Practice Final Solutions

Math 252

Exercise 1:

a) What is the order of a differential equation?

The highest order of derivative taken — for example, the order of $y''' + y' - 2x = 2$ is 3.

b) Given a continuous function f, what is an antiderivative of f?

A function F such that $F' = f$. Note: it is **not** enough to say the indefinite integral of f — we only know what integrals are in terms of antiderivatives, and we only know what those are in terms of derivatives.

c) What does it mean for $\int_0^\infty f(x)\, dx$ to converge?

By definition, $\int_0^\infty f(x)\, dx = \lim_{b\to\infty} \int_0^b f(x)\, dx$, so saying the integral converges means the limit does.

Exercise 2: Let R be the region bounded by the x-axis and the curve $y = \cos(x)$ on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Find the centroid of R.

We need to find M_x, M_y, and m. For all of these, we need to handle the region a little more carefully than we might be used to — since the function is below the x-axis on the entire interval, we really have a region bounded above by $y = 0$ and below by $y = \cos(x)$. If you didn't notice this and just integrated $\cos(x)$, you'll wind up with negative mass, which is a good indication something is wrong. Instead, we'll integrate $0 - \cos(x)$ for all three integrals. Specifically,

$$
\begin{aligned}
M_x &= \rho \int_{\pi/2}^{3\pi/2} \frac{1}{2}(0^2 - \cos^2(x))\ dx \\
&= -\rho \frac{1}{2} \int_{\pi/2}^{3\pi/2} \cos^2(x)\ dx \\
&= -\rho \frac{1}{2} \int_{\pi/2}^{3\pi/2} \frac{1}{2} + \frac{1}{2}\cos(2x)\ dx \\
&= -\rho \frac{1}{2} \left[\frac{1}{2}x + \frac{1}{4}\sin(2x) \right] \Bigg|_{\pi/2}^{3\pi/2} \\
&= -\rho \frac{1}{2} \left(\frac{1}{2}\frac{3\pi}{2} + \frac{1}{4}\sin(3\pi) - \frac{1}{2}\frac{\pi}{2} - \frac{1}{4}\sin(\pi) \right) \\
&= -\rho \frac{1}{2} \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) \\
&= -\rho \frac{\pi}{4}.
\end{aligned}
$$

$$
\begin{aligned}
M_y &= \rho \int_{\pi/2}^{3\pi/2} (x \cdot 0 - x\cos(x))\ dx \\
&= -\rho \int_{\pi/2}^{3\pi/2} x\cos(x)\ dx \\
&\qquad u = x \qquad\qquad v = \sin(x) \\
&\qquad du = dx \qquad\quad dv = \cos(x)dx \\
&= -\rho \left[x\sin(x) - \int \sin(x)\ dx \right] \Bigg|_{\pi/2}^{3\pi/2} \\
&= -\rho \left[x\sin(x) + \cos(x) \right] \Big|_{\pi/2}^{3\pi/2} \\
&= -\rho \left(\frac{3\pi}{2}\sin\left(\frac{3\pi}{2}\right) + \cos\left(\frac{3\pi}{2}\right) - \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) \right) \\
&= -\rho \left(-\frac{3\pi}{2} - \frac{\pi}{2} \right) \\
&= 2\pi\rho.
\end{aligned}
$$

$$
\begin{aligned}
m &= \rho \int_{\pi/2}^{3\pi/2} (0 - \cos(x))\ dx \\
&= -\rho \int_{\pi/2}^{3\pi/2} \cos(x)\ dx \\
&= -\rho \left[\sin(x) \right] \Big|_{\pi/2}^{3\pi/2} \\
&= -\rho \left(\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) \right) \\
&= 2\rho.
\end{aligned}
$$

Now we have $\overline{x} = \dfrac{M_y}{m} = \pi$ and $\overline{y} = \dfrac{M_x}{m} = -\dfrac{\pi}{8}$. As with all problems, it's a good idea to take a moment here and make sure these make sense. The region is horizontally symmetric, so the x-coordinate of the center of mass should be in the middle, and the y-coordinate should be just slightly negative, since that's where most of the mass is.

Exercise 3: A thin rope of length 1 meter has linear mass density of $\rho(x) = x^2 e^{5x}$ milligrams per meter, x meters from the left endpoint of the rope. What is the mass of the rope? Include units.

