I. Show that the matrix form of the Crank-Nicolson method for solving the heat equation with Dirichlet conditions is

$$\left(I-\frac{k}{2}A\right)\mathbf{U}^{n+1}=\left(I+\frac{k}{2}A\right)\mathbf{U}^{n}+\mathbf{b}^{n}$$

where

$$\mathbf{b}^n=\frac{r}{2}\begin{bmatrix} g_0(t_n)+g_0(t_{n+1}) \\ 0 \\ \ddots \\ 0 \\ g_0(t_n)+g_0(t_{n+1}) \end{bmatrix}$$

Solution:

$$\begin{aligned} &-rU_{i-1}^{n+1}+2(1+r)U_i^{n+1}-rU_{i+1}^{n+1}=rU_{i-1}^{n}+2(1-r)U_i^{n}+rU_{i+1}^{n} \\ \Longleftrightarrow &U_i^{n+1}-\frac{k}{2}\frac{\nu}{h^2}(U_{i-1}^{n+1}-2U_i^{n+1}+U_{i+1}^{n+1})=U_i^n+\frac{k}{2}\frac{\nu}{h^2}(U_{i-1}^{n}-2U_i^{n}+U_{i+1}^{n}) \end{aligned}$$

那么可以得到 $\forall i\in\mathbb{N},\ 1<i<m, ((I-\frac{k}{2}A)\,\mathbf{U}^{n+1})_i=((I+\frac{k}{2}A)\,\mathbf{U}^{n})_i$

对于 $i=1,m$ 的情况，$U_0^n=g_0(t_n)$，$U_0^{n+1}=g_0(t_{n+1})$，$U_{m+1}^n=g_1(t_n)$，$U_{m+1}^{n+1}=g_1(t_{n+1})$

带回上式得

$$\begin{aligned} U_1^{n+1}-\frac{k}{2}\frac{\nu}{h^2}(-2U_1^{n+1}+U_2^{n+1})&=U_1^n+\frac{k}{2}\frac{\nu}{h^2}(-2U_1^{n}+U_2^{n})+\frac{r}{2}(g_0(t_n)+g_0(t_{n+1})) \\ U_m^{n+1}-\frac{k}{2}\frac{\nu}{h^2}(U_{m-1}^{n+1}-2U_m^{n+1})&=U_m^n+\frac{k}{2}\frac{\nu}{h^2}(U_{m-1}^{n}-2U_m^{n})+\frac{r}{2}(g_1(t_n)+g_1(t_{n+1})) \end{aligned}$$

即验证了 $\left(I-\frac{k}{2}A\right)\mathbf{U}^{n+1}=\left(I+\frac{k}{2}A\right)\mathbf{U}^{n}+\mathbf{b}^{n}$ 成立

II. Prove Lemma 11.25 via the stability function of one-step methods

Solution:

对于 $\theta-method$ 其一步形式可以写成 $U^{n+1}=U^n+k(\theta f(U^{n+1},t_{n+1})+(1-\theta)f(U^n,t_n))$

其 stability function 为

$$R(k\lambda)=\frac{1+(1-\theta)k\lambda}{1-\theta k\lambda}$$

要满足

$$\left|\frac{1+(1-\theta)k\lambda}{1-\theta k\lambda}\right|\le 1+O(k)$$

$$\xLongrightarrow{drop\ O(k)}(\theta-1)\frac{4k\nu}{h^2}\le\theta\frac{k\nu}{h^2},\ and\ (1-2\theta)\frac{4k\nu}{h^2}\le 2$$

因为 $\theta\in[0,1]$，所以前一条式子一定成立

当 $\theta\in[\frac{1}{2},1]$ 时，$(1-2\theta)\frac{4k\nu}{h^2}\le 0\le 2$ 一定成立，所以此时 $\theta-method$ is unconditionally stable

当 $\theta\in[0,\frac{1}{2})$ 时，第二条式子成立则需要满足 $k\le\frac{h^2}{2(1-2\theta)\nu}$

III. Show that any grid function in $L^1(h\mathbb{Z})$ can be recovered by a Fourier transform followed by an inverse Fourier transform

Solution:

只需证明 $U_m = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}mh\xi}\hat{U}(\xi)\mathrm{d}\xi$ 确实成立即可

