Tropical Root System Polytopes

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Abstract

A root system is a collection of vectors with certain structural properties. The study of root systems originates from Lie Theory, in which every complex semisimple Lie algebra can be associated to a unique root system, but they have since grown into study in their own right. We discuss root systems and their associated polytopes, particularly focusing on the results proven by Marietti in "Root Polytopes of Crystallographic Root Systems". We then give a brief introduction to tropical geometry and define tropical polytopes. Finally, we give a tentative definition for a specific case of tropical reflection and a potential definition of tropical root system, and we attempt to tropicalize Marietti's results by making conjectures toward definitions of basic root system constructions.

1 Root Systems

We begin by defining root systems and outlining many of their properties. Though closely related to Lie theory, the study of root systems relies mostly on linear algebra, making it relatively simple in comparison.

Definition 1.1: Let V be a finite-dimensional, Euclidean vector space. A **root system** in V is a set of vectors Φ in V such that four conditions hold:

1. The vectors in Φ span V.
2. For any vector $\alpha \in \Phi$, Φ contains $-\alpha$, but no other scalar multiple of α.
3. For every $\alpha \in \Phi$, Φ is closed under the hyperplane perpendicular to α.
4. (*Integrality*) For all $\alpha, \beta \in \Phi$, the projection of β onto α is either an integer or half-integer multiple of α; or, equivalently, $\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer or half an integer, where (x, y) is the usual dot product on $\mathbb{R}^n$.

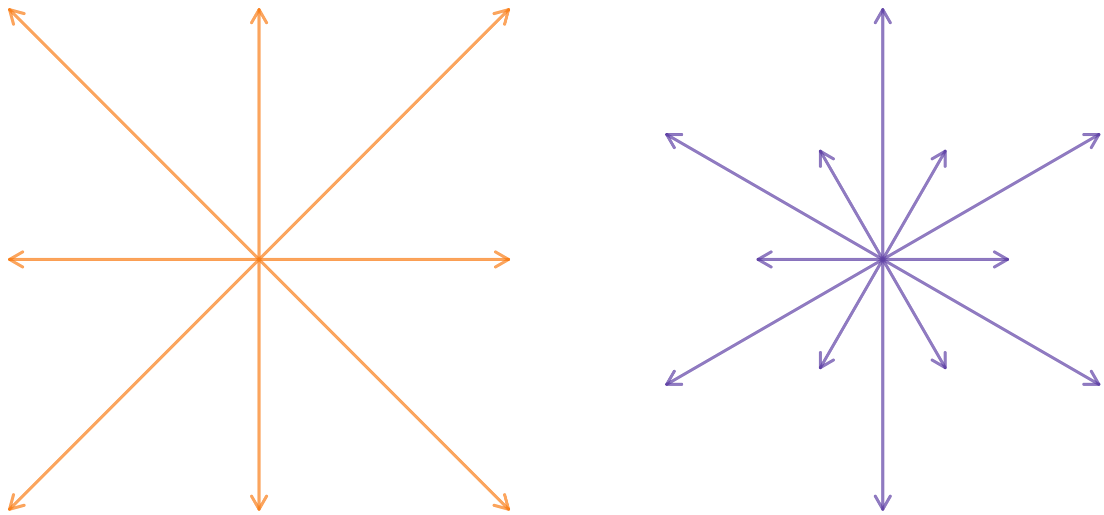

Figure 1: Two rank-2 root systems. Left: B_2. Right: G_2.

It is not uncommon for the definition to be given with only the first three conditions. In this case, a root system satisfying the fourth — integrality — is called **crystallographic**. We will assume integrality in the root systems we discuss, and we will also only consider irreducible root systems — those that are not the Cartesian product of lower-rank root systems.

Example: There are four irreducible root systems of rank 2 — A_2, B_2, C_2, and G_2. The first and last are depicted in Figure 1 (rank simply means dimension for root systems).

Since every root system lives in a vector space, we have the concept of bases at our disposal, and it is useful to have a basis for the root system in particular. To properly define this concept, we must first restrict to a certain set of roots.

Definition 1.2: Let Φ be a root system. A set of **positive roots**, denoted Φ^+, is a subset of Φ such that

1. For all $\alpha \in \Phi$, either α or $-\alpha$ is contained in Φ^+, but not both.
2. For all $\alpha, \beta \in \Phi^+$, if $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi^+$.

With a choice of Φ^+, we can define a **simple root** as any positive root that is not the sum of two others. The set of simple roots is called a **root basis** for Φ. The simple roots tend to appear "spread out" as much as possible in the set of positive roots; this is apparent in the examples in Figure 2.

The term root basis is appropriate, for a decomposition into simple roots is unique.

