,\pi]$ satisfies $\frac{\pi}{6}(1+\sqrt{3}) \leq A \leq \frac{\pi}{6}(3+\sqrt{3})$

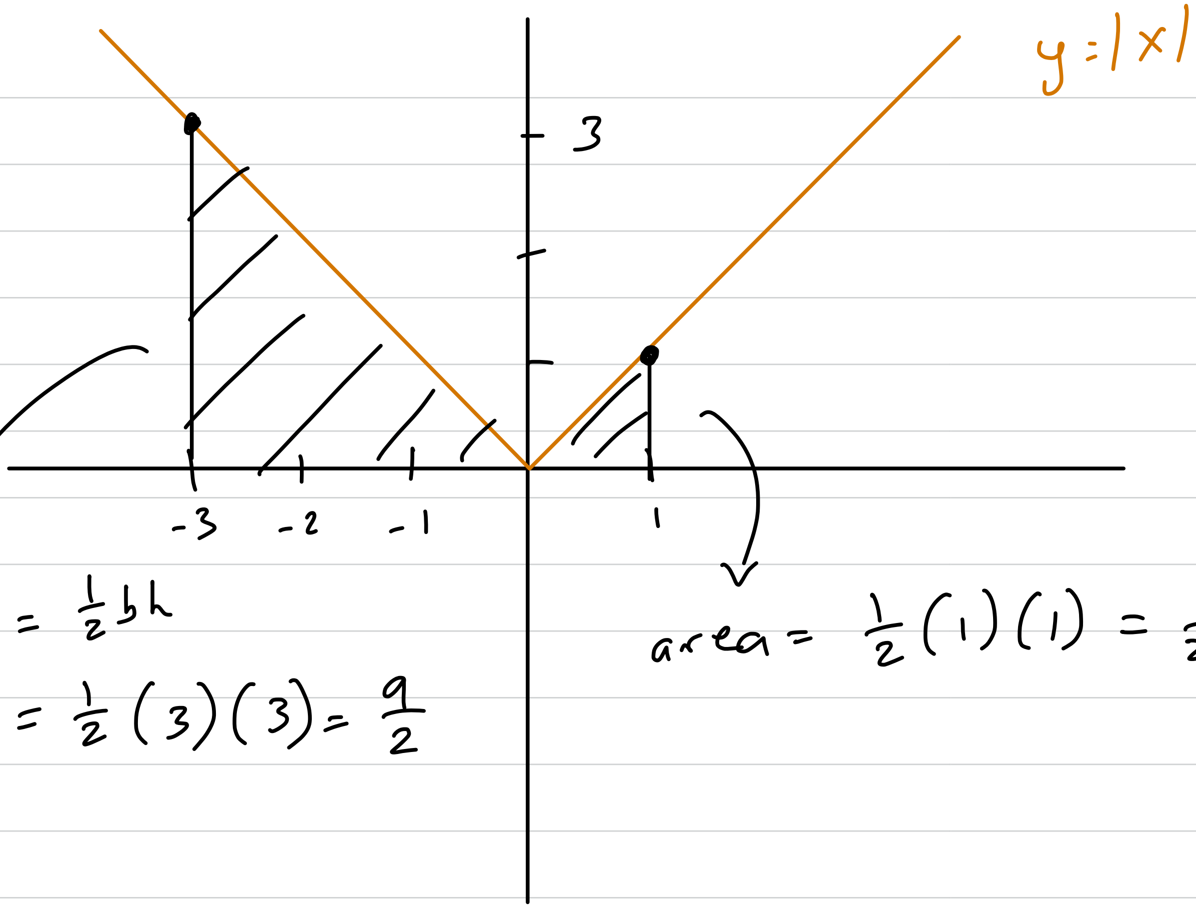
Thm: The area A under the graph of f on $[a,b]$ is $A = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\,\Delta x\right)$

Def: Let f be a nonnegative function on $[a,b]$. The <u>definite integral</u> of f on $[a,b]$ is the area under the graph of f on $[a,b]$, denoted $\int_a^b f(x)\,dx$, if the limit $\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\,\Delta x\right)$ exists. If so, we say f is <u>integrable</u> on $[a,b]$. We call a and b the <u>limits of integration</u> and we read $\int_a^b f(x)\,dx$ as "integral from a to b of $f(x)\,dx$" (or "of f")

Thm: Continuous functions are integrable.

