Machine Learning SS2013

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Assignment 05

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Exercise 1

Task

Write the given linear program in the standard form by determining **A, b, c**.

Answer

Substitute x_3 with $x_3' = -x_3$, invert the first equation and subtracting 4 from the third equation

$$\begin{aligned} \text{Minimize} \quad x_1 - 2x_2 - 4x_3' \\ \text{subject to} \quad x_1 - x_2 \leq -1 \\ 3x_1 - 2x_3' \leq -1 \\ -2x_1 + 5x_3' \leq -4 \\ x_1, x_2, x_3' \leq 0 \end{aligned}$$

Standard form:

Minimize $c^T x$
subject to $Ax \leq b$
and $x \leq 0$

with

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & -2 \\ -2 & 0 & 5 \end{bmatrix}, b = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}, c = \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$$

Exercise 2

Task 1

Find the derivative of the Lagrangian with respect to the vector x and set it to zero ($\nabla_x L(x, \lambda_1, \lambda_2) = 0$). Solve it with respect to λ_2.

Solution

$$
\begin{aligned}
\nabla_x L(x, \lambda_1, \lambda_2) &= c^T + \lambda_1^T A + \lambda_2^T = 0 \\
&\Leftrightarrow c + A\lambda_1 + \lambda_2 = 0 \\
&\Leftrightarrow \lambda_2 = -c - A\lambda_1
\end{aligned}
$$

Task 2

Replace λ_2 from part 1 in $L(x, \lambda_1, \lambda_2)$ and simplify it.

Solution

$$
\begin{aligned}
L(x, \lambda_1, \lambda_2) &= c^T x + \lambda_1^T (Ax - b) + \lambda_2^T x \\
&= c^T x + \lambda_1^T (Ax - b) + (-c - A\lambda_1)^T x \\
&= c^T x + \lambda_1^T (Ax - b) - c^T x - \lambda_1^T A^T x \\
&= \lambda_1^T ((A - A^T)x - b)
\end{aligned}
$$

Exercise 3

Task

Derive the dual problem of the hard margin SVM.

Solution

$$
\begin{aligned}
f_0(w, b) &= \frac{1}{2}\|w\|^2 \\
f_i(w, b) &= 1 - y_i(w^T x_i - b)
\end{aligned}
$$

$$
\Rightarrow L(w, b, \lambda) = \frac{1}{2}\|w\|^2 + \sum_{i=1}^{m} \lambda_i \cdot (1 - y_i(w^T x_i - b))
$$

We now define $g(\lambda) := \inf\limits_{(w,b)} L(w, b, \lambda)$, which gives us the dual problem

Exercise 4

Task

For any $\gamma > 0$ the hyperplane defined by the solution (w, b) to **(I)** $\min\limits_{w,b} \frac{1}{2}\|w\|^2$, subject to $y_i \cdot (\langle w, x_i \rangle - b) \geq 1$, is the same as the hyperplane defined by the solution (w', b') defined by **(II)** $\min\limits_{w',b'} \frac{1}{2}\|w'\|^2$, subject to $y_i \cdot (\langle w', x_i \rangle - b') \geq \gamma$.

Answer

For $\gamma' > 0$, minimizing $\frac{1}{2}\|w\|^2$ is equivalent to minimizing $\gamma' \cdot \frac{1}{2}\|w\|^2$. This especially applies for $\gamma' = \frac{1}{2}\gamma^{-2}$.

Given the problem **(II)** we know that

$$\begin{aligned}
&y_i \cdot (\langle w', x_i \rangle - b) \geq \gamma \\
\Leftrightarrow &\gamma^{-1} \cdot y_i \cdot (\langle w', x_i \rangle - b') \geq 1 \\
\Leftrightarrow &y_i \cdot (\langle \gamma^{-1} \cdot w', x_i \rangle - \gamma^{-1} \cdot b') \geq 1.
\end{aligned}$$

We also can see that $\frac{1}{2}\|\gamma^{-1} \cdot w\|^2 = \frac{1}{2}\gamma^{-2} \cdot \frac{1}{2}\|w\|^2$. And using the assumption made at the beginning, we can conclude that **(II)** can be solved by the solution $(\gamma \cdot w, \gamma \cdot b)$.

Now we need to show that the two hyperplanes H, H' defined by (w, b) and $(\gamma \cdot w, \gamma \cdot b)$ coincide. Given a point $x \in H$ we know that

$$\begin{aligned}
&\langle w, x \rangle + b = 0 \\
\Leftrightarrow\ &\langle \gamma \cdot w, x \rangle + \gamma \cdot b = \gamma \cdot (\langle w, x \rangle + b) = \gamma \cdot 0 = 0.
\end{aligned}$$

This means that also $x \in H'$ and because x was arbitrary $H \subset H'$. And because the last implication is an equivalence, we know that $H = H'$.