92.2

Analysis H – Hahn / Hlasek / Tantod
Unit 6: Matrices Exam
No Calculators

71
77 pts

If Michi Tanaka isn't sure how to solve it, you can always just Gauss and check! Per: 6

Questions 1 – 4 are multiple choice. Circle the BEST answer. (4 pts each)

1. If the dimensions of A is 4 x 3, the dimensions of B is 4 x 5, and the dimensions of C is 7 x 3, then the dimensions of $(A^T B)^T C^T$ is

   a) 5 x 3 b) 4 x 5 c) 5 x 7 d) 4 x 3

   $(3\times4 \cdot 4\times5)^T \to (3\times5)^T \to 5\times3 \cdot 3\times7 \to 5\times7$

2. If $A^2 - 2A - I = 0$, and the inverse of A exists, then the inverse of A is

   a) I b) $A + 2I$ c) $A - 2I$ d) A

   $(A - I)^2 = 0$

   $A = I$

   $A^2 - 2A = I$

   $A(A - 2A) = I$

   $A - 2A = A^{-1}I$

   $A - 2 = A^{-1}I$

   $(A-2)I = A^{-1}$

   $AI - 2I = A^{-1}$

   $\begin{matrix} a^2 & 0 \\ 0 & a^2 \end{matrix} - \begin{matrix} 2a & 0 \\ 0 & 2a \end{matrix} - \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} = 0$

   $a^2 - 2a - 1 \quad 0$

   $0 \quad a^2 - 2a - 1 = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$

   $(a-1)^2 = 0$

   $a = 1$

   $A = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \quad A^{-1} = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} = I$

3. Which statement is NOT always true for matrix multiplication?

   a) $A(BC) = (AB)(C)$ ✓ b) $A(B + C) = AB + AC$ ✓

   c) $AB = 0$ if either A or B is 0 ? ✓ d) $AB = BA$

   $[1] [0] = [0]$

   not commutative

4. For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ one of the eigenvalues is -2. Which of the following is the corresponding eigenvector?

   a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ c) $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

   $3x - 2y + 2z = -2x$ $\quad -2y + z = -2y$ $\quad z = -2z$

   $5x - 2y + 2z = 0$ $\quad z = 0$ $\quad z = 0$

   $5x - 2y = 0$

   $5x = 2y$

   $\langle 2, 5, 0 \rangle$

-4

Questions 5-12 are FREE RESPONSE. Show all your work.

5. Consider the following matrices

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 1 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

Find each of the following, or explain why the operation cannot be performed: [3 pts a-c, 4 pts d, e]

a. $BA =$

1×3 3×3 → 1×3 ✓

$$\begin{bmatrix} 2 & 8 & -6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 4 \\ 1 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -18 & 20 \end{bmatrix}$$ ✓

d. A^{-1} ✓

+ − +
− + −
+ − +

$$\begin{bmatrix} 1 & -3 & 4 \\ 1 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & -4 & 4 \\ 2 & -6 & 4 \\ 3 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 4 & 4 \\ -2 & -6 & -4 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 3 \\ 4 & -6 & 1 \\ 4 & -4 & 0 \end{bmatrix} \xrightarrow{1/\det} \begin{bmatrix} 0 & -1/2 & 3/4 \\ 1 & -3/2 & 1/4 \\ 1 & -1 & 0 \end{bmatrix}$$ ✓

$\det A = 1(0) - 1(2) + 2(3)$
$= 4$

$$\begin{bmatrix} 1 & -3 & 4 \\ 1 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1/2 & 3/4 \\ 1 & -3/2 & 1/4 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓

b. $2C - 6A =$

−1

3×3 3×3 ✓

$$\begin{bmatrix} 0 & 12 & -42 \\ 4 & 8 & -18 \\ 10 & -14 & 2 \end{bmatrix} - \begin{bmatrix} 6 & -18 & 24 \\ 6 & -18 & 18 \\ 12 & -12 & 12 \end{bmatrix} = \begin{bmatrix} -6 & 30 & -66 \\ -2 & 26 & 5 \\ -2 & -2 & -10 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 30 & -66 \\ -2 & 26 & -36 \\ -2 & -2 & -10 \end{bmatrix}$$

e. D^{-1}

not possible because not a square matrix. ✓

Inverse of non-square matrix is undefined.

