leastsquares

March 22, 2024

[74]:
```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import fetch_data
import math
```

[2]:
```
ls_data = fetch_data.getStandings()
```

[3]:
```
pts_array = ls_data['pts'].values
gd_array = ls_data['gd'].values
```

0.0.1 Least Squares Applications

In statistics, it is common to represent the variable of interest Y as a linear combination of other variables (factors) $x_1, \dots x_n$ as $Y = \beta_1 X_1 + \beta_2 X_2 + \dots\ + \beta_n X_n + \varepsilon$ where ε is a random variable that accounts for the missing factors.

Before we begin, lets talk about Least Squares.

In some cases $Ax = b$ may not have a solution, if the system is not compatible $(b \notin img(A))$

In such case, we solve the least squares problem. That is, we want the vector $x \in \mathbb{R}^n$ that minimizes $\parallel Ax - b \parallel^2$

0.0.2 Geometry of Least Squares

We can think about it like this: Lets say we have a plane for the $img(A)$ and we make a vector $\hat{w} = A\hat{x}$, this vector would be in the $img(A)$ because we are multiplying any vector in $\mathbb{R}^n$ times our matrix A. Now since our vector $(b \notin img(A))$, we need to find a vector that minimizes the distance between our vector w and b. We call this the residual, $\hat{r} = b - A\hat{x}$, the resiudal is what measures how close $A\hat{x}$ is to b.

$$\left\| \begin{matrix} b_1 - w_1 \\ b_2 - w_2 \\ \dots \\ b_n - w_n \end{matrix} \right\|^2 = (b_1 - v_1)^2 + (b_2 - v_2)^2 + \dots + (b_n - v_n)^2 = \hat{r}$$

This example is where the least squares terminology comes from, we are trying to find the least squares estimate (least squared distances). Now the closest vector in any subspace to a vector that is not in the subspace is called the projection. So, $\hat{w} = A\hat{x}$ is called the projection of b onto $img(A)$. Now through the fundamental theorem of linear algebra, our residual $\hat{r}$ lies in the $coker(A)$ because the residual is orthogonal to the $img(A)$. Now we know, the "best" solution is one for which the residual is orthogonal to the $img(A)$, since the $coker(A) = (img(A))^{\perp}$.

0.0.3 Derivation

Through the information we now know, we can derive the "normal equation".

Since, $b - A\hat{x} \in coker(A)$ then $A^{\top}(b - A\hat{x}) = 0$

$A^{\top}b - A^{\top}A\hat{x}$ then translates to $(A^{\top}A\hat{x} = A^{\top}b)$

In the case for Linear Regression, the normal equation would then just be $(X^{\top}X\hat{\beta} = X^{\top}y)$

Furthermore, solving for β we get $\beta = (X^{\top}X)^{-1}X^{\top}y$

0.0.4 Now lets do Linear Regression

[30]:
```
#constructing our first vector (Goal Differences)
x1 = np.array(gd_array).reshape(-1,1)

#constructing our design matrix
ones_array = np.ones_like(x1)
X = np.hstack((ones_array, x1))

#constructing our column response vector (Points)
y = np.array(pts_array)
```

[49]:
```
print('The design matrix is:')
print(X[:5]) #showing an example of the first 5 rows of the design matrix

print('The data vector is:')
print(y[:5]) #showing an example of the first 5 observations of the data vector
```

```
The design matrix is:
[[ 1 73]
 [ 1 50]
 [ 1 61]
 [ 1 49]
 [ 1 68]]
The data vector is:
[93 88 89 90 92]
```

