I. Suppose a grid function $\mathbf{g}:\mathbf{X}\to\mathbb{R}$ has $\mathbf{X}:=\{x_1,x_2,\cdots,x_N\}, g_1=O(h), g_N=O(h)$ and $g_j=O(h^2)$ for all $j=2,3,\cdots,N-1$. Show that $\|\mathbf{g}\|_{L_\infty}=O(h), \|\mathbf{g}\|_{L_1}=O(h^2), \|\mathbf{g}\|_{L_2}=O(h^{\frac{3}{2}})$

Solution:

由定义可知

$$\|\mathbf{g}\|_{L_\infty}=max_{1\le i\le N}|g_i|=O(h)$$

$$\|\mathbf{g}\|_{L_1}=h\sum_{i=1}^{N}|g_i|=hO(h)=O(h^2)$$

$$\|\mathbf{g}\|_{L_1}=(h\sum_{i=1}^{N}|g_i|^2)^{\frac{1}{2}}=h^{\frac{1}{2}}\sqrt{O(h^2)}=h^{\frac{1}{2}}O(h)=O(h^{\frac{3}{2}})$$

II. $w_{k,j}=\sin\frac{jk\pi}{m+1}$, show that

$$\forall i\neq k,\ \langle\mathbf{w_i},\mathbf{w_k}\rangle=0;\ \forall k=1,2,\cdots,m,\ \langle\mathbf{w_k},\mathbf{w_k}\rangle=\frac{m+1}{2}$$

Solution:

$$\begin{aligned}\langle\mathbf{w_i},\mathbf{w_k}\rangle&=\sum_{j=1}^{m}\sin\frac{jk\pi}{m+1}\cdot\sin\frac{ji\pi}{m+1}\\&=\frac{1}{2}\sum_{j=1}^{m}\left(\cos\frac{(k-i)j}{m+1}\pi-\cos\frac{(k+i)j}{m+1}\pi\right)\end{aligned}\tag{1}$$

因为 $k-i$ 和 $k+i$ 同奇偶，那么先考虑下面的求和

$$\sum_{j=1}^{m}\cos\frac{nj\pi}{m+1}$$

1. n 为奇数

$$\begin{aligned}\sum_{j=1}^{m}\cos\frac{nj\pi}{m+1}&=\begin{cases}\sum_{j=1}^{\lfloor\frac{m}{2}\rfloor}\left(\cos\frac{nj\pi}{m+1}+\cos\frac{n(m+1-j)\pi}{m+1}\right) & m\ is\ even\\ \sum_{j=1}^{\lfloor\frac{m}{2}\rfloor}\left(\cos\frac{nj\pi}{m+1}+\cos\frac{n(m+1-j)\pi}{m+1}\right)+\cos\frac{n\pi}{2} & m\ is\ odd\end{cases}\\&=\sum_{j=1}^{\lfloor\frac{m}{2}\rfloor}\left(\cos\frac{nj\pi}{m+1}+\cos(n\pi-\frac{nj\pi}{m+1})\right)\\&=\sum_{j=1}^{\lfloor\frac{m}{2}\rfloor}\left(\cos\frac{nj\pi}{m+1}-\cos\frac{nj\pi}{m+1}\right)\\&=0\end{aligned}\tag{2}$$

2. n 为偶数

$$\begin{aligned}
\sum_{j=1}^{m}\cos\frac{nj\pi}{m+1} &= \sum_{j=1}^{m+1}\cos\frac{nj\pi}{m+1} - \cos\frac{n(m+1)\pi}{m+1} \\
&= \sum_{j=1}^{m+1}\cos\frac{nj\pi}{m+1} - 1 \\
&= \frac{1}{2}\sum_{j=1}^{m+1}\left(e^{i\frac{nj\pi}{m+1}} + e^{-i\frac{nj\pi}{m+1}}\right) - 1 \\
&= \frac{1}{2}\sum_{j=1}^{m+1}\left((e^{i\frac{n\pi}{m+1}})^j + (e^{-i\frac{n\pi}{m+1}})^j\right) - 1 \\
&= \frac{1}{2}\left(\frac{e^{i\frac{n\pi}{m+1}}(1-e^{in\pi})}{1-e^{i\frac{n\pi}{m+1}}} + \frac{e^{-i\frac{n\pi}{m+1}}(1-e^{-in\pi})}{1-e^{-i\frac{n\pi}{m+1}}}\right) - 1 \\
&= -1
\end{aligned} \tag{3}$$

将 (2) 式和 (3) 式代回 (1) 式以及 $k-i$ 和 $k+i$ 同奇偶得

$$\langle \mathbf{w_i}, \mathbf{w_k}\rangle = \begin{cases} 0 & \text{if } i \neq k; \\ \frac{1}{2}\left(m - \sum_{j=1}^{m}\cos\frac{2kj\pi}{m+1}\right) = \frac{m+1}{2} & \text{if } i = k; \end{cases}$$