c. $B^T + D =$

3×1 3×1 ✓

$$\begin{bmatrix} 2 \\ 8 \\ -6 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$$ ✓

−1

6. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & x \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of A, then x is:[4 pts]

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \times \frac{1}{10}\begin{bmatrix} 2 \\ x \\ 3 \end{bmatrix} = 0$$

$$\left(1 \times \frac{2}{10}\right) + \left(-1 \times \frac{x}{10}\right) + \left(1 \times \frac{3}{10}\right) = 0$$

$$\frac{2}{10} - \frac{x}{10} + \frac{3}{10} = 0$$

$$2 - x + 3 = 0$$

$$-x = -5$$

$$x = 5$$ ✓

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 4/10 & 2/10 & 2/10 \\ -5/10 & 0 & 5/10 \\ 1/10 & -2/10 & 3/10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ✓

7. If $A = \begin{bmatrix} sinb & 0 \\ 0 & sinb \end{bmatrix}$ and $det\left(A^2 - \frac{1}{2}I\right) = 0$, then find one possible value of b. [4 pts]

$$A^2 = \begin{bmatrix} \sin^2 b & 0 \\ 0 & \sin^2 b \end{bmatrix}$$

$$\frac{1}{2}I = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\sin^2 b = 1/2$$

$$\sin b = \frac{\sqrt{2}}{2}$$

$$A^2 - \frac{1}{2}I = \begin{bmatrix} \sin^2 b - \frac{1}{2} & 0 \\ 0 & \sin^2 b - \frac{1}{2} \end{bmatrix}$$

$$\det(\ ) = \left(\sin^2 b - \tfrac{1}{2}\right)^2 = 0$$

$$\left(\sin b - \tfrac{\sqrt{2}}{2}\right)^2\left(\sin b + \tfrac{\sqrt{2}}{2}\right)^2 = 0 \quad b = \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{7\pi}{4}$$

$$\sin b = \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2} \quad b = 45°, 135°, 225°, 315°$$ ✓

let $\sin^2 b = u^2$

$$\left(u^2 - \tfrac{1}{2}\right) = \left(u + \tfrac{1}{\sqrt{2}}\right)\left(u - \tfrac{1}{\sqrt{2}}\right)$$

8. Write the system as a 3x4 matrix, then use Gauss-Jordan elimination to solve the linear system:[4 pts]

$$x + y + z = 3$$
$$2x + 3y + 4z = 11$$
$$4x + 9y + 16z = 41$$

$$\begin{matrix} 41 \\ -22 \\ \hline 19 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 4 & 11 \\ 4 & 9 & 16 & 41 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 4 & 11 \\ 0 & 3 & 8 & 19 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 3 & 8 & 19 \end{array}\right] \xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right] \xrightarrow[R_3/2]{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 - 2R_3]{R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \boxed{\begin{matrix} x = 0 \\ y = 1 \\ z = 2 \end{matrix}}$$ ✓

$(0) + (1) + (2) = 3$ ✓

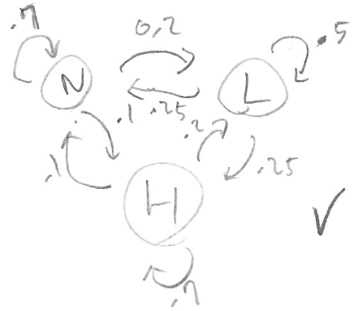
$2(0) + 3(1) + 4(2) = 11$ ✓

$4(0) + 9(1) + 16(2) = 41$ ✓

$\angle 0$

9. A credit card company classifies its customers in a given month in three groups: nonusers, light users, and heavy users. Over the course of a month 70% of nonusers remain nonusers and 20% become light users, 25% of light users become nonusers and 50% remain light users, and 10% of heavy users become nonusers and 20% become light users.

