Analysis Notes

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I — A Construction of $\mathbb{R}$

Definition 1.1: A **Dedekind cut** is a set $A \subseteq \mathbb{Q}$ such that

1. $A \neq \varnothing$ and $A \neq \mathbb{Q}$.
2. If $r \in A$, then $q \in A$ for all $q \in \mathbb{Q}$ with $q < r$.
3. A does not have a maximum element — that is, if $r \in A$, then $r < s$ for some $s \in A$.

Definition 1.2: The **real numbers**, $\mathbb{R}$, are the set of all Dedekind cuts.

Definition 1.3: Let $A, B \in \mathbb{R}$. A is **less than** B, written $A \leq B$, if $A \subseteq B$.

Proposition 1.4: $\leq$ is a total order on $\mathbb{R}$.

Proof: Clearly, $\leq$ is reflexive, antisymmetric, and transitive, since $\subseteq$ is. Thus $\leq$ is a partial order on $\mathbb{R}$. To show that it is a total order, suppose $A \nleq B$. Then $A \nsubseteq B$, so there is an $a \in A$ with $a \notin B$. Let $b \in B$. Since $a \notin B$, $b \in B$, and B is a cut, $a > b$ (where $\leq$ here is the standard order on $\mathbb{Q}$), and since A is a cut, $b \in A$. Thus $B \subseteq A$, so $B \leq A$.

Definition 1.5: Let $A, B \in \mathbb{R}$. The **sum** of A and B is $A + B$ $\{a + b \mid a \in A, b \in B\}$.

Theorem 1.6: $\mathbb{R}$ is closed under addition.

Proof: Let $A, B \in \mathbb{R}$. To show $A + B \in \mathbb{R}$, we need to verify each of the three Dedekind cut axioms.

(1) Since $A \neq \varnothing$ and $B \neq \varnothing$, $A + B \neq \varnothing$. Since $A \neq \mathbb{Q}$ and $B \neq \mathbb{Q}$, there is an $s \in \mathbb{Q} \setminus A$ and a $t \in \mathbb{Q} \setminus B$, and since A and B are cuts, $a < s$ and $b < t$ for all $a \in A$ and $b \in B$. Thus $a + b < s + t$ for all $a \in A$ and $b \in B$, or equivalently, for all $a + b \in A + B$. Thus $s + t \notin A + B$, so $A + B \neq \mathbb{Q}$.

(2) Let $a + b \in A + B$ and let $s \in \mathbb{Q}$ such that $s < a + b$. Then $s - b < a$, so $s - b \in A$, since A is a cut. Thus $(s - b) + b = s \in A + B$.

(3) Let $a + b \in A + B$. Since A and B are cuts, there is an $s \in A$ and a $t \in B$ such that $a < s$ and $b < t$. Then $s + t \in A + B$ and $a + b < s + t$.

Proposition 1.7: Let $A, B, C \in \mathbb{R}$. Then $A + B = B + A$ and $(A + B) + C = A + (B + C)$.

Definition 1.8: The real numbers **zero** and **one** are defined as $\mathbf{0} = \{q \in \mathbb{Q} \mid q < 0\}$ and $\mathbf{1} = \{q \in \mathbb{Q} \mid q < 1\}$.

Proposition 1.9: For all $A \in \mathbb{R}$, $A + \mathbf{0} = A$.

Proof: ($\subseteq$) Let $a + x \in A + \mathbf{0}$. Since $x < 0$, $a + x < a$, and since A is a cut, $a + x \in A$. Thus $A + \mathbf{0} \subseteq A$.

($\supseteq$) Let $a \in A$. Since A is a cut, there is an $s \in A$ such that $s > a$. Then $a - s < 0$, so $a - a \in \mathbf{0}$. Thus $a = s + (a - s) \in A + \mathbf{0}$, so $A \subset A + \mathbf{0}$.

Definition 1.10: Let $A \in \mathbb{R}$. The **additive inverse** of A is $-A = \{r \in \mathbb{Q} \mid r < -t \text{ for some } t \notin A\}$.

Proposition 1.11: Let $A \in \mathbb{R}$. Then $-A \in \mathbb{R}$.

Proposition 1.12: Let $A \in \mathbb{R}$. Then $A + (-A) = \mathbf{0}$.

Proof: ($\subseteq$) Let $a + n \in A + (-A)$. Since $n \in -A$, there is a $t \notin A$ such that $n < -t$, and since $a \in A$ and $t \notin A$, $a < t < -n$, so $a + n < 0$. Thus $a + n \in \mathbf{0}$, so $A + (-A) \subseteq \mathbf{0}$.

($\supseteq$) Let $x \in \mathbf{0}$, let $\varepsilon = \frac{|x|}{2} = -\frac{x}{2}$, and let $t \in \mathbb{Q}$ such that $t \notin A$ but $t - \varepsilon \in A$. Since $t \notin A$, $-(t + \varepsilon) \in -A$, since $t < -(-(t + \varepsilon))$ and therefore $-(t + \varepsilon) < -t$. Then $x = -2\varepsilon = -(t + \varepsilon) + (t - \varepsilon) \in A + (-A)$, so $\mathbf{0} \subseteq A + (-A)$.

Definition 1.13: Let $A, B \in \mathbb{R}$. If $A \geq \mathbf{0}$ and $B \geq \mathbf{0}$, then the **product** of A and B is

$$AB = \{ab \mid a \in A, b \in B, a \geq 0, b \geq 0\} \cup \mathbf{0}.$$

If $A \geq \mathbf{0}$ and $B < \mathbf{0}$, then $AB = -(A(-B))$, if $A < \mathbf{0}$ and $B \geq \mathbf{0}$, then $AB = -((-A)B)$, and if $A < \mathbf{0}$ and $B < \mathbf{0}$, then $AB = (-A)(-B)$.

