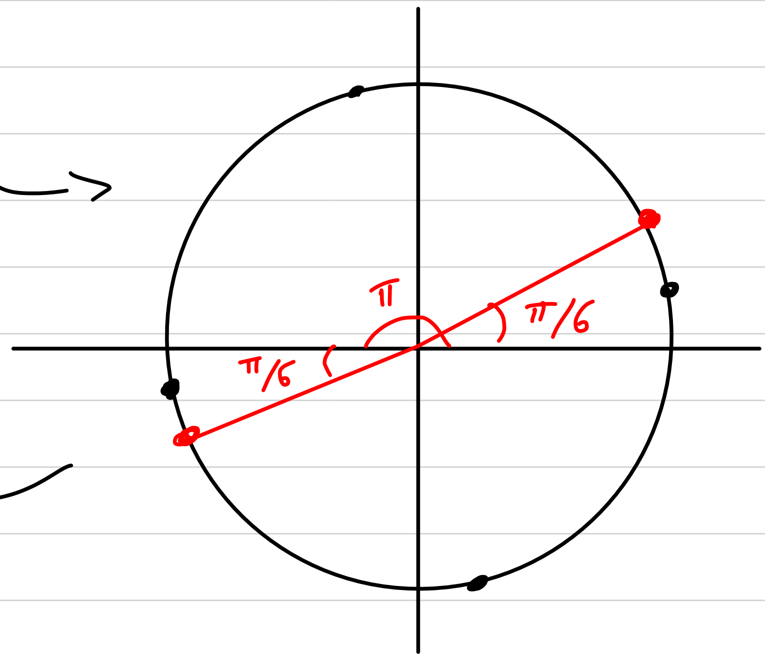
Ex: Find all the values of x such that

$$3\tan(2x) - \sqrt{3} = 0.$$

$$3\tan(2x) = \sqrt{3}$$

$$\tan(2x) = \frac{\sqrt{3}}{3}$$

$\arctan\left(\frac{\sqrt{3}}{3}\right) = \pi/6$



$$2x = \pi/6 + 2\pi n$$

or

$$2x = 7\pi/6 + 2\pi n$$

$$x = \pi/12 + \pi n$$

or

$$x = 7\pi/12 + \pi n$$

for any integer n.

$$x = \pi/12,\ 13\pi/12,\ 25\pi/12,\ -11\pi/12,\ -23\pi/12,\ \ldots$$

Sinusoidal Functions

Def: A function f is sinusoidal if $f(x)$ is a transformation of $\sin(x)$; that is, $f(x) = A\sin(B(x-h)) + k$, where $A > 0$, $B > 0$, and h and k are real numbers.

Prop: Let $f(x) = A\sin(B(x-h)) + k$. Then:

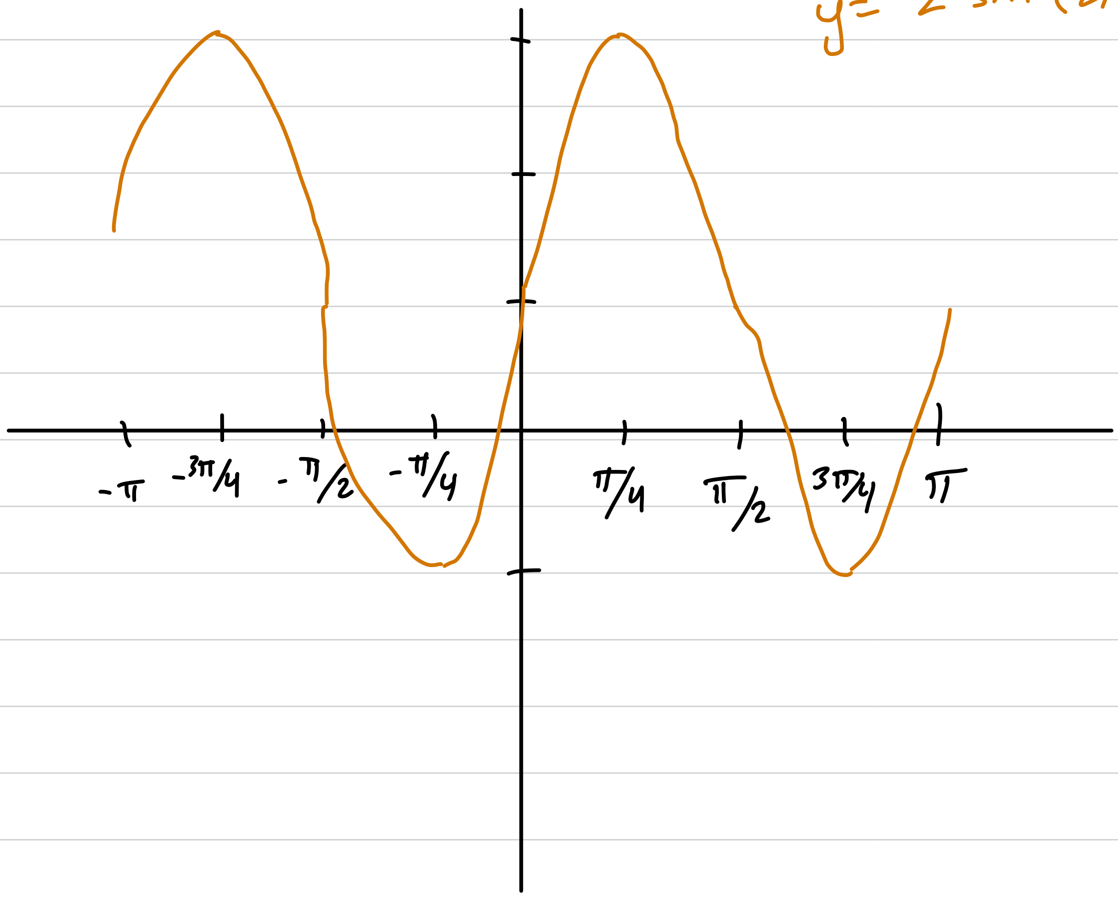
1. f is periodic with period $\frac{2\pi}{B}$
2. The amplitude of f is A.
3. The midline of f is k.

