Tidy Time Series & Forecasting in R

7. Exponential smoothing

robjhyndman.com/workshop2020

Outline

Outline

Historical perspective

- Developed in the 1950s and 1960s as methods (algorithms) to produce point forecasts.
- Combine a “level”, “trend” (slope) and “seasonal” component to describe a time series.
- The rate of change of the components are controlled by “smoothing parameters”: α, β and γ respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Equivalent ETS state space models developed in the 1990s and 2000s.

A model for levels, trends, and seasonalities

We want a model that captures the level (ℓ_t), trend (b_t) and seasonality (s_t).

How do we combine these elements?

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$y_t = \ell_{t-1} b_{t-1} s_{t-m}(1 + \varepsilon_t)$

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Perhaps a mix of both?

$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

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How do the level, trend and seasonal components evolve over time?

ETS models

General notation

$$\begin{array}{c} \text{E T S} : \textbf{E}\text{xponen}\textbf{T}\text{ial }\textbf{S}\text{moothing} \\ \nearrow \uparrow \nwarrow \\ \textbf{E}\text{rror }\textbf{T}\text{rend }\textbf{S}\text{eason} \end{array}$$

Error: Additive (`"A"`) or multiplicative (`"M"`)

ETS models

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$\nearrow \quad \uparrow \quad \nwarrow$

Error **T**rend **S**eason

Error: Additive (`"A"`) or multiplicative (`"M"`)

Trend: None (`"N"`), additive (`"A"`), multiplicative (`"M"`), or damped (`"Ad"` or `"Md"`).

ETS models

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Error: Additive (`"A"`) or multiplicative (`"M"`)

Trend: None (`"N"`), additive (`"A"`), multiplicative (`"M"`), or damped (`"Ad"` or `"Md"`).

Seasonality: None (`"N"`), additive (`"A"`) or multiplicative (`"M"`)

ETS(A,N,N): SES with additive errors

Forecast equation	$\hat{y}_{T+h\|T} = \ell_T$
Measurement equation	$y_t = \ell_{t-1} + \varepsilon_t$
State equation	$\ell_t = \ell_{t-1} + \alpha\varepsilon_t$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

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where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- "innovations" or "single source of error" because equations have the same error process, ε_t.
- Measurement equation: relationship between observations and states.
- Transition/state equation(s): evolution of the state(s) through time.

ETS(M,N,N): SES with multiplicative errors

Forecast equation	$\hat{y}_{T+h\|T} = \ell_T$
Measurement equation	$y_t = \ell_{t-1}(1 + \varepsilon_t)$
State equation	$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

ETS(M,N,N): SES with multiplicative errors

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where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.

Outline

Holt's linear trend

Additive errors: ETS(A,A,N)

$$
\begin{aligned}
\text{Forecast equation} \quad \hat{y}_{T+h|T} &= \ell_T + hb_T \\
\text{Measurement equation} \quad y_t &= \ell_{t-1} + b_{t-1} + \varepsilon_t \\
\text{State equations} \quad \ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= b_{t-1} + \beta\varepsilon_t
\end{aligned}
$$

Holt's linear trend

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$$
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b_t &= b_{t-1} + \beta\varepsilon_t
\end{aligned}
$$

Multiplicative errors: ETS(M,A,N)

$$
\begin{aligned}
\text{Forecast equation} \quad \hat{y}_{T+h|T} &= \ell_T + hb_T \\
\text{Measurement equation} \quad y_t &= (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t) \\
\text{State equations} \quad \ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= b_{t-1} + \beta\varepsilon_t
\end{aligned}
$$

Example: Australian population

```r
aus_economy <- global_economy %>% filter(Code == "AUS") %>%
  mutate(Pop = Population/1e6)
fit <- aus_economy %>% model(AAN = ETS(Pop))
report(fit)
```

```
## Series: Pop
## Model: ETS(A,A,N)
##   Smoothing parameters:
##     alpha = 1
##     beta  = 0.327
##
##   Initial states:
##     l      b
##  10.1 0.222
##
##   sigma^2:  0.0041
##
##   AIC  AICc   BIC
## -77.0 -75.8 -66.7
```