Theorem 1.14: Let $A, B, C \in \mathbb{R}$. Then $AB \in \mathbb{R}$, $AB = BA$, $(AB)C = A(BC)$, $\mathbf{1}A = A$, and if $A \neq \mathbf{0}$, then there is an $A^{-1} \in \mathbb{R}$ with $AA^{-1} = \mathbf{1}$.

Definition 1.15: A set $U \subseteq \mathbb{R}$ is **bounded above** if there is a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in U$. We call B an **upper bound** for U, and define **bounded below** and **lower bound** similarly.

Definition 1.16: Let $U \in \mathbb{R}$ such that $U \neq \varnothing$ and U is bounded above. We define $S(U) = \bigcup_{A \in U} A$.

Theorem 1.17: Let $U \subset \mathbb{R}$ be nonempty and bounded above. Then $S(U)$ is a cut.

Proof: (1) Since $U \neq \varnothing$ and $U \subseteq S(U)$, $S(U) \neq \varnothing$. Since U is bounded above, there is a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in U$. Then $A \subseteq B$ for all $A \in U$, so $S(U) = \bigcup A \subseteq B$. Since $B \neq \mathbb{Q}$, $S(U) \neq \mathbb{Q}$.

(2) Let $a \in S(U)$ and $q < a$. Then $a \in A$ for some $A \in U$, and since A is a cut and $q < a$, $q \in A \subseteq S(U)$.

(3) Let $a \in S(U)$. Then $a \in A$ for some $A \in U$, and since A is a cut, there is a $q \in A \subseteq S(U)$ with $a < q$.

Proposition 1.18: Let $U \subseteq \mathbb{R}$ be nonempty and bounded above. Then $S(U)$ is an upper bound for U.

Proof: For all $A \in U$, $A \subseteq \bigcup A = S(U)$, so $A \leq S(U)$.

Definition 1.19: A set $U \subseteq \mathbb{R}$ has a **supremum**, or least upper bound, if there is a $B \in \mathbb{R}$ such that B is an upper bound for U and $B \leq C$ for any upper bound C for U. We define the **infimum**, or greatest lower bound, similarly, and write $\sup U$ and $\inf U$ for the supremum and infimum.

Proposition 1.20: Let $U \subseteq \mathbb{R}$ be nonempty and bounded above. Then $S(U) = \sup U$.

Proof: Let C be an upper bound for U. Then $A \leq C$ for all $A \in U$, so $A \subseteq C$ for all $A \in U$. Then $S = \bigcup A \subset C$, so $S \leq C$.

Theorem 1.21: (The Completeness of the Reals) Every nonempty, bounded above subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R}$.

II — The Reals

Proposition 2.1: Let $A \subseteq \mathbb{R}$. If $\sup A \in A$, then $\sup A = \max A$.

Proposition 2.2: If $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$, then $\sup A \leq \sup B$.

Proof: Since $A \subseteq B$, $a \in B$ for all $a \in A$, and so since $\sup B \geq b$ for all $b \in B$, $\sup B \geq a$ for all $a \in A$. Then $\sup B$ is an upper bound for A, so $\sup A \leq \sup B$.

Theorem 2.3: Let s be an upper bound for $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for all $\varepsilon > 0$, there is an $a \in A$ with $s - \varepsilon < a$.

Proof: ($\Rightarrow$) Assume $s = \sup A$ and let $\varepsilon > 0$. Since $s - \varepsilon < s = \sup A$, $s - \varepsilon$ cannot be an upper bound for A. Thus there must be an $a \in A$ with $a > s - \varepsilon$.

Assume s is an upper bound for A and that for every $\varepsilon > 0$, there is an $a \in A$ such that $a > s - \varepsilon$. Let b be an upper bound for A and suppose $b < s$. Let $\varepsilon = \frac{s-b}{2}$. Since $a < b$ for all $a \in A$, there is no $a \in A$ such that $a > s - \varepsilon$, since $s - \varepsilon$ is the midpoint of s and b, and is therefore greater than b. $\lightning$

Theorem 2.4: (The Nested Interval Theorem) For each $n \in \mathbb{N}$, let $I_n = [a_n, b_n]$ be an interval such that $I_n \subseteq I_{n-1}$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof: Let $A = \{a_n \mid n \in \mathbb{N}\}$. Since A is nonempty and bounded above (by b_1, for instance), A has a least upper bound. In fact, each b_i is an upper bound for A, since otherwise the intervals would not be nested.

Let $s = \sup A$ and let $n \in \mathbb{N}$. Since s is an upper bound for A, $s \geq a_n$, and since b_n is an upper bound for A, $s \leq b_n$. Thus $s \in I_n$ for all $n \in \mathbb{N}$, so $s \in \bigcap I_n$.

Theorem 2.5: (The Well-Ordering Principle) Every nonempty subset of $\mathbb{N}$ has a minimum element.

Proposition 2.6: (The Archimedean Property) Let $x \in \mathbb{R}$. Then there is a $y \in \mathbb{N}$ with $y > x$.

Corollary 2.6.1: Let $x \in \mathbb{R}^+$. Then there is a $y \in \mathbb{N}$ with $\frac{1}{y} < x$.

Theorem 2.7: (The Density of $\mathbb{Q}$ in $\mathbb{R}$) Let $a, b \in \mathbb{R}$ with $a < b$. Then there is a $q \in \mathbb{Q}$ with $a < q < b$.

