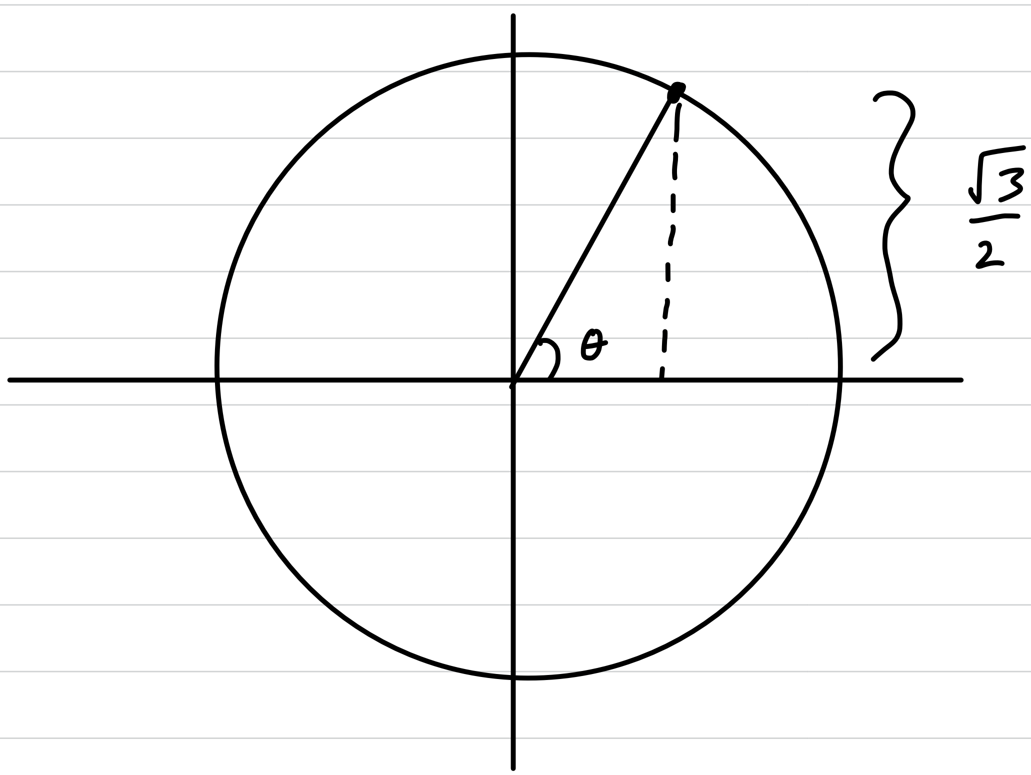
Recall: $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, and it's the slope of the line that passes through $(0,0)$ and the point on the unit circle with angle θ.

Inverse Trig Functions

Ex:



What is θ?

We know $\sin(60°) = \frac{\sqrt{3}}{2}$, so

$\theta = 60°$.

Comment: Recall inverse functions from algebra. If $y = f(x)$, then f^{-1} is the function with $f^{-1}(y) = x$.

Ex: $y = f(x) = x^3$, then $f^{-1}(y) = \sqrt[3]{y}$.

Why? $f^{-1}(y) = f^{-1}(x^3) = \sqrt[3]{x^3} = x$.

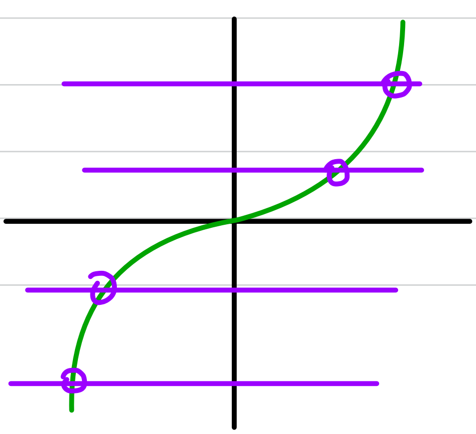
$f(2) = 8$

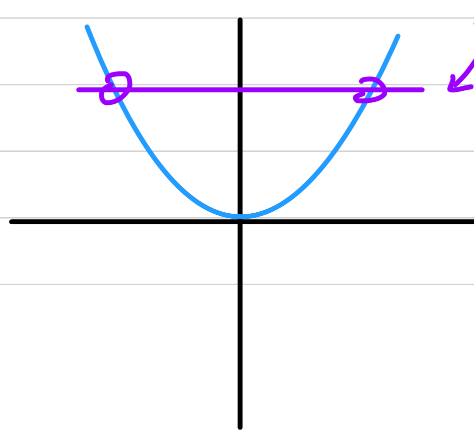
$f^{-1}(8) = 2$

Comment: Given a function f, f^{-1} only exists if f is one-to-one: for all a and b with $a \neq b$, $f(a) \neq f(b)$. Equivalently, if $f(a) = f(b)$, then $a = b$. This means no two x-values have the same y-value (i.e. f passes the horizontal line test).

