Analysis Honors – Deggeller /Gleason/ Tantod
Midterm 2019/20

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Date 2/4/20 Per D

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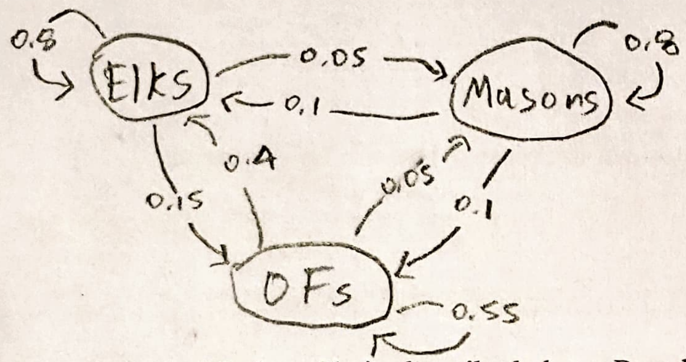
Calculator Section[32 pts]

***** You may use your calculator to do operations, but you must show your work to receive credit. Write down any matrix that you input into your calculator, as well as what the calculator returns to you, before you interpret the matrices to arrive at your answers.***

1. There are 3 different clubs in Walla Walla Washington, the Elks, the Masons, and the Odd Fellows. During the year, the Elks are expected to retain 80% of their members, lose 5% of its members to the Masons, and lose 15% of its members to the Odd Fellows. The Masons are expected to retain 80% of their members and lose 10% to each of the Elks and the Odd Fellows. The Odd Fellows are expected to retain 55% of its members, lose 40% to the Elks and lose 5% to the Masons.

a) Construct a transition **diagram** [3]

b) Construct a transition **matrix** [2]



$$\begin{array}{c|c} & \begin{matrix} E & M & O \end{matrix} \\ \begin{matrix} \text{Elks} \\ \text{Masons} \\ \text{OFs} \end{matrix} & \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.1 & 0.8 & 0.1 \\ 0.4 & 0.05 & 0.55 \end{bmatrix} \end{array}$$

c) Is the Markov Chain described above Regular, Absorbing, or Neither? Give evidence to support your answer. [3]

not absorbing; there are no absorbing states

it is $\boxed{\text{regular}}$ because none of the entries are 0 in the matrix.

d) Write a system of 3 equations and 3 unknowns that could be used to solve for the equilibrium distribution of club members in Walla Walla. YOU DO NOT NEED TO SOLVE THE SYSTEM. [3]

$$[E \ M \ O] \begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.1 & 0.8 & 0.1 \\ 0.4 & 0.05 & 0.55 \end{bmatrix} = $$

$$\boxed{\begin{aligned} 0.8E + 0.1M + 0.4O &= E \\ 0.05E + 0.8M + 0.05O &= M \\ E + M + O &= 1 \end{aligned}}$$

$$= [0.8E + 0.1M + 0.4O \quad 0.05E + 0.8M + 0.05O \quad 0.15E + 0.1M + 0.55O] = [E \ M \ O]$$

e) Use any method to determine the equilibrium distribution of club members in Walla Walla. [3]

take matrix from 1(b) to a large power

$$\begin{bmatrix} 0.8 & 0.05 & 0.15 \\ 0.1 & 0.8 & 0.1 \\ 0.4 & 0.05 & 0.55 \end{bmatrix}^{1000} \Rightarrow \begin{bmatrix} 0.56\overline{6} & 0.2 & 0.2\overline{3} \\ 0.56\overline{6} & 0.2 & 0.2\overline{3} \\ 0.56\overline{6} & 0.2 & 0.2\overline{3} \end{bmatrix}$$

Equilibrium dist : $[0.5\overline{6} \quad 0.2 \quad 0.2\overline{3}]$

f) In many years, there are projected to be 500,000 club members TOTAL in Walla Walla. How many would you predict will be Odd Fellows? Show your calculation. [2]

$500000[0.5\overline{6} \quad 0.2 \quad 0.2\overline{3}] = [283333 \quad 100000 \quad 116667]$

116667 would be at Odd Fellows

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2. Kareem and Bill are playing a strange version of scrabble. They have a bag full of tiles of the letters A, C, and T. There are an equal number of each of those letters. The player reaches in, picks a random tile, records the letter, and then puts the tile back in the bag. The goal is to draw the sequence C-A-T. The transition matrix is given.