Proof: First, suppose $a \geq 0$. By the Archimedean property, let $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. Let m be the smallest natural greater than na. Then $m - 1 \leq na < m$, so $m \leq na + 1 < m + 1$. Since $na < m$, $a < \frac{m}{n}$, and since $m \leq na + 1$ and $\frac{1}{n} < b - a$, $m < n\left(b - \frac{1}{n}\right) + 1 = nb$. Thus $\frac{m}{n} < b$, and so $a < \frac{m}{n} < b$.

If $a < 0$ and $b > 0$, then $a < \frac{0}{1} < b$, and if $a < 0$ and $b \leq 0$, then since $-b < -a$ (and $-b, -a > 0$), there is a $q \in \mathbb{Q}$ with $-b < q < -a$, so $a < -q < b$.

Theorem 2.8: There is an $\alpha \in \mathbb{R}$ with $\alpha^2 = 2$.

Proof: Let $T = \{t \in \mathbb{R} \mid t^2 < 2\}$, which is is nonempty and bounded above, and let $\alpha = \sup T$. Suppose $\alpha < 2$. By the Archimedean principle, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{2-\alpha^2}{2\alpha+1}$, or equivalently, $\frac{2\alpha+1}{n} < 2 - \alpha^2$. Then

$$\begin{aligned}
\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\
&< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\
&= \alpha^2 + \frac{2\alpha + 1}{n} \\
&< \alpha^2 + (2 - \alpha^2) \\
&= 2,
\end{aligned}$$

so $\alpha + \frac{1}{n} \in T$, but $\alpha + \frac{1}{n} > \alpha = \sup T$. ↯ Similarly, $a > 2$ gives a contradiction.

III — Sequences and Series

Definition 3.1: A **sequence** in a set S is a function $f : \mathbb{N} \longrightarrow S$. We write $a_n = f(n)$ and (a_n) for the entire sequence.

Definition 3.2: A sequence $(a_n) \subseteq \mathbb{R}$ **converges** to $a \in \mathbb{R}$, written $(a_n) \to a$, if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq \mathbb{N}$, then $|a_n - a| < \varepsilon$. A sequence **diverges** if it does not converge.

Example: Show $\left(\frac{1}{n}\right) \to 0$.

We want $\left|\frac{1}{n} - 0\right| < \varepsilon$, so $n > \frac{1}{\varepsilon}$. Therefore, let N be the first natural number greater than $\frac{1}{\varepsilon}$. Then if $n \geq N$, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

Definition 3.3: A sequence $(a_n) \subseteq \mathbb{R}$ is **bounded** if there is an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Proposition 3.4: Every convergent sequence is bounded.

Proof: Let $(a_n) \to a \in \mathbb{R}$. With $\varepsilon = 1$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - a| < 1$. Let $M = \max\{|a_1|, ..., |a_{N-1}|, |a| + 1\}$. Then if $k < N$, $|a_k| \leq |a_k| \leq M$, and if $k \geq N$, then $|a_k| - |a| \leq |a_k - a| < 1$, so $|a_k| < |a| + 1 \leq M$.

Theorem 3.5: Suppose $(a_n) \to a \in \mathbb{R}$ and $(b_n) \to b \in \mathbb{R}$. Then

1. $(a_n + b_n) \to a + b$.
2. $(ca_n) \to ca$.
3. $(a_n b_n) \to ab$.
4. $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$ if $b \neq 0$.

Proof: We will provide proofs for parts 1 and 3.

1. Let $\varepsilon > 0$. Since $(a_n) \to a$, there is an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|a_n - a| < \frac{\varepsilon}{2}$. Similarly, there is an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|b_n - b| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$, $|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, so $(a_n + b_n) \to a + b$.

3. Let $\varepsilon > 0$. Since (b_n) converges, it is bounded, so there is an $M > 0$ such that $|b_n| < M$ for all $n \in \mathbb{N}$. Since $(a_n) \to a$, there is an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|a_n - a| < \frac{\varepsilon}{2M}$. Similarly, since $(b_n) \to b$, there is an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|b_n - b| < \frac{\varepsilon}{2|a|}$ (if $a = 0$, simply omit this sentence). Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a b_n + a b_n - ab| \\
&\leq |a_n b_n - a b_n| + |a b_n - ab| \\
&= |b_n(a_n - a)| + |a(b_n - b)| \\
&= |b_n||a_n - a| + |a||b_n - b| \\
&\leq M|a_n - a| + |a||b_n - b| \\
&< M\left(\frac{\varepsilon}{2M}\right) + |a|\left(\frac{\varepsilon}{2|a|}\right) \\
&= \varepsilon.
\end{aligned}$$

Proposition 3.6: (The Order Limit Theorem) Suppose $(a_n) \to a \in \mathbb{R}$ and $(b_n) \to b \in \mathbb{R}$. Then

1. If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.
2. If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.
3. If there is a $c \in \mathbb{R}$ such that $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$, and if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

Proof: 1. Suppose $a < 0$ and let $\varepsilon = \frac{|a|}{2}$. Since $(a_n) \to a$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - a| < \varepsilon = \frac{|a|}{2}$. Then $a_N \in \left(\frac{3a}{2}, \frac{a}{2}\right)$, so $a_N < 0$. $\lightning$

2. Since $(b_n - a_n) \to b - a$ and $b_n - a_n \geq 0$, $b - a \geq 0$ by part 1.

3. Let $c_n = c$ for all $n \in \mathbb{N}$. Then part 2 gives both results.

Definition 3.7: A sequence $(a_n) \subseteq \mathbb{R}$ is **monotone increasing** if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, and **monotone decreasing** if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.

