I. Prove Theorem 10.117 by induction

Solution

设

$$A = \begin{bmatrix} -\alpha_{s-1} & -\alpha_{s-2} & \cdots & -\alpha_1 & -\alpha_0 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{bmatrix},\ \Theta_n = \begin{bmatrix} \theta_n \\ \theta_{n-1} \\ \vdots \\ \theta_{n-s+1} \end{bmatrix},\ Y_n = \begin{bmatrix} y_n \\ y_{n-1} \\ \vdots \\ y_{n-s+1} \end{bmatrix},\ \phi_n = \begin{bmatrix} \psi_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

对于线性差分方程

$$\theta_{n+s} = \sum_{i=0}^{s-1} \alpha_i \theta_{n+i} = 0$$

以及初值

$$\Theta_0 = \begin{bmatrix} \theta_0 \\ \theta_{-1} \\ \vdots \\ \theta_{-s+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

有唯一解 $\Theta_n = A^n\Theta_0 \Rightarrow \Theta_n = A\Theta_{n-1}$
而原方程等价于 $Y_{n+s} = AY_{n+s-1} + \phi_{n+s}$
设

$$\varPsi_n = \left[\Theta_{n-s+1}, \Theta_{n-s+2}, \cdots, \Theta_{n-1}, \Theta_n,\right] \Rightarrow A\varPsi_n = \varPsi_{n+1}$$

$$\Rightarrow \widetilde{Y}_{s-1} = \varPsi_{s-1}^{-1} Y_{s-1} \Leftrightarrow Y_{s-1} = \varPsi_{s-1}\widetilde{Y}_{s-1}$$

并且 $\phi_n = \psi_n\Theta_0$
下面将用数学归纳法证明

$$Y_n = \varPsi_n\widetilde{Y}_{s-1} + \sum_{i=s}^{n} \psi_i\Theta_{n-i}$$

1. $n = s$ 时, $Y_s = AY_{s-1} + \phi_s = A\varPsi_{s-1}\widetilde{Y}_{s-1} + \psi_s\Theta_0 = \varPsi_s\widetilde{Y}_{s-1} + \psi_s\Theta_0$ 成立
2. 假设当 $n = k$ 时, $Y_k = \varPsi_k\widetilde{Y}_{s-1} + \sum_{i=s}^{k} \psi_i\Theta_{k-i}$ 成立

   那么当 $n = k + 1$ 时

$$\begin{aligned} Y_{k+1} =& AY_k + \phi_{k+1} \\ =& A\varPsi_k\widetilde{Y}_{s-1} + \sum_{i=s}^{k} \psi_i A\Theta_{k-i} + \psi_{k+1}\Theta_0 \\ =& \varPsi_{k+1}\widetilde{Y}_{s-1} + \sum_{i=s}^{k} \psi_i\Theta_{k-i+1} + \psi_{k+1}\Theta_0 \\ =& \varPsi_{k+1}\widetilde{Y}_{s-1} + \sum_{i=s}^{k+1} \psi_i\Theta_{k+1-i} \end{aligned}$$

所以对于 $\forall n >= s, n \in \mathbb{N}$，成立

$$Y_n = \varPsi_n\widetilde{Y}_{s-1} + \sum_{i=s}^{n} \psi_i\Theta_{n-i} \Rightarrow y_n = \sum_{i=0}^{s-1} \theta_{n-i}\widetilde{y}_i + \sum_{i=s}^{n} \theta_{n-i}\psi_i$$

II. Prove that a convergent LMM is consistent, by considering the particular IVP promblems

$$u'(t) = f(t) = 0, u(0) = 1;$$

$$u'(t) = f(t) = 1, u(0) = 0.$$

Solution

- 考虑初值问题 $u'(t) = f(t) = 0, u(0) = 1$, 其真解为 $u(t) = 1$

  而步长为 k 的 LMM 对应的差分方程为 $\sum_{i=0}^{s} \alpha_i U_k^{n+i} = 0$，设其数值解为 $U_k^n = u_k(nk)$

  因为 LMM 收敛，所以对于任意的时间 T，$\lim_{k\to 0, Nk=T} U_k^N = u(T) = 1$

  所以

$$0 = \lim_{k\to 0, nk=T} \sum_{i=0}^{s} \alpha_i U_k^{n+i} = \sum_{i=0}^{s} \alpha_i \lim_{k\to 0, nk=T} U_k^{n+i} = \sum_{i=0}^{s} \alpha_i \lim_{k\to 0} u_k(T+ik) = \sum_{i=0}^{s} \alpha_i = \rho(1)$$

- 再考虑初值问题 $u'(t) = f(t) = 1, u(0) = 0$, 其真解为 $u(t) = t$

  步长为 k 的 LMM 对应的差分方程为 $\sum_{i=0}^{s} \alpha_i U_k^{n+i} = k \sum_{i=0}^{s} \beta_i$

  将 $U_k^n = nk\frac{\sigma(1)}{\rho'(1)}$ 代入上式，有

$$\begin{aligned}
LHS &= \sum_{i=0}^{s} \alpha_i U_k^{n+i} \\
&= \sum_{i=0}^{s} \alpha_i (n+i) k \frac{\sigma(1)}{\rho'(1)} \\
&= \sum_{i=0}^{s} i\alpha_i k \frac{\sigma(1)}{\rho'(1)} + (\sum_{i=0}^{s} \alpha_i) nk \frac{\sigma(1)}{\rho'(1)} \\
&= \rho'(1) k \frac{\sigma(1)}{\rho'(1)} + \rho(1) nk \frac{\sigma(1)}{\rho'(1)} \\
&= k\sigma(1) \\
&= k \sum_{i=0}^{s} \beta_i = RHS
\end{aligned}$$

  第五个等号由上面的 $\rho(1) = 0$ 可得，所以 $U_k^n = nk\frac{\sigma(1)}{\rho'(1)}$ 是该 LMM 的一个特解

  因为 LMM 收敛，所以对于任意的时间 T

$$\lim_{k\to 0, nk=T} U_k^n = \lim_{k\to 0, nk=T} nk \frac{\sigma(1)}{\rho'(1)} = T\frac{\sigma(1)}{\rho'(1)} = u(T) = T$$

  就能得到 $\frac{\sigma(1)}{\rho'(1)} = 1 \Rightarrow \sigma(1) = \rho'(1)$

综上就证明 a convergent LMM is consistent(即 $\rho(1) = 0$, $\sigma(1) = \rho'(1)$)

III. Write a program to reproduce the RAS plots in Figures 10.4 & 10.5

Solution

程序使用语言为 Python，调用了 numpy, sympy 以及 matplotlib 包（若没有，可用 pip 直接安装）

在作业目录下运行`python3 RAS.py`即可得到如下图像