因为 $\mathbf{U} \in L^1(h\mathbb{Z})$，所以 $\sum_{m\in\mathbb{Z}}|U_m| < \infty$

又有 $\forall m \in \mathbb{Z},\ \forall\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}],\ |e^{-\mathbf{i}mh\xi}U_m| \le |U_m|$

所以 $\forall\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}],\ \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}|e^{-\mathbf{i}mh\xi}U_m h| \le \sum_{m\in\mathbb{Z}}|U_m|h < \infty$

所以 $\hat{U}(\xi) := \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}} e^{-\mathbf{i}mh\xi}U_m h$ 在 $\xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$ 上一致收敛

所以可以逐项积分，即

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}mh\xi}\hat{U}(\xi)\mathrm{d}\xi &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}mh\xi}\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} e^{-\mathbf{i}nh\xi}U_n h\mathrm{d}\xi \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}(m-n)h\xi}U_n h\mathrm{d}\xi
\end{aligned}
$$

对于 $\forall m, n \in \mathbb{Z}$

$$
\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}(m-n)h\xi}U_n h\mathrm{d}\xi = \begin{cases} \frac{U_n}{\mathbf{i}(m-n)}e^{\mathbf{i}(m-n)h\xi}|_{-\frac{\pi}{h}}^{\frac{\pi}{h}} = \frac{U_n}{\mathbf{i}(m-n)}(e^{\mathbf{i}(m-n)\pi} - e^{-\mathbf{i}(m-n)\pi}) = 0, & m \ne n \\ \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} U_m h\mathrm{d}\xi = 2\pi U_m, & m = n \end{cases}
$$

所以

$$
\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}mh\xi}\hat{U}(\xi)\mathrm{d}\xi = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}(m-n)h\xi}U_n h\mathrm{d}\xi = \frac{1}{2\pi}2\pi U_m = U_m
$$

IV. Prove Lemma 11.25 via Von Neumann analysis. What can you say proof with that for Exercise 11.26

Solution:

$\theta - method:\ -\theta r U_{i-1}^{n+1} + (1+2\theta r)U_i^{n+1} - \theta r U_{i+1}^{n+1} = (1-\theta)rU_{i-1}^n + (1-2(1-\theta)r)U_i^n + (1-\theta)rU_{i+1}^n$

代入傅里叶逆变换得

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}(-\theta r e^{\mathbf{i}(j-1)h\xi} + (1+2\theta r)e^{\mathbf{i}jh\xi} - \theta r e^{\mathbf{i}(j+1)h\xi})\hat{U}^{n+1}(\xi)\mathrm{d}\xi \\
=&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}((1-\theta)re^{\mathbf{i}(j-1)h\xi} + (1-2(1-\theta)r)e^{\mathbf{i}jh\xi} + (1-\theta)re^{\mathbf{i}(j+1)h\xi})\hat{U}^n(\xi)\mathrm{d}\xi \\
\Longleftrightarrow&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi}\frac{1}{e^{\mathbf{i}jh\xi}}(-\theta r e^{\mathbf{i}(j-1)h\xi} + (1+2\theta r)e^{\mathbf{i}jh\xi} - \theta r e^{\mathbf{i}(j+1)h\xi})\hat{U}^{n+1}(\xi)\mathrm{d}\xi \\
=&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi}\frac{1}{e^{\mathbf{i}jh\xi}}((1-\theta)re^{\mathbf{i}(j-1)h\xi} + (1-2(1-\theta)r)e^{\mathbf{i}jh\xi} + (1-\theta)re^{\mathbf{i}(j+1)h\xi})\hat{U}^n(\xi)\mathrm{d}\xi
\end{aligned}
$$

由傅里叶变换的唯一性可得

$$
\begin{aligned}
&\frac{1}{e^{\mathrm{i}jh\xi}}(-\theta re^{\mathrm{i}(j-1)h\xi}+(1+2\theta r)e^{\mathrm{i}jh\xi}-\theta re^{\mathrm{i}(j+1)h\xi})\hat{U}^{n+1}(\xi)\\
=&\frac{1}{e^{\mathrm{i}jh\xi}}((1-\theta)re^{\mathrm{i}(j-1)h\xi}+(1-2(1-\theta)r)e^{\mathrm{i}jh\xi}+(1-\theta)re^{\mathrm{i}(j+1)h\xi})\hat{U}^{n}(\xi)\\
\Longleftrightarrow&(-\theta re^{-\mathrm{i}h\xi}+(1+2\theta r)-\theta re^{\mathrm{i}h\xi})\hat{U}^{n+1}(\xi)=((1-\theta)re^{-\mathrm{i}h\xi}+(1-2(1-\theta)r)+(1-\theta)re^{\mathrm{i}h\xi})\hat{U}^{n}(\xi)\\
\Longleftrightarrow&(1+4\theta r\sin^2(\frac{h\xi}{2}))\hat{U}^{n+1}(\xi)=(1-4(1-\theta)r\sin^2(\frac{h\xi}{2}))\hat{U}^{n}(\xi)\\
\Longrightarrow&\hat{U}^{n+1}(\xi)=\frac{1-4(1-\theta)r\sin^2(\frac{h\xi}{2})}{1+4\theta r\sin^2(\frac{h\xi}{2})}\hat{U}^{n}(\xi)
\end{aligned}
$$

