Matrix & Markov Midterm 2017 Yes, Eigen do this! Hannah Kim
Period B

No Calculator Section:

True False: (2 each) -13 62/75

det (A - Iλ) = 0
(A - Iλ) x = 0

Let M and N be 2 x 2 matrices with non-zero determinants, let k be a non-zero scalar and let I be the 2 x 2 Identity matrix. Answer the following questions true or false.

1. $(M^{-1})^T = (M^T)^{-1}$ T

2. k(M+N) = (N+M)k T

3. If Mx + I = N then x $=(N-I)M^{-1}$ T

4. If M and N are transposes of one another then they will have the same eigenvalues. F

5. Weezie is solving a 3 x 3 system of equations using Gauss Jordan Elimination. In her fourth step she ends up with the following augmented matrix: [4]

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 2 \\ 0 & 0 & 0 & 1 \\ 2 & -3 & 4 & 4 \end{array}\right]$$

What can you conclude about the system that she is trying to solve?

a) There are an infinite number of solutions

b) There is no real solution

c) The origin is a solution to the system

d) 1 is an eigenvalue

e) Either a or b could be true.

6. Given the matrix $\begin{bmatrix} 2 & 7 \\ -1 & -6 \end{bmatrix}$, which of the following is an eigenvector? [4]

a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

c) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

e) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

-8

7. Perform the following matrix multiplication: [6]

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 & 0 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ 0 & -7 & -1 \\ 14 & -7 & 2 \\ 2 & 4 & 1 \end{bmatrix}$$

8. Find the inverse of matrix M below by hand: [6]

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -2 & 1 \\ -1 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 & 1 \\ 1 & 1 & -1 \\ -1 & -2 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$M^{-1} = -1\begin{bmatrix} -2 & 1 & -1 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & -1 & 1 \\ -2 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}}$$

$$\det M = 1(-2) - 0(2) + 1(1)$$
$$= -2 + 1 = -1$$

9. Consider matrix A below where $A^2 = A$? Name two different possible values for the ordered pair (x,y). (many answers possible) [4]

$$A = \begin{bmatrix} 1 & x & y \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & x & y \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1(x) & (y)+2x+y \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$y = 2x + 2y$

$-y = 2x$

$y = -2x$

$x = 2, \; y = -4$

$(2, -4)$, $(1, -2)$

/-0

10. Given the following 3x3 matrix: [8 total]

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix}$$

$\det(A - I\lambda) = 0$

$(A - I\lambda)x = 0$

a) Find all the eigenvalues of the matrix.

$$\det\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & -1 & 4-\lambda \end{bmatrix} = 1-\lambda(2-\lambda)(4-\lambda) - 1(0-0) + 0(0-0)$$

$$= (1-\lambda)(8-6\lambda+\lambda^2) = 8-6\lambda+\lambda^2-8\lambda+6\lambda^2-\lambda^3$$

$$= -\lambda^3 + 7\lambda^2 - 14\lambda + 8$$

$$(\lambda-1)(-\lambda^2+6\lambda-8)$$

$$-(\lambda-1)(\lambda-4)(\lambda-2)$$

$\boxed{\lambda = 1, 4, 2}$

-1

$$\begin{array}{r|rrrr} 1 & -1 & 7 & -14 & 8 \\ & & -1 & 6 & -8 \\ \hline & -1 & 6 & -8 & 0 \end{array}$$

b) Find the family (eigenspace) of eigenvectors for the **smallest** eigenvalue that you found from part (a).

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & -1 & 4-\lambda \end{bmatrix} \xrightarrow{\lambda=1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} y = 0 \\ y = 0 \\ -y + 3z = 0 \end{matrix}$$

$$\boxed{\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x = \text{anything} \\ y = 0 \\ z = 0 \end{matrix}}$$

11. Answer "Always", "Sometimes", or "Never" for each statement about matrix T, which is a non-absorbing, 3x3 transition matrix. [2 each]

In matrix T, each row adds up to 1. Always

In matrix T, each column adds up to 1. sometimes

In matrix T, some of the entries are 0. Sometimes

-1

$\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}$

Calculator Section: Wow, ~~like this test is so absorbing~~!! Hannah Kim

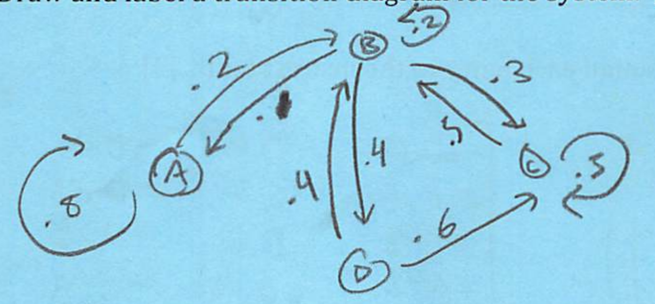
← no it's not

$F = (Q - I)^{-1}$

12. The transition matrix shows the airline routes between 4 different cities (Austin, Brussels, Cleveland, and Delhi), and the probabilities of travelling from one city to the next if you go to the airport and buy a ticket for the next flight out.