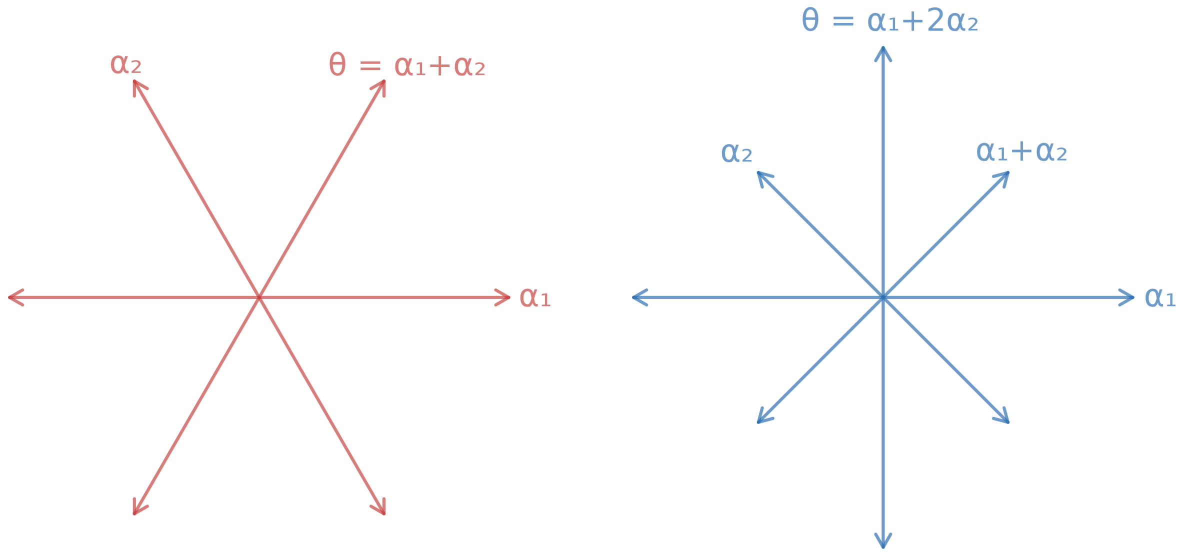

Figure 2: Positive roots, simple roots, and the unique highest root. Left: A_2. Right: C_2.

Proposition 1.3: Let Φ be a root system and $\Pi = \{\alpha_1, ..., \alpha_n\}$ a root basis for Φ. Then if $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$, the c_i are unique.

Another consequence of a choice of root basis is the existence of a highest, or maximal, root among the set of positive roots.

Proposition 1.4: Let Φ be a root system and $\Pi = \{\alpha_1, ..., \alpha_n\}$ a root basis for Φ. Then there is a root $\theta = m_1\alpha_1 + \cdots + m_n\alpha_n$ that satisfies $m_i \geq c_i$ for every root $c_1\alpha_1 + \cdots + c_n\alpha_n \in \Phi^+$, called the **highest root** of Φ.

This proposition is far from trivial, and we encourage the interested reader to examine the proof in [2] (178).

There is one more concept we must discuss before moving to polytopes — fundamental coweights, a particular type of vector that is highly relevant to [4].

Definition 1.5: Let Φ be a root system and $\Pi = \{\alpha_1, ..., \alpha_n\}$ a root basis for Φ. The **fundamental coweight** corresponding to α_i is $\check{\omega}_i \in V$, defined such that $(\alpha_i, \check{\omega}_i) = 1$ and $(\alpha_j, \check{\omega}_i) = 0$ for all $j \neq i$.

Fundamental coweights and highest roots play central roles in root system polytopes, which we will explore in Section 2.

Exercises for Section 1

Exercise 1.1: Choose a set of positive roots and determine the simple roots, highest root, and fundamental coweights for the root systems B_2 and G_2.

Exercise 1.2: A common construction is to define the **dual** of a vector α to be $\check{\alpha} = \frac{2}{(\alpha,\alpha)}\alpha$, and the **dual root system** of Φ as $\check{\Phi} = \{\check{\alpha} \mid \alpha \in \Phi\}$. Show that A_2 and G_2 are self-dual, and that B_2 and C_2 are dual to one another.

2 Root System Polytopes

Root system polytopes are obtained simply by taking a convex hull.

Definition 2.1: Let Φ be a root system. The **root polytope** of Φ, denoted $\mathcal{P}_\Phi$, is the convex hull of the vectors in Φ; that is, the smallest convex polytope that contains all the roots.

Certain faces of the root polytope, closely related to the highest root and the fundamental coweights, have special significance.

Proposition 2.2: Let Φ be a root system, $\mathcal{P}_\Phi$ its polytope, and $\{\alpha_1, ..., \alpha_n\}$ a root basis, with $\{\check{\omega}_1, ..., \check{\omega}_n\}$ the corresponding fundamental coweights. If $\theta = m_1\alpha_1 + \cdots + m_n\alpha_n$, then the hyperplane $\{x \in V \mid (x, \check{\omega}_i) = 1\}$ supports a face F_i of $\mathcal{P}_\Phi$ containing θ.

Some clarification might be needed. A **face** of the root polytope is, informally, a lower-dimensional, extreme subset of the polytope itself. A solid cube's faces, for example, are the six squares, twelve lines, and eight vertices that make up its boundary. A hyperplane **supports** a face if the hyperplane contains a point of the face and the face is entirely contained on one of the two sides of the hyperplane.