要满足 $|\frac{1-4(1-\theta)r\sin^2(\frac{h\xi}{2})}{1+4\theta r\sin^2(\frac{h\xi}{2})}|\leq 1$

即满足 $-4(1-\theta)r\sin^2(\frac{h\xi}{2})\leq 4\theta r\sin^2(\frac{h\xi}{2})$ 和 $2(1-2\theta)r\sin^2(\frac{h\xi}{2})\leq 1,\forall\xi\in[-\frac{\pi}{h},\frac{\pi}{h}]$

因为 $\theta\in[0,1]$，所以前一条式子一定成立

当 $\theta\in[\frac{1}{2},1]$ 时，$2(1-2\theta)r\sin^2(\frac{h\xi}{2})\leq 0\leq 1$，一定成立

当 $\theta\in[0,\frac{1}{2})$ 时，$2(1-2\theta)r\sin^2(\frac{h\xi}{2})\leq 2(1-2\theta)r\leq 1$ 成立 $\Longrightarrow k\leq\frac{h^2}{2(1-2\theta)\nu}$

11.26 的方法需要先将写出 one-step 的形式，失去了空间离散的信息

而 Von Neumann 分析的方法直接从迭代格式本身出发

V. Show that the Beam-Warming method is second-order accurate both in time and in space

Solution:

(11.86)

$$
\begin{aligned}
\tau(x,t)=&\frac{u(x,t+k)-u(x,t)}{k}+a\frac{3u(x,t)-4u(x-h,t)+u(x-2h,t)}{2h}-ka^2\frac{u(x,t)-2u(x-h,t)+u(x-2h,t)}{2h^2}\\
=&u_t+\frac{k}{2}u_{tt}+\frac{k^2}{6}u_{ttt}+au_x-\frac{ah^2}{3}u_{xxx}-\frac{ka^2}{2}u_{xx}+\frac{kha^2}{2}u_{xxx}+O(k^3+kh^2+h^3)\\
=&\frac{au_{xxx}}{6}(3akh-a^2k^2-2h^2)+O(k^3+kh^2+h^3)\\
=&O(k^2+h^2)
\end{aligned}
$$

(11.87)

$$
\begin{aligned}
\tau(x,t)=&\frac{u(x,t+k)-u(x,t)}{k}-a\frac{3u(x,t)-4u(x+h,t)+u(x+2h,t)}{2h}-ka^2\frac{u(x,t)-2u(x+h,t)+u(x+2h,t)}{2h^2}\\
=&u_t+\frac{k}{2}u_{tt}+\frac{k^2}{6}u_{ttt}+au_x-\frac{ah^2}{3}u_{xxx}-\frac{ka^2}{2}u_{xx}-\frac{kha^2}{2}u_{xxx}+O(k^3+kh^2+h^3)\\
=&-\frac{au_{xxx}}{6}(3akh+a^2k^2+2h^2)+O(k^3+kh^2+h^3)\\
=&O(k^2+h^2)
\end{aligned}
$$

第三个等式由 $u_t=-au_x$，$u_{tt}=-au_{tx}=a^2u_{xx}$，$u_{ttt}=a^2u_{txx}=-a^3u_{xxx}$ 得到

VI. Show that the Beam-Warming methods are stable for $\mu \in [0,2]$ and $\mu \in [-2,0]$, respectively. Reproduce the plots.