$$
\begin{array}{c|cccc}
 & \text{start} & C & CA & CAT \\
\hline
\text{start} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\
C & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
CA & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
CAT & 0 & 0 & 0 & 1
\end{array}
$$

$$
\Rightarrow
\begin{array}{c}
 \\ S \\ C \\ CA \\ CAT
\end{array}
\begin{array}{cccc}
CAT & S & C & CA \\
\hline
0 & \frac{2}{3} & \frac{1}{3} & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
1 & 0 & 0 & 0
\end{array}
\rightarrow
\begin{array}{c}
 \\ CAT \\ S \\ C \\ CA
\end{array}
\begin{array}{cccc}
CAT & S & C & CA \\
\hline
1 & 0 & 0 & 0 \\
0 & \frac{2}{3} & \frac{1}{3} & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{array}
$$

$$
\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{array}{c|ccc} & S & C & CA \\ \hline S & \frac{2}{3} & \frac{1}{3} & 0 \\ C & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ CA & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right)^{-1} = \begin{array}{c|ccc} & S & C & CA \\ \hline S & 15 & 9 & 3 \\ C & 12 & 9 & 3 \\ CA & 9 & 6 & 3 \end{array}
$$

a) What does the "1/3" in the third row and first column mean in context? Use the "1/3" in your sentence. Be very specific. [3] There is a $\frac{1}{3}$ chance that when one of the players already has the sequence 'CA', they end up not drawing a 'T' and end up having to start over.

and drawing a C an A

-1

b) How many tiles will it take on average for a player until the sequence C-A-T is produced? [4]
(use the space above for your work)

$S \Rightarrow [15 \quad 9 \quad 3]$ $\quad 15+9+3 = \boxed{27 \text{ tiles}}$

3. At a certain unnamed school district, the employees are constantly switching between 5 positions (Teacher, Director , Principal, Specialist, and Associate).
Once you become a Director or Associate you never switch (the pay is too good).
If you're a Teacher, you have a 20% chance of moving to Director and a 30% chance of moving to principal)
If you're a principal, you have a 50% chance of moving to Teacher and a 30% chance of moving to Specialist.
If you're a Specialist, you have a 10% chance of moving to Teacher, 40% chance of becoming a director, and 40% chance to go to Associate. ***Assume each transition represents 5 years at that position.***

a) Write the transition matrix for this situation in Canonical Form. [3]

$$
\begin{array}{c|ccccc}
 & D & A & T & P & S \\
\hline
D & 1 & 0 & 0 & 0 & 0 \\
A & 0 & 1 & 0 & 0 & 0 \\
T & 0.2 & 0 & 0.5 & 0.3 & 0 \\
P & 0 & 0 & 0.5 & 0.2 & 0.3 \\
S & 0.4 & 0.4 & 0.1 & 0 & 0.1
\end{array}
$$

b) If you start your career as a Teacher, how many 5 year blocks would you expect to spend as a Principal over the course of your career? (show how) [3]

$$
\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{array}{c|ccc} & T & P & S \\ \hline T & 0.5 & 0.3 & 0 \\ P & 0.5 & 0.2 & 0.3 \\ S & 0.1 & 0 & 0.1 \end{array}\right)^{-1} = \begin{array}{c|ccc} & T & P & S \\ \hline T & 3.3\bar{3} & \boxed{1.25} & 0.41\bar{6} \\ P & 2.2\bar{2} & 2.08\bar{3} & 0.69\bar{4} \\ S & 0.370 & 0.138 & 1.15741 \end{array}
$$

$\boxed{1.25 \text{ 5-year blocks}}$

c) What is the probability that a Specialist will end their career as a Director? [3] (again show how you got your answer)

$$
\begin{bmatrix} 3.33 & 1.25 & 0.416 \\ 2.22 & 2.083 & 0.694 \\ 0.37 & 0.132 & 1.157 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \\ 0.4 & 0.4 \end{bmatrix} = \begin{array}{c|cc} & D & A \\ \hline T & \frac{5}{6} & \frac{1}{6} \\ P & 0.722 & 0.277 \\ S & 0.537037 & 0.4629 \end{array}
$$

$S \to D = 0.537037\ldots = \frac{14.5}{27} = \boxed{\frac{29}{54}}$

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No Calculator Section [35 pts] **Name again please:** Alan Lee

1. Consider two matrices M, N: $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, N=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$

Answer Always, Sometimes or Never: [2 each]

MN = NM sometimes M+N = N+M always 3M+3N = 3(M+N) always

2. Consider the system of equations $M\begin{bmatrix} x \\ y \end{bmatrix}=B$, where M is a 2x2 matrix and B is a 2x1.

Below, circle all pairs of matrices for M, B that would result in a system with no real solution. [3]

a) $M=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ b) $M=\begin{bmatrix} 2 & 5 \\ 3 & 7.5 \end{bmatrix}, B=\begin{bmatrix} 10 \\ 20 \end{bmatrix}$

c) $M=\begin{bmatrix} 2 & 5 \\ 3 & 7.5 \end{bmatrix}, B=\begin{bmatrix} 10 \\ 15 \end{bmatrix}$ d) $M=\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, B=\begin{bmatrix} 10 \\ 15 \end{bmatrix}$

$2x+5y=10$
$3x+7.5y=15$

$2x+5y=10$
$3x+7.5y=20$ X

3. Below are two matrices A and B. Perform both AB and BA or state that they are not possible. [4]

$A=\begin{bmatrix} 1 & 0 & -2 & 3 \end{bmatrix}$

$B=\begin{bmatrix} -2 \\ 3 \\ 0 \\ 5 \end{bmatrix}$

AB: $\begin{bmatrix} 1 & 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 0 \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} 13 \end{bmatrix}}$

