Analysis Notes

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Math 412, 413, and 414, taught by Emily Hamilton

I — A Construction of $\mathbb{R}$

Definition: A **Dedekind cut** is a set $A \subseteq \mathbb{Q}$ such that

a) $A \neq \varnothing$ and $A \neq \mathbb{Q}$.

b) If $r \in A$, then $q \in A$ for all $q \in \mathbb{Q}$ with $q < r$.

c) A does not have a maximum element — that is, if $r \in A$, then $r < s$ for some $s \in A$.

Definition: The **real numbers**, $\mathbb{R}$, are the set of all Dedekind cuts.

Definition: Let $A, B \in \mathbb{R}$. A is **less than** B, written $A \leq B$, if $A \subseteq B$.

Proposition: $\leq$ is a total order on $\mathbb{R}$.

Proof: Clearly, $\leq$ is reflexive, antisymmetric, and transitive, since $\subseteq$ is. Thus $\leq$ is a partial order on $\mathbb{R}$. To show that it is a total order, suppose $A \nleq B$. Then $A \nsubseteq B$, so there is an $a \in A$ with $a \notin B$. Let $b \in B$. Since $a \notin B$, $b \in B$, and B is a cut, $a > b$ (where $\leq$ here is the standard order on $\mathbb{Q}$), and since A is a cut, $b \in A$. Thus $B \subseteq A$, so $B \leq A$.

Definition: Let $A, B \in \mathbb{R}$. The **sum** of A and B is $A + B\ \{a + b \mid a \in A, b \in B\}$.

Theorem: $\mathbb{R}$ is closed under addition.

Proof: Let $A, B \in \mathbb{R}$. To show $A + B \in \mathbb{R}$, we need to verify each of the three Dedekind cut axioms.

(1) Since $A \neq \varnothing$ and $B \neq \varnothing$, $A + B \neq \varnothing$. Since $A \neq \mathbb{Q}$ and $B \neq \mathbb{Q}$, there is an $s \in \mathbb{Q} \setminus A$ and a $t \in \mathbb{Q} \setminus B$, and since A and B are cuts, $a < s$ and $b < t$ for all $a \in A$ and $b \in B$. Thus $a + b < s + t$ for all $a \in A$ and $b \in B$, or, equivalently, for all $a + b \in A + B$. Thus $s + t \notin A + B$, so $A + B \neq \mathbb{Q}$.

(2) Let $a + b \in A + B$ and let $s \in \mathbb{Q}$ such that $s < a + b$. Then $s - b < a$, so $s - b \in A$, since A is a cut. Thus $(s - b) + b = s \in A + B$.

(3) Let $a + b \in A + B$. Since A and B are cuts, there is an $s \in A$ and a $t \in B$ such that $a < s$ and $b < t$. Then $s + t \in A + B$ and $a + b < s + t$.

Proposition: Let $A, B, C \in \mathbb{R}$. Then $A + B = B + A$ and $(A + B) + C = A + (B + C)$.

Definition: The real numbers **zero** and **one** are defined as $\mathbf{0} = \{q \in \mathbb{Q} \mid q < 0\}$ and $\mathbf{1} = \{q \in \mathbb{Q} \mid q < 1\}$.

Proposition: For all $A \in \mathbb{R}$, $A + \mathbf{0} = A$.

Proof: ($\subseteq$) Let $a + x \in A + \mathbf{0}$. Since $x < 0$, $a + x < a$, and since A is a cut, $a + x \in A$. Thus $A + \mathbf{0} \subseteq A$.

($\supseteq$) Let $a \in A$. Since A is a cut, there is an $s \in A$ such that $s > a$. Then $a - s < 0$, so $a - a \in \mathbf{0}$. Thus $a = s + (a - s) \in A + \mathbf{0}$, so $A \subset A + \mathbf{0}$.

Definition: Let $A \in \mathbb{R}$. The **additive inverse** of A is $-A = \{r \in \mathbb{Q} \mid r < -t \text{ for some } t \notin A\}$.

Proposition: Let $A \in \mathbb{R}$. Then $-A \in \mathbb{R}$.

Proposition: Let $A \in \mathbb{R}$. Then $A + (-A) = \mathbf{0}$.

