$\frac{86}{88}$

Analysis Matrices Test 2018/19
Deggeller, Gleason, Hahn
**No Calculation Section:** -2

Eigen go for that!! Timothy Heriken
Period: 6

1. [4 pts] Solve for r in terms of other constants in this equation. $k\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$

$r = ka - (2a+3c) = \boxed{(k-2)a - 3c}$

2. Does there exist a 2x3 matrix M that satisfies the equation below? If so, find it. If not clearly justify mathematically why it does not exist. [4]

pseudoinverse??

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 2a+3d & 2b+3e & 2c+3f \\ -a+d & -b+e & -c+f \\ 2d & 2e & 2f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer y/n No

Justification: Consider the desired multiplication above, where we wish to find a, b, c, d, e, f so the resultant matrix is $I_3$. Thus, we have $2d=0$, so $d=0$, but $-a+d=0$, so $a=0$. Finally, $2a+3d=1$, but we have a contradiction since $2(0)+3(0)=0$ and $1 \neq 0$. Thus, M does not exist

3. A certain transformation matrix T has eigenvalues $\lambda = -2$ (with eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$) and $\lambda = 3$ (with eigenvector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$). What is $T\begin{bmatrix} 6 \\ 10 \end{bmatrix}$? [3 points]

$$\begin{bmatrix} 6 \\ 10 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$\vec{t}$. eigenspace

$$6 \cdot -2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \cancel{3} \cdot \frac{2}{\cancel{3}}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ -24 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} -12 \\ -30 \end{bmatrix}}$$

4. Prove algebraically that the following matrix has no real eigenvalues. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ [3]

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda I\right) = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda^2 = -1 \longrightarrow \lambda = \pm i \notin \mathbb{R}$$

5. Find the eigenvalues for the matrix below along with the corresponding set of eigenvectors. Clearly display your answers. $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$ [5]

$$\det\left(\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}\right) = 0$$

$$(7-\lambda)(-1-\lambda) - 3(3) = 0 \Rightarrow -7 - 7\lambda + \lambda + \lambda^2 - 9 = 0$$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$(\lambda - 8)(\lambda + 2) = 0$$

$$\lambda = 8, -2$$

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8x \\ 8y \end{bmatrix}$$

$7x + 3y = 8x$
$3x - y = 8y$
$y = \frac{1}{3}x \longrightarrow$

$7x + 3y = -2x$
$3x - y = -2y$
$y = -3x \rightarrow \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Answers

| eigenvector | associated $\lambda$ |
|---|---|
| $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ | 8 |
| $\cancel{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}$ | -2 |

-1

6. Find the inverse of matrix A= $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ using either of the methods we learned in class. [5]

$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ minors $= \begin{bmatrix} 4-2 & 4-1 & 2-1 \\ -4-2 & 4-1 & 2-(-1) \\ -1-1 & 1-1 & 1-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ -6 & 3 & 3 \\ -2 & 0 & 2 \end{bmatrix}$

cofactor $= \begin{bmatrix} 2 & -3 & 1 \\ 6 & 3 & -3 \\ -2 & 0 & 2 \end{bmatrix}$ transposed $= \begin{bmatrix} 2 & 6 & -2 \\ -3 & 3 & 0 \\ 1 & -3 & 2 \end{bmatrix}$ (adjoint)

det $= 1 \cdot 2 + (-1)(-3) + 1(1) = 6$

$$\boxed{A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 6 & -2 \\ -3 & 3 & 0 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} \end{bmatrix}}$$

7. In the fine city of Sunnyvale (where the weather always transitions from Sunny to Sunny) there are 3 Starbucks within a mile of downtown, let's call them A,B and C. The following 3 x 3 matrix M captures their average number of coffees sold by size (Grande, Tall, and Venti) during the morning commute hours. You can call this M

$$\begin{array}{c c} & \begin{array}{ccc} G & T & V \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 100 & 180 & 60 \\ 120 & 150 & 100 \\ 30 & 70 & 110 \end{bmatrix} \end{array} = M$$

If the price of a Grande is \$1.85, a Tall is \$2.10 and a Venti is \$2.45 we can create a price matrix.

a) Circle the product below that would yield meaningful information: [3]

$$[1.85 \quad 2.10 \quad 2.45]\begin{bmatrix} 100 & 180 & 60 \\ 120 & 150 & 100 \\ 30 & 70 & 110 \end{bmatrix} \quad OR \quad \underbrace{\begin{bmatrix} 100 & 180 & 60 \\ 120 & 150 & 100 \\ 30 & 70 & 110 \end{bmatrix}}_{3\times3}\underbrace{\begin{bmatrix} 1.85 \\ 2.10 \\ 2.45 \end{bmatrix}}_{3\times1} \rightarrow 3\times1$$

b) Write a sentence explaining what your product (from above) would represent. YOU DO NOT NEED TO ACTUALLY MULTIPLY!! [3]