BA $\begin{bmatrix} -2 \\ 3 \\ 0 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & 0 & 4 & -6 \\ 3 & 0 & -6 & 9 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & -10 & 15 \end{bmatrix}}$

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4. Prove either the Associative Property for Multiplication, or the Distributive Property, for the matrices x, y, and z below: [5]

$$x=\begin{bmatrix}1&2\\3&4\end{bmatrix}, y=\begin{bmatrix}2&0\\1&3\end{bmatrix} z=\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

$$z(x+y)=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}3&2\\4&7\end{bmatrix}=\begin{bmatrix}3a+4b&2a+7b\\3c+4d&2c+7d\end{bmatrix}=$$

$$=\begin{bmatrix}a+3b&2a+4b\\c+3d&2c+4d\end{bmatrix}+\begin{bmatrix}2a+b&3b\\2c+d&3d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}$$

$$+\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}2&0\\1&3\end{bmatrix}=z\cdot x+z\cdot y \quad \square$$

5. Are $\begin{bmatrix}1&0&0\\-b&1&0\\0&0&1\end{bmatrix}$ and $\begin{bmatrix}1&0&0\\b&1&0\\0&0&1\end{bmatrix}$ inverses of one another? Show work. Write a sentence justifying your answer. [3]

$$\begin{bmatrix}1&0&0\\-b&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\b&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-b+b&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$=$ identity(3).

Since their product is the identity matrix, they are inverses.

6. Solve for x, y, and z. [4]

$$\begin{bmatrix}8&0\\-1&5\end{bmatrix}\cdot\begin{bmatrix}x+1&y\\4&0\end{bmatrix}+3\begin{bmatrix}y&-10\\z&0\end{bmatrix}=\begin{bmatrix}7&10\\-12&-y\end{bmatrix}$$

$$\begin{bmatrix}8x+8&8y\\-x-1+20&-y\end{bmatrix}+\begin{bmatrix}3y&-30\\3z&0\end{bmatrix}=\begin{bmatrix}7&10\\-12&-y\end{bmatrix}$$

$$\begin{bmatrix}8&0\\-1&5\end{bmatrix}\begin{bmatrix}-1&5\\4&0\end{bmatrix}$$

$$+\begin{bmatrix}15&-30\\-33&0\end{bmatrix}=\begin{bmatrix}7&10\\-12&-5\end{bmatrix}$$

$8y-30=10$

$y=5$

$8x+8+3y=7$

$8x+23=7$

$8x=-16$

$x=-2$

$-x-1+20+3z=-12$

$2-1+20+3z=-12$

$3z=-33$

$z=-11$

$\boxed{(-2,5,-11)}$

7. Find

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x, y.

7. Find the inverse of the 3 by 3 inverse using any method. [5]

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & 1 & 1 \end{bmatrix} \qquad \det(A) = 2$$

$$\begin{bmatrix} -0.5 & 0.5 & -0.5 \\ -3 & 2 & 1 \\ 2.5 & -1.5 & -0.5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\mathrm{adj}(A) = \begin{bmatrix} -1 & 6 & 5 \\ -1 & 4 & 3 \\ -1 & -2 & -1 \end{bmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \rightarrow \begin{bmatrix} -1 & -6 & 5 \\ 1 & 4 & -3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\text{transpose} \Rightarrow \begin{bmatrix} -1 & 1 & -1 \\ -6 & 4 & 2 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\frac{1}{\det} \cdot \text{transpose} = \boxed{\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -3 & 2 & 1 \\ \frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}}$$

8. Solve the following system of equations by **clearly demonstrating the use of inverse matrices.** [5]

$ax + by = c$

$dx - ey = f$

$$\begin{bmatrix} a & b \\ d & -e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

inverse

$$= -\frac{1}{ae+bd}\begin{bmatrix} -e & -b \\ -d & a \end{bmatrix}$$

$$-\frac{1}{ae+bd}\begin{bmatrix} -e & -b \\ -d & a \end{bmatrix}\begin{bmatrix} a & b \\ d & -e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{ae+bd}\begin{bmatrix} -e & -b \\ -d & a \end{bmatrix}\begin{bmatrix} c \\ f \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{ae+bd}\begin{bmatrix} -ce-bf \\ -cd+af \end{bmatrix} = \frac{1}{ae+bd}\begin{bmatrix} ce+bf \\ cd-af \end{bmatrix}$$

$x = \dfrac{ce+bf}{ae+bd}$

$y = \dfrac{cd-af}{ae+bd}$

Check

$$\frac{ace+\cancel{baf}}{ae+bd} + \frac{bcd-\cancel{baf}}{ae+bd} = c\left(\frac{ae+bd}{ae+bd}\right) = c$$

$$\frac{\cancel{dce}+dbf}{ae+bd} + \frac{afe-\cancel{ced}}{ae+bd} = f\left(\frac{ae+bd}{ae+bd}\right) = f \quad ✓$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix} =$$