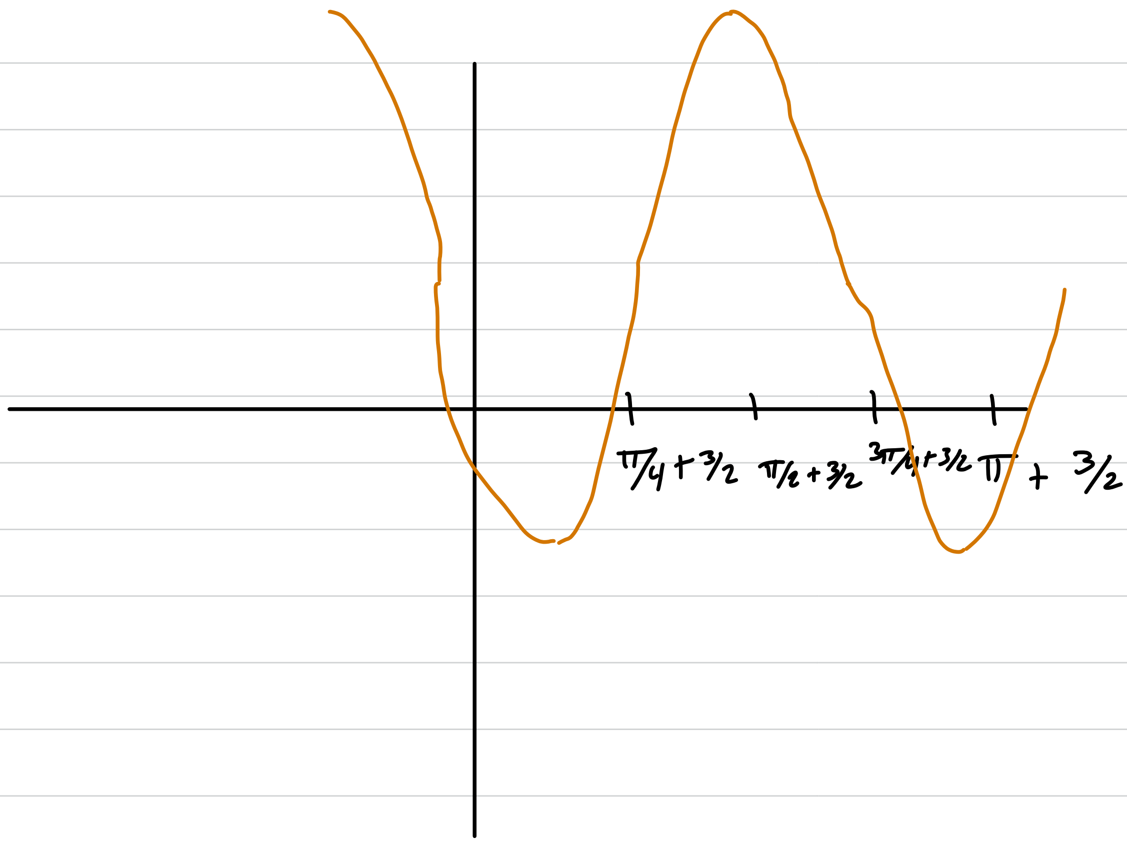
Ex: Let $f(x) = 2\sin(2x-3) + 1$.

$$f(x) = 2\sin\left(2\left(x - \tfrac{3}{2}\right)\right) + 1$$

$A = 2 \Rightarrow$ amplitude 2
$B = 2 \Rightarrow$ period $\frac{2\pi}{2} = \pi$
$h = 3/2 \Rightarrow$ horizontal shift $3/2$ to the right
$k = 1 \Rightarrow$ midline 1

$y = 2\sin(2x) + 1$





Ex: Find a formula for a sinusoidal function $f(x)$ with period 4, midline 2, and amplitude 3, that passes through $(1,2)$ and is increasing there.