III. Show that all elements of the first column of $\mathbf{B_E} = \mathbf{A_E^{-1}}$ are O(1) where

$$\mathbf{A_E} = \frac{1}{h^2}\begin{bmatrix} -h & h & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 0 & h^2 \end{bmatrix}$$

Solution:

将矩阵 $\mathbf{A_E}$ 分块

$$\mathbf{A_E} = \left[\begin{array}{c|c} \mathbf{C} & \mathbf{a_E} \\ \hline \mathbf{0}^T & 1 \end{array}\right],\ \mathbf{C} = \frac{1}{h^2}\begin{bmatrix} -h & h & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix},\ \mathbf{a_E} = \frac{1}{h^2}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

就有

$$\mathbf{B_E} = \mathbf{A_E}^{-1} = \left[\begin{array}{c|c} \mathbf{C}^{-1} & -\mathbf{C}^{-1}\mathbf{a_E} \\ \hline \mathbf{0}^T & 1 \end{array}\right]$$

所以只用计算 $\mathbf{C}^{-1}$ 的第一列，并验证其元素为 $O(1)$ 即可
再将 $\mathbf{C}$ 分块

$$\mathbf{C} = \left[\begin{array}{c|c} -\frac{1}{h} & h\mathbf{a_0}^T \\ \hline \mathbf{a_0} & \mathbf{A} \end{array}\right],\ \mathbf{A} = \frac{1}{h^2}diag(1,-2,1),\ \mathbf{a_0} = \frac{1}{h^2}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$$

记 $\mathbf{D} = (d_{ij})$ 为 $\mathbf{C}^{-1}$, 就有

$$\mathbf{D} = \mathbf{C}^{-1} = \left[\begin{array}{c|c} (-\frac{1}{h} - h\mathbf{a_0}^T\mathbf{A}^{-1}\mathbf{a_0})^{-1} & \mathbf{X} \\ \hline -\mathbf{A}^{-1}\mathbf{a_0}(-\frac{1}{h} - h\mathbf{a_0}^T\mathbf{A}^{-1}\mathbf{a_0})^{-1} & \mathbf{X} \end{array}\right]$$

由讲义中的引理可知

$$\mathbf{A}^{-1} = \mathbf{B} = h\begin{bmatrix} x_1(x_1-1) & x_1(x_2-1) & \cdots & x_1(x_m-1) \\ x_1(x_2-1) & x_2(x_2-1) & \cdots & x_2(x_m-1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(x_m-1) & x_2(x_m-1) & \cdots & x_m(x_m-1) \end{bmatrix},\ x_i = ih$$

记 $\mathbf{B}_j$ 为 $\mathbf{B}$ 的第 j 列, 那么

$$\begin{aligned} d_{11} &= (-\frac{1}{h} - h\mathbf{a_0}^T\mathbf{A}^{-1}\mathbf{a_0})^{-1} \\ &= (-\frac{1}{h} - \frac{1}{h}\mathbf{a_0}^T\mathbf{B}_1)^{-1} \\ &= (-\frac{1}{h} - \frac{1}{h^3}hx_1(x_1-1))^{-1} \\ &= (-\frac{1}{h} - \frac{h-1}{h})^{-1} \\ &= -1 \\ &= O(1) \end{aligned}$$

$$\begin{bmatrix} d_{2,1} \\ d_{3,1} \\ \vdots \\ d_{m,1} \end{bmatrix} = -\mathbf{A}^{-1}\mathbf{a_0}(-\frac{1}{h} - h\mathbf{a_0}^T\mathbf{A}^{-1}\mathbf{a_0})^{-1} = -\frac{1}{h^2}\mathbf{B}_1 d_{11} = \begin{bmatrix} h-1 \\ 2h-1 \\ \vdots \\ mh-1 \end{bmatrix} = \begin{bmatrix} O(1) \\ O(1) \\ \vdots \\ O(1) \end{bmatrix}$$

综上，$\mathbf{B_E}$ 的第一列的所有元素都是 $O(1)$

IV. Show that the LET of the FD method in two-dimensional BVP

$$\begin{cases} -\frac{\partial^2}{\partial x^2}u(x,y) - \frac{\partial^2}{\partial y^2}u(x,y) = f(x,y) \\ u(x,y)|_{\partial\Omega} = 0 \end{cases}$$

is

$$\tau_{i,j} = -\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\bigg|_{(x_i,y_j)} + O\left(h^4\right)$$

