Infinite families of k-vertex-critical (P_5, C_5)-free graphs

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1 New infinite families of k-vertex-critical graphs

There are surprisingly few such constructions for H-free graphs when H is the disjoint union of paths. The most commonly used is the $2P_2$-free infinite family of k-vertex-critical graphs from [9], as this family has turned out to have much richer structure than was originally identified. Another very recent construction is Chudnovsky et al.'s [6] infinite family of 4-vertex-critical P_7-free graphs. My careful analysis of these two constructions with Hoàng led to the following generalization:

Let $G(q,k)$ be a graph on vertex set $\{v_0, v_1, ..., v_{kq}\}$ where, with each integer taken modulo $kq+1$, the neighbourhood of vertex v_i is $\{v_{i-1}, v_{i+1}\} \cup \{v_{i+kj+m} : m = 2, 3, ...k-1 \text{ and } j = 0, 1, ..., q-1\}$.

The graph $G(q,k)$ can be shown to be $(k+1)$-vertex-critical for all $q \geq 1$ and $k \geq 3$ and $\{G(q,3) : q \geq 1\}$ is equivalent to the infinite family in [**?**] and $\{G(q,4) : q \geq 1\}$ is equivalent to the infinite family in [9]. Further, $G(q,k)$ is $(2P_2, K_3+P_1)$-free (where K_n is the complete graph of order n) for all $k \geq 4$ thus answering an open question posed in [9]. Our new construction is also very natural in the sense that $G(1,k) \cong K_{k+1}$ and $G(2,k) \cong \overline{C_{2k+1}}$ for all k, the prototypical $(k+1)$-vertex-critical graphs from the Strong Perfect Graph Theorem [7].

For a given q and k and for $0 \leq i \leq k-1$, let $V_i = \{v_t : t \equiv\ i \pmod k\}$. It is clear that the V_i's partition the vertex set of $G(q,k)$.

Lemma 1.1. *For $1 \leq i \leq k$, V_i is a stable set of $G(q,k)$ and the only edge in V_0 is v_0v_{qk}.*

Proof. Since $2 \leq kj+m \leq k(q-1)$, it follows that $i \not\equiv i+kj+m \pmod{qk+1}$ for all i. Further, $qk+1 \geq 3$ for all q, k for which inf qk is defined, so the only edge between v_i and v_{i+1} (respectively v_i and v_{i-1}) where $i \equiv i+1 \pmod{qk+1}$ (respectively $i \equiv i-1 \pmod{qk+1}$) is v_0v_{qk}. □

Lemma 1.2. *For all $k \geq 4$ and $q \geq 1$, $G(q,k)$ is $2K_2$-free.*

Proof. Let $G = G(q,k)$ and suppose G is not $2K_2$-free. By symmetry, we may suppose without loss of generality that v_0 belongs to an induced $2K_2$. Let $v_\ell, x, y \in V(G)$ such that $\{v_0, v_\ell, x, y\}$ induces a $2K_2$ in G such that $v_0 \sim v_\ell$ and $x \sim y$. It is clear by the definition of G that $N(v_0) = \{v_1, v_{qk}\} \cup V_2 \cup V_3 \cup \cdots \cup V_{k-1}$. Therefore, if $\{x,y\} \not\subseteq (V_0 \setminus \{v_0, v_{qk}\}) \cup V_1$, then v_0 will be adjacent to x or y, contradicting our assumption. Since $x \sim y$, $\{x,y\} \subseteq (V_0 \setminus \{v_0, v_{qk}\}) \cup (V_1 \setminus \{v_1\})$, and both $V_1 \setminus \{v_1\}$ and $V_0 \setminus \{v_0, v_{qk}\}$ are stable sets from Lemma 1.1, it follows that (without loss of generality) $x \in V_0 \setminus \{v_0, v_{qk}\}$ and $y \in V_1 \setminus \{v_1\}$. There are now only three cases to consider.

Case 1: $v_\ell \in V_2 \cup V_3 \cup \cdots \cup V_{k-2}$. Let $\ell' = \ell \pmod k$. By assumption, $2 \leq \ell' \leq k-2$. For $v_{hk} \in V_0$ where $\ell < hk \leq qk$, we have that $hk = \ell + k(h-1) + (k-\ell')$, so $v_{hk} \in N(v_\ell)$. For $v_{hk} \in V_0$ where $qk < hk < \ell$, we have that $hk = \ell' + k(h-1) + (k-\ell'+1) \pmod{qk+1}$, so $v_{hk} \in N(v_\ell)$. It follows that $V_0 \subseteq N(v_\ell)$ and, thus, $v_\ell \sim x$. This contradicts our assumption on the induced $2K_2$ in G.