Proof: ($\subseteq$) Let $a + n \in A + (-A)$. Since $n \in -A$, there is a $t \notin A$ such that $n < -t$, and since $a \in A$ and $t \notin A$, $a < t < -n$, so $a + n < 0$. Thus $a + n \in \mathbf{0}$, so $A + (-A) \subseteq \mathbf{0}$.

($\supseteq$) Let $x \in \mathbf{0}$, let $\varepsilon = \frac{|x|}{2} = -\frac{x}{2}$, and let $t \in \mathbb{Q}$ such that $t \notin A$ but $t - \varepsilon \in A$. Since $t \notin A$, $-(t + \varepsilon) \in -A$, since $t < -(-(t + \varepsilon))$ and therefore $-(t + \varepsilon) < -t$. Then $x = -2\varepsilon = -(t + \varepsilon) + (t - \varepsilon) \in A + (-A)$, so $\mathbf{0} \subseteq A + (-A)$.

Definition: Let $A, B \in \mathbb{R}$. If $A \geq \mathbf{0}$ and $B \geq \mathbf{0}$, then the **product** of A and B is

$$AB = \{ab \mid a \in A, b \in B, a \geq 0, b \geq 0\} \cup \mathbf{0}.$$

If $A \geq \mathbf{0}$ and $B < \mathbf{0}$, then $AB = -(A(-B))$, if $A < \mathbf{0}$ and $B \geq \mathbf{0}$, then $AB = -((-A)B)$, and if $A < \mathbf{0}$ and $B < \mathbf{0}$, then $AB = (-A)(-B)$.

Theorem: Let $A, B, C \in \mathbb{R}$. Then $AB \in \mathbb{R}$, $AB = BA$, $(AB)C = A(BC)$, $\mathbf{1}A = A$, and if $A \neq \mathbf{0}$, then there is an $A^{-1} \in \mathbb{R}$ with $AA^{-1} = \mathbf{1}$.

Definition: A set $U \subseteq \mathbb{R}$ is **bounded above** if there is a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in U$. We call B an **upper bound** for U, and define **bounded below** and **lower bound** similarly.

Definition: Let $U \in \mathbb{R}$ such that $U \neq \varnothing$ and U is bounded above. We define $S(U) = \bigcup_{A \in U} A$.

Theorem: Let $U \subset \mathbb{R}$ be nonempty and bounded above. Then $S(U)$ is a cut.

Proof: (1) Since $U \neq \varnothing$ and $U \subseteq S(U)$, $S(U) \neq \varnothing$. Since U is bounded above, there is a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in U$. Then $A \subseteq B$ for all $A \in U$, so $S(U) = \bigcup A \subseteq B$. Since $B \neq \mathbb{Q}$, $S(U) \neq \mathbb{Q}$.

(2) Let $a \in S(U)$ and $q < a$. Then $a \in A$ for some $A \in U$, and since A is a cut and $q < a$, $q \in A \subseteq S(U)$.

(3) Let $a \in S(U)$. Then $a \in A$ for some $A \in U$, and since A is a cut, there is a $q \in A \subseteq S(U)$ with $a < q$.

Proposition: Let $U \subseteq \mathbb{R}$ be nonempty and bounded above. Then $S(U)$ is an upper bound for U.

Proof: For all $A \in U$, $A \subseteq \bigcup A = S(U)$, so $A \leq S(U)$.

Definition: A set $U \subseteq \mathbb{R}$ has a **supremum**, or least upper bound, if there is a $B \in \mathbb{R}$ such that B is an upper bound for U and $B \leq C$ for any upper bound C for U. We define the **infimum**, or greatest lower bound, similarly, and write $\sup U$ and $\inf U$ for the supremum and infimum.

Proposition: Let $U \subseteq \mathbb{R}$ be nonempty and bounded above. Then $S(U) = \sup U$.

Proof: Let C be an upper bound for U. Then $A \leq C$ for all $A \in U$, so $A \subseteq C$ for all $A \in U$. Then $S = \bigcup A \subset C$, so $S \leq C$.

Theorem: (The Completeness of the Reals) Every nonempty, bounded above subset of $\mathbb{R}$ has a supremum in $\mathbb{R}$.

II — The Reals

Proposition: Let $A \subseteq \mathbb{R}$. If $\sup A \in A$, then $\sup A = \max A$.