Comment: Right now, we have two ways to calculate an integral: use geometry if the function is simple, or we can use the limit def directly. This is similar to when you first learned about derivatives: for very simple functions like lines, you know the derivative already, and there is a limit def. There are a bunch of properties that make derivatives easier, and we'll have those for integrals eventually.

Ex: Find $\int_{-3}^{1} |x|\, dx$.

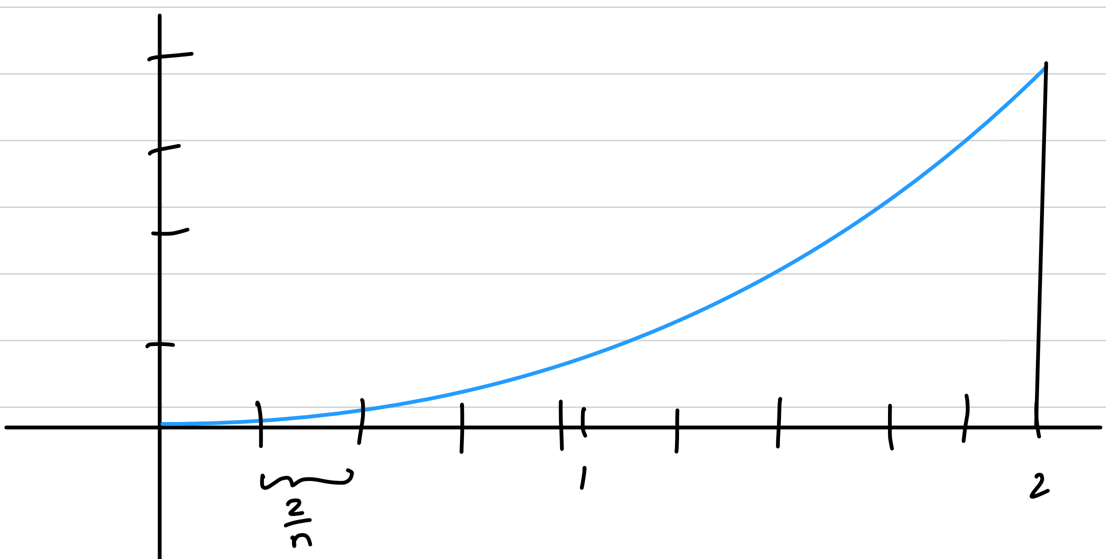


$$\text{area} = \tfrac{1}{2}bh$$
$$= \tfrac{1}{2}(3)(3) = \tfrac{9}{2}$$

$$\text{area} = \tfrac{1}{2}(1)(1) = \tfrac{1}{2}$$

$$\int_{-3}^{1} |x|\,dx = \frac{9}{2} + \frac{1}{2} = \frac{10}{2} = 5.$$

Ex: Evaluate $\int_0^2 t^2\,dt$



$$\underbrace{}_{\frac{2}{n}}$$

First, get a regular partition of $[0,2]$ with n subintervals.

The width of each subinterval is $\frac{2}{n}$.

The ith subinterval is $\left[\frac{2}{n}(i-1), \frac{2}{n}i\right]$

Can pick x_i^* to be any point in that interval — let's pick the right endpoint $\frac{2}{n}i$.

So, $$\int_0^2 x^2\,dx \overset{\text{DEF}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2}{n}i\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}i\right)^2 \frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n^2} i^2 \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n \to \infty} \left(\frac{8}{n^3} \cdot \frac{2n^3 + n^2 + 2n^2 + n}{6} \right)$$

$$= \lim_{n \to \infty} \left(\frac{2 \cdot 8}{6} + \frac{8 \cdot 3}{6n} + \frac{8}{6n^2} \right)$$

$$= \frac{2 \cdot 8}{6}$$

$$= 8/3.$$