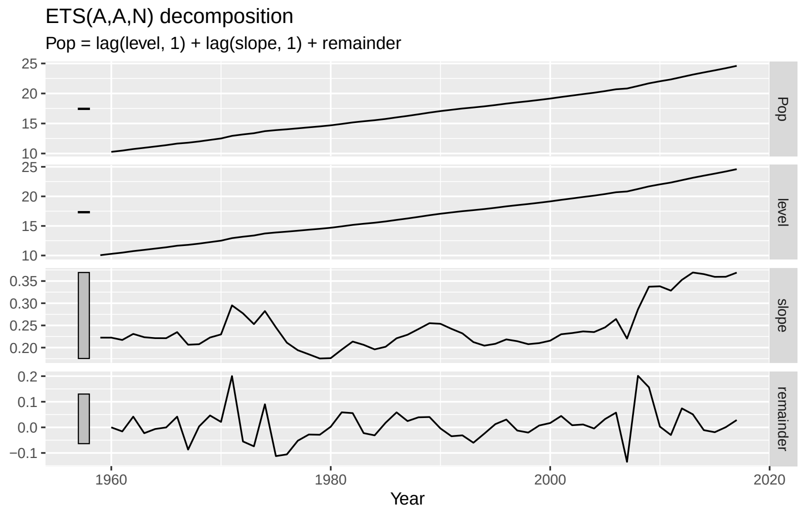
Example: Australian population

```r
components(fit)
```

```
## # A dable:                       59 x 7 [1Y]
## # Key:                           Country, .model [1]
## # ETS(A,A,N) Decomposition: Pop = lag(level, 1) + lag(slope, 1)
## #   remainder
##    Country   .model  Year   Pop level slope remainder
##    <fct>     <chr>  <dbl> <dbl> <dbl> <dbl>     <dbl>
##  1 Australia AAN     1959  NA    10.1 0.222 NA
##  2 Australia AAN     1960  10.3  10.3 0.222 -0.000145
##  3 Australia AAN     1961  10.5  10.5 0.217 -0.0159
##  4 Australia AAN     1962  10.7  10.7 0.231  0.0418
##  5 Australia AAN     1963  11.0  11.0 0.223 -0.0229
##  6 Australia AAN     1964  11.2  11.2 0.221 -0.00641
##  7 Australia AAN     1965  11.4  11.4 0.221 -0.000314
##  8 Australia AAN     1966  11.7  11.7 0.235  0.0418
##  9 Australia AAN     1967  11.8  11.8 0.206 -0.0869
## 10 Australia AAN     1968  12.0  12.0 0.208  0.00350
```

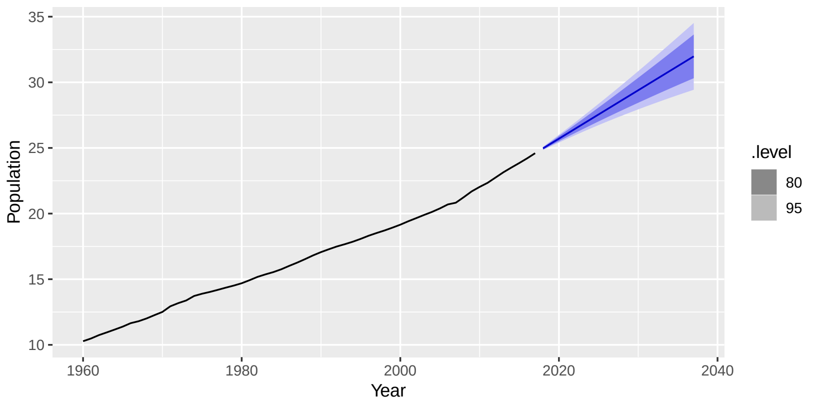
Example: Australian population

```
components(fit) %>% autoplot()
```



Example: Australian population

```
fit %>%
  forecast(h = 20) %>%
  autoplot(aus_economy) +
  ylab("Population") + xlab("Year")
```



