I. Prove Theorem 9.4 $\frac{1}{cond(A)}\frac{\|\mathbf{r}\|_2}{\|\mathbf{b}\|_2} \le \frac{\|\mathbf{e}\|_2}{\|\mathbf{x}\|_2} \le cond(A)\frac{\|\mathbf{r}\|_2}{\|\mathbf{b}\|_2}$

Solution:

$$
\begin{aligned}
\frac{1}{cond(A)}\frac{\|\mathbf{r}\|_2}{\|\mathbf{b}\|_2} &= \frac{1}{\|A\|_2\left\|A^{-1}\right\|_2}\frac{\|A\mathbf{e}\|_2}{\|\mathbf{b}\|_2} \\
&\le \frac{1}{\|A\|_2\left\|A^{-1}\right\|_2}\frac{\|A\|_2\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2} \\
&= \frac{\|\mathbf{e}\|_2}{\left\|A^{-1}\right\|_2\|\mathbf{b}\|_2} \\
&\le \frac{\|\mathbf{e}\|_2}{\left\|A^{-1}\mathbf{b}\right\|_2} \\
&= \frac{\|\mathbf{e}\|_2}{\|\mathbf{x}\|_2} \\
&= \frac{\left\|A^{-1}\mathbf{r}\right\|_2}{\|\mathbf{x}\|_2} \\
&\le \left\|A^{-1}\right\|_2\|A\|_2\frac{\|\mathbf{r}\|_2}{\|A\|_2\|\mathbf{x}\|_2} \\
&\le cond(A)\frac{\|\mathbf{r}\|_2}{\|A\mathbf{x}\|_2} = cond(A)\frac{\|\mathbf{r}\|_2}{\|\mathbf{b}\|_2}
\end{aligned}
$$

II. What are the values of cond(A) for A in (7.13) for $n = 8$ and $n = 1024$

Solution:

A 的特征值为

$$\lambda_k(A) = \frac{4}{h^2}\sin^2\frac{k\pi}{2n},\ k = 1, 2, \cdots, n-1$$

$$
\begin{aligned}
cond(A) &= \|A\|_2\left\|A^{-1}\right\|_2 \\
&= \frac{\max_{k=1,2,\cdots,n-1}\lambda_k(A)}{\min_{k=1,2,\cdots,n-1}\lambda_k(A)} \\
&= \frac{\sin^2\frac{(n-1)\pi}{2n}}{\sin^2\frac{\pi}{2n}}
\end{aligned}
$$

所以 $n = 8$ 时 $cond(A) = \frac{\sin^2\frac{7\pi}{16}}{\sin^2\frac{\pi}{16}} \approx 48.991$

所以 $n = 1024$ 时 $cond(A) = \frac{\sin^2\frac{1023\pi}{2048}}{\sin^2\frac{\pi}{2048}} \approx 1046267.343$

III. Prove Lemma 9.16

Solution:

$$A = diag(-1, 2, -1) = D - L - U$$

$$L + U = \frac{1}{h^2}\begin{bmatrix} & -1 & & & \\ -1 & & -1 & & \\ & \ddots & & \ddots & \\ & & -1 & & -1 \\ & & & -1 & \end{bmatrix},\ D = \frac{2}{h^2} I$$

所以

$$\begin{aligned} T_\omega &= (1-\omega)I + \omega D^{-1}(L+U) \\ &= I - \omega\frac{h^2}{2}D + \omega\frac{h^2}{2}(L+U) \\ &= I - \frac{\omega h^2}{2}(D - L - U) \\ &= I - \frac{\omega h^2}{2}A \end{aligned}$$

设 $\lambda_k(A)$ 所对应的特征向量为 $\mathbf{w_k}$, 设 $Q = [\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_{n-1}}]$, 则 $A = Q\ diag(\lambda_k(A))\ Q^T$ 而

$$\begin{aligned} T_\omega &= I - \frac{\omega h^2}{2}A \\ &= Q\ I\ Q^T + \frac{\omega h^2}{2}Q\ diag(\lambda_k(A))\ Q^T \\ &= Q\ diag(1 - \frac{\omega h^2}{2}\lambda_k(A))\ Q^T \end{aligned}$$

