МИНИСТЕРСТВО НАУКИ И ВЫСШЕГО ОБРАЗОВАНИЯ РФ

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Кафедра: 806 «Вычислительная математика и программирование»

Лабораторная работа № 5
по курсу «Численные
методы»

Группа: М8О-407Б-21

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Оценка:

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In [85]:
```
import numpy as np
from main import *
import matplotlib.pyplot as plt
from matplotlib import cm
from mpl_toolkits.mplot3d import Axes3D
```

Вариант 7:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 0.5 \cdot exp(-0.5t) \cdot cos(x), \quad (1)$$
$$U_x(0,t) = exp(-0.5t), \quad (2)$$
$$U_x(\frac{\pi}{2},t) = -exp(-0.5t), \quad (3)$$
$$U(x,0) = sin(x) \quad (4)$$

Аналитическое решение:

$$U(x,t) = exp(-0.5t) \cdot sin(x) \quad (5)$$

Для решения задачи используются 3 вида конечно-разностных схем: 1. явная схема

$$\frac{u_i^{k+1} - u_i^k}{\tau} = a\frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{h^2} + b\frac{u_{i+1}^k - u_{i-1}^k}{2h} + cu_i^k + f(x_i, t^{k+1})$$

2. неявная схема

$$\frac{u_i^{k+1} - u_i^k}{\tau} = a\frac{u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}}{h^2} + b\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h} + cu_i^{k+1} + f(x_i, t^k)$$

3. схема Кранка-Николсона ($\theta = 0.5$)

$$\frac{u_i^{k+1} - u_i^k}{\tau} = \theta\left(a\frac{u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}}{h^2} + b\frac{u_{i+1}^{k+1} - u_{i-1}^{k+1}}{2h} + cu_i^{k+1} + f(x_i, t^k)\right)$$
$$+ (1-\theta)\left(a\frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{h^2} + b\frac{u_{i+1}^k - u_{i-1}^k}{2h} + cu_i^k + f(x_i, t^{k+1})\right)$$

Для решения задачи используются 3 вида аппроксимации граничных условий: 1. двухточечная аппроксимация с первым порядком

$$\left.\frac{du}{dx}\right|_{j=0}^{k+1} = \frac{u_1^{k+1} - u_0^{k+1}}{h}$$

$$\left.\frac{du}{dx}\right|_{j=N}^{k+1} = \frac{u_N^{k+1} - u_{N-1}^{k+1}}{h}$$

2. трехточечная аппроксимация со вторым порядком

$$\left.\frac{du}{dx}\right|_{j=0}^{k+1} = \frac{-3u_0^{k+1} + 4u_1^{k+1} - u_2^{k+1}}{2h}$$

$$\left.\frac{du}{dx}\right|_{j=N}^{k+1} = \frac{u_{N-2}^{k+1} - 4u_{N-1}^{k+1} + 3u_N^{k+1}}{2h}$$

3. двухточечная аппроксимация со вторым порядком

Разложим в граничных узлах на точном решении значение u_1^{k+1} и u_{N-1}^{k+1} в окрестности точки $x = 0$ в ряд Тейлора по переменной x до третьей производной включительно

$$u_1^{k+1} = u(0+h, t^{k+1}) = u_0^{k+1} + \left.\frac{du}{dx}\right|_0^{k+1} \cdot h + \left.\frac{d^2u}{dx^2}\right|_0^{k+1} \cdot \frac{h^2}{2}$$

$$u_{N-1}^{k+1} = u(l-h, t^{k+1}) = u_N^{k+1} - \left.\frac{du}{dx}\right|_N^{k+1} \cdot h + \left.\frac{d^2u}{dx^2}\right|_N^{k+1} \cdot \frac{h^2}{2}$$

Input equation type (example: explicit / implicit)