Theorem 3.8: (Monotone Convergence) If a sequence is monotone increasing and bounded above, then it converges.

Proof: Let $(a_n) \subseteq \mathbb{R}$ be monotone increasing and bounded above, and let $A = \{a_n \mid n \in \mathbb{N}\}$. Since A is nonempty and bounded above, $s = \sup A$ exists. Let $\varepsilon > 0$. Then there is an $a_N \in A$ such that $s - \varepsilon < a_N$. Then if $n \geq N$, $s - \varepsilon < a_N \leq a_n \leq s < s + \varepsilon$, so $|a_n - s| < \varepsilon$. Thus $(a_n) \to s$.

Corollary 3.8.1: If a sequence is monotone decreasing and bounded below, it converges.

Definition 3.9: Let (b_n) be a sequence. An **infinite series** is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \cdots.$$

The corresponding sequence of partial sums is $(s_m) = (b_1 + \cdots + b_m)$.

Definition 3.10: The series $\sum b_n$ **converges** if (s_m) converges, and **diverges** otherwise.

Proposition 3.11: If $b_n \geq 0$, then $\sum b_n$ converges if and only if (s_m) is bounded above.

Proof: Since $b_n \geq 0$, (s_m) is monotone increasing, so by the Monotone Convergence Theorem, (s_m) converges if and only if (s_m) is bounded above.

Example: Show $\sum \frac{1}{n^2}$ converges.

We want an upper bound for (s_m). To find one, notice that

$$\begin{aligned} s_m &= a + \frac{1}{(2)(2)} + \frac{1}{(3)(3)} + \frac{1}{(4)(4)} + \cdots + \frac{1}{(m)(m)} \\ &< 1 + \frac{1}{(2)(1)} + \frac{1}{(3)(2)} + \frac{1}{(4)(3)} + \cdots + \frac{1}{(m)(m-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= 1 + 1 - \frac{1}{m} \\ &= 2 - \frac{1}{m} \\ &< 2. \end{aligned}$$

Example: Show $\sum \frac{1}{n}$ diverges.

We want to show that (s_m) is unbounded. To do this, note that

$$\begin{aligned} s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right) \\ &= (k+2)\left(\frac{1}{2}\right) \\ &> \frac{k}{2}. \end{aligned}$$

Definition 3.12: Let (a_n) be a sequence and let $n_1 < n_2 < \cdots$ be a strictly increasing sequence of naturals. Then $a_{n_1}, a_{n_2}, \ldots$ is a **subsequence** of (a_n), denoted (a_{n_k}).

Proposition 3.13: Subsequences of a convergent sequence converge to the same limit.

Example: Show that $\left(\left(\frac{3}{4}\right)^n\right) \to 0$.

Since the sequence is bounded below and decreasing, it converges, say to x. Since $\left(\left(\frac{3}{4}\right)^{2n}\right)$ is a subsequence of $\left(\left(\frac{3}{4}\right)^n\right)$, $\left(\left(\frac{3}{4}\right)^{2n}\right) \to x$. But $\left(\left(\frac{3}{4}\right)^{2n}\right) = \left(\left(\frac{3}{4}\right)^n \left(\frac{3}{4}\right)^n\right) \to x^2$, so $x = x^2$. Thus $x = 0$ or $x = 1$, and since $\left(\left(\frac{3}{4}\right)^n\right)$ is monotone decreasing and $\frac{3}{4} < 1$, $x = 0$.

Theorem 3.14: (Bolzano-Weierstrass) Every bounded sequence contains a convergent subsequence.

Proof: Let (a_n) be a bounded sequence. We wish to show that (a_n) has a monotone subsequence. First, define a peak index to be an $m \in \mathbb{N}$ such that $a_n \leq a_m$ for all $n \geq m$.

Suppose there are only finitely many peak indices. Then there is an $N \in \mathbb{N}$ such that there are no peak indices greater than N. Let $n_1 = N + 1$. Since n_1 is not a peak index, there is an $n_2 \in \mathbb{N}$ with $n_2 > n_1$ and $a_{n_2} \geq a_{n_1}$. Repeat this inductively. Then (a_{n_k}) is monotone increasing and bounded above, since (a_n) is, so it converges.

If there are infinitely many peak indices, then let n_k be the kth one. Then (a_{n_k}) is monotone decreasing, so it converges.

Definition 3.15: A sequence (a_n) is **Cauchy** if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|a_m - a_n| < \varepsilon$.

Proposition 3.16: Every convergent sequence is Cauchy.

Proof: Suppose $(a_n) \to a$ and let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - a| < \frac{\varepsilon}{2}$. Then if $m, n \geq N$, $|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a - a_n| < \varepsilon$.

Proposition 3.17: Every Cauchy sequence is bounded.

Proof: Let (a_n) be Cauchy. With $\varepsilon = 1$, there is an $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|a_m - a_n| < 1|$. Thus $|a_n| \leq |a_N| + 1$ for all $n \in \mathbb{N}$, so $\max\{|a_1|, ..., |a_{N-1}|, |a_N| + 1\}$ is a bound (a_n).

Theorem 3.18: Every Cauchy sequence in $\mathbb{R}$ converges.

