CSCI-570 Homework 1

Chunho Lin, 3226964170

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1. **Problem 1. [Think through]**

   As presented in the lecture, the Gale-Shapley algorithm requires that the preference orders of all men and women do not have any ties. How could we adapt the algorithm for the scenario where ties are allowed?

   [As an example of a tie, imagine that there are 5 men and 5 women, and m_2's preference order is w_4, w_1, w_2 tied with w_3, w_5. Thus, if m_2 cannot be married to neither w_4 nor w_1, but can do "better" than w_5, he is indifferent whether he is married to w_2 or w_3.]

   Think. As we know, the Gale-Shapley algorithm works with the preference orders without the ties.

   Once we would like to add the ties in the preference orders. What we need to do is to break the ties arbitrarily. Literally, we can randomly assign the order of the tied candidates.

   For example, if

   $$m_2 : w_4 > w_1 > \{w_2, w_3\} > w_5,$$

   then one possible strict refinement is

   $$m_2 : w_4 > w_1 > w_2 > w_3 > w_5,$$

   and another is

   $$m_2 : w_4 > w_1 > w_3 > w_2 > w_5.$$

   Once all ties have been resolved in this way, we apply the Gale-Shapley algorithm to the resulting strict preference profiles, which produces a stable matching with respect to the chosen refinements. ∎

2. **Problem 2.**

   Prove by induction that $n^3 + 2n$ is divisible by 3 for all natural numbers. Indicate the structure of your proof clearly.

Proof.

Base case. If $n = 1$, then $n^3 + 2n = 1^3 + 2 \cdot 1 = 3$, which is divisible by 3. (If one includes $n = 0$ as natural, then $0^3 + 2 \cdot 0 = 0$ is also divisible by 3.)

Inductive hypothesis. Assume that for some $k \in \mathbb{N}$, the number $k^3 + 2k$ is divisible by 3, i.e.,

$$k^3 + 2k = 3m \quad \text{for some } m \in \mathbb{Z}.$$

Inductive step. We must show that $(k+1)^3 + 2(k+1)$ is also divisible by 3:

$$(k+1)^3 + 2(k+1) = \left(k^3 + 3k^2 + 3k + 1\right) + 2k + 2$$
$$= \underbrace{(k^3 + 2k)}_{\text{multiple of 3 by IH}} + \underbrace{3k^2 + 3k + 3}_{=3(k^2+k+1)}.$$

By the inductive hypothesis, $k^3 + 2k$ is divisible by 3, and clearly $3(k^2 + k + 1)$ is also divisible by 3. Thus, their sum is divisible by 3.

Conclusion. By the principle of mathematical induction, $n^3 + 2n$ is divisible by 3 for all natural numbers n.

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3. **Problem 3.**

Prove by induction that $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ holds for all natural numbers. Indicate the structure of your proof clearly.

For example, for $n = 3$ the above formula gives

$$\sum_{i=0}^{3} 2^i = 2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 16 - 1 = 2^4 - 1.$$

Proof.

Base case. Prove for $n = 0$. LHS $= \displaystyle\sum_{i=0}^{0} 2^i = 2^0 = 1$. RHS $= 2^{0+1} - 1 = 2^1 - 1 = 1$. LHS $=$ RHS, so the base case holds.

Inductive hypothesis. We assume that the $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ is true for some $n \in \mathbb{N}$.

Inductive step. Proof for $n + 1$: $\displaystyle\sum_{i=0}^{n+1} 2^i = 2^{(n+1)+1} - 1$

$$\sum_{i=0}^{n+1} 2^i = 2^0 + 2^1 + 2^2 + \ldots + 2^n + 2^{n+1}$$
$$= \sum_{i=0}^{n+1} 2^i + 2^{n+1}$$
$$= (2^{n+1} - 1) + 2^{n+1}$$
$$= 2(2^{n+1}) - 1 = 2^{n+2} - 1$$

We shows that LSH = RHS.

<u>Conclusion.</u> $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ holds for all natural numbers $n \in \mathbb{N}$.

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4. **Problem 4.**

Suppose $x_1, x_2, x_3, x_4 \in \mathbb{R}$, $x_1 + x_2 = x_3 + x_4 = 1$, and $x_1 x_3 + x_2 x_4 > 1$. Use proof by contradiction to show that at least one of x_1, x_2, x_3, x_4 is negative.

[Hint: What do you know about the product $(x_1 + x_2) \cdot (x_3 + x_4)$?]