In [86]:
```
equation_type = str(input())
```

In [87]:
```
N = 15
K = 400
T = 1
curr_time = 0

params = {
    'l': np.pi,
    'psi':  lambda x: np.sin(x),
    'f':    lambda x, t: 0.5 * np.exp(-0.5 * t) * np.cos(x),
    'phi0': lambda t: np.exp(-0.5 * t),
    'phi1': lambda t: -np.exp(-0.5 * t),
    'solution': lambda x, t: np.exp(-0.5 * t) * np.sin(x),
    'bound_type': 'a1p1',
}
```

In [88]:
```
params['bound_type'] = 'a1p1'
```

In [89]:
```
def implicit_solver(self, N, K, T):
    lst = get_zeros(N, K)
    a = lst[0]
    b = lst[1]
    c = lst[2]
    d = lst[3]
    u = lst[4]

    for i in range(1, N - 1):
        u[0][i] = self.data.psi(i * self.h)
    u[0][-1] = 0

    for k in range(1, K):
```

```
        self.calculate(a, b, c, d, u, k, N, T, K)
        u[k] = tma(a, b, c, d)

    return u

def explicit_solver(self, N, K, T):
    u = np.zeros((K, N))
    t = np.arange(0, T, T / K)
    x = np.arange(0, np.pi / 2, np.pi / 2 / N)
    for j in range(1, N - 1):
        u[0][j] = self.data.psi(j * self.h)

    for k in range(1, K):
        for j in range(1, N - 1):


            u[k][j] = (u[k - 1][j + 1] * (self.a * self.tau / self.h ** 2
                        + u[k - 1][j] * (-2 * self.a * self.tau / self.h
                        + u[k - 1][j - 1] * (self.a * self.tau / self.h *
                        + self.tau * self.data.f(x[j], t[k]))

        if self.data.bound_type == 'a1p1':
            u[k][0] = (self.data.phi0(t[k]) - self.alpha / self.h * u[k][
            u[k][-1] = (self.data.phi1(t[k]) + self.gamma / self.h * u[k]
        elif self.data.bound_type == 'a1p2':
            u[k][0] = (((2.0 * self.alpha * self.a / self.h / (2.0 * self
                        (self.alpha * self.h / self.tau / (2.0 * self.a -
                        (self.alpha * self.h / (2.0 * self.a - self.h * s
                        self.data.phi0(t[k]) /
                        ((2.0 * self.alpha * self.a / self.h / (2.0 * sel
                                self.alpha * self.h / self.tau / (2.0 * s
                            (self.alpha * self.h / (2.0 * self.a - self.h
            u[k][-1] = (((2.0 * self.gamma * self.a / self.h / (2.0 * sel
                            (self.gamma * self.h / self.tau / (2.0 * self
                            (self.gamma * self.h * self.c / (2.0 * self.a
                        self.data.l, t[k]) + self.data.phi1(t[k])) / (
                                (2.0 * self.gamma * self.a / self.h / (2.
                                self.gamma * self.h / self.tau / (2.0 * s
                                        self.gamma * self.h * self.c / (
                                        2.0 * self.a + self.h * self.b))
        elif self.data.bound_type == 'a1p3':
            u[k][-1] = (self.data.phi1(k * self.tau) + u[k][-2] / self.h
                        (1 / self.h + 2 * self.tau / self.h)
    return u
```