While we will not prove Proposition 2.2 in its entirety, θ's presence in the hyperplane is not as difficult a fact to show, and we leave it as an exercise.

Root system polytopes and their coordinate faces have numerous interesting properties, many of which are detailed in [4]. We list several.

Theorem 2.3: Again, let Φ be a root system, $\mathcal{P}_\Phi$ its polytope, and $\{\alpha_1, ..., \alpha_n\}$ a root basis, with $\{\check{\omega}_1, ..., \check{\omega}_n\}$ the fundamental coweights. Then the following hold:

1. Coordinate faces corresponding to distinct coweights are distinct; that is, if $i \neq j$, then $F_i \neq F_j$.
2. The sum of two roots in a coordinate face is never another root.
3. There is a root $\eta_i \in \Phi^+$, analogous to θ, satisfying $\Phi \cap F_i = [\eta_i, \theta]$ (the roots in F_i are totally ordered).
4. The dimension of F_i is the number of c_i not equal to m_i, where $\eta_i = c_1\alpha_1 + \cdots + c_n\alpha_n$.
5. The barycenter of each F_i is contained in the line through $\check{\omega}_i$.

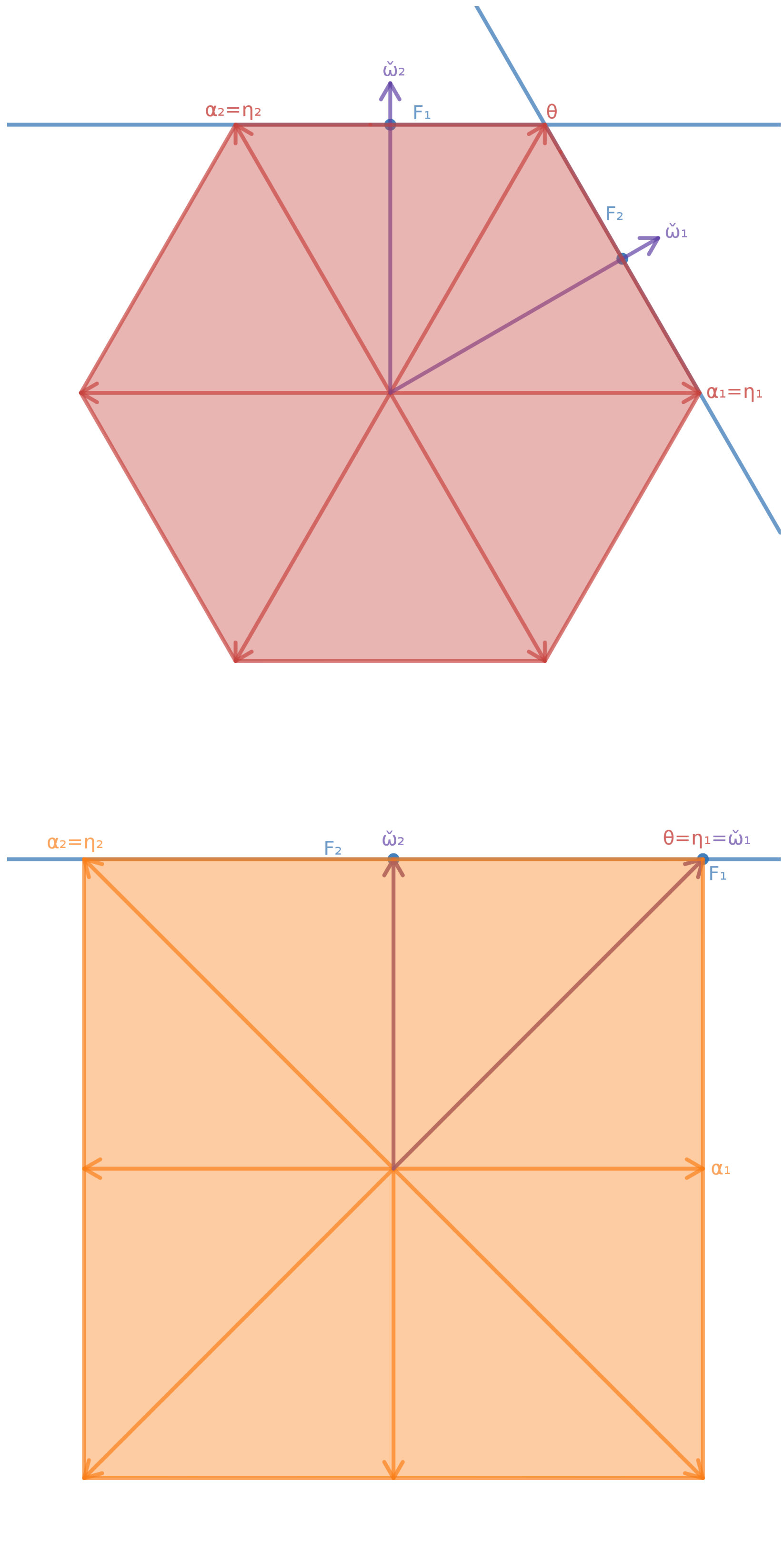

Figure 3: Polytopes, bases, coweights, highest roots, and coordinate faces. Top: A_2. Bottom: B_2.

