I. Write down the Butcher tableaux of the modified Euler method, the improved Euler method, and Heun's third-order method

Solution:

improved Euler method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

modified Euler method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

Heun's third-order method

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

II. Verify that the RK method can be rewritten as

$$\begin{cases} \xi_i = U^n + k\sum_{j=1}^{s} a_{i,j} f(\xi_j, t_n + c_j k), \\ U^{n+1} = U^n + k\sum_{j=1}^{s} b_j f(\xi_j, t_n + c_j k), \end{cases}$$

wehre $i = 1, 2, \cdots, s$

Solution:

令 $\xi_i = U^n + k\sum_{j=1}^{s} a_{i,j} f(\xi_j, t_n + c_j k),\ i = 1, 2, \cdots, s$
那么就有 $y_i = f(U^n + k\sum_{j=1}^{s} a_{i,j} y_j, t_n + c_j k) = f(\xi_i, t_n + c_i k)$
所以 $U^{n+1} = U^n + k\sum_{j=1}^{s} b_j y_j = U^n + k\sum_{j=1}^{s} b_j f(\xi_j, t_n + c_j k)$

III. There are three one-parameter families of third-order three-stage ERK methods, one of which is

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
\frac{2}{3} & \frac{2}{3} & 0 & 0 \\
\frac{2}{3} & \frac{2}{3}-\frac{1}{4\alpha} & \frac{1}{4\alpha} & 0 \\
\hline
 & \frac{1}{4} & \frac{3}{4}-\alpha & \alpha
\end{array}
$$

where α is the free parameter. Derive the above family. Does Heun's third-order formula belong to this family?

Solution:

设 three-stage ERK

$$
\begin{cases}
y_1 = f(U^n, t_n) \\
y_2 = f(U^n + ka_{2,1}y_1, t_n + c_2k) \\
y_3 = f(U^n + ka_{3,1}y_1 + ka_{3,2}y_2, t_n + c_3k) \\
U^{n+1} = U^n + k(b_1y_1 + b_2y_2 + b_3y_3)
\end{cases}
$$

再设

$$
\begin{aligned}
y_1(t_n) &= f(u(t_n), t_n) \\
y_2(t_n) &= f(u(t_n) + ka_{2,1}y_1(t_n), t_n + c_2k) \\
y_3(t_n) &= f(u(t_n) + ka_{3,1}y_1(t_n) + ka_{3,2}y_2(t_n), t_n + c_3k)
\end{aligned}
$$

那么 $\mathcal{L}u(t_n) = u(t_{n+1}) - u(t_n) - k(b_1y_1(t_n) + b_2y_2(t_n) + b_3y_3(t_n))$
将 $u(t_{n+1})$ 展开到 k^3 项，$y_1(t_n) + y_2(t_n) + y_3(t_n)$ 展开到 k^2 项，并用 f 表示 $f(u(t_n), t_n)$, 其他同理，得

$$
\begin{aligned}
u(t_{n+1}) =& u + ku' + \frac{k^2}{2}u'' + \frac{k^3}{6}u''' + O(k^4) \\
y_1(t_n) =& f \\
y_2(t_n) =& f + kc_2(f_uf + f_t) + \frac{k^2}{2}c_2^2(f_{uu}f^2 + 2f_{ut}f + f_tt) + O(k^3) \\
y_3(t_n) =& f + k(a_{3,1} + a_{3,2})(f_uf + f_t) \\
&+ k^2(\frac{1}{2}(a_{3,1} + a_{3,2})^2(f_{uu}f^2 + 2f_{ut}f + f_tt) + a_{3,2}c_2(f_u^2f + f_uf_t)) + O(k^3)
\end{aligned}
$$

代回 $\mathcal{L}u(t_n)$ 得

$$
\begin{aligned}
\mathcal{L}u(t_n) = & k(1 - b_1 - b_2 - b_3)u' + k^2(\frac{1}{2} - b_2c_2 - b_3(a_{3,1} + a_{3,2}))u'' \\
&+ k^3(\frac{1}{6} - \frac{1}{2}b_2c_2^2 - \frac{1}{2}b_3(a_{3,1} + a_{3,2})^2)(f_{uu}f^2 + 2f_{ut}f + f_tt) \\
&+ k^3(\frac{1}{6} - b_3a_{3,2}c_2)(f_u^2f + f_uf_t) + O(k^4)
\end{aligned}
$$