Case 2: $v_\ell \in V_{k-1}$. Similarly to the previous case, we find that $V_1 \subseteq N(v_{ell})$ since

$$\{v_{(k-1)+qj+2} : j = 0, 1, \ldots, q-1\} \cup \{v_{(k-1)+qj+3} : j = 0, 1, \ldots, q-1\} \subseteq N(y)$$

by definition. Therefore, $y \sim v_\ell$ which again contradicts our assumption on the induced $2K_2$ in G.

Case 3: $v_\ell \in \{v_{qk}, v_1\}$. If $v_\ell = v_1$, then since $1 + jk + m \mod qk + 1 = 1 + jk + m$ for all $0 \leq j \leq q-1$ and $2 \leq m \leq k-1$, we can obtain all v_h such that $v_h \in V_0$ in the set $\{v_{1+(q-1)j+(k-1)} : j = 0, 1, \ldots, q-1\}$. Thus, Since $1 + jk + m \mod qk + 1 = 1 + jk + m$ for all $0 \leq j \leq q-1$ and $2 \leq m \leq k-1$, we can obtain all v_h such that $v_h \in V_{k-1}$ in the set $\{v_{1+(q-1)j+(k-2)} : j = 0, 1, \ldots, q-1\}$. Thus, $x \sim v_\ell = v_1$ which contradicts our assumption on the induced $2K_2$ in G. Similarly, for $v_\ell = v_{qk}$, we get that $y \sim v_\ell$ which again contradicts our assumption on the induced $2K_2$ in G.

All three cases together show that G is $2K_2$-free. □

Lemma 1.3. *For all $k \geq 5$ and $q \geq 1$, $G(q,k)$ is C_5-free.*

Proof. Let $k \geq 5$, $q \geq 1$, and $G = G(q,k)$. Suppose by way of contradiction that G contains an induced C_5. Without loss of generality suppose v_0 is on an induced C_5 in G and let $\{v_0, v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\}$ induce a C_5 in G such that $v_0 \sim v_{i_1}$, $v_0 \sim v_{i_4}$, and $v_{i_j} \sim v_{i_{j+1}}$ for $j = 1, 2, 3$. By similar arguments to the proof of Lemma 1.2, we must have, without loss of generality, $v_{i_2} \in V_0 \setminus \{v_0, v_{qk}\}$ and $v_{i_3} \in V_1 \setminus \{v_1\}$. Further, since $N(v_0) = \{v_1, v_{qk}\} \cup V_2 \cup V_3 \cup \cdots \cup V_{k-1}$, we must have $\{v_{i_4}, v_{i_5}\} \subseteq V_2 \cup V_3 \cup \cdots \cup V_{k-1}$. By a similar argument to that of Case 1 of the proof of Lemma 1.2 and since $v_1 \sim v_{qk}$, we must have $v_{i_4} \in V_{k-1}$ (or else $v_{i_2} \sim v_{i_4}$) and $v_{i_3} \in V_j$ for some $2 \leq j \leq k-2$. Further, $j = k-2$, or else $v_{i_1} \sim v_{i_4}$. But now since $k \geq 5$, and therefore $k-2 \geq 3$, we have $v_{i_3} \sim v_{i_1}$. This contradicts our assumption on the induced C_5. Therefore, G is C_5-free. □

Note our proof of the following lemma is adapted directly from Theorem 2.4 of [9] where it was shown that $G(q, 4,)$ (under the name G_p) is 4-vertex-critical for all q.

Lemma 1.4. *For all $k \geq 3$ and $q \geq 1$, $G(q,k)$ is k-vertex-critical.*

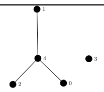
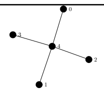
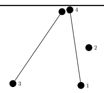
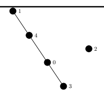
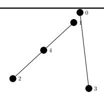
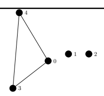
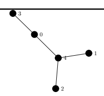
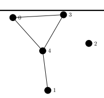
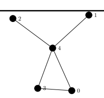
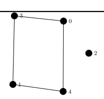
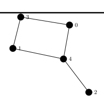
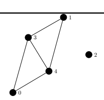
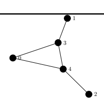
Proof. Let $G = G(q,k)$. Note that for all $0 \leq i \leq (q-1)k+1$, the set $\{v_{i+j} : j = 0, 1, \ldots, k-1\}$ induces a K_{k-1}. Thus, $\chi(G) \geq k-1$. Suppose G is $(k-1)$-colourable, and without loss of generality, let vertices v_i be assigned colour i for $i = 0, 1, \ldots, k-1$. We now must have vertex v_j be assigned colour $j \mod k$ for all $k \leq j \leq qk-1$. But now $N(v_{qk})$ contains all $k-1$ colours. So G is not $(k-1)$-colourable. From Lemma 1.1, however, this partial colouring is valid and we may give vertex v_{qk} colour k to get that $\chi(G) = k$. Since v_{qk} is the only vertex with colour k, it follows by the symmetry of G that G is k-vertex-critical. □

