1.4 Converse / Inverse / Contrapositive

Only if / if and only if

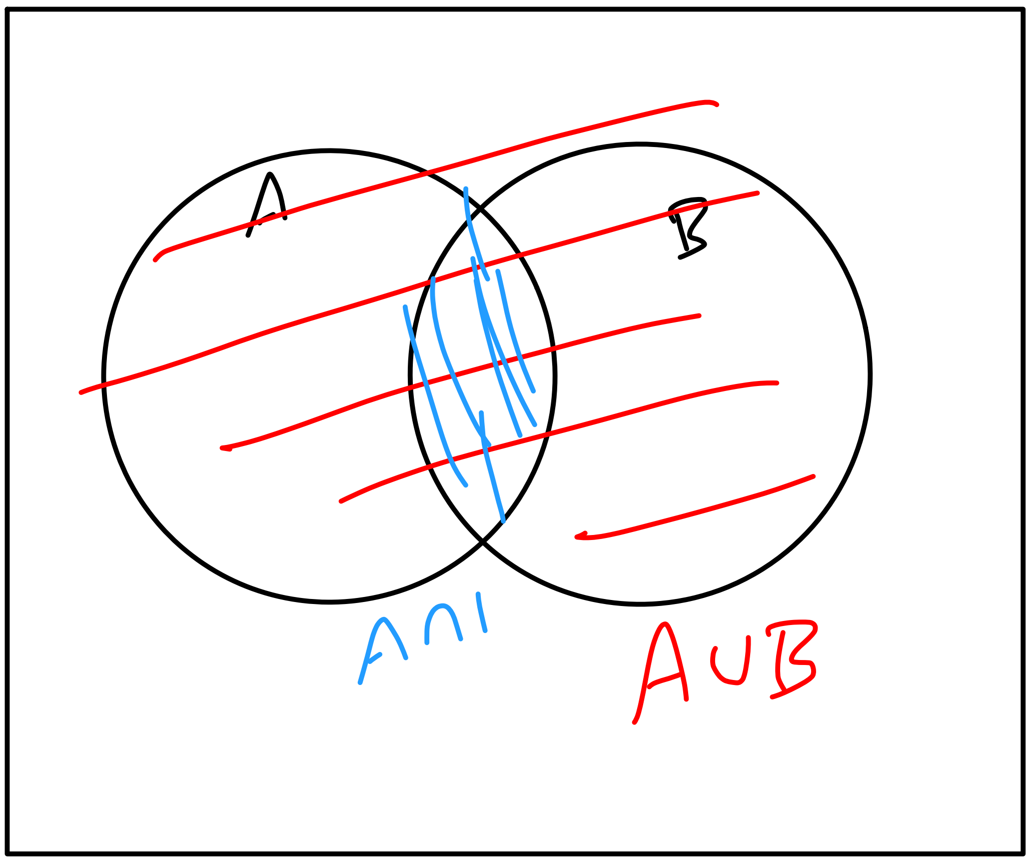
"p only if q" means "if p, then q"

$$p \to q$$

"p if and only if q" means "if p, then q, and if q, then p" $p \leftrightarrow q$

