1.4.3 Calculus, when calculus is possible

When the utility function is differentiable (nice and smooth) we can use calculus to find that bundle, *. (Note that differentiability –in the sense that we know– is a property that cannot be even defined unless the goods are perfectly divisible.) We can do it directly, now that we know that the best choice, if interior, is characterized by MRS equal to the ratio of prices. But let us do it from scratch. Let us use the tools that calculus offers us to solve problem (5). First, if we see that the function $u(x_1, x_2)$ that represents the preferences of the consumer is increasing (more is better), then we know that the solution will surely satisfy that $p_1 \times x_1 + p_2 \times x_2 = y$: the consumer will not leave money unspent (then burnt). So, we may substitute this constraint for the one we have in (5). How do we use calculus tools to solve an optimization problem with **one equality constraint**?

We may use:

The Lagrange method

For our purposes (if you do not want to dig into why), treat this method as a black box, as a routine, one of the few that you should consider in this course. I explain this tool now.

You know that, at a point at which a function attains an (interior) maximum, its slope must be zero. But what if the problem has a constraint added to it? Here is what you do, in that case.

1.- Construct another function, called Lagrangean, by adding to the original function, $u(x_1, x_2)$ in this case, the constraint after you transport all the terms to one side of the equality, leaving 0 in the other, multiplied by some fictitious variable, call it λ. That is, we construct the function $L(x_1, x_2, \lambda)$ as

$$L(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda(p_1 \times x_1 + p_2 \times x_2 - y).$$

2.- Note that this is a function of three variables now: x_1, x_2 and λ. (Remember that p_1, p_2, and y are not variables that the consumer chooses, but numbers that she faces.) Then, the condition that the solution to problem (5) needs to satisfy is the condition that x_1, x_2 and λ need to satisfy to maximize $L(x_1, x_2, \lambda)$. That is, the partial derivatives with respect to all the three variables must equal 0. (I.e., L is flat in all directions.) In our case:

$$\begin{aligned}
\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} &= \frac{\partial u(x_1, x_2)}{\partial x_1} - \lambda \times p_1 = 0, \\
\frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} &= \frac{\partial u(x_1, x_2)}{\partial x_2} - \lambda \times p_2 = 0, \\
\frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} &= p_1 \times x_1 + p_2 \times x_2 - y = 0.
\end{aligned}$$

These are three equations with three unknowns which hopefully will identify the maximum, just as the first order conditions in an unconstraint maximization problem do.

Take the first two of these conditions. Write them as

$$\begin{aligned}
\frac{\partial u(x_1, x_2)}{\partial x_1} &= \lambda \times p_1, \\
\frac{\partial u(x_1, x_2)}{\partial x_2} &= \lambda \times p_2.
\end{aligned}$$

Now, if the left hand side of the first line is equal to the right hand side of that line, and the same goes for the second line, then if we divide the left hand side of the first line by the left hand side of the second line, we should get something that is equal to what we get when we divide the right hand side of the first line by the right hand side of the second line. That is, since λ cancels out,

$$\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = \frac{p_1}{p_2}.$$

Notice that, in our case, this takes care of this "fictitious" variable λ. So, our solution, according to the Lagrange method, if interior, needs to satisfy that

$$\begin{aligned}
\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} &= \frac{p_1}{p_2}, \\
p_1 \times x_1 + p_2 \times x_2 &= y.
\end{aligned}$$

This gives us two equations with two unknowns.

Consider the first of these two conditions. (The second simply says that we must be on the budget constraint!) We have seen the expression on the left before. Yes!, exactly: it is simply the marginal rate of substitution!

Thus, the Lagrange method tells us what we saw before graphically: if the choice is interior (the bundle contains a positive amount of each good), then it must be that at that bundle the MRS coincides with the ratio of prices.

Exercise: Why don't you try to find the choice for a consumer with preferences represented by the utility function (1) when she has an income $y = 30$ and the prices are $p_1 = 2$ and $p_2 = 5$?