要达到三阶精度，那么 k, k^2, k^3 的系数为 0 得到

$$\begin{cases} b_1 + b_2 + b_3 = 0 \\ b_2 c_2 + b_3 c_3 = \frac{1}{2} \\ b_2 c_2^2 + b_3 c_3^2 = \frac{1}{3} \\ b_3 a_{3,2} c_2 = \frac{1}{6} \end{cases}$$

四个方程六个未知量，所以有两个自由参数，设 $b_3 = \alpha$，$a_{3,2} = \beta$ 代入解得

$$\begin{cases} b_1 = -3\alpha\beta \mp \frac{\alpha}{2}\sqrt{48\alpha\beta^2 - 12\beta + 1} - \frac{\alpha}{2} + 1 \\ b_2 = 3\alpha\beta \pm \frac{\alpha}{2}\sqrt{48\alpha\beta^2 - 12\beta + 1} - \frac{\alpha}{2} \\ c_2 = \frac{1}{6\alpha\beta} \\ c_3 = \mp\frac{1}{12\alpha\beta}\sqrt{48\alpha\beta^2 - 12\beta + 1} + \frac{1}{12\alpha\beta} \end{cases}$$

将 $\beta = \frac{1}{4\alpha}$ 代入即得

$$\begin{cases} b_1 = \frac{1}{4} \\ b_2 = \frac{3}{4} - \alpha \\ b_3 = \alpha \\ a_{2,1} = c_2 = \frac{2}{3} \\ a_{3,1} = \frac{2}{3} - \frac{1}{4\alpha} \\ a_{3,2} = \frac{1}{4\alpha} \\ c_3 = \frac{2}{3} \end{cases}, \quad \begin{cases} b_1 = \frac{1}{4} - \alpha \\ b_2 = \frac{3}{4} \\ b_3 = \alpha \\ a_{2,1} = c_2 = \frac{2}{3} \\ a_{3,1} = -\frac{1}{4\alpha} \\ a_{3,2} = \frac{1}{4\alpha} \\ c_3 = 0 \end{cases}$$

前面的即为所求的 family of third-order three-stage ERK

取 $\alpha = \frac{3}{4}$，得 $a_{2,1} = \frac{2}{3}$，$a_{3,1} = a_{3,2} = \frac{1}{3}$, Heun's third-order formula 与其不符，所以 Heun's third-order formula 不属于该 famliy

IV. Show that the quadrature formula of a RK method is exact for all polynomials f of degree < r,i.e.,

$$\forall f \in \mathcal{P}_{r-1}, \qquad I_s(f) = \int_{t_n}^{t_n+k} f(t)\mathrm{d}t,$$

if and only if the RK method is $B(r)$

Solution:

首先有 $\forall q \in \mathbb{N}$，$f(t) = t^q$

$$\begin{aligned} I(f) &= \int_{t_n}^{t_n+k} t^q \mathrm{d}t \\ &= \frac{1}{q+1} t_n^{q+1}|_{t_n}^{t_n+k} \\ &= \frac{1}{q+1}((t_n+k)^{q+1} - t_n^{q+1}) \\ &= \frac{1}{q+1}\sum_{l=1}^{q+1} C_{q+1}^l k^l t_n^{q+1-l} \end{aligned}$$

以及

$$\begin{aligned}
I_s(f) &= k\sum_{j=1}^{s} b_j(t_n+c_jk)^q \\
&= k\sum_{j=1}^{s} b_j \sum_{l=0}^{q} C_q^l c_j^l k^l t_n^{q-l} \\
&= \sum_{l=0}^{q}(\sum_{j=1}^{s} b_j c_j^l) C_q^l k^{l+1} t_n^{q-l} \\
&= \sum_{l=1}^{q+1}(\sum_{j=1}^{s} b_j c_j^{l-1}) C_q^{l-1} k^l t_n^{q+1-l} \\
&= \frac{1}{q+1}\sum_{l=1}^{q+1}(\sum_{j=1}^{s} b_j c_j^{l-1}) l C_{q+1}^l k^l t_n^{q+1-l}
\end{aligned}$$

必要性：

quadrature formula 对于阶数小于 r 多项式严格成立，那么 $\forall q\in\mathbb{N},\ q<r,\ f(t)=t^q,\ I(f)=I_s(f)$
依次取 $q=0,1,2,\cdots,r-1$，代入上面得式子就能得到

$$\forall l=1,2,\cdots,r,\ (\sum_{j=1}^{s} b_j c_j^{l-1})l=1 \Rightarrow \sum_{j=1}^{s} b_j c_j^{l-1}=\frac{1}{l}$$