a) Construct a transition **diagram** [2]



✓

b) Construct a transition **matrix.** Label your rows and columns using N, L, H [2]

End

$$\text{Start}\quad \begin{array}{c|ccc} & N & L & H \\ \hline N & .7 & .2 & .1 \\ L & .25 & .5 & .25 \\ H & .1 & .2 & .7 \end{array}$$

✓

.25 × .2 = .050

.7 × .7 = .49

.07 + .05 + .07 = .19

.49 + .05 + .1 = .64

.14 + .10 + .025 = .265

$$\begin{bmatrix} .7 & .2 & .1 \\ .25 & .5 & .25 \\ .1 & .2 & .7 \end{bmatrix}^2 \rightarrow$$

.64 .265 .19

c) Is the Markov Chain described above Regular, Absorbing, or Neither? Give evidence to support your answer.[2] not Absorbing → no term only go to themselves

Regular there aren't any 0 terms ✓

d) Write a system of 3 equations and 3 unknowns that could be used to solve for the equilibrium distribution, where x, y, and z are the number of users in each category, out of a population of 5000 people. YOU DO NOT NEED TO SOLVE THE SYSTEM. [2]

-1

$$[x\ y\ z]\ T = [x\ y\ z]$$

$$.7x + .25y + .1z = x$$
$$.2x + .5y + .2z = y$$
$$.1x + .25y + .7z = z$$

$$\rightarrow \begin{cases} -.3x + .25y + .1z = 0 \\ .2x - .5y + .2z = 0 \\ .1x + .25y - .3z = 0 \end{cases} \rightarrow \text{distribution of equilibrium}$$

$$x + y + z = 5000$$

-1

10. You are playing a game with a spinner which lands green 75% of the time and yellow 25% of the time. You win as soon as you spin **the ordered sequence** of **green, yellow, yellow.** The transition matrix is given on the right.

a) Partition the matrix into canonical form, and label the submatrices Q, R, I and O.[1]

Q

$$\begin{array}{c|c} & \begin{array}{cccc} 0 & g & gy & gyy \end{array} \\ \begin{array}{c} 0 \\ g \\ gy \\ gyy \end{array} & \left[\begin{array}{ccc|c} .25 & .75 & 0 & 0 \\ 0 & .75 & .25 & 0 \\ 0 & .75 & 0 & .25 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

R

O I

✓

b) Compute the Fundamental Matrix N. [4]

$$\begin{bmatrix} 3/4 & -3/4 & 0 \\ 0 & 1/4 & -1/4 \\ 0 & -3/4 & 1 \end{bmatrix}\begin{bmatrix} 4/3 & 16 & 4 \\ 0 & 16 & 4 \\ 0 & 12 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$N = (I - Q)^{-1}$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .25 & .75 & 0 \\ 0 & .75 & .25 \\ 0 & .75 & 0 \end{bmatrix}\right)^{-1} = \begin{bmatrix} .75 & -.75 & 0 \\ 0 & .25 & -.25 \\ 0 & -.75 & 1 \end{bmatrix}^{-1}$$

$$\left[\begin{array}{ccc|ccc} 3/4 & -3/4 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & -1/4 & 0 & 1 & 0 \\ 0 & -3/4 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} 4R_1 \\ 4R_2 \\ 4R_3 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 3 & -3 & 0 & 4 & 0 & 0 \\ 0 & 1 & -1 & 0 & 4 & 0 \\ 0 & -3 & 4 & 0 & 0 & 4 \end{array}\right] \xrightarrow{R_1/3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 4/3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 4 & 0 \\ 0 & -3 & 4 & 0 & 0 & 4 \end{array}\right] \xrightarrow{R_3 + 3R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 4/3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 12 & 4 \end{array}\right] \xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 4/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 16 & 4 \\ 0 & 0 & 1 & 0 & 12 & 4 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4/3 & 16 & 4 \\ 0 & 1 & 0 & 0 & 16 & 4 \\ 0 & 0 & 1 & 0 & 12 & 4 \end{array}\right]$$

