Vertex-critical graphs in $2P_2$-free graphs

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Abstract

1 Introduction

[4]

1.1 Notation

For a vertex v, $N(v)$, $N[v]$ and $\overline{N[v]}$ denote the open neighbourhood, closed neighbourhood, and set of nonneighbours of v, respectively. We let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of G, respectively. We let $\alpha(G)$ denote the independence number of G. For subsets A and B of $V(G)$, we say A is *(anti)complete* to B if a is (non)adjacent to b for all $a \in A$ and $b \in B$. If $A = \{a\}$ then we say a is (anti)complete to B if $\{a\}$ is. If vertices u and v are adjacent we write $u \sim v$ and if they are nonadjacent we write $u \nsim v$.

2 Structure

We will make extensive use of the following lemma, in particular when $m = 1$ throughout the paper.

Lemma 2.1 ([6])**.** *Let G be a graph with chromatic number k. If G contains two disjoint m-cliques $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ such that $N(a_i) \setminus A \subseteq N(b_i) \setminus B$ for all $1 \leq i \leq m$, then G is not k-vertex-critical.*

Lemma 2.2. *If G is a k-vertex-critical $2P_2$-free graph, then for every nonuniversal vertex $v \in V(G)$, $\overline{N[v]}$ induces a connected graph with at least two vertices.*

Proof. Let G be a k-vertex-critical $2P_2$-free graph, $v \in V(G)$ be nonuniversal, and H be the graph induced by $\overline{N[v]}$. If $u \in \overline{N[v]}$ such that u is an isolated vertex in the graph induced by H, then $N(u) \subseteq N(v)$ contradicting G being k-vertex-critical by Lemma 2.1. Therefore, if H has at least two components, then each component has at least one edge and therefore taking an edge from each component induces a $2P_2$. This contradicts G being $2P_2$-free. $\square$

Lemma 2.3. *Let G be a $2P_2$-free graph that contains an induced $P_3 + \ell P_1$ for some $\ell \geq 1$. Let $S \cup \{v_1, v_2, v_3\} \subseteq V(G)$ induce a $P_3 + \ell P_1$ where $v_1v_2v_3$ is the induced P_3 and S contains the vertices in the ℓP_1. Then for ever vertex $u \in \overline{N[v_2]}$ such that u has a neighbour in S, u is complete to $\{v_1, v_3\}$.*

Proof. Let $s \in S$ and $u \in \overline{N[v_2]}$ with $u \sim s$. Let $i \in \{1, 3\}$. If $u \nsim v_i$, then $\{s, u, v_i, v_2\}$ induces a $2P_2$ in G, a contradiction. Thus, u is complete to $\{v_1, v_3\}$. $\square$

A part of this work will be exhaustively generating all k-vertex-critical graphs in certain families for small values of k. While there are excellent exhaustive generation algorithms that exist like the one introduced in [6] and then optimized and expanded in [5], these still rarely terminate for and values of $k \geq 6$. The small independence number of some of the critical graphs in our results allow us to use simpler exhaustive afforded by the implied bound (proven in the next lemma) on their order and the invaluable tool `nauty` [7] to generate all for values of k up to 7 in some cases.

Lemma 2.4. *If G is a k-vertex-critical graph with $\alpha(G) = c$ for some constant c, then $|V(G)| \leq c(k-1)+1$.*

Proof. Let G be a k-vertex-critical graphs with $\alpha(G) = c$, $v \in V(G)$, and let $n = |V(G)|$. Since G is k-vertex-critical, $G - v$ is $(k-1)$-colourable and has order $n-1$ and $\alpha(G-v) \leq c$. Since no colour-class of any $(k-1)$-colouring of $G - v$ can have more than c vertices, it follows that $G - v$ can have at most $(k-1)c$ vertices. Thus, $n = n - 1 + 1 \leq (k-1)c + 1$. $\square$

3 ($2P_2$, ℓ-*squid*)-free

The ℓ-*squid* is the graph obtained from a C_4 by adding ℓ leaves to one vertex, see Figure 1.

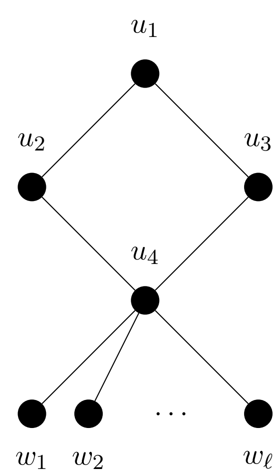

Figure 1: The ℓ-*squid* graph.

