Name: ________________________

Midterm 2

Math 256

Spring 2023

You have 50 minutes to complete this exam and turn it in. You may use a 3x5 inch two-sided handwritten index card and a scientific calculator, but not a graphing one, and you may not consult the internet or other people. If you have a question, don't hesitate to ask — I just may not be able to answer it. **Enough work should be shown that there is no question about the mathematical process used to obtain your answers.**

You should expect to spend about one minute per question per point it's worth — there are 50 points possible on the exam and 50 minutes total.

Part I (9 points) Multiple choice. You don't need to show your work.

1. (3 points) Only one of the following four matrices is invertible. Which one is it?

A) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Not square

B) $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

C) $\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Terrible

det = 0

D) A 3×3 matrix $\mathbf{D}$ with eigenvectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, corresponding to eigenvalues 1, 0, and −6.

2. (3 points) To solve the following four nonhomogeneous DEs, we can use the methods of undetermined coefficients or variation of parameters. Which one can **only** be solved with variation of parameters?

A) $y'' + 2y = \cancel{e^t}$.

B) $y'' + 4y' = \sin(t) \cancel{+ \cos(2t)}$.

C) $y''' - y' + y = \cancel{t^2 e^{-3t}}$.

D) $y' + y = \csc(t)$.

3. (3 points) Matrix **A** has 4 rows and 3 columns, and matrix **C** has 4 rows and 2 columns. For the product **AB** = **C** to be defined, what must be the shape of **B**?

A) 3×2.

B) 4×4.

C) 3×3.

D) There is no shape that makes the product defined.

$(4 \times \underbrace{3)(3}_{} \times 2) = \underbrace{4 \times 2}_{}$

Part II (12 points) Short-answer. Explain your reasoning and show your work for each question.

1. (4 points) One of the eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ is $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda = 3$. What is $\mathbf{A}^8\mathbf{v}$ — i.e. the result of multiplying $\mathbf{A}$ by $\mathbf{v}$ eight times.

$A\vec{v} = \lambda\vec{v} = 3\vec{v}$

$A^8\vec{v} = 3^8\vec{v} = \begin{bmatrix} 0 \\ 3^8 \\ 3^8 \end{bmatrix}$

2. (4 points) Let $\mathbf{B} = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix}$. Find $\mathbf{B}^{-1}$.

$\left[\begin{array}{cc|cc} 5 & -4 & 1 & 0 \\ -6 & 5 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{cc|cc} 1 & -1 & -1 & -1 \\ 0 & 1 & 6 & 5 \end{array}\right]$

$\left[\begin{array}{cc|cc} 5 & -4 & 1 & 0 \\ -1 & 1 & 1 & 1 \end{array}\right]$ $\left[\begin{array}{cc|cc} 1 & 0 & 5 & 4 \\ 0 & 1 & 6 & 5 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & -1 & -1 & -1 \\ 5 & -4 & 1 & 0 \end{array}\right]$

3. (4 points) Give an example of a differential equation whose general solution is

$$y = c_1 \cos(2t) + c_2 \sin(2t) + c_3 t \cos(2t) + c_4 t \sin(2t).$$

Need roots of $r = \pm 2i, \pm 2i$

So $(r^2+4)^2 = r^4 + 8r^2 + 16$

i.e $y'''' + 8y'' + 16y = 0$

Part III (29 points) More involved questions with multiple parts.

1. (14 points) Let's look at a few variations of a DE.

a) (2 points) Solve $y'' - 4y = 0$.

$$r^2 - 4 = 0$$

$$r = \pm 2$$

$$y = c_1 e^{2t} + c_2 e^{-2t}$$

b) (6 points) Solve $y'' - 4y = e^{2t}$ using undetermined coefficients.

$$Y = Ate^{2t}$$

$$Y' = Ae^{2t} + 2Ate^{2t}$$

$$Y'' = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t}$$

$$Y'' - 4Y = 4Ae^{2t} + 4Ate^{2t} - 4Ate^{2t} = e^{2t}$$

$$4Ae^{2t} = e^{2t}$$

$$A = \frac{1}{4} \Rightarrow Y = \frac{1}{4}te^{2t}$$

c) (6 points) Solve $y'' - 4y = e^{2t}$ (the same as in part b) using variation of parameters.

$$W[e^{2t}, e^{-2t}] = \det \begin{bmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{bmatrix}$$

$$= -4$$

$$Y_1 = -e^{2t} \int \frac{e^{-2t} e^{2t}}{-4}\, dt + e^{-2t} \int \frac{e^{2t} e^{2t}}{4}\, dt$$

$$= e^{2t} \cdot \frac{t}{4} + \underbrace{e^{-2t} \cdot \frac{e^{4t}}{16}}$$

$$= \frac{1}{16} e^{2t}, \text{ already in } y_c$$

so $Y_1 = e^{2t} \cdot \frac{t}{4}$

2. (15 points) Consider the system of equations

$$3x - 2y = 54x - 3y \qquad = 8.$$

a) (2 points) Write this system as a matrix equation of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$.

$$\left[\begin{array}{cc|c} 3 & -2 & 5 \\ 4 & -3 & 8 \end{array}\right]$$

b) (5 points) Solve for $\mathbf{x}$ by row-reducing $\mathbf{A}$. Clearly indicate every row operation.

$$\left[\begin{array}{cc|c} 3 & -2 & 5 \\ 1 & -1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 3 & -2 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -4 \end{array}\right]$$

c) (3 points) Find the eigenvalues of **A**.

$$\det \begin{bmatrix} 3-\lambda & -2 \\ 4 & -3-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(-3-\lambda) + 8 = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

d) (5 points) Find the corresponding eigenvectors of **A**.

$\lambda = 1$: $\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 4 & -4 & 0 \end{array}\right]$

$$v_1 - v_2 = 0$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -1$: $\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 4 & -2 & 0 \end{array}\right]$

$$2v_1 = v_2$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$