I. Does the length of a short thick line segment in Figure 10.9 represent the one-step error in Definition 10.159?

Solution:

粗短黑线的长度为

$$\begin{aligned}(u(t_{n+1})-u(t_n)+U^n)-U^{n+1} &= u(t_{n+1})-u(t_n)-(U^{n+1}-U^n)\\ &= u(t_{n+1})-u(t_n)-k\Phi(U^n,t^n;k)\end{aligned}$$

因为 $U^n \neq u(t^n)$，所以与定义 10.159 中的 $\mathcal{L}u(t_n) := u(t_{n+1})-u(t_n)-k\Phi(u(t_n),t^n;k)$ 不符
所以不是 one-step error

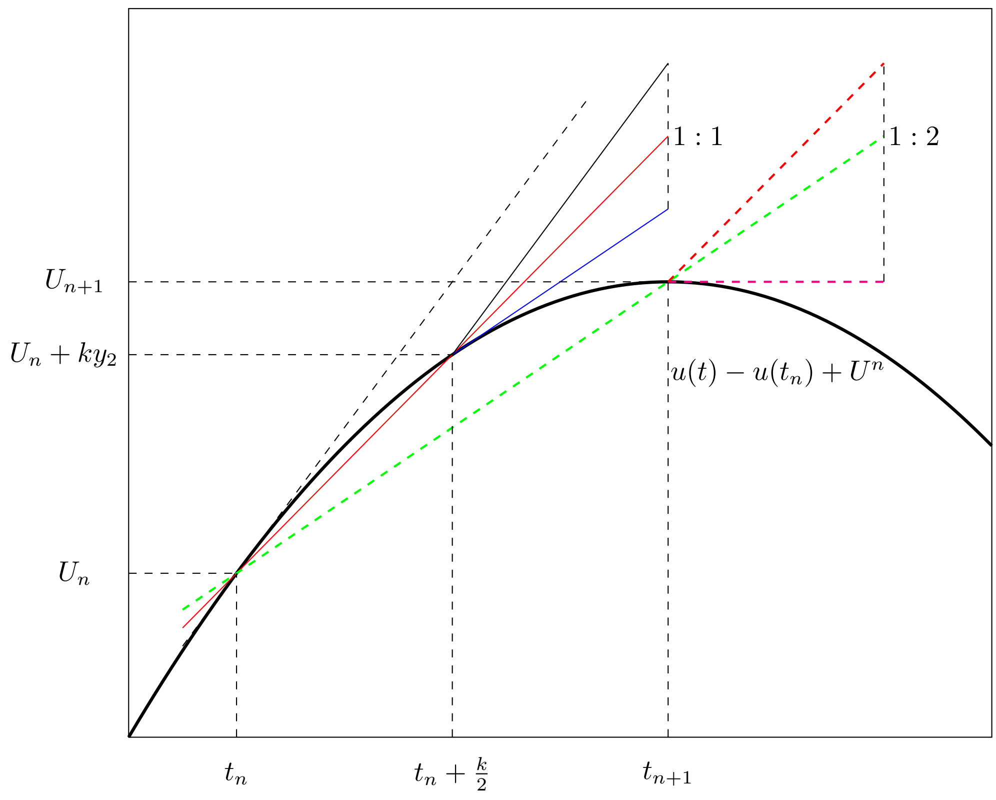
II. Give a geometric interpretation of TR-BDF2

Solution:

首先将公式改写成如下格式：

$$\begin{cases} y_1 = f(U^n, t_n) \\ y_2 = \frac{1}{2}(y_1 + f(U^n + \frac{k}{2}y_2, t_n + \frac{k}{2})) \\ U^{n+1} = U^n + k(\frac{2}{3}y_2 + \frac{1}{3}f(U^{n+1}, t_{n+1})) \end{cases}$$

(ps: 图中相同颜色的实线和虚线相互平行，画图所用的方程为 $u'(t) = 1-2t$, 是严格成立，所以图中看不到误差)



III. Use resursive Taylor expansions to derive the k^3 term in the one-step error of explicit midpoint method

Solution:

