Analysis H – Hahn / Tantod
Midterm Exam: Matrices and Markov Chains
No Calculators!

Here we go eigen: Chris Lee

Period: 3

74

80 points

↳ my fav one so far :)

For problems #1-3, refer to matrices $A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 5 \\ -3 & 2 & 1 \end{bmatrix}$, $B = [-2 \quad 3 \quad 6]$, and $C = \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$,

1. Find each. If not possible, write "nice cap, boomer": [3 points each]

a) BA

$$\underset{1\times 3}{[-2 \quad 3 \quad 6]} \underset{3\times 3}{\begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 5 \\ -3 & 2 & 1 \end{bmatrix}} = [-19 \quad 24 \quad 15]$$

b) BC

$$\underset{1\times 3}{[-2 \quad 3 \quad 6]} \underset{3\times 1}{\begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}} = [38]$$

c) CA

3×1 3×3

nice cap, boomer...

d) $B^T B$

$$\begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} [-2 \quad 3 \quad 6] = \cancel{[49]}$$

-2

e) detA

$$\det A = 2 \begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 4 \\ -3 & 2 \end{vmatrix}$$

$$= 2 \cdot -6 + 3 \cdot 14$$

$$= -12 + 42 \qquad \boxed{\det A = 30}$$

f) detB

nice cap boomer...

2. $A - 4D = I$. Find matrix D. [4 points]

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 5 \\ -3 & 2 & 1 \end{bmatrix} - 4D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4D = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 3 & 5 \\ -3 & 2 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{5}{4} \\ \frac{-3}{4} & \frac{1}{2} & 0 \end{bmatrix}$$

-2

3. CE is a 3x8 matrix. What are the dimensions of matrix E? 1 × 8 [2 points]

$$3\times 1 \quad 1\times 8 = 3\times 8$$

4. Neo is trying to solve a 3x3 system using Gauss-Jordan Elimination. His partial work is shown below.

$$\begin{bmatrix}0&3&1&7\\1&-2&1&6\\8&1&4&5\end{bmatrix}\rightarrow\begin{bmatrix}1&-2&1&6\\0&3&1&7\\8&1&4&5\end{bmatrix}\xrightarrow{-8\mathrm{I}}\begin{bmatrix}1&-2&1&6\\0&0&5&32\\0&-15&-4&-43\end{bmatrix}\rightarrow\begin{bmatrix}1&-2&1&6\\0&-15&-4&-43\\0&0&5&32\end{bmatrix}$$

0 3 1 7   0 0 5 32   5II − III   0 15 5 35

8 1 4 5   0 −15 −4 −43   −15 −4 −43

One of his numbers is wrong. Find and circle his mistake, then write the corrected number here: $\boxed{3}$. [3]

0 6 2 14

3 −6 3 18

3 0 5 32

5. A and B are 4x4 matrices (but are not the same matrix). Neither A nor B are identity matrices, and both A and B have inverses. Circle "TRUE" or "FALSE" for each statement. [2 each]

a) $A + B = B + A$ TRUE or FALSE

b) $AB = BA$ TRUE or FALSE

c) $\det A = 0$ TRUE or FALSE

d) $B^TA^T = (AB)^T$ TRUE or ~~FALSE~~

e) $BI = B^{-1}$ TRUE or FALSE

f) $AB + A = A(B + I)$ TRUE or FALSE

g) $ABA^{-1} = B$ TRUE or FALSE

h) $AB^{-1}BA^{-1} = I$ TRUE or ~~FALSE~~

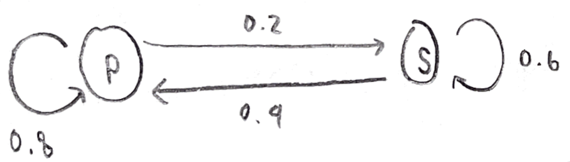
$$\begin{bmatrix}2&4\\3&1\end{bmatrix}\begin{bmatrix}5&4\\2&3\end{bmatrix}=\begin{bmatrix}18&20\\17&15\end{bmatrix}\rightarrow\begin{bmatrix}18\\20\end{bmatrix}$$