P_1	$p \to q$
P_2	$\sim q$
C	$\sim p$

$$P_1 \wedge P_2 \to C$$



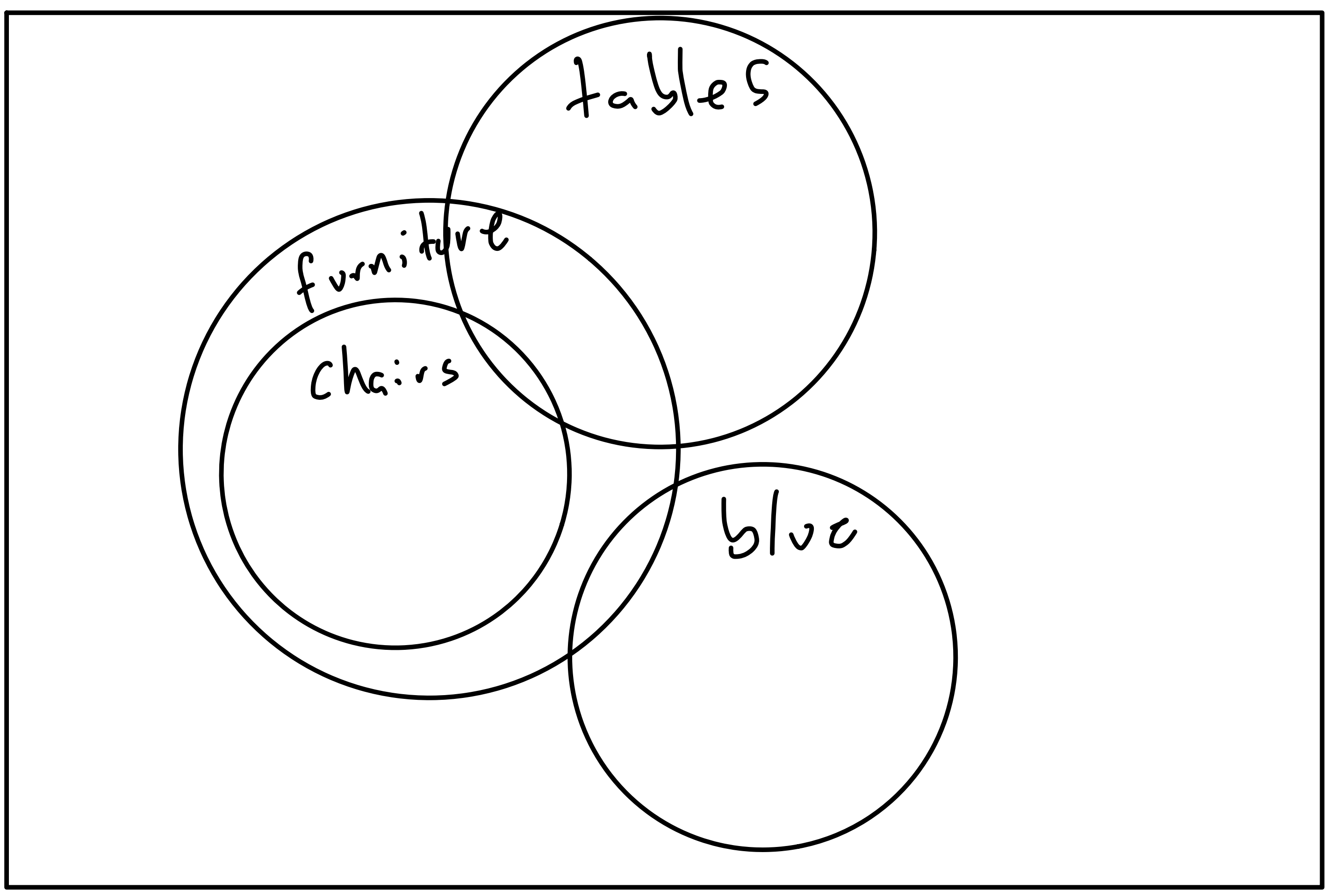
HW 1 solutions

① general facts
1. All chairs are furniture.
2. Some furniture is blue.
3. Nothing blue is a table.

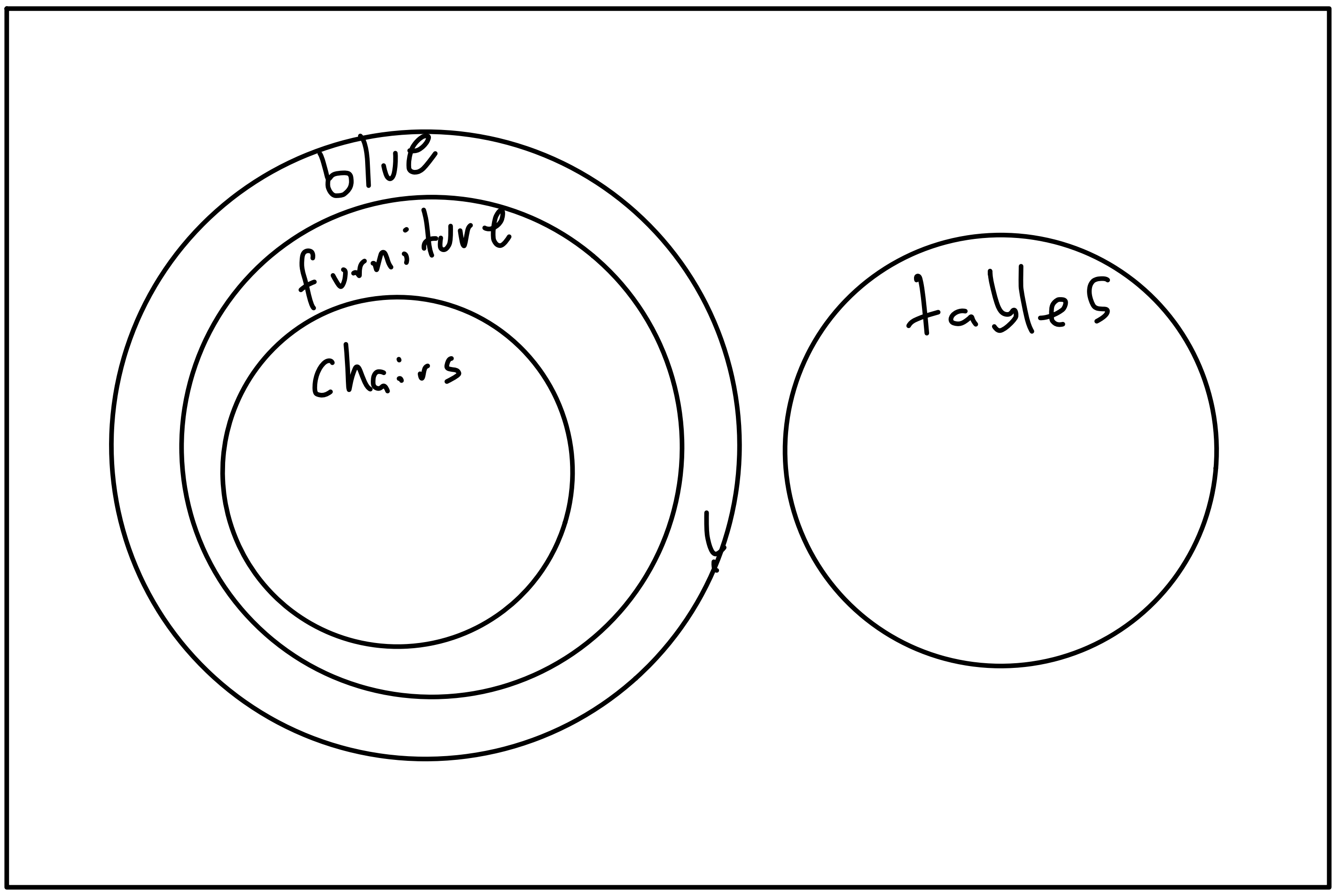
No chairs are tables.