Example: Figure 3 gives two complete examples of Theorem 2.3 on A_2 and B_2. Notice that F_1 in B_2 is merely a point — it is a 0-dimensional face.

Further study of root systems and their polytopes leads to a complete classification of root system polytopes, which is the direction Maretti pursues in [4]. We will take a different path — attempting to tropicalize these results.

Exercises for Section 2

Exercise 2.1: Determine the root polytopes of C_2 and G_2, and verify that Theorem 2.3 holds on them.

Exercise 2.2: Let Φ be a root system. Give a direct proof that the highest root, θ, is contained in every coordinate face F_i.

Exercise 2.3: Let Φ be a root system. Show that $\displaystyle\bigcap_{i=1}^{n} F_i = \{\theta\}$.

3 Tropical Geometry

Tropical geometry is a still-developing subfield of algebraic geometry that attempts to replace algebraic arguments with combinatorial ones. Many tropical theorems have been shown to apply in the affine (that is, nontropical) case as well — but the tropical proof is often far easier than the corresponding affine one. Because of this close relationship, it is also useful to translate affine theorems and constructions to a tropical setting, laying the groundwork for more tropical theorems. Unfortunately, tropical algebra has many quirks and anomalies that often make it difficult to work with, and the main challenge of tropicalization is, in many cases, to deal with these oddities.

Definition 3.1: The tropical numbers are the semiring $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ equipped with the binary operations $\oplus$ and $\odot$, defined as $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$. $-\infty$ is a formal symbol, acting as the tropical zero, that satisfies $-\infty \oplus a = a \oplus -\infty = a$ and $-\infty \odot a = a \odot -\infty = -\infty$ for all $a \in \mathbb{T}$.

The tropical numbers do not form a field, or even a ring, because additive inverses are not defined. This is where much of the difficulty and strangeness of tropical geometry arises — there is no subtraction. Additionally, the strange operations can make parsing expressions difficult — for example, $0 \odot x^2 = 2x$.

This last example is rather important. Since tropical exponents become real coefficients, every term of a tropical polynomial will become linear in the reals, and so *tropical graphs will have no curves*, giving tropical geometry its underlying skeleton of combinatorics.

To have a hope of bringing root systems and their polytopes to the tropical world, we must first discuss tropical graphs. Let's take $0 \odot x^2 \oplus 1 \odot xy \oplus 0 \odot y^2 \oplus 1 \odot x \oplus 1 \odot y \oplus \frac{1}{2}$ as an example. The standard notion

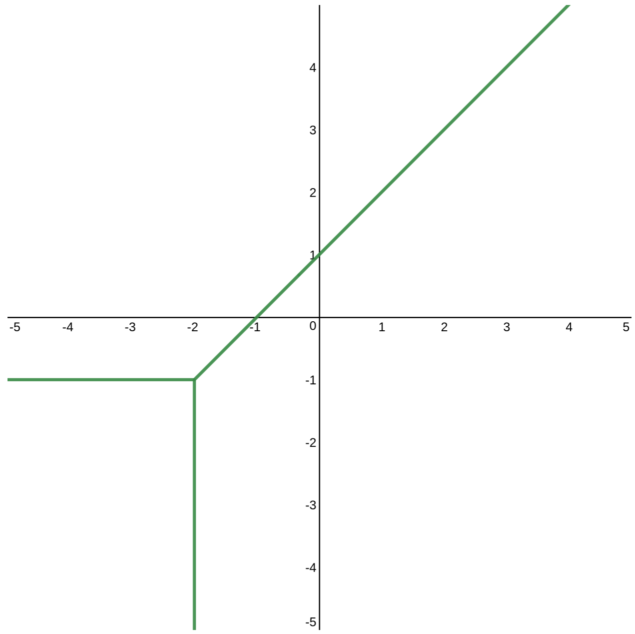

Figure 4: The bend locus of $3 \odot x \oplus 2 \odot y \oplus 1$.

of graph would tell us to plot the (x, y) pairs that make the expression zero, but *no points will*, since we are taking a maximum with $\frac{1}{2}$. This, then, is not an acceptable notion of graph.

In the end, the definition we choose is unintuitive at best, but it serves our purposes and is widely considered standard.

Definition 3.2: Let $f \in \mathbb{T}[x_1, ..., x_n]$ be a polynomial. The **bend locus** of f, denoted $\text{Bend}(f)$, is the set of points $(a_1, ..., a_n) \in \mathbb{T}^n$ for which at least two terms of $f(a_1, ..., a_n)$ are tied for the maximum. In other words, at least two terms in $f(a_1, ..., a_n)$ are equal to $f(a_1, ..., a_n)$.