$$N = \begin{array}{c|c} & \begin{array}{ccc} 0 & g & gy \end{array} \\ \begin{array}{c} 0 \\ g \\ gy \end{array} & \begin{bmatrix} 4/3 & 16 & 4 \\ 0 & 16 & 4 \\ 0 & 12 & 4 \end{bmatrix} \end{array} \checkmark$$

c) Explain what entry $n_{1,3}$ means in your fundamental matrix. [2]

expected number of times you get green, yellow before getting green, yellow, yellow given that you start at 0. ✓

d) Suppose the game is structured so that it costs $3 for every time you hit the spinner. And for a win in the game (**one green and two yellows**), the prize is $10. Suppose you decide to start playing the game (at state 0), and you have $150 in your pocket. You spin until you win the $10, and then stop. How much money do you expect you have now?[4]

$$NR = \begin{bmatrix} 4/3 & 16 & 4 \\ 0 & 16 & 4 \\ 0 & 12 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Start at 0, expected visits before gyy is $4/3 + 16 + 4 = \frac{64}{3}$

$3 per spin $10 win subtotal $= 10 - 64 = -54$

$\frac{64}{3} \cdot 3 = 64$

total $= 150 - 54 = 96$

$\boxed{\$96}$ ✓

/10

11. Matrix T has eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ with $\lambda = 2$ and $\begin{bmatrix}2\\-1\end{bmatrix}$ with $\lambda = -3$. Find $T\begin{bmatrix}15\\6\end{bmatrix}$. [4]

$\begin{bmatrix}15\\6\end{bmatrix} = av_1 + bv_2$

$= a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}2\\-1\end{bmatrix}$

$15 = a + 2b$

$6 = a - b$

$a = 6 + b$

$15 = 6 + 3b$

$9 = 3b$

$b = 3$

$a = 9$

$\begin{bmatrix}15\\6\end{bmatrix} = 9\begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}2\\-1\end{bmatrix}$

$T\begin{bmatrix}15\\6\end{bmatrix} = T(av_1 + bv_2)$

$= Tav_1 + Tbv_2$

$= \lambda_1 a v_1 + \lambda_2 b v_2$

$= 2\cdot 9\begin{bmatrix}1\\1\end{bmatrix} + (-3)(3)\begin{bmatrix}2\\-1\end{bmatrix}$

$= \begin{bmatrix}18\\18\end{bmatrix} - \begin{pmatrix}18\\-9\end{pmatrix}$

$= \begin{bmatrix}0\\27\end{bmatrix}$ ✓

12. Find both eigenvalues for $\begin{bmatrix}1 & 3\\4 & 2\end{bmatrix}$. You don't need to find the eigenvectors [5]

$\det\begin{bmatrix}1-\lambda & 3\\4 & 2-\lambda\end{bmatrix} = 0$

$(1-\lambda)(2-\lambda) - 12 = 0$

$2 - 3\lambda + \lambda^2 - 12 = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$(\lambda + 2)(\lambda - 5) = 0$

$\lambda = -2, 5$ ✓

A B C

D E

Question 13 is a Multiple Choice Problem. Choose the best answer. [4 pts]

13. People moving between booths at a convention can be modeled by an Absorbing Markov Chain where booths A, B, and C are transition states, and booths D and E are absorbing states. The NR matrix is shown below.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{cc} D & E \end{array} \\ \begin{bmatrix} .2 & .8 \\ .7 & .3 \\ .4 & .6 \end{bmatrix} \end{array} = NR$$

Which of the following is true?

a) Over the long term, 20% of the people will end up at booth D
b) Over the long term, everyone will end up at booth E
c) If you start at booth A, you will never visit booth B
d) Booth E will end up with more people than booth D
e) If you start at booth C, you are more likely to end up at booth E than booth D.

✓

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