对于 expilit midpoint method

$$\Phi(u(t_n),t_n;k)=f(u(t_n)+\frac{k}{2}f(u(t_n),t_n),t_n+\frac{k}{2})$$

将其泰勒展开（为表示简单，这里用 f 代表 $f(u(t_n),t_n)$，f_u 等同理）

$$\Phi(u(t_n),t_n;k)=f+(f_u\frac{k}{2}f+f_t\frac{k}{2})+\frac{1}{2}(f_{uu}(\frac{k}{2}f)^2+2f_{ut}\frac{k}{2}f\frac{k}{2}+f_{tt}(\frac{k}{2})^2)+\cdots$$

那么

$$\begin{aligned}
\mathcal{L}(u(t_n)) =& u(t_{n+1})-u(t_n)-k\Phi(u(t_n),t^n;k)\\
=& ku_n'+\frac{k^2}{2}u_n''+\frac{k^3}{6}u_n'''-k\Phi(u(t_n),t^n;k)+O(k^4)\\
=& ku_n'+\frac{k^2}{2}u_n''+\frac{k^3}{6}u_n'''-(kf+\frac{k^2}{2}(f_uf+f_t)+\frac{k^3}{8}(f_{uu}f^2+2f_{ut}f+f_{tt}))+O(k^4)\\
\xlongequal{u_n'=f,\ u_n''=f_uf+f_t}& \frac{k^3}{6}u_n'''-\frac{k^3}{8}(f_{uu}f^2+2f_{ut}f+f_{tt})+O(k^4)\\
=& \frac{k^3}{6}(f_u^2f+f_{uu}f^2+f_uf_t+2f_{ut}f+f_{tt})-\frac{k^3}{8}(f_{uu}f^2+2f_{ut}f+f_{tt})+O(k^4)\\
=& (\frac{1}{24}(f_{uu}f^2+2f_{ut}f+f_{tt})+\frac{1}{6}f_u^2f+f_uf)k^3+O(k^4)\\
=& \Theta(k^3)
\end{aligned}$$

所以 expilit midpoint method 是二阶精度

IV. Show that the TR-BDF2 method satisfies

$$R(z) = \frac{1 + \frac{5}{12}z}{1 - \frac{7}{12} + \frac{1}{12}z^2}$$

and $R(z) - e^z = O(z^3) asz \to 0$

Solution:

首先将公式改写成如下格式：

$$\begin{cases} y_1 = f(U^n, t_n) \\ y_2 = \frac{1}{2}(y_1 + f(U^n + \frac{k}{2}y_2, t_n + \frac{k}{2})) \\ U^{n+1} = U^n + k(\frac{2}{3}y_2 + \frac{1}{3}f(U^{n+1}, t_{n+1})) \end{cases}$$

对于方程 $u'(t) = \lambda u$，$f(U^n, t_n) = \lambda U^n$
那么

$$\begin{aligned} & y_1 = \lambda U^n \\ \Longrightarrow & y_2 = \frac{1}{2}(\lambda U^n + \lambda U^n + \frac{\lambda k}{2}y_2) \Rightarrow y_2 = \frac{4\lambda}{4 - \lambda k}U^n \\ \Longrightarrow & U^{n+1} = U^n + \frac{8\lambda k}{12 - 3\lambda k}U^n + \frac{\lambda k}{3}U^{n+1} \\ \Longrightarrow & U^{n+1} = \frac{12 + 5\lambda k}{12 - 7\lambda k + (\lambda k)^2}U^n \\ \xRightarrow{z:=\lambda k} & U^{n+1} = \frac{12 + 5z}{12 - 7z + z^2}U^n \end{aligned}$$

所以

$$R(z) = \frac{1 + \frac{5}{12}z}{1 - \frac{7}{12} + \frac{1}{12}z^2} = 1 + z + \frac{z^2}{2} + \frac{5}{24}z^3 + O(z^4) \quad z \to 0$$