In [90]:
```
def calculate(self, a, b, c, d, u, k, N, T, K):
        t = np.arange(0, T, T / K)
        for j in range(1, N):
            a[j] = self.sigma
            b[j] = -(1 + 2 * self.sigma)
            c[j] = self.sigma
            d[j] = -u[k - 1][j]

        if self.data.bound_type == 'a1p1':
            a[0] = 0
            b[0] = -(self.alpha / self.h) + self.beta
            c[0] = self.alpha / self.h
```

```
            d[0] = self.data.phi0(t[k])
            a[-1] = self.gamma / self.h
            b[-1] = self.gamma / self.h + self.delta
            c[-1] = 0
            d[-1] = self.data.phi1(t[k])
        elif self.data.bound_type == 'a1p2':
            a[0] = 0
            b[0] = -(1 + 2 * self.sigma)
            c[0] = self.sigma
            d[0] = -(u[k - 1][0] + self.sigma * self.data.phi0(k * self.t
                   self.tau * self.data.f(0, k * self.tau)
            a[-1] = self.sigma
            b[-1] = -(1 + 2 * self.sigma)
            c[-1] = 0
            d[-1] = -(u[k - 1][-1] + self.sigma * self.data.phi1(k * self
                    self.tau * self.data.f((N - 1) * self.h, k * self.tau
        elif self.data.bound_type == 'a1p3':
            a[0] = 0
            b[0] = -(1 + 2 * self.sigma)
            c[0] = self.sigma
            d[0] = -((1 - self.sigma) * u[k - 1][1] + self.sigma / 2 * u[
                   * self.data.f(0, k * self.tau) - self.sigma * self.dat
                k * self.tau)
            a[-1] = self.sigma
            b[-1] = -(1 + 2 * self.sigma)
            c[-1] = 0
            d[-1] = self.data.phi1(k * self.tau) + self.data.f((N - 1) *
                    * self.h / (2 * self.tau) * u[k - 1][-1]


def crank_nicolson_solver(self, N, K, T):
    theta = 0.5
    lst = get_zeros(N, K)
    a = lst[0]
    b = lst[1]
    c = lst[2]
    d = lst[3]
    u = lst[4]
    for i in range(1, N - 1):
        u[0][i] = self.data.psi(i * self.h)

    for k in range(1, K):
        self.calculate(a, b, c, d, u, k, N, T, K)

        tmp_imp = tma(a, b, c, d)

        tmp_exp = np.zeros(N)
        tmp_exp[0] = self.data.phi0(self.tau)
        for j in range(1, N - 1):
            tmp_exp[j] = self.sigma * u[k - 1][j + 1] + (1 - 2 * self.sig
                            self.sigma * u[k - 1][j - 1] + self.tau * sel
        tmp_exp[-1] = self.data.phi1(self.tau)

        for j in range(N):
            u[k][j] = theta * tmp_imp[j] + (1 - theta) * tmp_exp[j]
    return u
solver = ParabolicSolver(params, equation_type)
```

In [91]:
```
dict_ans = {
        'numerical': solver.solve(N, K, T).tolist(),
        'analytic': solver.analyticSolve(N, K, T).tolist()
    }
```

