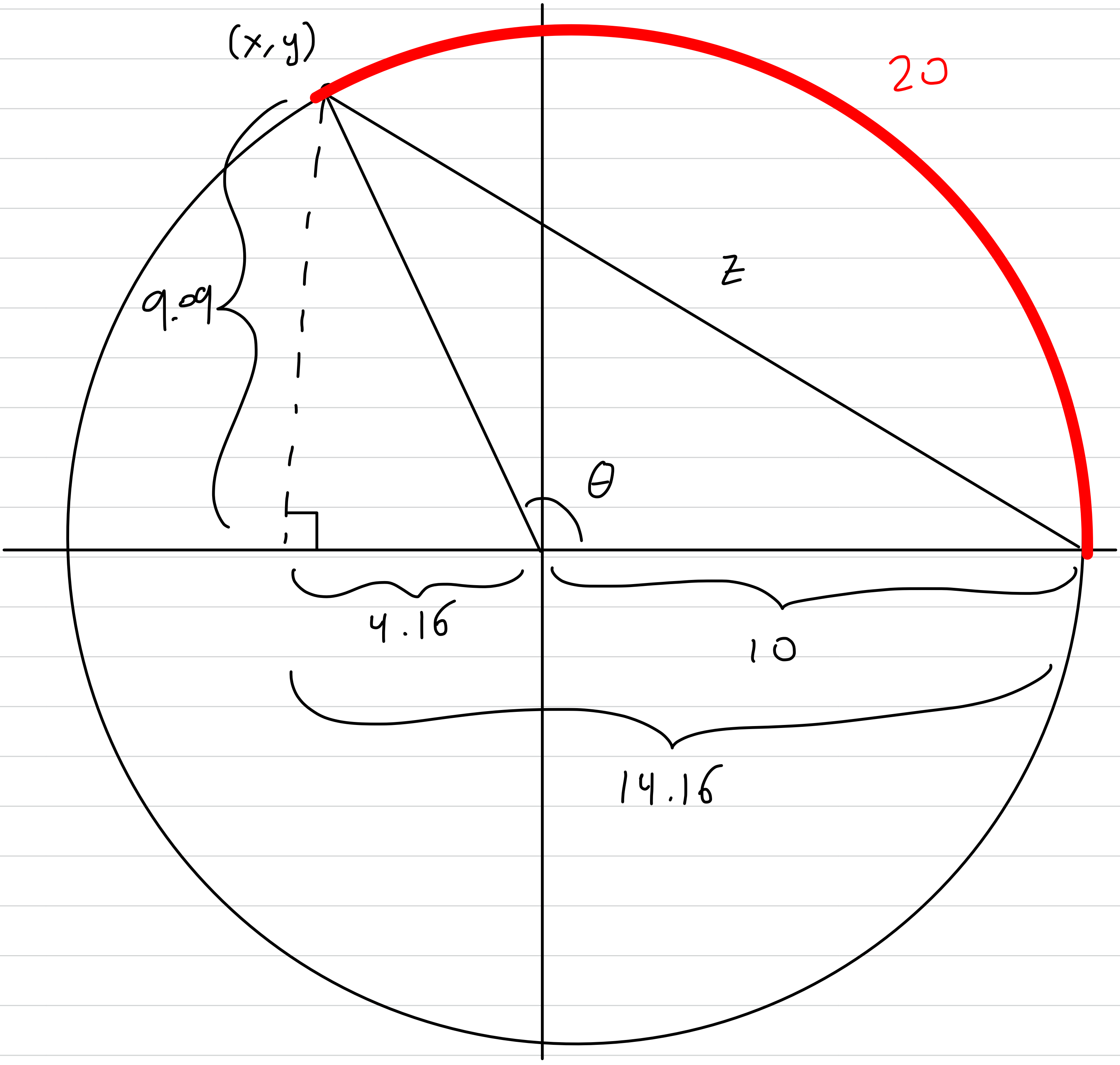
Ex: Find x, y, and z:



$$10\theta = 20$$

$$\theta = 2 \text{ (in radians)}$$

$x = 10\cos(2)$ $y = 10\sin(2)$

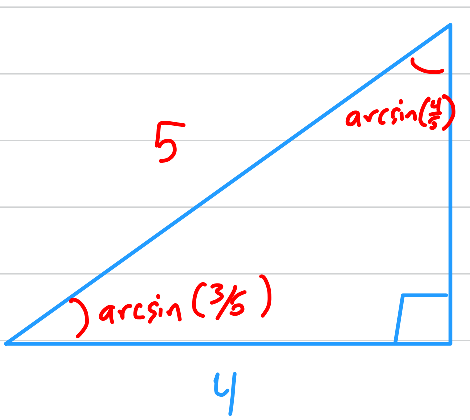
$x = -4.16$ $y = 9.09$

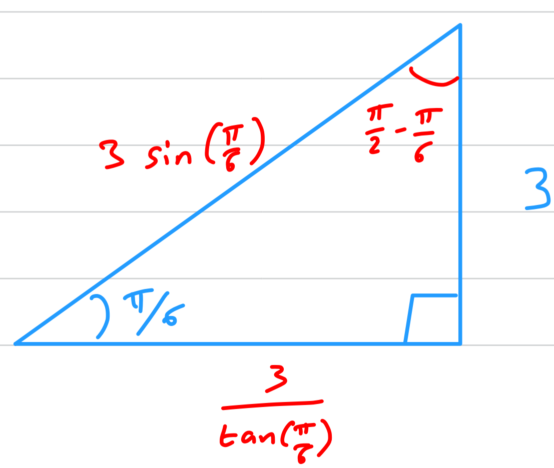
$$14.16^2 + 9.09^2 = z^2$$

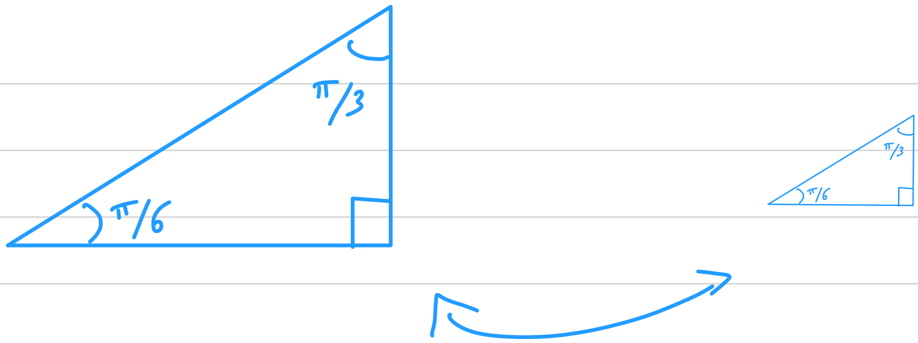
$$\Rightarrow z = 16.83$$

Trig in Non-right Triangles

<u>Comment</u>: In any triangle, there are six pieces of information: three sides and three angles. In a right triangle, we know one of the six: one angle. If we know two more pieces of information and one is a side, we can find all six.





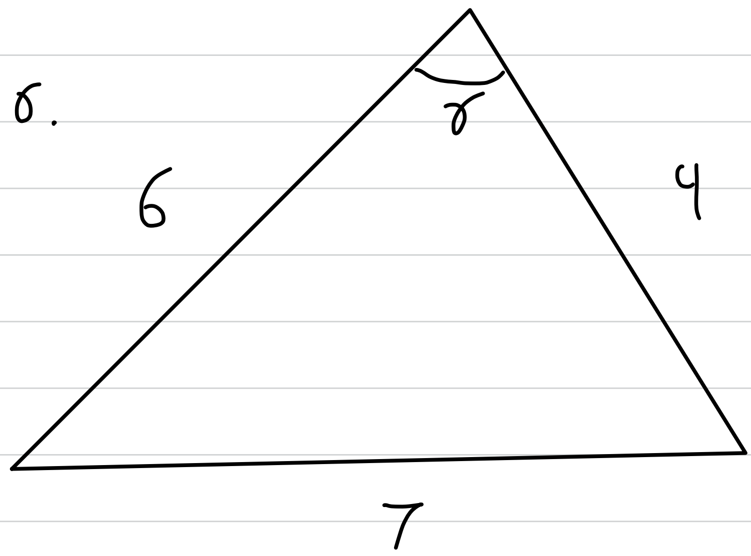


same angles
but different sides

In general, we'll need three pieces of information about a non-right triangle, and they can't all be angles.

<u>Theorem</u> (The Law of Cosines) In any triangle with sides a, b, and c, and angle γ (gamma) opposite side c, $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$

Ex: Find γ.



$c = 7$

$a = 6$
$b = 4$

$7^2 = 6^2 + 4^2 - 2(6)(4)\cos(\gamma)$

$49 = 52 - 48\cos(\gamma)$

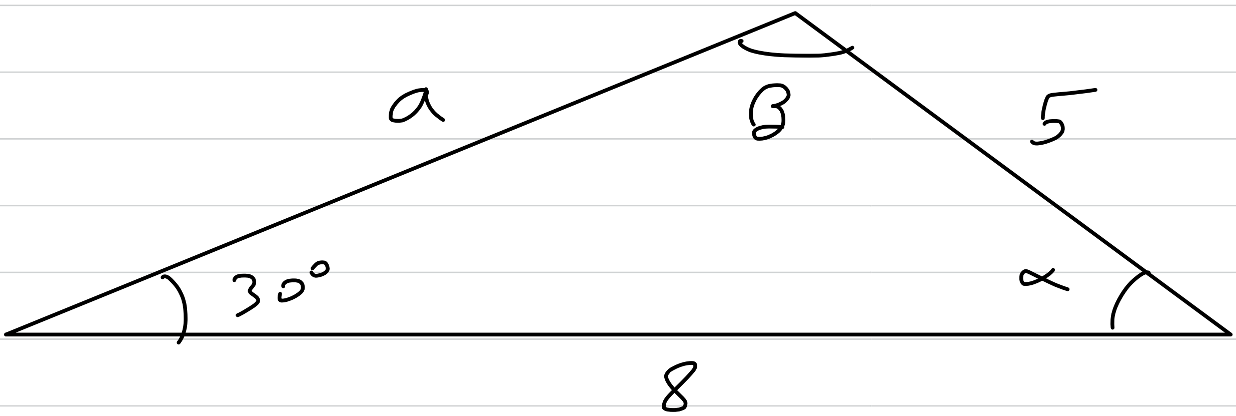
$-48\cos(\gamma) = -3$

$\cos(\gamma) = 1/16$

Since $0 \le \gamma \le \pi$, $\gamma = \arccos\left(\frac{1}{16}\right)$

$\gamma = 1.51$

Ex: Find a, α, and β.