Proof: Let $(a_n) \subseteq \mathbb{R}$ be a Cauchy sequence. Then (a_n) is bounded, so it contains a convergent subsequence $(a_{n_k}) \to a$. Let $\varepsilon > 0$. Since $(a_{n_k}) \to a$, there is a $k_0 \in \mathbb{N}$ such that $n_{k_0} \geq N$ and $\left|a_{n_{k_0}} - a\right| < \frac{\varepsilon}{2}$, and since (a_n) is Cauchy, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $\left|a_n - a_{n_{k_0}}\right| < \frac{\varepsilon}{2}$. Then if $n \geq N$, $|a_n - a| = \left|a_n - a_{n_{k_0}} + a_{n_{k_0}} + a\right| \leq \left|a_n - a_{n_{k_0}}\right| + \left|a_{n_{k_0}} - a\right| < \varepsilon$.

Proposition 3.19: Suppose $\sum a_k = a$ and $\sum b_k = b$. Then $\sum ca_k = ca$ and $\sum a_k + b_k = a + b$.

Proposition 3.20: A series $\sum a_k$ for $a_k \in \mathbb{R}$ converges if and only if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n > m \geq N$, then $|a_{m+1} + \cdots + a_n| < \varepsilon$.

Proof: $\sum a_k$ converges if and only if (s_n) converges, if and only if it is Cauchy, if and only if there is an $N \in \mathbb{N}$ such that if $n > m \geq N$, then $|s_n - s_m| = |a_{m+1} + \cdots + a_n| < \varepsilon|$.

Corollary 3.20.1: If $\sum a_k$ converges, then $(a_k) \to 0$.

Proposition 3.21: (The Ratio Test) Let (a_k) and (b_k) be sequences such that $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then if $\sum b_k$ converges, so does $\sum a_k$, and if $\sum a_k$ diverges, then $\sum b_k$ does too.

Proof: The second statement is just the contrapositive of the first, so we need only prove one. Suppose $\sum b_k$ converges and let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that if $n > m \geq N$, $|b_{m+1} + \cdots + b_n| < \varepsilon$. Since $0 \leq a_k \leq b_k$, $|a_{m+1} + \cdots + a_n| < \varepsilon$, so $\sum a_k$ converges.

Definition 3.22: A series is **geometric** if it is of the form $\sum ar^k$, where $r \neq 0$.

Theorem 3.23: The series $\sum ar^k$ converges if and only if $|r| < 1$, and if $|r| < 1$, then $\sum ar^k = \frac{a}{1-r}$.

Proof: If $r \neq 1$, then since $(1-r)\left(1+r+r^2+\cdots+r^{m-1}\right) = 1-r^m$, $s_{m-1} = a+ar+ar^2+\cdots+ar^{m-1} = \frac{a(1-r^m)}{1-r}$. If $|r| > 1$, then (s_m) is not bounded, so it does not converge. If $|r| < 1$, then $(r^m) \to 0$, so $(s_m) \to \frac{1}{1-r}$. Finally, if $|r| = 1|$, then either $r = 1$, in which case $(s_m) = (ma)$ is unbounded, or $r = -1$, in which case $(s_m) = \left(\frac{a(1-(-1)^m)}{2}\right) = (a, 0, a, 0, \ldots)$, which does not converge. Either way, (s_m) diverges.

Theorem 3.24: If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Proof: Suppose $\sum |a_n|$ converges and let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that if $n > m \geq N$, $\big|\, |a_{m+1}| + \cdots + |a_n| \,\big| = |a_{m+1}| + \cdots + |a_n| < \varepsilon$. Thus $|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \varepsilon$.

Theorem 3.25: Let (a_n) be a sequence with $a_1 \geq a_2 \geq \cdots$ and $(a_n) \to 0$. Then $\sum(-1)^{n+1}a_n$ converges.

Definition 3.26: A series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges, and it **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem 3.27: Suppose $\sum a_k$ converges conditionally. Then for any $A \in \mathbb{R}$, there is a permutation σ of $\mathbb{N}$ such that $\sum a_{\sigma(k)} = A$.

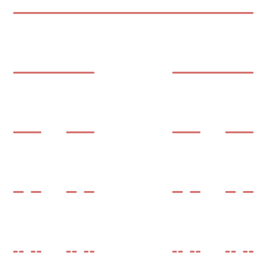
Theorem 3.28: If $\sum a_k$ converges absolutely, then $\sum a_{\sigma(k)} = \sum a_k$ for any permutation σ.

Proof: Suppose $\sum a_k$ converges absolutely to A and let $\sigma : \mathbb{N} \twoheadrightarrow \mathbb{N}$. Let $s_m = a_1 + \cdots + a_m$ and $t_m = a_{\sigma(1)} + \cdots + a_{\sigma(n)}$ and let $\varepsilon > 0$. Since $(s_m) \to A$, there is an $N_1 \in \mathbb{N}$ such that if $m \geq N_1$, then $|s_m - a| < \frac{\varepsilon}{2}$, and since $\sum |a_k|$ converges, there is an $N_2 \in \mathbb{N}$ such that if $n > m \geq N_2$, then $|a_{m+1}| + \cdots + |a_n| < \frac{\varepsilon}{2}$. Let $M = \max\{N_1, N_2\}$ and call the subsequence $(a_{M+1}, a_{M+2} \ldots)$ the *tail* of (a_k). Then the sum of the absolute values of any finite collection of elements in the tail is less than $\frac{\varepsilon}{2}$, since $M \geq N_2$. Let $N \in \mathbb{N}$ such that $\{1, \ldots, M\} \subseteq \{\sigma(1), \ldots, \sigma(N)\}$. Then if $n \geq N$, $t_n - s_M$ is the sum of a finite number of terms in the tail of (a_k), so by the triangle inequality, $|t_n - s_M| < \frac{\varepsilon}{2}$. Also, since $N \geq M \geq N_1$, $|s_N - A| < \frac{\varepsilon}{2}$. Thus if $n \geq N$,

$$\begin{aligned} |t_n - A| &= |t_n - s_M + s_M - A| \\ &\leq |t_n - s_M| + |s_M - A| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

IV — The Topology of $\mathbb{R}$

Example: Let $C_0 = [0,1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, and similarly for C_i for all $i \in \mathbb{N}$. The *Cantor set* is $C = \bigcap C_i$.