所以 RK method 是 $B(r)$

充分性：

RK method 是 $B(r)$, 那么有

$$\forall l=1,2,\cdots,r,\ \sum_{j=1}^{s} b_j c_j^{l-1}=\frac{1}{l}$$

就可以得到 $\forall q\in\mathbb{N},\ q<r,\ f(t)=t^q,\ I(f)=I_s(f)$
那么 $\forall f\in\mathbb{P}_{r-1},\ f(t)=\sum_{i=0}^{r-1}a_it^i$, 也成立 $I(f)=I_s(f)$

V. Show that an s-stage collocation method is at least s-order accurate

Solution:

设 s 阶插值节点为 $t_n, t_n+c_1k, t_n+c_2k,\cdots,t_n+c_sk$ 的插值多项式 $p(t)$ 满足

$$\begin{cases} p(t_n)=u(t_n) \\ \forall i=1,2,\cdots,s,\quad p'(t_n^{(i)})=f(p(t_n^{(i)}),t_n^{(i)})=u'(t_n^{(i)}) \end{cases},\text{where}\quad t_n^{(i)}=t_n+c_ik$$

那么

$$u(t_n)+k\Phi(u(t_n),t_n;k)=p(t_{n+1})$$

所以

$$
\begin{aligned}
\mathcal{L}u(t_n) &= u(t_{n+1}) - u(t_n) - k\Phi(u(t_n), t_n; k) \\
&= u(t_{n+1}) - p(t_{n+1}) \\
&= R_s(u; t_{n+1}) \\
&= \frac{u^{(s+1)}(\xi)}{(s+1)!}((t_n+k)-t_n)\prod_{i=1}^{s}((t_n+k)-(t_n+c_ik)) \\
&= \frac{u^{(s+1)}(\xi)}{(s+1)!}(\prod_{i=1}^{s}(1-c_i))k^{s+1}
\end{aligned}
$$

所以 s-stage collocation method 至少是 s 阶精度的

VI. Prove that the collocation method viewed as an RK method satisfies

$$c_i = \sum_{j=1}^{s} a_{i,j},\ \ i = 1, 2, \cdots, s \quad \textbf{and} \quad \sum_{i=1}^{s} b_i = 1$$

Solution:

Lagrange 插值多项式基函数 $\ell_j(\tau) = \prod_{i\neq j;i=1}^{s}\frac{\tau - c_i}{c_j - c_i}$ 满足 $\sum_{j=1}^{s}\ell_j(\tau) = 1$

$$\forall i = 1, 2, \cdots, s, \sum_{j=1}^{s} a_{i,j} = \sum_{j=1}^{s}\int_0^{c_i}\ell_j(\tau)\mathrm{d}\tau = \int_0^{c_i}\sum_{j=1}^{s}\ell_j(\tau)\mathrm{d}\tau = \int_0^{c_i}\mathrm{d}\tau = c_i$$

$$\sum_{j=1}^{s} b_j = \sum_{j=1}^{s}\int_0^{1}\ell_j(\tau)\mathrm{d}\tau = \int_0^{1}\sum_{j=1}^{s}\ell_j(\tau)\mathrm{d}\tau = \int_0^{1}\mathrm{d}\tau = 1$$

VII. Derive the three-stage IRK method that corresponds to the collocation points $c_1 = \frac{1}{4},\ c_2 = \frac{1}{2},\ c_3 = \frac{3}{4}$

Solution:

对于所给节点 $c_1 = \frac{1}{4},\ c_2 = \frac{1}{2},\ c_3 = \frac{3}{4}$, 有

$$\ell_1(\tau) = \frac{\tau - \frac{1}{2}}{\frac{1}{4} - \frac{1}{2}}\frac{\tau - \frac{3}{4}}{\frac{1}{4} - \frac{3}{4}} = 8\tau^2 - 10\tau + 3$$

$$\ell_2(\tau) = \frac{\tau - \frac{1}{4}}{\frac{1}{2} - \frac{1}{4}}\frac{\tau - \frac{3}{4}}{\frac{1}{2} - \frac{3}{4}} = -16\tau^2 + 16\tau - 3$$

$$\ell_3(\tau) = \frac{\tau - \frac{1}{4}}{\frac{3}{4} - \frac{1}{4}}\frac{\tau - \frac{1}{2}}{\frac{3}{4} - \frac{1}{2}} = 8\tau^2 - 6\tau + 1$$