Proposition: If $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$, then $\sup A \leq \sup B$.

Proof: Since $A \subseteq B$, $a \in B$ for all $a \in A$, and so since $\sup B \geq b$ for all $b \in B$, $\sup B \geq a$ for all $a \in A$. Then $\sup B$ is an upper bound for A, so $\sup A \leq \sup B$.

Theorem: Let s be an upper bound for $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for all $\varepsilon > 0$, there is an $a \in A$ with $s - \varepsilon < a$.

Proof: ($\Rightarrow$) Assume $s = \sup A$ and let $\varepsilon > 0$. Since $s - \varepsilon < s = \sup A$, $s - \varepsilon$ cannot be an upper bound for A. Thus there must be an $a \in A$ with $a > s - \varepsilon$.

Assume s is an upper bound for A and that for every $\varepsilon > 0$, there is an $a \in A$ such that $a > s - \varepsilon$. Let b be an upper bound for A and suppose $b < s$. Let $\varepsilon = \frac{s-b}{2}$. Since $a < b$ for all $a \in A$, there is no $a \in A$ such that $a > s - \varepsilon$, since $s - \varepsilon$ is the midpoint of s and b, and is therefore greater than b. $\lightning$

Theorem: (The Nested Interval Theorem) For each $n \in \mathbb{N}$, let $I_n = [a_n, b_n]$ be an interval such that $I_n \subseteq I_{n-1}$. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof: Let $A = \{a_n \mid n \in \mathbb{N}\}$. Since A is nonempty and bounded above (by b_1, for instance), A has a least upper bound. In fact, each b_i is an upper bound for A, since otherwise the intervals would not be nested.

Let $s = \sup A$ and let $n \in \mathbb{N}$. Since s is an upper bound for A, $s \geq a_n$, and since b_n is an upper bound for A, $s \leq b_n$. Thus $s \in I_n$ for all $n \in \mathbb{N}$, so $s \in \bigcap I_n$.

Theorem: (The Well-Ordering Principle) Every nonempty subset of $\mathbb{N}$ has a minimum element.

Proposition: (The Archimedean Property) Let $x \in \mathbb{R}$. Then there is a $y \in \mathbb{N}$ with $y > x$.

Corollary: Let $x \in \mathbb{R}^+$. Then there is a $y \in \mathbb{N}$ with $\frac{1}{y} < x$.

Theorem: (The Density of $\mathbb{Q}$ in $\mathbb{R}$) Let $a, b \in \mathbb{R}$ with $a < b$. Then there is a $q \in \mathbb{Q}$ with $a < q < b$.

Proof: First, suppose $a \geq 0$. By the Archimedean property, let $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. Let m be the smallest natural greater than na. Then $m - 1 \leq na < m$, so $m \leq na + 1 < m + 1$. Since $na < m$, $a < \frac{m}{n}$, and since $m \leq na + 1$ and $\frac{1}{n} < b - a$, $m < n\left(b - \frac{1}{n}\right) + 1 = nb$. Thus $\frac{m}{n} < b$, and so $a < \frac{m}{n} < b$.

If $a < 0$ and $b > 0$, then $a < \frac{0}{1} < b$, and if $a < 0$ and $b \leq 0$, then since $-b < -a$ (and $-b, -a > 0$), there is a $q \in \mathbb{Q}$ with $-b < q < -a$, so $a < -q < b$.

Theorem: There is an $\alpha \in \mathbb{R}$ with $\alpha^2 = 2$.

Proof: Let $T = \{t \in \mathbb{R} \mid t^2 < 2\}$, which is is nonempty and bounded above, and let $\alpha = \sup T$. Suppose $\alpha < 2$. By the Archimedean principle, there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{2-\alpha^2}{2\alpha+1}$, or, equivalently, $\frac{2\alpha+1}{n} < 2 - \alpha^2$. Then

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n} \\ &< \alpha^2 + (2 - \alpha^2) \\ &= 2,\end{aligned}$$

so $\alpha + \frac{1}{n} \in T$, but $\alpha + \frac{1}{n} > \alpha = \sup T$. ↯ Similarly, $a > 2$ gives a contradiction.