ETS(A,Ad,N): Damped trend method

Additive errors

$$\begin{aligned}
\text{Forecast equation} \quad \hat{y}_{T+h|T} &= \ell_T + (\phi + \cdots + \phi^{h-1})b_T \\
\text{Measurement equation} \quad y_t &= (\ell_{t-1} + \phi b_{t-1}) + \varepsilon_t \\
\text{State equations} \quad \ell_t &= (\ell_{t-1} + \phi b_{t-1}) + \alpha\varepsilon_t \\
b_t &= \phi b_{t-1} + \beta\varepsilon_t
\end{aligned}$$

ETS(A,Ad,N): Damped trend method

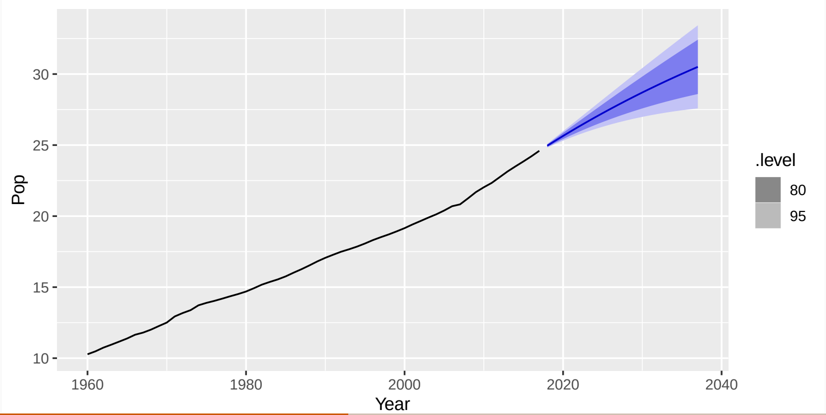
Additive errors

Forecast equation	$\hat{y}_{T+h\|T} = \ell_T + (\phi + \cdots + \phi^{h-1})b_T$
Measurement equation	$y_t = (\ell_{t-1} + \phi b_{t-1}) + \varepsilon_t$
State equations	$\ell_t = (\ell_{t-1} + \phi b_{t-1}) + \alpha\varepsilon_t$
	$b_t = \phi b_{t-1} + \beta\varepsilon_t$

- Damping parameter $0 < \phi < 1$.
- If $\phi = 1$, identical to Holt's linear trend.
- As $h \to \infty$, $\hat{y}_{T+h|T} \to \ell_T + \phi b_T/(1 - \phi)$.
- Short-run forecasts trended, long-run forecasts constant.

Example: Australian population

```
aus_economy %>%
  model(holt = ETS(Pop ~ trend("Ad"))) %>%
  forecast(h = 20) %>%
  autoplot(aus_economy)
```



Example: National populations

```r
fit <- global_economy %>%
  mutate(Pop = Population/1e6) %>%
  model(ets = ETS(Pop))
fit
```

```
## # A mable: 263 x 2
## # Key:     Country [263]
##    Country             ets
##    <fct>               <model>
##  1 Afghanistan         <ETS(A,A,N)>
##  2 Albania             <ETS(M,A,N)>
##  3 Algeria             <ETS(M,A,N)>
##  4 American Samoa      <ETS(M,A,N)>
##  5 Andorra             <ETS(M,A,N)>
##  6 Angola              <ETS(M,A,N)>
##  7 Antigua and Barbuda <ETS(M,A,N)>
##  8 Arab World          <ETS(M,A,N)>
##  9 Argentina           <ETS(A,A,N)>
## 10 Armenia             <ETS(M,A,N)>
## # ... with 253 more rows
```

Example: National populations

```r
fit %>%
  forecast(h = 5)
```

```
## # A fable: 1,315 x 5 [1Y]
## # Key:     Country, .model [263]
##    Country     .model  Year   Pop .distribution
##    <fct>       <chr>  <dbl> <dbl> <dist>
##  1 Afghanistan ets     2018 36.4  N(36, 0.012)
##  2 Afghanistan ets     2019 37.3  N(37, 0.059)
##  3 Afghanistan ets     2020 38.2  N(38, 0.164)
##  4 Afghanistan ets     2021 39.0  N(39, 0.351)
##  5 Afghanistan ets     2022 39.9  N(40, 0.644)
##  6 Albania     ets     2018  2.87 N(2.9, 0.00012)
##  7 Albania     ets     2019  2.87 N(2.9, 0.00060)
##  8 Albania     ets     2020  2.87 N(2.9, 0.00169)
##  9 Albania     ets     2021  2.86 N(2.9, 0.00362)
```

