橢圓擬合
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任意位置大小的橢圓方程式

$$\left(\frac{X-X_0}{a}\right)^2+\left(\frac{Y-Y_0}{b}\right)^2=1$$

上式可分解成圓形對 X,Y 軸做變形

$$\begin{cases}\begin{pmatrix}X-X_0\\Y-Y_0\end{pmatrix}=\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}X'\\Y'\end{pmatrix}\\(X')^2+(Y')^2=1\end{cases}$$

對 X,Y 軸做旋轉

$$\begin{cases}\begin{pmatrix}X-X_0\\Y-Y_0\end{pmatrix}=\begin{pmatrix}Cos[\theta]&Sin[\theta]\\-Sin[\theta]&Cos[\theta]\end{pmatrix}\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}X'\\Y'\end{pmatrix}\\(X')^2+(Y')^2=1\end{cases}$$

求得 X',Y'

$$\begin{pmatrix}X'\\Y'\end{pmatrix}=\begin{pmatrix}a\mathrm{Cos}[\theta]&b\mathrm{Sin}[\theta]\\-a\mathrm{Sin}[\theta]&b\mathrm{Cos}[\theta]\end{pmatrix}^{-1}\begin{pmatrix}X-X_0\\Y-Y_0\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}X'\\Y'\end{pmatrix}=\begin{pmatrix}\frac{\mathrm{Cos}[\theta]}{a}&-\frac{\mathrm{Cos}[\theta]}{a}\cdot\frac{\mathrm{Sin}[\theta]}{\mathrm{Cos}[\theta]}\\\frac{\mathrm{Cos}[\theta]}{b}\cdot\frac{\mathrm{Sin}[\theta]}{\mathrm{Cos}[\theta]}&\frac{\mathrm{Cos}[\theta]}{b}\end{pmatrix}\begin{pmatrix}X-X_0\\Y-Y_0\end{pmatrix}$$

令 $\alpha=\frac{\mathrm{Cos}[\theta]}{a},\ \beta=\frac{\mathrm{Cos}[\theta]}{b},\ \gamma=\mathrm{Tan}[\theta]$

$$\Rightarrow\begin{pmatrix}X'\\Y'\end{pmatrix}=\begin{pmatrix}\alpha&-\alpha\cdot\gamma\\\beta\cdot\gamma&\beta\end{pmatrix}\begin{pmatrix}X-X_0\\Y-Y_0\end{pmatrix}=\begin{pmatrix}\alpha(X-X_0)-\alpha\gamma(Y-Y_0)\\\beta\gamma(X-X_0)+\beta(Y-Y_0)\end{pmatrix}$$

代入圓形方程&整理

$$(\alpha(X-X_0)-\alpha\gamma(Y-Y_0))^2+(\beta\gamma(X-X_0)+\beta(Y-Y_0))^2=1$$
$$\Rightarrow(\alpha^2+\beta^2\gamma^2)(X-X_0)^2+2(\beta^2\gamma-\alpha^2\gamma)(X-X_0)(Y-Y_0)+(\alpha^2\gamma^2+\beta^2)(Y-Y_0)^2=1$$

令 $\mathrm{U}=\alpha^2+\beta^2\gamma^2,\ \mathrm{V}=2(\beta^2\gamma-\alpha^2\gamma),\ \mathrm{W}=\alpha^2\gamma^2+\beta^2$

$$(U)X^2+(V)XY+(W)Y^2+(-2UX_0-VY_0)X+(-VX_0-2WY_0)Y+(UX_0^2+VX_0Y_0+WY_0^2)=1$$
$$\Rightarrow X^2+\left(\frac{V}{U}\right)XY+\left(\frac{W}{U}\right)Y^2+\left(\frac{-2UX_0-VY_0}{U}\right)X+\left(\frac{-VX_0-2WY_0}{U}\right)Y+\left(\frac{UX_0^2+VX_0Y_0+WY_0^2}{U}\right)=1$$
$$\Rightarrow X^2+\left(\frac{V}{U}\right)XY+\left(\frac{W}{U}\right)Y^2+\left(-2X_0-\frac{V}{U}Y_0\right)X+\left(-\frac{V}{U}X_0-2\frac{W}{U}Y_0\right)Y+\left(X_0^2+\frac{V}{U}X_0Y_0+\frac{W}{U}Y_0^2\right)=\frac{1}{U}$$