Law of Cosines to:

$$\left.\begin{array}{ll} \alpha: & a^2 = 8^2 + 5^2 - 2(8)(5)\cos(\alpha) \\ \beta: & 13^2 = a^2 + 5^2 - 2(5)(a)\cos(\beta) \end{array}\right\} \text{two unknowns}$$

$$30°: \quad 5^2 = a^2 + 8^2 - 2(8)(a)\cos(30°)$$

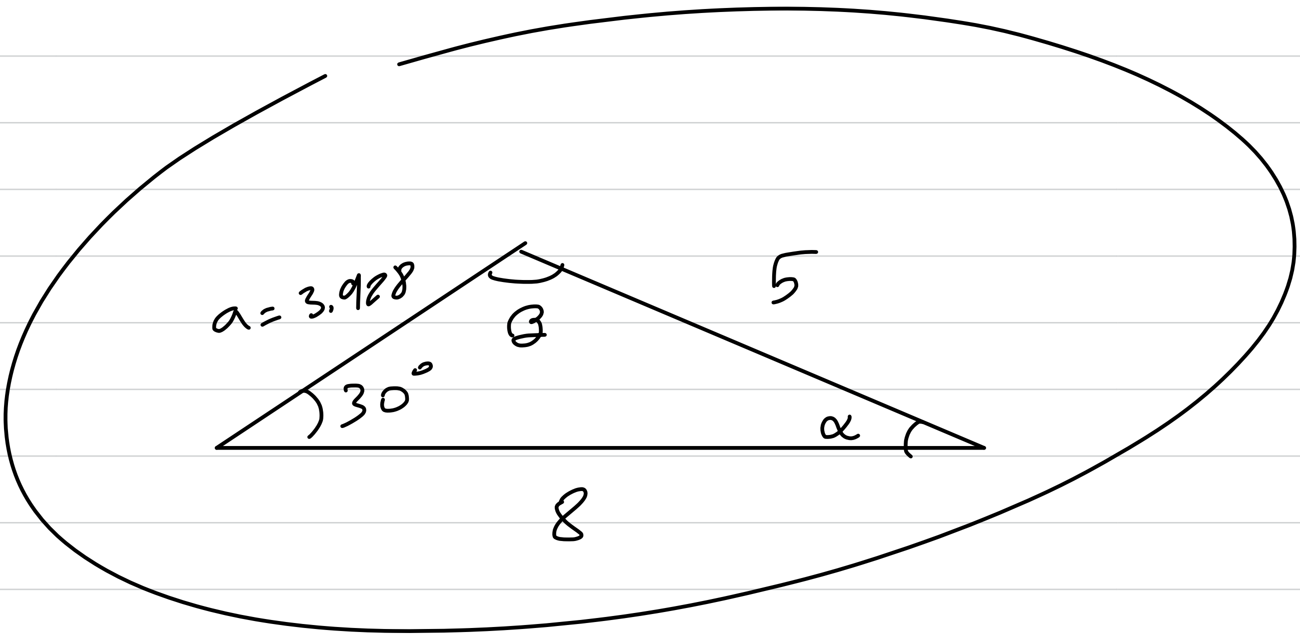
$$25 = a^2 + 8^2 - 16a\frac{\sqrt{3}}{2}$$