In [92]:
```
print("Sigma: ",solver.sigma)
```

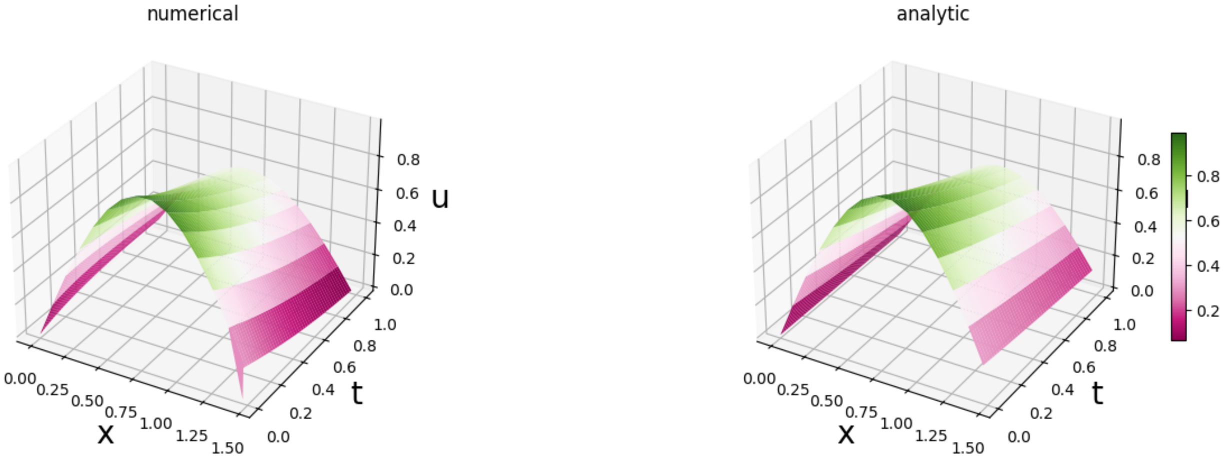
```
Sigma:  0.05699316579881500б
```

In [93]:
```
def draw(dict_, N, K, T, save_file="plot.png"):
    fig = plt.figure(figsize=plt.figaspect(0.3))
    # Make data
    x = np.arange(0, np.pi / 2, np.pi / 2 / N)
    t = np.arange(0, T, T / K)
    x, t = np.meshgrid(x, t)
    z1 = np.array(dict_['numerical'])
    z2 = np.array(dict_['analytic'])

    # Plot the surface.
    ax = fig.add_subplot(1, 2, 1, projection='3d')
    plt.title('numerical')
    ax.set_xlabel('x', fontsize=20)
    ax.set_ylabel('t', fontsize=20)
    ax.set_zlabel('u', fontsize=20)
    ax.plot_surface(x, t, z1, cmap=cm.PiYG,
                    linewidth=0, antialiased=True)

    ax = fig.add_subplot(1, 2, 2, projection='3d')
    ax.set_xlabel('x', fontsize=20)
    ax.set_ylabel('t', fontsize=20)
    ax.set_zlabel('u', fontsize=20)
    plt.title('analytic')
    surf = ax.plot_surface(x, t, z2, cmap=cm.PiYG,
                           linewidth=0, antialiased=True)
    # # Customize the z axis
    # ax.set_zlim(-1.01, 1.01)

    # # Add a color bar which maps values to colors.
    fig.colorbar(surf, shrink=0.5, aspect=15)

    plt.savefig(save_file)
    plt.show()
```

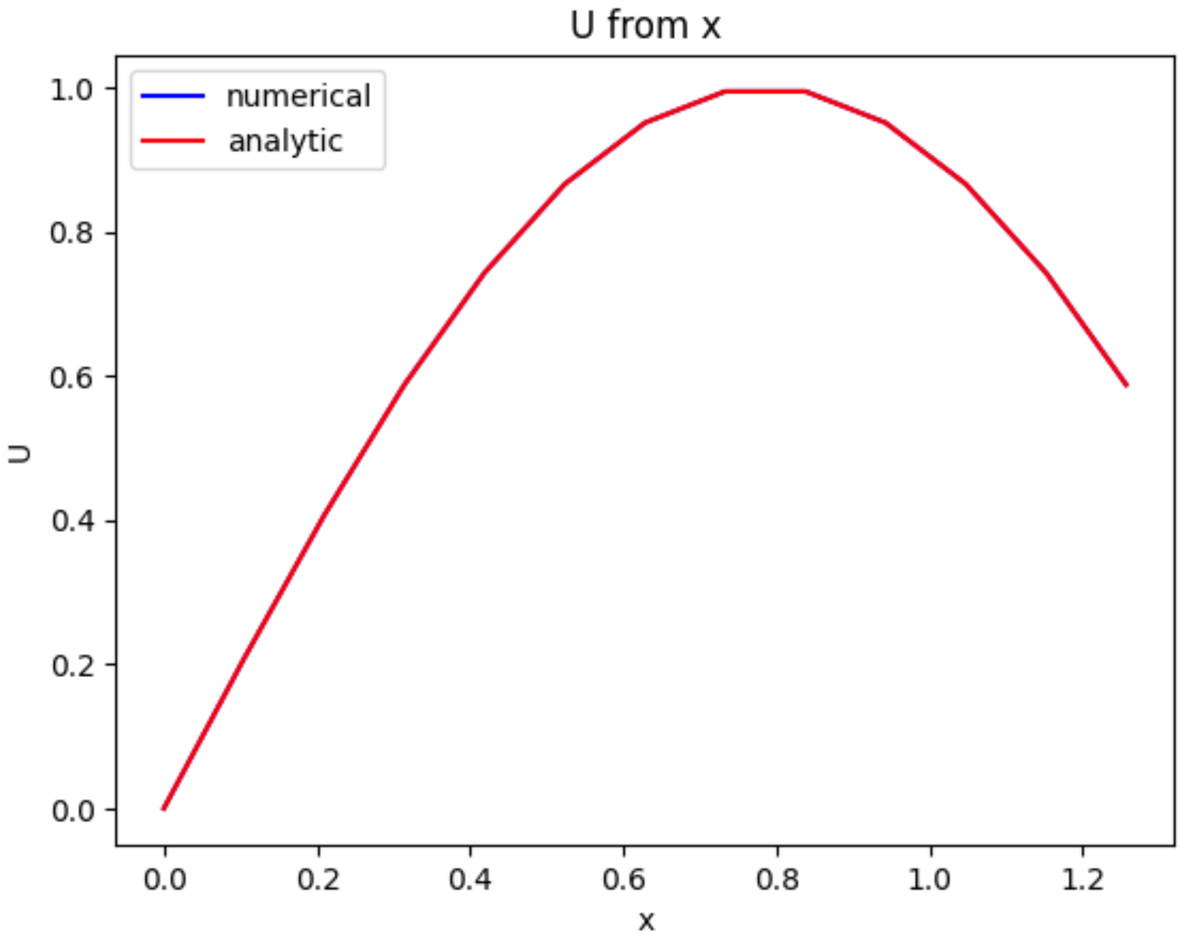
In [94]:
```
def draw_u_x(dict_, N, K, T, time=0, save_file="plot_u_x.png"):
    fig = plt.figure()
    x = np.arange(0, np.pi / 2, np.pi / 2 / N)
    t = np.arange(0, T, T / K)
    z1 = np.array(dict_['numerical'])
    z2 = np.array(dict_['analytic'])

#     print(z1)

    plt.title('U from x')
    plt.plot(x[0:-2], z1[time][0:-2], color='b', label='numerical')
    plt.plot(x[0:-2], z2[time][0:-2], color='r', label='analytic')
    plt.legend(loc='best')
```

