Practice Midterm 1

Math 252

Exercise 1: Let $f(x)$ be a continuous function on $[a, b]$. Write both parts of the Fundamental Theorem of Calculus.

Part I: $\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$. Part II: $\int_a^b f(x)\ dx = F(b) - F(a)$.

Exercise 2: Find $\displaystyle\int_1^4 (s^3 + s^2)\ ds$.

We have

$$\begin{aligned} \int_1^4 (s^3 + s^2)\ ds &= \left[\frac{s^4}{4} + \frac{s^3}{3}\right]\Bigg|_1^4 \\ &= \frac{4^4}{4} + \frac{4^3}{3} - \frac{1^4}{4} - \frac{1^3}{3} \\ &= 64 + \frac{64}{3} - \frac{1}{4} - \frac{1}{3}. \end{aligned}$$

Exercise 3: Evaluate $\int_0^5 x^3\ dx$ by taking a limit of Riemann sums.

First, take a partition of $[0, 5]$ into n subintervals. Then the width of each subinterval is $\frac{5}{n}$, and the right endpoint of the ith subinterval is $\frac{5}{n}i$. The Right Riemann sum is therefore

$$\begin{aligned}\sum_{i=1}^{n} f(x_i^*)\Delta x &= \sum_{i=1}^{n}\left(\frac{5}{n}i\right)^3\left(\frac{5}{n}\right)\\ &= \sum_{i=1}^{n}\left(\frac{125}{n^3}i^3\right)\left(\frac{5}{n}\right)\\ &= \sum_{i=1}^{n}\frac{625}{n^4}i^3\\ &= \frac{625}{n^4}\sum_{i=1}^{n} i^3\\ &= \left(\frac{625}{n^4}\right)\left(\frac{n^2(n+1)^2}{4}\right)\\ &= \frac{625(n+1)^2}{4n^2}\\ &= \frac{625(n^2+2n+1)}{4n^2}.\end{aligned}$$

Now taking the limit as $n \to \infty$, the ratio of the leading terms is $\frac{625n^2}{4n^2} = \frac{625}{4}$, which is our final answer.

Exercise 4: Evaluate $\int (x+1)^{10}\,dx$.

This could be foiled out, but that is a massive amount of work. Instead, let's try u-sub. Let $u = x+1$, since that's the inside part. Then $du = dx$, and so we get

$$\int u^{10}\,du = \frac{u^{11}}{11} + C = \frac{(x+1)^{11}}{11} + C.$$

Exercise 5: If an object's total distance traveled over 5 seconds is 12 meters, can its net displacement be 2 meters?

Yes — if it moves 7 meters to the right and then 5 to the left, it travels 12 meters total but has net displacement 2 meters.

Exercise 6: What is the average value of $\sin(x)$ on $\left[-\pi, \frac{\pi}{2}\right]$?

This is

$$\begin{aligned}\frac{1}{\frac{\pi}{2}-(-\pi)}\int_{-\pi}^{\pi/2}\sin(x)\,dx &= \frac{1}{\frac{\pi}{2}-(-\pi)}\left[-\cos(x)\right]\Big|_{-\pi}^{\pi/2}\\ &= \frac{1}{\frac{3\pi}{2}}\left(-\cos\left(\frac{\pi}{2}\right)+\cos(-\pi)\right)\\ &= \frac{2}{3\pi}(0-1)\\ &= -\frac{2}{3\pi}.\end{aligned}$$