而

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + O(k^4) \quad z \to 0$$

所以

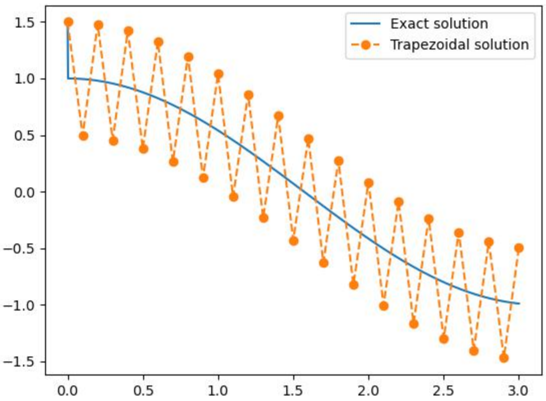
$$R(z) - e^z = \frac{1}{24}z^3 + O(k^4) = O(k^3) \quad z \to 0$$

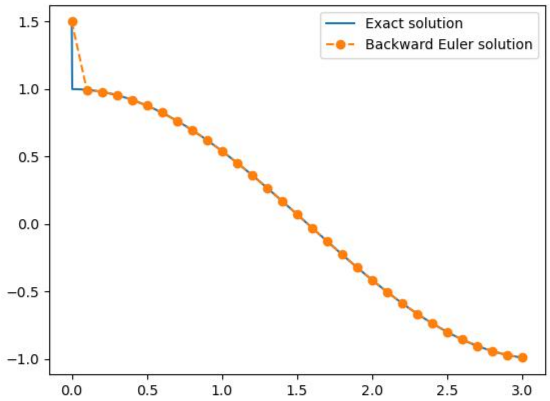
V. Reproduce the results in Example 10.175 and explain in your own language why the first-order backward Euler is superior to the second-order trapezoidal method

Solution:

计算与绘图使用的语言为 python，在该目录下运行`python3 Lstability.py`
由于 python 的计算精度问题，计算所得的误差会讲义中的精度有差距，但是可以复现讲义中出现的情况
结果如下：
分别用 Backward Euler method 和 Trapezoidal method 去计算可以得到递推的数值解

	k	Backward Euler	Trapezoidal
$\eta = 1$	0.2	1.2288439763530334e-06	1.3188459746515946e-07
	0.1	1.1803365906892793e-06	2.130032792813097e-06
	0.05	1.1557991848043514e-06	2.1263552584249723e-06
$\eta = 1.5$	0.2	1.2288439763530334e-06	-0.49984989061314766
	0.1	1.1803365906892793e-06	0.49940248988875685
	0.05	1.1557991848043514e-06	0.4976078771490346





- **Backward Euler:**

$$U^{n+1} = \frac{U^n}{1-\lambda k} - \frac{\lambda k \cos t_{n+1} + k \sin t_{n+1}}{1-\lambda k}$$

- **Trapezoidal:**

$$U^{n+1} = \frac{1+\frac{\lambda k}{2}}{1-\frac{\lambda k}{2}} U^n - \frac{\frac{\lambda k}{2}\cos t_n + \frac{k}{2}\sin t_n + \frac{\lambda k}{2}\cos t_{n+1} + \frac{k}{2}\sin t_{n+1}}{1-\frac{\lambda k}{2}}$$

当 λk 很大时 (图中为 $\lambda k = -10^5$)，初值有一个较大的误差，设 $U^0 = U + E$(E 为误差)，代入递推式得

- **Backward Euler:**

$$U^1 = \frac{U}{1-\lambda k} + \frac{E}{1-\lambda k} - \frac{\lambda k \cos t_1 + k \sin t_1}{1-\lambda k}$$

- **Trapezoidal:**

$$U^1 = \frac{1+\frac{\lambda k}{2}}{1-\frac{\lambda k}{2}} U + \frac{1+\frac{\lambda k}{2}}{1-\frac{\lambda k}{2}} E - \frac{\frac{\lambda k}{2}\cos t_0 + \frac{k}{2}\sin t_0 + \frac{\lambda k}{2}\cos t_1 + \frac{k}{2}\sin t_1}{1-\frac{\lambda k}{2}}$$

Backward Euler method 会快速将误差 ($\frac{E}{1-\lambda k} \approx 10^{-5}E$) 消去
而 Trapezoidal method 几乎将误差保留 ($\frac{1+\frac{\lambda k}{2}}{1-\frac{\lambda k}{2}}E \approx -0.99996E$)
所以在这个情况下 Backward Euler 会比 Trapezoidal 好很多