任意橢圓方程式

$$X^2+A\cdot XY+B\cdot Y^2+C\cdot X+D\cdot Y+E=0$$

其中 $A=\frac{V}{U}$, $B=\frac{W}{U}$, $C=-2X_0-AY_0$, $D=-AX_0-2BY_0$, $E=X_0^2+AX_0Y_0+BY_0^2-\frac{1}{U}$

$$\Rightarrow X^2+AXY+BY^2+(-2X_0-AY_0)X+(-AX_0-2BY_0)Y+\left(X_0^2+AX_0Y_0+BY_0^2-\frac{1}{U}\right)=0$$

使用最小平方法做橢圓擬合，其中 N 為取樣數目

$$Q=\sum_{i=1}^{N}{F_i}^2=\sum_{i=1}^{N}({X_i}^2+A\cdot X_iY_i+B\cdot {Y_i}^2+C\cdot X_i+D\cdot Y_i+E)^2$$

欲使 Q 為最小值，必有

$$\frac{\partial Q}{\partial A}=\frac{\partial Q}{\partial B}=\frac{\partial Q}{\partial C}=\frac{\partial Q}{\partial D}=\frac{\partial Q}{\partial E}=0$$

展開

$$\frac{\partial Q}{\partial A}=2\sum_{i=1}^{N}\left({X_i}^2+A\cdot X_iY_i+B\cdot {Y_i}^2+C\cdot X_i+D\cdot Y_i+E\right)(X_iY_i)=0$$
$$\frac{\partial Q}{\partial B}=2\sum_{i=1}^{N}\left({X_i}^2+A\cdot X_iY_i+B\cdot {Y_i}^2+C\cdot X_i+D\cdot Y_i+E\right)({Y_i}^2)=0$$
$$\frac{\partial Q}{\partial C}=2\sum_{i=1}^{N}\left({X_i}^2+A\cdot X_iY_i+B\cdot {Y_i}^2+C\cdot X_i+D\cdot Y_i+E\right)(X_i)=0$$
$$\frac{\partial Q}{\partial D}=2\sum_{i=1}^{N}\left({X_i}^2+A\cdot X_iY_i+B\cdot {Y_i}^2+C\cdot X_i+D\cdot Y_i+E\right)(Y_i)=0$$
$$\frac{\partial Q}{\partial E}=2\sum_{i=1}^{N}\left({X_i}^2+A\cdot X_iY_i+B\cdot {Y_i}^2+C\cdot X_i+D\cdot Y_i+E\right)(1)=0$$

矩陣形式

$$\begin{pmatrix}
\sum_{i=1}^{N}{X_i}^2{Y_i}^2 & \sum_{i=1}^{N}X_i{Y_i}^3 & \sum_{i=1}^{N}{X_i}^2Y_i & \sum_{i=1}^{N}X_i{Y_i}^2 & \sum_{i=1}^{N}X_iY_i \\
\sum_{i=1}^{N}X_i{Y_i}^3 & \sum_{i=1}^{N}{Y_i}^4 & \sum_{i=1}^{N}X_i{Y_i}^2 & \sum_{i=1}^{N}{Y_i}^3 & \sum_{i=1}^{N}{Y_i}^2 \\
\sum_{i=1}^{N}{X_i}^2Y_i & \sum_{i=1}^{N}X_i{Y_i}^2 & \sum_{i=1}^{N}{X_i}^2 & \sum_{i=1}^{N}X_iY_i & \sum_{i=1}^{N}X_i \\
\sum_{i=1}^{N}X_i{Y_i}^2 & \sum_{i=1}^{N}{Y_i}^3 & \sum_{i=1}^{N}X_iY_i & \sum_{i=1}^{N}{Y_i}^2 & \sum_{i=1}^{N}Y_i \\
\sum_{i=1}^{N}X_iY_i & \sum_{i=1}^{N}{Y_i}^2 & \sum_{i=1}^{N}X_i & \sum_{i=1}^{N}Y_i & N
\end{pmatrix}
\begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix}
=-\begin{pmatrix}
\sum_{i=1}^{N}{X_i}^3Y_i \\
\sum_{i=1}^{N}{X_i}^2{Y_i}^2 \\
\sum_{i=1}^{N}{X_i}^3 \\
\sum_{i=1}^{N}{X_i}^2Y_i \\
\sum_{i=1}^{N}{X_i}^2
\end{pmatrix}$$