Definition 4.1: Let $a \in \mathbb{R}$ and $\varepsilon > 0$. The **open ε-ball** centered at a is $V_\varepsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon)$.

Definition 4.2: A set $U \subseteq \mathbb{R}$ is **open** if for all $x \in U$, there is an $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq U$.

Theorem 4.3: Arbitrary unions and finite intersections of open sets are open.

Proof: Let $\{U_i \mid i \in I\}$ be a collection of open sets and let $U = \bigcup U_i$. Let $a \in U$. Then $a \in U_i$ for some $i \in I$, so there is an $\varepsilon > 0$ such that $V_\varepsilon(a) \subseteq U_i \subseteq U$. Thus U is open.

Now let $\{U_1, \ldots, U_n\}$ be a finite collection of open sets and let $U = \bigcap U_i$. Let $a \in U$. Then $a \in U_i$ for all i, so there is an $\varepsilon_i > 0$ such that $V_{\varepsilon_i}(a) \subseteq U_i$ for each $i \in \{1, \ldots, n\}$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $V_\varepsilon(a) \subseteq V_{\varepsilon_i} \subseteq U_i$ for all i, so $V_\varepsilon \subseteq U$.

Definition 4.4: Let $A \subseteq \mathbb{R}$. The **compliment** of A is $A^c = \{x \in \mathbb{R} \mid x \notin A\}$.

Definition 4.5: A set $F \in \mathbb{R}$ is **closed** if F^c is open.

Example: $\mathbb{R}$ is open and closed.

$\mathbb{Q}$ is neither open nor closed.

$\mathbb{Z}$ is closed but not open.

$(0,1)$ is open but not closed.

Proposition 4.6: Arbitrary intersections and finite unions of closed sets are closed.

Example: The Cantor set C is closed, since each C_i is closed and $C = \bigcap C_i$.

Definition 4.7: A point $x \in \mathbb{R}$ is a **limit point** of a set A if for all $\varepsilon > 0$, $V_\varepsilon(x)$ contains a point of A other than x. The set of limit points of A is denoted $L(A)$.

Example: $L((2,3)) = [2,3]$, $L(\mathbb{Q}) = \mathbb{R}$, $L(\mathbb{Z}) = \varnothing$, and for any finite set A, $L(A) = \varnothing$.

Proposition 4.8: Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a limit point of A if and only if $(a_n) \to x$ for some sequence $(a_n) \subseteq A \setminus \{x\}$.

Proof: ($\Rightarrow$) Suppose x is a limit point of A. Then for all $\varepsilon > 0$, there is an $a \in V_\varepsilon(x) \cap A$ with $a \neq x$. Let $n \in \mathbb{N}$ be arbitrary. Then with $\varepsilon = \frac{1}{n}$, there is an $a_n \in V_{\frac{1}{n}}(x) \cap A$ with $a_n \neq x$. Then $a_n \in A$ and $|a_n - x| < \frac{1}{n}$, so $(a_n) \to x$.

($\Leftarrow$) Suppose $(a_n) \to x$ for some sequence $(a_n) \subseteq A \setminus \{x\}$. Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that if $n \geq N$, $|a_n - x| < \varepsilon$, so in particular, $|a_N - x| < \varepsilon$. Then $a_N \in V_\varepsilon(x) \cap A$ and $a_N \neq x$.

Definition 4.9: Let $A \subseteq \mathbb{R}$. A point $x \in A$ is an **isolated point** of A if x is not a limit point of A.

Example: $(2,3)$ and $\mathbb{Q}$ contain no isolated points, but every point in $\mathbb{Z}$ is isolated.

Theorem 4.10: A set $F \subseteq \mathbb{R}$ is closed if and only if F contains its limit points.

Proof: The set F is closed if and only if F^c is open, if and only if for all $x \in F^c$, there is an $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq F^c$. But $V_\varepsilon(x) \subseteq F^c$ if and only if $V_\varepsilon(x)$ does not intersect F, so F is closed if and only if for all $x \in F^c$, there is an $\varepsilon > 0$ such that $V_\varepsilon(x)$ does not intersect F, or, equivalently, no element of F^c is a limit point of F. But no element of F^c is a limit point of F if and only if no limit point of F is outside F, which is equivalent to $L(F) \subseteq F$.

Theorem 4.11: A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence in F has a limit in F.

Proof: ($\Rightarrow$) Suppose F is closed and let $(a_n) \subseteq F$ be Cauchy. Then $(a_n) \to a$ for some $a \in \mathbb{R}$. If $a = a_{n_0}$ for some $n_0 \in \mathbb{N}$, then $a \in F$, so assume no $a_n = a$. Then (a_n) is a sequence in $F \setminus \{a\}$ that converges to a, so $a \in L(F)$. Since F is closed, $a \in F$.

($\Leftarrow$) Suppose every Cauchy sequence in F converges to an element of F. Let a be a limit point of F. Then $(a_n) \to a$ for some sequence $(a_n) \in F$. Since (a_n) converges, it is Cauchy, so it converges to an element of F. Thus $a \in F$, so $L(F) \subseteq F$, and so F is closed.

