1.2 Utility

Take another look at Table 1. There, each indifference curve, or rather each bundle, has what we can call a **label**: the number corresponding to the place that the bundle (or all the bundles in the indifference curve) occupy in the ranking. Thus, the table defines a **fuction**: to each bundle it assigns a number (label). That is, a function from the set of bundles to the set of (integer) numbers.

Observe that knowing the label of each bundle is all that is required to know the ranking. Indeed, if I do not show you the table but I tell you that the label of bundle $(1,4)$ is 3, and the label of bundle $(4,0)$ is 5, you will know that this consumer prefers the bundle $(1,4)$ to the bundle $(4,0)$. Alternatively, if I tell you that this consumer prefers the bundle $(1,4)$ to the bundle $(0,4)$, then you know that the label of $(1,4)$ must be lower than the label of $(4,0)$. That is, these labels **represent** the consumer's preferences, just as the table did.

We are going to do this from now on, that is, we are going to represent the preferences of a consumer with a set of labels, similarly as we did in Table 1, **except that we are going to invert** the direction of the labels. When we have infinitely (a continuum, perhaps) of possible bundles, the set of integers may not suffice to assign the necessary labels. Moreover, there may not be a "number 1" bundle. (What would be the consumption bundle that you like the most among all the conceivable ones?) So, first we may assign labels in the reverse order, so that a **higher** label actually represents a **higher** position in the ranking, not a lower one (this is just a convention, not an important change). Second, we are going to –possibly– use all real numbers instead of only the (positive) integers. Note that there is no highest (nor lowest) real number.

When we assign labels in that way, we are still defining a function, of course. Any such function, that is, any function from the set of all bundles to the set of real numbers so that **when** the consumer prefers one bundle to another one **then** the first is assigned a larger number than the latter is what we call a **utility function** that represents the consumer's preferences. For instance, the following table shows **a** utility function that represents the preferences of our old friend behind Table 1.

Utility (label)						
312	(4,4)					
8	(3,4)	(4,3)				
3	(1,4)	(2,4)	(3,3)	(4,2)		
2.4	(0,4)	(0,3)	(1,3)	(2,3)	(3,2)	(4,1)
1	(1,2)	(2,2)	(3,1)	(4,0)	(2,1)	
0	(0,2)	(1,1)	(3,0)			
-7	(0,1)	(1,0)	(2,0)			
-8	(0,0)					

Table 2

Just a function, again with the property that we have assigned a higher label to bundles that are preferred by the consumer and the same label to bundles among which the consumer is indifferent. Would you be able to construct a different function that represented the same preferences? I bet you would. How many ways are there to do so?

We usually write a utility function (defined on bundles of two goods) as $U(x_1, x_2)$,for instance.[1] What is important is that you remember what that fancy piece of notation means: nothing but the same type of information that is contained in Table 2. For instance, in this particular case, we have that $U(3, 2) = 2.4$ or that $U(4, 4) = 3, 123$.

When we have a continuum of bundles (say, all points in the positive quadrant) we will typically work with examples so nice that a simple mathematical expression will describe all the labels (all the info in the huge equivalent to Table 2). For instance, we may say

$$U(x_1, x_2) = 2x_1 + (x_2)^2. \tag{1}$$

This simply means that the preferences of this consumer may be represented by a set of labels where, for instance, the bundle $(1, 2)$ has label 6, the bundle $(3.5, 3)$ has label 16, etc. Make sure that you understand this, and always remember this meaning.

In any case, as when we had Table 1, once we have all these labels (the utility function) we know whether bundle (x_1, x_2) is preferred or not to bundle (y_1, y_2) by just looking at the label of the former, $U(x_1, x_2)$ and the label of

[1] In fact, to be more rigorous, we should write: $U : \mathbb{R}^2_+ \to \mathbb{R}$, where $U(x_1, x_2)$ is the image of bundle $(x_1, x_2) \in \mathbb{R}^2$.

the latter, $U(y_1, y_2)$ and seeing which one is higher. That is a very convenient way of representing preferences, one that we will use.

