

四元數

中原大學 CYPH 王文宏

1. 方向餘弦矩陣 DCM

先建立旋轉矩陣 R_x , R_y , R_z ,

繞 X, Y, Z 軸旋轉角度分別為 θ , ϕ , ψ

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\theta] & \sin[\theta] \\ 0 & -\sin[\theta] & \cos[\theta] \end{pmatrix}, \quad R_y = \begin{pmatrix} \cos[\phi] & 0 & -\sin[\phi] \\ 0 & 1 & 0 \\ \sin[\phi] & 0 & \cos[\phi] \end{pmatrix}, \quad R_z = \begin{pmatrix} \cos[\psi] & \sin[\psi] & 0 \\ -\sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

設一向量 $P(X, Y, Z)$, 並依 Z-Y-X 順序對向量旋轉

對 Z 軸旋轉, P' 為 P 對 Z 軸旋轉後的向量

$$P' = R_z \cdot P$$

對 Y 軸旋轉, P'' 為 P' 對 Y 軸旋轉後的向量

$$P'' = R_y \cdot P' = R_y \cdot R_z \cdot P$$

對 X 軸旋轉, P''' 為 P'' 對 X 軸旋轉後的向量

$$P''' = R_x \cdot P'' = R_x \cdot R_y \cdot R_z \cdot P$$

令 $R_{xyz} = R_x \cdot R_y \cdot R_z$, 則

$$R_{xyz} = \begin{pmatrix} \cos[\phi]\cos[\psi] & \cos[\phi]\sin[\psi] & -\sin[\phi] \\ \cos[\psi]\sin[\theta]\sin[\phi] - \cos[\theta]\sin[\psi] & \cos[\theta]\cos[\psi] + \sin[\theta]\sin[\phi]\sin[\psi] & \cos[\phi]\sin[\theta] \\ \cos[\theta]\cos[\psi]\sin[\phi] + \sin[\theta]\sin[\psi] & -\cos[\psi]\sin[\theta] + \cos[\theta]\sin[\phi]\sin[\psi] & \cos[\theta]\cos[\phi] \end{pmatrix}$$

2. 四元數 Quaternions

四元數可以解決 DCM 旋轉所產生的萬向節鎖(Gimbal lock)問題, 計算量也比 DCM 更為少
四元數由一個實數與三個虛數組成, 可表示成

$$Q = q_0 + q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k} = \cos\left[\frac{\theta}{2}\right] + \hat{n} \cdot \sin\left[\frac{\theta}{2}\right]$$

其中 \hat{n} 為轉軸方向, $\theta/2$ 為轉角大小, $\hat{i}, \hat{j}, \hat{k}$ 關係為

$$\begin{aligned} \hat{i} \cdot \hat{i} &= \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = -1 \\ \hat{i} \cdot \hat{j} &= -\hat{j} \cdot \hat{i} = \hat{k}, \quad \hat{j} \cdot \hat{k} = -\hat{k} \cdot \hat{j} = \hat{i}, \quad \hat{k} \cdot \hat{i} = -\hat{i} \cdot \hat{k} = \hat{j} \end{aligned}$$

設有兩四元數 Q_A, Q_B , 其乘法關係為

$$\begin{aligned} Q &= Q_A \cdot Q_B = (q_{A0} + q_{A1}\hat{i} + q_{A2}\hat{j} + q_{A3}\hat{k}) \cdot (q_{B0} + q_{B1}\hat{i} + q_{B2}\hat{j} + q_{B3}\hat{k}) \\ &= \begin{pmatrix} q_{A0} \cdot q_{B0} - q_{A1} \cdot q_{B1} - q_{A2} \cdot q_{B2} - q_{A3} \cdot q_{B3} \\ q_{A1} \cdot q_{B0} + q_{A0} \cdot q_{B1} - q_{A3} \cdot q_{B2} + q_{A2} \cdot q_{B3} \\ q_{A2} \cdot q_{B0} + q_{A3} \cdot q_{B1} + q_{A0} \cdot q_{B2} - q_{A1} \cdot q_{B3} \\ q_{A3} \cdot q_{B0} - q_{A2} \cdot q_{B1} + q_{A1} \cdot q_{B2} + q_{A0} \cdot q_{B3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \end{aligned}$$