−4

$$\begin{bmatrix}2&3\\4&1\end{bmatrix}\begin{bmatrix}5&2\\4&3\end{bmatrix}=\begin{bmatrix}22&13\\24&11\end{bmatrix}$$

6. Each year, 20% of the population of Palo Alto moves to Sunnyvale, and 40% of the population of Sunnyvale moves to Palo Alto. People don't move anywhere else, because there are no other good places to live in the United States.

a) Draw a transition diagram to represent the situation. [3]



b) Translate (a) into a transition matrix. Label the rows and columns with city names. [3]

$$\begin{array}{c|cc} & \text{PA} & \text{Sunny.} \\ \text{Palo Alto} & 0.8 & 0.2 \\ \text{Sunnyvale} & 0.4 & 0.6 \end{array}$$

explanation for c: 1000 years means raising the tran. matrix to high power which will give us eq vector. So, we can find the eq vector algebraically then mult by 2M ppl to get population

$\frac{2}{5}\cdot\frac{1}{3}=\frac{2}{15}$

c) If the combined population of the two cities is 2 million people, what will the population of each city be in 1000 years? Show work to explain your thinking and justify your answer. [4]

$$2{,}000{,}000\begin{bmatrix}\frac{10}{15}&\frac{5}{15}\end{bmatrix}$$ (PA, SV) $\rightarrow$ pop Palo Alto: $2M\cdot\frac{10}{15}$ ppl ✓

pop Sunnyvale: $2M\cdot\frac{5}{15}$ ppl

START HERE: $\rightarrow \begin{bmatrix}a&b\end{bmatrix}\begin{bmatrix}0.8&0.2\\0.4&0.6\end{bmatrix}=\begin{bmatrix}a&b\end{bmatrix}$

$0.8a + 0.4b = a$ ✓
$0.2a + 0.6b = b$

$-0.2a + 0.4b = 0$
$0.2a - 0.4b = 0$

$$\left[\begin{array}{cc|c}0.2&-0.4&0\\1&1&1\end{array}\right]\xrightarrow{-5\mathrm{I}}\left[\begin{array}{cc|c}0.2&-0.4&0\\0&3&1\end{array}\right]$$

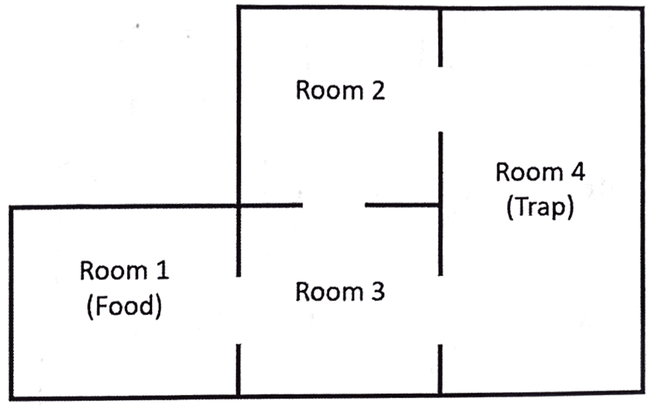
$$\left[\begin{array}{cc|c}0.2&-0.4&0\\0&1&\frac{1}{3}\end{array}\right]\xrightarrow{+0.4\mathrm{II}}\left[\begin{array}{cc|c}0.2&0&\frac{2}{15}\\0&1&\frac{5}{15}\end{array}\right]$$

$$\left[\begin{array}{cc|c}1&0&\frac{10}{15}\\0&1&\frac{5}{15}\end{array}\right]$$

$\boxed{a=\frac{10}{15} \quad b=\frac{5}{15}}$

−4

7. A mouse is placed in the maze shown on the right, and it moves from room to room randomly. From any room, the mouse will choose a door to the next room with equal probabilities. If the mouse reaches room 1, it finds food and never leaves the room. Yay mouse! If the mouse reaches room 4, it is trapped and cannot leave that room. Sad mouse.



