Analysis H – Hahn / Hlasek / Tantod
Unit 6: Matrices Exam
No Calculators

97.4

75
77 pts

If Anderson isn't sure how to solve it, you can always just Gauss and check! Per: 6

Questions 1 – 4 are multiple choice. Circle the BEST answer. (4 pts each)

1. If the dimensions of A is 4 x 3, the dimensions of B is 4 x 5, and the dimensions of C is 7 x 3, then the dimensions of $(A^T B)^T C^T$ is

   a) 5 x 3 b) 4 x 5 c) 5 x 7 d) 4 x 3

2. If $A^2 - 2A - I = 0$, and the inverse of A exists, then the inverse of A is

   a) I b) $A + 2I$ c) $A - 2I$ d) A

3. Which statement is NOT always true for matrix multiplication?

   a) $A(BC) = (AB)(C)$ b) $A(B + C) = AB + AC$

   c) $AB = 0$ if either A or B is 0 d) $AB = BA$

4. For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ one of the eigenvalues is -2. Which of the following is the corresponding eigenvector?

   a) $\cancel{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}$ b) $\cancel{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}$ c) $\cancel{\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}}$ d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

$Av = \lambda v$

$$\begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \\ -2z \end{bmatrix}$$

$5x - 2y + 2z = 0$
$z = 0$

/20

Questions 5-12 are FREE RESPONSE. Show all your work.

5. Consider the following matrices

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 1 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

3×3   1×3   3×3   3×1

Find each of the following, or explain why the operation cannot be performed: [3 pts a-c, 4 pts d, e]

a. $BA =$

$\begin{bmatrix} -2 & -18 & 20 \end{bmatrix}$ ✓

0 412 16

d. A^{-1}

$\begin{bmatrix} 0 & -4 & 4 \\ -2 & -6 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 4 & 4 \\ -2 & -6 & -4 \\ 3 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -2 & 3 \\ 4 & -6 & 1 \\ 4 & -4 & 0 \end{bmatrix} \cdot \frac{1}{4}$ $\begin{bmatrix} 0 & -\frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{3}{2} & \frac{1}{4} \\ 1 & -1 & 0 \end{bmatrix}$ ✓

$0 - 12 + 16 = 4.$

b. $2C - 6A =$

$\begin{bmatrix} 0 & 12 & -42 \\ 4 & 8 & -18 \\ 10 & -14 & 2 \end{bmatrix} - \begin{bmatrix} 6 & -18 & 24 \\ 6 & -18 & 18 \\ 12 & -12 & 12 \end{bmatrix}$

$= \begin{bmatrix} -6 & 30 & -66 \\ -2 & 26 & -36 \\ -2 & -2 & -10 \end{bmatrix}$ ✓

e. D^{-1}

-1 not possible

WHY?

c. $B^T + D =$

$\begin{bmatrix} 2 \\ 8 \\ -6 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix}$ ✓

-1

6. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & x \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of A, then x is:[4 pts]

$$\begin{bmatrix} 4 & 5 & 1 \\ -2 & 0 & 2 \\ 2 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \cdot \frac{1}{10}$$

$\boxed{x=5}$ ✓

$4+5+1=10$

7. If $A = \begin{bmatrix} sinb & 0 \\ 0 & sinb \end{bmatrix}$ and $det\left(A^2 - \frac{1}{2}I\right) = 0$, then find one possible value of b. [4 pts]

$$\begin{bmatrix} \sin b & 0 \\ 0 & \sin b \end{bmatrix}\begin{bmatrix} \sin b & 0 \\ 0 & \sin b \end{bmatrix} = \begin{bmatrix} \sin^2 b - 0 & \\ 0 & \sin^2 b \end{bmatrix} - \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} =$$

$$\det \begin{bmatrix} \sin^2 b - \frac{1}{2} & 0 \\ 0 & \sin^2 b - \frac{1}{2} \end{bmatrix} \quad (\sin^2 b - \tfrac{1}{2})(\sin^2 b - \tfrac{1}{2}) = \sin^4 b - \sin^2 b + \tfrac{1}{4} = 0$$

$x = \sin^2 b$ $\quad x^2 - x + 1/4 = 0 \quad x = \frac{1 \pm \sqrt{1-4/4}}{2} = \frac{1}{2}$

$\sin^2 b = \frac{1}{2} \quad \sin b = \frac{\sqrt{2}}{2} \quad \boxed{b = 45°}$ ✓