Lemma 3.1. *Let $\ell, k \geq 1$ and $c = ((\ell-1)(k-1)+1)$. If G is k-vertex-critical ($2P_2$, ℓ-squid)-free, then G is $(P_3 + cP_1)$-free.*

Proof. Let G be a k-vertex-critical ($2P_2$, ℓ-*squid*)-free graph for some $\ell \geq 0$ and let $c = (\ell-1)(k-1)+1$. Suppose by way of contradiction that G contains an induced $P_3 + cP_1$ with $\{v_1, v_2, v_3\}$ inducing the P_3 in that order and $S = \{s_1, s_2 \dots, s_c\}$ the cP_1 of the induced $P_3 + cP_1$.

By Lemma 2.2, each s_i must have a neighbour in $\overline{N[v_2]}$. Further, by Lemma 2.3, for every $u \in \overline{N[v_2]} - S$, such that $u \sim s_i$ for some $s_i \in S$, we must have that u is complete to $\{v_1, v_3\}$. Therefore, each $u \in \overline{N[v_2]} - S$ has at most $\ell - 1$ neighbours in S, else u, v_1, v_2, v_3 together with any ℓ of u's neighbours in S would induce an ℓ-*squid*. Let $U = \{u_1, u_2, \dots u_m\}$ be a subset of $N(S) \cap \overline{N[v_2]}$ such that $(\bigcup_{i=1}^{m} N(u_i)) \cap S = S$ and such that $N(u_i) \cap S \not\subseteq N(u_j) \cap S$ for all $i \neq j$. Such a set U exists since each vertex in S has at least one neighbour in $N(S) \cap \overline{N[v_2]}$. Since each u_i can have at most $\ell - 1$ neighbours in S we must have that $|U| \geq k$ by the Pigeonhole Principle. Let $u_i, u_j \in U$ for $i \neq j$ and without loss of generality let $s_i \in N(u_i) - N(u_j)$ and $s_j \in N(u_j) - N(u_i)$. If $u_i \nsim u_j$, then $\{s_i, u_i, s_j, u_j\}$ induces a $2P_2$ in G, a contradiction. Therefore, $u_i \sim u_j$ for all $i \neq j$ and therefore U induces a clique with at least k vertices in G. However, U is a proper subgraph of G and requires at leat k colours, which contradicts G being k-vertex-critical. Therefore, G must be $(P_3 + cP_1)$-free. □

The following theorem follows directly from Lemma 3.1 and Theorem 6.11.

Theorem 3.2. *There are only finitely many k-vertex-critical ($2P_2$, ℓ-squid)-free graphs for all $k, \ell \geq 1$.*

Since *chair* is an induced subgraph of 1-*squid*, *claw* + P_1 is an induced subgraph of 3-*squid*, and more generally $K_{1,\ell} + P_1$ is an induced subgraph of ℓ-*squid* for all $\ell \geq 1$, we get the following immediate corollaries of Theorem 3.2

Corollary 3.3. *There are only finitely many k-vertex-critical ($2P_2$, chair)-free graphs for all $k \geq 1$.*

Corollary 3.4. *There are only finitely many k-vertex-critical ($2P_2$, claw $+ P_1$)-free graphs for all $k \geq 1$.*

Corollary 3.5. *There are only finitely many k-vertex-critical ($2P_2$, $K_{1,\ell} + P_1$)-free graphs for all $k \geq 1$.*

We note as well that *banner* is isomorphic to 1-*squid*, so our results also imply that there are only finitely many k-vertex-critical ($2P_2$, *banner*)-free graphs for all k. This result is not new though as it was recently shown in [2] that every k-vertex-critical (P_5, *banner*)-free graph has interdependence number less than 3 and therefore there only finitely many such graphs. However, a special case Lemma 3.1 implies that every k-vertex-critical ($2P_2$, *banner*)-free graph is $(P_3 + P_1)$-free and therefore by Theorem 3.1 in [3], it follows that every such graph has independence number at most 2. Thus our results give an alternate short proof (in fact the proof of Lemma 3.1 only requires the first three sentences if restricted to *banner*-free graphs) of a slightly weaker version of the result in [2].

