Analysis H – Hahn / Hlasek / Tantod
Unit 8: Limits, Quiz 1 on Sequences
No Calculators

25
26 points

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Period: 6

1. Answer True or False for each statement. [1 pt each]

a) If a sequence is always decreasing and bounded below, it must converge. True ✓

b) If a sequence has an upper and a lower bound, then it the sequence must have a limit. false ✓

c) If a sequence is always increasing and does NOT have an upper bound, it must be divergent. True ✓

d) If a sequence $\{a_n\}$ is everywhere decreasing, then $a_n \geq a_{n+1}$ for all values of n. True ✓

e) If $a_n \leq b_n \leq c_n$ for all n, and a_n converges to -1 and c_n converge to 1, then b_n converges to 0. False ✓

f) If $a_n > -n$ for all n, this proves that $\{a_n\}$ converges. False ✓

2. a) Is the sequence $\left\{\frac{2^n}{n!}\right\}$ always decreasing? If so, prove it algebraically. If not determine (and justify) the interval over which it is decreasing. [3 pts]

-0.5

$\frac{2^n}{n!} > \frac{2^{n+1}}{(n+1)!}$ $\quad n > 2$

$2^n \cdot n \cdot n! > 2 \cdot 2^n \cdot n!$

sequence $\left\{\frac{2^n}{n!}\right\}$ is decreasing after $n=2$

Interval : $(2, \infty)$

b) Do one additional thing to prove that the sequence above converges. Include a conclusion statement. [3 pts]

-0

$\frac{2^n}{n!} > 0$

$2^n > 0$

True for all $n \in \mathbb{N}$

Sequence $\left\{\frac{2^n}{n!}\right\}$ is everywhere decreasing after ~~$n > 2$~~, and it is bounded below, therefore it converges. ✓

3. Find the limit of each sequence, or say "diverges" if the sequence diverges. No formal proof is required. [2 pts each]

a) $a_n = 2n - \frac{7}{n}$

diverges ✓

b) $b_n = \frac{b_{n-1}}{2} - 1;\ \ b_1 = 3$

$b_n = \frac{b_n}{2} - 1$

$\frac{b_n}{2} = -1$

$b_n = -2$

$\boxed{\text{limit } -2}$ ✓

c) $c_n = \frac{n}{3^n}$

limit : 0 ✓

3^n dominates n

-0.5

4. a) For the sequence $\left\{\frac{2n+1}{5n+4}\right\}$, state the value to which the sequence converges, or state that the sequence diverges. CLEARLY justify your answer with a neighborhood proof for general ε. If your work is correct but is difficult to interpret, you may not receive full credit. [4 pts]

-0.5

the **value** to which the sequence converges (or say "diverges"): $\frac{2}{5}$

Show your work here:

$\varepsilon > 0$

$\frac{2}{5} - \varepsilon < \frac{2n+1}{5n+4} < \frac{2}{5} + \varepsilon$

$\frac{2-5\varepsilon}{5} < \frac{2n+1}{5n+4} \qquad \frac{2n+1}{5n+4} < \frac{2+5\varepsilon}{5}$

$\cancel{10n} + 8 - 25\varepsilon n - 20\varepsilon < \cancel{10n} + 5 \qquad 10n + 5 < 10n + 25\varepsilon n + 8 + 20\varepsilon$

$-25\varepsilon n - 20\varepsilon < -3 \qquad -3 < 25\varepsilon n + 20\varepsilon$

$-25n - 20 < -\frac{3}{\varepsilon} \qquad -\frac{3}{\varepsilon} < 25n + 20 \quad \varepsilon > 0$

$25n < \frac{3 + 20\varepsilon}{\varepsilon} \qquad -\frac{3}{\varepsilon} - 20 < 25n$

$\boxed{n > \frac{-3+20\varepsilon}{25\varepsilon}} \qquad \frac{-3-20\varepsilon}{25\varepsilon} < n$ always True for $\varepsilon > 0$ and $n \in \mathbb{N}$

$M = \left\lceil \frac{3+20\varepsilon}{25\varepsilon} \right\rceil + 1$

For all $n \geq M$, the sequence $\left\{\frac{2n+1}{5n+4}\right\}$ lies within the neighborhood therefore the sequence converges to $\frac{2}{5}$.

b) For the sequence $\left\{\frac{\cos(4n)}{5^n}\right\}$, state the value to which the sequence converges, or state that the sequence diverges. CLEARLY justify your answer with a proof. For your proof, you MUST use one of the following tests: the squeeze theorem OR the big theorem (bounded above/below and always increasing/decreasing theorem) OR the comparison principle. If your work is correct but is difficult to interpret, you may not receive full credit. [4 pts]

Test/Principle used: Squeeze Theorem

the **value** to which the sequence converges (or say "diverges"): 0

Show your work here:

$-\frac{1}{5^n} < \frac{\cos(4n)}{5^n} < \frac{1}{5^n}$

$-\frac{5^n}{5^n} < \cos(4n) < \frac{5^n}{5^n}$

$-1 < \cos(4n) < 1$

range of cosine is $[-1, 1]$

BUT that is only when θ is 0 or $k\pi$ $(k \in \mathbb{Z})$

$n \in \mathbb{N}$ which means $4n \in \mathbb{N}$ which means

$\cos(4n)$ will never reach its maximum or minimum

$\therefore$ always true

Since $-\frac{1}{5^n}$ and $\frac{1}{5^n}$ both converge to 0 and $-\frac{1}{5^n} < \frac{\cos(4n)}{5^n} < \frac{1}{5^n}$ as proved to the left, the sequence $\left\{\frac{\cos(4n)}{5^n}\right\}$ must converge to 0.

QED

-0.5