8. Write the system as a 3x4 matrix, then use Gauss-Jordan elimination to solve the linear system:[4 pts]

$x + y + z = 3$
$2x + 3y + 4z = 11$
$4x + 9y + 16z = 41$

$$\begin{array}{c} A \\ B \\ C \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & 4 & 11 \\ 4 & 9 & 16 & 41 \end{array}\right] \xrightarrow[B-2A]{C-4A} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 3 & 8 & 19 \end{array}\right] \xrightarrow{C-3B} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right] \quad \begin{array}{r} 3\cdot 5 \\ -22 \\ \hline 19 \end{array}$$

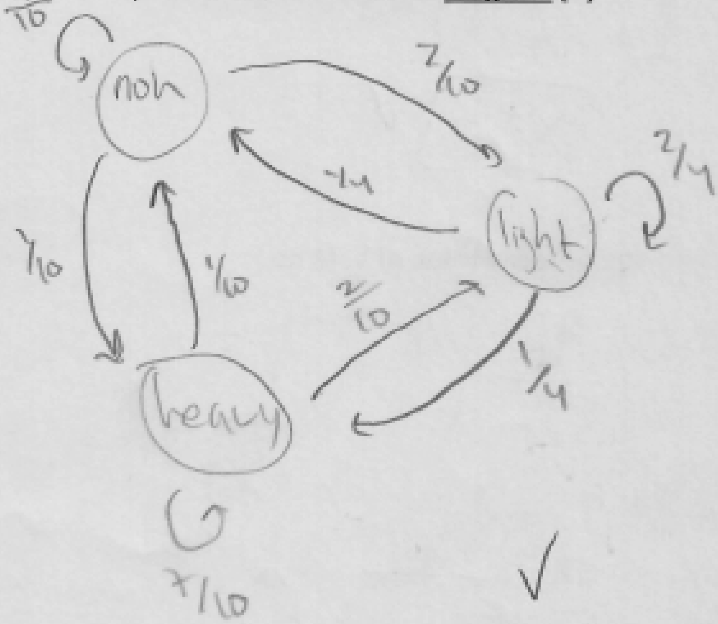
$$\xrightarrow{B-C} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right] \xrightarrow{A-B} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right] \xrightarrow[C/2]{A-\frac{C}{2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x=0$
$y=1$
$z=2$
✓

$\cancel{\ }0$

9. A credit card company classifies its customers in a given month in three groups: nonusers, light users, and heavy users. Over the course of a month 70% of nonusers remain nonusers and 20% become light users, 25% of light users become nonusers and 50% remain light users, and 10% of heavy users become nonusers and 20% become light users.

a) Construct a transition **diagram** [2]



b) Construct a transition **matrix.** Label your rows and columns using N, L, H [2]

$$T = \begin{array}{c} \\ N \\ L \\ H \end{array} \begin{array}{c} \begin{array}{ccc} N & L & H \end{array} \\ \begin{bmatrix} \frac{7}{10} & \frac{2}{10} & \frac{1}{10} \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ \frac{1}{10} & \frac{2}{10} & \frac{7}{10} \end{bmatrix} \end{array}$$

✓

c) Is the Markov Chain described above Regular, Absorbing, or Neither? Give evidence to support your answer.[2] Not Absorbing (no absorbing states)

Yes regular (any exponent of T will have all positive non-zero)
(all values are already + no zero) ✓

-1 d) Write a system of 3 equations and 3 unknowns that could be used to solve for the equilibrium distribution, where x, y, and z are the number of users in each category, out of a population of 5000 people. YOU DO NOT NEED TO SOLVE THE SYSTEM. [2]

$vT = v$ 1×3 3×3 1×3

$$[x\ y\ z]\begin{bmatrix} \frac{7}{10} & \frac{2}{10} & \frac{1}{10} \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ \frac{1}{10} & \frac{2}{10} & \frac{7}{10} \end{bmatrix} = [x\ y\ z]$$

$$-\frac{3x}{10} + \frac{y}{4} + \frac{z}{10} = 0$$

$$\frac{x}{5} - \frac{y}{2} + \frac{z}{5} = 0 \quad \xrightarrow{\text{solved}} [x\ y\ z]\cdot 5000$$

$$\frac{x}{10} + \frac{y}{4} - \frac{3z}{10} = 0$$

$$x + y + z = 1$$

-1

10. You are playing a game with a spinner which lands green 75% of the time and yellow 25% of the time. You win as soon as you spin **the ordered sequence** of **green, yellow, yellow.** The transition matrix is given on the right.

$$
\begin{array}{c}
\\ 0 \\ g \\ gy \\ gyy
\end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & g & gy & gyy \end{array} \\
\left[\begin{array}{ccc|c}
.25 & .75 & 0 & 0 \\
0 & .75 & .25 & 0 \\
0 & .75 & 0 & .25 \\ \hline
0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

Q, R, O, I

a) Partition the matrix into canonical form, and label the submatrices Q, R, I and O.[1]

$$\left[\begin{array}{c|c} Q & R \\ \hline O & I \end{array}\right] \checkmark$$

b) Compute the Fundamental Matrix N. [4]