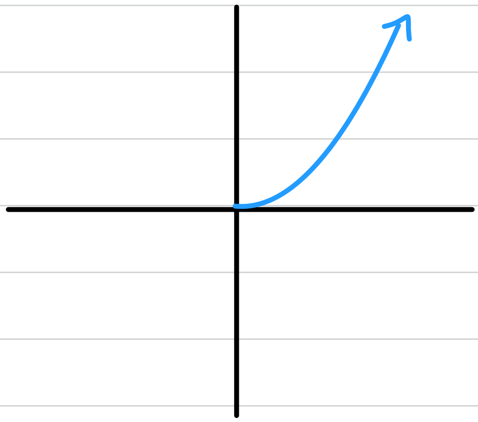
Ex: $y = x^3$ is one-to-one, but $y = x^2$ is not.

fails HLT



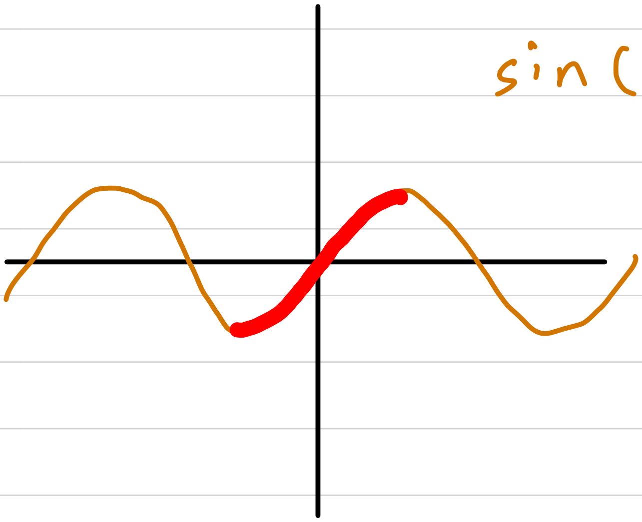


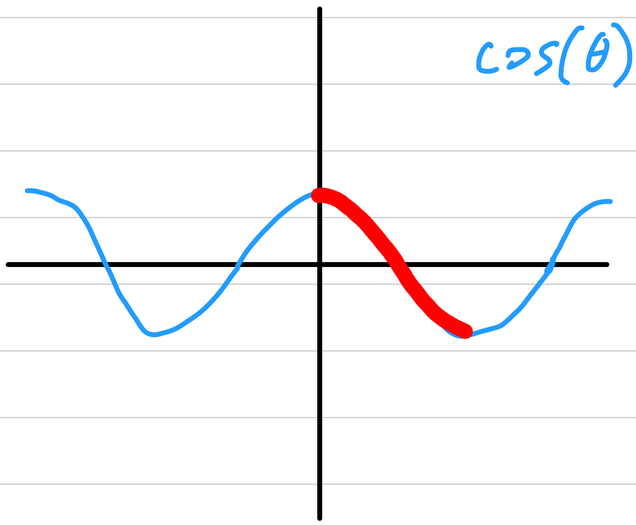
Although $y = x^2$ is not one-to-one, we can shrink its domain so that it is:

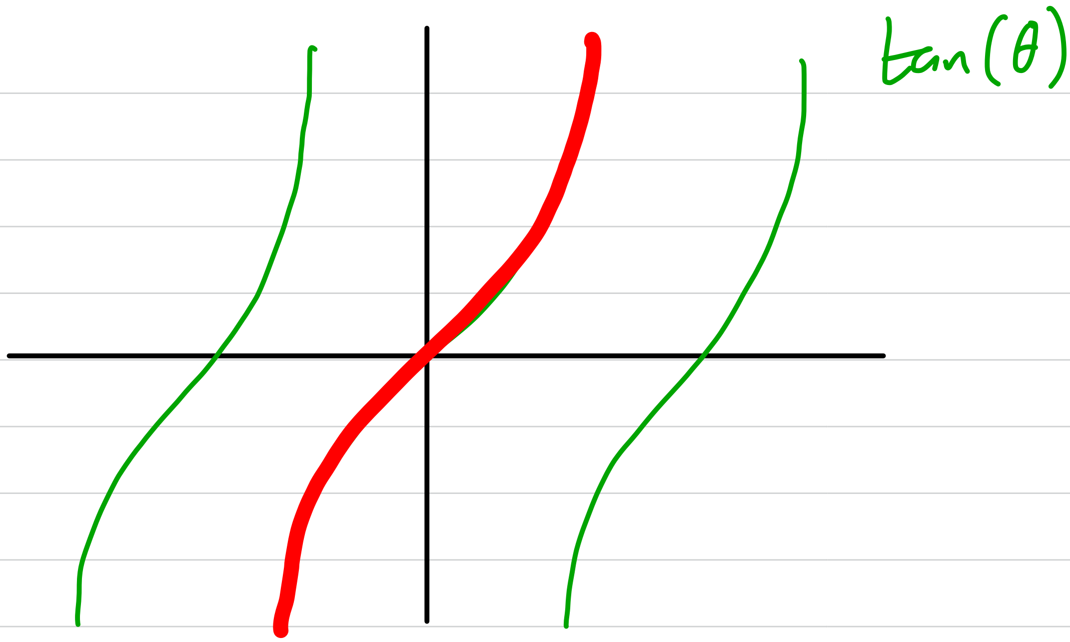


$y = x^2$ is one-to-one when $0 \le x < \infty$. Then, $f^{-1}(y) = \sqrt{y}$

We can similarly restrict the domain of sin, cos, and tan to create inverse functions for them.