Outline

Lab Session 14

Consider the data set `fma::eggs`, the price of a dozen eggs in the United States from 1900-1993. Experiment with the various options in the `ETS()` function to see how much the forecasts change with damped trend, or with a Box-Cox transformation. Try to develop an intuition of what each is doing to the forecasts. How can we stop the forecasts becoming negative?

[Hint: use `h=100` when forecasting, so you can clearly see the differences between the various options when plotting the forecasts.]

Outline

ETS(A,A,A): Holt-Winters additive method

$$
\begin{aligned}
\text{Forecast equation} \quad \hat{y}_{t+h|t} &= \ell_t + hb_t + s_{t+h-m(k+1)} \\
\text{Observation equation} \quad y_t &= \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t \\
\text{State equations} \quad \ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= b_{t-1} + \beta\varepsilon_t \\
s_t &= s_{t-m} + \gamma\varepsilon_t
\end{aligned}
$$

- k = integer part of $(h-1)/m$.
- $\sum_i s_i \approx 0$.
- Parameters: $0 \le \alpha \le 1$, $0 \le \beta^* \le 1$, $0 \le \gamma \le 1-\alpha$ and m = period of seasonality (e.g. $m = 4$ for quarterly data).

ETS(M,A,M): Holt-Winters multiplicative method

$$
\begin{aligned}
\text{Forecast equation} \quad \hat{y}_{t+h|t} &= (\ell_t + hb_t)s_{t+h-m(k+1)} \\
\text{Observation equation} \quad y_t &= (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t) \\
\text{State equations} \quad \ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= b_{t-1}(1 + \beta\varepsilon_t) \\
s_t &= s_{t-m}(1 + \gamma\varepsilon_t)
\end{aligned}
$$

- k is integer part of $(h-1)/m$.
- $\sum_i s_i \approx m$.
- Parameters: $0 \le \alpha \le 1$, $0 \le \beta^* \le 1$, $0 \le \gamma \le 1 - \alpha$ and m = period of seasonality (e.g. $m = 4$ for quarterly data).

Example: Australian holiday tourism

```r
holidays <- tourism %>%
  filter(Purpose == "Holiday")
fit <- holidays %>% model(ets = ETS(Trips))
fit
```

```
## # A mable: 76 x 4
## # Key:     Region, State, Purpose [76]
##    Region                     State              Purpose ets
##    <chr>                      <chr>              <chr>   <model>
##  1 Adelaide                   South Australia    Holiday <ETS(A,N,A~
##  2 Adelaide Hills             South Australia    Holiday <ETS(A,A,N~
##  3 Alice Springs              Northern Territo~ Holiday <ETS(M,N,A~
##  4 Australia's Coral Coast    Western Australia Holiday <ETS(M,N,A~
##  5 Australia's Golden Outba~ Western Australia Holiday <ETS(M,N,M~
##  6 Australia's North West     Western Australia Holiday <ETS(A,N,A~
##  7 Australia's South West     Western Australia Holiday <ETS(M,N,M~
##  8 Ballarat                   Victoria           Holiday <ETS(M,N,A~
##  9 Barkly                     Northern Territo~ Holiday <ETS(A,N,A~
## 10 Barossa                    South Australia    Holiday <ETS(A,N,N~
```

Example: Australian holiday tourism

```r
fit %>% filter(Region=="Snowy Mountains") %>% report()
```

```
## Series: Trips
## Model: ETS(M,N,A)
##   Smoothing parameters:
##     alpha = 0.157
##     gamma = 1e-04
##
##   Initial states:
##     l   s1   s2     s3     s4
##   142 -61 131 -42.2 -27.7
##
##   sigma^2:  0.0388
##
##   AIC  AICc   BIC
##   852   854   869
```