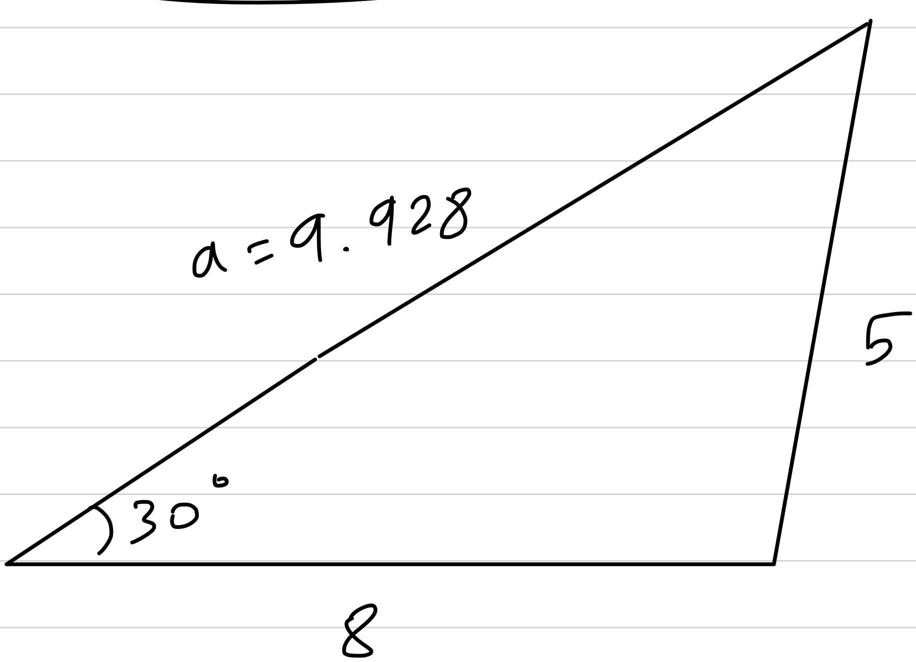
$$a^2 - 8\sqrt{3}\,a + 39 = 0$$

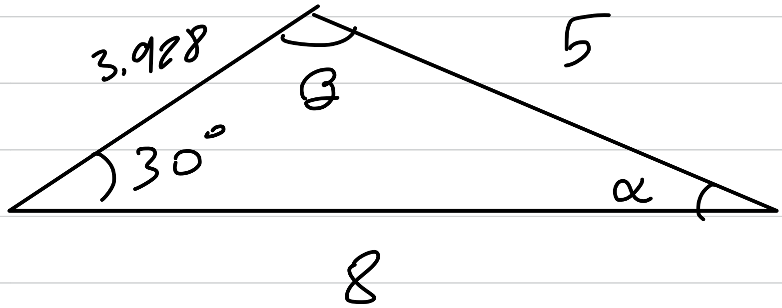
$$a = \frac{8\sqrt{3} \pm \sqrt{64\cdot 3 - 4\cdot 39}}{2} = 9.928 \text{ or } 3.928$$

$\Rightarrow$ there are actually two different triangles that satisfy the picture.

Matches our picture







Apply Law of Cosines to α

$$3.928^2 = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cos(\alpha)$$

$$15.431 = 25 + 64 - 80 \cos(\alpha)$$

$$-73.569 = -80 \cos(\alpha)$$

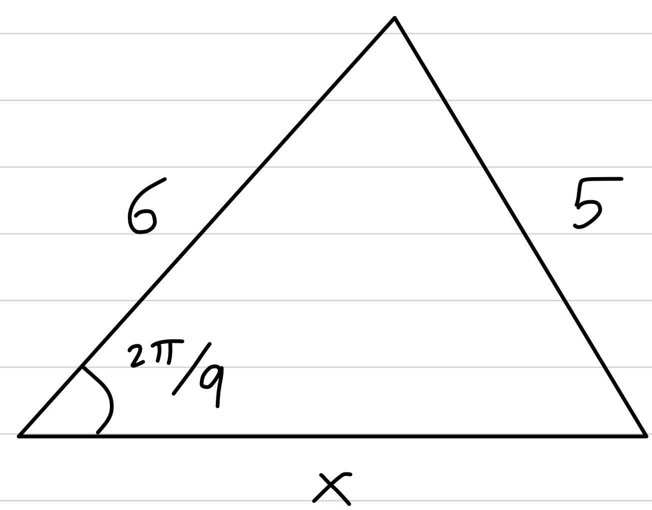
$$\cos(\alpha) = .9196$$

Since $0^\circ \leq \alpha \leq 180^\circ$, $\alpha = \arccos(.9196) = 23.1^\circ$

$$30^\circ + 23.1^\circ + \beta = 180^\circ$$

$$\beta = 126.9^\circ.$$

Ex: Find x, where x is the longest side in the triangle.



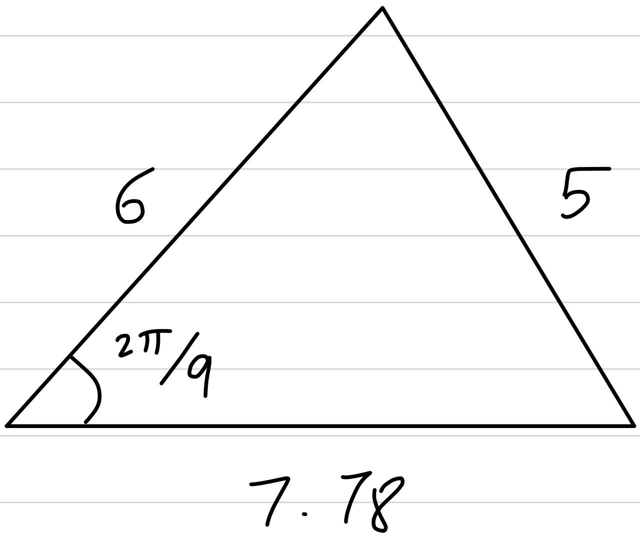
$$5^2 = 6^2 + x^2 - 2 \cdot 6 \cdot x \cos(2\pi/9)$$