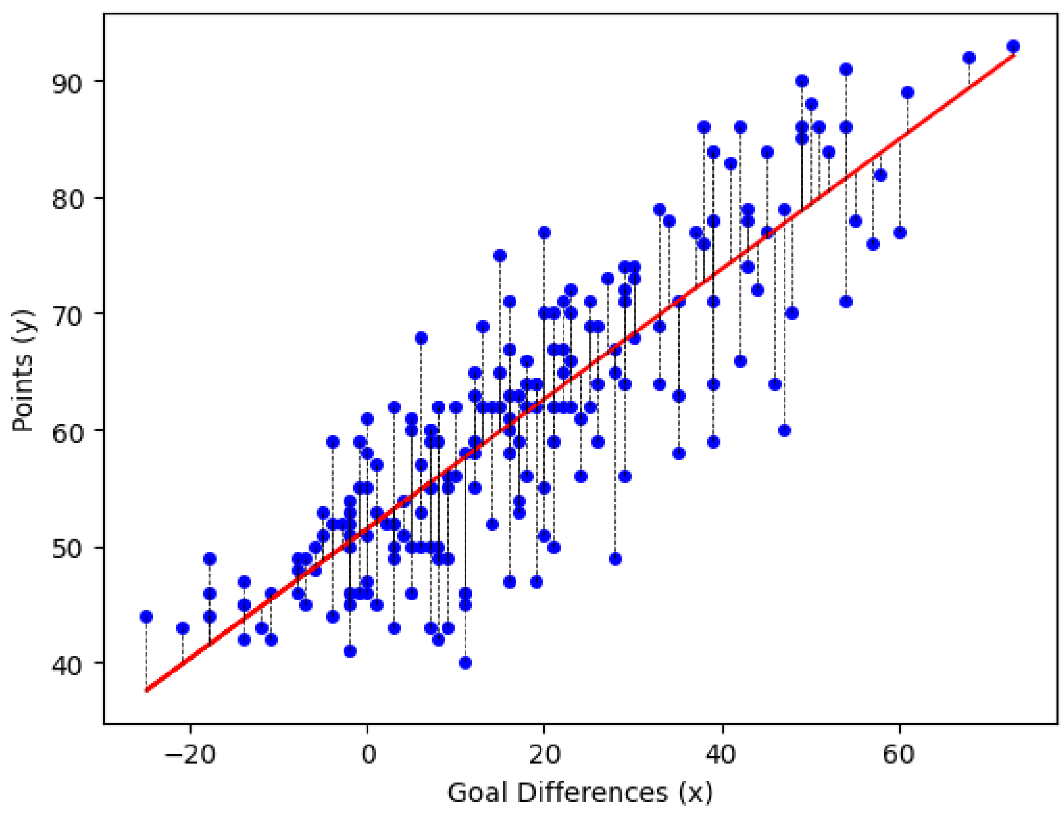
Computing the equation $\beta = (X^{\top}X)^{-1}X^{\top}y$

[50]:
```
beta = np.dot((np.dot(np.linalg.inv(np.dot(X.T, X)), X.T)), y)
print('The least squares coefficients are \n')
print(beta)
```

```
The least squares coefficients are

[51.51670337  0.55603074]
```

[86]:
```
fig, ax = plt.subplots()

#getting our residuals vector
residual_vector = y - np.dot(X, beta)

ax.plot(x1, y, linestyle='none', markersize=4, marker='o', color='blue')

#plotting our least squares line
ax.plot(x1, beta[0] + beta[1]*x1, 'r')
#plotting residual lines
for i in range(len(x1)):
    ax.plot([x1[i], x1[i]], [y[i], y[i] - residual_vector[i]], 'k--',␣
 ↪linewidth=0.5)

ax.set_xlabel('Goal Differences (x)')
ax.set_ylabel('Points (y)')
```

[86]:
```
Text(0, 0.5, 'Points (y)')
```



0.0.5 Nows lets see how many points a team with a goal difference of +48 can expect

[55]:
```
print('y = %d points' % math.floor(beta[0] + beta[1]*49))
```

```
y = 78 points
```

[91]:
```
#calculate the mean of our response vector
mean_y = np.mean(y)
#calculate the total sum of squares (TSS)
TSS = np.sum((y - mean_y)**2)
#calculate R squared
R_squared = 1 - (residual_vector**2).sum() / TSS
print("R squared:", R_squared)
```

```
R squared: 0.7218755174333396
```