Solution:

(11.86)

代入傅里叶逆变换得

$$\begin{aligned}U_j^{n+1} =&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi}[(1-\frac{3\mu}{2}+\frac{\mu^2}{2})+(2\mu-\mu^2)e^{-\mathbf{i}h\xi}+\frac{\mu^2-\mu}{2}e^{-2\mathbf{i}h\xi}]\hat{U}^n(\xi)\mathrm{d}\xi\\ =&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi}\hat{U}^{n+1}(\xi)\mathrm{d}\xi\end{aligned}$$

由傅里叶变换的唯一性可得

$$\begin{aligned}\hat{U}^{n+1}(\xi) &= [(1-\frac{3\mu}{2}+\frac{\mu^2}{2})+(2\mu-\mu^2)e^{-\mathbf{i}h\xi}+\frac{\mu^2-\mu}{2}e^{-2\mathbf{i}h\xi}]\hat{U}^n(\xi)\\ &= e^{-\mathbf{i}h\xi}(1-2(1-\mu)^2\sin^2\frac{h\xi}{2}+\mathbf{i}(1-\mu)\sin h\xi)\hat{U}^n(\xi)\end{aligned}$$

那么

$$\begin{aligned}&|e^{-\mathbf{i}h\xi}(1-2(1-\mu)^2\sin^2\frac{h\xi}{2}+\mathbf{i}(1-\mu)\sin h\xi)| \le 1\\ \iff&|e^{-\mathbf{i}h\xi}(1-2(1-\mu)^2\sin^2\frac{h\xi}{2}+\mathbf{i}(1-\mu)\sin h\xi)|^2 \le 1\\ \iff&1-4\mu(2-\mu)(1-\mu)^2\sin^4\frac{h\xi}{2} \le 1\end{aligned}$$

就可知 $\forall\mu\in[0,2]$，使得上式成立

(11.87)

$$\begin{aligned}U_j^{n+1} =&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi}[(1+\frac{3\mu}{2}+\frac{\mu^2}{2})-(2\mu+\mu^2)e^{\mathbf{i}h\xi}+\frac{\mu^2+\mu}{2}e^{2\mathbf{i}h\xi}]\hat{U}^n(\xi)\mathrm{d}\xi\\ =&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi}\hat{U}^{n+1}(\xi)\mathrm{d}\xi\end{aligned}$$

由傅里叶变换的唯一性可得

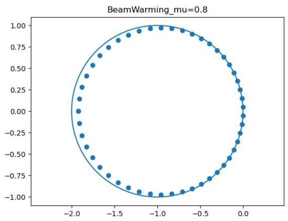
$$\begin{aligned}\hat{U}^{n+1}(\xi) &= [(1+\frac{3\mu}{2}+\frac{\mu^2}{2})-(2\mu+\mu^2)e^{\mathbf{i}h\xi}+\frac{\mu^2+\mu}{2}e^{2\mathbf{i}h\xi}]\hat{U}^n(\xi)\\ &= e^{\mathbf{i}h\xi}(1-2(1+\mu)^2\sin^2\frac{h\xi}{2}-\mathbf{i}(1+\mu)\sin h\xi)\hat{U}^n(\xi)\end{aligned}$$

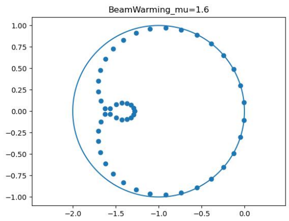
那么

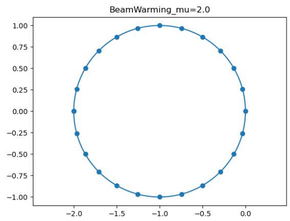
$$\begin{aligned}&|e^{\mathbf{i}h\xi}(1-2(1+\mu)^2\sin^2\frac{h\xi}{2}-\mathbf{i}(1+\mu)\sin h\xi)\hat{U}^n(\xi)| \le 1\\ \iff&|e^{\mathbf{i}h\xi}(1-2(1+\mu)^2\sin^2\frac{h\xi}{2}-\mathbf{i}(1+\mu)\sin h\xi)\hat{U}^n(\xi)|^2 \le 1\\ \iff&1-4\mu(2+\mu)(1+\mu)^2\sin^4\frac{h\xi}{2} \le 1\end{aligned}$$