Alternative method

There is another way to solve our problem (5). Note that the budget constraint may be read as follows: if the consumer buys x_1 units of good 1, then she can only buy (at most)

$$x_2 = \frac{y - p_1 \times x_1}{p_2} \tag{6}$$

units of good 2. We have just manipulaed the budget constraint by "solving" for x_2. That is, even though the consumer chooses both x_1 and x_2, the budget constraint **kills** one of these two degrees of freedom: once x_1 is chosen, there is little to choose with respect to good 2. Therefore we can substitute (6) for x_2 in (5) to write our problem as

$$\max_{x_1} \; u\left(x_1, \frac{y - p_1 \times x_1}{p_2}\right).$$

The constraint has been already taken into account, so we got rid of it! Also, instead of creating a problem without constraints but with three variables, as in Lagrange, we have created a problem without constraint but with one variable! Using the chain rule, you should be able to see that the condition for the maximization of this function is

$$\frac{\partial u(x_1, x_2)}{\partial x_1} + \frac{\partial u(x_1, x_2)}{\partial x_2}\left(\frac{-p_1}{p_2}\right) = 0,$$

with x_2 given by (6). That is, the same condition as in the Lagrange method.

1.4.4 Corner solutions

Consider the following example: the consumer's preferences can be represented by the function

$$U(x_1, x_2) = (x_1)^2 + (x_2)^2.$$

There is nothing anomalous with those preferences: you can check that they satisfy our three axioms. Also, suppose that $y = 10$ and prices are $p_1 = 1$ and $p_2 = 2$. Then, if we use the Lagrange method, for instance, we solve

$$\max_{x_1, x_2, \lambda} (x_1)^2 + (x_2)^2 - \lambda(1 \times x_1 + 2 \times x_2 - 10).$$

The bundle (and λ) at which this function is flat satisfies:

$$\begin{aligned} 2x_1 - \lambda &= 0 \\ 2x_2 - \lambda \times 2 &= 0 \\ x_1 + 2x_2 &= 10 \end{aligned}$$

which means $x_1 = 2$ and $x_2 = 4$, a bundle with label $u(2, 4) = 20$. Now, consider the bundles $(0, 5)$ and $(10, 0)$, both affordable for the consumer The first has label 25 and the second label 100, both higher than 20! Thus $(2, 4)$ is certainly not the best choice for the consumer!

What happened here and what do we learn from it? First, we may represent the indifference curves for a consumer with the preferences that we are considering. By doing so we see that they look like the ones in Figure 5A. (This would be a good exercise for calculus, if you want to try.) We have represented the problem for the consumer in Figure 10. At the bundle $(2, 4)$ the objective function (the Lagrangearn, if you wish) is indeed flat (has zero derivatives)... because there the objective function is at its lowest point among the ones on the budget constraint! Do you remember: the function is also flat at its minima! It is true that the MRS coincides with the ratio of prices at that point, but any feasible trade –any movement along the budget constraint– would actually lead the consumer to bundles on higher indifference curves, as you can see in the figure.

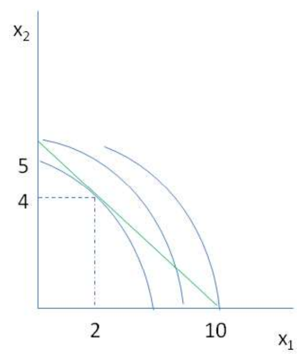

Figure 10

Why? At bundle $(2, 4)$ the consumer is indeed **willing**, but just so, to engage in a very small trade at a rate of $1/2$ units of good 2 per unit of good 1, the rate of prices, i.e., the rate at which she **can** trade in the market. However, the minute she gets a little more good 1 in exchange for some good 2 she begins to strictly prefer to keep trading more and more units of good 2 for units of 1. And vice versa: if she gets a little more good 2 in exchange for good 1, she becomes more eager to keep obtaining more good 2 in exchange for good 1. In a sense, the more of one good she has and the less of the other, the more she values the former with respect to the other.

When the curves bend in the other way (are convex, instead of concave) the opposite is true: the more the consumer has of one good and the less of the other, the less she values the former in relation to the latter. This latter case may sound more plausible, but remember: the consumer's tastes are her privilege! In any case, we will typically deal with examples with "convex" indifference curves, but it is useful to understand the "concave" indifference curves cases.

