

# Analysis Notes

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## I — A Construction of $\mathbb{R}$

**Definition 1.1:** A **Dedekind cut** is a set  $A \subseteq \mathbb{Q}$  such that

1.  $A \neq \emptyset$  and  $A \neq \mathbb{Q}$ .
2. If  $r \in A$ , then  $q \in A$  for all  $q \in \mathbb{Q}$  with  $q < r$ .
3.  $A$  does not have a maximum element — that is, if  $r \in A$ , then  $r < s$  for some  $s \in A$ .

**Definition 1.2:** The **real numbers**,  $\mathbb{R}$ , are the set of all Dedekind cuts.

**Definition 1.3:** Let  $A, B \in \mathbb{R}$ .  $A$  is **less than**  $B$ , written  $A < B$ , if  $A \subsetneq B$ .

**Proposition 1.4:**  $\leq$  is a total order on  $\mathbb{R}$ .

**Proof:** Clearly,  $\leq$  is reflexive, antisymmetric, and transitive, since  $\subseteq$  is. Thus  $\leq$  is a partial order on  $\mathbb{R}$ . To show that it is a total order, suppose  $A \not\leq B$ . Then  $A \not\subseteq B$ , so there is an  $a \in A$  with  $a \notin B$ . Let  $b \in B$ . Since  $a \notin B$ ,  $b \in B$ , and  $B$  is a cut,  $a > b$  (where  $\leq$  here is the standard order on  $\mathbb{Q}$ ), and since  $A$  is a cut,  $b \in A$ . Thus  $B \subseteq A$ , so  $B \leq A$ .

**Definition 1.5:** Let  $A, B \in \mathbb{R}$ . The **sum** of  $A$  and  $B$  is  $A + B = \{a + b \mid a \in A, b \in B\}$ .

**Theorem 1.6:**  $\mathbb{R}$  is closed under addition.

**Proof:** Let  $A, B \in \mathbb{R}$ . To show  $A + B \in \mathbb{R}$ , we need to verify each of the three Dedekind cut axioms.

(1) Since  $A \neq \emptyset$  and  $B \neq \emptyset$ ,  $A + B \neq \emptyset$ . Since  $A \neq \mathbb{Q}$  and  $B \neq \mathbb{Q}$ , there is an  $s \in \mathbb{Q} \setminus A$  and a  $t \in \mathbb{Q} \setminus B$ , and since  $A$  and  $B$  are cuts,  $a < s$  and  $b < t$  for all  $a \in A$  and  $b \in B$ . Thus  $a + b < s + t$  for all  $a \in A$  and  $b \in B$ , or, equivalently, for all  $a + b \in A + B$ . Thus  $s + t \notin A + B$ , so  $A + B \neq \mathbb{Q}$ .

(2) Let  $a + b \in A + B$  and let  $s \in \mathbb{Q}$  such that  $s < a + b$ . Then  $s - b < a$ , so  $s - b \in A$ , since  $A$  is a cut. Thus  $(s - b) + b = s \in A + B$ .

(3) Let  $a + b \in A + B$ . Since  $A$  and  $B$  are cuts, there is an  $s \in A$  and a  $t \in B$  such that  $a < s$  and  $b < t$ . Then  $s + t \in A + B$  and  $a + b < s + t$ .

**Proposition 1.7:** Let  $A, B, C \in \mathbb{R}$ . Then  $A + B = B + A$  and  $(A + B) + C = A + (B + C)$ .

**Definition 1.8:** The real numbers **zero** and **one** are defined as  $\mathbf{0} = \{q \in \mathbb{Q} \mid q < 0\}$  and  $\mathbf{1} = \{q \in \mathbb{Q} \mid q < 1\}$ .

**Proposition 1.9:** For all  $A \in \mathbb{R}$ ,  $A + \mathbf{0} = A$ .

**Proof:** ( $\subseteq$ ) Let  $a + x \in A + \mathbf{0}$ . Since  $x < 0$ ,  $a + x < a$ , and since  $A$  is a cut,  $a + x \in A$ . Thus  $A + \mathbf{0} \subseteq A$ .

( $\supseteq$ ) Let  $a \in A$ . Since  $A$  is a cut, there is an  $s \in A$  such that  $s > a$ . Then  $a - s < 0$ , so  $a - s \in \mathbf{0}$ . Thus  $a = s + (a - s) \in A + \mathbf{0}$ , so  $A \subset A + \mathbf{0}$ .

**Definition 1.10:** Let  $A \in \mathbb{R}$ . The **additive inverse** of  $A$  is  $-A = \{r \in \mathbb{Q} \mid r < -t \text{ for some } t \notin A\}$ .

**Proposition 1.11:** Let  $A \in \mathbb{R}$ . Then  $-A \in \mathbb{R}$ .

**Proposition 1.12:** Let  $A \in \mathbb{R}$ . Then  $A + (-A) = \mathbf{0}$ .

**Proof:** ( $\subseteq$ ) Let  $a + n \in A + (-A)$ . Since  $n \in -A$ , there is a  $t \notin A$  such that  $n < -t$ , and since  $a \in A$  and  $t \notin A$ ,  $a < t < -n$ , so  $a + n < 0$ . Thus  $a + n \in \mathbf{0}$ , so  $A + (-A) \subseteq \mathbf{0}$ .