Proof. Based on the hint, we know that $(x_1 + x_2) \cdot (x_3 + x_4) = 1$ because $x_1 + x_2 = 1$ and $x_3 + x_4 = 1$. Then we can expand the left-hand side:

$$(x_1 + x_2)(x_3 + x_4) = x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 = 1$$
$$= (x_1 x_3 + x_2 x_4) + (x_1 x_4 + x_2 x_3) = 1$$

Since each $x_i \geq 0$, we have $x_1 x_4 + x_2 x_3 \geq 0$. Hence

$$x_1 x_3 + x_2 x_4 = 1 - (x_1 x_4 + x_2 x_3) \leq 1,$$

which contradicts the given hypothesis $x_1 x_3 + x_2 x_4 > 1$.

<u>Conclusion.</u> The assumption that all four numbers are nonnegative is false; at least one of x_1, x_2, x_3, x_4 must be negative.

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5. **Problem 5.**

Order the following functions from smallest asymptotic running time to greatest. Additionally, identify all pairs of functions f_i and f_j where $f_i(n) = \Theta(f_j(n))$, or explicitly state that none exist. Explain your answers.

(a) $f_a(n) = n!$

(b) $f_b(n) = \sqrt{\log n^{20}}$

(c) $f_c(n) = 2^{n^3}$

(d) $f_d(n) = n \cdot \frac{\ln n}{\ln 2}$

(e) $f_e(n) = n(\log n)^{20}$

(f) $f_f(n) = n^{\log n}$

(g) $f_g(n) = \log \sqrt{n^{20}}$

(h) $f_h(n) = \lfloor \pi e \rfloor!$

(i) $f_i(n) = \Pi_{i=1}^{n} \frac{i+1}{i}$

(j) $f_j(n) = \frac{n}{f_h(n)}$

Proof.

(a) $f_a(n) = n! \Rightarrow f_h(n) = \Theta(n!)$

(b) $f_b(n) = \sqrt{\log n^{20}} = \sqrt{20 \cdot logn} = \sqrt{20} \cdot \sqrt{logn} \Rightarrow \Theta(\sqrt{logn})$

(c) $f_c(n) = 2^{n^3} \Rightarrow \Theta(2^{n^3})$

(d) $f_d(n) = n \cdot \frac{\ln n}{\ln 2} = n \cdot log_2 n \Rightarrow \Theta(nlogn)$

(e) $f_e(n) = n(logn)^{20} \Rightarrow \Theta(n(logn)^{20})$

(f) $f_f(n) = n^{logn} \Rightarrow \Theta(n^{logn})$

(g) $f_g(n) = log\sqrt{n^{20}} = logn^{\frac{20}{2}} = 10logn \Rightarrow \Theta(logn)$

(h) $f_h(n) = \lfloor \pi e \rfloor ! \approx \lfloor 3.14 * 2.71 \rfloor ! = 8!$, which is a constant function. $\Rightarrow f_h(n) = \Theta(1)$

(i) $f_i(n) = \Pi_{i=1}^{n} \frac{i+1}{i} = \frac{1+1}{1} \cdot \frac{2+1}{2} \cdot \frac{3+1}{3} \cdots \frac{n+1}{n} = n+1$ by telescoping. $\Rightarrow f_i(n) = \Theta(n)$

(j) $f_j(n) = \frac{n}{f_h(n)}$ by $(h.)$ $f_j(n) = \frac{n}{8!} \Rightarrow \Theta(n)$

$$h < b < g < j, i < d < e < f < a < c$$

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6. **Problem 6.**

$f(n) \in O(s(n))$, and $g(n) \in O(r(n))$. Prove or disprove (by giving a counter-example) the following claims:

(a) if $g(n) \in O(f(n))$, then $f(n) + g(n) \in O(s(n))$

Proof. Since $f(n) \in O(s(n))$, there exists $C_1 > 0$ and $n_1 > 0$ such that for all $n \geq n_1$, $f(n) \leq C_1 s(n)$. Since $g(n) \in O(f(n))$, there exists $C_2 > 0$ and $n_2 > 0$ such that for all $n \geq n_2$, $g(n) \leq C_2 f(n)$.

$$\Rightarrow f(n) + g(n) \leq f(n) + C_2 f(n) = (1 + C_2)f(n) \leq (1 + C_2)C_1 s(n)$$

Conclustion. Thus, $f(n) + g(n) \in O(s(n))$ which the constant is $(1 + C_2)C_1$ and $n_0 = \max(n_1, n_2)$.

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(b) if $r(n) \in O(s(n))$, then $g(n) \in O(f(n))$

Proof. Take counterexample to pick

$$s(n) = 1, \qquad r(n) = 1, \qquad f(n) = \frac{1}{n}, \qquad g(n) = 1.$$

Then

- $f(n) \in O(s(n))$ because $\frac{1}{n} \leq 1$ for $n \geq 1$.
- $g(n) \in O(r(n))$ because $1 \leq 1 \cdot r(n)$.
- $r(n) \in O(s(n))$ trivially (they are equal).