4 ($2P_2$, *bull*)-free

Lemma 4.1. *Let $k \geq 1$. If G is k-vertex-critical ($2P_2$, bull)-free, then G is $(P_3 + P_1)$-free.*

Proof. Let G be a k-vertex-critical $(2P_2, \mathit{bull})$-free graph and by way of contradiction let $\{v_1, v_2, v_3, s_1\}$ induce a $P_3 + P_1$ in G where $\{v_1, v_2, v_3\}$ induces the P_3, in that order, of the $P_3 + P_1$. We will show that v_1 and v_3 are comparable which will contradict Lemma 2.1.

We first show that s_1 has no neighbours in $(N(v_1) \cap N(v_2)) - N(v_3)$ (and by symmetry no neighbours in $(N(v_3) \cap N(v_2)) - N(v_1)$). Suppose $n \in (N(v_1) \cap N(v_2)) - N(v_3)$ such that $s_1 \sim n$. Now, $\{s_1, v_1, v_2, v_3, n\}$ induces a *bull* in G, a contradiction. If s_1 is not comparable with v_1, then it must have a neighbour u_1 such that $u_1 \nsim v_1$. Since s_1 has no neighbours in $(N(v_1) \cap N(v_2)) - N(v_3)$, it must be that $u_1 \nsim v_3$.

If v_1 and v_3 are not comparable, then they must have distinct neighbours. Let v_1' be such that $v_1 \sim v_1'$ and $v_1' \nsim v_3$ and v_3' be such that $v_3 \sim v_3'$ and $v_3' \nsim v_1$.

Suppose $v_1' \in N[v_2]$ or $v_3' \in N[v_2]$. Without loss of generality assume $v_1' \in N[v_2]$. Since we have already argued above that s_1 is anticomplete to $(N(v_2) \cup N(v_1)) - N(v_3)$, it follows that $s_1 \nsim v_1'$. Thus we must have $u_1 \sim v_1'$, else $\{s_1, u_1, v_1, v_1'\}$ induces a $2P_2$. But now, $\{s_1, u_1, v_1', v_2, v_3\}$ induces a *bull*, a contradiction. Therefore, $v_1' \in \overline{N[v_2]} - \{s_1, s_2\}$ and $v_3' \in \overline{N[v_2]} - \{s_1, s_2\}$. Further, by Lemma 2.3 we must have $\{v_1', v_3'\}$ is anticomplete to s_1. Now, $v_1' \sim v_3'$, else $\{v_1, v_1', v_3, v_3'\}$ induces a $2P_2$. Further u_1 must be complete to $\{v_1', v_3'\}$, else $\{s_1, u_1, v_1, v_1'\}$ or $\{s_1, u_1, v_3, v_3'\}$ will induce a $2P_2$ in G. But now, $\{u_1, v_1', v_3', v_1, v_3\}$ induces a *bull* in G, a contradiction. Thus in this case, v_1 and v_3 are comparable, contradicting G being k-vertex-critical.

□

Should probably add a picture or two to aid with the proof above.

5 $(2P_2, \overline{\mathit{diamond} + P_1})$-free

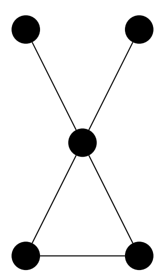

Figure 2: A thing

Lemma 5.1. *Let $\ell, k \geq 1$. If G is k-vertex-critical $(2P_2, \overline{\mathit{diamond} + P_1})$-free, then G is $(\ell$-squid$)$-free.*

Proof. Let G be a k-vertex-critical $(2P_2, \overline{K_3 + 2P_1})$-free graph. Suppose by way of contradiction that G contains an induced $\ell - \mathit{squid}$ with the same construction as Figure 1 for some $2\ell \geq 1$. u_2, u_3 must have unique neighbours, or else they are comparable and the graph is not k-vertex-critical, a contradiction. Further, u_2', u_3' cannot be adjacent to u_4 else it forms the forbidden graph. We will call these vertices u_2', u_3'. Now, $\{u_2, u_3, u_2', u_3'\}$ induce a $2P_2$ unless $u_2' \sim u_3'$. Then $\{u_2', u_3', w_i, u_4\}$ for any $w_i \in w$ induces a $2P_2$ unless u_2', u_3' are combined complete to w. Further, $\{u_1, u_2, u_3', w_i\}$ and $\{u_1, u_3, u_2', w_i\}$ both induce a $2P_2$ unless $u_3', u_2' \sim u_1$. Then by pigeon-hole principle, either u_3' or u_2' are adjacent to ℓ vertices, inducing the forbidden graph. Thus, $(2P_2, \overline{K_3 + 2P_1})$-free graphs are finite. □