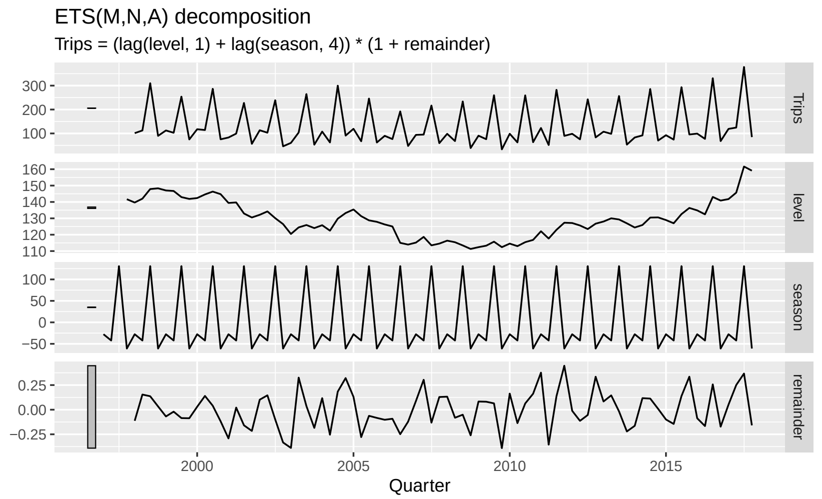
Example: Australian holiday tourism

```r
fit %>% filter(Region=="Snowy Mountains") %>% components(fit)
```

```
## # A dable:                    84 x 9 [1Q]
## # Key:                        Region, State, Purpose, .model [1]
## # ETS(M,N,A) Decomposition: Trips = (lag(level, 1) + lag(season,
## #   4)) * (1 + remainder)
##    Region State Purpose .model    Quarter Trips level season
##    <chr>  <chr> <chr>   <chr>       <qtr> <dbl> <dbl>  <dbl>
##  1 Snowy~ New ~ Holiday ets       1997 Q1   NA     NA   -27.7
##  2 Snowy~ New ~ Holiday ets       1997 Q2   NA     NA   -42.2
##  3 Snowy~ New ~ Holiday ets       1997 Q3   NA     NA   131.
##  4 Snowy~ New ~ Holiday ets       1997 Q4   NA    142.  -61.0
##  5 Snowy~ New ~ Holiday ets       1998 Q1 101.    140.  -27.7
##  6 Snowy~ New ~ Holiday ets       1998 Q2 112.    142.  -42.2
##  7 Snowy~ New ~ Holiday ets       1998 Q3 310.    148.  131.
##  8 Snowy~ New ~ Holiday ets       1998 Q4  89.8   148.  -61.0
##  9 Snowy~ New ~ Holiday ets       1999 Q1 112.    147.  -27.7
## 10 Snowy~ New ~ Holiday ets       1999 Q2 103.    147.  -42.2
## # ... with 74 more rows, and 1 more variable: remainder <dbl>
```

Example: Australian holiday tourism

```r
fit %>% filter(Region=="Snowy Mountains") %>%
  components(fit) %>% autoplot()
```