Is it always possible to do so, to represent preferences with labels, with a utility function? If not, what can go wrong? The answer: even though there are infinitely many real numbers, we may run out of them, out of labels! You should not worry much about this, but just for reference: this will never be a problem if the preferences of the consumer satisfy completeness, transitivity... and something called continuity. We will always work with such type of preferences. In fact, when we deal with perfectly divisible goods, we will always assume that the preferences are such that they could be represented with a *differentiable* utility function. So, good news: you will be able to take derivatives!

Now that we know what a utility function is, we have another (equivalent, of course) way of saying what an indifference curve is. Indeed, if the preferences are represented by a utility function, then an indifference curve is the set of all bundles that (occupy the same position in the ranking and so) sport the same label! That is, all the bundles (x_1, x_2) for which $U(x_1, x_2)$ is the same number.

Suppose that the consumer's preferences may be represented by the utility function (1). Let us obtain an indifference curve. For instance, the one that contains the bundle $(2, 3)$. That is, the set of all bundles that the consumer considers just as good as the bundle $(2, 3)$. We know that all of them must have the same label, which also coincides with the label of bundle $(2, 3)$. The latter is

$$U(2, 3) = 2 \times 2 + (3)^2 = 13.$$

Thus, the indifference curve we are looking for is the set of all bundles (x_1, x_2) that has label 13, that is, that satisfy

$$2x_1 + (x_2)^2 = 13. \tag{2}$$

Simple, right? For instance, the bundle in that indifference curve which has 4 units of of good 1 is the bundle $(4, \sqrt{5})$. Make sure that you see this.

Note that (2) is an equation, a condition that links two "unknowns": x_1 and x_2. If we give values to one of the unknowns, then we have an equation with only one unknown left. That is what we did to find that if the bundle (in that indifference curve, that is, satisfying (2)) with 4 units of of good 1 has $\sqrt{5}$ units of good 2. In general, the bundle in that curve with x_1 units of good Í is then the bundle, with

$$x_2 = \sqrt{13 - 2x_1}, \tag{3}$$

units of good 2. (We may detect a function there: a mapping from values of good 1 to values of good 2. Yes, that is another way of thinking of the indifference curve with label 13. But keep in mind that an indifference curve is still a set, not a function.)

Let us now talk about **marginal rate of substitution**. If the consumer is sitting at bundle $(2, 3)$, what is her willingness to trade good 2 for good 1? That is, how much is she willing to give up of good 2 per extra unit of good 1... (or how much would she require of good 2 per unit of good 1 she gives away...) when we talk about very small trades? Whatever the answer, that trade should not take the consumer off the indifference curve, right?: that trade should leave the consumer indifferent, so the new bundle should still have label 13. Thus, the question is how should x_2 change (per unit) as x_1 changes when we move (a little) along the indifference curve.

That, in mathematics, is called the derivative of x_2 with respect to x_1... for the expression (considered as the definition of the function we have mentioned) shown in (3). Stop here and check that you understand this point. You probably want to read again the last couple of paragraphs.

Very good, in this case, we compute that derivative and get

$$\frac{dx_2}{dx_1} = -\left(13 - 2x_1\right)^{-\frac{1}{2}},$$

which, ignoring the sign (remember, that is how we defined MRS: increase for decrease), at the bundle $(2, 3)$, that is, when $x_1 = 2$, is $\frac{1}{3}$.

There is a more sophisticated and direct way of computing this, and one that does not take us into thinking of an indifference curve as a function (which is not, I insist!). Again, the equation (2) is a condition that all bundles (in the indifference curve) with label 13 satisfy. (It is the same equation (3) written is a different way.) Now, the trade we are talking about will mean introducing some change of a small amount of good 1, call it dx_1, for some small amount of good 2, call it dx_2. Both will be small, but still we want to know what is the rate of these changes. A change in good 1 by itself will change the label of the resulting bundle. Likewise, a change in good 2 will result in another change in the label of the resulting bundle. In fact, using the language of mathematics, per unit of increase in good 1, a small increase in that good will change the label by an amount $\frac{\partial U(x_1,x_2)}{\partial x_1}$ per unit of good 1. Similarly, the change in good 2 will change the label by $\frac{\partial U(x_1,x_2)}{\partial x_2}$ per unit of good 2.