所以 T_ω 和 A 具有相同的特征向量
且对应的特征值为 $\lambda_k(T_\omega) = 1 - \frac{\omega h^2}{2}\lambda_k(A) = 1 - \frac{\omega h^2}{2}\frac{4}{h^2}\sin^2\frac{k\pi}{2n} = 1 - 2\omega\sin^2\frac{k\pi}{2n},\ k = 1, 2, \cdots, n-1$

IV. Show that, for $\nu_1 = \nu_2 = 1$, the computational cost of an FMG cycle is less than $\frac{2}{(1-2^{-D})^2}WU$. Give upper bounds as tight as possible for computational costs of an FMG cycle for $D = 1, 2, 3$

Solution:

与 Lemma 9.33 一致，且由 FMG 的定义可知，FMG 的计算开销为

$$\begin{aligned}
2WU\sum_{k=0}^{m}\sum_{i=m-k}^{m}2^{-iD} &= 2WU\sum_{k=0}^{m}\frac{2^{-(m-k)D}(1-2^{-(k+1)D})}{1-2^{-D}} \\
&= 2WU\sum_{k=0}^{m}\left(\frac{2^{-(m-k)D}}{1-2^{-D}} - \frac{2^{-(m+1)D}}{1-2^{-D}}\right) \\
&= 2WU\left(\frac{1-2^{-(m+1)D}}{(1-2^{-D})^2} - (m+1)\frac{2^{-(m+1)D}}{1-2^{-D}}\right) \\
&< \frac{2WU}{(1-2^{-D})^2}
\end{aligned}$$

- D=1: $8WU$
- D=2: $\frac{7}{2}WU$
- D=3: $\frac{5}{2}WU$

V. Prove Lemma 9.44: The full-weighting operator satisfies

$$dim\mathcal{R}(I_h^{2h}) = \frac{n}{2} - 1, \qquad dim\mathcal{N}(I_h^{2h}) = \frac{n}{2}$$

Solution:

$I_h^{2h} : \mathbb{R}^{n-1} \to \mathbb{R}^{\frac{n}{2}-1}$ 是一个线性映射，就有 $dim\mathcal{R}(I_h^{2h}) = rank(I_h^{2h})$

而 $I_h^{2h} = \begin{bmatrix} 1 & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & 1 & 2 & 1 & \\ & & & & & & 1 & 2 & 1 \end{bmatrix}$, 设 C_j 为 I_h^{2h} 的第 j 列

按 $j = 2k-1, k = 1, 2, \cdots, \frac{n}{2}-1$ 的顺序作变换：$C_{j+1} - 2C_j$，$C_{j+2} - C_j$ 得

$$I_h^{2h} \xrightarrow[C_{j+1}-2C_j,\ C_{j+2}-C_j]{for\ j=1;j<n-1;j+=2} \begin{bmatrix} 1 & 0 & 0 & & & & & & \\ & & 1 & 0 & 0 & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & 1 & 0 & 0 & \\ & & & & & & 1 & 0 & 0 \end{bmatrix}$$

所以 $dim\mathcal{R}(I_h^{2h}) = rank(I_h^{2h}) = \frac{n}{2} - 1$

又有 $dim\mathcal{R}(I_h^{2h}) + dim\mathcal{N}(I_h^{2h}) = dim(\mathbb{R}^{n-1}) = n - 1 \Rightarrow dim\mathcal{N}(I_h^{2h}) = \frac{n}{2}$

VI. Exercise 9.11. For $\Omega = (0,1)$,plot to show that the maximum wavenumber that is representable on Ω^h is $n_{max} = \frac{1}{h}$. What if we require that the Fourier mode be 0 at boundary points?