Solution:

$$D_x^2 u(x_i,y_j) := \frac{u(x_{i-1},y_j) - 2u(x_i,y_j) + u(x_{i+1},y_j)}{h^2}$$
$$D_y^2 u(x_i,y_j) := \frac{u(x_i,y_{j-1}) - 2u(x_i,y_j) + u(x_i,y_{j+1})}{h^2}$$

有

$$D_x^2 u(x_i,y_j) - \frac{\partial^2}{\partial x^2}u(x_i,y_j) = \frac{h^2}{12}\frac{\partial^4}{\partial x^4}u(x_i,y_j) + O(h^4)$$
$$D_y^2 u(x_i,y_j) - \frac{\partial^2}{\partial y^2}u(x_i,y_j) = \frac{h^2}{12}\frac{\partial^4}{\partial y^4}u(x_i,y_j) + O(h^4)$$

$$\hat{\mathbf{U}} := [u(x_1,y_1), u(x_2,y_1), \cdots, u(x_m,y_1), u(x_1,y_2), u(x_2,y_2), \cdots, u(x_m,y_2), \cdots, u(x_m,y_m)]^T$$

由 $\boldsymbol{\tau} = \mathbf{A}_{2D}\hat{\mathbf{U}} - \mathbf{F}$ 以及 $f_{ij} = -(\frac{\partial^2}{\partial x^2}u(x_i,y_j) + \frac{\partial^2}{\partial y^2}u(x_i,y_j))$ 可得

$$\begin{aligned} \tau_{ij} &= -(D_x^2 u(x_i,y_j) + D_y^2 u(x_i,y_j)) - fij \\ &= -\left(D_x^2 u(x_i,y_j) - \frac{\partial^2}{\partial x^2}u(x_i,y_j) + D_y^2 u(x_i,y_j) - \frac{\partial^2}{\partial y^2}u(x_i,y_j)\right) \\ &= -\frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\bigg|_{(x_i,y_j)} + O\left(h^4\right) \end{aligned}$$

V. Show that, in Example 7.61, the LTE at an irregular equation-discretization point is $O(h)$ while the LTE at a regular equation discretization point is $O(h^2)$

先考虑用 $u(\bar{x}-\alpha h), u(\bar{x}), u(\bar{x}+\beta h)$ 的三点差分格式来近似 $u''(x)$ 在 $\bar{x}$ 处的值 $(0<\alpha,\beta\le 1)$

$$u(\bar{x}-\alpha h)=u(\bar{x})-\alpha h u'(\bar{x})+\frac{1}{2}\alpha^2h^2u''(\bar{x})-\frac{1}{6}\alpha^3h^3u'''(\bar{x})+\frac{1}{24}\alpha^4h^4u^{(4)}(\bar{x})+O(h^5) \tag{4}$$

$$u(\bar{x}+\beta h)=u(\bar{x})+\beta h u'(\bar{x})+\frac{1}{2}\beta^2h^2u''(\bar{x})+\frac{1}{6}\beta^3h^3u'''(\bar{x})+\frac{1}{24}\beta^4h^4u^{(4)}(\bar{x})+O(h^5) \tag{5}$$

$\beta\times(4)+\alpha\times(5)$ 得

$$\frac{\beta u(\bar{x}-\alpha h)-(\alpha+\beta)u(\bar{x})+\alpha u(\bar{x}+\beta h)}{\frac{1}{2}\alpha\beta(\alpha\varphi\theta+\beta)h^2}=u''(\bar{x})+\frac{1}{3}(\beta-\alpha)hu'''(\bar{x})+\frac{\alpha^3+\beta^3}{12(\alpha+\beta)}h^2u^{(4)}(\bar{x})+O(h^3)$$

可见只有当 $\alpha=\beta$ 时上述格式是二阶精度的，否则是一阶精度
用上述的差分格式来近似偏导数 $\frac{\partial^2 u}{\partial x^2}$ 和 $\frac{\partial^2 u}{\partial y^2}$ 记为 D_x^2u 和 D_y^2u
那么可以定义对应的二维五点差分格式
设 $P=(x,y)$，与之相邻的四个点为 $A=(x-\varphi h,y), B=(x+\theta h,y), C=(x,y+\alpha h), D=(x,y-\beta h)$
$(0<\alpha,\beta,\varphi,\theta\le 1)$

$$L_hU_p:=\frac{(\varphi+\theta)U_p-\theta U_A-\varphi U_B}{\frac{1}{2}\varphi\theta(\varphi+\theta)h^2}+\frac{(\alpha+\beta)U_P-\beta U_C-\alpha U_D}{\frac{1}{2}\alpha\beta(\alpha+\beta)h^2}$$