a) Inductive or $\boxed{\text{deductive}}$

b) Let's try to prove that it's invalid. ✓



c)



(2) p: "all chairs are furniture"

q: "no furniture is blue"

r: "some blue objects are tables"

s: "no chairs are tables"

a) p
$\sim q$
$\sim r$
$\overline{}$
s

$p \wedge \sim q \wedge \sim r \longrightarrow s$

p	q	r	s	$p \wedge \sim q \wedge \sim r$	$p \wedge \sim q \wedge \sim r \rightarrow s$
T	T	T	T	F	T
T	T	T	F	F	T
T	T	F	T	F	T
T	T	F	F	F	T
T	F	T	T	F	T
T	F	T	F	F	T
T	F	F	T	T	T
T	F	F	F	T	$\boxed{F}$
F	T	T	T	F	T
F	T	T	F	F	T
F	T	F	T	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	T	F	F	T
F	F	F	T	F	T
F	F	F	F	F	T

c) $x \to y \equiv \sim x \vee y$

$$p \wedge \sim q \wedge \sim r \to s$$

$$\equiv \sim (p \wedge \sim q \wedge \sim r) \vee s$$

$$\equiv \sim p \vee \sim (\sim q) \vee \sim (\sim r) \vee s$$

$$\equiv \sim p \vee q \vee r \vee s$$

2.2: More Venn Diagrams

Ex: The results of a survey tell us:

213 people have tablets

294 have cell phones

337 have Blu-Ray players

109 have all three

64 have none

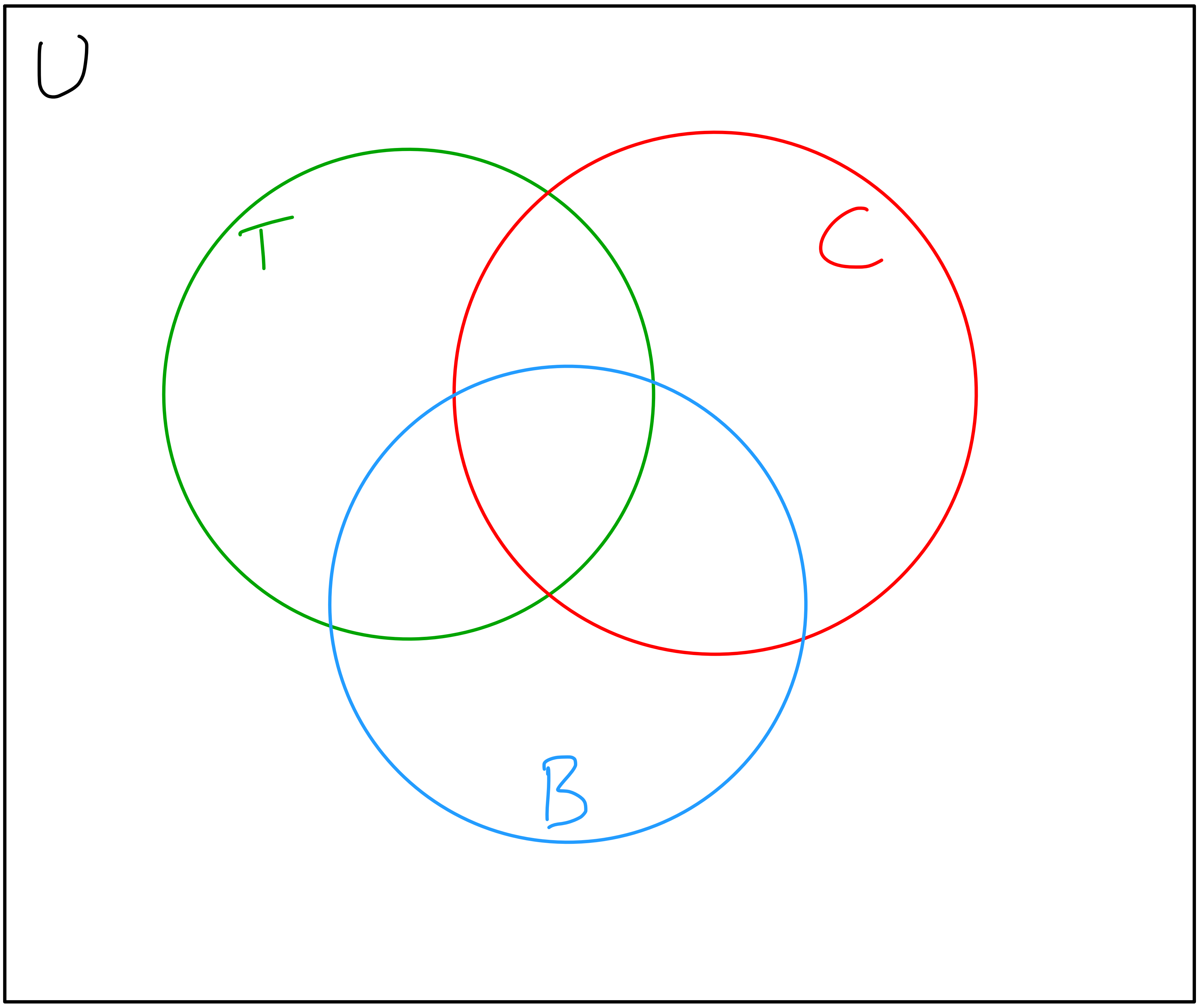
198 have cell phones and Blu-Ray players

