Analysis H – Hahn / Hlasek / Tantod
Unit 6: Matrices Exam
No Calculators

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Justin Oh is in school to get adjugated
Period: 7

**Questions 1 – 7 are multiple choice. Circle the BEST answer. (4 pts each)**

1. Which of the following properties apply to matrices?

   I. associative property of multiplication
   II. commutative property of multiplication
   III. distributive property of multiplication over addition

   a) I only  b) II only  c) I and II only  d) I and III only  e) I, II, and III.

2. The product of a 4x5 matrix and a 5x2 matrix is a __________ matrix.

   a) 4x2  b) 5x5  c) 4x5  d) 5x2  e) none of these

3. The sum of a 4x5 and a 5x2 matrix is a __________ matrix

   a) 4x2  b) 5x5  c) 4x5  d) 5x2  e) none of these

4. In the case of Absorbing Markov Chains, we wrote our canonical form as $\begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$, with N = the Fundamental Matrix. Which of the following shows the "expected number of visits to each transition state before being absorbed"?

   a) Q  b) R  c) $(I - Q)^{-1}$  d) $(Q - I)^{-1}$  e) $(I - Q)^{-1}R$

5. In the case of Absorbing Markov Chains, we wrote our canonical form as $\begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$, with N = the Fundamental Matrix. Which of the following shows the "probability of ending at each absorbing state, based on initial transition state"?

   a) QR  b) R  c) N  d) NR  e) RN

6. $\begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$ has $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ as one of its eigenvectors. The associated eigenvalue is $\underline{-5}$.

   $-4-4-2=-10$
   $-4+1-2=-5$

   a) -5  b) 5  c) -2  d) 2  e) -1

7. Given: A is a square matrix, $x$ is a column vector, and $\lambda$ is a scalar. *If* $Ax = \lambda x$ has a non-zero solution for $x$, then ______________.

   a) $Ax = 0$  b) $(A - \lambda I) = 0$  c) $Ax - \lambda I = 0$  d) $(A - \lambda I)^{-1} = 0$  e) $\det(A - \lambda I) = 0$

$$\overset{3\times 2}{E = \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ 6 & -4 \end{bmatrix}} \qquad \overset{2\times 2}{F = \begin{bmatrix} 2 & 6 \\ -1 & 8 \end{bmatrix}} \qquad \overset{3\times 3}{G = \begin{bmatrix} 1 & 0 & 2 \\ -4 & 0 & 3 \\ 7 & 0 & 4 \end{bmatrix}} \qquad \overset{3\times 3}{H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}}$$

8. Refer to the matrices E, F, G, and H above. Find each, if possible. If not possible, write "Not Possible" [(a) – (d) are 3 pts each, (e) and (f) are 4 pts each]

a) EF

$$= \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ -1 & 8 \end{bmatrix} = \boxed{\begin{bmatrix} -8 & -2 \\ -3 & 46 \\ 16 & 4 \end{bmatrix}}$$

b) $FE^T$

$$= \begin{bmatrix} 2 & 6 \\ -1 & 8 \end{bmatrix}\begin{bmatrix} -3 & 1 & 6 \\ 2 & 5 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 32 & -12 \\ 19 & 39 & -38 \end{bmatrix}}$$

c) $E^{-1}$

Not possible

d) $F^{-1}$

$$= \frac{1}{\det F}\begin{bmatrix} 8 & -6 \\ 1 & 2 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 8 & -6 \\ 1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{4}{11} & -\frac{3}{11} \\ \frac{1}{22} & \frac{1}{11} \end{bmatrix}}$$

e) $G^{-1}$

$\det G = 0$

$\Rightarrow$ Not possible

f) $H^{-1}$

$$A: \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_3 - R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 & -1 & 2 \end{array}\right] \xrightarrow[R_3/3]{3R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 6 & 0 & -4 & 4 & -2 \\ 0 & 0 & 1 & 1/3 & -1/3 & 2/3 \end{array}\right]$$

$$\xrightarrow[R_2/6]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 2/3 & -1/3 \\ 0 & 0 & 1 & 1/3 & -1/3 & 2/3 \end{array}\right] ; \quad \boxed{H^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{bmatrix}}$$

