Vertex-critical graphs in $2P_2$-free graphs

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Abstract

1 Introduction

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1.1 Notation

For a vertex v, $N(v)$, $N[v]$ and $\overline{N[v]}$ denote the open neighbourhood, closed neighbourhood, and set of nonneighbours of v, respectively. We let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of G, respectively. We let $\alpha(G)$ denote the independence number of G.

2 Structure

We will make extensive use of the following lemma, in particular when $m = 1$ throughout the paper.

Lemma 2.1 ([4])**.** *Let G be a graph with chromatic number k. If G contains two disjoint m-cliques $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_m\}$ such that $N(a_i) \setminus A \subseteq N(b_i) \setminus B$ for all $1 \leq i \leq m$, then G is not k-vertex-critical.*

Lemma 2.2. *If G is a k-vertex-critical $2P_2$-free graph, then for every vertex $v \in V(G)$, $\overline{N[v]}$ induces a connected graph with at least two vertices.*

Proof. Let G be a k-vertex-critical $2P_2$-free graph, $v \in V(G)$, and H be the graph induced by $\overline{N[v]}$. If $u \in \overline{N[v]}$ such that u is an isolated vertex in the graph induced by H, then $N(u) \subseteq N(v)$ contradicting G being k-vertex-critical by Lemma 2.1. Therefore, if H has at least two components, then each component has at least one edge and therefore taking an edge from each component induces a $2P_2$. This contradicts G being $2P_2$-free. $\square$

A part of this work will be exhaustively generating all k-vertex-critical graphs in certain families for small values of k. While there are excellent exhaustive generation algorithms that exist like the one introduced in [4] and then optimized and expanded in [3], these still rarely terminate for and values of $k \geq 6$. The small independence number of some of the critical graphs in our results allow us to use simpler exhaustive afforded by the implied bound (proven in the next lemma) on their order and the invaluable tool `nauty` [5] to generate all for values of k up to 7 in some cases.

Lemma 2.3. *If G is a k-vertex-critical graph with $\alpha(G) = c$ for some constant c, then $|V(G)| \leq c(k-1)+1$.*

Proof. Let G be a k-vertex-critical graphs with $\alpha(G) = c$, $v \in V(G)$, and let $n = |V(G)|$. Since G is k-vertex-critical, $G - v$ is $(k-1)$-colourable and has order $n-1$ and $\alpha(G-v) \leq c$. Since no colour-class of any $(k-1)$-colouring of $G-v$ can have more than c vertices, it follows that $G-v$ can have at most $(k-1)c$ vertices. Thus, $n = n-1+1 \leq (k-1)c+1$. $\square$

3 $(2P_2, K_3+P_1, P_4+P_1)$-free graphs

IDEA: If G is also P_3+2P_1-free then done by my previous paper. Thus, contains an induced P_3+2P_1. By Lemma 3.1, nonneighbourhood of s_1 (s_1, s_2 are the $2P_1$ vertices), its nonneighbourhood must be a complete bipartite graph, and thus s_2 must be complete to v_1 and v_2 (the two leaves of the P_3), contradicting the induced $P_3 + 2P_1$!.

Throughout this section, assume G is a k-vertex-critical non-complete $(2P_2, K_3+P_1, P_4+P_1)$-free graph.

For any $v \in V(G)$, G partitions into $\{v\}$, $N(v)$, and $\overline{N[v]}$. $N(v)$ further partitions into N_1, N_1', N_2. $\exists u_1 \in \overline{N[v]}$ such that $u_1 \sim N_1$, and $\exists u_2 \in \overline{N[v]}$ such that $u_2 \sim N_2$.

Lemma 3.1. *For every nonuniversal vertex $v \in V(G)$, $\overline{N[v]}$ induces a complete bipartite graph $K_{n,m}$ some $n, m \geq 1$.*

Proof. Let v be a nonuniversal vertex in G and let H be the subgraph of G induced by $\overline{N[v]}$. We first note that If $S \subseteq \overline{N[v]}$ induces P_4 or K_3, then $\{v\} \cup S$ induces a $P_4 + P_1$ or $K_3 + P_1$, a contradiction. Thus, H is $(P4, K_3)$-free. Since H is P_4-free it is either a join or disjoint union of graphs (since P_4-free graphs are co-graphs). By Lemma 2.2, H must be connected, so it therefore must be the join of graphs. Further, since H is K_3-free and the join of graphs, it must be a complete bipartite graph. $\square$

Now let S be the maximum independent set of G, and $v \in S$. Let $S - v = \{s_1, s_2, \ldots, s_\ell\}$. Also, let $V(G) - (S \cup N(v)) = \{y_1, y_2, \ldots, y_j\}$. Note that if y_1 is complete to $N(v)$, then $N(v) \subseteq N(y_1)$, contradicting the k-vertex-criticality of G by Lemma 2.1. So let $N' \subseteq N(v)$ be the set $\overline{N[y_1]} \cap N(v)$. Note also that we may assume $\ell \geq 2$ otherwise we are done by Ramsey's Theorem.

