Critical $(P_4+P_1,\ K_3+P_1, 2P_2)$-free graphs

Ben Cameron
Department of Computing Science
The King's University
Edmonton, AB Canada
ben.cameron@kingsu.ca

Thaler Knodel
Department of Computing Science
The King's University
Edmonton, AB Canada

Christopher Bury
Department of Computing Science
The King's University
Edmonton, AB Canada

Melvin Adekanye
Department of Computing Science
The King's University
Edmonton, AB Canada

David Supina
Department of Computing Science
The King's University
Edmonton, AB Canada

July 21, 2023

Abstract

1 Structure

Throughout this section, assume G is a k-vertex-critical non-complete $(P_4+P_1,\ K_3+P_1, 2P_2)$-free graph.

For any $v \in V(G)$, G partitions into $\{v\}$, $N(v)$, and $\overline{N[v]}$. $N(v)$ further partitions into N_1, N_1', N_2. $\exists u_1 \in \overline{N[v]}$ such that $u_1 \sim N_1$, and $\exists u_2 \in \overline{N[v]}$ such that $u_2 \sim N_2$.

Lemma 1.1. *For every vertex $v \in V(G)$, $G[\overline{N[v]}]$ is $(P4, K_3)$-free.*

Proof. If $S \subseteq \overline{N[v]}$ induces P_4 or K_3, then $\{v\} \cup S$ induces a P_4+P_1 or K_3+P_1, a contradiction. □

Lemma 1.2. *For every vertex $v \in V(G)$, $\overline{N[v]}$ induces $K_{n,m}$ some $n, m \geq 1$.*

Proof. By Lemma 1.1, $\overline{N[v]}$ is P_4-free and therefore either a join of disjoint union of graphs (since P_4-free graphs are co-graphs). Thus, every componenet is K_1 or a join. Further, since $\overline{N[v]}$ is K_3-free, each component must be a complete bipartite graph, since if it were the join of any graph with an edge, a triangle would be induced, contradicting Lemma 1.1. Now, if some component of $\overline{N[v]}$ is K_1, then the neighbourhood of this component is contained in the neighbourhood of v which makes comparable vertices and therefore contradicts G being vertex-critical. Thus, every component of $\overline{N[v]}$ contains at least one edge. If there are two components, then take any two vertices that are adjacent from two different components and these four vertices will induce a $2P_2$, contradicting G being $2P_2$-free. Thus, $\overline{N[v]}$ induces $K_{n,m}$ some $n, m \geq 1$. $\square$

Lemma 1.3. *$N_1 - N_1'$ is complete to N_2.*

Proof. If $N_1 - N_1'$ is not complete to N_2, then $n_1 \in N_1, n_2 \in N_2$, $\{n_1, u_2, n_2, u_1\}$ induces a $2K_2$. $\square$

Lemma 1.4. *N_2 is an independent set.*

Proof. By 1.2, the non-neighbours of N_1 are a complete bipartite graph. $\square$

Now let S be the maximum independent set of G, and $v \in S$. Let $S - v = \{u_1, u_2, \ldots, u_l\}$. Also, let $V(G) - S - N(v) = \{Y_1, Y_2, \ldots, Y_j\}$. Suppose that l is greater than some arbitrary constant, and that $j \geq 2$. If Y_1 is complete to $N(v)$, then Y_1, v are comparable, which contradicts the k-vertex criticality of the graph. So let $N' \subseteq N(v)$ such that Y_1 is anticomplete to N'.

Lemma 1.5. *Y_i is anticomplete to N' $\forall i \in \{1, \ldots, j\}$.*

Proof. If $\exists Y_i \in V(G) - S - N(v)$ such that $Y_i \sim n$, then $\{u_1, Y_i, n\} \subseteq \overline{N(Y_1)}$ contradicting that the non-neighbours are a complete biparite graph. $\square$

Lemma 1.6. *$S - v$ is complete to N'.*

Proof. If $\exists v_i \in S - v$ such that $u_i \not\sim n$ for some $n \in N'$, then $\{u_i, Y_1, v, n\}$ induces a $2K_2$, which contradicts our graph characterization. $\square$

So $N(Y_i) \subseteq N(Y_1) \forall i \in \{2, \ldots, j\}$, so Y_i, Y_1 are comparable vertices, making this not vertex critical and thus a contradiction. Further, since this applies for all $i \geq 2$, then the $|Y| \leq 1$. We also know (somehow) that this applies to the set of $\{u_1, \ldots, u_l\}$ so they are limited to ≤ 1. Now we can use Ramsay's Theorem to identify that since there is a maximum independent set, there is a finite amount of graphs.