旋轉四元數

$$\begin{aligned} Q_{xyz} &= Q_z \cdot Q_y \cdot Q_x = \left(\cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\psi}{2}\right]\hat{k}\right) \cdot \left(\cos\left[\frac{\phi}{2}\right] + \sin\left[\frac{\phi}{2}\right]\hat{j}\right) \cdot \left(\cos\left[\frac{\theta}{2}\right] + \sin\left[\frac{\theta}{2}\right]\hat{i}\right) \\ &\Rightarrow \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \cos\left[\frac{\psi}{2}\right]\cos\left[\frac{\phi}{2}\right]\cos\left[\frac{\theta}{2}\right] + \sin\left[\frac{\psi}{2}\right]\sin\left[\frac{\phi}{2}\right]\sin\left[\frac{\theta}{2}\right] \\ \cos\left[\frac{\psi}{2}\right]\cos\left[\frac{\phi}{2}\right]\sin\left[\frac{\theta}{2}\right] - \sin\left[\frac{\psi}{2}\right]\sin\left[\frac{\phi}{2}\right]\cos\left[\frac{\theta}{2}\right] \\ \cos\left[\frac{\psi}{2}\right]\sin\left[\frac{\phi}{2}\right]\cos\left[\frac{\theta}{2}\right] + \sin\left[\frac{\psi}{2}\right]\cos\left[\frac{\phi}{2}\right]\sin\left[\frac{\theta}{2}\right] \\ \sin\left[\frac{\psi}{2}\right]\cos\left[\frac{\phi}{2}\right]\cos\left[\frac{\theta}{2}\right] - \cos\left[\frac{\psi}{2}\right]\sin\left[\frac{\phi}{2}\right]\sin\left[\frac{\theta}{2}\right] \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \end{aligned}$$

設一向量 $R = x\hat{i} + y\hat{j} + z\hat{k}$, 及旋轉後向量 $R' = x'\hat{i} + y'\hat{j} + z'\hat{k}$, 四元數滿足

$$R' = Q^{-1} \cdot R \cdot Q$$

四元數倒數

$$Q^{-1} = \frac{1}{Q} = \frac{Q^*}{Q \cdot Q^*} = \frac{Q^*}{|Q|}$$

當四元數被歸一化後 $|Q| = 1$

$$Q^{-1} = Q^*$$

可得到

$$R' = Q^* \cdot R \cdot Q$$

將其展開

$$\begin{aligned} \hat{x'i} + \hat{y'j} + \hat{z'k} &= (q_0 - \hat{q_1i} - \hat{q_2j} - \hat{q_3k}) \cdot (\hat{xi} + \hat{yj} + \hat{zk}) \cdot (\hat{q_0} + \hat{q_1i} + \hat{q_2j} + \hat{q_3k}) \\ \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} &= \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 \cdot q_2 + q_0 \cdot q_3) & 2(q_1 \cdot q_3 - q_0 \cdot q_2) \\ 2(q_1 \cdot q_2 - q_0 \cdot q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 \cdot q_3 + q_0 \cdot q_1) \\ 2(q_1 \cdot q_3 + q_0 \cdot q_2) & 2(q_2 \cdot q_3 - q_0 \cdot q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned}$$

令四元數旋轉矩陣為 M_q

$$M_q = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 \cdot q_2 + q_0 \cdot q_3) & 2(q_1 \cdot q_3 - q_0 \cdot q_2) \\ 2(q_1 \cdot q_2 - q_0 \cdot q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 \cdot q_3 + q_0 \cdot q_1) \\ 2(q_1 \cdot q_3 + q_0 \cdot q_2) & 2(q_2 \cdot q_3 - q_0 \cdot q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix}$$

將 Q_{xyz} 帶入 M_q 會得到與 R_{xyz} 相同結果

$$\begin{aligned} M_q &= \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \\ &= \begin{pmatrix} \cos[\phi]\cos[\psi] & \cos[\phi]\sin[\psi] & -\sin[\phi] \\ \cos[\psi]\sin[\theta]\sin[\phi] - \cos[\theta]\sin[\psi] & \cos[\theta]\cos[\psi] + \sin[\theta]\sin[\phi]\sin[\psi] & \cos[\phi]\sin[\theta] \\ \cos[\theta]\cos[\psi]\sin[\phi] + \sin[\theta]\sin[\psi] & -\cos[\psi]\sin[\theta] + \cos[\theta]\sin[\phi]\sin[\psi] & \cos[\theta]\cos[\phi] \end{pmatrix} \end{aligned}$$

將尤拉角轉成四元數, 由 M_q 可得出

$$\theta = \text{ArcTan} \left[\frac{M_{23}}{M_{33}} \right], \quad \phi = -\text{ArcSin}[M_{13}], \quad \psi = \text{ArcTan} \left[\frac{M_{12}}{M_{11}} \right]$$

3. 四元數微分

四元數微分, 已知一四元數 $Q = \cos[\frac{\theta}{2}] + \hat{n} \cdot \sin[\frac{\theta}{2}]$, 對時間微分

$$\frac{dQ}{dt} = -\frac{1}{2} \sin[\frac{\theta}{2}] \cdot \frac{d\theta}{dt} + \frac{d\hat{n}}{dt} \cdot \sin[\frac{\theta}{2}] + \hat{n} \cdot \frac{1}{2} \cos[\frac{\theta}{2}] \cdot \frac{d\theta}{dt}$$