This product will be a 3x1 matrix (column matrix) with rows A, B, C and giving the total sales (in terms of money) at each location.

$$\begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} - \\ - \\ - \end{bmatrix}$$

-0

8. Let M be a random 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = M$

a) True/False: $M(M^T) = M^T(M)$ for any M ____false____ [3]

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

b) True/False: If detM = 7, then the equation M$\begin{bmatrix} x \\ y \end{bmatrix}$=$\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ has exactly 1 solution. ____True____ [3]

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{M^{-1}}_{\text{exists}}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

c) If M=$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ find $M^{700}$

$$\boxed{\begin{bmatrix} 2^{700} & 0 \\ 0 & 2^{700} \end{bmatrix}}$$ [3]

$$M^2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4I$$

$$M^{700} = (M^2)^{350} = (4I)^{350} = 2^{700} I^{350} = 2^{700} I$$

9. Professor Eigentdoit has made an error in his Gauss Jordan Elimination below! Find his error, fix it, and finish the problem correctly (making changes along the way) to find the true solution using G-J. [5]

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \to R_2 - 2R_1 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right] \to R_3 - 4R_1 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right] \to$$

$$R_3 + 4R_2 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right] \to \div 13 \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \to R_1 - R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -2 & \cancel{5}\ 7 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \to R_1 + 2R_3 \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \cancel{1}\ 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \to R_2 - 3R_3 \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & \cancel{1}\ 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Correct final answer: $\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$

10. **Circle all** matrices below which are **regular transition matrices** [5]

a. $\begin{bmatrix} .3 & .4 & .3 \\ .2 & .6 & .2 \\ .1 & .1 & .8 \end{bmatrix}$ $\to 1$, $\to 1$, $\to 1$

b. $\cancel{\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}}$ $\to \frac{7}{5} > 1$

c. $\cancel{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$ $\to 2 > 1$

d. $\cancel{\begin{bmatrix} .4 & -.6 \\ .6 & .4 \end{bmatrix}}$ $\to -.2 < 0$

e. $\cancel{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$ not regular (nor absorbing)

GO

11. **<u>Circle all</u>** matrices below that would generate **absorbing Markov Chains**: [5]

a. $\begin{bmatrix} .2 & .3 & .5 \\ 1 & 0 & 0 \\ .5 & .2 & .3 \end{bmatrix}$

b. $\begin{bmatrix} .1 & .3 & .6 \\ 0 & .2 & .8 \\ 0 & 0 & 1 \end{bmatrix}$ → yes

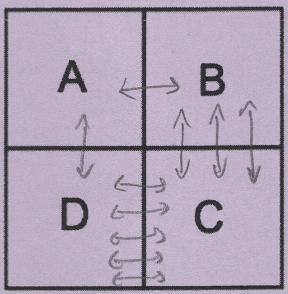
c. $\begin{bmatrix} 0 & .5 & .5 & 0 \\ .2 & 0 & 0 & .8 \\ .3 & 0 & 0 & .7 \\ 0 & .2 & .8 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ .3 & .5 & .2 & 0 \\ .5 & .2 & .3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

can't ~~necessarily~~ absorb all starting states

12. A rat is roaming through a square maze with four rooms. There is one door between rooms A and B, 3 doors between B and C, 5 doors between C and D, and one door between A and D. A step in the Markov process consists of moving from one room to a different room. **The rat must change rooms each move. Each door in a given room is equally likely to be taken.**

a) Draw the correct number of doors between rooms on the diagram to the right. [2]




b) Write the transition matrix which represents the probabilities of the rat moving from room to room in the maze: [3]

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{8} & 0 & \frac{5}{8} \\ \frac{1}{6} & 0 & \frac{5}{6} & 0 \end{bmatrix} \end{array}$$

c) Roxie the Rat is in room C right now. Given that she takes exactly 1 step, what is the probability she will end up in room D? [2]

$\boxed{\frac{5}{8}}$

d) Roxie the Rat is in C right now. Given that she takes exactly 2 steps, what is the probability she will end up back in room C? NO need to simplify. [3]

$$\langle 0, \tfrac{3}{8}, 0, \tfrac{5}{8} \rangle \cdot \langle 0, \tfrac{3}{4}, 0, \tfrac{5}{6} \rangle = \frac{3}{8}\cdot\frac{3}{4} + \frac{5}{8}\cdot\frac{5}{6}$$
$$= \boxed{\frac{9}{32} + \frac{25}{48}}$$

e) In the long run, Roxie's chances of being in a room is directly proportional to the number of doors in that room. Write the number of doors per room as a 1x4 matrix and then state the probability of the rat ending up in room B in the long run, given the rat started in any room. [3]

$$\frac{1}{\sum M_i} M = \frac{1}{20}[2\ 4\ 8\ 6) = \frac{1}{10}(1\ 2\ 4\ 3) = [.1\ .2\ .4\ .3)$$