Definition 4.12: Let $A \subseteq \mathbb{R}$. The **closure** of A is $\overline{A} = A \cup L(A)$.

Example: $\overline{(0,1)} = [0,1]$, $\overline{\mathbb{Q}} = \mathbb{R}$, and $\overline{\mathbb{Z}} = \mathbb{Z}$.

Theorem 4.13: Let $A \subseteq \mathbb{R}$. Then $\overline{A}$ is closed, and it is the smallest closed set containing A.

Proof: Let x be a limit point of $\overline{A}$. If $x \in A$, then $x \in \overline{A}$. Otherwise, $x \notin A$. Let $\varepsilon > 0$. Since $x \in L(\overline{A})$, there is a $p \in \overline{A}$ with $p \in V_\varepsilon(x)$. If $p \in A$, then there is a point of A in $V_\varepsilon(x)$, so $x \in L(A)$, and so $x \in \overline{A}$. Otherwise, $p \notin A$, so $p \in L(A)$. Since $p \in V_\varepsilon(x)$ and $V_\varepsilon(x)$ is an open set, there is a $\delta > 0$ such that $V_\delta(p) \subseteq V_\varepsilon(x)$. Since $p \in L(A)$, $V_\delta(p)$ contains a point of A, so $V_\varepsilon(x)$ does too. Thus $x \in L(A)$, to $x \in \overline{A}$. In all cases, $\overline{A}$ contains its limit points, so it is closed.

To show $\overline{A}$ is the smallest closed set containing A, let F be a closed set with $A \subseteq F$. Then $L(A) \subseteq L(F) \subseteq F$, and so $A \cup L(A) = \overline{A} \subseteq F$.

Definition 4.14: Let $A \subseteq \mathbb{R}$. The **interior** of A is $A^o = \{x \in A \mid V_\varepsilon(x) \subseteq A \text{ for some } \varepsilon > 0\}$.

Theorem 4.15: Let $A \subseteq \mathbb{R}$. Then A^o is open and it is the largest open set contained in A.

Definition 4.16: A set $K \subseteq \mathbb{R}$ is **compact** if every sequence in K has a subsequence that converges to an element of K.

Example: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are not compact, since $(1, 2, 3, ...)$ has no convergent subsequence.

$(0,1)$ is not compact either, since every subsequence of $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right)$ converges to $0 \in (0,1)$.

Definition 4.17: Let $A \subseteq \mathbb{R}$. A is **bounded** if there is an $M \in \mathbb{R}^+$ such that $|a| \leq M$ for all $a \in A$.

Theorem 4.18: (Heine-Borel) A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof: ($\Rightarrow$) Suppose A is compact but not bounded. Then for all $n \in \mathbb{N}$, there is an $a_n \in A$ with $|a_n| > n$. Since A is compact, there is a convergent subsequence $(a_{n_k}) \to a \in A$. But since $|a_{n_k}| > n_k \geq k$, (a_{n_k}) is unbounded, so it cannot converge. $\lightning$ Thus A is bounded.

Now let $a \in L(A)$. Then there is a sequence $(a_n) \subseteq A$ with $(a_n) \to A$. Since A is compact, there is a subsequence $(a_{n_k}) \to a$ and $a \in A$. Thus $L(A) \subseteq A$, so A is closed.

($\Leftarrow$) Suppose A is closed and bounded and let $(a_n) \subset A$. Since A is bounded, (a_n) is bounded, so by the Bolzano-Weierstrass Theorem, there is a convergent subsequence $(a_{n_k}) \to a$. Since $(a_{n_k}) \subseteq (a_n) \subseteq A$ and A is closed, $a \in A$. Thus (a_n) has a subsequence that converges to an element of A, so A is compact.

Example: $[0,1]$ and the Cantor set are both compact.

Theorem 4.19: (The Nested Compact Set Theorem) Let $K_1 \supseteq K_2 \supseteq \cdots$ be a chain of nonempty compact sets. Then $\bigcap K_i \neq \varnothing$.

Proof: For each $n \in \mathbb{N}$, choose $x_n \in K_n$. Then $(x_n) \subseteq K_1$, and since K_1 is compact, (x_n) has a convergent subsequence $(x_{n_k}) \to x \in K_1$. Let $i \in \mathbb{N}$. Then $(x_i, x_{i+1}, ...) \subseteq K_i$ by definition. Let $(x_{n_k})_{k \geq i}$ be the subsequence of (x_{n_k}) with indices greater than or equal to i. Then $(x_{n_k})_{k \geq i} \subseteq (x_i, x_{i+1}, ...) \subseteq K_i$, and since K_i is closed, $x = \lim (x_{n_k}) = \lim (x_{n_k})_{k \geq i} \in K_i$. Thus $x \in K_i$ for all $i \in \mathbb{N}$, so $x \in \bigcap K_i$. Thus $\bigcap K_i \neq \varnothing$.

Definition 4.20: An **open cover** of a set A is a collection of open sets $\{U_i \mid i \in I\}$ such that $A \subseteq \bigcup U_i$.

Example: Open covers of $[0,1]$: $\{\mathbb{R}\}$, $\{(-1,1),(0,2)\}$, and $\left\{\left(\frac{1}{2},2\right),\left(-1,\frac{1}{2}\right),\left(-1,\frac{3}{4}\right),\left(-1,\frac{7}{8}\right),...\right\}$.

Definition 4.21: Let C be an open cover of a set A. A **finite subcover** of C is a finite subcollection of C whose union still contains A.