III — Sequences and Series

Definition: Let $S \subseteq \mathbb{R}$. A **sequence in S** is a function $f : \mathbb{N} \longrightarrow S$. We write x_n instead of $f(n)$ and (x_n) to refer to the whole sequence.

Definition: Let $(a_n) \subseteq \mathbb{R}$. (a_n) **converges** to $a \in \mathbb{R}$ if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - a| < \varepsilon$. A sequence **diverges** if it does not converge.

Example: Show that $\left(\frac{1}{n}\right) \longrightarrow 0$.

Let $\varepsilon > 0$. We want $\left|\frac{1}{n} - 0\right| < \varepsilon$, so $n \geq \frac{1}{\varepsilon}$. Therefore, let N be the first natural number greater than ε. Then if $n \geq N$, $\left|\frac{1}{n} - 0\right| < \varepsilon$.

Example: Show that $\left(\frac{\sqrt{n^2+1}}{n!}\right) \longrightarrow 0$.

Let $\varepsilon > 0$. Since $\frac{\sqrt{n^2+1}}{n!} \leq \frac{\sqrt{n^2+n^2}}{n!} = \sqrt{2}\left(\frac{n}{n!}\right) = \sqrt{2}\left(\frac{1}{(n-1)!}\right) \leq \frac{2}{n-1}$, let $N > \frac{2+\varepsilon}{\varepsilon}$. Then if $n \geq N$, $\frac{2}{n-1} < \varepsilon$, so $\frac{\sqrt{n^2+1}}{n!} < \frac{2}{n-1} < \varepsilon$.

Definition: A sequence $(a_n) \subseteq \mathbb{R}$ is **bounded** if there is an $M \in \mathbb{R}^+$ such that $|a_n| < M$ for all $n \in \mathbb{N}$.

Proposition: Every convergent sequence in $\mathbb{R}$ is bounded.

Proof: Let $(a_n) \subseteq \mathbb{R}$ and suppose $(a_n) \longrightarrow a$. With $\varepsilon = 1$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - a| < 1$. Thus $|a_n| < |a| + 1$. Let $M = \max\{|a_1|, |a_2|, ..., |a_{N-2}|, |a_{N-1}|, |a| + 1\}$. Then $|a_n| < M$ for all $n \in \mathbb{N}$.

Theorem: (The Algebraic Limit Theorem) Let $(a_n), (b_n) \subseteq \mathbb{R}$ such that $(a_n) \longrightarrow a$ and $(b_n) \longrightarrow b$. Then

a) For all $c \in \mathbb{R}$, $(ca_n) \longrightarrow ca$.

b) $(a_n + b_n) \longrightarrow a + b$.

c) $(a_n b_n) \longrightarrow ab$.

d) If $b \neq 0$, then $\left(\frac{a_n}{b_n}\right) \longrightarrow \frac{a}{b}$.

Proof: We will prove parts 2 and 3. Part 1 is a simple exercise, and part 4 proceeds similarly to part 3.

b) Let $\varepsilon > 0$. Since $(a_n) \longrightarrow a$, there is an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|a_n - a| < \frac{\varepsilon}{2}$, and since $(b_n) \longrightarrow a$, there is an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|b_n - b| < \frac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$, $|(a_n + b_n) - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

c) Let $\varepsilon > 0$ and suppose that $a \neq 0$. Since (b_n) converges, it is bounded, so there is an $M > 0$ such that $|b_n| < M$ for all $n \in \mathbb{N}$. Since $(a_n) \longrightarrow a$, there is an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|a_n - a| < \frac{\varepsilon}{2M}$. Since $(b_n) \longrightarrow b$, there is an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|b_n - b| < \frac{\varepsilon}{2|a|}$. Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b| \\ &< M\left(\frac{\varepsilon}{2M}\right) + |a|\left(\frac{\varepsilon}{2|a|}\right) \\ &= \varepsilon. \end{aligned}$$

Thus $(a_n b_n) \longrightarrow ab$. If $a = 0$, the proof is very similar.

Theorem: (The Order Limit Theorem) Let $(a_n), (b_n) \subseteq \mathbb{R}$ with $(a_n) \longrightarrow a$ and $(b_n) \longrightarrow b$. Then

a) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

b) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

c) If there is a $c \in \mathbb{R}$ such that $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$, and similarly, if $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$.