We know that mass is the integral of density over the entire object. Here, the rope runs from $x = 0$ to $x = 1$, so we have

$$\begin{aligned}
m &= \int_0^1 x^2 e^{5x}\, dx \\
&\qquad u = x^2 \qquad\quad v = \frac{1}{5}e^{5x} \\
&\qquad du = 2x dx \qquad dv = e^{5x} dx \\
&= \left[\frac{1}{5}x^2 e^{5x} - \frac{2}{5}\int x e^{5x}\, dx\right]\Bigg|_0^1 \\
&\qquad u = x \qquad\quad v = \frac{1}{5}e^{5x} \\
&\qquad du = dx \qquad dv = e^{5x} dx \\
&= \left[\frac{1}{5}x^2 e^{5x} - \frac{2}{5}\left(\frac{1}{5}x e^{5x} - \frac{1}{5}\int e^{5x}\, dx\right)\right]\Bigg|_0^1 \\
&= \left[\frac{1}{5}x^2 e^{5x} - \frac{2}{5}\left(\frac{1}{5}x e^{5x} - \frac{1}{25}e^{5x}\right)\right]\Bigg|_0^1 \\
&= \left[\frac{1}{5}x^2 e^{5x} - \frac{2}{25}x e^{5x} + \frac{2}{125}e^{5x}\right]\Bigg|_0^1 \\
&= \frac{1}{5}e^5 - \frac{2}{25}e^5 + \frac{2}{125}e^5 - \frac{2}{125} \\
&= \frac{17}{125}e^5 - \frac{2}{125}.
\end{aligned}$$

Note the contrast to the previous problem, where density was constant but the area of each slice varied, and this one, where slice size is constant but density varied. Although we didn't talk about it and won't have questions on it, you could just as easily have both varying at once, and integrate density times area to find mass. This illustrates one of the most important takeaways from the class: integrals add up things that are changing by taking lots of little slices. If something's constant, like density in the previous exercise, it usually doesn't need to be integrated.

Exercise 4: Is $y = e^{3x}$ a solution to the differential equation $\frac{y'}{y} = x + y - e^{3x}$?

This isn't obvious from just looking at it, so let's work it through. We have $y' = 3e^{3x}$, so the equation becomes

$$\frac{3e^{3x}}{e^{3x}} = x + e^{3x} - e^{3x}.$$

This simplifies to $x = 3$, which is not necessarily a true statement, so no — $y = e^{3x}$ is not a solution to this differential equation.

Exercise 5: A mug of tea is initially at $210°F$ and is placed in a room at $72°F$. The temperature in the mug is initially declining at a rate of $4°F$ per minute. Find the temperature of the mug after 15 minutes.

Hint: think carefully about how to turn the initial cooling rate into an equation.

Newton's law of cooling tells us that $T'(t) = k(T(t) - T_s)$, where $T(t)$ is temperature at time t and T_s is the ambient temperature. Here, that means $T_s = 72$. The initial temperature being 210 means $T(0) = 210$, but the other piece of information is a little harder to turn into an equation. The rate at which the temperature is changing at time t is $T'(t)$ by definition, so if the initial rate of decline is 4, then $T'(0) = -4$ (note the negative). Now we can proceed with solving the DE:

$$\begin{aligned}
\frac{dT}{dt} &= k(T - 72) \\
\frac{1}{T-72}\ dT &= k\ dt \\
\int \frac{1}{T-72}\ dT &= \int k\ dt \\
\ln(T - 72) &= kt + C \\
T - 72 &= e^{kt+C} \\
T &= e^{kt+C} + 72.
\end{aligned}$$

Now $T(0) = 210$, so $210 = e^C + 72$. Thus $C = \ln(138)$. To use the information about T', we solve $T'(t) = ke^{kt+C} = ke^{kt+\ln(138)}$ and set $T'(0) = -4$: $-4 = ke^{\ln(138)} = 138k$, so $k = -\frac{2}{69}$. Thus in total, $T(t) = e^{-2t/69+\ln(138)} + 72$, so $T(15) \approx 161.3$. This doesn't seem ridiculous — the boiling point of water is $212°F$, and if you leave some near-boiling water out for 15 minutes, it cools down a fair bit but not so much that it's no longer hot at all.

Exercise 6: Find the general solution to the differential equation $y'(t) = \frac{\ln|t|}{ty^4}$.

This is the same process as the previous question: the equation is separable, since we can get the t and y on opposite side of the equation, so we do exactly that.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{\ln|t|}{ty^4} \\
y^4\ dy &= \frac{\ln|t|}{t}\ dt \\
\int y^4\ dy &= \int \frac{\ln|t|}{t}\ dt \\
u &= \ln|t| \\
du &= \frac{1}{t}\ dt \\
\frac{y^5}{5} &= \int u\ du \\
\frac{y^5}{5} &= \frac{u^2}{2} + C \\
y^5 &= \frac{5}{2}\ln|t| + C \\
y &= \sqrt[5]{\frac{5}{2}\ln|t| + C}
\end{aligned}$$

Exercise 7: Find the arc length of the curve $f(x) = \sqrt{1-x^2}$ on $\left[0, \frac{\sqrt{3}}{2}\right]$.