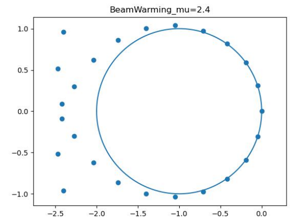
就可知 $\forall\mu\in[-2,0]$，使得上式成立

Plots



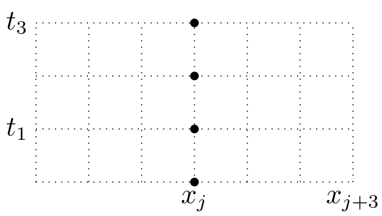




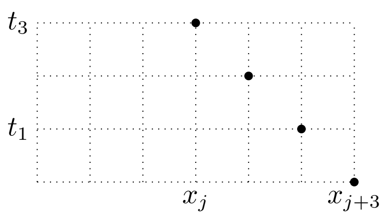


VII. Plot the numerical domains of dependence of grid point (x_j, t_3) for the upwind method with $a < 0$ and $\mu = 0, -1, -2$

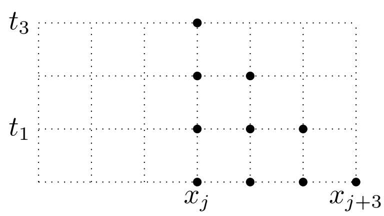
Solution:



(a) $\mu = 0$



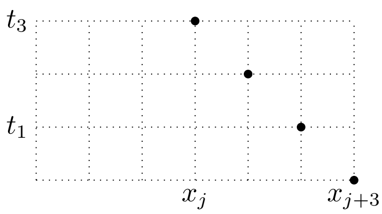
(b) $\mu = -1$



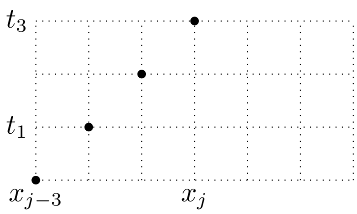
(c) $\mu = -2$

VIII. Plot the numerical domains of dependence of grid point (x_j, t_3) for the Lax-Wendroff method $\mu = +1, -1$

Solution:



(d) $\mu = -1$



(e) $\mu = 1$

IX. Show that the modified equation of the leapfrog method is also (11.96). However, if one more term of higher-order derivative had been retained, the modified equation of the leapfrog method would have been

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} = \epsilon_f v_{xxxxx}$$

while that of the Lax-Wendroff method would been

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} = \epsilon_w v_{xxxx}$$

Solution:

Leapfrog:

$$\begin{aligned}
&\frac{v(x,t+k)-v(x,t-k)}{2k} + a\frac{v(x+h,t)-v(x-h,t)}{2h} = 0 \\
\xLeftrightarrow{assumed\ h=O(k)} & v_t + \frac{k^2}{6}v_{ttt} + \frac{k^4}{120}v_{ttttt} + a(v_x + \frac{h^2}{6}v_{xxx} + \frac{h^4}{120}v_{xxxxx}) + O(k^5) = 0 \\
\Longleftrightarrow & v_t + av_x = -\frac{1}{6}(k^2 v_{ttt} + ah^2 v_{xxx}) - \frac{1}{120}(k^4 v_{ttttt} + ah^4 v_{xxxxx}) + O(k^5)
\end{aligned}$$

可以得到

$$\begin{aligned}
v_{tt} &= -av_{xt} - \frac{k^2}{6}v_{tttt} - \frac{ah^2}{6}v_{xxxt} + O(k^3) \\
v_{xt} &= -av_{xx} - \frac{k^2}{6}v_{xttt} - \frac{ah^2}{6}v_{xxxx} + O(k^3) \\
v_{tt} &= a^2 v_{xx} - \frac{k^2}{6}(v_{tttt} - av_{xttt}) - \frac{ah^2}{6}(v_{xxxt} - av_{xxxx}) + O(k^3) \\
v_{ttt} &= a^2 v_{xxt} - \frac{k^2}{6}(v_{ttttt} - av_{xtttt}) - \frac{ah^2}{6}(v_{xxxtt} - av_{xxxxt}) + O(k^3) \\
v_{xxt} &= -av_{xxx} - \frac{k^2}{6}v_{xxttt} - \frac{ah^2}{6}v_{xxxxx} + O(k^3) \\
v_{ttt} &= -a^3 v_{xxx} - \frac{k^2}{6}(v_{ttttt} - av_{xtttt} + a^2 v_{xxttt}) - \frac{ah^2}{6}(v_{xxxtt} - av_{xxxxt} + a^2 v_{xxxxx}) + O(k^3) \\
v_{xxxxt} &= -av_{xxxxx} + O(k) \\
v_{xxxtt} &= a^2 v_{xxxxx} + O(k) \\
v_{xxttt} &= -a^3 v_{xxxxx} + O(k) \\
v_{xtttt} &= -av_{xxttt} + O(k) = a^4 v_{xxxxx} + O(k) \\
v_{ttttt} &= a^2 v_{xxttt} + O(k) = -a^5 v_{xxxxx} + O(k)
\end{aligned}$$