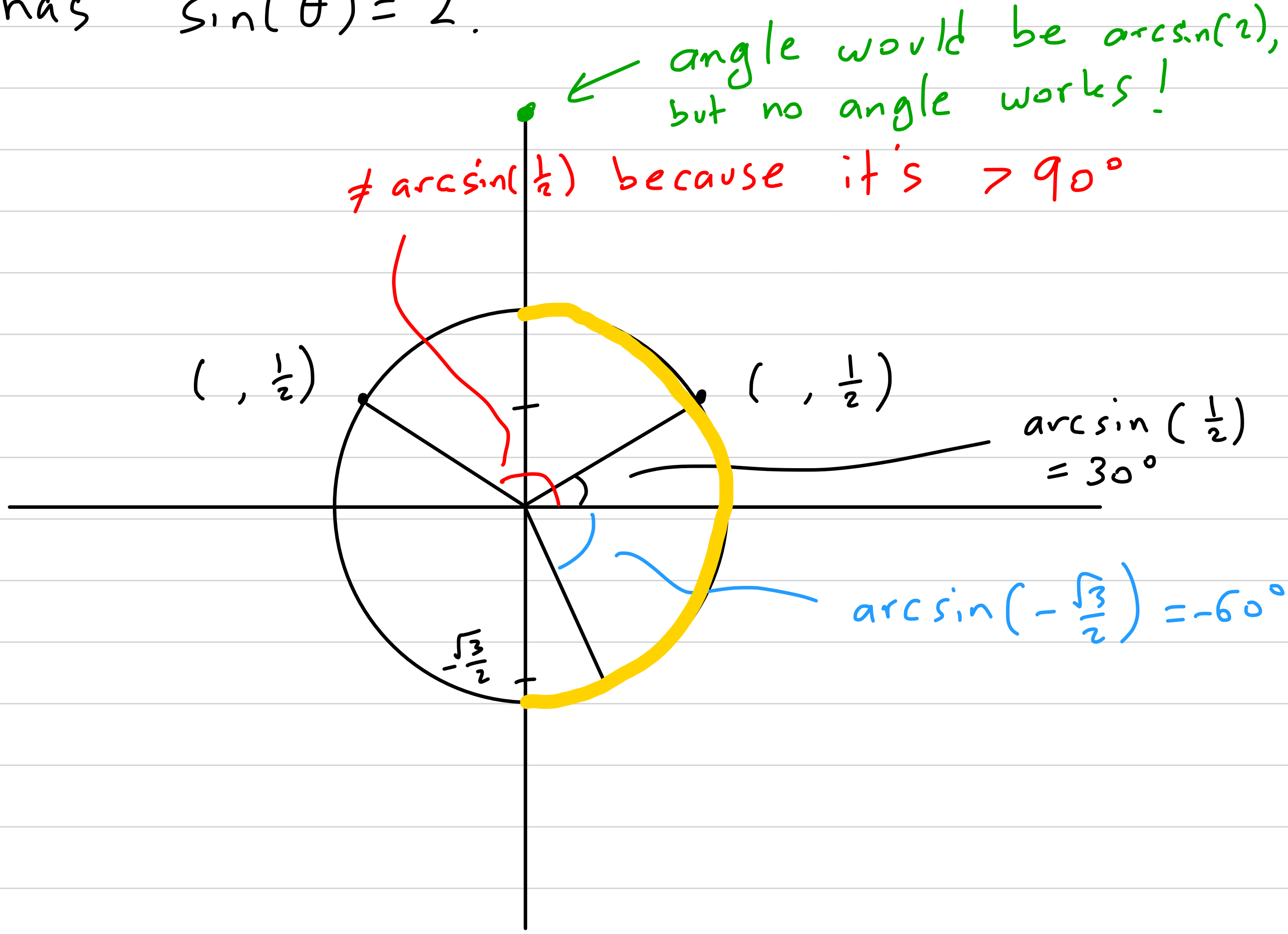


Def: Let x be in $[-1, 1]$. The arcsine of x is $\arcsin(x) = \theta$, where θ is the angle in $[-90°, 90°]$ such that $\sin(\theta) = x$.

Ex: $\arcsin\left(\frac{1}{2}\right) = 30°$ since $\sin(30°) = \frac{1}{2}$ and $-90° \le 30° \le 90°$

$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -60°$ since $\sin(-60°) = -\frac{\sqrt{3}}{2}$ and $-90° \le -60° \le 90°$

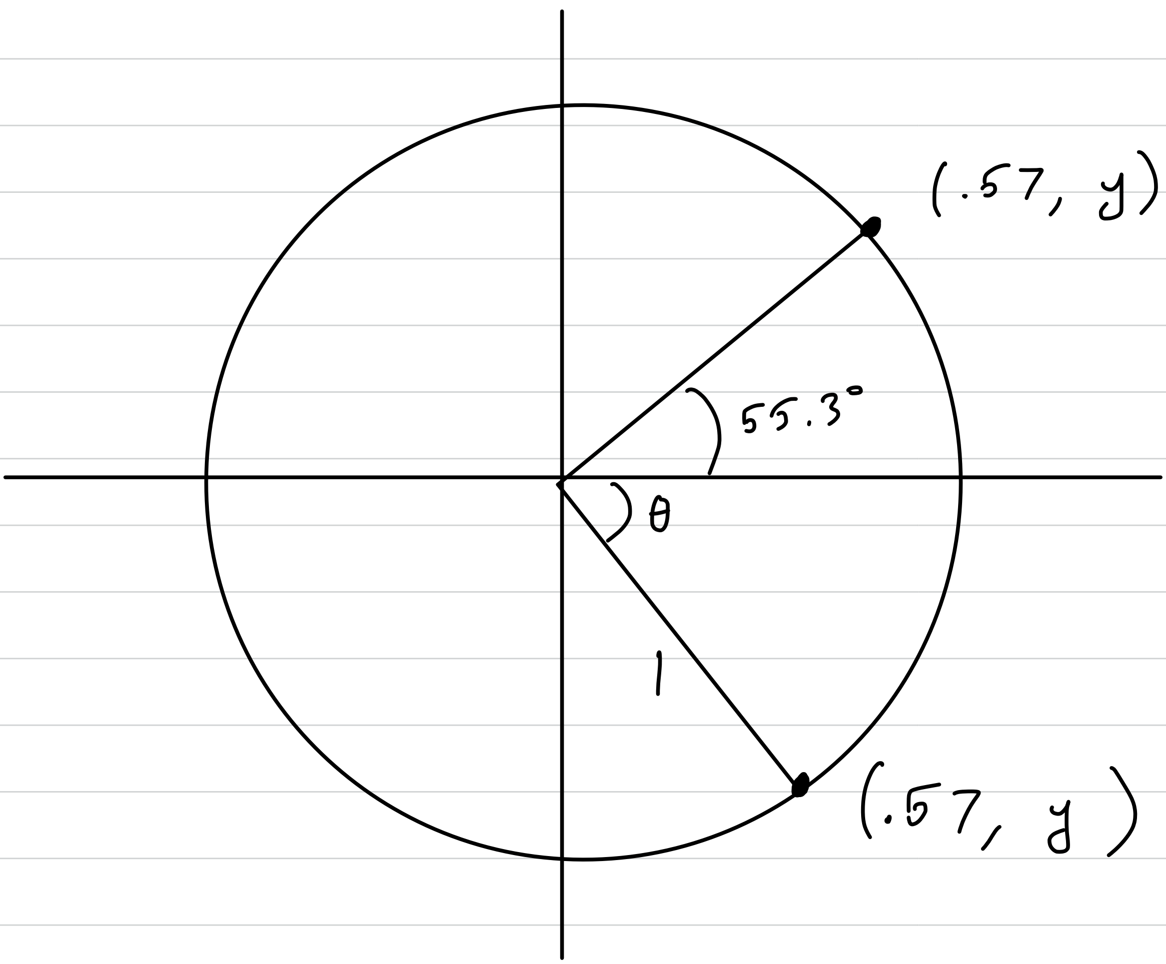
arcsin(2) DNE because no angle θ has $\sin(\theta) = 2$.