(取 $\varphi=\beta=1$ 即和讲义中的 (7.88) 式一致)
由 $\boldsymbol{T}=L_h\hat{\mathbf{U}}-\mathbf{F}$ 以及 $f_P=-(\frac{\partial^2}{\partial x^2}u(P)+\frac{\partial^2}{\partial y^2}u(P))$ 可得
那么

$$\begin{aligned}T_p&=-\left(D_x^2u(P)-\frac{\partial^2}{\partial x^2}u(P)+D_y^2u(P)-\frac{\partial^2}{\partial y^2}u(P)\right)\\&=-\frac{1}{3}(\alpha-\beta)h\frac{\partial^3}{\partial x^3}u(P)-\frac{1}{3}(\theta-\varphi)h\frac{\partial^3}{\partial y^3}u(P)-\frac{\alpha^3+\beta^3}{12(\alpha+\beta)}h^2\frac{\partial^4}{\partial x^4}u(P)-\frac{\varphi^3+\theta^3}{12(\varphi+\theta)}h^2\frac{\partial^4}{\partial y^4}u(P)+O(h^3)\end{aligned}$$

1. 当 P 是 regular equation-discretization point 时 $\varphi=\theta=\alpha=\beta=1$，$\Rightarrow \tau_p=O(h^2)$
2. 当 P 是 irregular equation-discretization point 时 φ,θ 和 α,β 至少有一组满足其中一个为 1 另一个小于 1 $\Rightarrow \tau_p=O(h)$

VI. Prove Theorem 7.63 by choosing a function ψ to DMP applies

取

$$\psi_P := \begin{cases} E_P + \frac{T_1}{C_1}\phi_P & P \in \mathbf{X_1} \\ E_P + \frac{T_2}{C_2}\phi_P & P \in \mathbf{X_2} \end{cases}$$

有 $L_h E_P = -T_P$，那么

$$\begin{aligned} L_h\psi_P &= \begin{cases} -T_P + \frac{T_1}{C_1}L_h\phi_P & P \in \mathbf{X_1} \\ -T_P + \frac{T_2}{C_2}L_h\phi_P & P \in \mathbf{X_2} \end{cases} \\ &= \begin{cases} -T_P - T_1 & P \in \mathbf{X_1} \\ -T_P - T_2 & P \in \mathbf{X_2} \end{cases} \\ &\leq 0 \end{aligned}$$

所以 $\forall P \in \mathbf{X_1}, \psi_P \leq max_{Q\in\mathbf{X}_{\partial 1}}\psi_Q$; $\forall P \in \mathbf{X_2}, \psi_P \leq max_{Q\in\mathbf{X}_{\partial 2}}\psi_Q$

且 $\forall Q \in \mathbf{X}_\partial = \mathbf{X}_{\partial\mathbf{1}} \cup \mathbf{X}_{\partial\mathbf{2}}$，$E_Q = 0$

进一步由 ϕ 非负可以推出 $max_{Q\in\mathbf{X}_\partial}\psi_Q$ 非负

所以

$$\begin{aligned} \forall P \in \mathbf{X},\ E_P \leq \psi_P &\leq \max\left\{\max_{P\in\mathbf{X_1}\cup\mathbf{X}_{\partial 1}}\psi_P, \max_{P\in\mathbf{X_2}\cup\mathbf{X}_{\partial 2}}\psi_P\right\} \\ &\leq \max\left\{\max_{Q\in\mathbf{X}_{\partial 1}}(E_Q + \frac{T_1}{C_1}\phi_Q), \max_{Q\in\mathbf{X}_{\partial 2}}(E_Q + \frac{T_2}{C_2}\phi_Q)\right\} \\ &= max\left\{\frac{T_1}{C_1}\max_{Q\in\mathbf{X}_{\partial 1}}\phi_Q, \frac{T_2}{C_2}\max_{Q\in\mathbf{X}_{\partial 2}}\phi_Q\right\} \\ &\leq \left(\max_{Q\in\mathbf{X}_\partial}\phi(Q)\right)\max\left\{\frac{T_1}{C_1}, \frac{T_2}{C_2}\right\} \end{aligned}$$

取

$$\psi_P := \begin{cases} -E_P + \frac{T_1}{C_1}\phi_P & P \in \mathbf{X_1} \\ -E_P + \frac{T_2}{C_2}\phi_P & P \in \mathbf{X_2} \end{cases}$$

同理可得

$$-E_P \leq \left(\max_{Q\in\mathbf{X}_\partial}\phi(Q)\right)\max\left\{\frac{T_1}{C_1}, \frac{T_2}{C_2}\right\}$$

综上,

$$\forall P \in \mathbf{X},\quad |E_P| \leq \left(\max_{Q\in\mathbf{X}_\partial}\phi(Q)\right)\max\left\{\frac{T_1}{C_1}, \frac{T_2}{C_2}\right\}$$