解線性方程組後，可得到 A,B,C,D,E 的值

由 $C=-2X_0-AY_0,\ D=-AX_0-2BY_0$ ，可解得 $X_0,\ Y_0$

$$X_0=\frac{2BC-AD}{A^2-4B},\qquad Y_0=\frac{2D-AC}{A^2-4B}$$

由 $\mathrm{U}=\alpha^2+\beta^2\gamma^2,\ \mathrm{V}=2(\beta^2\gamma-\alpha^2\gamma),\ \mathrm{W}=\alpha^2\gamma^2+\beta^2$ ，可解得 $\gamma,\ \theta$

$$\frac{W-U}{V}=\frac{1-\gamma^2}{2\gamma}$$
$$\Longrightarrow V\gamma^2+2(W-U)\gamma-\mathrm{V}=0$$
$$\Longrightarrow A\gamma^2+2(B-1)\gamma-A=0$$

$$\gamma=\frac{1-B}{A}\pm\sqrt{(\frac{1-B}{A})^2+1}$$

$$\theta=\mathrm{ArcTan}[\gamma]$$

由 $E=X_0^2+AX_0Y_0+BY_0^2-\frac{1}{U}$，可解得 U

$$U=\frac{1}{(X_0^2+AX_0Y_0+BY_0^2)-E}$$

由 $\mathrm{U}=\alpha^2+\beta^2\gamma^2,\ \mathrm{V}=2(\beta^2\gamma-\alpha^2\gamma)$ ，可解得 a,b

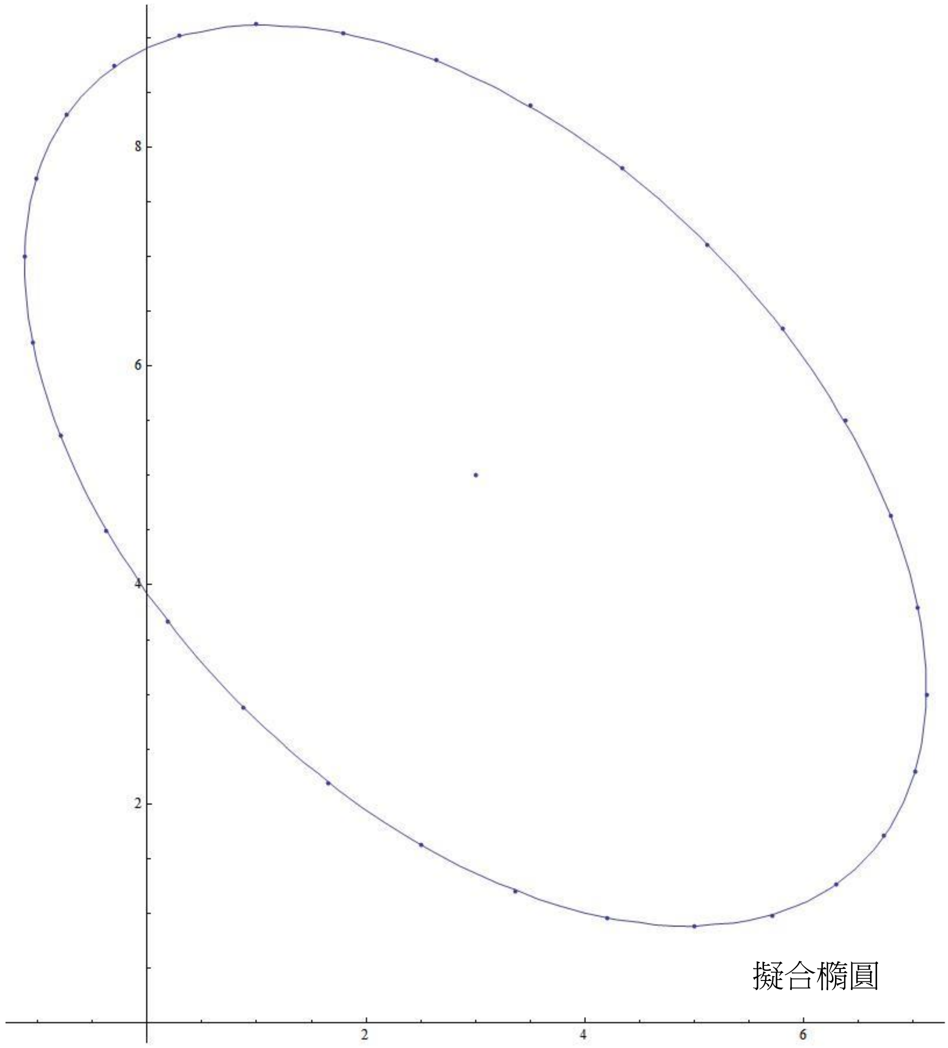
$$\alpha=\sqrt{\frac{\mathrm{U}}{\gamma^4-1}(\mathrm{B}\gamma^2-1)},\qquad \beta=\sqrt{\frac{\mathrm{U}}{\gamma^4-1}(\gamma^2-\mathrm{B})}$$
$$\Longrightarrow a=\frac{\mathrm{Cos}[\theta]}{\alpha},\qquad b=\frac{\mathrm{Cos}[\theta]}{\beta}$$