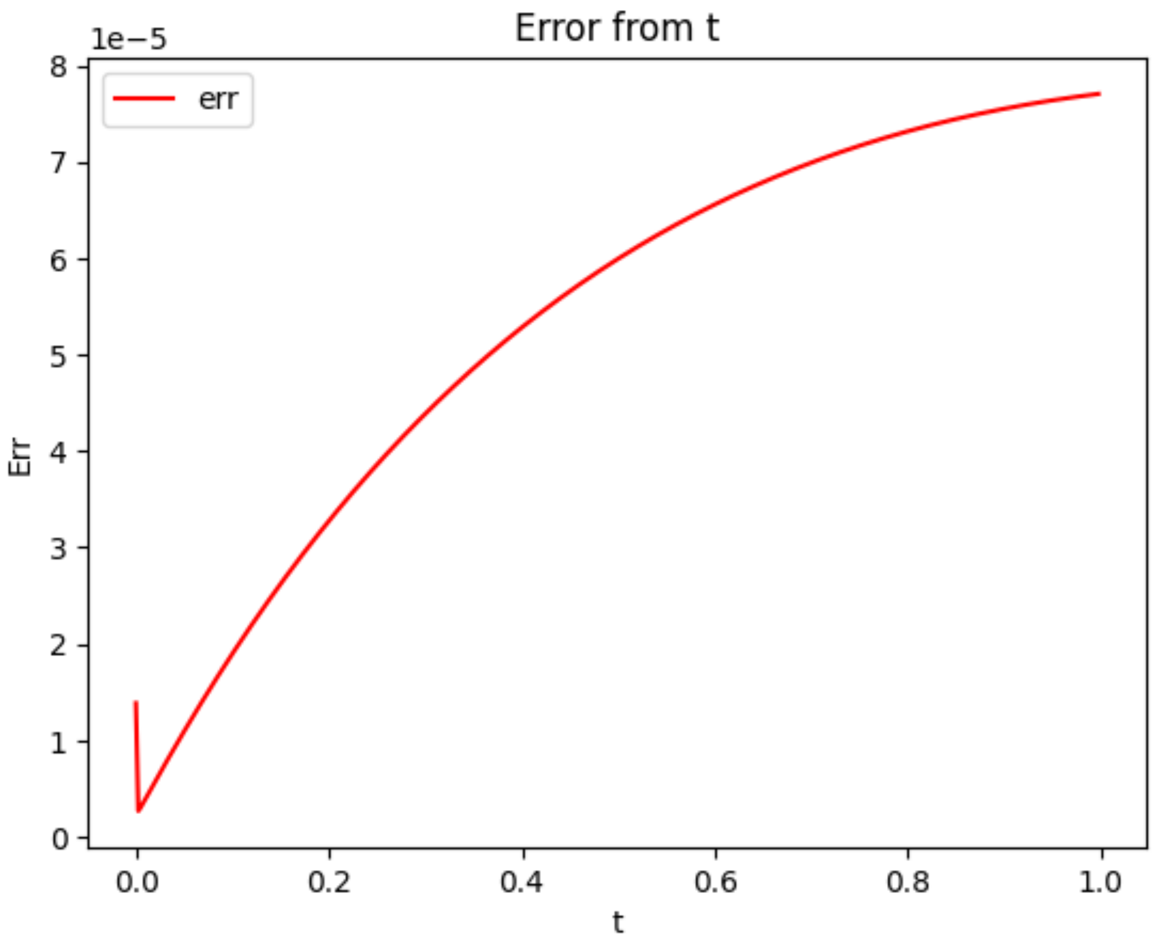
```
    plt.ylabel('U')
    plt.xlabel('x')
    plt.savefig(save_file)
    plt.show()

    err = []
    error = compare_error(dict_ans)
    for i in range(len(error)):
        tmp = 0
        for j in error[i]:
            tmp += j
        err.append(tmp/len(error[i])/1000)
    plt.title('Error from t')
    plt.plot(t, err, color='r', label='err')
    plt.legend(loc='best')
    plt.ylabel('Err')
    plt.xlabel('t')
    plt.savefig('err.png')
    plt.show()
```