Example: Let's examine $a \odot x \oplus b \odot y \oplus c$, a general linear equation. We have three possibilities for points $(x, y) \in \text{Bend}(a \odot x \oplus b \odot y \oplus c)$:

1. $a \odot x = c$ and $c \geq b \odot y$,
2. $b \odot y = c$ and $c \geq a \odot x$, or
3. $a \odot x = b \odot y$ and $a \odot x \geq c$.

Translating to real operations, we have $x = a - c$ with $y \leq b - c$, $y = b - c$ with $x \leq a - c$, and $y = x + a - b$ with $x \geq c - a$. Plotting these three lines, we see that they describe a strange tripartite shape that has a vertex at $(-a, -b)$, as depicted in Figure 4.

Bend loci become increasingly complicated as the polynomial's degree increases, as Figure 5 shows. It becomes far more difficult to predict where the "tentacles" will lie, and the center regions become rapidly complex.

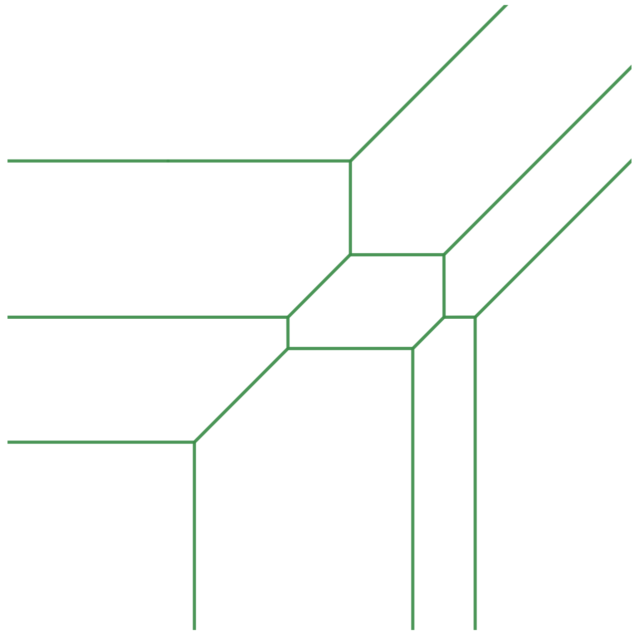

Figure 5: $\text{Bend}(1 \odot x^3 \oplus 6 \odot x^2 y \oplus 8 \odot xy^2 \oplus 4 \odot y^3 \oplus 6 \odot x^2 \oplus 10 \odot xy \oplus 9 \odot y^2 \oplus 9 \odot x \oplus 9 \odot y \oplus 5)$.

We will now turn our attention to tropical polytopes and convex hulls, in the interest of tropical root system polytopes.

Exercises for Section 3

Exercise 3.1: Simplify $0 \odot x^5 \odot y \oplus 0 \odot 1 \odot x \oplus 3 \odot y^0 \oplus x^{-2}$ and convert it to real operations.

Exercise 3.2: Produce a graph of $\text{Bend}(0 \odot x^5 \odot y \oplus 0 \odot 1 \odot x \oplus 3 \odot y^0 \oplus x^{-2})$ by considering each pair of terms separately.

Exercise 3.3: At how many points do two tropical lines (linear bend loci) generally intersect? What about a line and the cubic in Figure 4? Is this analogous to the affine case? How could we define intersection to account for the pathological case in which two line segments overlap? These questions and others are explored further in [1].

4 Tropical Polytopes and Reflection

A convex tropical polytope is just the convex hull of a finite set of points, so all we need understand is this tropical convex hull. To do so, however, we must introduce a new way of viewing $\mathbb{T}$ — projectively.

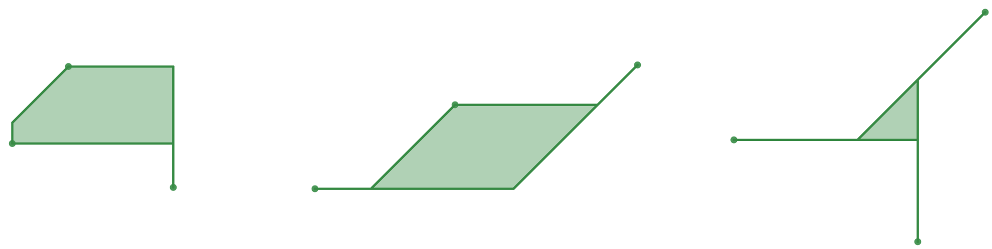

Figure 6: Three tropical triangles in $\mathbb{TP}^2$.

Definition 4.1: The **tropical projective space** is the semiring $\mathbb{TP}^n = \{[x_1 : \cdots : x_{n+1}] \mid x_i \in \mathbb{T}\}$, where we identify two points $[x_1 : \cdots : x_{n+1}]$ and $[y_1 : \cdots : y_{n+1}]$ if $[x_1 : \cdots : x_{n+1}] = c \odot [y_1 : \cdots : y_{n+1}]$ for some $c \in \mathbb{T}$.