a) Write the transition matrix for this situation in canonical form. [4]

$$\begin{array}{cc} & \begin{array}{cccc} 2 & 3 & 1 & 4 \end{array} \\ \begin{array}{c} Q \\ \\ O \end{array} \begin{array}{c} 2 \\ 3 \\ 1 \\ 4 \end{array} & \left[\begin{array}{cc|cc} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R \\ \\ I \end{array} \end{array}$$

b) Find the fundamental matrix, and clearly label its rows and columns. [4]

$N = (I - Q)^{-1}$

$I - Q = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{3} & 1 \end{bmatrix}$ $\quad 1 - \frac{1}{6}$

$(I-Q)^{-1} = \frac{6}{5} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & 1 \end{bmatrix} =$

$$\boxed{N = \begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} \frac{6}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{6}{5} \end{bmatrix} \end{array}}$$

c) If the mouse is initially placed in room 3, what is the expected number of visits the mouse will make to room 2? [2]

$\frac{2}{5}$ visits

d) If the mouse is initially placed in room 3, what is the expected number of moves the mouse will make before it ends up in room 1 or 4? [2]

$\frac{8}{5}$ moves

e) If the mouse is initially placed in room 3, what is the probability that the mouse will find the food? [4]

$$NR = \begin{bmatrix} \frac{6}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{6}{5} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{array}{cc} & \begin{array}{cc} 1 & 4 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} \frac{3}{15} & \frac{12}{15} \\ \frac{6}{15} & \frac{9}{15} \end{bmatrix} \end{array}$$

$\frac{9}{15} + \frac{3}{15}$

$\frac{3}{15} + \frac{6}{15}$

$\boxed{\frac{6}{15} \text{ probability mouse will find the food}}$

8. Given matrix $A = \begin{bmatrix} \frac{1}{4} & 0 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -3 \end{bmatrix}$, find A^{-1} using any method. Show all your work. [4]

$$\begin{bmatrix} \frac{1}{4} & 0 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 0 & 0 \\ 0 & -\frac{3}{4} & 0 \\ -3 & \frac{1}{8} & \frac{1}{4} \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 0 & 0 \\ 0 & -\frac{3}{4} & 0 \\ -3 & -\frac{1}{8} & \frac{1}{4} \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 0 & -3 \\ 0 & -\frac{3}{4} & -\frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \rightarrow$$

$$\det A = -\frac{3}{4} \longrightarrow -\frac{4}{3}\begin{bmatrix} -3 & 0 & -3 \\ 0 & -\frac{3}{4} & -\frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & 0 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 4 & 0 & 4 \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ✓$$

$$\boxed{A^{-1} = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}}$$

9. Given matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix}$, find the eigenvalues for A (you don't need to find the eigenvectors). [4]

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} (1-\lambda) & 1 & 0 \\ 0 & (2-\lambda) & 0 \\ 0 & -1 & (4-\lambda) \end{bmatrix}$$

$$(1-\lambda)(8 - 6\lambda + \lambda^2)$$

$$8 - 6\lambda + \lambda^2 - 8\lambda + 6\lambda^2 - \lambda^3$$

$$-\lambda^3 + 7\lambda^2 - 14\lambda + 8$$

$$\begin{array}{r|rrrr} 2 & -1 & 7 & -14 & 8 \\ & & -2 & 10 & -8 \\ \hline & -1 & 5 & -4 & 0 \end{array}$$

$$(x-2)(x^2 - 5x + 4) = 0$$

$$(x-2)(x-4)(x-1)$$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= 4 \end{aligned}}$$

10. Given matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{bmatrix}$ has an eigenvalue $\lambda = 2$. Find the corresponding eigenvector for A. [3]

$$(A - 2I)x = 0$$

$$\begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-x + 2y + z = 0$$

$$2x - 4y - 2z = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} t}$$

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