382 have cell phones or tablets

61 have tablets and Blu-Ray players, but not cell phones

a) How many people surveyed own tablets but neither Blu-Ray players or cell phones?

b) How many own a Blu-Ray player but not a tablet or cell phone?



If U is the set of people surveyed, C is the set of people with cell phones, T is the set of people with tablets, and B is the set with Blu-Ray players, then:

$$n(T) = 213$$

$$n(C) = 294$$

$$n(B) = 337$$

$$n(C \cap T \cap B) = 109$$

$$n(C' \cap T' \cap B') = 64$$

$$n(C \cap B) = 198$$

$$n(C \cup T) = 382$$

$$n(T \cap B \cap C') = 61$$

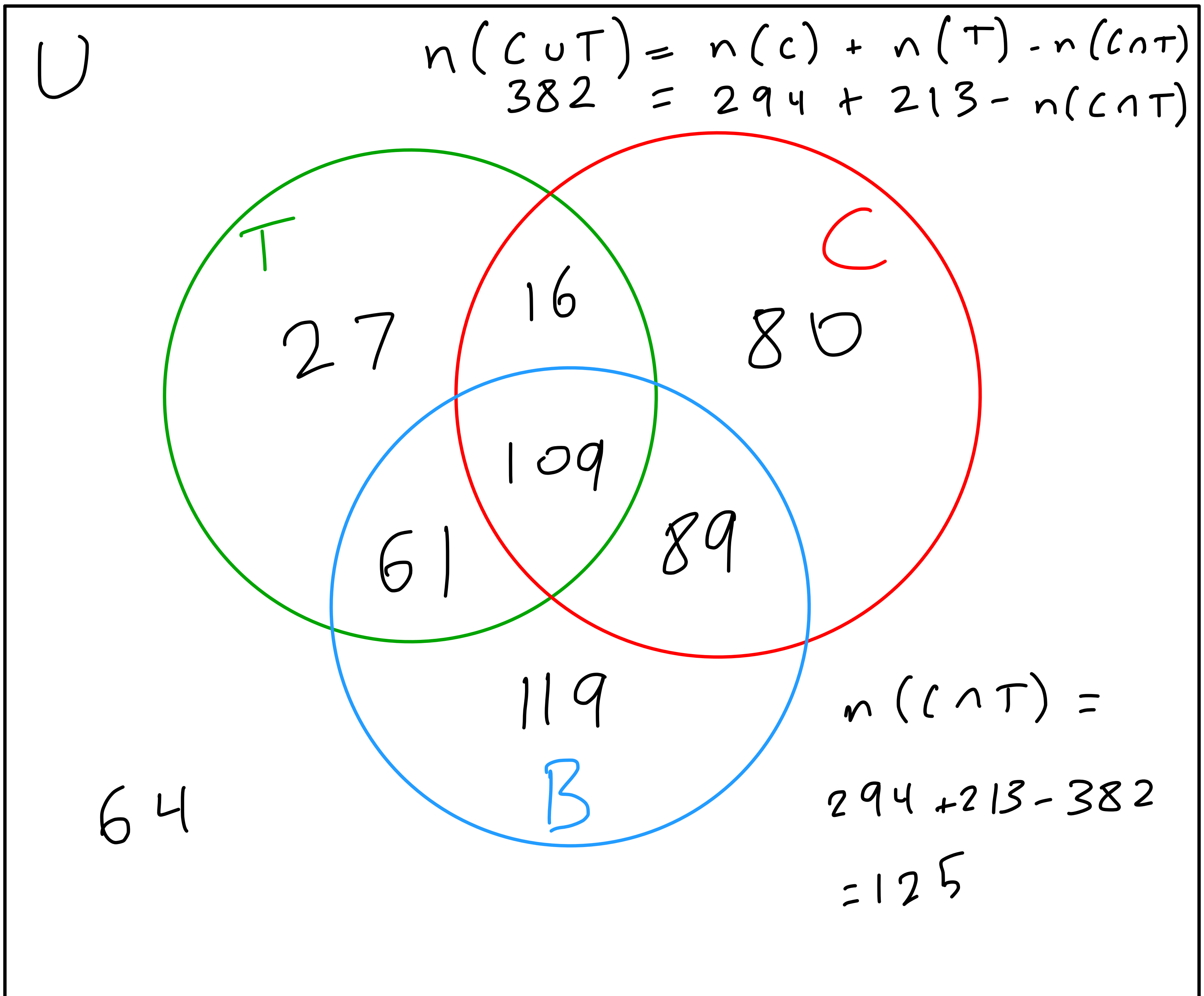
Want: $n(T \cap B' \cap C')$

$n(B \cap T' \cap C')$

To solve these problems, fill in sections of the Venn diagram with cardinality when we know them.

Important: only write in numbers for sets that are not split into smaller sets. E.g. don't write the cardinality of B or $C \cap T$. Then use

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ to solve for the rest.



$n(C \cup T) = n(C) + n(T) - n(C \cap T)$

$382 = 294 + 213 - n(C \cap T)$

$n(C \cap T) =$

$294 + 213 - 382$

$= 125$

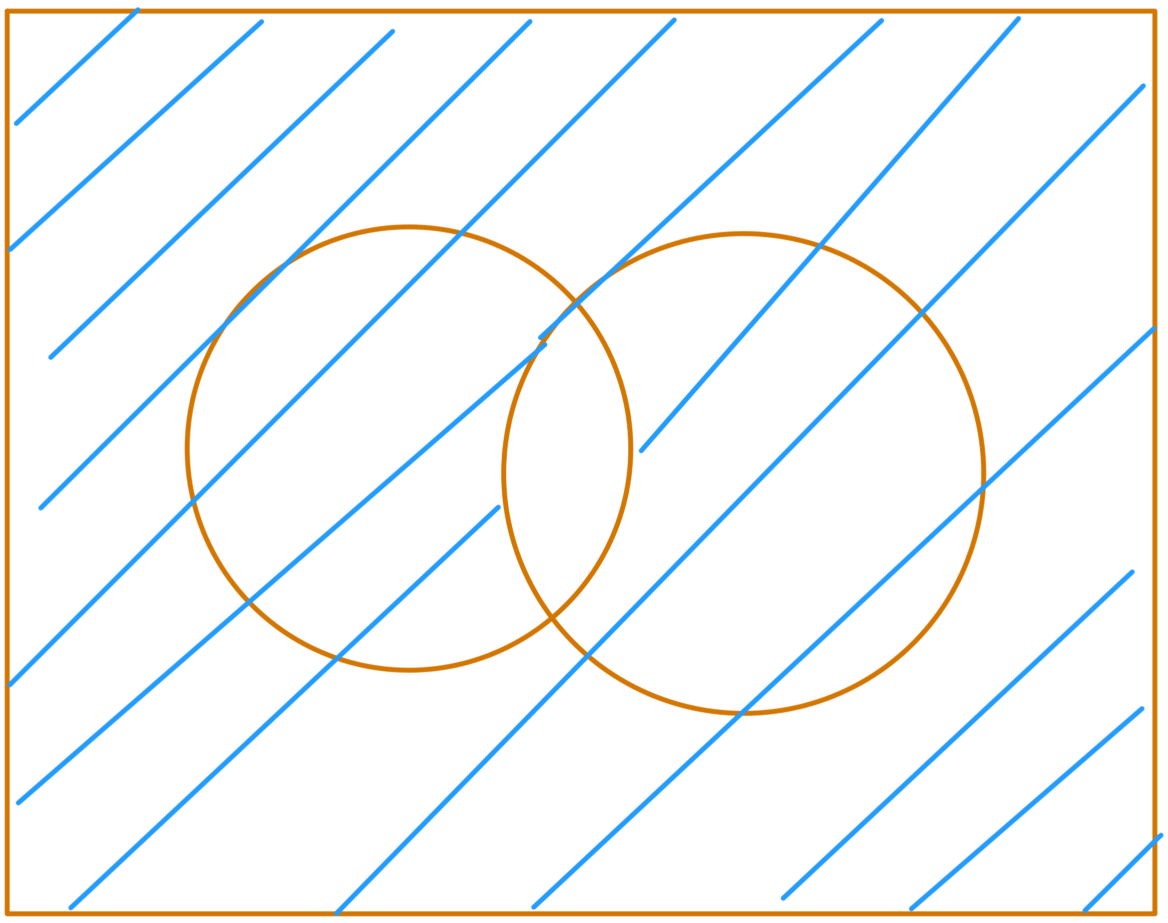
Comment: Recall De Morgan's Laws:

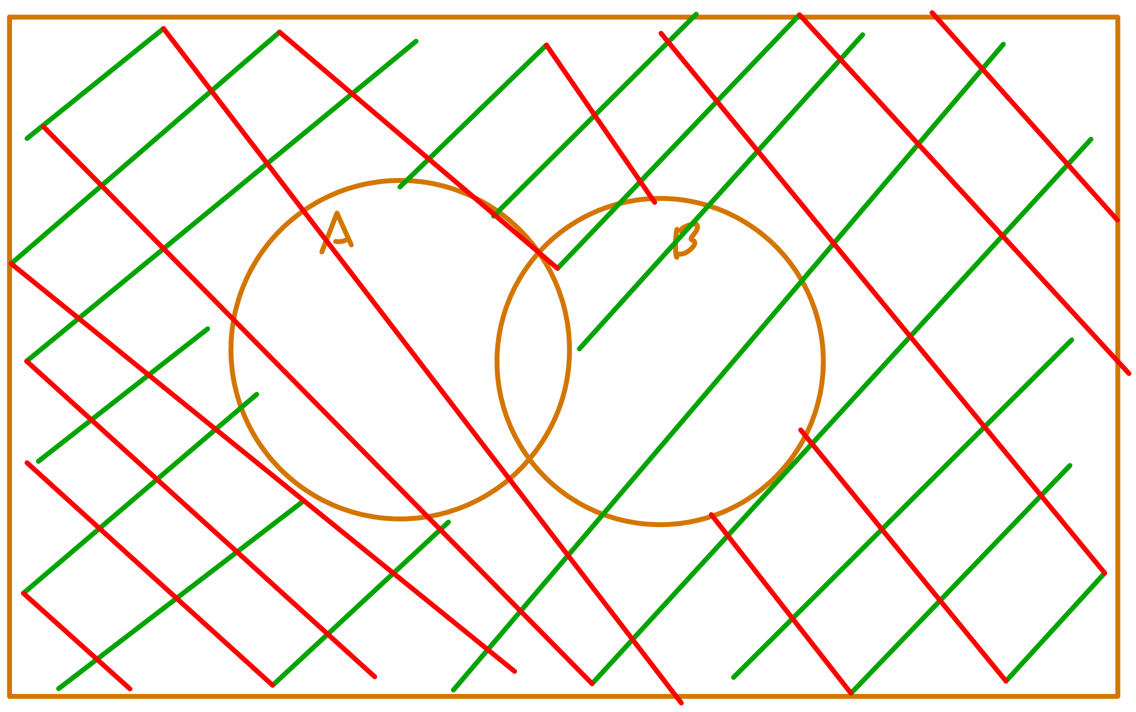
$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