6 Old Stuff I'm keeping just in case

Lemma 6.1. *Let $k \geq 1$. If G is k-vertex-critical ($2P_2$, bull)-free, then G is $(P_3 + 2P_1)$-free.*

Proof. Let G be a k-vertex-critical ($2P_2$, *bull*)-free graph and by way of contradiction let $\{v_1, v_2, v_3, s_1, s_2\}$ induce a $P_3 + 2P_1$ in G where $\{v_1, v_2, v_3\}$ induces the P_3 in that order and $\{s_1, s_2\}$ induces the $2P_1$ of the $P_3 + P_1$. We will show that $N(s_1) \subseteq N(s_2)$ or $N(s_2) \subseteq N(s_1)$ and therefore contradicts Lemma 2.1.

We first show that for $i \in \{1, 2\}$, s_i has no neighbours in $(N(v_1) \cap N(v_2)) - N(v_3)$ (and by symmetry no neighbours in $(N(v_3) \cap N(v_2)) - N(v_1)$). Suppose $n \in (N(v_1) \cap N(v_2)) - N(v_3)$ such that $s_i \sim n$. Now, $\{s_i, v_1, v_2, v_3, n\}$ induces a *bull* in G, a contradiction. Therefore the neighbours of s_i must belong to one of the following three sets, $\overline{N[v_2]} - \{s_1, s_2\}$, $N(v_2) - (N(v_1) \cup N(v_3)$ and $N(v_1) \cap N(v_2) \cap N(v_3)$.

If $N(s_1) \not\subseteq N(s_2)$ and $N(s_2) \not\subseteq N(s_1)$, there must be a $u_1 \in N(s_1) \underline{\subseteq} \overline{N(s_2)}$ and $u_2 \in N(s_2) \subseteq N(s_1)$. We note that just as in the proof of Lemma 3.1, for every $u \in \overline{N[v_2]} - S$, such that $u \sim s_i$ for some $i \in \{1, 2\}$, we must have that u is complete to $\{v_1, v_3\}$, else $\{u, s_i, v_1, v_2\}$ or $\{u, s_i, v_3, v_2\}$ induces a $2P_2$ in G. We now consider BLANK cases. We note that in each of the cases, we must have $u_1 \sim u_2$, or else $\{s_1, u_1, s_2, u_2\}$ induces a $2P_2$ in G.

Case 1: $u_1, u_2 \in \overline{N[v_2]} - \{s_1, s_2\}$.

From above, we also know that $\{u_1, u_2\}$ is complete to $\{v_1, v_3\}$ and therefore, $\{s_1, s_2, u_1, u_2, v_1\}$ induces a *bull*, a contradiction.

Case 2: $u_1, u_2 \in N(v_2)$.

In this case, $\{s_1, s_2, u_1, u_2, v_2\}$ induces a *bull*, a contradiction.

Thus the only remaining case is when one of the u_i's is in $\overline{N[v_2]} - \{s_1, s_2\}$ and the other is in $N(v_2)$. Without loss of generality we may assume the following.

Case 3: $u_1 \in N(v_2)$ and $u_2 \in \overline{N[v_2]} - \{s_1, s_2\}$.

In this case we now must consider the fact that v_1 and v_3 must have distinct neighbours, else v_1 and v_3 are comparable. Let v_1' be such that $v_1 \sim v_1'$ and $v_1' \nsim v_3$ and v_3' be such that $v_3 \sim v_3'$ and $v_3' \nsim v_1$.