Def: Let x be in $[-1, 1]$. The arc cosine of x is $\arccos(x) = \theta$, where θ is the angle in $[0°, 180°]$ such that $\cos(\theta) = x$.

Def: Let x be in $[-1, 1]$. The arctangent of x is $\arctan(x) = \theta$, where θ is the angle in $[-90°, 90°]$ such that $\tan(\theta) = x$.

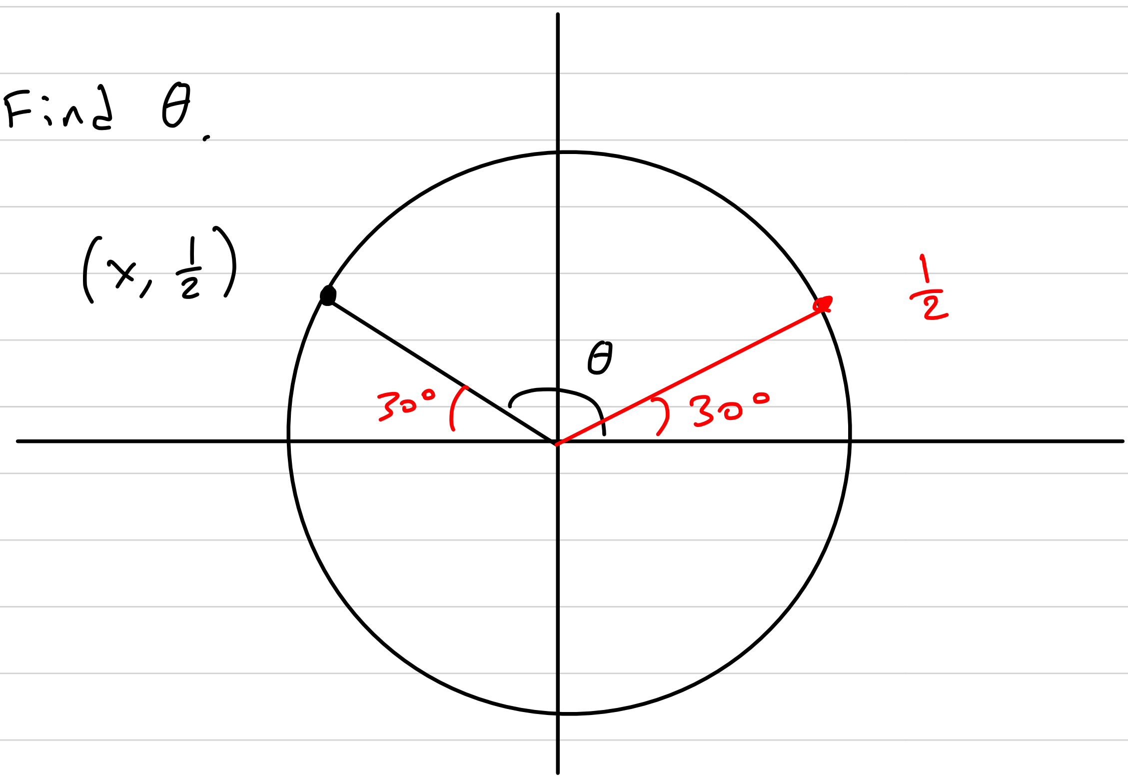
Ex: find θ.