In [95]:
```
draw(dict_ans, N, K, T)
```



In [96]:
```
draw_u_x(dict_ans, N, K, T, curr_time)
```





In [97]:
```
error = compare_error(dict_ans)
avg_err = 0.0
```

```python
for i in error:
    for j in i:
        avg_err += j
    avg_err /= N
```

First elements in error array:

In [98]:
```python
print(error[0])
```

```
[0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2
0791169081775973]
```

Middle elements in error array:

In [99]:
```python
print(error[int(K/2)])
```

```
[0.08095185139751016, 0.08214171835941295, 0.07259736480814499, 0.05578028
877416352, 0.034827738907620365, 0.012410764974746247, 0.00935904926914987
7, 0.028978246775185013, 0.04556153075816716, 0.05881917224222655, 0.06900
656894318624, 0.0768491053366892, 0.08344422456718442, 0.0901423240608418
3, 0.09840895012861858]
```

Last elements in error array:

In [100…
```python
print(error[-1])
```

```
[0.09055482532632883, 0.09148265370892378, 0.08140044343220618, 0.06320148
022433109, 0.03969934574890266, 0.013478931111025494, 0.01322554415250076
8, 0.03862100498260236, 0.061411831095585456, 0.0808211577094683, 0.09657
802524501752, 0.10887280343931482, 0.11828472256599448, 0.125686575141908
8, 0.13213268259724725]
```

In [101…
```python
print(f'Average error in each N: {avg_err}')
```

```
Average error in each N: 0.08252989664468588
```