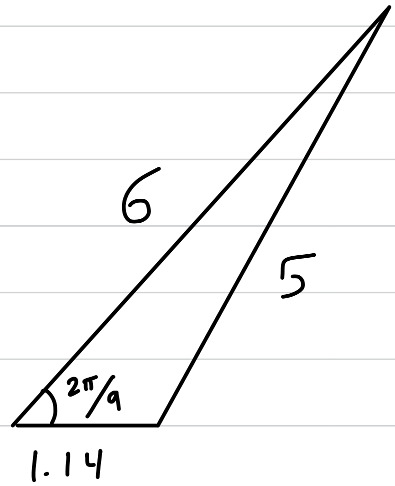
$$25 = 36 + x^2 - 12x\,(.766)$$

$$x^2 - 9.193x + 11 = 0$$

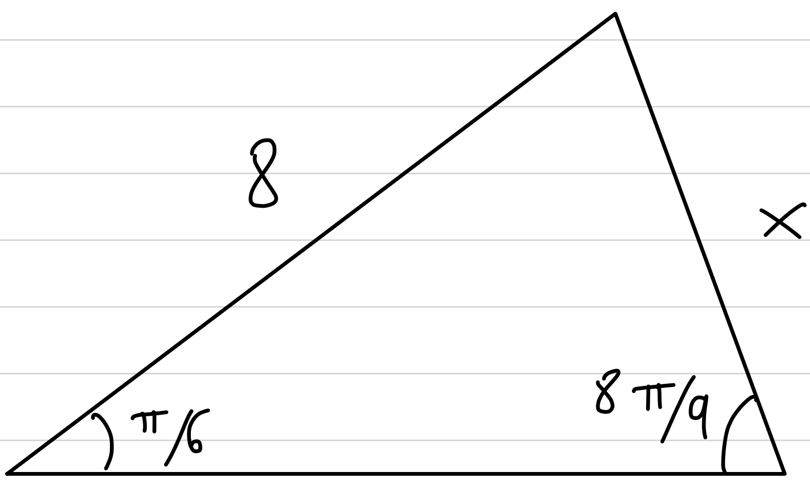
$$x = \frac{9.193 \pm \sqrt{9.193^2 - 44}}{2} = 1.414 \text{ or } 7.78$$

Since x is the longest side in the triangle, $x = 7.78$.



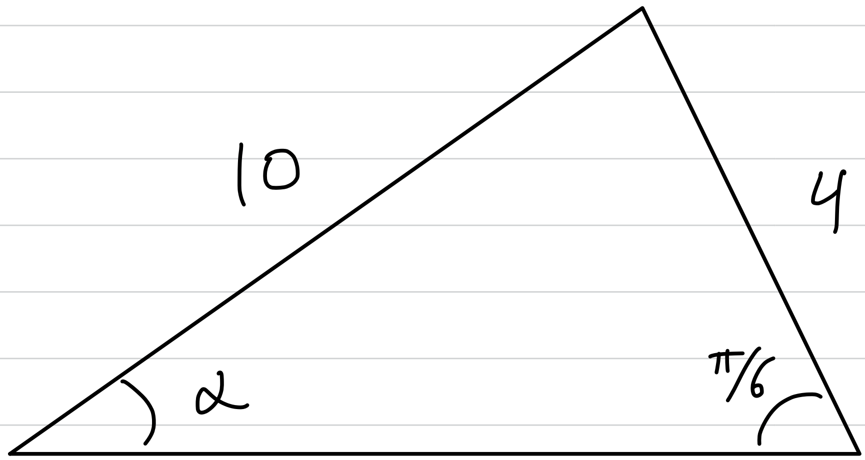


Ex: Find x.



We can't use LoC, because we only know one side.

Ex: Find α.

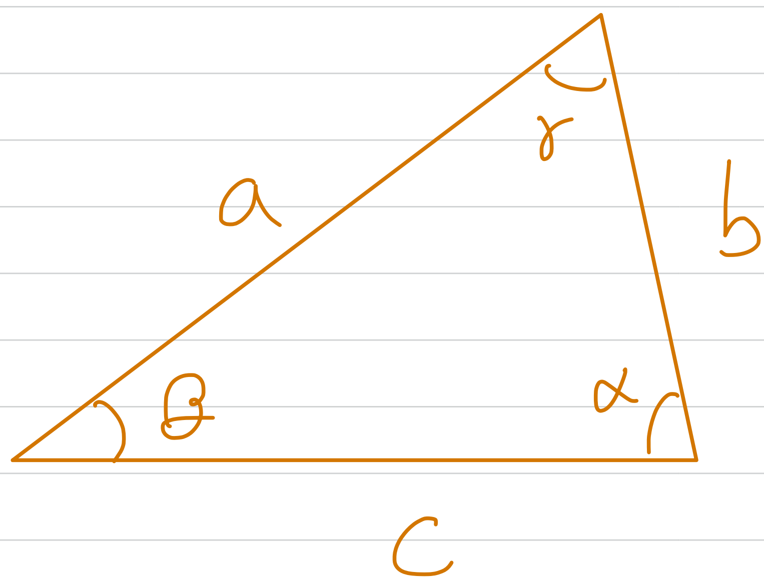


It's possible to use LoC, but you have to go out of your way to find the bottom side first.

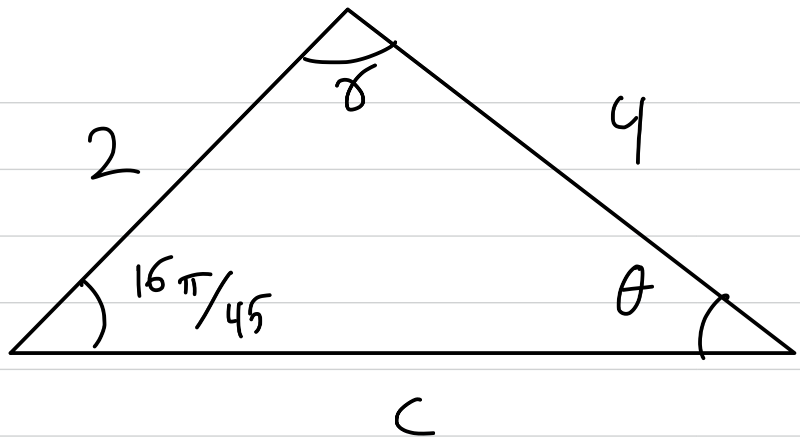
In both of these examples, there's a better way.

Theorem (The Law of Sines): In any triangle with sides a, b, and c, and angles α, β, and γ opposite a, b, and c, respectively,

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$



Ex: Find θ.