$\cos(\theta) = .57$, so we want to say $\theta = \arccos(.57) = 55.3°$. So $\theta = -55.3°$.

Comment: arcsin, arccos, and arctan are also called $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$.

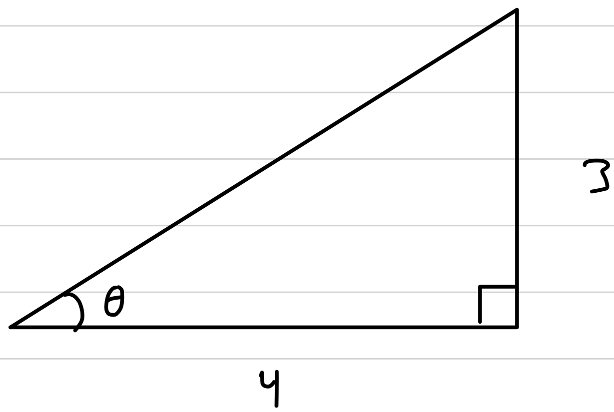
Ex: Find θ.



$\sin(\theta) = \frac{1}{2}$

$\arcsin\left(\frac{1}{2}\right) = 30° \implies \theta = 150°$

Ex: Find θ.



$\tan(\theta) = 3/4 \implies$ use arctan

Since $-90^\circ \le \theta \le 90^\circ$, $\theta = \arctan(3/4)$

(calc) $= 36.87^\circ$

Recap of chapter 2:

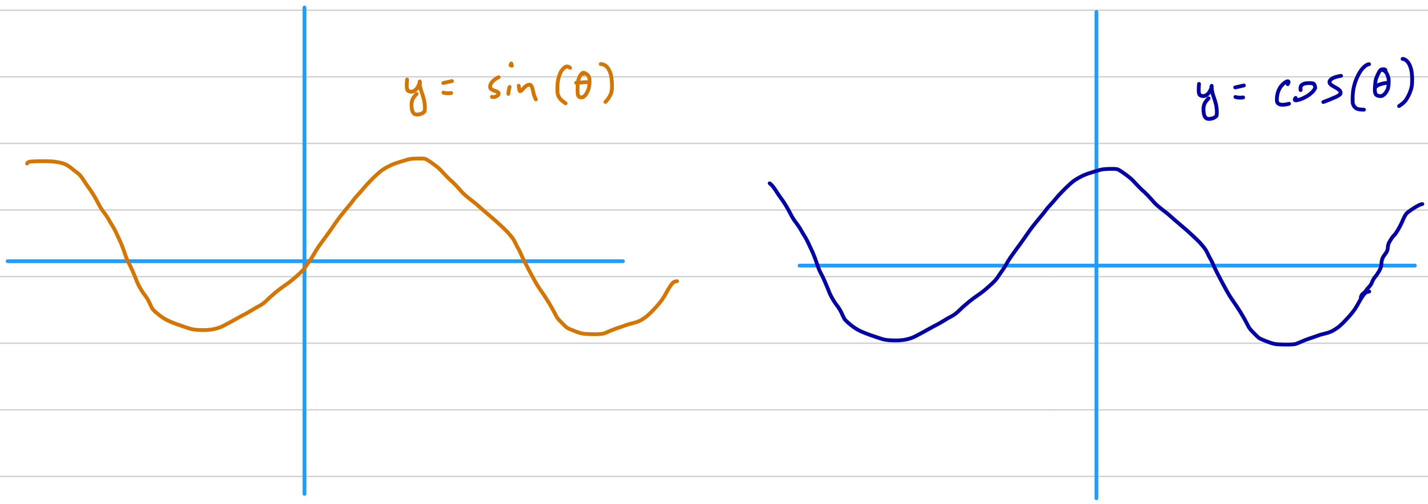
- sin and cos take in angles and give y and x coordinates of points on the unit circle with those angles.

- sin and cos also give ratios of side

lengths of right triangles

- $0°$, $30°$, $45°$, $60°$, and $90°$, and any angle with one of those as a reference angle is an angle whose sin and cos we know exactly.

- The graphs of sin and cos are waves.