Lemma 3.2. *$S - v$ is complete to N'.*

Proof. Suppose there is a $s_i \in S - v$ and $n \in N'$ such that $s_i \nsim n$. From Lemma 3.1, $S - v$ is complete to $V(G) - (S \cup N(v))$, so $y_1 \sim s_i$, and by definition $y_1 \nsim n$. Therefore, $\{y_i, s_i, n, v\}$ induces a $2P_2$ in G, a contradiction. Thus, $S - v$ is complete to N'. $\square$

We originally thought the next result was *anticomplete*, so we need to figure out how to proceed. One option is to also forbid the dart as then we must a dart induced by s_1, s_1's unique neighbour in $N(v)$ and y_1 and y_2. Thus y_1 is the only y_i and a similar argument gets the $S - v$ set down to one element. May also be possible if chair is forbidden as well.

Lemma 3.3. *y_i is complete to N' $\forall i \in \{2, \ldots, j\}$.*

Proof. Let $2 \leq i \leq j$ and let $n \in N'$. By Lemma 3.1, $y_i \in \overline{N[y_1]}$, so $\{v, n, y_i\} \subseteq \overline{N[y_1]}$. Since $v \sim n$ and $v \nsim y_i$ by definition, it follows that $n \sim y_i$ otherwise $\overline{N[y_1]}$ would not be complete bipartite, contradicting Lemma 3.1. Therefore, y_i is complete to N'. □

So $N(Y_i) \subseteq N(Y_1) \forall i \in \{2, \ldots, j\}$, so Y_i, Y_1 are comparable vertices, making this not vertex critical and thus a contradiction. Further, since this applies for all $i \geq 2$, then the $|Y| \leq 1$. We also know (somehow) that this applies to the set of $\{u_1, \ldots, u_l\}$ so they are limited to ≤ 1. Now we can use Ramsay's Theorem to identify that since there is a maximum independent set, there is a finite amount of graphs.

4 $(2P_2, K_3 + P_1, claw + P_1)$-free

Throughout this section, assume G is a k-vertex-critical non-complete $(2P_2, K_3 + P_1, \text{claw} + P_1)$-free graph.

Lemma 4.1. *For every nonuniversal vertex $v \in V(G)$, $\overline{N[v]}$ induces a P_j or C_m for $2 \leq j \leq 4$ and $4 \leq m \leq 5$.*

Proof. Let v be a nonuniversal vertex in G and let H be the subgraph of G induced by $\overline{N[v]}$. If $S \subseteq V(H)$ such that S induces a K_3 or claw, then $S \cup \{v\}$ induces a claw+P_1 or $K_3 + P_1$, a contradiction. Therefore, H is $(K_3 + P_1, \text{claw} + P_1)$-free. Suppose there is a vertex $u \in \overline{N[v]}$ such that $|N(u) \cap \overline{N[v]}| \geq 3$ and let $\{u_1, u_2, u_3\} \subseteq N(u) \cap \overline{N[v]}$. If the u_i's are all pairwise disjoint, then $\{u, u_1, u_2, u_3\}$ induces a claw, a contradiction. So there must be at least one edge in the graph induced by $\{u_1, u_2, u_3\}$, without loss of generality say $u_1 \sim u_2$. But now $\{u, u_1, u_2\}$ induces a K_3, a contradiction. Thus, we must have $|N(u) \cap \overline{N[v]}| \leq 2$ for all $u \in \overline{N[v]}$ and thus $\Delta(H) \leq 2$. From Lemma 2.2, H must be connected and have at least two vertices, so H must be a P_j or a C_m. Since H is $(2P_2, K_3 + P_1)$-free, it follows that $2 \leq j \leq 4$ and $4 \leq m \leq 5$. □

Corollary 4.2. $\alpha(G) \leq 3$.

Proof. Let $S = \{s_1, s_2, \ldots s_\ell\}$ be a maximum independent set in G. By Lemma 4.1, it follows that the graph induced by $\overline{N[s_1]}$ must have independence number at most 2. Since $s_i \in \overline{N[s_1]}$ for all $2 \leq i \leq \ell$, it follows that $\ell \leq 3$. □

Theorem 4.3. *There are only finitely many k-vertex-critical $(2P_2, K_3 + P_1, claw + P_1)$-free graphs for any given k.*

Proof. Fix k and let G be a k-vertex-critical $(2P_2, K_3 + P_1, \text{claw} + P_1)$-free graph. Since G is k-vertex-critical, $\omega(G) \leq k$. From Corollary 4.2, $\alpha(G) \leq 3$. Thus, by Ramsey's Theorem, G has order bounded above by $R(4, k + 1)$, a constant. □

Questions: (1) can we remove the $K_3 + P_1$-free restriction and still get only finitely many of these graphs? (2) can we show that $\overline{N[v]}$ cannot induce a cycle and therefore every such critical graph has independence number at most 2? (3) Can we exhaustively generate all for small k even without the further restriction on independence number give the -k argument in nauty v2.8.6?