We think of $\mathbb{TP}^n$ as $\mathbb{T}^n$ along with a copy of $\mathbb{TP}^{n-1}$ "at infinity." The $\mathbb{T}^n$ can be seen by taking every point with last coordinate not equal to $-\infty$ and tropically dividing by it, effectively restricting to n-dimensional tropical space. When last first coordinate is $-\infty$, though, we cannot remove it, and so, with one coordinate fixed, we have a copy of $\mathbb{TP}^{n-1}$ at infinity. As a side note, this concept of points at infinity comes much more naturally with the tropical numbers, as opposed to the reals, where we must create metaphors for dividing by 0.

We now move to tropical polytopes, the main focus of this section.

Definition 4.2: Let $\mathbf{x_1}, ..., \mathbf{x_k} \in \mathbb{TP}^n$. The **tropical convex hull** of $\mathbf{x_1}, ..., \mathbf{x_k}$ is the set $\{c_1 \odot \mathbf{x_1} \oplus \cdots \oplus c_k \odot \mathbf{x_k} \mid c_i \in \mathbb{R}\}$. A **tropical polytope** is the tropical convex hull of finitely many points in $\mathbb{TP}^n$.

The use of homogeneous coordinates is important — without them, all polytopes would be infinitely long cylinders in $\mathbb{TP}^n$ (Exercise 4.1). Instead, our definition produces interestingly-shaped tropical objects. We give several examples in Figure 6.

A more detailed investigation of tropical polytopes can be found in [3]. This is all that is required for our purposes, however, and we will now turn our attention to tropical reflection. On the surface, this seems to present a major problem, since it is impossible to directly tropicalize the standard notion of reflection — in two dimensions, reflecting through the y-axis negates the x-coordinate, an operation that has no equivalent in $\mathbb{T}$. While we the definition we ultimately choose is by no means general, and by no means universally applicable, it is sufficient to define tropical root systems.

Definition 4.3: Let $\mathbf{x} \in \mathbb{TP}^n$ be a point not at infinity. The **tropical reflection** of $\mathbf{x}$ through the origin is the set of points given by the affine reflection of $\mathbf{x}$ through each component hyperplane of $\mathrm{Bend}(x_1 \oplus \cdots \oplus x_{n+1})$, the n-dimensional tropical hyperplane with vertex at the origin.

Example: Figure 7 shows the reflection of $[0 : 2 : 1] \in \mathbb{TP}^2$ into $\binom{2+1}{2} = 3$ points, given by reflecting $(2, 1)$ across the lines $y = 0$, $x = 0$, and $y = x$. The lines are obtained by extending the tentacles of $\mathrm{Bend}(x \oplus y \oplus z)$, the tropical line centered at the origin.

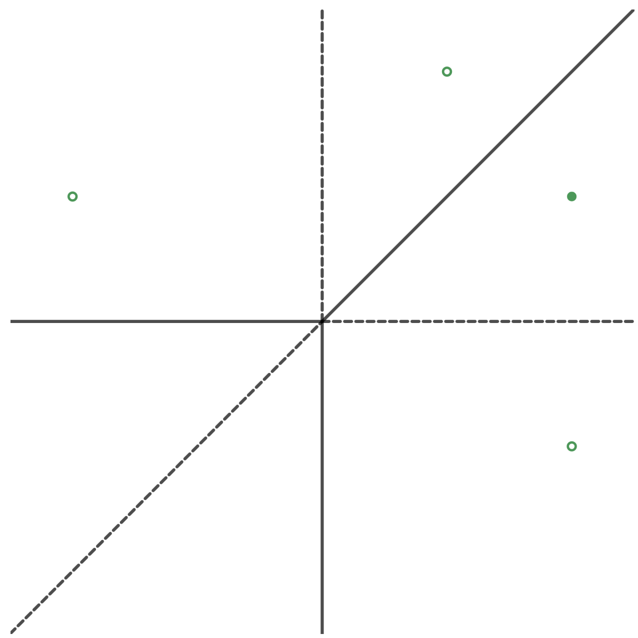

Figure 7: The tropical reflection of $[0 : 2 : 1]$ through the origin.

By repeatedly reflecting the newly reflected points, we can generate a total of $2^{\binom{n+1}{2}}$ points, arranged symmetrically about the origin. These points lay the foundation for tropical root systems.

Exercises for Section 4

Exercise 4.1: Show that if we use nonhomogeneous coordinates in Definition 4.2, tropical polytopes will be infinitely long cylinders.

Exercise 4.2: Given a point $\mathbf{x} \in \mathbb{TP}^n$ not in $\mathrm{Bend}(x_1 \oplus \cdots \oplus x_{n+1})$, prove that we have N reflections of $\mathbf{x}$, and 2^N total points by repeated reflection, where $N = \binom{n+1}{2}$.

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5 Tropical Root Systems

We are now ready to define a tropical root system. The proposed definition is similar to the affine one, but with one condition removed and the others tropicalized.