( $\supseteq$ ) Let  $x \in \mathbf{0}$ , let  $\varepsilon = \frac{|x|}{2} = -\frac{x}{2}$ , and let  $t \in \mathbb{Q}$  such that  $t \notin A$  but  $t - \varepsilon \in A$ . Since  $t \notin A$ ,  $-(t + \varepsilon) \in -A$ , since  $t < -(t + \varepsilon)$  and therefore  $-(t + \varepsilon) < -t$ . Then  $x = -2\varepsilon = -(t + \varepsilon) + (t - \varepsilon) \in A + (-A)$ , so  $\mathbf{0} \subseteq A + (-A)$ .

**Definition 1.13:** Let  $A, B \in \mathbb{R}$ . If  $A \geq \mathbf{0}$  and  $B \geq \mathbf{0}$ , then the **product** of  $A$  and  $B$  is

$$AB = \{ab \mid a \in A, b \in B, a \geq 0, b \geq 0\} \cup \mathbf{0}.$$

If  $A \geq \mathbf{0}$  and  $B < \mathbf{0}$ , then  $AB = -(A(-B))$ , if  $A < \mathbf{0}$  and  $B \geq \mathbf{0}$ , then  $AB = -((-A)B)$ , and if  $A < \mathbf{0}$  and  $B < \mathbf{0}$ , then  $AB = (-A)(-B)$ .

**Theorem 1.14:** Let  $A, B, C \in \mathbb{R}$ . Then  $AB \in \mathbb{R}$ ,  $AB = BA$ ,  $(AB)C = A(BC)$ ,  $\mathbf{1}A = A$ , and if  $A \neq \mathbf{0}$ , then there is an  $A^{-1} \in \mathbb{R}$  with  $AA^{-1} = \mathbf{1}$ .

**Definition 1.15:** A set  $U \subseteq \mathbb{R}$  is **bounded above** if there is a  $B \in \mathbb{R}$  such that  $A \leq B$  for all  $A \in U$ . We call  $B$  an **upper bound** for  $U$ , and define **bounded below** and **lower bound** similarly.

**Definition 1.16:** Let  $U \in \mathbb{R}$  such that  $U \neq \emptyset$  and  $U$  is bounded above. We define  $S(U) = \bigcup_{A \in U} A$ .

**Theorem 1.17:** Let  $U \subset \mathbb{R}$  be nonempty and bounded above. Then  $S(U)$  is a cut.

**Proof:** (1) Since  $U \neq \emptyset$  and  $U \subseteq S(U)$ ,  $S(U) \neq \emptyset$ . Since  $U$  is bounded above, there is a  $B \in \mathbb{R}$  such that  $A \leq B$  for all  $A \in U$ . Then  $A \subseteq B$  for all  $A \in U$ , so  $S(U) = \bigcup A \subseteq B$ . Since  $B \neq \mathbb{Q}$ ,  $S(U) \neq \mathbb{Q}$ .

(2) Let  $a \in S(U)$  and  $q < a$ . Then  $a \in A$  for some  $A \in U$ , and since  $A$  is a cut and  $q < a$ ,  $q \in A \subseteq S(U)$ .

(3) Let  $a \in S(U)$ . Then  $a \in A$  for some  $A \in U$ , and since  $A$  is a cut, there is a  $q \in A \subseteq S(U)$  with  $a < q$ .

**Proposition 1.18:** Let  $U \subseteq \mathbb{R}$  be nonempty and bounded above. Then  $S(U)$  is an upper bound for  $U$ .

**Proof:** For all  $A \in U$ ,  $A \subseteq \bigcup A = S(U)$ , so  $A \leq S(U)$ .

**Definition 1.19:** A set  $U \subseteq \mathbb{R}$  has a **supremum**, or least upper bound, if there is a  $B \in \mathbb{R}$  such that  $B$  is an upper bound for  $U$  and  $B \leq C$  for any upper bound  $C$  for  $U$ . We define the **infimum**, or greatest lower bound, similarly, and write  $\sup U$  and  $\inf U$  for the supremum and infimum.

**Proposition 1.20:** Let  $U \subseteq \mathbb{R}$  be nonempty and bounded above. Then  $S(U) = \sup U$ .

**Proof:** Let  $C$  be an upper bound for  $U$ . Then  $A \leq C$  for all  $A \in U$ , so  $A \subseteq C$  for all  $A \in U$ . Then  $S = \bigcup A \subseteq C$ , so  $S \leq C$ .

**Theorem 1.21: (The Completeness of the Reals)** Every nonempty, bounded above subset of  $\mathbb{R}$  has a supremum in  $\mathbb{R}$ .