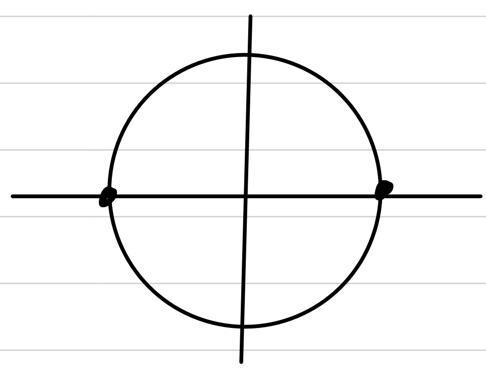
$$\frac{2\pi}{B} = 4 \qquad B = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$f(x) = 3\sin\left(\frac{\pi}{2}(x-h)\right) + 2$$

In general, finding h means solving a trig equation.

Since $f(1) = 2$, $\quad 2 = 3\sin\left(\frac{\pi}{2}(1-h)\right) + 2$

$$\sin\left(\frac{\pi}{2}(1-h)\right) = 0$$



$$\frac{\pi}{2}(1-h) = 0 + 2\pi n$$

or

$$\frac{\pi}{2}(1-h) = \pi + 2\pi n$$

$1 - h = 4n$

or

$1 - h = 2 + 4n$

$h = 1 - 4n$

or

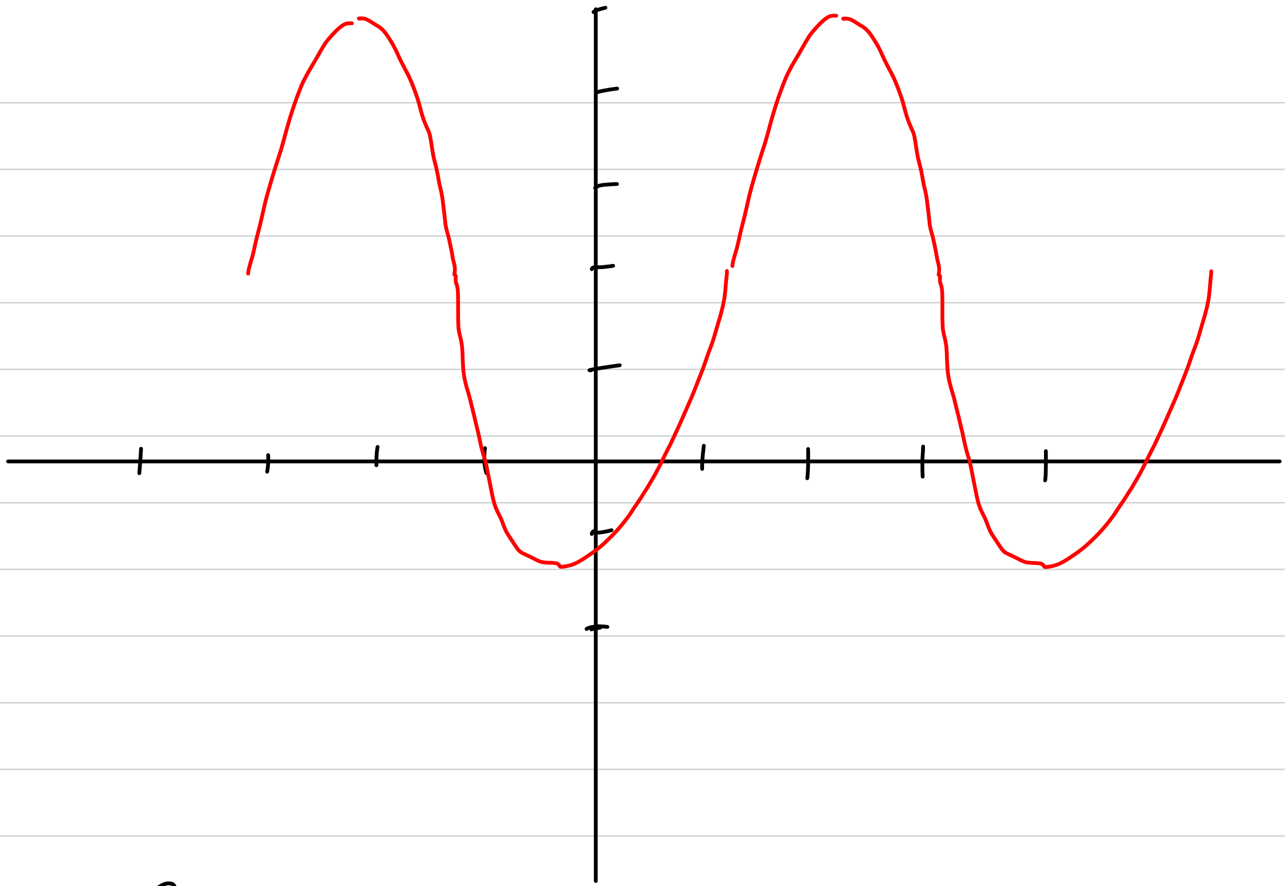
$h = -1 - 4n$

In general, since the n term is because of sin being periodic, we can take n to be whatever we want (usually zero).