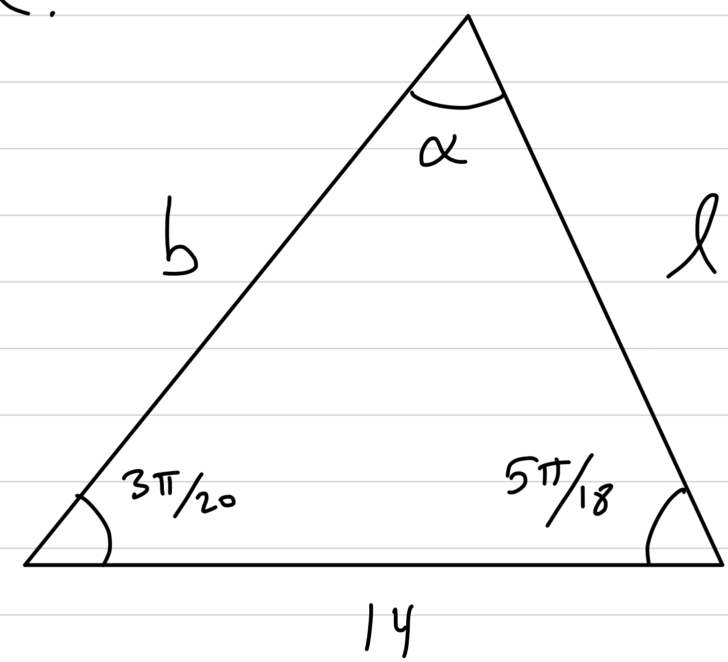
$$\boxed{\frac{\sin(16\pi/45)}{4} = \frac{\sin(\theta)}{2}} = \frac{\sin(\gamma)}{c}$$

$$\sin(\theta) = \frac{\sin(16\pi/45)}{2} = .449$$

With LoS, just try to take the arc sin. If it's positive, everything's fine, and if not, use reference angles as usual.

$$\theta = \arcsin(.449) = .467$$

Ex: Find ℓ.



$$\boxed{\frac{\sin(\alpha)}{14}} = \frac{\sin(5\pi/18)}{b} = \boxed{\frac{\sin(3\pi/20)}{\ell}}$$

This looks bad, because we have two unknowns no matter which two sides of the equation we pick.

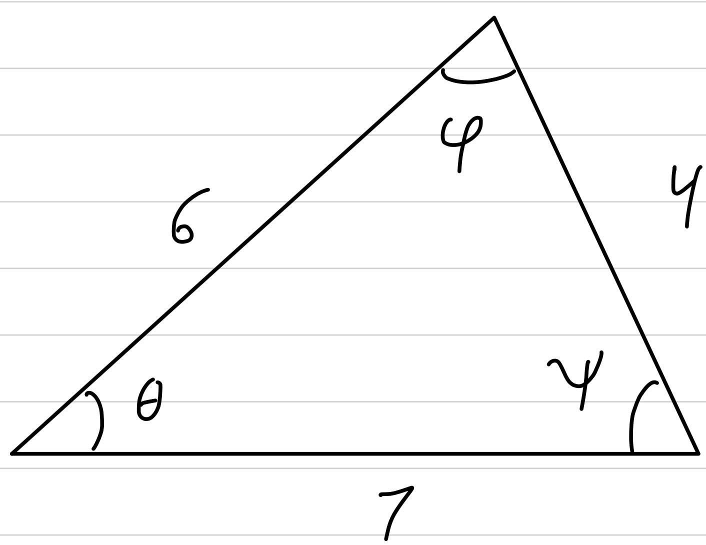
But! We can solve for α, because

$3\pi/20 + 5\pi/18 + \alpha = \pi$, so $\alpha = \dfrac{103\pi}{180}$

$$\frac{\sin(103\pi/180)}{14} = \frac{\sin(3\pi/20)}{\ell}$$

$$\ell = \frac{14\sin(3\pi/20)}{\sin(103\pi/180)} = 6.52.$$

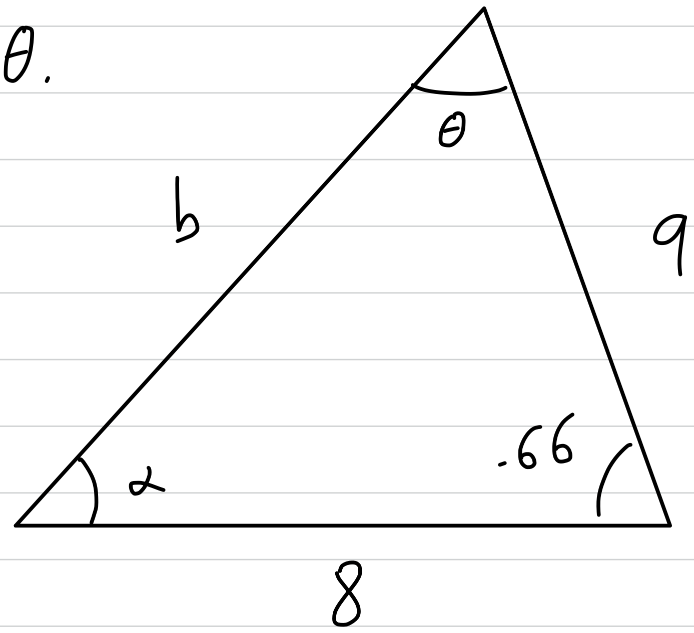
Ex: Find all the angles of this triangle:



LoC: $4^2 = 6^2 + 7^2 - 2\cdot 6\cdot 7\cdot\cos(\theta)$ helpful

LoS: $\frac{\sin(\theta)}{4} = \frac{\sin(\varphi)}{7} = \frac{\sin(\psi)}{6}$ not helpful

Ex: Find θ.