Proof:

a) Suppose $a_n \geq 0$ for all $n \in \mathbb{N}$, but $a < 0$. Let $\varepsilon = \frac{|a|}{2}$. Then there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - a| < \varepsilon = \frac{|a|}{2}$. Then $a_N \in \left(\frac{a}{2}, \frac{3a}{2}\right)$, so $a_N < 0$. $\lightning$

b) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $b_n - a_n \geq 0$, and $(b_n - a_n) \longrightarrow b - a$. By part 1, $b - a \geq 0$, so $a \leq b$.

c) Let $c_n = c$. Then we are done by part 2.

Definition: A sequence $(a_n) \subseteq \mathbb{R}$ is **monotone increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, and **monotone decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition: A sequence $(a_n) \subset \mathbb{R}$ is **monotone** if it is either increasing or decreasing.

Theorem: (The Monotone Convergence Theorem) Let $(a_n) \subseteq \mathbb{R}$. If (a_n) is monotone increasing and bounded above, then it converges.

Proof: Let $\varepsilon > 0$ and let $A = \{a_n \mid n \in \mathbb{N}\}$. Since A is nonempty and bounded above, $\sup A$ exists. Let $s = \sup A$. Then there is an $a_N \in A$ such that $s - \varepsilon < a_N$, so $s - \varepsilon < a_n$ for all $n \geq N$, since (a_n) is increasing. Thus $s - \varepsilon < a_n \leq s < s + \varepsilon$, so $|a_n - s| < \varepsilon$. Thus $(a_n) \longrightarrow s$.

Definition: Let (a_n) be a sequence. A **subsequence** of (a_n) is a sequence (a_{n_k}), where $n_1 < n_2 < \cdots$ is a strictly increasing sequence of natural numbers.

Proposition: Subsequences of a convergent sequence converge to the limit.

Theorem: (The Bolzano-Weierstrass Theorem) Every bounded sequence in $\mathbb{R}$ contains a convergent subsequence.

Proof: Let $(a_n) \subseteq \mathbb{R}$ be bounded (above and below). Define a *peak index* of (a_n) to be a value of $m \in \mathbb{N}$ such that $a_m \geq a_n$ for all $n \geq m$. Either there are finitely many peak indices or infinitely many.

If there are only finitely many peak indices, then there is an $N \in \mathbb{N}$ such that there are no peak indices greater than N. Let $n_1 = N + 1$. Since n_1 is not a peak index, there is an $n_2 \geq n_1$ such that $a_{n_2} \geq a_{n_1}$. Repeat this inductively to create a sequence (a_{n_k}) that is monotone increasing. Since it is bounded above, it must converge.

If there are infinitely many peak indices, then let n_k be the kth peak index. Then (a_{n_k}) is monotone decreasing, and since it is bounded below, it converges.

Definition: A sequence $(a_n) \subseteq \mathbb{R}$ is **Cauchy** if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|a_n - a_m| < \varepsilon$.

Proposition: Every convergent sequence in $\mathbb{R}$ is Cauchy.

Proof: Suppose $(a_n) \subseteq \mathbb{R}$ with $(a_n) \longrightarrow a$ and let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - a| \leq \frac{\varepsilon}{2}$. Now if $m, n \geq N$, $|a_m - a_n| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < \varepsilon$. Thus (a_m) is Cauchy.

Proposition: Every Cauchy sequence in $\mathbb{R}$ is bounded.

Proof: Let $(a_n) \subseteq \mathbb{R}$ be Cauchy. With $\varepsilon = 1$, there is an $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|a_n - a_m| < 1$. Thus $|a_n| \leq |a_N| + 1$ for all $n \geq N$, so $\max\{|a_1|, |a_2|, ..., |a_{N-1}|, |a_N| + 1\}$ is a bound for (a_n).

Theorem: Every Cauchy sequence in $\mathbb{R}$ converges.

Proof: Let $(a_n) \subseteq \mathbb{R}$ be Cauchy. Then (a_n) is bounded, so it contains a convergent subsequence $(a_{n_k}) \longrightarrow a$. We claim that $(a_n) \longrightarrow a$.

Let $\varepsilon > 0$. Since (a_n) is Cauchy, there is an $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|a_n - a_m| < \frac{\varepsilon}{2}$.