$n = 0$:

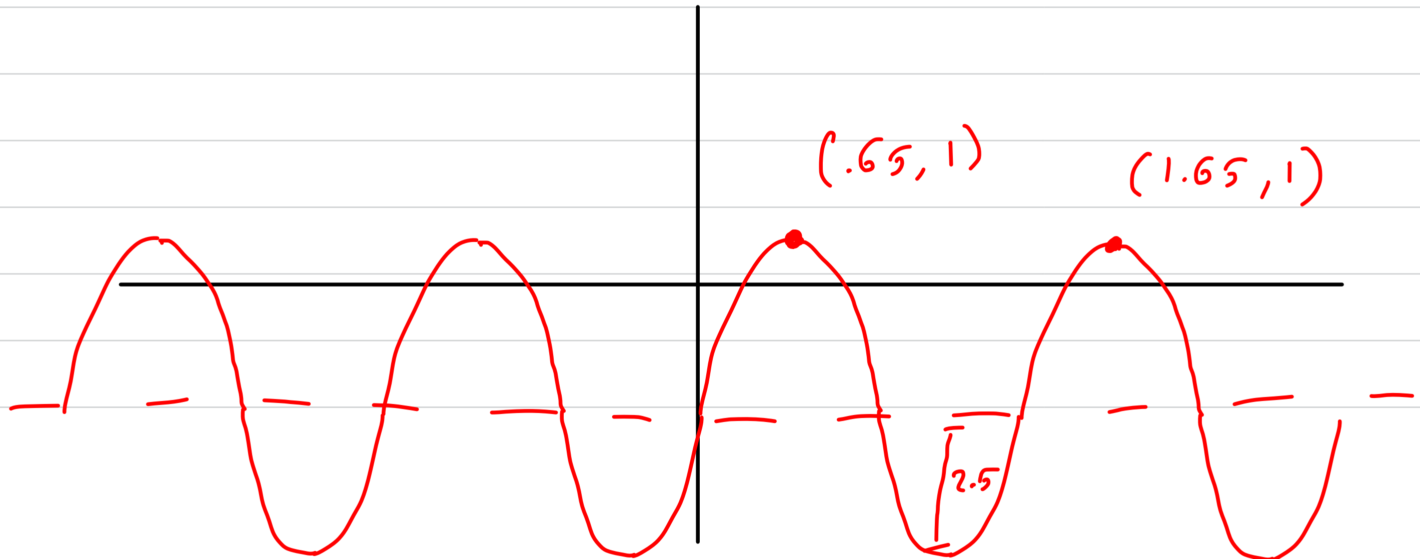
$h = 1$ or $h = -1$.

Try $h = 1$. $f(x) = 3 \sin\left(\frac{\pi}{2}(x-1)\right) + 2$



Therefore, $h=1$ was correct.

Ex: Find an equation for g, given that $g(x)$ is sinusoidal.



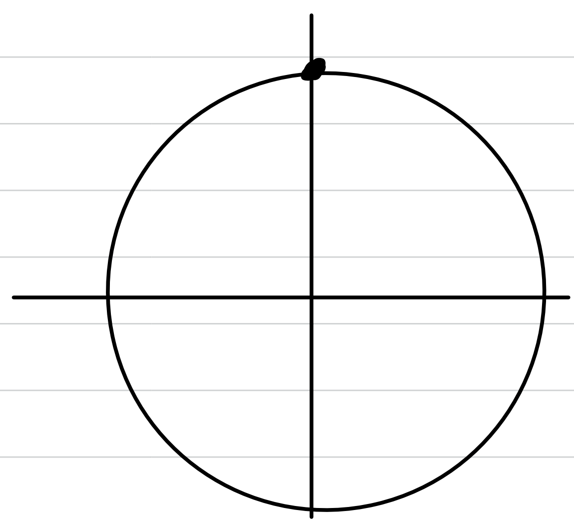
amplitude: 2.5
midline: -1.5
period: 1
horizontal shift: ??

$\frac{2\pi}{B} = 1 \qquad B = 2\pi$

$g(x) = 2.5 \sin(2\pi(x-h)) - 1.5$

$1 = 2.5 \sin(2\pi(.65-h)) - 1.5$

$1 = \sin(2\pi(.65-h))$



$2\pi(.65-h) = \pi/2 + 2\pi n$

$.65 - h = 1/4 + n$

$h = .65 - 1/4 - n$

$h = .4 - n$

Any n works, so with $n=0$, $h = .4$.

