CSCI-570 Homework 1 Solutions

Due Date: Tuesday, September 9th, 11:59pm

Problem 1. [Think through]
As presented in the lecture, the Gale-Shapley algorithm requires that the preference orders of all men and women do not have any ties. How could we adapt the algorithm for the scenario where ties are allowed?

[As an example of a tie, imagine that there are 5 men and 5 women, and m_2's preference order is w_4, w_1, w_2 *tied with* w_3, w_5. Thus, if m_2 cannot be married to neither w_4 nor w_1, but can do "better" than w_5, he is indifferent whether he is married to w_2 or w_3.]

[2 points]

Problem 2.
Prove by induction that $n^3 + 2n$ is divisible by 3 for all natural numbers. Indicate the structure of your proof clearly.

[4 points]

Problem 3.
Prove by induction that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ holds for all natural numbers. Indicate the structure of your proof clearly.

For example, for $n = 3$ the above formula gives

$$\sum_{i=0}^{3} 2^i = 2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 16 - 1 = 2^4 - 1.$$

[4 points]

Problem 4.
Suppose $x_1, x_2, x_3, x_4 \in \mathbb{R}$, $x_1 + x_2 = x_3 + x_4 = 1$, and $x_1x_3 + x_2x_4 > 1$. Use proof by contradiction to show that at least one of x_1, x_2, x_3, x_4 is negative.

[Hint: What do you know about the product $(x_1 + x_2) \cdot (x_3 + x_4)$?]

[4 points]

Problem 5.
Order the following functions from smallest asymptotic running time to greatest. Additionally, identify all pairs of functions f_i and f_j where $f_i(n) = \Theta(f_j(n))$, or explicitly state that none exist. Explain your answers.

(a) $f_a(n) = n!$

(b) $f_b(n) = \sqrt{log\ n^{20}}$

(c) $f_c(n) = 2^{n^3}$

(d) $f_d(n) = n \cdot \frac{ln\ n}{ln\ 2}$

(e) $f_e(n) = n(log\ n)^{20}$

(f) $f_f(n) = n^{log\ n}$

(g) $f_g(n) = log\sqrt{n^{20}}$

(h) $f_h(n) = \lfloor \pi e \rfloor !$

(i) $f_i(n) = \prod_{i=1}^{n} \frac{i+1}{i}$

(j) $f_j(n) = \frac{n}{f_h(n)}$

[10 points]

Problem 6.
$f(n) \in O(s(n))$, and $g(n) \in O(r(n))$. Prove or disprove (by giving a counter-example) the following claims:

(a) if $g(n) \in O(f(n))$, then $f(n) + g(n) \in O(s(n))$

(b) if $r(n) \in O(s(n))$, then $g(n) \in O(f(n))$

(c) $\left(\dfrac{f(n)}{g(n)}\right) \in O\left(\dfrac{s(n)}{r(n)}\right)$

[8 points]

Problem 7.
Consider a binary counter where the cost of flipping the kth bit is $k+1$ units. That is, flipping the lowest order bit costs 1 unit (0+1), the next bit costs 2 units (1+1), the next bit costs 3 units (2+1), and so on. Use the aggregate method to calculate the amortized cost (via a sequence of n increments starting from 0) of incrementing the counter!

[4 points]

Problem 8. [Think through]
Provide a list of all non-negative numbers whose value is the same both as a binary and as a decimal number. That is, all numbers n for which $n_2 = n_{10}$. Provide a proof that you listed all of them.

For example, 101 does not fit the above description, as $101_2 = 5_{10} \neq 101_{10}$. Also, numbers like 213 are not valid numbers.

[4 points]