Critical $(P_4 + P_1,\ K_3 + P_1, 2P_2)$-free graphs

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Abstract

1 Structure

Throughout this section, assume G is a k-vertex-critical non-complete $(P_4 + P_1,\ K_3 + P_1, 2P_2)$-free graph.

For any $v \in V(G)$, G partitions into $\{v\}$, $N(v)$, and $\overline{N[v]}$. $N(v)$ further partitions into N_1, N_1', N_2. $\exists u_1 \in \overline{N[v]}$ such that $u_1 \sim N_1$, and $\exists u_2 \in \overline{N[v]}$ such that $u_2 \sim N_2$.

Lemma 1.1. *For every vertex $v \in V(G)$, $G[\overline{N[v]}]$ is $(P4, K_3)$-free.*

Proof. If $S \subseteq \overline{N[v]}$ induces P_4 or K_3, then $\{v\} \cup S$ induces a $P_4 + P_1$ or $K_3 + P_1$, a contradiction. □

Lemma 1.2. *For every vertex $v \in V(G)$, $\overline{N[v]}$ induces $K_{n,m}$ some $n, m \geq 1$.*

Proof. By Lemma 1.1, $\overline{N[v]}$ is P_4-free and therefore either a join of disjoint union of graphs (since P_4-free graphs are co-graphs). Thus, every componenet is K_1 or a join. Further, since $\overline{N[v]}$ is K_3-free, each component must be a complete bipartite graph, since if it were the join of any graph with an edge, a triangle would be induced, contradicting Lemma 1.1. Now, if some component of $\overline{N[v]}$ is K_1, then the neighbourhood of this component is contained in the neighbourhood of v which makes comparable vertices and therefore contradicts G being vertex-critical. Thus, every component of $\overline{N[v]}$ contains at least one edge. If there are two components, then take any two vertices that are adjacent from two different components and these four vertices will induce a $2P_2$, contradicting G being $2P_2$-free. Thus, $\overline{N[v]}$ induces $K_{n,m}$ some $n, m \geq 1$. □

Lemma 1.3. *$N_1 - N_1'$ is complete to N_2.*

Proof. If $N_1 - N_1'$ is not complete to N_2, then $n_1 \in N_1, n_2 \in N_2$, $\{n_1, u_2, n_2, u_1\}$ induces a $2K_2$. □

Lemma 1.4. *N_2 is an independent set.*

Proof. By 1.2, the non-neighbours of N_1 are a complete bipartite graph. □

Now let S be the maximum independent set of G, and $v \in S$. Let $S - v = \{u_1, u_2, \ldots, u_l\}$. Also, let $V(G) - S - N(v) = \{Y_1, Y_2, \ldots, Y_j\}$. Suppose that l is greater than some arbitrary constant, and that $j \geq 2$. If Y_1 is complete to $N(v)$, then Y_1, v are comparable, which contradicts the k-vertex criticality of the graph. So let $N' \subseteq N(v)$ such that Y_1 is anticomplete to N'.

Lemma 1.5. *Y_i is anticomplete to N' $\forall i \in \{1, \ldots, j\}$.*

Proof. If $\exists Y_i \in V(G) - S - N(v)$ such that $Y_i \sim n$, then $\{u_1, Y_i, n\} \subseteq \overline{N(Y_1)}$ contradicting that the non-neighbours are a complete biparite graph. □

Lemma 1.6. *$S - v$ is complete to N'.*

Proof. If $\exists v_i \in S - v$ such that $u_i \not\sim n$ for some $n \in N'$, then $\{u_i, Y_1, v, n\}$ induces a $2K_2$, which contradicts our graph characterization. □

So $N(Y_i) \subseteq N(Y_1) \forall i \in \{2, \ldots, j\}$, so Y_i, Y_1 are comparable vertices, making this not vertex critical and thus a contradiction. Further, since this applies for all $i \geq 2$, then the $|Y| \leq 1$. We also know (somehow) that this applies to the set of $\{u_1, \ldots, u_l\}$ so they are limited to ≤ 1. Now we can use Ramsay's Theorem to identify that since there is a maximum independent set, there is a finite amount of graphs.