Theorem: $(A \cap B)' = A' \cup B'$

$(A \cup B)' = A' \cap B'$





2.3: Intro to Combinatorics

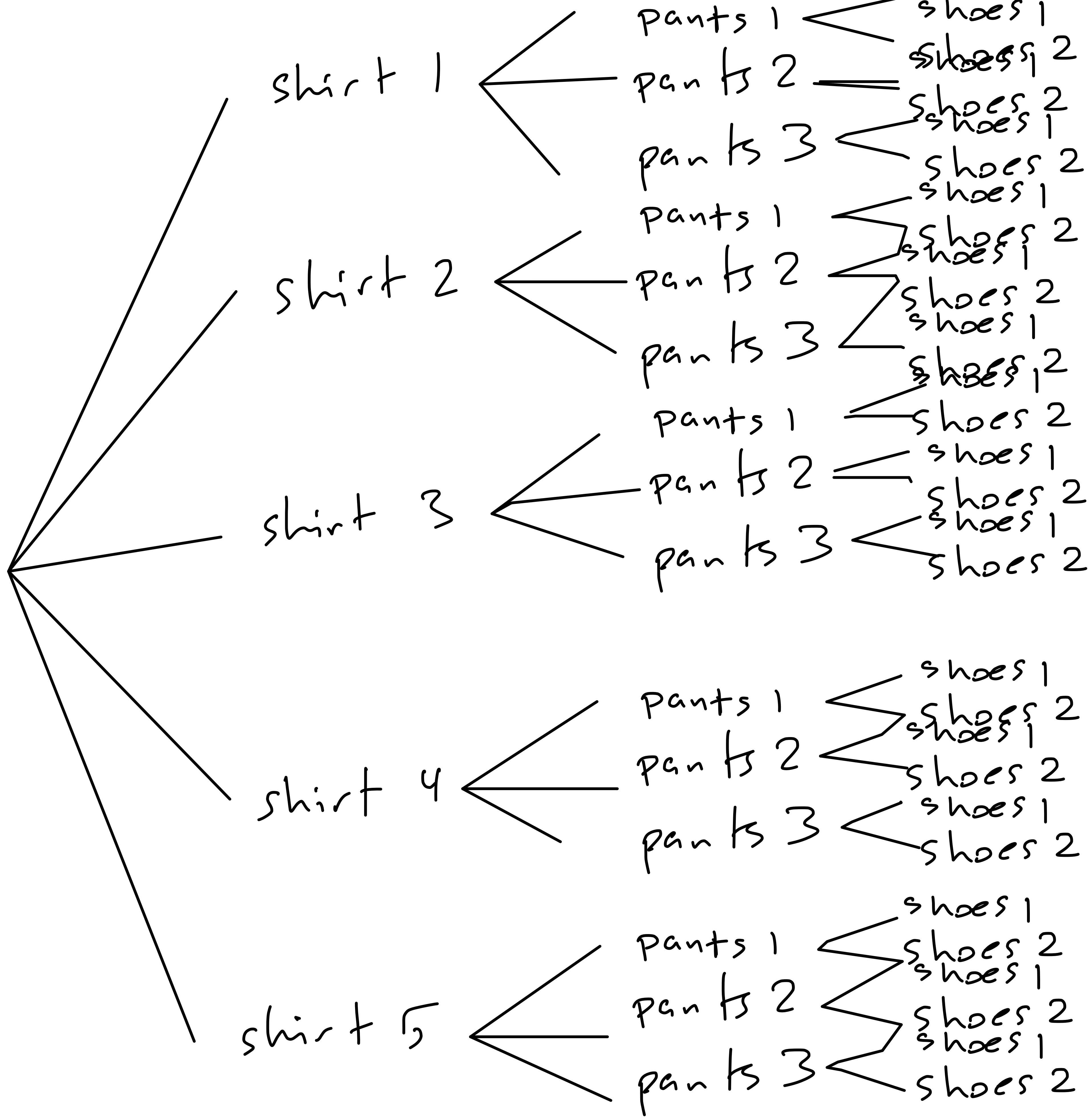
<u>Comment</u>: Combinatorics is the study of counting — answering "how many" questions.

<u>Ex</u>: How many ways can seven people sit in a row if there are two people who refuse to sit next to one another?

Theorem (The Fundamental Principle of Counting): The number of ways to do something is the product of the number of choices you have for each property, as long as those choices have no effect on one another.

Ex: You have five shirts, three pairs of pants, and two pairs of shoes. How many total outfits do you have?

$$5 \cdot 3 \cdot 2 = 30$$


shirt 1
pants 1
shoes 1
shoes 2
pants 2
shoes 1
shoes 2
pants 3
shoes 1
shoes 2
shirt 2
pants 1
shoes 1
shoes 2
pants 2
shoes 1
shoes 2
pants 3
shoes 1
shoes 2
shirt 3
pants 1
shoes 1
shoes 2
pants 2
shoes 1
shoes 2
pants 3
shoes 1
shoes 2
shirt 4
pants 1
shoes 1
shoes 2
pants 2
shoes 1
shoes 2
pants 3
shoes 1
shoes 2
shirt 5
pants 1
shoes 1
shoes 2
pants 2
shoes 1
shoes 2
pants 3
shoes 1
shoes 2

Ex: How many ways can you make a string of five letters so that no letter is repeated?

A	D	E	X	C
26	25	24	23	22

choices

- first pick the first letter
- then pick the second letter from the remaining 25
- then pick the third from the remaining 24

⋮

Now, although the first letter picked affects the choices possible for the second letter, it doesn't affect the number of choices.

Therefore, there are $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$
$= 8{,}252{,}400$ possible strings.

Def: Let n be a positive integer.

n factorial, written $n!$, is the

number $n! = n(n-1)(n-2)\cdots(3)(2)(1)$.

Ex: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$2! = 2 \cdot 1 = 2$

$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$

Theorem: The number of ways to place n different objects in n different boxes with 1 object per box is $n!$.

Ex: The number of ways to arrange the letters A, B, C, D, and E is 5!

The number of ways a 52-card deck can be arranged is $52! \approx 8 \cdot 10^{67}$

The number of seconds since the Big Bang is 10^{18}

Def $0! = 1$.