Theorem 4.22: A set K is compact if and only if every open cover of K has a finite subcover.

Example: $\left\{\left(\frac{1}{3},\frac{2}{3}\right),\left(\frac{1}{4},\frac{3}{4}\right),\left(\frac{1}{5},\frac{4}{5}\right),...\right\}$ is an open cover of $(0,1)$, but it contains no finite subcovers.

Definition 4.23: A set is **perfect** if it is closed and contains no isolated points.

Example: $[0,1]$, $\mathbb{R}$, and $[0,\infty)$ are perfect.

Theorem 4.24: The Cantor set is perfect.

Proof: We already know that C is closed, so we need only show it has no isolated points. Let $x \in C$, and for each $n \in \mathbb{N}$, let x_n be the an endpoint of the subinterval of C_n that contains x. If x is itself an endpoint, pick the other endpoint, so that $x_n \neq x$ for any n. Then $(x_n) \subseteq C$, and since each subinterval of C_n has length $\frac{1}{3^n}$, $|x - x_n| < \frac{1}{3^n}$. Thus $(x_n) \to x$, so $x \in L(C)$. Thus $C = L(C)$, so it is perfect.

Theorem 4.25: Nonempty perfect sets are uncountable.

Proof: Let P be a nonempty perfect set. P cannot be finite, since then $L(P) = \varnothing \neq P$, so either P is countably infinite or uncountable. Suppose it is the former. Then $P = \{x_1, x_2, ...\}$ for some $x_n \in \mathbb{R}$. Let $I_1 = [x_1 - 1, x_1 + 1]$. Since x_1 is not isolated, $x_1 \in L(P)$, so using the definition of limit point with $\varepsilon = 1$, there is a $y_2 \in P \cap (x_1 - 1, x_1 + 1)$ such that $y_2 \neq x_1$. Let I_2 be a closed subinterval centered at y_2 such that $I_2 \subseteq I_1$ but $x_1 \notin I_2$. Since $y_2 \in P$ and y_2 is not isolated, there is a $y_3 \in P$ such that $y_3 \in (I_2)^o$, $y_3 \neq y_2$, and $y_2 \neq x_2$. Let I_3 be a closed interval centered at y_3 such that $I_3 \subseteq I_2$ and $x_2 \notin I_3$. Repeat this process inductively for all $n \in \mathbb{N}$ to produce $I_1 \supseteq I_2 \supseteq \cdots$ with $x_n \notin I_{n+1}$ and $I_n \cap P \neq \varnothing$ (since $y_n \in I_n \cap P$).

Let $K_n = I_n \cap P$. Since I_n and P are closed and I_n is bounded, each K_n is compact, so by the nested compact set theorem, $\bigcap K_n \neq \varnothing$. But $P = \{x_1, x_2, ...\}$ and $x_n \notin K_{n+1}$, so $\bigcap K_n = \varnothing$. $\lightning$ Thus P is uncountable.

Corollary 4.25.1: The Cantor set is uncountable.

Definition 4.26: Two nonempty sets $A, B \subseteq \mathbb{R}$ are **separated** if $\overline{A} \cup B = A \cup \overline{B} = \varnothing$.

Definition 4.27: A set $E \subseteq \mathbb{R}$ is **disconnected** if $E = A \cup B$ for separated sets A and B.

Definition 4.28: A set is **connected** if it is not disconnected.

Example: $\mathbb{Q}$ is disconnected: $A = (-\infty, \pi) \cap \mathbb{Q}$ and $B = (\pi, \infty) \cap \mathbb{Q}$.

Theorem 4.29: A set $E \subseteq \mathbb{R}$ is connected if and only if $[a,b] \subseteq E$ for all $a, b \in E$ with $a < b$.

Proof: ($\Rightarrow$) Suppose $E \subseteq \mathbb{R}$ is connected and let $a, b \in E$ with $a < b$. Suppose there is a $c \in (a,b)$ with $c \notin E$. Let $A = (-\infty, c) \cap E$ and $B = (c, \infty) \cap E$. Then $a \in A$ and $b \in B$, and A and B are separated, so E is disconnected. $\lightning$

($\Leftarrow$) Suppose that for all $a, b \in E$ with $a < b$, $[a,b] \subseteq E$, but that E is disconnected. Then $E = A \cup B$ for two separated sets A and B. By definition, A and B must be nonempty, so there is an $x \in A$ and a $y \in B$. Without loss of generality, suppose $x < y$. Then $x, y \in E$, so $[x,y] \subseteq E$ by assumption.

Let $z = \sup(A \cap [x,y])$. Since $x \in A \cap [x,y]$ and z is an upper bound, $x \leq z$. And since $z = \sup(A \cap [x,y]) \leq \sup[x,y] = y$, $z \in [x,y]$. Since $z = \sup(A \cap [x,y]) \in \overline{A \cap [x,y]} \subseteq \overline{A}$ and $\overline{A} \cup B = \varnothing$, $z \notin B$. And since $z \in [x,y] \subseteq E = A \cup B$, $z \in A$. Since $z \in A$, $y \in B$, and $A \cap B \subseteq \overline{A} \cap B = \varnothing$, $z \neq y$. Thus $(z,y] \subseteq B$, since $(z,y]$ is contained in E, but not in A. Thus for all $\varepsilon > 0$, $V_\varepsilon(z)$ contains a point of B other than z, so $z \in L(B)$ by definition. Thus $z \in \overline{B}$, but $z \in A$ and $A \cap \overline{B} = \varnothing$. $\lightning$ Thus E is connected.