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0.5 & .5 & 0 \\ 0 & 0.4 & 0.6 & 0 \end{bmatrix} \end{array}$$

a) Draw and label a transition diagram for the system. [3]



b) If you start in Brussels, what is the probability that you will be in Delhi after taking 2 trips? [3]

M^2 $\boxed{.08}$

c) What is the probability of finding yourself in Cleveland on a random day in the distant future? [2]

$\boxed{.3624}$

13. Luis and Roger collected baseball cards in the 80's. Luis had 2 Fleer sets and 3 Topps sets. Roger had 4 Fleer sets and 5 Topps sets. Currently, the Fleer sets are worth \$15 and the Topps sets are worth \$20. Experience says that in a decade the cards will grow in value by 12%. Write out a matrix multiplication problem whose answer would be the amount of money each person's collection is worth in a decade. You do not need to actually multiply. [5]

$$\begin{array}{c} \\ L \\ R \end{array} \begin{array}{c} \begin{array}{cc} F & T \end{array} \\ \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \end{array} \begin{bmatrix} 15 \\ 20 \end{bmatrix} 1.12 = \begin{bmatrix} \text{L's collection \$} \\ \text{R's collection \$} \end{bmatrix}$$

/-0

14. The following matrix represents an absorbing system, written in canonical form. There are 6 different states: A, B, C, D, E, and F in that order. For these problems show how you arrived at your answers for possible partial credit.

$$\begin{array}{c|cccc|cc} & A & B & C & D & E & F \\ \hline A & 0.2 & 0.1 & 0.1 & 0.3 & 0.1 & 0.2 \\ B & 0.5 & 0.2 & 0.1 & 0.2 & 0 & 0 \\ C & 0.6 & 0.2 & 0 & 0 & 0 & 0.2 \\ D & 0 & 0 & 0.4 & 0.3\ Q & 0.3\ R & 0 \\ \hline E & 0 & 0 & 0 & 0\ 0 & 1\ I & 0 \\ F & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

a) Name the absorbing states E, F [2] (nothing leaves)

b) Find the Fundamental Matrix N. Round each entry to the nearest tenth. [3]

negatives???

$N = (Q-I)^{-1}$ -2 $(I-Q)^{-1}$

$$\left(\begin{bmatrix} .2 & .1 & .1 & .3 \\ .5 & .2 & .1 & .2 \\ .6 & .2 & 0 & 0 \\ 0 & 0 & .4 & .3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} -2.0 & -0.4 & -0.6 & -1.0 \\ -1.6 & -1.7 & -0.8 & -1.2 \\ -1.5 & -0.6 & -1.5 & -0.8 \\ -0.9 & -0.3 & -0.9 & -1.9 \end{bmatrix}$$

c) If you start in state B, how many cycles will it take until being absorbed (on average)? [3] (you may use your rounded values from part a)

$|-1.6 - 1.7 - 0.8 - 1.2|$

$= |-5.3| = 5.3$

can't have negative cycles so $|a|$ is applied

$\boxed{5.3 \text{ cycles}}$

d) If you start in state D, what is the probability that you will end in state F? [3]

$$NR = \begin{bmatrix} -.5 & -.5 \\ -.5 & -.5 \\ -.4 & -.6 \\ -.7 & -.3 \end{bmatrix}$$

$\boxed{-0.3469}$ ecf

BUT probability can't be negative so $|a|$ is applied, leading to

$\boxed{0.3469}$

15. Matrix A is a 2x2 transformation matrix with eigenvalues $\lambda = 2$ (with corresponding eigenvectors $\begin{bmatrix}1\\2\end{bmatrix}t$) and $\lambda = -3$ (with corresponding eigenvectors $\begin{bmatrix}-1\\1\end{bmatrix}t$). Express the following products as a single vector: [2 each]

a) $A\begin{bmatrix}4\\8\end{bmatrix} = A4\begin{bmatrix}1\\2\end{bmatrix}$

$= 2\cdot4\begin{bmatrix}1\\2\end{bmatrix}$

$\boxed{= \begin{bmatrix}8\\16\end{bmatrix}}$

b) $A\begin{bmatrix}0\\3\end{bmatrix} = A\begin{bmatrix}1\\2\end{bmatrix} + A\begin{bmatrix}-1\\1\end{bmatrix}$

$= 2\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}-1\\1\end{bmatrix}$

$= \boxed{\cancel{\begin{bmatrix}-1\\7\end{bmatrix}}}$ -2

-4