带回原方程忽略高阶项就得到了

$$\begin{aligned}
& v_t + av_x + \frac{1}{6}(ah^2 - a^3k^2)v_{xxx} = \epsilon_f v_{xxxxx} \\
\Longleftrightarrow & v_t + av_x + \frac{ah^2}{6}(1 - (\frac{ak}{h})^2)v_{xxx} = \epsilon_f v_{xxxxx} \\
\Longleftrightarrow & v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} = \epsilon_f v_{xxxxx}
\end{aligned}$$

若直接忽略 $O(k^3)$ 项即为

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} = 0$$

Lax-Wendroff:

$$\begin{aligned}
&\frac{v(x,t+k)-v(x,t)}{k} + a\frac{v(x+h,t)-v(x-h,t)}{2h} \\
&= ka^2\frac{v(x+h,t)-2v(x,t)+v(x+h,t)}{2h^2} \\
\xLeftrightarrow{assumed\ h=O(k)} & v_t + \frac{k}{2}v_{tt} + \frac{k^2}{6}v_{ttt} + \frac{k^3}{24}v_{tttt} + av_x + a\frac{h^2}{6}v_{xxx} = \frac{ka^2}{2}v_{xx} - \frac{kh^2a^2}{24}v_{xxxx} + O(k^4) \\
\Longleftrightarrow & v_t + av_x = \frac{k}{2}(a^2v_{xx} - v_{tt}) - \frac{1}{6}(k^2v_{ttt} - ah^2v_{xxx}) - \frac{k}{24}(k^2v_{tttt} + h^2a^2v_{xxxx}) + O(k^4)
\end{aligned}$$

可以得到

$$\begin{aligned}
&v_{tt} = -av_{xt} - \frac{k}{2}v_{ttt} + \frac{k}{2}a^2v_{xxt} - \frac{k^2}{6}v_{tttt} + \frac{ah^2}{6}v_{xxxt} + O(k^3) \\
&v_{xt} = -av_{xx} - \frac{k}{2}v_{xtt} + \frac{k}{2}a^2v_{xxx} - \frac{k^2}{6}v_{xttt} + \frac{ah^2}{6}v_{xxxx} + O(k^3) \\
&v_{tt} = a^2v_{xx} - \frac{k}{2}(v_{ttt} - av_{xtt}) + \frac{k}{2}a^2(v_{xxt} - av_{xxx}) - \frac{k^2}{6}(v_{tttt} - av_{xttt}) + \frac{ah^2}{6}(v_{xxxt} - v_{xxxx}) + O(k^3) \\
&v_{ttt} = a^2v_{xxt} - \frac{k}{2}(v_{tttt} - av_{xttt}) + \frac{k}{2}a^2(v_{xxtt} - av_{xxxt}) + O(k^2) \\
&v_{xxt} = -av_{xxx} - \frac{k}{2}v_{xxtt} + \frac{k}{2}a^2v_{xxxx} + O(k^2) \\
&v_{xtt} = a^2v_{xxx} - \frac{k}{2}(v_{xttt} - av_{xxtt}) + \frac{k}{2}a^2(v_{xxxt} - av_{xxxx}) + O(k^2) \\
&v_{ttt} = -a^3v_{xxx} - \frac{k}{2}(v_{tttt} - av_{xttt} + a^2v_{xxtt}) + \frac{k}{2}a^2(v_{xxtt} - av_{xxxt} + a^2v_{xxxx}) + O(k^2) \\
&v_{xxxt} = -av_{xxxx} + O(k) \\
&v_{xxtt} = a^2v_{xxxx} + O(k) \\
&v_{xttt} = -a^3v_{xxxx} + O(k) \\
&v_{tttt} = a^2v_{xxtt} + O(k) = a^4v_{xxxx} + O(k)
\end{aligned}$$