But $g(n) \notin O(f(n))$: $\frac{g(n)}{f(n)} = \frac{1}{1/n} = n \to \infty$, so there is no constant C with $1 \leq C \cdot \frac{1}{n}$ for all large n. Hence the implication fails.

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(c) $(\frac{f(n)}{g(n)}) \in O(\frac{s(n)}{r(n)})$

Proof. Take counterexample to pick

$$f(n) = 1, \qquad s(n) = 1, \qquad g(n) = \frac{1}{n^2}, \qquad r(n) = \frac{1}{n}.$$

- $f \in O(s)$ since $1 \le 1 \cdot 1$.
- $g \in O(r)$ because $\frac{1}{n^2} \le \frac{1}{n}$ for all $n \ge 1$.
- But

$$\frac{f(n)}{g(n)} = \frac{1}{1/n^2} = n^2, \qquad \frac{s(n)}{r(n)} = \frac{1}{1/n} = n,$$

and n^2 is **not** $O(n)$. So the claimed relation can fail.

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7. **Problem 7.**

Consider a binary counter where the cost of flipping the *kth* bit is $k+1$ units. That is, flipping the lowest order bit costs 1 unit $(0+1)$, the next bit costs 2 units $(1+1)$, the next bit costs 3 units $(2+1)$, and so on. Use the aggregate method to calculate the amortized cost (via a sequence of n increments starting from 0) of incrementing the counter!

Proof. Let the binary counter be incremented n times from 0. For each $k \ge 0$, denote by b_k the bit at position k (with the least significant bit at $k = 0$). During the n increments, bit b_k flips exactly

$$\left\lfloor \frac{n}{2^k} \right\rfloor$$

times.

Hence, the total cost of performing all n increments is

$$\begin{aligned}
T(n) &= \sum_{k=0}^{\lfloor \log n \rfloor} (k+1) \cdot \left\lfloor \tfrac{n}{2^k} \right\rfloor \\
&\le \sum_{k=0}^{\lfloor \log n \rfloor} (k+1) \cdot \frac{n}{2^k} \\
&= n \sum_{k=0}^{\lfloor \log n \rfloor} \frac{k+1}{2^k} \\
&\le n \sum_{k=0}^{\infty} \frac{k+1}{2^k}.
\end{aligned}$$

The infinite sum evaluates to

$$\sum_{k=0}^{\infty} \frac{k+1}{2^k} = 4,$$

so

$$T(n) \le 4n.$$

On the other hand, each increment costs at least 1, so $T(n) \geq n$. Thus,

$$T(n) = \Theta(n).$$

Finally, the amortized cost per increment is

$$\frac{T(n)}{n} = \Theta(1),$$

with an explicit upper bound of 4 units per increment. ∎

8. **Problem 8. [Think through]**

Provide a list of all non-negative numbers whose value is the same both as a binary and as a decimal number. That is, all numbers n for which $n_2 = n_{10}$. Provide a proof that you listed all of them.

For example, 101 does not fit the above description, as $101_2 = 5_{10} \neq 101_{10}$. Also, numbers like 213 are not valid numbers.

Proof. Let a non-negative integer n have the same string of digits when written in base 10 and in base 2. Write that common string as

$$n = d_k d_{k-1} \ldots d_1 d_0, \qquad d_i \in \{0, 1\}, \ d_k = 1.$$

where d_k is the leading digit (so the representation has no leading zeros).

Interpreting this string in base 10 gives:

$$n_{10} = \sum_{i=0}^{k} d_i \, 10^i,$$

and in base 2 gives

$$n_2 = \sum_{i=0}^{k} d_i \, 2^i.$$

We require $n_{10} = n_2 \Rightarrow n_{10} - n_2 = 0$, so

$$0 = n_{10} - n_2 = \sum_{i=0}^{k} d_i (10^i - 2^i).$$

For $i = 0$, the term is $d_0(10^0 - 2^0) = d_0(1 - 1) = 0$.

However, for $i \geq 1$, we have $10^i - 2^i > 0$ and since $d_i \in \{0, 1\}$, each term $d_i(10^i - 2^i) \geq 0$.

Thus, the only way for the RHS to be zero is if each term is zero, which requires $d_i = 0$ for all $i \geq 1$.

In other words, the only valid digit strings are d_0 and $1d_0$, which correspond to the numbers 0 and 1.

To sum up, the only non-negative numbers whose value is the same both as a binary and as a decimal number are 0 and 1.

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