Matrix: [2, 4, 8, 6]

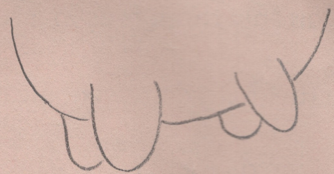
probability: .2

-0

**ator Section:** I hope Timothy H. doesn't get Mark'ed Ov.

1. Consider the transformation matrix for a Markov chain:

$$M = \begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{bmatrix}$$



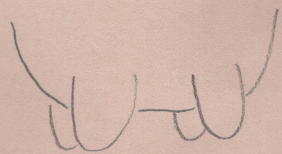
***M is a unique transition matrix in that it is neither absorbing or regular.*** oh yes!

a) Suppose the initial distribution vector of the Markov Chain is: [.1 .6 .3]

Explore what would happen over time to the distribution, considering consecutive "big" powers like $M^{200}$ *and* $M^{201}$. Write a sentence summarizing your findings: [4]

-2

It appears that $M^{2n} = \begin{bmatrix} .5 & 0 & .5 \\ 0 & 1 & 0 \\ .5 & 0 & .5 \end{bmatrix}$, while $M^{2n+1} = M$, where $n \in \mathbb{Z}^+$.



b) Surprisingly there **does** exist an equilibrium distribution for this Markov Chain! Set up and solve a system of equations to find it. Show your work and clearly state your answer. [5]

$$[x\ y\ z] \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{bmatrix}}^{M} = [.5y\ \ x+z\ \ .5y]$$

1x3 3x3

because it's a prob. distrib.

$x = .5y,\ y = x+z,\ z = .5y,\ \boxed{x+y+z=1}$

$\Downarrow$

$x = .25,\ y = .5,\ z = .25$

$$\rightarrow \boxed{[.25, .5, .25]}$$

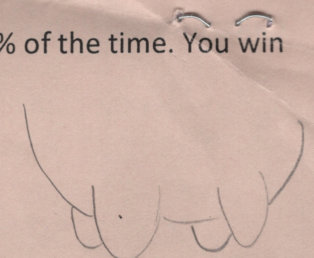
-2

FLIP OVER FOR THE THRILLING LAST PROBLEM!!!!

THRILLING!

2. You are playing a game with a spinner which lands red 70% of the time and blue 30% of the time. You win as soon as you spin **three reds in a row**. The transition matrix is given below.

a) Partition the matrix into canonical form, and label the sub matrices Q, R, I and O.



$$
\begin{array}{c} \\ 0 \\ r \\ rr \\ rrr \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & r & rr & rrr \end{array} \\
\left[\begin{array}{ccc|c} .3 & .7 & 0 & 0 \\ .3 & 0 & .7 & 0 \\ .3 & 0 & 0 & .7 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
\end{array}
(Q)\ (R)\ (O)\ (I)
\longrightarrow
\begin{array}{c} \\ rrr \\ 0 \\ r \\ rr \end{array}
\begin{array}{c}
\begin{array}{cccc} rrr & 0 & r & rr \end{array} \\
\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & .3 & .7 & 0 \\ 0 & .3 & 0 & .7 \\ .7 & .3 & 0 & 0 \end{array}\right]
\end{array}
$$

I, O, R, Q

b) Compute the Fundamental Matrix F, showing how you used your calculator to arrive at your answer. [4]

$$F = (I - Q)^{-1} = \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .3 & .7 & 0 \\ .3 & 0 & .7 \\ .3 & 0 & 0 \end{bmatrix} \right)^{-1} \xrightarrow{\text{calculator}} \boxed{\begin{bmatrix} 2.915 & 2.041 & 1.429 \\ 1.487 & 2.041 & 1.429 \\ 0.875 & 0.612 & 1.429 \end{bmatrix}}$$

$\Downarrow$

(calc input : identity (3) − . . .

c) Explain what entry $F_{3,2}$ means in your fundamental matrix. [3]

this means that starting from a state of rr, what is the expected number of visits to the state r before being absorbed.

It's less than 1 because it's pretty likely to be absorbed immediately (intuitively speaking), in which case it won't pass through r at all.

d) Suppose the game is structured so that it costs $1 for every time you hit the spinner. And for a win in the game (3 reds in a row), the prize is $10. Suppose Dr. Eigendoit decides to start playing the game (at state 0), and he has $100 in his pocket. He spins until he wins the $10, and then stops. How much money do you expect he has now?

because there is a linear relationship between turns & money left we can use F's expected value. We have to assume the Doctor can go in debt, though.

Starting at 0, expected turns until winning = 2.915 + 2.041 + 1.429

= 6.38484 turns

→ 100 − 6.38484 + 10 = $\boxed{\$103.62}$ (rounded to nearest ¢)

10