2 $claw + P_1, K_3 + P_1, 2P_2$-free

We can use a similar technique to find a finite amount of these graphs. Assume we have a $claw + P_1, K_3 + P_1, 2K_2$-free graph. Then we have the maximum independent set S. Let $v \in S$. We will reuse our definition of $\overline{N[v]}$. We know that $\overline{N[v]}$ is claw free, else we take the claw plus v to make $claw + p1$. We also know it's triangle free for the same reason.

Lemma 2.1. $\forall u \in \overline{N[v]} - S, |N(u) \cap S| < 3$

Proof. Assume $|N(u) \cap S| \geq 3$. $N(u) \cap S = \{s_1, s_2, s_3, \ldots, s_j\}$. Then $\{u, s_1, s_2, s_3, v\}$ induces a $claw + P_1$, a contradiction. □

Lemma 2.2. $\forall s \in S, \{u, u'\} \in N(s) \cap \overline{N[v]} - S, u \not\sim u'$

Proof. This induces a $K3 + P1$. □

Thus no two neighbours of s in noneighbours are adjacent.

Lemma 2.3. $|S| \leq 2$

Proof. Assume $|S| > 2$. Let $u \in N(v)$ and $u' \in \overline{N[v]}$ and $s \in S - v$. $s \sim u'$ else s is disconnected and makes the graph not k-vertex-critical. We now have 3 cases. Case 1: $u \sim u', s \not\sim u$. In this case, u, u', v, s induces a P_4, and any additional member of $S - v \not\sim u'$ induces a $P_4 + P_1$. □

Lemma 2.4. s $\forall v \in \overline{N[v]}$,

3 $(2P_2, P_4 + P_1, chair, bull)$-free

Let G be a $(2P_2, P_4 + P_1, chair, bull)$-free k-vertex-critical graph. Let us have a vertex v and $N(v)$ and $\overline{N[v]}$. Let S be the maximum independent set of G where $v \in S$. Let A be $\overline{N[v]} - S$. Let s_1 and s_2 be any two separate vertices in S not equal to v.

Lemma 3.1. *For $u \in A$, $u \nsim u'$ where $u' \in N(v)$.*

Proof. Assume $u \sim u'$. This means that $\{u', u, v, u_2\}$ creates an induced P_4 where $u_2 \in N(v)$. Thus u' must be complete to S in order to not create an induced $P_4 + P_1$. This creates an induced chair with the vertex set $\{u', u, v, s_1, s_2\}$ unless $s_2 \sim u$. With this edge in place, $\{u', u, v, s_1, s_2\}$ is an induced bull. ¡This somehow covers every case; fill this in¿ □

Lemma 3.2. *The length of S is bounded to a maximum of 2 vertices.*

Proof. With $u \nsim u'$, $\{s_1, u'\}$ and $\{v, u\}$ form an induced $2P_2$ where $s_1 \in S$ unless $s_1 \sim u$. With this edge present, $\{s_1, u', s_2, u\}$ creates an induced P_4, meaning u must be complete to S to avoid creating an induced $P_4 + P_1$. However, this makes every vertex in S comparable to each other, contradicting the assumption that G is k-vertex-critical. This means that S must have a maximum length of 2 in order for G to exist, containing at most v and s_1. □

The length of the maximum independent set of G is bounded to some constant value, meaning that there are finitely many k-vertex-critical graphs that meet the criteria of G. Results
$P_4 + \ell_1 P_1, 2P_2, \ell_2 squid$
$claw + P_1, K_3 + P_1, 2P_2$
$2P_2, P_4 + P_1, chair, ragingbull$
$P_4 + \ell_1 P_1, 2K_2, K_{1,\ell_2} + P_1$-free