Ex Simplify $\dfrac{8!}{3!\cdot 5!}$

$$\frac{8!}{3!\,5!} = \frac{8\cdot 7\cdot 6\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{3\cdot 2\cdot 1\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}$$

$$= \frac{8\cdot 7\cdot \cancel{6}}{\cancel{3}\cdot \cancel{2}\cdot 1} = \frac{8\cdot 7}{1} = 56.$$

2.4: Combinations and Permutations

Def: We can choose objects from a large supply in two ways: with replacement or without replacement.

$\underbrace{\text{with replacement}}$: the same object can be drawn multiple times

$\underbrace{\text{without replacement}}$: once an object is drawn, it cannot be drawn again

Ex: How many 4-digit bank PINs are there? This is drawing from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with replacement.

Here, there are 10 possibilities for each digit, so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10000$ PINs total.

<u>Ex</u>: How many ways can you form a group of five people in a room with ten people?

- Drawing without replacement, because one person can't appear multiple times in a single group.

- Order doesn't matter: if we number the people 1 – 10, then pulling 1, 7, 8, 2, 10 gives the same group as pulling 2, 10, 8, 1, 7.

<u>Theorem</u>: The number of ways to choose k objects from a set of n without replacement, where order doesn't matter, is ${}_nC_k = \frac{n!}{k!\,(n-k)!}$.

Ex: The number of ways to choose 5 people from a set of 10 is ${}_{10}C_5 =$

$$\frac{10!}{5!\,(10-5)!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}$$

$$= \frac{10\cdot 9\cdot 8\cdot 7\cdot \cancel{6}}{5\cdot 4\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = \frac{10\cdot 9\cdot \overset{2}{\cancel{8}}\cdot 7}{5\cdot \cancel{4}} = \frac{\overset{2}{\cancel{10}}\cdot 9\cdot 2\cdot 7}{\cancel{5}}$$

$$= 2\cdot 9\cdot 2\cdot 7 = 252$$

Def: Pascal's Triangle is the triangle of numbers formed when each number is the sum of the two above it.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

<u>Theorem</u>: ${}_nC_k$ is the k^{th} entry of the n^{th} row of Pascal's Triangle, where we start counting both the rows and the entries from 0.

<u>Ex</u>: ${}_4C_2 = 6$, ${}_5C_0 = 1$, ${}_6C_1 = 6$

Ex: How many ways can we form an ordered line of 5 people when we're choosing them from a group of 10?

- Drawing without replacement, because one person can't appear multiple times in a single group.
- Order **does** matter.

Theorem: The number of ways to choose and arrange k objects from a set of n is ${}_nP_k = ({}_nC_k)(k!)$.

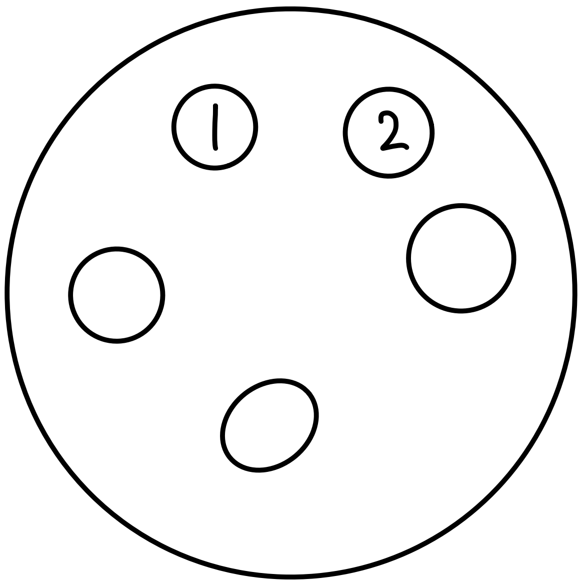
Ex: in our previous example, we have

$${}_{10}P_5 = ({}_{10}C_5)(5!) = (252)(120) = 30240$$

Ex: Now suppose there are two people in the room who refuse to be in the same group. How many orderings of 5 people are there now?

Let's find the number of ways these two people can be the same group. Let's say they're #1 and #2.

Any group with both 1 and 2 has 3 free spots left, and there are no restrictions on the other three slots.



Therefore, there are ${}_8C_3$ ways to fill the rest of the group: 8 people to choose from, because we already used 1 and 2, and 3 people to choose. And ${}_8C_3 = \frac{8!}{3!5!} = 56$.

Therefore, the number of ways to form an unordered group of 5 people without both #1 and #2 at the same time is $252 - 56 = 196$. We want this to be ordered, so we multiply by 5! to get $196 \cdot 5! = 23520$.

$p \to q$

"If $\overbrace{\text{you are a lion}}^{p}$, then $\overbrace{\text{you are a cat}}^{q}$"

$p \to q$ $\equiv$

"You are a cat if you are a lion"

$p \to q$ $\equiv$

"You are a lion only if you are a cat"

$p \to q$

"$\overbrace{\text{You learn about set theory at UO}}^{r}$ if and only if $\underbrace{\text{you take 105}}_{s}$"

$r \longleftrightarrow s$