Summary Notes for Week 7: Coalitional Games

Coalitional Games

Coalitional games are concerned with how to split the utility such that the split is fair and stable. Fairness is calculated using the shapley value while stability is calculated with the core.

Shapley Value

Rewards each agent according to his contribution. The payoff to each player is his marginal contribution to the coalition of his predecessors.

$$x_i = v(\{1, \ldots, i-1, i\}) - v(\{1, \ldots, i-1\}).$$

Properties of the Shapley value:

- Efficiency: $\Phi_1 + \ldots + \Phi_n = v(N)$
- Dummy: if i is a dummy, $\Phi_i = 0$
- Symmetry: if i and j are symmetric, $\Phi_i = \Phi_j$
- Additivity: $\Phi_i(\Gamma_1 + \Gamma_2) = \Phi_i((\Gamma_1) + \Phi_i(\Gamma_2)$

Image taken from https://www.cs.upc.edu/~mjserna/docencia/agt-miri/slides/AGT13-coop-GT.pdf

There is a single capitalist (c) and a group of 2 workers ($w1$ and $w2$).

The production function is such that total output is 0 if the firm (coalition) is composed only of the capitalist or of the workers (a coalition between the capitalist and a worker is required to produce positive output).

The production function satisfies:

- $F(c \cup w1) = F(c \cup w2) = 3$
- $F(c \cup w1 \cup w2) = 4$

First person	2nd addition	3rd addition	V(c)	V(w1 = w2)
c	c, w1	c, w1, w2	V(c) = 0 (since output is 0 if firm is composed only of capitalist)	V(w1) = V(c, w1) - V(c) = 3 - 0 = 3

c	c, w2	c, w1, w2	V(c) = 0	V(w1) = V(c, w1, w2) - V(c, w2) = 4 - 3 = 1
w1	c, w1	c, w1, w2	V(c) = V(c, w1) - V(w1) = 3 - 0 = 3	V(w1) = 0
w1	w1, w2	c, w1, w2	V(c) = V(c, w1, w2) - V(w1, w2) = 4 - 0 = 4 (V(w1, w2) is 0 since output is 0 if firm does not have a capitalist)	V(w1) = 0
w2	c, w2	c, w1, w2	V(c) = V(c, w2) - V(w2) = 3 - 0 = 3	V(w1) = V(c, w1, w2) - V(c, w2) = 4 - 3 = 1
w2	w1, w2	c, w1, w2	V(c) = V(c, w1, w2) - V(w1, w2) = 4 - 0 = 4	V(w1) = V(w2, w1) - V(w2) = 0 - 0 = 0

Shapley value of c = (0+0+3+4+3+4) / 3! = 7/3. [3!=6 which is the number of ways the people being added can be ordered]
Shapley value of w1 = w2 = (3+1+1) / 3! = 5/6.
Note that 7/3 + 5/6 + 5/6 = 4 (output when all three work together)

<u>Core</u>
The core of a game is the set of all stable outcomes, i.e., outcomes that no coalition wants to deviate from.

$x_1 + x_2 >= v(1, 2)$; $x_1 + x_3 >= v(1, 3)$; $x_2 + x_3 >= v(2, 3)$

$x_1 + x_2 + x_3 >=1$

Given

$N = 3$ and $v(1) = v(2) = v(3) = 0$, $v(1,2) = v(2,3) = v(3,1) = 0.8$, $v(1,2,3) = 1$.

- By definition, the core of this game is formed by a triplet $(x_1, x_2, x_3) \in R_+^3$ that satisfies:
- $x_i + x_j \geq 0.8$ for $i \neq j$
- $x_1 + x_2 + x_3 \geq 1$
- There is no triplet (x_1, x_2, x_3) that satisfies all inequalities. Then, the core is empty.

Given

$N = 3$ and $v(1) = v(2) = v(3) = 0$, $v(1,2) = v(2,3) = v(3,1) = 2/3$, $v(1,2,3) = 1$.

- By definition, the core of this game is formed by a triplet $(x_1, x_2, x_3) \in R^3_+$ that satisfies:
- $x_i + x_j \geq 2/3$ for $i \neq j$
- $x_1 + x_2 + x_3 \geq 1$
- Then, the core is a singleton with $(x_1, x_2, x_3) = (1/3, 1/3, 1/3)$.