Suppose $v_1' \in N[v_2]$ or $v_3' \in N[v_2]$. Without loss of generality assume $v_1' \in N[v_2]$. Since we have already argued above that s_1 is anticomplete to $(N(v_2) \cup N(v_1)) - N(v_3)$, it follows that $s_1 \nsim v_1'$. Thus we must have $u_1 \sim v_1'$, else $\{s_1, u_1, v_1, v_1'\}$ induces a $2P_2$. But now, $\{s_1, u_1, v_1', v_2, v_3\}$ induces a *bull*, a contradiction. Therefore, $v_1' \in \overline{N[v_2]} - \{s_1, s_2\}$ and $v_3' \in \overline{N[v_2]} - \{s_1, s_2\}$. Further, since every vertex in $v_3' \in \overline{N[v_2]} - \{s_1, s_2\}$ adjacent to s_1 or s_2 must be complete to $\{v_1, v_3\}$ it follows that $\{v_1', v_3'\}$ is anticomplete to $\{s_1, s_2\}$. Now, $v_1' \sim v_3'$, else $\{v_1, v_1', v_3, v_3'\}$ induces a $2P_2$. Further u_1 must be complete to $\{v_1', v_3'\}$, else $\{s_1, u_1, v_1, v_1'\}$ or $\{s_1, u_1, v_3, v_3'\}$ will induce a $2P_2$ in G. But now, $\{u_1, v_1', v_3', v_1, v_3\}$ induces a *bull* in G, a contradiction. Thus in this case, v_1 and v_3 are comparable, contradicting G being k-vertex-critical.

Since we already argued above that any vertex in $\overline{N[v_2]}$ adjacent to s_1 or s_2 must be complete to $\{v_1, v_3\}$ we must have v_1' is anticomplete to $\{s_1, s_2\}$. Thus, $v_1' \neq u_2$.

If $u_1 \sim v_1$, then $\{s_1, s_2, u_1, u_2, v_1\}$ induces a *bull*, a contradiction. Therefore, $u_1 \in N(v_2) - (N(v_1) \cup N(v_3))$.

From Lemma 2.1, s_2 must have a neighbour that is not adjacent to v_1, else $N(s_2) \subseteq N(v_1)$. Let u_1' be such a neighbour of s_2. Since every every vertex in $\overline{N[v_2]} - \{s_1, s_2\}$ adjacent to s_2 must also be complete to $\{v_1, v_3\}$ from a above, it follows that u_1' must be in $N(v_2)$. Since $u_1 \notin N(s_2)$, $u_1 \neq u_1'$. If $u_1' \sim u_1$, then $\{u_1, u_1', v_2, s_2, v_1\}$ induces a *bull*, a contradiction. Therefore, $s_1 \sim u_1'$ or else $\{s_1, u_1, s_2, u_1'\}$ induces a $2P_2$.

Again from Lemma 2.1, s_1 must have a neighbour that is not adjacent to v_2, else $N(s_1) \subseteq N(v_2)$. Let u_2' be such a neighbour of s_1. We note that u_2' must also be complete to $\{v_1, v_3\}$ and so if $u_2 \sim u_2'$, then $\{s_1, s_2, u_2, u_2', v_1\}$ induces a *bull*. Therefore, we must have $s_2 \sim u_2'$, else $\{s_1, u_1, s_2, u_1'\}$ induces a $2P_2$.

Thus, if s_1 and s_2 are not comparable, we have $N(s_1) \cap (\overline{N[v_2]} - \{s_1, s_2\}) \subset N(s_2) \cap (\overline{N[v_2]} - \{s_1, s_2\})$ and $N(s_2) \cap N(v_2) \subset N(s_1) \cap N(v_2)$.

FROM HERE¡ Change to be a $P_3 + 3P_1$ and bring s_3 which must contain the neighbourhoods of both s_1 and s_3 (I think!)

Now s_3 must have a neighbour

□

Lemma 6.2. *If G is k-vertex-critical $(2P_2, claw + P_1)$-free, then G is $(P_3 + 3P_1)$-free.*

Proof. Suppose by way of contradiction that G contains an induced $P_3 + 3P_1$, and let $\{v_1, v_2, v_3\} \cup \{s_1, s_2, s_3\}$ induce a $P_3 + 2P_1$ in G with $v_1v_2v_3$ the induced P_3, in that order. Since $\overline{N[v_2]}$ must be connected from Lemma 2.2, it follows that s_1 has a neighbour in $\overline{N[v_2]}$, say u_1. If $u_1 \nsim v_i$ for $i \in \{1, 3\}$, then $\{s_1, u_1, v_i, v\}$ induces a $2P_2$ in G, a contradiction. Thus, u_1 is complete to $\{v_1, v_3\}$. If $s_i \nsim u_1$ for $i \in \{2, 3\}$, then $\{s_1, u_1, v_1, v_3, s_i\}$ induces a $claw + P_1$ in G, a contradiction. Thus, u_1 is complete to $\{s_2, s_3\}$. But now, $\{s_1, s_2, s_3, u_1, v_2\}$ induces a $claw + P_1$, a contradiction. Therefore, G must be $(P_3 + 3P_1)$-free. □