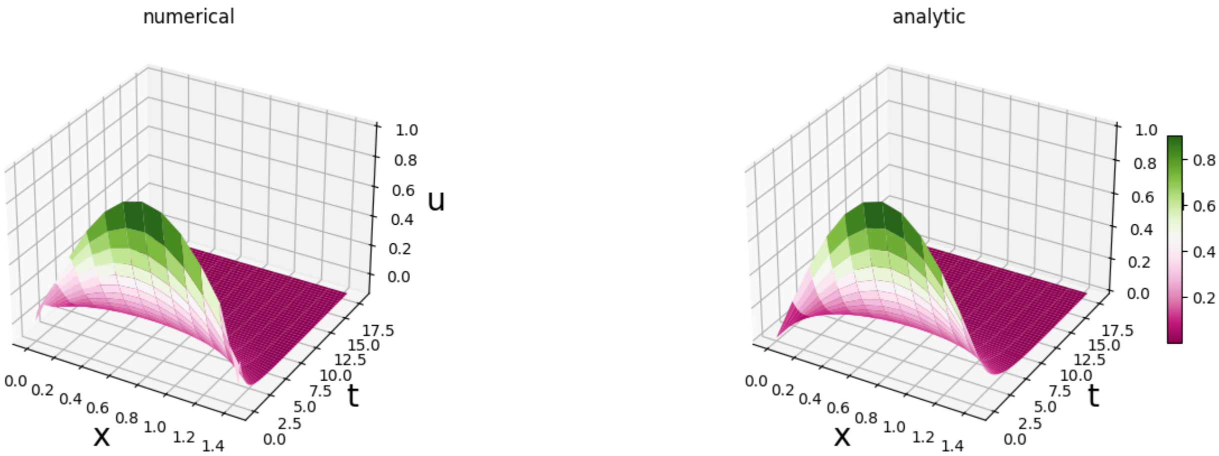
In [102…
```python
print(f'Average error\t\t: {avg_err / K}')
```

```
Average error           : 0.0002063247416117147
```

In [103…
```python
equation_type = str(input())
```

In [105…
```python
N = 12
K = 800
T = 18
curr_time = 0
params['bound_type'] = 'a1p1'
solver = ParabolicSolver(params, equation_type)
dict_ans = {
        'numerical': solver.solve(N, K, T).tolist(),
        'analytic': solver.analyticSolve(N, K, T).tolist()
    }
print("Sigma: ",solver.sigma)
draw(dict_ans, N, K, T)
```

```
Sigma:  0.32828063500117444
```



draw_u_x(dict_ans, N, K, T, curr_time)

In [61]:
```
error = compare_error(dict_ans)
avg_err = 0.0
for i in error:
    for j in i:
        avg_err += j
    avg_err /= N
```

In [62]:
```
print(error[int(K/2)])
```

```
[0.15004323820867924, 0.15003908897981438, 0.14936905048260124, 0.14802353
229739648, 0.1459959414158312, 0.14328297214661226, 0.1398848734691995, 0.
13580568803885318, 0.1310534574451474, 0.12564038882426248, 0.119582978514
40504, 0.11290208911315253, 0.10562297703359104, 0.09777526844980879, 0.08
939288235767477, 0.0805139003391925, 0.071180383492943, 0.0614381378638916
1, 0.05133643055787989, 0.040927659544548405, 0.030266980922935578, 0.0194
11898133123107, 0.008421818232685858, 0.002642419092700911]
```

In [63]:
```
print(error[-1])
```

```
[0.012334146727334022, 0.012334100112082183, 0.012272310337819916, 0.01214
8946166527067, 0.011964485133377644, 0.0117197139928831, 0.011415727524155
636, 0.011053925651563613, 0.010636008848490695, 0.010163971804324615, 0.0
09640095348056793, 0.009066936635819759, 0.00844731762416014, 0.0077843118
65663499, 0.007081229678526632, 0.006341601756619593, 0.00556916130129713
6, 0.00476782477051005, 0.0039416713544368955, 0.003094921299717477, 0.002
2319132162416586, 0.001357080511163347, 0.00047492710421674275, 0.00040999
741362547854]
```

In [64]:
```
print(f'Average error in each N: {avg_err}')
```

```
Average error in each N: 0.008100242309372108
```