带回原方程忽略高阶项就得到了

$$\begin{aligned}
&v_t + av_x + \frac{1}{6}(ah^2 - a^3k^2)v_{xxx} = \epsilon_w v_{xxxx} \\
\Longleftrightarrow & v_t + av_x + \frac{ah^2}{6}(1 - (\frac{ak}{h})^2)v_{xxx} = \epsilon_w v_{xxxx} \\
\Longleftrightarrow & v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} = \epsilon_w v_{xxxx}
\end{aligned}$$

X. Show that the modified equation of the Beam-Warming method applied to the advection equation with $a \geq 0$ is

$$v_t + av_x + \frac{ah^2}{6}(-2+3\mu-\mu^2)v_{xxx} = 0$$

Thus we have

$$C_p(\xi) = a + \frac{ah^2}{6}(\mu-1)(\mu-2)\xi^2$$
$$C_g(\xi) = a + \frac{ah^2}{2}(\mu-1)(\mu-2)\xi^2$$

How does these facts answers Question (e) of Example 11.87

Solution:

$$\begin{aligned}
&\frac{v(x,t+k)-v(x,t)}{k} + a\frac{3v(x,t)-4v(x-h,t)+v(x-2h,t)}{2h} \\
&= ka^2\frac{v(x,t)-2v(x-h,t)+v(x-2h,t)}{2h^2} \\
\xLeftrightarrow{assumed\ h=O(k)} & v_t + \frac{k}{2}v_{tt} + \frac{k^2}{6}v_{ttt} + av_x - \frac{ah^2}{3}v_{xxx} = \frac{ka^2}{2}v_{xx} - \frac{kha^2}{2}v_{xxx} + O(k^3) \\
\Longleftrightarrow & v_t + av_x = \frac{k}{2}(a^2v_{xx} - v_{tt}) - \frac{k^2}{6}v_{ttt} + (\frac{ah^2}{3} - \frac{kha^2}{2})v_{xxx} + O(k^3)
\end{aligned}$$

可以得到

$$\begin{aligned}
v_{tt} &= -av_{xt} - \frac{k}{2}v_{ttt} + \frac{k}{2}a^2v_{xxt} + O\left(k^2\right) \\
v_{xt} &= -av_{xx} - \frac{k}{2}v_{xtt} + \frac{k}{2}a^2v_{xxx} + O\left(k^2\right) \\
v_{xxt} &= -av_{xxx} + O(k) \\
v_{xtt} &= -av_{xxt} + O(k) = a^2v_{xxx} + O(k) \\
v_{ttt} &= -av_{xtt} + O(k) = -a^3v_{xxx} + O(k) \\
v_{tt} &= a_v^2xx + O(k^2)
\end{aligned}$$

带回原方程忽略高阶项得

$$\begin{aligned}
& v_t + av_x + \frac{a}{6}(-2h^2 + 3akh - a^2k^2)v_{xxx} = 0 \\
\Longleftrightarrow & v_t + av_x + \frac{ah^2}{6}(-2 + 3\frac{ak}{h} - (\frac{ak}{h})^2)v_{xxx} = 0 \\
\Longleftrightarrow & v_t + av_x + \frac{ah^2}{6}(-2 + 3\mu - \mu^2)v_{xxx} = 0
\end{aligned}$$

那么由 $C_p(\xi)$ 和 $C_g(\xi)$ 定义就可以得到数值解的 phase velocity 和 group velocity 为

$$C_p(\xi) = a + \frac{ah^2}{6}(\mu-1)(\mu-2)\xi^2$$
$$C_g(\xi) = a + \frac{ah^2}{2}(\mu-1)(\mu-2)\xi^2$$

因为在 Example 11.86 Question(e) 中 $\mu = 0.8 < 1 < 2$
所以数值解的 phase velocity 和 group velocity 都比真实解的 phase velocity 和 group velocity a 大
所以数值解的波动相较于真实解有前移

XI. What if $\mu = 1$? Discuss this case for both Lax-Wendroff and leapfrog methods to answer Question (f) of Example 11.87

Solution:

当 $\mu = 1$ 时对于 Lax-Wendroff and leapfrog methods 都抹去了他们的 $C_p(\xi)$ 和 $C_g(\xi)$ 中的 $-\frac{ah^2}{6}(1-\mu^2)\xi^2$ 项，使得数值解和真实解的 phase velocity 和 group velocity 相差很小，此时数值解的准确度会高很多。