## II — The Reals

**Proposition 2.1:** Let  $A \subseteq \mathbb{R}$ . If  $\sup A \in A$ , then  $\sup A = \max A$ .

**Proposition 2.2:** If  $A, B \subseteq \mathbb{R}$  such that  $A \subseteq B$ , then  $\sup A \leq \sup B$ .

**Proof:** Since  $A \subseteq B$ ,  $a \in B$  for all  $a \in A$ , and so since  $\sup B \geq b$  for all  $b \in B$ ,  $\sup B \geq a$  for all  $a \in A$ . Then  $\sup B$  is an upper bound for  $A$ , so  $\sup A \leq \sup B$ .

**Theorem 2.3:** Let  $s$  be an upper bound for  $A \subseteq \mathbb{R}$ . Then  $s = \sup A$  if and only if for all  $\varepsilon > 0$ , there is an  $a \in A$  with  $s - \varepsilon < a$ .

**Proof:** ( $\Rightarrow$ ) Assume  $s = \sup A$  and let  $\varepsilon > 0$ . Since  $s - \varepsilon < s = \sup A$ ,  $s - \varepsilon$  cannot be an upper bound for  $A$ . Thus there must be an  $a \in A$  with  $a > s - \varepsilon$ .

Assume  $s$  is an upper bound for  $A$  and that for every  $\varepsilon > 0$ , there is an  $a \in A$  such that  $a > s - \varepsilon$ . Let  $b$  be an upper bound for  $A$  and suppose  $b < s$ . Let  $\varepsilon = \frac{s-b}{2}$ . Since  $a < b$  for all  $a \in A$ , there is no  $a \in A$  such that  $a > s - \varepsilon$ , since  $s - \varepsilon$  is the midpoint of  $s$  and  $b$ , and is therefore greater than  $b$ .  $\nexists$

**Theorem 2.4: (The Nested Interval Theorem)** For each  $n \in \mathbb{N}$ , let  $I_n = [a_n, b_n]$  be an interval such that  $I_n \subseteq I_{n-1}$ . Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

**Proof:** Let  $A = \{a_n \mid n \in \mathbb{N}\}$ . Since  $A$  is nonempty and bounded above (by  $b_1$ , for instance),  $A$  has a least upper bound. In fact, each  $b_i$  is an upper bound for  $A$ , since otherwise the intervals would not be nested.

Let  $s = \sup A$  and let  $n \in \mathbb{N}$ . Since  $s$  is an upper bound for  $A$ ,  $s \geq a_n$ , and since  $b_n$  is an upper bound for  $A$ ,  $s \leq b_n$ . Thus  $s \in I_n$  for all  $n \in \mathbb{N}$ , so  $s \in \bigcap I_n$ .

**Theorem 2.5: (The Well-Ordering Principle)** Every nonempty subset of  $\mathbb{N}$  has a minimum element.

**Proposition 2.6: (The Archimedean Property)** Let  $x \in \mathbb{R}$ . Then there is a  $y \in \mathbb{N}$  with  $y > x$ .

**Corollary 2.6.1:** Let  $x \in \mathbb{R}^+$ . Then there is a  $y \in \mathbb{N}$  with  $\frac{1}{y} < x$ .

**Theorem 2.7: (The Density of  $\mathbb{Q}$  in  $\mathbb{R}$ )** Let  $a, b \in \mathbb{R}$  with  $a < b$ . Then there is a  $q \in \mathbb{Q}$  with  $a < q < b$ .

**Proof:** First, suppose  $a \geq 0$ . By the Archimedean property, let  $n \in \mathbb{N}$  such that  $\frac{1}{n} < b - a$ . Let  $m$  be the smallest natural greater than  $na$ . Then  $m - 1 \leq na < m$ , so  $m \leq na + 1 < m + 1$ . Since  $na < m$ ,  $a < \frac{m}{n}$ , and since  $m \leq na + 1$  and  $\frac{1}{n} < b - a$ ,  $m < n(b - \frac{1}{n}) + 1 = nb$ . Thus  $\frac{m}{n} < b$ , and so  $a < \frac{m}{n} < b$ .

If  $a < 0$  and  $b > 0$ , then  $a < \frac{0}{1} < b$ , and if  $a < 0$  and  $b \leq 0$ , then since  $-b < -a$  (and  $-b, -a > 0$ ), there is a  $q \in \mathbb{Q}$  with  $-b < q < -a$ , so  $a < -q < b$ .

**Theorem 2.8:** There is an  $\alpha \in \mathbb{R}$  with  $\alpha^2 = 2$ .

**Proof:** Let  $T = \{t \in \mathbb{R} \mid t^2 < 2\}$ , which is nonempty and bounded above, and let  $\alpha = \sup T$ . Suppose  $\alpha < 2$ . By the Archimedean principle, there is an  $n \in \mathbb{N}$  such that  $\frac{1}{n} < \frac{2-\alpha^2}{2\alpha+1}$ , or, equivalently,  $\frac{2\alpha+1}{n} < 2 - \alpha^2$ . Then

$$\begin{aligned} \left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha+1}{n} \\ &< \alpha^2 + (2 - \alpha^2) \\ &= 2, \end{aligned}$$

so  $\alpha + \frac{1}{n} \in T$ , but  $\alpha + \frac{1}{n} > \alpha = \sup T$ .  $\nexists$  Similarly,  $a > 2$  gives a contradiction.