- The tangent function is $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.

It gives the slope of the line that passes through (0, 0) and the point on the unit circle with angle θ. It also gives a ratio of side lengths in a right triangle.

- The inverse trig functions arcsin, arccos, and arctan take in $\underbrace{\text{coordinates on the unit circle}}_{\text{arcsin and arccos}}$, $\underbrace{\text{slopes}}_{\text{arctan}}$, or $\underbrace{\text{ratios of sides in right triangles}}_{\text{arcsin, arccos, arctan}}$, and they output the corresponding angles.

Chapter III

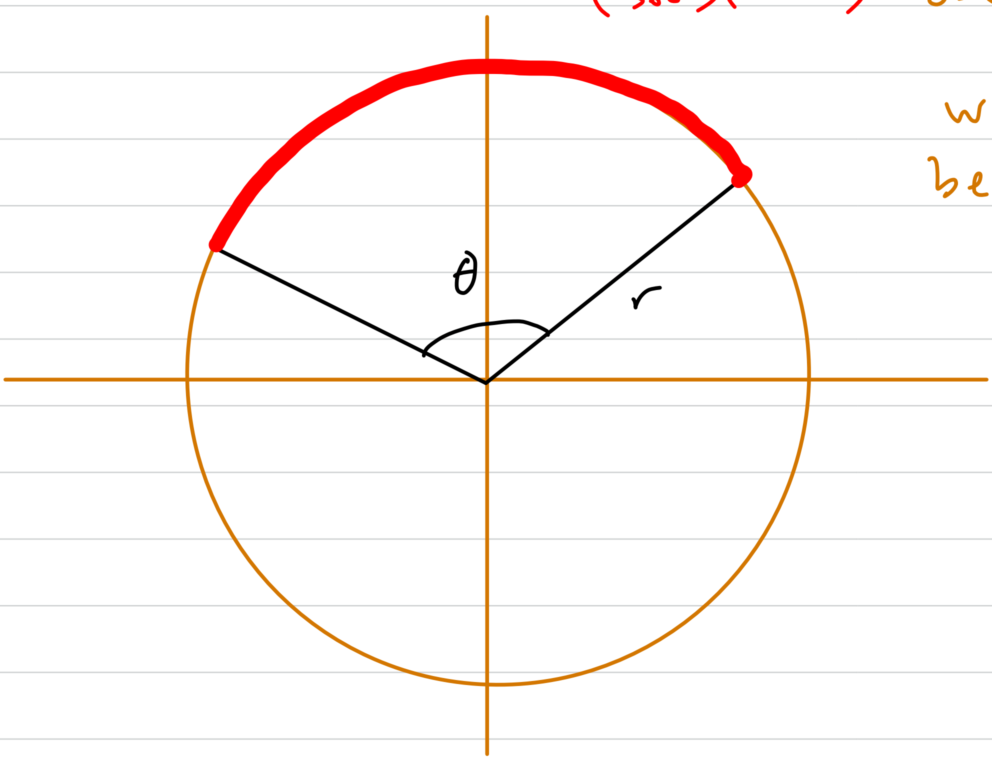
- Is there a better way to measure angles?
- How can we use trig in non-right triangles?
- How can we solve equations involving trig functions?
- What properties do transformations of $\sin(\theta)$ have?
- What relationships do the trig functions have with one another?

Radians

Prop: A circle with radius r has area πr^2 and circumference $2\pi r$.

Prop: An arc of a circle with radius r whose angle measure is θ degrees has arc length $\left(\frac{\theta}{360}\right)(2\pi r)$. ← this is somewhat awkward to use/remember

$\left(\frac{\theta}{360}\right)(2\pi r)$



we can do better

Def: Let θ be an angle. The radian measure of θ is the arc length of an arc with angle θ in the unit circle.

Ex: $360° = 2\pi$ in radians, since the arc with angle $360°$ is the entire unit circle, and its circumference is $2\pi(1) = 2\pi$.

Ex:

degrees	radians	
$0°$	0	
$30°$	$\pi/6$	*
$45°$	$\pi/4$	
$60°$	$\pi/3$	*
$90°$	$\pi/2$	
$180°$	π	
$270°$	$3\pi/2$	
$360°$	2π	

be careful!

Comment: Radians technically have no units, so if you don't see a degree symbol, assume that angle is given in radians.