Definition 5.1: A **root system** in $\mathbb{TP}^n$ is a set of tropical vectors (i.e., points) Φ in $\mathbb{TP}^n$ such that two conditions hold:

1. For any vector $\alpha \in \Phi$, Φ contains α^{-1}, but no other power of α.
2. Φ is closed under tropical reflection through the origin.
3. (*Integrality*) For any two vectors $\alpha, \beta \in \Phi$, the number $(\alpha, \beta) \odot (\alpha, \alpha)^{-1}$ is an integer or half an integer.

Noticeably, one condition is missing from the affine definition. We abandon the spanning requirement, as the tropical span of the points in Φ is the tropical convex hull — it would be pointless to insist this be all of $\mathbb{TP}^n$ when the very thing we want to study is the polytope itself.

This definition produces interesting objects. Unlike in the affine case, there are not finitely many for a given rank — in fact, for any positive $[x : y : 0] \in \mathbb{TP}^2$ with $y = x + 1$, the tropical reflection of $[x : y : 0]$ through the origin satisfies the definition of a tropical root system (Exercise 5.1). We give some rank 2 examples in Figure 8. The lines are given by the tropical convex hull of each root and the origin.

In the interest of studying tropical root system polytopes, we next attempt to define positive roots.

Definition 5.2: Let Φ be a tropical root system. A set of **positive roots**, denoted Φ^+, is a subset of Φ such that

1. For all $\alpha \in \Phi$, either α or α^{-1} is contained in Φ^+, but not both.
2. For all $\alpha, \beta \in \Phi^+$, if $\alpha \oplus \beta \in \Phi$, then $\alpha \oplus \beta \in \Phi^+$.

Our previous definition of simple root is no longer satisfactory, since the set of positive roots which are not the tropical sum of two others often contains only one root (this is the case in every example in Figure 8). Additionally, if we allow adding any root to itself, which results in the same root, then the set of simple roots will be empty. If we are to have simple roots, then, we will need a better definition.

Definition 5.3: Let Φ be a tropical root system and Φ^+ a choice of positive roots. A set of **simple roots**, or a **root basis**, for Φ is a subset Π of Φ^+ such that any root in Φ^+ can be expressed as a linear combination of the simple roots. Equivalently, the tropical convex hull of the simple roots contains every positive root.

In the affine case, this is an alternate definition of simple roots. Tropically, though, it is different: it may give more than n roots for a rank n root system, and additionally, unlike Proposition 1.3 in the affine case, we are not guaranteed uniqueness if we decompose into tropical simple roots (Exercise 5.3). We provide a choice of positive roots and simple roots for three tropical root systems in Figure 8.

We would like a statement similar to Proposition 1.4 regarding a highest root, but this proves to be very difficult regardless of the definition we choose, since the nature of tropical algebra guarantees that a decomposition into a linear combination of simple roots will not be unique. Instead, we leave the highest root and move to fundamental coweights.

Definition 5.4: Let Φ be a tropical root system and $\Pi = \{\alpha_1, ..., \alpha_n\}$ a root basis. The **fundamental coweight** corresponding to α_i is $\check{\omega}_i \in \mathbb{TP}^n$, defined such that $(\alpha_i, \check{\omega}_i) = 0$ and for all $j \neq i$, $(\alpha_j, \check{\omega}_i)$ achieves a double maximum among its terms, where (x, y) is the tropical dot product on $\mathbb{TP}^n$, defined by