以 $h = 0.1$ 为例，绿色的为实际波形

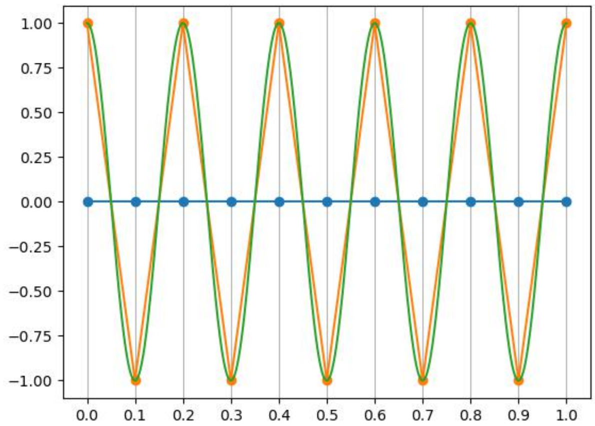


图 1: $k = 10,\ w_{k,j} = \cos(kx_j\pi),\ x_j = jh,\ j = 1, 2, \cdots, 9$

从上图可以看出最大能够表示的波数 $n_{max} = \frac{1}{h}$

若限制边界点为 0，那么最大能表示的波数不再为 $\frac{1}{h}$ 而是 $\frac{1}{h} - 1$, 同样以 $h = 0.1$ 为例

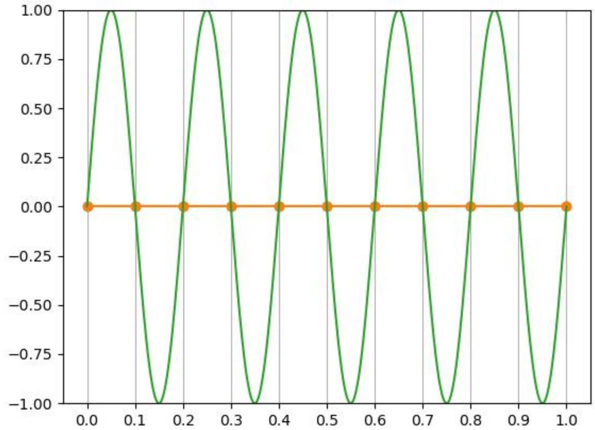
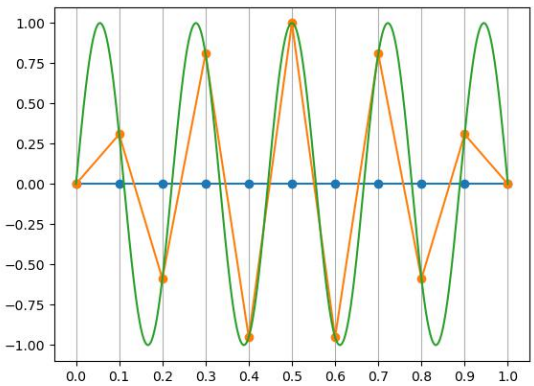
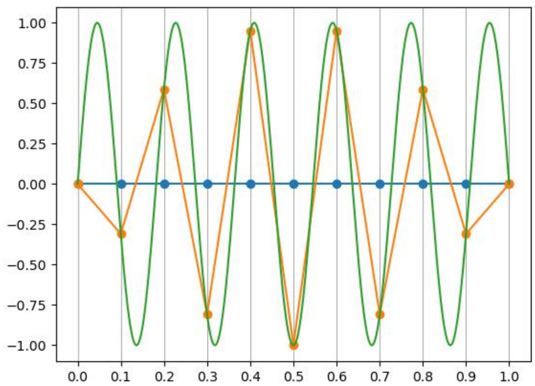


图 2: $k = 10,\ w_{k,j} = \sin(kx_j\pi),\ x_j = jh,\ j = 1, 2, \cdots, 9$

可见当 $k = \frac{1}{h}$ 时所表示的波数为 0



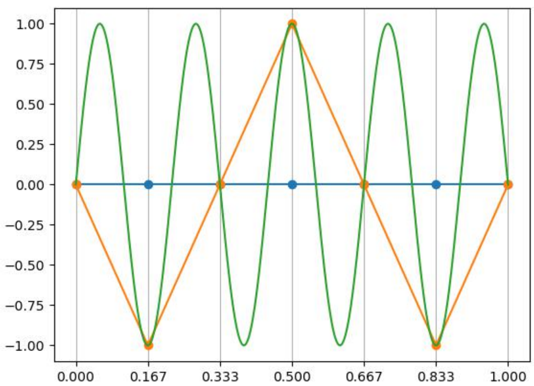
(a) $k = 9,\ w_{k,j} = \sin(kx_j\pi),\ x_j = jh,\ j = 1, 2, \cdots, 9$