Ex: $30° = \pi/6$, not $30° = \pi/6$ radians

Prop: If θ is measured in degrees, then the radian measure of θ is $\left(\frac{\pi}{180°}\right)\theta$.
If θ is measured in radians, then the degree measure of θ is $\left(\frac{180°}{\pi}\right)(\theta)$.

Ex: What is $120°$ in radian measure?

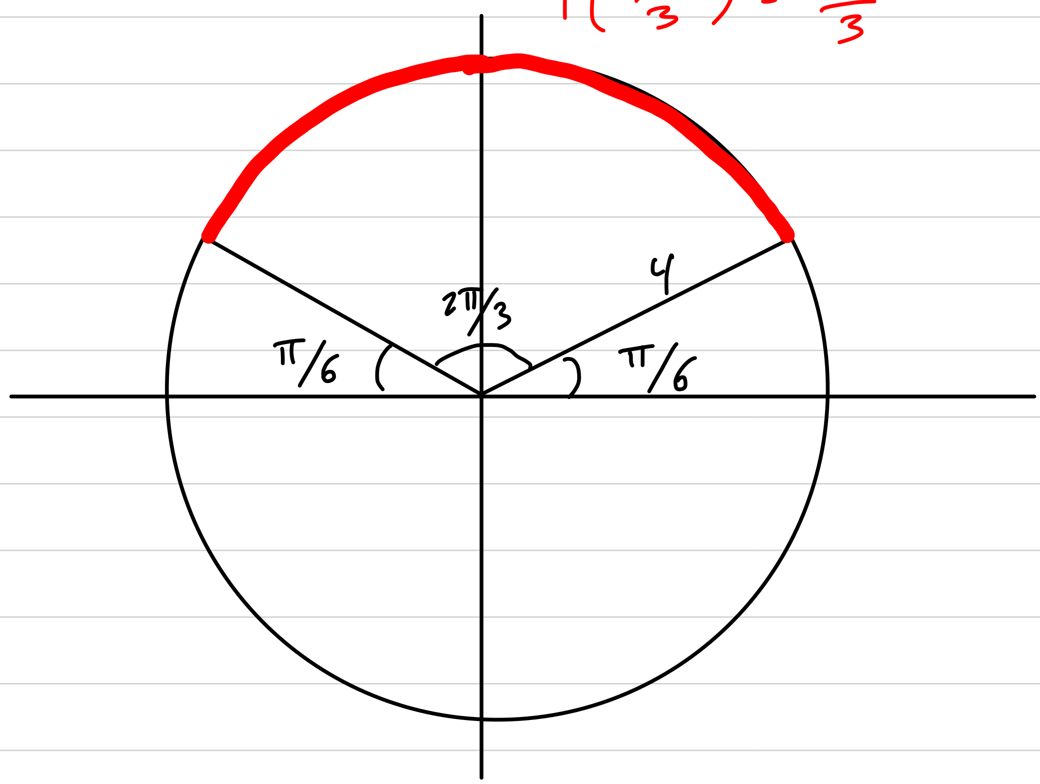
$$\left(\frac{\pi}{180°}\right)(120°) = \frac{120}{180}\pi = \frac{4\cdot 30}{6\cdot 30}\pi = \frac{4}{6}\pi = \frac{2\pi}{3}$$

Comment: Just like (e.g. $\frac{\sqrt{3}}{2}$), always leave radian measures in exact form (with π)

Theorem: If a circle of radius r has an arc with angle θ (in radian measure), then the arc length is $r\theta$.

$4\left(\frac{2\pi}{3}\right) = \frac{8\pi}{3}$

Ex:

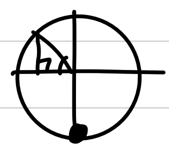


Ex: Find all the quantities listed, with exact and radian values whenever possible.

$\cos(\pi/3) = 1/2$ $\quad\quad$ $\pi/3 = 60°$

$\sin(\pi/2) = 1$ $\quad\quad$ $\pi/2 = 90°$

$\sin(3\pi/4) = \sqrt{2}/2$ $\quad\quad$ $3\pi/4 = 135°$



$\tan(3\pi/2) = \text{undefined}$ $\quad\quad$ $3\pi/2 = 270°$

$\cos(-\pi/6) = \frac{\sqrt{3}}{2}$ $\quad\quad$ $-\pi/6 = -30°$

$\arcsin(1/2) = \pi/6$