所以

$$a_{1,1} = \int_0^{\frac{1}{4}}\ell_1(\tau)\mathrm{d}\tau = \frac{23}{48},\ a_{1,2} = \int_0^{\frac{1}{4}}\ell_2(\tau)\mathrm{d}\tau = -\frac{1}{3},\ a_{1,3} = \int_0^{\frac{1}{4}}\ell_3(\tau)\mathrm{d}\tau = \frac{5}{48}$$

$$a_{2,1} = \int_0^{\frac{1}{2}}\ell_1(\tau)\mathrm{d}\tau = \frac{7}{12},\ a_{2,2} = \int_0^{\frac{1}{2}}\ell_2(\tau)\mathrm{d}\tau = -\frac{1}{6},\ a_{2,3} = \int_0^{\frac{1}{2}}\ell_3(\tau)\mathrm{d}\tau = \frac{1}{12}$$

$$a_{3,1}=\int_0^{\frac{3}{4}}\ell_1(\tau)\mathrm{d}\tau=\frac{9}{16},\ a_{3,2}=\int_0^{\frac{3}{4}}\ell_2(\tau)\mathrm{d}\tau=0,\ a_{3,3}=\int_0^{\frac{3}{4}}\ell_3(\tau)\mathrm{d}\tau=\frac{3}{16}$$

$$b_1=\int_0^{1}\ell_1(\tau)\mathrm{d}\tau=\frac{2}{3},\ b_2=\int_0^{1}\ell_2(\tau)\mathrm{d}\tau=-\frac{1}{3},\ b_3=\int_0^{1}\ell_3(\tau)\mathrm{d}\tau=\frac{2}{3}$$

所以其对应的 three-stage IRK method 的 Butcher tubleau 为

$$\begin{array}{c|ccc}
\frac{1}{4} & \frac{23}{48} & -\frac{1}{3} & \frac{5}{48} \\
\frac{1}{2} & \frac{7}{12} & -\frac{1}{6} & \frac{1}{12} \\
\frac{3}{4} & \frac{9}{16} & 0 & \frac{3}{16} \\
\hline
 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3}
\end{array}$$

VIII. Show $B(s+r)$ and $C(s)$ imply $D(r)$ via multiplying the two vectors $u_j := \sum_{i=1}^{s} b_i c_i^{m-1} a_{i,j}$ and $v_j := \frac{1}{m} b_j(1-c_j^m)$ by the Vandermonde matrix $V(c_1, c_2, \cdots, c_s)$ in Definition 2.3.

Solution:

由 $B(s+r)$ 和 $C(r)$ 可知

$$\forall m=1,2,\cdots,s+r,\ s.t. \sum_{j=1}^{s} b_j c_j^{m-1}=\frac{1}{m}$$

$$\begin{cases} \forall i=1,2,\cdots,s, \\ \forall m=1,2,\cdots,s, \end{cases} s.t. \sum_{j=1}^{s} a_{i,j} c_j^{m-1}=\frac{c_i^m}{m}$$

设 $\mathbf{u} := \left[\sum_{j=1}^{s} b_j c_j^{m-1} a_{j,1}, \sum_{j=1}^{s} b_j c_j^{m-1} a_{j,2}, \cdots, \sum_{j=1}^{s} b_j c_j^{m-1} a_{j,s}\right]^T$

以及 $\mathbf{v} := \left[\frac{1}{m} b_1(1-c_1^m), \frac{1}{m} b_2(1-c_2^m), \frac{1}{m} b_2(1-c_2^m)\right]^T$

那么

$$\begin{aligned}
(V(c_1,c_2,\cdots,c_s)\mathbf{u})_i &= \sum_{k=1}^{s} c_k^{i-1} \sum_{j=1}^{s} b_j c_j^{m-1} a_{j,k} \\
&= \sum_{j=1}^{s} b_j c_j^{m-1} \sum_{k=1}^{s} c_k^{i-1} a_{j,k} \\
&= \sum_{j=1}^{s} b_j c_j^{m-1} \frac{c_j^i}{i} \\
&= \frac{1}{i} \sum_{j=1}^{s} b_j c_j^{i+m-1} \\
&= \frac{1}{i(i+m)}
\end{aligned}$$

