Name: ________________________ Homework 4 | Math 342 | Cruz Godar

*Due Wednesday of Week 5 at the start of class*

Complete the following problems and submit them as a pdf to Canvas. 8 points are awarded for thoroughly attempting every problem, and I'll select three problems to grade on correctness for 4 points each. Enough work should be shown that there is no question about the mathematical process used to obtain your answers.

## Section 5

In problems 1–6, determine if the linear transformation $T$ is one-to-one, if it is onto, and if it is invertible. If it is invertible, find the inverse transformation.

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix}$.

2. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y + z \end{bmatrix}$.

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -10 \end{bmatrix}.$$

4. $T : \mathbb{R}^3 \to \mathbb{R}^3$, where

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \qquad T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 16 \\ 7 \end{bmatrix}.$$

5. $T : \mathbb{R}^3 \to \mathbb{R}$, where

$$T\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = -3 \qquad T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = 1 \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = 4.$$

6. $T : \mathbb{R}^5 \to \mathbb{R}^2$, where

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

7. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by the matrix $\begin{bmatrix} 1 & 7 & 2 & -1 \\ -3 & 6 & 3 & 12 \\ 0 & 3 & 1 & 1 \end{bmatrix}$.

a) Is $T$ one-to-one? Is it onto? Is it invertible?

b) Find the vectors $\vec{v}$ for which $T(\vec{v}) = \vec{0}$ and express it as a span of one or more vectors, i.e. $\text{span}\{\vec{v_1}, ..., \vec{v_n}\}$.

c) Find vectors $\vec{w_1}, ..., \vec{w_m} \in \mathbb{R}^4$ that are linearly independent to $\vec{v_1}, ..., \vec{v_n}$ so that all together,

$$\text{span}\{\vec{v_1}, ..., \vec{v_n}, \vec{w_1}, ..., \vec{w_m}\} = \mathbb{R}^4.$$

<span style="width: 32px"></span>Hint: $n + m$ should equal 4. (Why?)

a) Show that $T(\vec{w_1}), ..., T(\vec{w_m})$ are linearly independent. Briefly explain why this has to be the case.

8. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be a function (not necessarily a linear transformation) defined by rotating its inputs $90°$ counterclockwise. For example, $R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $R\left(\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}\right) = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$.

a) Explain with a picture why $R$ is in fact a linear transformation, i.e. $R(\vec{v_1} + \vec{v_2}) = R(\vec{v_1}) + R(\vec{v_2})$ and $R(c\vec{v}) = cR(\vec{v})$.

b) Find the matrix for $R$.

c) Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates its inputs an angle $\theta$ counterclockwise, where $\theta$ is a variable. Find a matrix for $R_\theta$.