Name: ____________________ Homework 5 | Math 342 | Cruz Godar

Due Wednesday of Week 6 at the start of class

Complete the following problems and submit them as a pdf to Canvas. 8 points are awarded for thoroughly attempting every problem, and I'll select three problems to grade on correctness for 4 points each. Enough work should be shown that there is no question about the mathematical process used to obtain your answers.

Section 4

In problems 1–3, use the Gram-Schmidt process to produce an orthonormal basis for the given subspace X. Then find the orthogonal decomposition of given vector $\vec{v}$ as $\vec{v} = \vec{x} + \vec{x}'$ for $\vec{x} \in X$ and $\vec{x'} \in X^{\perp}$.

1. $X = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \right\}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

2. $X = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ -4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 6 \end{bmatrix} \right\}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$.

3. $X = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$.

In problems 4–5, find the closest vector $\vec{x} \in X$ to $\vec{v}$, and compute the distance between the two.

4. $X = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$.

5. $X = \operatorname{span}\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ and $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 4 \end{bmatrix}$ be the matrix whose columns are the basis vectors from problem 1. By using the data from the Gram-Schmidt process, write $A = QR$ for a 3×2 unitary matrix Q and a 2×2 upper triangular matrix R whose eigenvalues are all positive. This is known as a **QR factorization** of A — write a brief sentence explaining why this is always possible for a matrix with linearly independent columns.

7. Find a QR factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & 1 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 6 \end{bmatrix}$.

8. Find a QR factorization of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$.

9. Let X be a subspace of $\mathbb{R}^n$. What is the kernel of the map proj_X? (This should be a brief answer.)

10. True or false: for any subspace X of $\mathbb{R}^n$ and any $\vec{v} \in \mathbb{R}^n$, $||\mathrm{proj}_X(\vec{v})|| \leq ||\vec{v}||$. If true, briefly explain why, and if not, provide a short counterexample.