Therefore, the changes dx_1 and dx_2 of the amount of goods 1 and 2 respectively will together cause the label of the (resulting) bundle to change by

$$dx_1 \times \frac{\partial U(x_1, x_2)}{\partial x_1} + dx_2 \times \frac{\partial U(x_1, x_2)}{\partial x_2}. \tag{4}$$

So, if the change we are talking about need to leave the consumer in the same indifference curve (i.e., takes her to a bundle with the same label), then dx_1 and dx_2 must be such that the label does not change. That is, so that (4) equals 0. Or, rearranging terms,

$$\frac{dx_2}{dx_1} = -\frac{\frac{\partial U(x_1,x_2)}{\partial x_1}}{\frac{\partial U(x_1,x_2)}{\partial x_2}}.$$

In our example, the right hand side of the expression above is simply (the negative of)

$$\frac{2}{2x_2},$$

which evaluated at our bundle $(2, 3)$ is, indeed, $\frac{1}{3}$.

(This is no magic, but simple math. Making sure you understand what is going on here is a good math exercise. Also, as you could see, I did not care much for "marginal utility", and jargon like that. It is a fancy term, but we can do without it. It is better not to lose sight of the fact that we are talking about labels all the time, so that numbers like the 13 above carry no meaning by itself. The $\frac{1}{3}$ above, the MRS, **does** mean something, though.)

Note that, as illustrated in Figure 4, the MRS at one bundle is just the slope of the indifference curve (e.g., (3)) that contains the bundle evaluated at that bundle. It coincides with the rate of the partial derivatives of the utility function with respect to the two goods, again evaluated at the bundle.

But the most important thing is still the economic meaning: the MRS is the **rate** at which the consumer is **willing to trade** one good for another given what the consumer has.

What about the shape of the indifference curves, and so the way the MRS behaves as we move along one given indifference curve? The indifference curves represented in Figure 3, for instance, are convex. What this means is that the MRS (the slope of the indifference curve, as we know) gets smaller as we move **along** the indifference curve from one bundle to another bundle with more good 1 and less good 2 (sliding to the right along the indifference curve). Therefore, that consumer is willing to give less and less good 2 per extra unit of good 1 as she has less and less good 2 (and more and more good 1). It seems reasonable. But remember: tastes are tastes, and the consumer could perfectly well have preferences that resulted in indifference curves like the ones represented in Figure 5-A or Figure 5-B.

In Figure 5-A, the MRS behaves in the opposite way. That is, the more good 1 the consumer has (and the less good 2, other things equal) the larger the MRS. That is, the less inclined she is to give up good 1 for good 2 (the more good 2 per unit of good 1 she requires to stay indifferent), and vice versa. She seems to prefer either one good or the other, but cares less for mixing the two. In Figure 5-B, the MRS is constant along an indifferent curve. That is, the consumer is willing to give up a fixed amount of good 2 (say, α units, the slope of those straight lines) per unit of good 1 no matter what bundle she sits at. That is, from the point of view of that consumer, α units of good 2 perfectly substitute for one unit of good 1. That is why we would call these two goods **perfect substitutes** for **that consumer**.

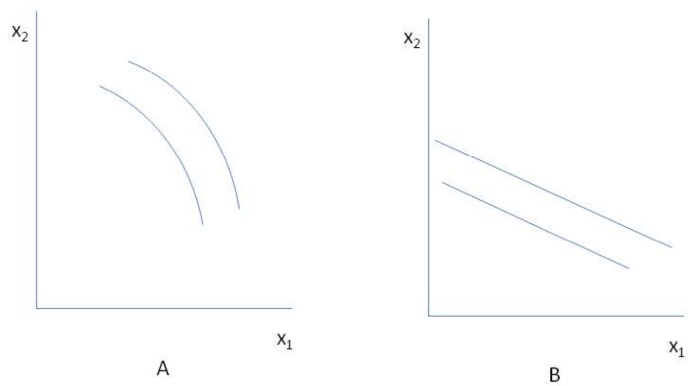

Figure 5

Now imagine that the indifference curves were L-shaped lines. Why would we call the gods **perfect complements** for the consumer?