5 $(2P_2, P_4 + P_1, chair, bull)$-free

NOTE!! The main result of this section is now a corollary of the $P_4 + \ell P_1$ stuff.

Throughout this section let G be a $(2P_2, P_4 + P_1, chair, bull)$-free k-vertex-critical graph. Let S be the maximum independent set of G, $v \in S$, and let $S - v = \{s_1, s_2, \dots, s_\ell\}$. Let A be $\overline{N[v]} - S$.

Lemma 5.1. $\alpha(G) \leq 2$.

Proof. Suppose by way of contradiction that $\alpha(G) \geq 3$, so that $\ell \geq 2$. Since $s_1 \nsim v$ by definition, we must have $v_1 \in N(v)$ such that $s_1 \nsim v_1$, otherwise we would have $N(v) \subseteq N(s_1)$, contradicting G being k-vertex-critical by Lemma 2.1. Further, by Lemma 2.2, s_1 must have a neighbour $u \in A$, otherwise s_1 would be an isolated vertex in the graph induced by $\overline{N[v]}$. If $u \nsim v_1$, then $\{s_1, u, v_1, v\}$ induces a $2P_2$ in G, a contradiction. Therefore, $u \sim v_1$. Now, $s_1 \nsim s_2$ and $s_2 \nsim v$ by definition, so if $u \nsim s_2$ and $v_1 \nsim s_2$, $\{s_2, s_1, u, v_1, v\}$ induces a $P_4 + P_1$ in G, a contradiction. Therefore, $u \sim s_2$ or $v \sim s_2$. If exactly one of u and v is adjacent to s_2, then $\{s_2, s_1, u, v_1, v\}$ induces a chair in G, a contradiction. Therefore, both u and v must be adjacent to s_2. However, we now have $\{s_2, s_1, u, v_1, v\}$ inducing a bull in G, a contradiction. This completes the proof. □

Theorem 5.2. *There are only finitely many k-vertex-critical $(2P_2, P_4 + P_1, chair, bull)$-free graphs for any given k.*

Proof. Fix k and let G be a k-vertex-critical $(2P_2, P_4 + P_1, \text{chair}, \text{bull})$-free graph. Since G is k-vertex-critical, $\omega(G) \leq k$. From Lemma 5.1, $\alpha(G) \leq 2$. Thus, by Ramsey's Theorem, G has order bounded above by $R(3, k+1)$, a constant. □

6 $(2P_2, P_4 + \ell P_1, m$-squid$)$-free

For this section we need the following result:

Theorem 6.1 ([1])**.** *There are only finitely many k-vertex-critical $(P_3 + \ell P_1)$-free graphs for all k and ℓ.*

Theorem 6.2. *There are only finitely many k-vertex-critical $(2P_2, P_4 + \ell P_1, m$-squid$)$-free for all k, ℓ, and m.*

Proof. Fix k, ℓ, and m and let G be a k-vertex-critical $(2P_2, P_4 + \ell P_1, m\text{-squid})$-free graph. If G is $(P_3 + cP_1)$-free for $c = \ell + m$, then we are done by Theorem 6.1. Thus we may assume G has an induced $P_3 + cP_1$. Let v be the centre of the P_3, v_1 and v_2 be its leaves, and $s_1, s_2 \dots s_c$ be the c isolated vertices of an induced $P_3 + cP_1$ in G. Let $S = \{s_1, s_2, \dots, s_c\}$. If $N(s_1) \subseteq N(v)$, then we contradict G being k-vertex-critical by Lemma 2.1. So there must be a vertex $u \in V(G) - (\overline{N[v]} \cup \{v_1, v_2\})$ such that $s_1 \sim u$. If $u \nsim v_1$ or $u \nsim v_2$, then $\{s_1, u, v_1, v\}$ or $\{s_1, u, v_2, v\}$ induces a $2P_2$ in G, a contradiction. Therefore, $u \sim v_1$ and $u \sim v_2$. If u has at least ℓ nonneighbours in $S - s_1$, then $\{s_1, u, v_1, v\}$ together with any ℓ nonneighbours of u in $S - s_1$ induces a $P_4 + \ell P_1$, a contradiction. Thus, u has at least $c - \ell = m$ neighbours in S (including s_1). But now $\{u, v, v_1, v_2\}$ and any m of u's neighbours in S induce an m-squid, a contradiction. This completes the proof. □

Since both $K_{1,m} + P_1$ and chair are an induced subgraphs of the m-squid, we get the following corollarie immediately.

Corollary 6.3. *There are only finitely many k-vertex-critical $(2P_2, P_4 + \ell P_1, K_{1,m} + P_1)$-free for all k, ℓ, and m.*

Corollary 6.4. *There are only finitely many k-vertex-critical $(2P_2, P_4 + \ell P_1$, chair$)$-free for all k, ℓ,*

Question: Can we get rid of the $2P_2$-free restriction in the theorem?

7 Conclusion

References

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