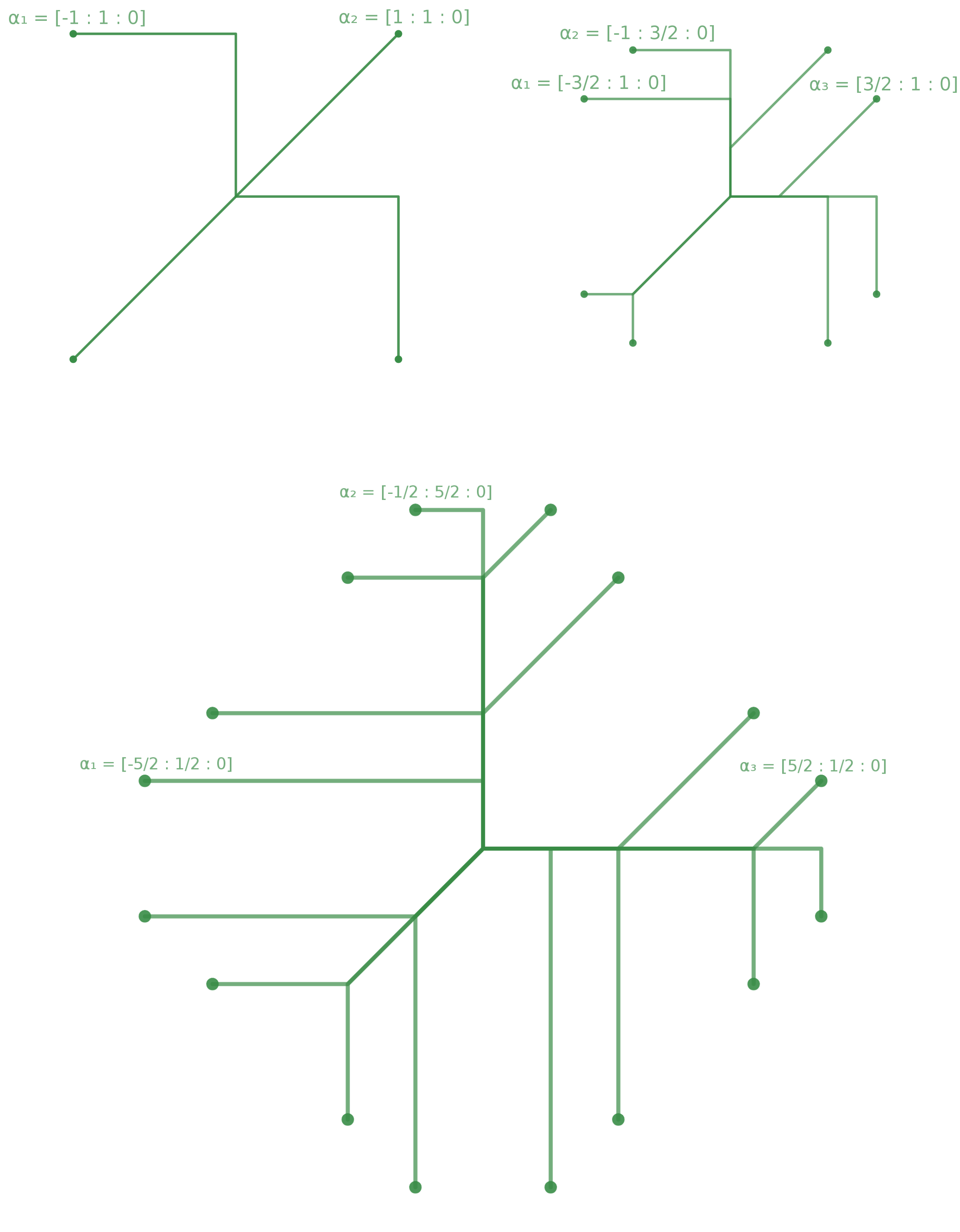

Figure 8: Three tropical root systems. The positive roots are those above the x-axis and the simple roots are labeled.

$$([x_1 : \cdots : x_{n+1}], [y_1 : \cdots : y_{n+1}]) = x_1 \odot y_1 \oplus \cdots \oplus x_{n+1} \odot y_{n+1}.$$

It is unclear that such a vector exists, though in many cases it certainly does. It is possible, though, to construct a set of simple roots — $\{[2 : 4 : 0], [1 : 1 : 0]\}$, for instance — for which there is no fundamental coweight corresponding to one of the roots ($[2 : 4 : 0]$, here). What is not clear, however, is whether these pathological simple roots can in fact appear in a tropical root system at all. More investigation is required.

Ultimately, we would like a tropical analogue of Theorem 2.3, but many obstacles stand in the way. The definition of an affine root system lends itself to being tropicalized, and the result is surprisingly robust and nontrivial, but the definitions built on root systems — positive roots, simple roots, and fundamental coweights — are not quite so simple. The study of tropical root systems is virtually nonexistent, and interest in it — and research — could prove useful to tropical geometry and Lie theory alike.

Exercises for Section 5

Exercise 5.1: Verify that the tropical reflection of $[x : x + 1 : 0]$ through the origin gives a tropical root system.

Exercise 5.2: For each of the three tropical root system axioms, construct a set of points that fails to satisfy that axiom only.

Exercise 5.3: Show that a decomposition into tropical simple roots may not be unique.

Exercise 5.4: For each tropical root system given in Figure 8, verify that the positive roots and simple roots meet the definition and determine all possible fundamental coweights, if any exist.

Further Investigations

There is much more work to be done. Potential research topics include:

1. **Simple roots:** is there an equivalent affine definition that better fits the tropical case? Can we determine how few simple roots we will need for a given root system? If we insist on having the minimum possible number of simple roots, will they be unique? If a root system can be decomposed into two smaller root systems (i.e., it is reducible), should we refuse to consider it?

2. **Polytopes:** is there a suitable definition of supporting tropical hyperplane? Does each simple root correspond with a unique tropical face of the root system polytope? If so, is the intersection of the faces nonempty? Does it consist of a single point, and if so, can we define the highest root to be that point, a la Exercise 2.3?

3. **Fundamental coweights:** is there always one for each simple root? Is it unique? If not, does restricting to only irreducible root systems guarantee existence or uniqueness? If the fundamental coweights are generally not unique, is there a particular one we should choose? For example, the condition that a dot product achieves a double maximum gives a tropical hyperplane of valid points, and the vertex seems to stand out if we were to choose a single point on the hyperplane.

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