LoC: $8^2 = 9^2 + b^2 - 2 \cdot 9 \cdot b \cdot \cos(\theta)$

LoS: $\dfrac{\sin(\theta)}{8} = \dfrac{\sin(.66)}{b} = \dfrac{\sin(\alpha)}{9}$

Neither of these lets us solve for θ!

The problem isn't impossible, we just can't do it directly. The solution: use LoC on a different angle.

LoC on α: $9^2 = b^2 + 8^2 - 2 \cdot 8 \cdot b \cdot \cos(\alpha)$

LoC on .66: $b^2 = 9^2 + 8^2 - 2 \cdot 9 \cdot 8 \cdot \cos(.66)$

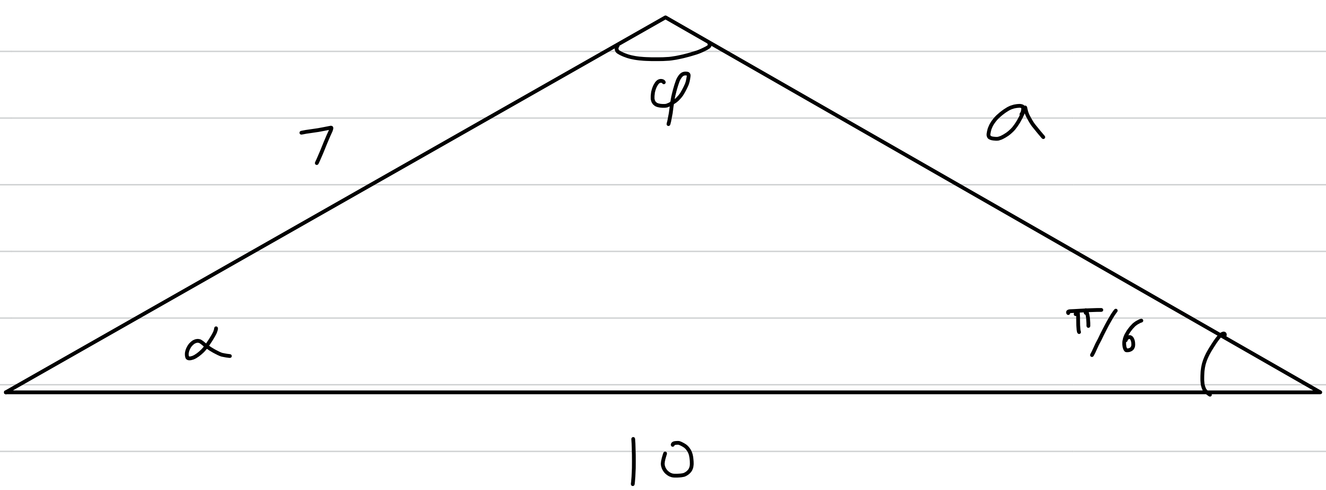
$$b = 5.59$$

LoS: $$\frac{\sin(\theta)}{8} = \frac{\sin(.66)}{5.59}$$

$$\sin(\theta) = \frac{8\sin(.66)}{5.59} = .877$$

$$\arcsin(.877) = 1.41 \text{ (in radians)}$$

Ex: Find φ.



LoC on φ: $10^2 = 7^2 + a^2 - 2 \cdot 7 \cdot a \cdot \cos(\varphi)$

LoS: $\dfrac{\sin(\varphi)}{10} = \dfrac{\sin(\pi/6)}{7} = \dfrac{\sin(\alpha)}{a}$

$$\sin(\varphi) = \frac{10 \sin(\pi/6)}{7} = \frac{5}{7}$$

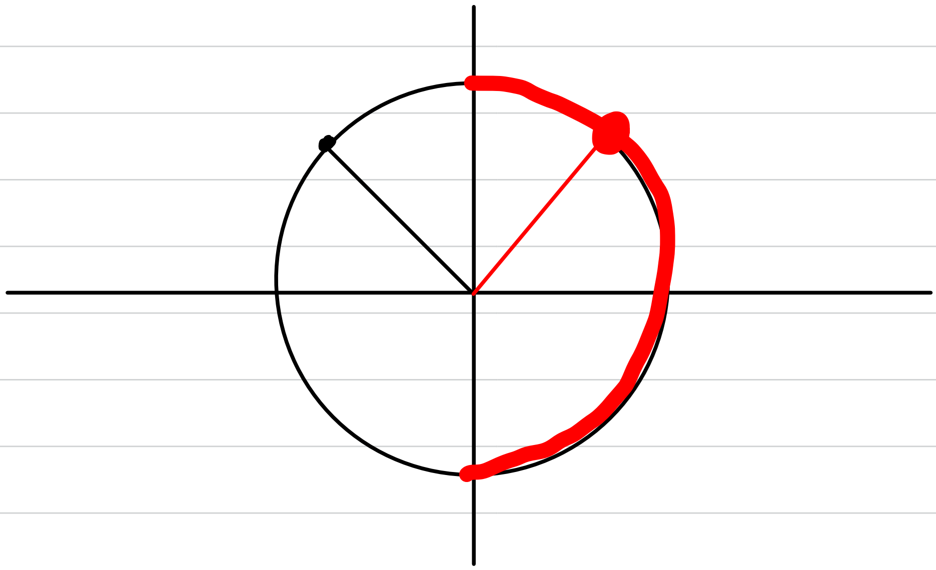
$\arcsin(5/7) = .795$ (in radians)

or $= 45.6°$.