We have $f'(x) = \frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x)$, so $(f'(x))^2 = \dfrac{x^2}{1-x^2}$. Now the arc length is

$$\begin{aligned}
\int_0^{\sqrt{3}/2} \sqrt{1 + \frac{x^2}{1-x^2}}\ dx &= \int_0^{\sqrt{3}/2} \sqrt{\frac{1-x^2+x^2}{1-x^2}}\ dx \\
&= \int_0^{\sqrt{3}/2} \sqrt{\frac{1}{1-x^2}}\ dx \\
&= \int_0^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\ dx \\
&= \left[\sin^{-1}(x)\right]\Big|_0^{\sqrt{3}/2} \\
&= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}(0) \\
&= \frac{\pi}{3}.
\end{aligned}$$

Exercise 8: Find the average value of $\dfrac{1}{\sqrt{1-x^2}}$ on $\left[0, \frac{1}{2}\right]$.

$$\frac{1}{\frac{1}{2}-0}\int_0^{1/2}\frac{1}{\sqrt{1-x^2}}\,dx = 2\left[\sin^{-1}(x)\right]\Big|_0^{1/2}$$
$$= 2\left(\frac{\pi}{6}-0\right)$$
$$= \frac{\pi}{3}.$$

Exercise 9: Compute $\int_0^\infty xe^{-x^2}\,dx$. Show all your work and use good notation.

This is an improper integral, so we have

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx$$
$$u = -x^2$$
$$du = -2x\,dx$$
$$= \lim_{b\to\infty} -\frac{1}{2}\int_0^b e^u\,du$$
$$= \lim_{b\to\infty} -\frac{1}{2}\left[e^u\right]\Big|_0^b$$
$$= \lim_{b\to\infty} -\frac{1}{2}\left[e^{-x^2}\right]\Big|_0^b$$
$$= \lim_{b\to\infty} -\frac{1}{2}\left(e^{-b^2}-e^0\right)$$
$$= \lim_{b\to\infty} -\frac{1}{2}\left(e^{-b^2}-1\right)$$
$$= \lim_{b\to\infty} -\frac{1}{2}(-1), \text{ since } \lim_{b\to\infty} e^{-b^2}=0$$
$$= \frac{1}{2}.$$

"Good notation" means, among other things, not writing $e^{-\infty}=0$. If you stick with math for a while longer, you'll have functions properly defined on $[-\infty,\infty]$, but we're not there yet, and plugging in infinity to a function has no well-defined meaning to us. Stick to limits.

Exercise 10: (Adapted from a bonus problem) In a branch of math called Number Theory, a famous theorem states that for a fixed large number x, the probability of a random positive integer less than or equal to x being a prime number is approximately $\frac{1}{\ln(x)}$. Therefore, to count the number of primes less than or equal to x, a good approximation is $\int_2^x \frac{1}{\ln(t)}\,dt$. Unfortunately, there isn't an exact solution for this integral, so we just call it $\mathrm{Li}(x)$.

a) Show that

$$\mathrm{Li}(x) = \frac{x}{\ln(x)} - \frac{2}{\ln(2)} + \int_2^x \frac{1}{\ln^2(t)}\ dt.$$

b) The term $\int_2^x \frac{1}{\ln^2(t)}\ dt$ is called the error term because it is small compared to $\mathrm{Li}(x)$. Specifically,

$$\lim_{x\to\infty} \frac{\int_2^x \frac{1}{\ln^2(t)}\ dt}{\mathrm{Li}(x)} = 0.$$

Show this (hint: use L'Hôpital).

This problem looks scary, but it's not that bad. For part a), that result looks a whole lot like an application of integration by parts, so let's try that.

$$\begin{aligned}
\mathrm{Li}(x) &= \int_2^x \frac{1}{\ln(t)}\ dt \\
u &= \frac{1}{\ln(t)} \qquad\qquad v = t\ dt \\
du &= -\frac{1}{\ln^2(t)} \cdot \frac{1}{t} \qquad dv = dt \\
&= \left[\frac{t}{\ln(t)} + \int \frac{1}{\ln^2(t)}\ dt\right]\Bigg|_2^x \\
&= \frac{x}{\ln(x)} - \frac{2}{\ln(2)} + \int_2^x \frac{1}{\ln^2(t)}\ dt.
\end{aligned}$$

For part b), we'll take it from the hint that L'Hôpital applies. Then

$$\begin{aligned}
\lim_{x\to\infty} \frac{\int_2^x \frac{1}{\ln^2(t)}\ dt}{\mathrm{Li}(x)} &= \lim_{x\to\infty} \frac{\int_2^x \frac{1}{\ln^2(t)}\ dt}{\int_2^x \frac{1}{\ln(t)}\ dt} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}\int_2^x \frac{1}{\ln^2(t)}\ dt}{\frac{d}{dx}\int_2^x \frac{1}{\ln(t)}\ dt} \\
&= \lim_{x\to\infty} \frac{\frac{1}{\ln^2(x)}}{\frac{1}{\ln(x)}} \text{ by FTC part I} \\
&= \lim_{x\to\infty} \frac{\ln(x)}{\ln^2(x)} \\
&= \lim_{x\to\infty} \frac{1}{\ln(x)} \\
&= 0.
\end{aligned}$$

If you got lost in the sauce there, this is a really cool result! For example, there are 78498 primes less than one million, but without knowing that, we can estimate it by taking our formula from part a) and dropping the last term because part b) says it's very small for large x: $\dfrac{1000000}{\ln(1000000)} - \dfrac{2}{\ln(2)} \approx 72380$. Obviously this isn't the closest, but finding primes is a very difficult problem, and having any approximation at all is impressive.