而 Example 11.87 中 $k = h$ 时 $\mu = 1$ 会明显好于 $k = 0.8h$ 时 $\mu = 0.8$

XII. Apply the von Neumann analysis to the Lax-Friedrichs method to derive its amplification factor as

$$g(\xi h) = \cos(\xi h) - \mu \mathbf{i} \sin(\xi h)$$

For which values of μ would the method be stable

Solution:

Lax-Friedrichs method：$U_j^{n+1} = \frac{1}{2}(U_{j+1}^n + U_{j-1}^n) - \frac{\mu}{2}(U_{j+1}^n - U_{j-1}^n)$

代入傅里叶逆变换得

$$\begin{aligned}
U_j^{n+1} =& \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi} \left(\frac{e^{\mathbf{i}h\xi} + e^{-\mathbf{i}h\xi}}{2} - \mu \frac{e^{\mathbf{i}h\xi} - e^{-\mathbf{i}h\xi}}{2}\right) \hat{U}^n(\xi) \mathrm{d}\xi \\
=& \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi} \hat{U}^{n+1}(\xi) \mathrm{d}\xi
\end{aligned}$$

由傅里叶变换的唯一性可得

$$\begin{aligned}
\hat{U}^{n+1}(\xi) =& \left(\frac{e^{\mathbf{i}h\xi} + e^{-\mathbf{i}h\xi}}{2} - \mu \frac{e^{\mathbf{i}h\xi} - e^{-\mathbf{i}h\xi}}{2}\right) \hat{U}^n(\xi) \\
=& [\cos(\xi h) - \mu \mathbf{i} \sin(\xi h)] \hat{U}^n(\xi)
\end{aligned}$$

所以 $g(\xi h) = \cos(\xi h) - \mu \mathbf{i} \sin(\xi h)$

$$\begin{aligned}
& |g(\xi h)| \le 1 \\
\Longleftrightarrow & |g(\xi h)|^2 = 1 + (\mu^2 - 1)\sin^2(\xi h) \le 1 \\
\Longrightarrow & |\mu| \le 1
\end{aligned}$$

XIII. Apply the von Neumann analysis to the Lax-Wendroff method to derive its amplification factor as

$$g(\xi h) = 1 - 2\mu^2 \sin^2 \frac{\xi h}{2} - \mu \mathbf{i} \sin(\xi h)$$

For which values of μ would the method be stable

Solution:

Lax-Wendroff method: $U_j^{n+1} = U_j^n - \frac{\mu}{2}(U_{j+1}^n - U_{j-1}^n) + \frac{\mu^2}{2}(U_{j+1}^n - U_j^n + U_{j-1}^n)$
代入傅里叶逆变换得

$$\begin{aligned}
U_j^{n+1} =& \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi}(1 - \mu^2 + \mu^2 \frac{e^{\mathbf{i}h\xi} + e^{-\mathbf{i}h\xi}}{2} - \mu \frac{e^{\mathbf{i}h\xi} - e^{-\mathbf{i}h\xi}}{2}) \hat{U}^n(\xi) \mathrm{d}\xi \\
=& \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathbf{i}jh\xi} \hat{U}^{n+1}(\xi) \mathrm{d}\xi
\end{aligned}$$

由傅里叶变换的唯一性可得

$$\begin{aligned}
\hat{U}^{n+1}(\xi) =& (1 - \mu^2 + \mu^2 \frac{e^{\mathbf{i}h\xi} + e^{-\mathbf{i}h\xi}}{2} - \mu \frac{e^{\mathbf{i}h\xi} - e^{-\mathbf{i}h\xi}}{2}) \hat{U}^n(\xi) \\
=& [1 - \mu^2(1 - \cos(\xi h)) - \mu \mathbf{i} \sin(\xi h)] \hat{U}^n(\xi) \\
=& [1 - 2\mu^2 \sin^2 \frac{\xi h}{2} - \mu \mathbf{i} \sin(\xi h)] \hat{U}^n(\xi)
\end{aligned}$$

所以 $g(\xi h) = 1 - 2\mu^2 \sin^2 \frac{\xi h}{2} - \mu \mathbf{i} \sin(\xi h)$

$$\begin{aligned}
& |g(\xi h)| \leq 1 \\
\Longleftrightarrow & |g(\xi h)|^2 = 1 + \mu^2(\mu^2 - 1) \sin^4 \frac{\xi h}{2} \leq 1 \\
\Longrightarrow & |\mu| \leq 1
\end{aligned}$$