$N = (I-Q)^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/4 & 3/4 & 0 \\ 0 & 3/4 & 1/4 \\ 0 & 3/4 & 0 \end{bmatrix} = \begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|ccc} \frac{3}{4} & -\frac{3}{4} & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 0 & 1 & 0 \\ 0 & -\frac{3}{4} & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} A+3B \\ \\ C+3B \end{array} \left[\begin{array}{ccc|ccc} \frac{3}{4} & 0 & -3/4 & 1 & 3 & 0 \\ 0 & 1/4 & -1/4 & 0 & 1 & 0 \\ 0 & 0 & 1/4 & 0 & 3 & 1 \end{array}\right]$$

$$\begin{array}{c} A+3C \\ B+C \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 3/4 & 0 & 0 & 1 & 12 & 3 \\ 0 & 1/4 & 0 & 0 & 4 & 1 \\ 0 & 0 & 1/4 & 0 & 3 & 1 \end{array}\right] \rightarrow \begin{array}{c} A\cdot\frac{4}{3} \\ B\cdot 4 \\ C\cdot 4 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{3} & 16 & 4 \\ 0 & 1 & 0 & 0 & 16 & 4 \\ 0 & 0 & 1 & 0 & 12 & 4 \end{array}\right]$$

$$N = \begin{array}{c} \\ 0 \\ g \\ gy \end{array} \begin{array}{c} \begin{array}{ccc} 0 & g & gy \end{array} \\ \begin{bmatrix} \frac{4}{3} & 16 & 4 \\ 0 & 16 & 4 \\ 0 & 12 & 4 \end{bmatrix} \end{array} \checkmark$$

c) Explain what entry $n_{1,3}$ means in your fundamental matrix. [2]

Starting with state 0, you get to the gy state avege of 4 times before absorbing ✓

d) Suppose the game is structured so that it costs \$3 for every time you hit the spinner. And for a win in the game (**one green and two yellows**), the prize is \$10. Suppose you decide to start playing the game (at state 0), and you have \$150 in your pocket. You spin until you win the \$10, and then stop. How much money do you expect you have now?[4]

$\frac{4}{3} + 20 = \frac{64}{3}$ steps before absorbing

$\frac{64}{3} \cdot \$3 = \$64$

$$\begin{array}{r} 150 \\ -\ 64 \\ \hline 86 \end{array}$$

$+\$10 = \$96$ ✓

/10

11. Matrix T has eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ with $\lambda = 2$ and $\begin{bmatrix}2\\-1\end{bmatrix}$ with $\lambda = -3$. Find $T\begin{bmatrix}15\\6\end{bmatrix}$. [4]

$V = aV_1 + bV_2$

$\begin{bmatrix}15\\6\end{bmatrix} = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}2\\-1\end{bmatrix}$

$a + 2b = 15$

$a - b = 6$

$3b = 9$

$b = 3$

$a = 9$

$AV = a\lambda_1 v_1 + b\lambda_2 v_2$

$T\begin{bmatrix}15\\6\end{bmatrix} = 18\begin{bmatrix}1\\1\end{bmatrix} + -9\begin{bmatrix}2\\-1\end{bmatrix}$

$\begin{bmatrix}18\\18\end{bmatrix} + \begin{bmatrix}-18\\9\end{bmatrix}$

$\boxed{\begin{bmatrix}0\\27\end{bmatrix}}$ ✓

12. Find both eigenvalues for $\begin{bmatrix}1 & 3\\4 & 2\end{bmatrix}$. You don't need to find the eigenvectors [5]

$\det\begin{bmatrix}1-\lambda & 3\\4 & 2-\lambda\end{bmatrix} = 0$

$(1-\lambda)(2-\lambda) - 12 = 0$

$2 - 3\lambda + \lambda^2 - 12 = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$(\lambda+2)(\lambda-5) = 0$

$\boxed{\lambda = -2, \ \lambda = 5}$ ✓

Question 13 is a Multiple Choice Problem. Choose the best answer. [4 pts]

13. People moving between booths at a convention can be modeled by an Absorbing Markov Chain where booths A, B, and C are transition states, and booths D and E are absorbing states. The NR matrix is shown below.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cc} D & E \end{array} \\ \begin{bmatrix} .2 & .8 \\ .7 & .3 \\ .4 & .6 \end{bmatrix} \end{array} = NR$$

Which of the following is true?

a) Over the long term, 20% of the people will end up at booth D
b) Over the long term, everyone will end up at booth E
c) If you start at booth A, you will never visit booth B
d) Booth E will end up with more people than booth D
e) If you start at booth C, you are more likely to end up at booth E than booth D.

✓

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