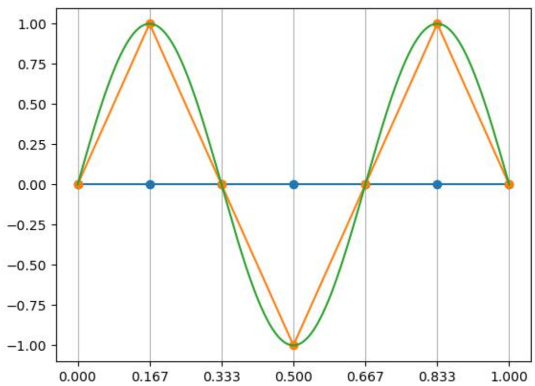
(b) $k = 11,\ w_{k,j} = \sin(kx_j\pi),\ x_j = jh,\ j = 1, 2, \cdots, 9$

如上图所示，限制边界点为 0 时，最大能表示的波数 $n_{max} = \frac{1}{h} - 1$

VII. Exercise 9.14.Plot the case of $n = 6$ for the mode with $k = \frac{3}{2}n$ is represented by $k = \frac{1}{2}n$



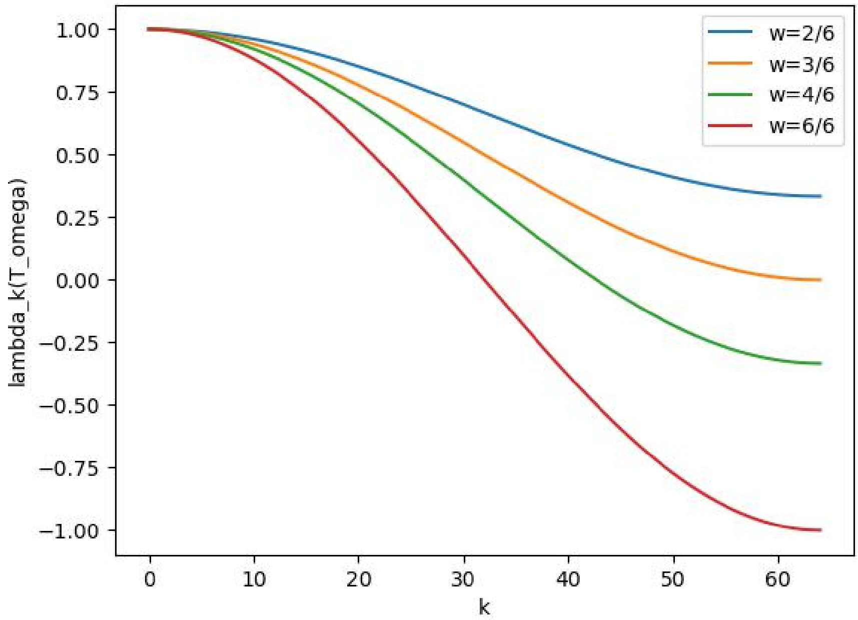
(c) $k = \frac{3}{2}n$



(d) $k = \frac{1}{2}n$

可见 $k = \frac{3}{2}n$ 和 $k = \frac{1}{2}n$ 的 Fourier mode 除了相差一个正负号是一样的

VIII. Exercise 9.18. Write a program to reproduce Fig. 2.7 in the book by Briggs et al. [2000]. For $n = 64,\ \omega \in [0, 1]$, verify $\rho(T_\omega) \geq 0.9986$ and hence slow convergence.



由上图可见

$$\begin{aligned}
\rho(T_\omega) \geq \min_{\omega} \rho(T_\omega) &= \min_{\omega} \max_{k=1,2,\cdots,63} |\lambda_k(T_\omega)| \\
&= \min_{\omega} \max \{|\lambda_1(T_\omega)|, |\lambda_{63}(T_\omega)|\} \\
&= \min_{\omega} \lambda_1(T_\omega) \\
&= \lambda_1(T_1) \\
&= 1 - 2\sin^2 \frac{\pi}{128} \\
&\approx 0.9988 \geq 0.9986
\end{aligned}$$