One border-line case is worth mentioning. Suppose that the consumer's preferences can be represented with the utility function

$$U(x_1, x_2) = x_1 + 2 \times x_2.$$

Compute the MRS for a bundle (x_1, x_2):

$$\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = \frac{1}{2}.$$

As you can see, the MRS does not depend on what particular bundle we are talking about. That is, the MRS is the same no matter where the consumer is sitting: she is always willing to exchange one unit of good 2 for two units of good 1. Indeed, the indifference curve (with a label k) is

$$x_1 + 2 \times x_2 = k,$$

or, "solving" for x_2,

$$x_2 = \frac{k}{2} - \frac{1}{2}x_1.$$

As you can see, this is the equation of a straight line with intercept $k/2$, and so a curve with constant slope of, yes, $\frac{1}{2}$, the MRS we have already computed.

We can call these goods, when the indifference curves are as in Figure 5B, **perfect substitutes**: in any circumstances, one unit of good 2 is, from the point of view of the consumer, perfectly replaceable with 2 units of good 1.

Now suppose that we apply the Lagrange method to find where our objective function is flat. That is, we solve

$$\max_{x_1,x_2,\lambda} x_1 + 2 \times x_2 - \lambda(p_1 \times x_1 + p_2 \times x_2 - y).$$

The bundle (and λ) at which this function is flat satisfies:

$$\begin{aligned} 1 - \lambda p_1 &= 0 \\ 2 - \lambda \times p_2 &= 0 \\ p_1 \times x_1 + p_2 \times x_2 &= y, \end{aligned}$$

and substituting for λ in the first two equations as before we get:

$$\frac{p_1}{p_2} = \frac{1}{2}.$$

What?! Where did the variables we are looking for (remember: x_1 and x_2) go!?

OK, let us go back and review what the equations tell us. They say: at any x_1, x_2 (and λ) that satisfy these three equations, the slope of the objective function is 0. So, the answer they give us is: the function is flat at any bundle (x_1, x_2) where $p_1 \times x_1 + p_2 \times x_2 = y$ (i.e., on the budget line) AND $\frac{p_1}{p_2} = \frac{1}{2}$. But look, the latter conditions has nothing to do with the consumer's choice: either the ratio of prices is $1/2$ or is not, whatever the bundle! Nothing the consumer can do about it! Thus, if the ratio of prices happen to be indeed $1/2$, then the objective function is flat at ANY bundle on the budget constraint. AND if the ratio of prices is not equal to $1/2$, then the function is NOT flat at any bundle on the budget line (and so only CORNER solutions are possible).

A graph to visualize that? OK. Here:

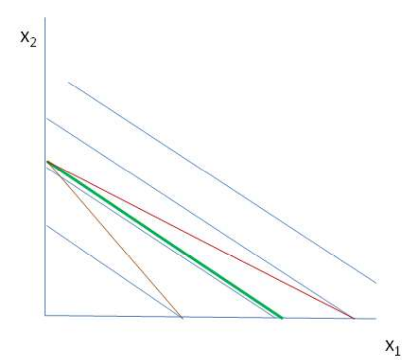

Figure11

Figure 11 shows a few indifference curves for our consumer (in blue) and three budget constraints for different price ratios. Let's begin by assuming that income and prices are such that the budget constraint is the green line. As you can see, every bundle on the budget constraint is on the same indifference curve, which also coincides with the budget constraint. This is so when the ratio of prices, i.e., the slope of the green line, coincides with the (constant, in this case of perfect substitutes) MRS on ALL bundles.This is the $\frac{p_1}{p_2} = \frac{1}{2}$ case before. The red line may represent the budget constraint for the same income and price of good 2 when the price of good 1 falls. (Try to understand this.) In that case, as you can see, there is no "tangency" of the budget line with any indfference curve. That is, the objective function in the Lagrange problem is never flat. What is the best choice here? Clearly, the corner where the consumer only buys good 1: $(\frac{y}{p_1}, 0)$. Finally, the brown line may represent the original case but when the price of good 1 increases. Again, in that case the solution is the corner where the consumer only buys the now relatively cheaper good 2: $(0, \frac{y}{p_2})$.

Question: What shape would the indifference curves have if the two goods are perfect complements, from the point of view of the consumer?