Practice Midterm 1 Solutions

Math 252

1.

a) Version I: Let f be a continuous function defined on $[a,b]$. Then $\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$.

Version II: Let f be a continuous function defined on $[a,b]$ and let F be an antiderivative of f. Then $\int_a^b f(x)\ dx = F(b) - F(a)$.

b) The integrand is $s^3 + s^2$, the limits are 1 to 4, and the variable is s. (This is good to know, but you don't need to know these exact definitions for the midterm.)

2.

a) True. The upper sum is taken by choosing x_i^* to be the largest value of $f(x)$ in the subinterval, so the area of the rectangle will overestimate the area under the graph along the subinterval. Since every rectangle is an overestimate, the entire function is an overestimate.

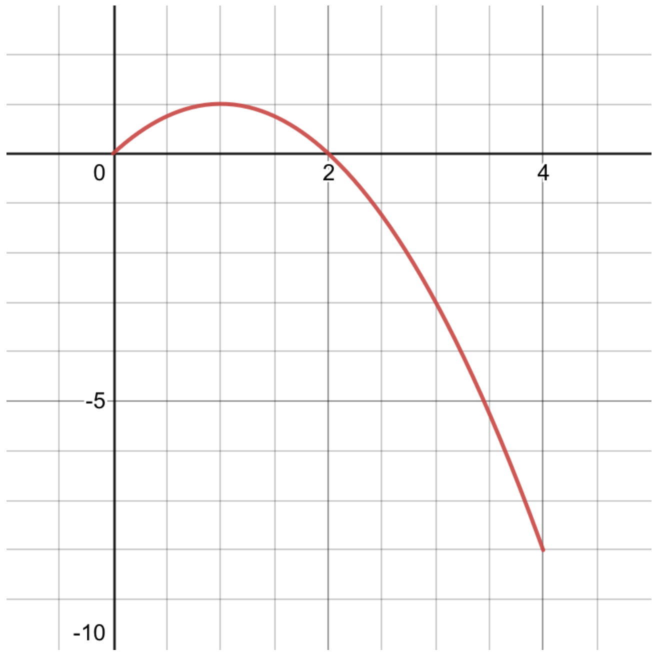
b) False. The definition of an antiderivative is that $A'(x) = f(x)$, so $A(x) - \pi$ doesn't necessarily meet this definition (although that has nothing to do with the π).

c) False. A function like $y = x^3$ shows this to be false: for example, $a = -1$, $b = 0$, and $c = 10$.

d) True. For example, move 8 meters to the right, 1 meter to the left, and 1 meter back to the right.

3. This integral is missing its dt. An integral is the limit of a Riemann sum, where the height of the rectangles approaches the height of the function (here $t^2 + 1$), and the width of the subintervals becomes arbitrarily small (denoted by dt). Without the dt term, we're taking more and more rectangles in the limit, but the widths aren't getting smaller (they're staying at 1), so the expression is always going to be infinitely large and not measure anything meaningful.

5. Something seems to have happened with the graphs. Regardless, this one can be solved in many ways, but here's one example:



6. We want to find s, which is an antiderivative of v. Because of that, one way to solve this problem is to set $s(t) = \int v(t)\ dt$. Then

$$\begin{aligned} s(t) &= \int v(t)\ dt \\ &= \int \frac{\ln(t+e)}{t+e}\ dt \\ &= \int u\ du, \text{ where } u = \ln(t+e) \text{ and therefore } du = \frac{1}{t+e}\ dt \\ &= \frac{u^2}{2} + C \\ &= \frac{(\ln(t+e))^2}{2} + C. \end{aligned}$$

Then we use $s(0) = 3$ to find that

$$\begin{aligned} 3 &= \frac{(\ln(0+e))^2}{2} + C \\ &= \frac{1^2}{2} + C \\ &= \frac{1}{2} + C. \end{aligned}$$

Thus $C = \frac{5}{2}$, so $s(t) = \frac{(\ln(t+e))^2}{2} + \frac{5}{2}$. Alternatively, we can use the Fundamental Theorem of Calculus (part II):

$$\begin{aligned} s(x) - s(0) &= \int_0^x v(t)\ dt \\ &\vdots \\ &= \left[\frac{(\ln(t+e))^2}{2}\right]\Bigg|_0^x \\ &= \frac{(\ln(x+e))^2}{2} - \frac{(\ln(0+e))^2}{2} \\ &= \frac{(\ln(x+e))^2}{2} - \frac{1}{2}. \end{aligned}$$

Since $s(0) = 3$, we find that $s(x) = \frac{(\ln(x+e))^2}{2} - \frac{5}{2}$, which is equivalent to the previous answer. Either method is fine to use.

7. Set $F(x) = \displaystyle\int_4^x \frac{t^4 - 2}{t^2 + 2t + t}\ dt$ (Amusing that the dt is missing here when there was a whole problem about it earlier). Then

$$\begin{aligned} \frac{d}{dx}\int_4^x \frac{t^4-2}{t^2+2t+1}\ dt &= \frac{d}{dx}F(e^{3x}) \\ &= F'(e^{3x})(e^{3x})', \text{ by the Chain Rule} \\ &= \frac{(e^{3x})^4 - 2}{(e^{3x})^2 + 2(e^{3x}) + 1} 3e^{3x}, \text{ by FTC I.} \end{aligned}$$

8. All antiderivatives differ by adding or subtracting a constant, so we just need to find one:

$$\begin{aligned}\int \sin^3(\pi x)\cos(\pi x)\ dx &= \frac{1}{\pi}\int u^3\ dx, \text{ where } u = \sin(\pi x) \text{ and therefore } du = \pi\cos(\pi x)\ dx, \text{ so } \frac{1}{\pi}du = \cos(\pi x)\ dx\\ &= \frac{u^4}{4}\\ &= \frac{\sin^4(\pi x)}{4} + C.\end{aligned}$$

So for example, taking $C = 1$ and $C = -3$, we get two antiderivatives: $\frac{\sin^4(\pi x)}{4} + 1$ and $\frac{\sin^4(\pi x)}{4} - 3$.

9. We have that the average value is

$$\begin{aligned}\frac{1}{6-2}\int_2^6 \frac{16}{4+t^2}\ dt &= \frac{16}{4}\int_2^6 \frac{1}{4+t^2}\ dt\\ &= 4\int_2^6 \frac{1}{4+t^2}\ dt\\ &= 4\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]\Bigg|_2^6\\ &= 4\left(\frac{1}{2}\tan^{-1}(3) - \frac{1}{2}\tan^{-1}(1)\right).\end{aligned}$$

And this answer is in units of thousand leaves.