Relationships Between Trig Functions

Def: The cosecant function is $\csc(\theta) = \frac{1}{\sin(\theta)}$.

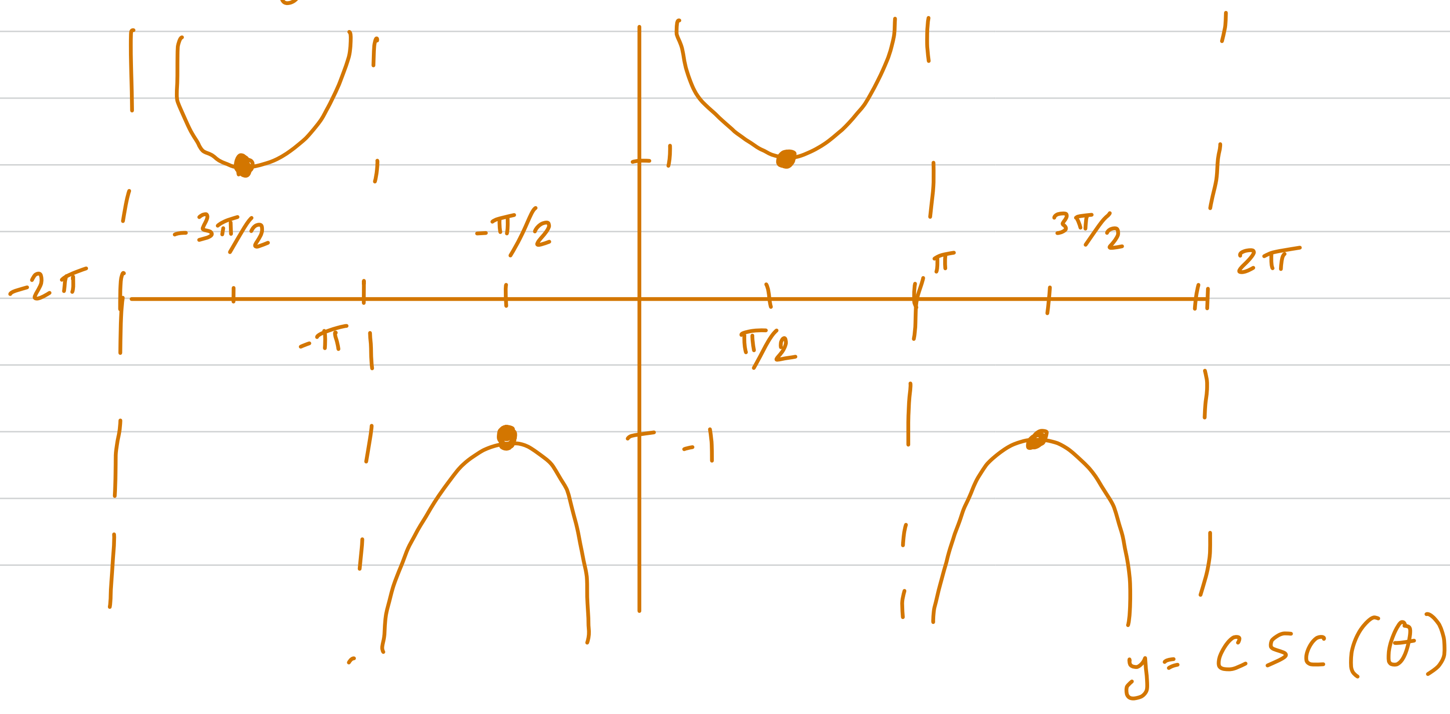
The secant function is $\sec(\theta) = \frac{1}{\cos(\theta)}$

