Critical $(P_4+P_1,\ K_3+P_1, 2P_2)$-free graphs

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Abstract

1 Structure

Throughout this section, assume G is a k-vertex-critical non-complete $(P_4+P_1,\ K_3+P_1, 2P_2)$-free graph.

For any $v \in V(G)$, G partitions into $\{v\}$, $N(v)$, and $\overline{N[v]}$.

Lemma 1.1. *For every vertex $v \in V(G)$, $G[\overline{N[v]}]$ is $(P4, K_3)$-free.*

Proof. If $S \subseteq \overline{N[v]}$ induces P_4 or K_3, then $\{v\} \cup S$ induces a P_4+P_1 or K_3+P_1, a contradiction. □

Lemma 1.2. *For every vertex $v \in V(G)$, $\overline{N[v]}$ induces $K_{n,m}$ some $n, m \geq 1$.*

Proof. By Lemma 1.1, $\overline{N[v]}$ is P_4-free and therefore either a join of disjoint union of graphs (since P_4-free graphs are co-graphs). Thus, every componenet is K_1 or a join. Further, since $\overline{N[v]}$ is K_3-free, each component must be a complete bipartite graph, since if it were the join of any graph with an edge, a triangle would be induced, contradicting Lemma 1.1. Now, if some component of $\overline{N[v]}$ is K_1, then the neighbourhood of this component is contained in the neighbourhood of v which makes comparable vertices and therefore contradicts G being vertex-critical. Thus, every component of $\overline{N[v]}$ contains at least one edge. If there are two components, then take any two vertices that are adjacent from two different components and these four vertices will induce a $2P_2$, contradicting G being $2P_2$-free. Thus, $\overline{N[v]}$ induces $K_{n,m}$ some $n, m \geq 1$. □

Lemma 1.3. *$N_1 - N_1'$ is complete to N_2.*

Proof. If $N_1 - N_1'$ is not complete to N_2, then $n_1 \in N_1, n_2 \in N_2$, $\{n_1, u_2, n_2, u_1\}$ induces a $2K_2$. □

Lemma 1.4. *N_2 is an independent set.*

Proof. By 1.2, the non-neighbours of N_1 are a complete bipartite graph. □

Let S be the maximum independent set of G, and $v \in S$. Let $S - v = \{u_1, u_2, \dots, u_l\}$. Also, let $V(G) - S - N(v) = \{Y_1, Y_2, \dots, Y_j\}$. Suppose that l is greater than some arbitrary constant, and that $j \geq 2$. If Y_1 is complete to $N(v)$, then Y_1, v are comparable, which contradicts the k-vertex criticality of the graph. So let $N' \subseteq N(v)$ such that Y_1 is anticomplete to N'.

Lemma 1.5. *$S - v$ is complete to N'.*

Proof. If $\exists v_i \in S - v$ such that $u_i \not\sim n$ for some $n \in N'$, then $\{u_i, Y_1, v, n\}$ induces a $2K_2$, which contradicts our graph characterization. $\square$

Lemma 1.6. *Y_i is anticomplete to N' $\forall i \in \{1, \dots, j\}$.*

Proof. If $\exists Y_i \in V(G) - S - N(v)$ such that $Y_i \sim n$, then $\{u_1, Y_i, n\} \subseteq \overline{N(Y_1)}$ contradicting that the non-neighbours are a complete biparite graph. $\square$

So $N(Y_i) \subseteq N(Y_1) \forall i \in \{2, \dots, j\}$, so Y_i, Y_1 are comparable vertices, making this not vertex critical and thus a contradiction. Further, since this applies for all $i \geq 2$, then the $|Y| \leq 1$. We also know (somehow) that this applies to the set of $\{u_1, \dots, u_l\}$ so they are limited to ≤ 1