已知 $\hat{n} \cdot \hat{n} = -1$, $\frac{d\hat{n}}{dt} = 0$, $\frac{d\theta}{dt} = \omega_{Eb}^E$, E 為地理座標系, b 為飛行器坐標系

$$\frac{dQ}{dt} = \frac{1}{2} \hat{n} \cdot \omega_{Eb}^E \cdot (\cos[\frac{\theta}{2}] + \hat{n} \cdot \sin[\frac{\theta}{2}]) = \frac{1}{2} \vec{\omega}_{Eb}^E \cdot Q$$

因為陀螺儀在飛行器上測到的角速度為 $\vec{\omega}_{Eb}^b = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$, 故將 $\vec{\omega}_{Eb}^E$ 轉換成 $\vec{\omega}_{Eb}^b$ 會較為方便

$$\begin{aligned} \vec{\omega}_{Eb}^b &= Q^* \vec{\omega}_{Eb}^E \cdot Q \Rightarrow Q \cdot \vec{\omega}_{Eb}^b = Q \cdot Q^* \cdot \vec{\omega}_{Eb}^E \cdot Q = \vec{\omega}_{Eb}^E \cdot Q \\ &\Rightarrow \frac{dQ}{dt} = \frac{1}{2} \vec{\omega}_{Eb}^E \cdot Q = \frac{1}{2} Q \cdot \vec{\omega}_{Eb}^b \end{aligned}$$

將 $\frac{dQ}{dt} = \frac{1}{2} Q \cdot \vec{\omega}_{Eb}^b$ 展開

$$\frac{dQ}{dt} = \frac{1}{2} (q_0 + q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k}) \cdot (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) = \frac{1}{2} \begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{pmatrix} \cdot \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

整理為

$$\frac{dQ}{dt} = \Omega_b \cdot Q$$

其中

$$\Omega_b = \frac{1}{2} \begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{pmatrix}$$

4. 更新四元數

使用一階 Runge-Kutta 更新四元數, 假設有一微分方程

$$\frac{dX}{dt} = f[X[t], \omega[t]]$$

則其解為

$$X[t + \Delta t] = X[t] + \Delta t \cdot f[X[t], \omega[t]]$$

其中 Δt 為取樣週期, 將套用至四元數

$$Q[t + \Delta t] = Q[t] + \Delta t \cdot \Omega_b[t] \cdot Q[t]$$

展開上式

$$\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}_{t+\Delta t} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}_t + \frac{\Delta t}{2} \begin{pmatrix} -\omega_x \cdot q_1 - \omega_y \cdot q_2 - \omega_z \cdot q_3 \\ +\omega_x \cdot q_0 - \omega_y \cdot q_3 + \omega_z \cdot q_2 \\ +\omega_x \cdot q_3 + \omega_y \cdot q_0 - \omega_z \cdot q_1 \\ -\omega_x \cdot q_2 + \omega_y \cdot q_1 + \omega_z \cdot q_0 \end{pmatrix}$$

只需利用角速度即可更新四元數

5. 使用四元數融合加速度計

利用四元數將地理的重力加速度旋轉至飛行器上面，
再與加速度計讀出的值(已歸一化的)做外積，得出誤差，
用此誤差對角速度做校正融合

重力加速度

$$\vec{g} = g\hat{z}$$

做歸一化

$$\vec{g} \rightarrow \hat{g} = \hat{z}$$

設旋轉至飛行器上的重力加速度為 \hat{g}_b

$$\begin{aligned}\hat{g}_b &= M_q \cdot \hat{g} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \Rightarrow \hat{g}_b &= \begin{pmatrix} g_{bx} \\ g_{by} \\ g_{bz} \end{pmatrix} = \begin{pmatrix} M_{13} \\ M_{23} \\ M_{33} \end{pmatrix}\end{aligned}$$

飛行器上的加速度計測得的加速度 \vec{a}_b 做歸一化

$$\vec{a}_b \rightarrow \hat{a}_b$$

做外積，計算誤差 \vec{e}

$$\begin{aligned}\vec{e} &= \hat{a}_b \times \hat{g}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_{bx} & a_{by} & a_{bz} \\ g_{bx} & g_{by} & g_{bz} \end{vmatrix} = \begin{vmatrix} a_{by} & a_{bz} \\ g_{by} & g_{bz} \end{vmatrix} \hat{x} + \begin{vmatrix} a_{bz} & a_{bx} \\ g_{bz} & g_{bx} \end{vmatrix} \hat{y} + \begin{vmatrix} a_{bx} & a_{by} \\ g_{bx} & g_{by} \end{vmatrix} \hat{z} \\ \Rightarrow \vec{e} &= \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} = \begin{pmatrix} a_{by} \cdot g_{bz} - a_{bz} \cdot g_{by} \\ a_{bz} \cdot g_{bx} - a_{bx} \cdot g_{bz} \\ a_{bx} \cdot g_{by} - a_{by} \cdot g_{bx} \end{pmatrix}\end{aligned}$$

將 e 融合至角速度上

參考書籍：

鄭正隆, "慣性技術 INERTIAL TECHNOLOGY", 崧博出版事業有限公司, 2011