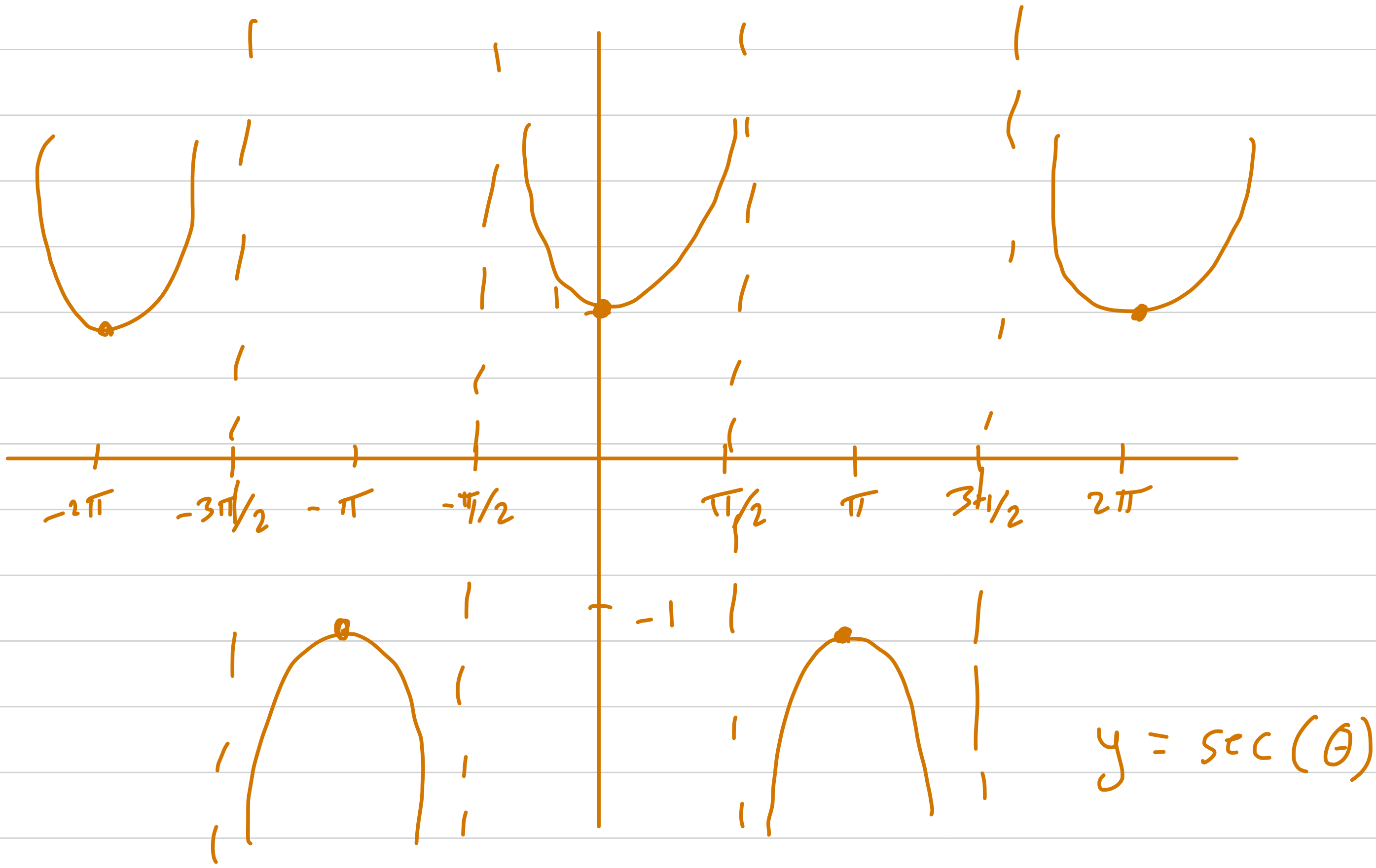
The cotangent function is $\cot(\theta) = \frac{1}{\tan(\theta)}$

Ex: $\csc(\pi/4) = 1/\sin(\pi/4) = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} = \sqrt{2}$

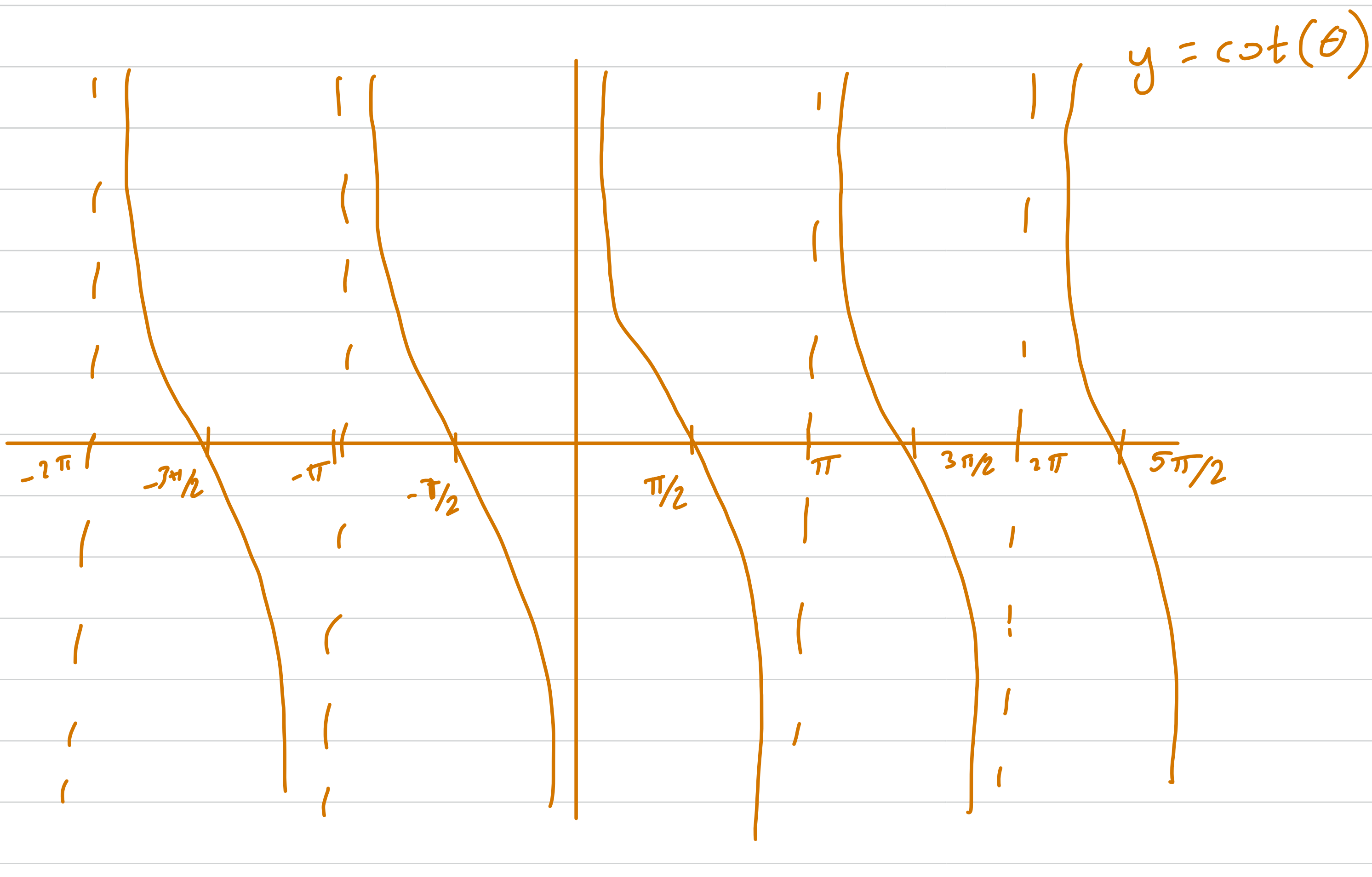
$\cot(\pi/6) = \frac{1}{\tan(\pi/6)} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$