9. Matrix A and B are distinct, non-zero, 3x3 matrices. Matrix C and D are distinct, non-zero 3x1 matrices. Determine if each statement is Always True, Sometimes True, or Never True. Then CIRCLE the appropriate word for each statement. [1 pt each → 6 pts total]

| | |
|---|---|
| a) $AC = B$ | a) Always or Sometimes or Never |
| b) $AB = A$ | b) Always or Sometimes or Never |
| c) $A + B$ is a 3x3 matrix | c) Always or Sometimes or Never |
| d) $DC^T = CD^T$ | d) Always or Sometimes or Never |
| e) $(AB)^T = B^TA^T$ | e) Always or Sometimes or Never |
| f) $AC = C$ | f) Always or Sometimes or Never |

$D \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad C \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

$DC^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} = \begin{matrix} a\alpha & a\beta & a\gamma \\ b\alpha & b\beta & b\gamma \\ c\alpha & c\beta & c\gamma \end{matrix}$

$CD^T = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{matrix} a\alpha \\ a\beta \\ a\gamma \end{matrix}$

-0

10. Consider the system:

$$\begin{cases} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{cases}$$

Write the system as a 3x4 augmented matrix A, and then use Gauss-Jordan row operations to change entries $a_{2,1}$, $a_{3,1}$, and $a_{3,2}$ into 0's (<u>stop once you've achieved this result</u>). Then state the value of z in the system. [4 pts]

$$A = \left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 0 & -3 & -3 & 3 \\ 4 & 10 & -1 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2/3} \left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & -6 & -9 & -3 \end{array}\right] \xrightarrow{R_3 - 6R_2} \boxed{\left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -3 & -9 \end{array}\right]}$$

$-3z = -9$

$\boxed{z = 3}$

-2 11. Identify each matrix as "Regular", "Absorbing", or "Neither" [2 pts each]

a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0.5 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$ d) $\begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.1 & 0.5 & 0.4 \\ 0.2 & 0.7 & 0.2 \end{bmatrix}$

a) neither b) neither c) absorbing d) ~~regular~~

12. The "We are Chess Kids" (WaCK) club at Gunn currently has 500 members, while the "Thankful if Kids Take Over Chess" (TiKTOC) club at Gunn has 200 members, but there's constant shuffling of members between the two. $\frac{1}{3}$ of the WaCK members switch to the TiKTOC club each week, while $\frac{1}{4}$ of the TiKTOC members switch to the WaCK club every week.

a) Write a transition matrix for this scenario. Label the rows and columns of the matrix. [4 pts]

$$T = \begin{array}{c} \\ \text{WaCK} \\ \text{TiKTOC} \end{array} \begin{array}{c} \begin{array}{cc} W & T \end{array} \\ \begin{bmatrix} 2/3 & 1/3 \\ 1/4 & 3/4 \end{bmatrix} \end{array}$$

b) Write a 2x3 augmented matrix that, if you put it into a calculator and use the "Reduced Row Echelon Form" function, would return a matrix that you could use to easily find out how many members each club would have after many weeks. [4 pts]

$[x\ y]\,T = [x\ y]$ $\quad 2/3\,x + \frac{y}{4} = x$

$\frac{y}{4} = \frac{x}{3}$

$x + y = 1$

$$\text{rref}\left[\begin{array}{cc|c} 1/3 & -1/4 & 0 \\ 1 & 1 & 1 \end{array}\right]$$

-2

13. The transition matrix of a certain Markov Chain (where the states in the system are named A, B, C, D, E, and F):

$$T = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0.25 & 0 & 0.75 & 0 & 0 & 0 \\ 0 & 0.25 & 0.25 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \end{array}$$

a) Identify the absorbing state(s) in the system [2 pts]: C, E

b) Write matrix T in canonical form (Keep the states in alphabetical order within their respective categories, and label them on the left and top of your answer). [4 pts]

$$T = \begin{array}{c} \\ \text{non-absorbing}\left\{\begin{array}{c} A \\ B \\ D \\ F \end{array}\right. \\ \text{absorbing}\left\{\begin{array}{c} C \\ E \end{array}\right. \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & D & F & C & E \end{array} \\ \left[\begin{array}{cccc|cc} 0.25 & 0 & 0 & 0 & 0.75 & 0 \\ 0 & 0.25 & 0 & 0 & 0.25 & 0.5 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0.25 & 0.25 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \\ \begin{array}{cc} \underbrace{\hspace{6em}}_{\text{transient}} & \underbrace{\hspace{3em}}_{\text{absorbing}} \end{array} \end{array}$$