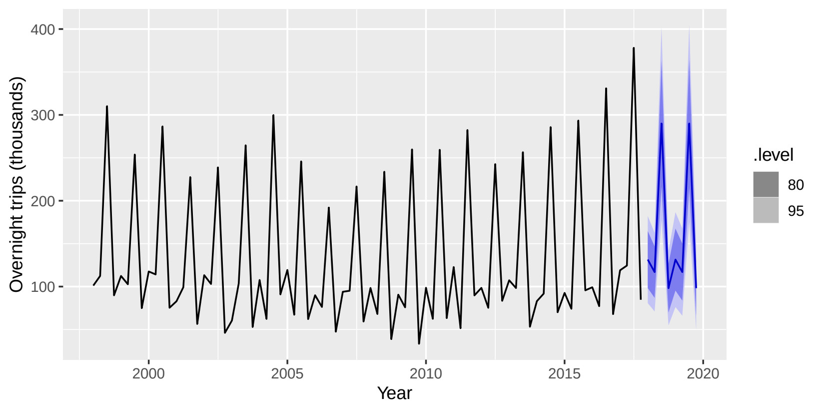
Example: Australian holiday tourism

```r
fit %>% forecast()
```

```
## # A fable: 608 x 7 [1Q]
## # Key:     Region, State, Purpose, .model [76]
##    Region    State     Purpose .model   Quarter Trips .distribution
##    <chr>     <chr>     <chr>   <chr>      <qtr> <dbl> <dist>
##  1 Adelaide  South A~ Holiday ets      2018 Q1 210.  N(210, 457)
##  2 Adelaide  South A~ Holiday ets      2018 Q2 173.  N(173, 473)
##  3 Adelaide  South A~ Holiday ets      2018 Q3 169.  N(169, 489)
##  4 Adelaide  South A~ Holiday ets      2018 Q4 186.  N(186, 505)
##  5 Adelaide  South A~ Holiday ets      2019 Q1 210.  N(210, 521)
##  6 Adelaide  South A~ Holiday ets      2019 Q2 173.  N(173, 537)
##  7 Adelaide  South A~ Holiday ets      2019 Q3 169.  N(169, 553)
##  8 Adelaide  South A~ Holiday ets      2019 Q4 186.  N(186, 569)
##  9 Adelaide~ South A~ Holiday ets      2018 Q1  19.4 N(19, 36)
## 10 Adelaide~ South A~ Holiday ets      2018 Q2  19.6 N(20, 36)
## # ... with 598 more rows
```

Example: Australian holiday tourism

```r
fit %>% forecast() %>%
  filter(Region=="Snowy Mountains") %>%
  autoplot(holidays) +
    xlab("Year") + ylab("Overnight trips (thousands)")
```



Outline

Exponential smoothing models

Additive Error		**Seasonal Component**		
Trend Component		N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	~~A,N,M~~
A	(Additive)	A,A,N	A,A,A	~~A,A,M~~
A_d	(Additive damped)	A,A_d,N	A,A_d,A	~~A,A_d,M~~

Multiplicative Error		**Seasonal Component**		
Trend Component		N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A_d	(Additive damped)	M,A_d,N	M,A_d,A	M,A_d,M

Estimating ETS models

- Smoothing parameters α, β, γ and ϕ, and the initial states ℓ_0, b_0, $s_0, s_{-1}, \dots, s_{-m+1}$ are estimated by maximising the "likelihood" = the probability of the data arising from the specified model.
- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

Model selection

Akaike's Information Criterion

$$\text{AIC} = -2\log(\text{L}) + 2k$$

where L is the likelihood and k is the number of parameters initial states estimated in the model.

Model selection

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$$\text{AIC} = -2\log(\text{L}) + 2k$$

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Corrected AIC

$$\text{AIC}_\text{c} = \text{AIC} + \frac{2(k+1)(k+2)}{T-k}$$

which is the AIC corrected (for small sample bias).

Model selection

Akaike's Information Criterion

$$\text{AIC} = -2\log(\text{L}) + 2k$$

where L is the likelihood and k is the number of parameters initial states estimated in the model.

Corrected AIC

$$\text{AIC}_\text{c} = \text{AIC} + \frac{2(k+1)(k+2)}{T-k}$$

which is the AIC corrected (for small sample bias).

Bayesian Information Criterion

$$\text{BIC} = \text{AIC} + k(\log(T) - 2).$$

AIC and cross-validation

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

Automatic forecasting

From Hyndman et al. (IJF, 2002):

1. Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE.
2. Select best method using AICc.
3. Produce forecasts using best method.
4. Obtain forecast intervals using underlying state space model.

- Method performed very well in M3 competition.
- Used as a benchmark in the M4 competition.

Outline

Lab Session 15

Find an ETS model for the Gas data from `aus_production`.

- Why is multiplicative seasonality necessary here?
- Experiment with making the trend damped. Does it improve the forecasts?