6.1 $(2P_2, K_3 + P_1, P_4 + P_1)$-free graphs

IDEA: If G is also P_3+2P_1-free then done by my previous paper. Thus, contains an induced P_3+2P_1. By Lemma 6.3, nonneighbourhood of s_1 (s_1, s_2 are the $2P_1$ vertices), its nonneighbourhood must be a complete bipartite graph, and thus s_2 must be complete to v_1 and v_2 (the two leaves of the P_3), contradicting the induced $P_3 + 2P_1$!.

Throughout this section, assume G is a k-vertex-critical non-complete $(2P_2, K_3 + P_1, P_4 + P_1)$-free graph.

For any $v \in V(G)$, G partitions into $\{v\}$, $N(v)$, and $\overline{N[v]}$. $N(v)$ further partitions into N_1, N_1', N_2. $\exists u_1 \in \overline{N[v]}$ such that $u_1 \sim N_1$, and $\exists u_2 \in \overline{N[v]}$ such that $u_2 \sim N_2$.

Lemma 6.3. *For every nonuniversal vertex $v \in V(G)$, $\overline{N[v]}$ induces a complete bipartite graph $K_{n,m}$ some $n, m \geq 1$.*

Proof. Let v be a nonuniversal vertex in G and let H be the subgraph of G induced by $\overline{N[v]}$. We first note that If $S \subseteq \overline{N[v]}$ induces P_4 or K_3, then $\{v\} \cup S$ induces a $P_4 + P_1$ or $K_3 + P_1$, a contradiction. Thus, H is $(P4, K_3)$-free. Since H is P_4-free it is either a join or disjoint union of graphs (since P_4-free g raphs are co-graphs). By Lemma 2.2, H must be connected, so it therefore must be the join of graphs. Further, since H is K_3-free and the join of graphs, it must be a complete bipartite graph. □

Now let S be the maximum independent set of G, and $v \in S$. Let $S - v = \{s_1, s_2, \ldots, s_\ell\}$. Also, let $V(G) - (S \cup N(v)) = \{y_1, y_2, \ldots, y_j\}$. Note that if y_1 is complete to $N(v)$, then $N(v) \subseteq N(y_1)$, contradicting the k-vertex-criticality of G by Lemma 2.1. So let $N' \subseteq N(v)$ be the set $\overline{N[y_1]} \cap N(v)$. Note also that we may assume $\ell \geq 2$ otherwise we are done by Ramsey's Theorem.

Lemma 6.4. *$S - v$ is complete to N'.*

Proof. Suppose there is a $s_i \in S - v$ and $n \in N'$ such that $s_i \nsim n$. From Lemma 6.3, $S - v$ is complete to $V(G) - (S \cup N(v))$, so $y_1 \sim s_i$, and by definition $y_1 \nsim n$. Therefore, $\{y_i, s_i, n, v\}$ induces a $2P_2$ in G, a contradiction. Thus, $S - v$ is complete to N'. □

We originally thought the next result was *anticomplete*, so we need to figure out how to proceed. One option is to also forbid the dart as then we must a dart induced by s_1, s_1's unique neighbour in $N(v)$ and y_1 and y_2. Thus y_1 is the only y_i and a similar argument gets the $S - v$ set down to one element. May also be possible if chair is forbidden as well.