c) The fundamental matrix is $\begin{array}{c} A \\ B \\ D \\ F \end{array} \begin{bmatrix} \frac{4}{3} & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 0 & 0 \\ 0 & \frac{4}{3} & 1 & 0 \\ 0 & \frac{4}{9} & \frac{1}{3} & \frac{4}{3} \end{bmatrix}$.

If you started in state D, how many total transitions would you expect to make before being absorbed? [2 pts]

$$\boxed{\frac{7}{3}}$$

d) If you started in state F, what is the probability that you will end up in state C? Show the matrix multiplication you did to justify your answer. [4 pts]

$$NR = \begin{bmatrix} 4/3 & 0 & 0 & 0 \\ 0 & 4/3 & 0 & 0 \\ 0 & 4/3 & 1 & 0 \\ 0 & 4/9 & 1/3 & 4/3 \end{bmatrix} \begin{bmatrix} 3/4 & 0 \\ 1/4 & 1/2 \\ 0 & 0 \\ 1/4 & 1/4 \end{bmatrix} = \begin{array}{c} \\ A \\ B \\ D \\ F \end{array} \begin{array}{c} \begin{array}{cc} C & E \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1/3 & 2/3 \\ 1/3 & 2/3 \\ 4/9 & 5/9 \end{bmatrix} \end{array}$$

$$\boxed{4/9}$$

-2

14. A certain 2x2 matrix G has 2 eigenvalues: 3 and 5.
An eigenvector for $\lambda_1 = 3$ is $\begin{bmatrix}1\\1\end{bmatrix}$, and an eigenvector for $\lambda_2 = 5$ is $\begin{bmatrix}2\\0\end{bmatrix}$. Find $G\begin{bmatrix}20\\8\end{bmatrix}$. [4 points]

$\vec{v}_1$ $\vec{v}_2$

$$\begin{bmatrix}20\\8\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}2\\0\end{bmatrix} \Rightarrow G\begin{bmatrix}20\\8\end{bmatrix} = \lambda_1\alpha\begin{bmatrix}1\\1\end{bmatrix} + \lambda_2\beta\begin{bmatrix}2\\0\end{bmatrix}$$

$\alpha = 8, \beta = 6$

$$= \begin{bmatrix}24\\24\end{bmatrix} + \begin{bmatrix}60\\0\end{bmatrix} = \boxed{\begin{bmatrix}84\\24\end{bmatrix}}$$

15. Find both eigenvalues for $\begin{bmatrix}1 & -4\\3 & -6\end{bmatrix}$. You don't need to find the eigenvectors. [5 pts]

$A\vec{v} = \lambda\vec{v}$

$\begin{vmatrix}1 & -4\\3 & -6\end{vmatrix} = -6 + 12 = 6$

$(1-\lambda)(-6-\lambda) + 12 = 0$

$\lambda^2 + 5\lambda + 6 = 0$

$\lambda = -\frac{5}{2} \pm \sqrt{\frac{25}{4} - 6}$

$= -\frac{5}{2} \pm \sqrt{\frac{1}{4}}$

$= -\frac{5}{2} \pm \frac{1}{2}$

$\boxed{\lambda = -2, -3}$

16. One of the eigenvalues of $\begin{bmatrix}4 & 0 & 1\\-1 & -6 & -2\\5 & 0 & 0\end{bmatrix}$ is $\lambda = 5$. Find the eigenspace associated with $\lambda = 5$. [5 pts]

$$\begin{bmatrix}4 & 0 & 1\\-1 & -6 & -2\\5 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = 5\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$4x + z = 5x$

$-x - 6y - 2z = 5y$

$z = 1 : x = 1, y = -\frac{3}{11}$

$\begin{bmatrix}t\\-\frac{3t}{11}\\t\end{bmatrix}$ $\boxed{\begin{bmatrix}11t\\-3t\\11t\end{bmatrix}}$

1.0