2 $claw + P_1, K_3 + P_1, 2P_2$-free

We can use a similar technique to find a finite amount of these graphs. Assume we have a $claw + P_1, K_3 + P_1, 2K_2$-free graph. Then we have the maximum independent set S. Let $v \in S$. We will reuse our definition of $\overline{N[v]}$. We know that $\overline{N[v]}$ is claw free, else we take the claw plus v to make $claw + p1$. We also know it's triangle free for the same reason.

Lemma 2.1. $\forall u \in \overline{N[v]} - S, |N(v) \cap S| < 3$

Proof. Assume $|N(v) \cap S| \geq 3$. $N(v) \cap S = \{s_1, s_2, s_3, \ldots, s_j\}$. Then $\{u, s_1, s_2, s_3, v\}$ induces a $claw + P_1$, a contradiction. $\square$

Lemma 2.2. $\forall s \in S, u, u' \in N(s) \cap \overline{N[v]} - S, u \not\sim u'$

Proof. This induces a $K3 + P1$ $\square$

Lemma 2.3. $|S| \leq 2$

Proof. Assume $|S| > 2$. Let $u \in N(v)$ and $u' \in \overline{N[v]}$ and $s \in S - v$. $s \sim u'$ else s is disconnected and makes the graph not k-vertex-critical. We now have 3 cases. Case 1: $u \sim u', s \not\sim u$. In this case, u, u', v, s induces a P_4, and any additional member of $S - v \not\sim u'$ induces a $P_4 + P_1$. $\square$

Lemma 2.4. s $\forall v \in \overline{N[v]}$,

3 $(2P_2, P_4 + P_1, chair, bull)$-free

Let G be a $(2P_2, P_4 + P_1, chair, bull)$-free k-vertex-critical graph. Let us have a vertex v and $N(v)$ and $\overline{N[v]}$. Let S be the maximum independent set of G where $v \in S$. Let A be $\overline{N[v]} - S$. Let s_1 and s_2 be any two separate vertices in S not equal to v.

Lemma 3.1. *For $u \in A$, $u \nsim u'$ where $u' \in N(v)$.*

Proof. Assume $u \sim u'$. This means that $\{u', u, v, u_2\}$ creates an induced P_4 where $u_2 \in N(v)$. Thus u' must be complete to S in order to not create an induced $P_4 + P_1$. This creates an induced chair with the vertex set $\{u', u, v, s_1, s_2\}$ unless $s_2 \sim u$. With this edge in place, $\{u', u, v, s_1, s_2\}$ is an induced bull. ¡This somehow covers every case; fill this in¿ □

Lemma 3.2. *The length of S is bounded to a maximum of 2 vertices.*

Proof. With $u \nsim u'$, $\{s_1, u'\}$ and $\{v, u\}$ form an induced $2P_2$ where $s_1 \in S$ unless $s_1 \sim u$. With this edge present, $\{s_1, u', s_2, u\}$ creates an induced P_4, meaning u must be complete to S to avoid creating an induced $P_4 + P_1$. However, this makes every vertex in S comparable to each other, contradicting the assumption that G is k-vertex-critical. This means that S must have a maximum length of 2 in order for G to exist, containing at most v and s_1. □

The length of the maximum independent set of G is bounded to some constant value, meaning that there are finitely many k-vertex-critical graphs that meet the criteria of G. Results
$P_4 + \ell_1 P_1, 2P_2, \ell_2 squid$
s $claw + P_1, K_3 + P_1, 2P_2$
$2P_2, P_4 + P_1, chair, ragingbull$