Prop: The graphs of csc, sec and cot:





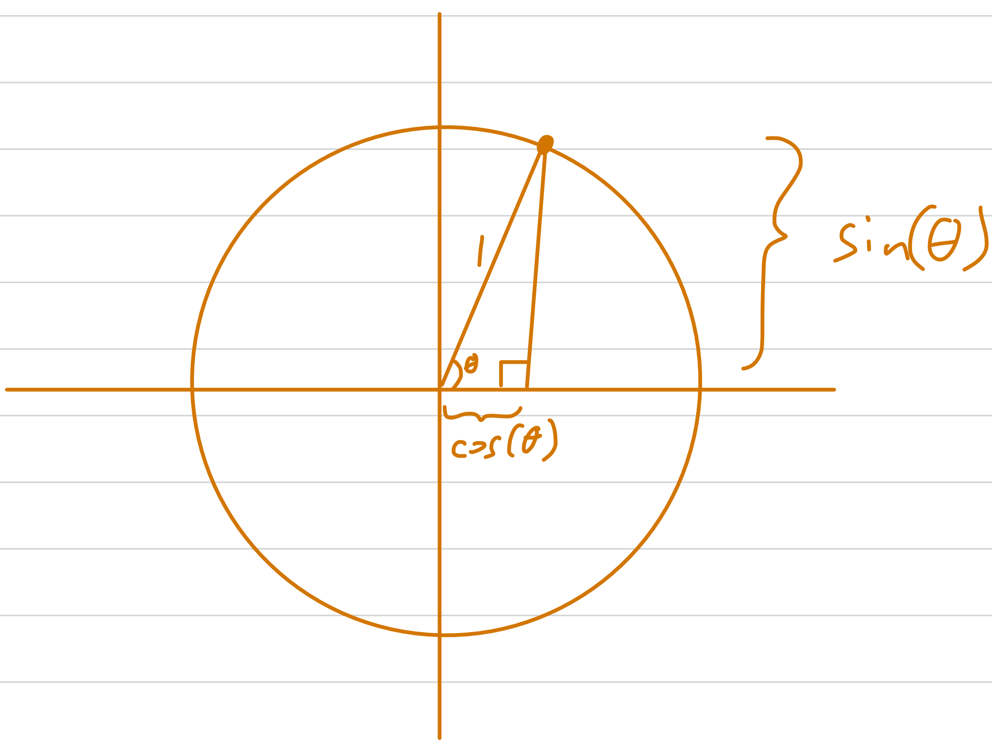
-2π
-3π/2
-π
-π/2
π/2
π
3π/2
2π
-1
y = sec(θ)



y = cot(θ)
-2π
-3π/2
-π
-π/2
π/2
π
3π/2
2π
5π/2

Theorem: The basic relationships:

1. $\sin(-\theta) = -\sin(\theta)$
2. $\cos(-\theta) = \cos(\theta)$
3. $\tan(-\theta) = -\tan(\theta)$
4. $(\sin(\theta))^2 + (\cos(\theta))^2 = 1$



Theorem (The Half-Angle Formulas):

① $\sin(\theta/2) = \pm\sqrt{\dfrac{1-\cos(\theta)}{2}}$

② $\cos(\theta/2) = \pm\sqrt{\dfrac{1+\cos(\theta)}{2}}$

③ $\tan(\theta/2) = \dfrac{\sin(\theta)}{1+\cos(\theta)}$

When there is a $\pm$, use which quadrant you're in to determine if it should be + or −.