Theorem 1.5. *There are infinitely many k-vertex-critical (P_5, C_5)-free graphs for all $k \geq 6$.*

Proof. The proof follows from Lemma 1.2, Lemma 1.3, and Lemma 1.4. □

2 State of (P_5, H)-free graphs when H is of order 5

In this section we detail in a table whether there are only finitely many or infinitely many k-critical (P_5, H)-free graphs for all nonisomorphic graphs H of order 5. This is to make progress on the open problem raised in [5]. For table, see this site for all graph names: https://www.graphclasses.org/smallgraphs.html#

Graph	Graph name	finite/infinite	reference
	$\overline{K_5}$	finite	Ramsey's Theorem [14]
	$P_2 + 3P_1$	finite	[4]
	$P_3 + 2P_1$	finite	[1]
	claw+P_1	unknown	[]
	$K_{1,4}$	finite	[12] (see also [13, Theorem 1])
	$2K_2 + P_1$	infinite	Contains $2K_2$ [9]
	$P_4 + P_1$	unknown	[]
	$P_3 + P_2$	infinite	Contains $2K_2$ [9]
	$K_3 + 2P_1$	infinite	Contains $K_3 + P_1$ [5, 9]
	chair	finite $k \leq 5$, unknown $k \geq 6$	[11]
	co-dart	infinite	Contains $K_3 + P_1$ [5, 9]
	$\overline{\text{diamond} + P_1}$	unknown	[]
	$C_4 + P_1$	unknown	[]
	banner	finite	[2, Theorem 3(i)]
	diamond+P_1	infinite	Contains $K_3 + P_1$ [5, 9]
	bull	finite $k = 5$, unknown $k \geq 6$	[10]

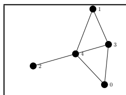
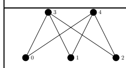
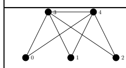
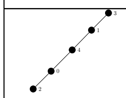
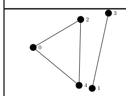
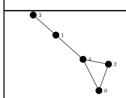
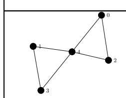
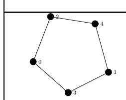
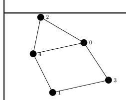
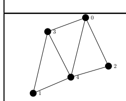
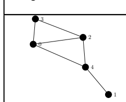
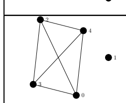
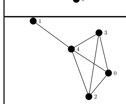
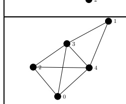
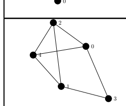
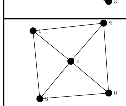
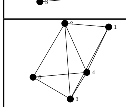
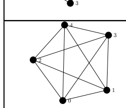
	dart	unknown	[]
	$K_{2,3}$	finite	[12]
	$\overline{K_3 + 2P_2}$	unknown	[]
	P_5	infinite	Contains $2K_2$ [9]
	$K_3 + P_2$	infinite	Contains $2K_2$ [9]
	co-banner	infinite	Contains $2K_2$ [9] (see also [2])
	butterfly	infinite	Contains $2K_2$ [9]
	C_5	infinite	this paper
	$\overline{P_5}$	finite	[8]
	gem	finite	[3] (see also our P_5, gem)-free paper)
	kite	infinite	Contains $K_3 + P_1$ [5, 9]
	$K_4 + P_1$	infinite	Contains $K_3 + P_1$ [5, 9]
	$\overline{\text{claw} + K_1}$	infinite	Contains $K_3 + P_1$ [5, 9]
	$\overline{P_3 + 2P_1}$	unknown	[]
	paraglider or $\overline{P_3 + P_2}$	finite	[3]
	W_4	unknown	[]
	$K_5 - e$	unknown	[]
	K_5	infinite $k = 5$, unknown $k \geq 6$	[9]

Table 1: The state-of-the-art for H of order 5.

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