而

$$
\begin{aligned}
(V(c_1,c_2,\cdots,c_s)\mathbf{v})_i &= \sum_{j=1}^{s} c_j^{i-1}\frac{1}{m}b_j(1-c_j^m) \\
&= \frac{1}{m}(\sum_{j=1}^{s} b_j c_j^{i-1} - \sum_{j=1}^{s} b_j c_j^{i+m-1}) \\
&= \frac{1}{im} - \frac{1}{(i+m)m} \\
&= \frac{1}{i(i+m)}
\end{aligned}
$$

所以

$$
V(c_1,c_2,\cdots,c_s)\mathbf{u} = V(c_1,c_2,\cdots,c_s)\mathbf{v} \Leftrightarrow V(c_1,c_2,\cdots,c_s)(\mathbf{u}-\mathbf{v}) = \mathbf{0}
$$

因为 $V(c_1,c_2,\cdots,c_s)$ 满秩，只有 0 解，所以 $\mathbf{u}=\mathbf{v}$

又因为上面的等式只对 $\forall m, i=1,2,\cdots,s,\ i+m<=s+r$ 成立，所以 $m=1,2,\cdots,r$

$$
\Rightarrow \begin{cases} \forall i=1,2,\cdots,s, \\ \forall m=1,2,\cdots,r, \end{cases} \sum_{j=1}^{s} b_j c_j^{m-1} a_{j,i} = \frac{b_i}{m}(1-c_i^m)
$$

IX. Determine the order of accuracy of the collocation method in Exercise 10.213

Solution:

由 Exercise 10.213 可知其 Butcher tubleau 为

$$
\begin{array}{c|ccc}
\frac{1}{4} & \frac{23}{48} & -\frac{1}{3} & \frac{5}{48} \\
\frac{1}{2} & \frac{7}{12} & -\frac{1}{6} & \frac{1}{12} \\
\frac{3}{4} & \frac{9}{16} & 0 & \frac{3}{16} \\
\hline
 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3}
\end{array}
$$

取 $l=1,2,3,4,5$ 可得

$$
\begin{aligned}
&\sum_{j=1}^{s} b_j c_j^0 = 1 \\
&\sum_{j=1}^{s} b_j c_j^1 = \frac{1}{2} \\
&\sum_{j=1}^{s} b_j c_j^2 = \frac{1}{3} \\
&\sum_{j=1}^{s} b_j c_j^3 = \frac{1}{4} \\
&\sum_{j=1}^{s} b_j c_j^4 = \frac{37}{192} \neq \frac{1}{5}
\end{aligned}
$$

可知该方法是 $B(4)$ 但是不是 $B(5)$ 的

由 RK order conditions 可知，若 RK method 是 p 阶精度的，那么一定是 $B(p)$ 的（取 $m=0$）

所以该方法最多为 4 阶精度

下面将验证该方法是 $C(3)$ 的，取 $m=1,2,3$

- $m=1$ 时，$\sum_{j=1}^{s} a_{i,j}=c_i$ 成立
- $m=2$ 时，
$$
\begin{aligned}
i=1, a_{1,1}c_1+a_{1,2}c_2+a_{1,3}c_3 &= \frac{1}{32}=\frac{c_1^2}{2}\\
i=2, a_{2,1}c_1+a_{2,2}c_2+a_{2,3}c_3 &= \frac{1}{8}=\frac{c_2^2}{2}\\
i=3, a_{3,1}c_1+a_{3,2}c_2+a_{3,3}c_3 &= \frac{9}{32}=\frac{c_3^2}{2}
\end{aligned}
$$
- $m=3$ 时，
$$
\begin{aligned}
i=1, a_{1,1}c_1^2+a_{1,2}c_2^2+a_{1,3}c_3^2 &= \frac{1}{192}=\frac{c_1^3}{3}\\
i=2, a_{2,1}c_1^2+a_{2,2}c_2^2+a_{2,3}c_3^2 &= \frac{1}{24}=\frac{c_2^3}{3}\\
i=3, a_{3,1}c_1^3+a_{3,2}c_2^2+a_{3,3}c_3^2 &= \frac{9}{64}=\frac{c_3^3}{3}
\end{aligned}
$$

所以该方法是 $C(3)$ 的，所以该方法是 $D(1)$ 的
又由 $4\leq 2*3+2$ 以及 $4\leq 1+3+1$ 可知该方法至少为 4 阶精度
综上所述，该方法是 4 阶精度的