Ex: Find the exact value of $\sin(\pi/8)$.

$\pi/8 = \frac{1}{2}(\pi/4)$, so $\sin(\pi/8) = \pm\sqrt{\dfrac{1-\cos(\pi/4)}{2}}$

$= \pm\sqrt{\dfrac{1-\sqrt{2}/2}{2}} = \pm\sqrt{\dfrac{\frac{2-\sqrt{2}}{2}}{2}} = \pm\sqrt{\dfrac{2-\sqrt{2}}{4}}$

$= \pm\dfrac{\sqrt{2-\sqrt{2}}}{2} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

since $\sin(\pi/8)$ is positive

Ex: Find the exact value of $\tan(15°)$.

$$15° = \tfrac{1}{2}(30°)$$

$$\tan(15°) = \frac{\sin(30°)}{1+\cos(30°)} = \frac{1/2}{1+\sqrt{3}/2}$$

$$= \frac{1}{2(1+\sqrt{3}/2)} = \frac{1}{2+\sqrt{3}}.$$

Theorem (Double-Angle Formulas):

1. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.
2. $\cos(2\theta) = (\cos(\theta))^2 - (\sin(\theta))^2$.
3. $\tan(2\theta) = \dfrac{2\tan(\theta)}{1-(\tan(\theta))^2}$.

Ex: Given that $\sin(\pi/10) = \frac{1}{4}(\sqrt{5}-1)$, find $\sin(\pi/5)$ exactly.

Since $\pi/5 = 2(\pi/10)$, we can use

$$\sin(\pi/5) = 2\sin(\pi/10)\cos(\pi/10)$$
$$= 2\left(\frac{1}{4}(\sqrt{5}-1)\right)\cos(\pi/10)$$

Since $(\sin(\pi/10))^2 + (\cos(\pi/10))^2 = 1$,

$$\left(\frac{1}{4}(\sqrt{5}-1)\right)^2 + (\cos(\pi/10))^2 = 1$$
$$\frac{1}{16}(5 - 2\sqrt{5} + 1) + (\cos(\pi/10))^2 = 1$$
$$\frac{6}{16} - \frac{2}{16}\sqrt{5} + (\cos(\pi/10))^2 = 1$$
$$\cos(\pi/10) = \sqrt{\frac{10}{16} + \frac{2}{16}\sqrt{5}}$$
$$\cos(\pi/10) = \sqrt{\frac{5}{8} + \frac{1}{8}\sqrt{5}}$$

$$\sin(\pi/5) = 2\left(\frac{1}{4}(\sqrt{5}-1)\right)\left(\sqrt{\frac{5}{8} + \frac{1}{8}\sqrt{5}}\right).$$

Ex Given that $\tan(\pi/7) = .4816$, find $\tan(4\pi/7)$.

$$4\pi/7 = 2\left(2\pi/7\right) = 2\left(2\left(\pi/7\right)\right)$$

$$\tan\left(2\pi/7\right) = \frac{2\tan(\pi/7)}{1-\left(\tan(\pi/7)\right)^2} = \frac{2(.4816)}{1-(.4816)^2} = 1.2539$$

$$\tan\left(4\pi/7\right) = \frac{2\tan(2\pi/7)}{1-\left(\tan(2\pi/7)\right)^2} = \frac{2(1.2539)}{1-(1.2539)^2} = -4.3813.$$

Theorem (The Sum and Difference Formulas):

1. $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
2. $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
3. $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
4. $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
5. $\tan(\alpha+\beta) = \dfrac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$
6. $\tan(\alpha-\beta) = \dfrac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$.

Ex: Find the exact value of $\cos(75^\circ)$.

$75^\circ = 30^\circ + 45^\circ$

$$\cos(75^\circ) = \cos(30^\circ)\cos(45^\circ) - \sin(30^\circ)\sin(45^\circ)$$

$$= \frac{\sqrt{3}}{2}\,\frac{\sqrt{2}}{2} - \frac{1}{2}\,\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

Could also use half-angle: we know $\cos(150^\circ)$

Ex: Find the exact value of $\tan(7\pi/12)$.

$7\pi/12 = \frac{4\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$, so

$$\tan(7\pi/12) = \frac{\tan(\pi/3) + \tan(\pi/4)}{1 - \tan(\pi/3)\tan(\pi/4)} = \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}}$$