But $\varphi \neq 45.6°$

Whenever this happens, $\varphi = 180° - 45.6° = 134.4$

Why?



Comment: If LoS gives an angle that is $< 90°$ when it is clearly $> 90°$ in triangle, take $180° - \text{angle}$.

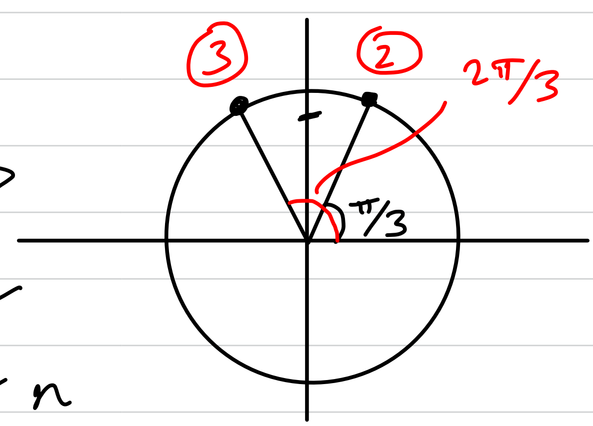
Trig Equations

Ex: Find all solutions to

$$2\sin(3x+1) + 4\sqrt{3} = 5\sqrt{3}$$

① $$2\sin(3x+1) = \sqrt{3}$$

$$\sin(3x+1) = \frac{\sqrt{3}}{2}$$



④ $$3x+1 = \frac{\pi}{3} + 2\pi n$$

or

$$3x+1 = \frac{2\pi}{3} + 2\pi n$$

for any integer n

$$3x = \frac{\pi}{3} - 1 + 2\pi n$$

or

$$3x = \frac{2\pi}{3} - 1 + 2\pi n$$

5

$$x = \frac{\pi}{9} - \frac{1}{3} + \frac{2\pi}{3} n$$

or

$$x = \frac{2\pi}{9} - \frac{1}{3} + \frac{2\pi}{3} n$$

for any integer n.

Method (Solving equations containing trig functions)

1. Solve as usual until one side has a trig function containing the variable you're solving for.

2. Draw a unit circle and label the angle that is the arc function of the other side of the equation.

(3) Depending on the trig function, draw a line to find the other point on the unit circle with the same value for the trig function.

- sin: horizontal line
- cos: vertical line
- tan: diagonal line through origin

(4) The inside of trig function is now equal to either of these two angles plus $2\pi n$ for any integer n.

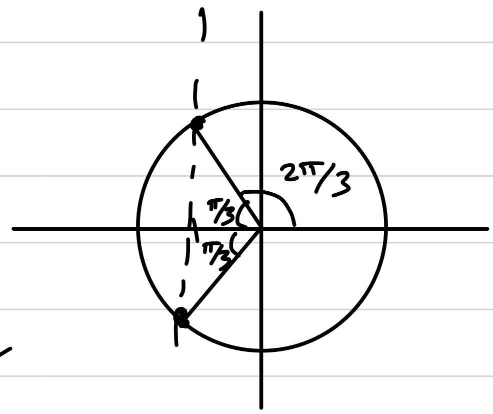
(5) Finish solving.

Ex: Find all values of x such that

$$\frac{8\cos\left(\frac{\pi}{4}(x-5)\right)+10}{3} = 2 \quad \text{and} \quad -8 \le x \le 2.$$

$$8\cos\left(\frac{\pi}{4}(x-5)\right) = -4$$

$$\cos\left(\frac{\pi}{4}(x-5)\right) = -\frac{1}{2}$$



$$\frac{\pi}{4}(x-5) = \frac{2\pi}{3} + 2\pi n$$

or

$$\frac{\pi}{4}(x-5) = \frac{4\pi}{3} + 2\pi n$$

$$x-5 = \frac{8}{3} + 8n$$

or

$$x-5 = \frac{16}{3} + 8n$$

$$x = \frac{23}{3} + 8n$$

or

$$x = \frac{31}{3} + 8n$$

We only want x in $[-8, 2]$.

Try $n=0$.

$$x = \frac{23}{3} = \cancel{7.67}$$

or

$$x = \frac{31}{3} = \cancel{10.3}$$

$n=-1$?

$$x = \frac{23}{3} - 8 = -\frac{1}{3} = -.33 \checkmark$$

or

$$x = \frac{31}{3} - 8 = \frac{7}{3} = \cancel{2.33}$$

$n=-2$

$$x = \frac{23}{3} - 16 = -\frac{25}{3} = \cancel{-8.33}$$

or

$$x = \frac{31}{3} - 16 = -\frac{17}{3} = 5.67 \checkmark$$

$n=-3$

$$x = \frac{23}{3} - 24 = -\frac{49}{3} = \cancel{-16.33}$$

or

$$x = \frac{31}{3} - 24 = -\frac{41}{3} = \cancel{-13.67}$$

$x = -\frac{1}{3}$ or $-\frac{17}{3}$