Lemma 6.5. *y_i is complete to N' $\forall i \in \{2, \dots, j\}$.*

Proof. Let $2 \leq i \leq j$ and let $n \in N'$. By Lemma 6.3, $y_i \in \overline{N[y_1]}$, so $\{v, n, y_i\} \subseteq \overline{N[y_1]}$. Since $v \sim n$ and $v \nsim y_i$ by definition, it follows that $n \sim y_i$ otherwise $\overline{N[y_1]}$ would not be complete bipartite, contradicting Lemma 6.3. Therefore, y_i is complete to N'. □

So $N(Y_i) \subseteq N(Y_1) \forall i \in \{2, \dots, j\}$, so Y_i, Y_1 are comparable vertices, making this not vertex critical and thus a contradiction. Further, since this applies for all $i \geq 2$, then the $|Y| \leq 1$. We also know (somehow) that this applies to the set of $\{u_1, \dots, u_l\}$ so they are limited to ≤ 1. Now we can use Ramsay's Theorem to identify that since there is a maximum independent set, there is a finite amount of graphs.

6.2 $(2P_2, K_3 + P_1, claw + P_1)$-free

Throughout this section, assume G is a k-vertex-critical non-complete $(2P_2, K_3 + P_1, \text{claw} + P_1)$-free graph.

Lemma 6.6. *For every nonuniversal vertex $v \in V(G)$, $\overline{N[v]}$ induces a P_j or C_m for $2 \leq j \leq 4$ and $4 \leq m \leq 5$.*

Proof. Let v be a nonuniversal vertex in G and let H be the subgraph of G induced by $\overline{N[v]}$. If $S \subseteq V(H)$ such that S induces a K_3 or claw, then $S \cup \{v\}$ induces a claw+P_1 or $K_3 + P_1$, a contradiction. Therefore, H is $(K_3 + P_1, \text{claw} + P_1)$-free. Suppose there is a vertex $u \in \overline{N[v]}$ such that $|N(u) \cap \overline{N[v]}| \geq 3$ and let $\{u_1, u_2, u_3\} \subseteq N(u) \cap \overline{N[v]}$. If the u_i's are all pairwise disjoint, then $\{u, u_1, u_2, u_3\}$ induces a claw, a contradiction. So there must be at least one edge in the graph induced by $\{u_1, u_2, u_3\}$, without loss of generality say $u_1 \sim u_2$. But now $\{u, u_1, u_2\}$ induces a K_3, a contradiction. Thus, we must have $|N(u) \cap \overline{N[v]}| \leq 2$ for all $u \in \overline{N[v]}$ and thus $\Delta(H) \leq 2$. From Lemma 2.2, H must be connected and have at least two vertices, so H must be a P_j or a C_m. Since H is $(2P_2, K_3 + P_1)$-free, it follows that $2 \leq j \leq 4$ and $4 \leq m \leq 5$. □

Corollary 6.7. *$\alpha(G) \leq 3$.*

Proof. Let $S = \{s_1, s_2, \dots s_\ell\}$ be a maximum independent set in G. By Lemma 6.6, it follows that the graph induced by $\overline{N[s_1]}$ must have independence number at most 2. Since $s_i \in \overline{N[s_1]}$ for all $2 \leq i \leq \ell$, it follows that $\ell \leq 3$. □

Theorem 6.8. *There are only finitely many k-vertex-critical $(2P_2, K_3 + P_1, claw + P_1)$-free graphs for any given k.*

Proof. Fix k and let G be a k-vertex-critical $(2P_2, K_3 + P_1, \text{claw} + P_1)$-free graph. Since G is k-vertex-critical, $\omega(G) \leq k$. From Corollary 6.7, $\alpha(G) \leq 3$. Thus, by Ramsey's Theorem, G has order bounded above by $R(4, k + 1)$, a constant. □

Questions: (1) can we remove the $K_3 + P_1$-free restriction and still get only finitely many of these graphs? (2) can we show that $\overline{N[v]}$ cannot induce a cycle and therefore every such critical graph has independence number at most 2? (3) Can we exhaustively generate all for small k even without the further restriction on independence number give the -k argument in nauty v2.8.6?

6.3 $(2P_2, P_4 + P_1, chair, bull)$-free

NOTE!! The main result of this section is now a corollary of the $P_4 + \ell P_1$ stuff.

Throughout this section let G be a $(2P_2, P_4 + P_1, chair, bull)$-free k-vertex-critical graph. Let S be the maximum independent set of G, $v \in S$, and let $S - v = \{s_1, s_2, \ldots, s_\ell\}$. Let A be $\overline{N[v]} - S$.

Lemma 6.9. $\alpha(G) \leq 2$.

Proof. Suppose by way of contradiction that $\alpha(G) \geq 3$, so that $\ell \geq 2$. Since $s_1 \nsim v$ by definition, we must have $v_1 \in N(v)$ such that $s_1 \nsim v_1$, otherwise we would have $N(v) \subseteq N(s_1)$, contradicting G being k-vertex-critical by Lemma 2.1. Further, by Lemma 2.2, s_1 must have a neighbour $u \in A$, otherwise s_1 would be an isolated vertex in the graph induced by $\overline{N[v]}$. If $u \nsim v_1$, then $\{s_1, u, v_1, v\}$ induces a $2P_2$ in G, a contradiction. Therefore, $u \sim v_1$. Now, $s_1 \nsim s_2$ and $s_2 \nsim v$ by definition, so if $u \nsim s_2$ and $v_1 \nsim s_2$, $\{s_2, s_1, u, v_1, v\}$ induces a $P_4 + P_1$ in G, a contradiction. Therefore, $u \sim s_2$ or $v \sim s_2$. If exactly one of u and v is adjacent to s_2, then $\{s_2, s_1, u, v_1, v\}$ induces a chair in G, a contradiction. Therefore, both u and v must be adjacent to s_2. However, we now have $\{s_2, s_1, u, v_1, v\}$ inducing a bull in G, a contradiction. This completes the proof. □

Theorem 6.10. *There are only finitely many k-vertex-critical $(2P_2, P_4 + P_1, chair, bull)$-free graphs for any given k.*

Proof. Fix k and let G be a k-vertex-critical $(2P_2, P_4 + P_1, \text{chair}, \text{bull})$-free graph. Since G is k-vertex-critical, $\omega(G) \leq k$. From Lemma 6.9, $\alpha(G) \leq 2$. Thus, by Ramsey's Theorem, G has order bounded above by $R(3, k + 1)$, a constant. □

6.4 $(2P_2, P_4 + \ell P_1, m$-squid$)$-free

For this section we need the following result:

Theorem 6.11 ([1])**.** *There are only finitely many k-vertex-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.*

Theorem 6.12. *There are only finitely many k-vertex-critical $(2P_2, P_4 + \ell P_1, m$-squid$)$-free for all k, ℓ, and m.*

Proof. Fix k, ℓ, and m and let G be a k-vertex-critical $(2P_2, P_4 + \ell P_1, m\text{-squid})$-free graph. If G is $(P_3 + cP_1)$-free for $c = \ell + m$, then we are done by Theorem 6.11. Thus we may assume G has an induced $P_3 + cP_1$. Let v be the centre of the P_3, v_1 and v_2 be its leaves, and $s_1, s_2 \ldots s_c$ be the c isolated vertices of an induced $P_3 + cP_1$ in G. Let $S = \{s_1, s_2, \ldots, s_c\}$. If $N(s_1) \subseteq N(v)$, then we contradict G being k-vertex-critical by Lemma 2.1. So there must be a vertex $u \in V(G) - (\overline{N[v]} \cup \{v_1, v_2\})$ such that $s_1 \sim u$. If $u \nsim v_1$ or $u \nsim v_2$, then $\{s_1, u, v_1, v\}$ or $\{s_1, u, v_2, v\}$ induces a $2P_2$ in G, a contradiction. Therefore, $u \sim v_1$ and $u \sim v_2$. If u has at least ℓ nonneighbours in $S - s_1$, then $\{s_1, u, v_1, v\}$ together with any ℓ nonneighbours of u in $S - s_1$ induces a $P_4 + \ell P_1$, a contradiction. Thus, u has at least $c - \ell = m$ neighbours in S (including s_1). But now $\{u, v, v_1, v_2\}$ and any m of u's neighbours in S induce an m-squid, a contradiction. This completes the proof. □

Since both $K_{1,m} + P_1$ and chair are an induced subgraphs of the m-squid, we get the following corollarie immediately.

Corollary 6.13. *There are only finitely many k-vertex-critical $(2P_2, P_4 + \ell P_1, K_{1,m} + P_1)$-free for all k, ℓ, and m.*

Corollary 6.14. *There are only finitely many k-vertex-critical $(2P_2, P_4 + \ell P_1$, chair$)$-free for all k, ℓ,*

Question: Can we get rid of the $2P_2$-free restriction in the theorem?

7 Conclusion

References

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