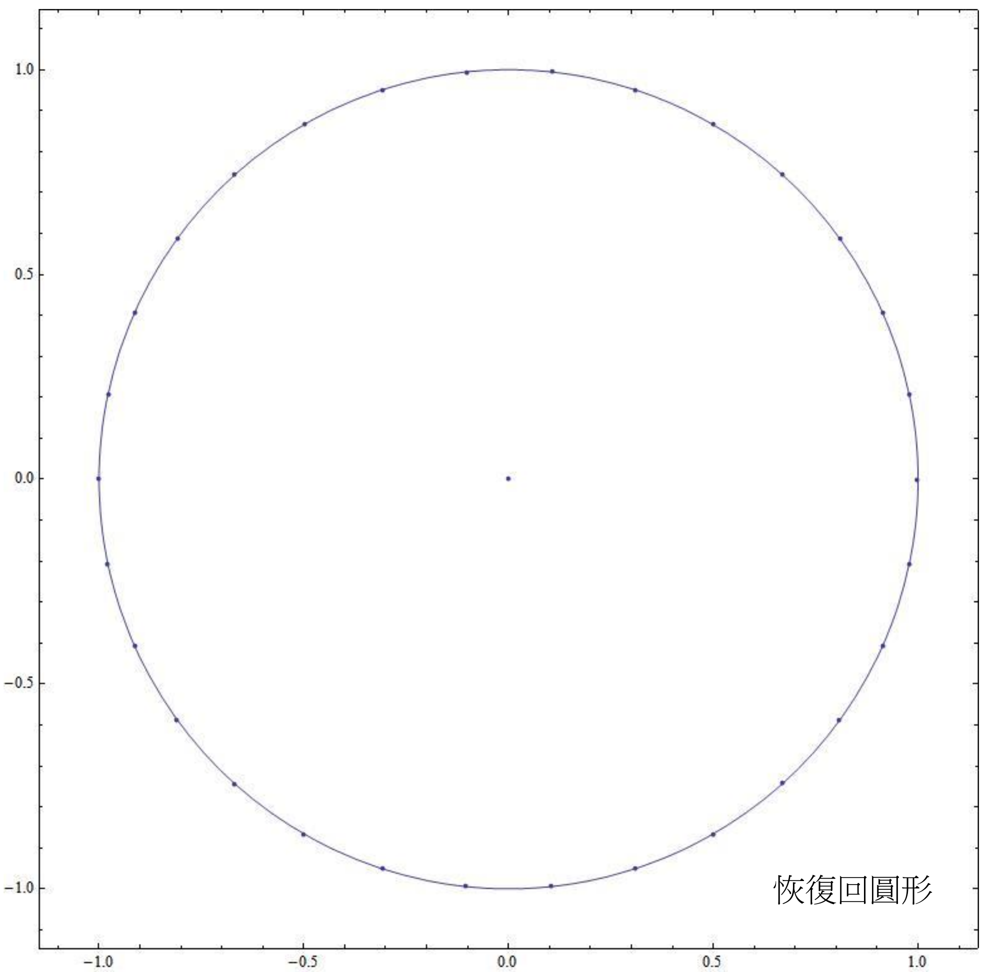
整理結果

$$\begin{cases} X_0=\dfrac{2BC-AD}{A^2-4B} \\ Y_0=\dfrac{2D-AC}{A^2-4B} \\ \theta=\mathrm{ArcTan}[\gamma] \\ a=\dfrac{\mathrm{Cos}[\theta]}{\sqrt{\dfrac{\mathrm{U}}{\gamma^4-1}(\mathrm{B}\gamma^2-1)}} \\ b=\dfrac{\mathrm{Cos}[\theta]}{\sqrt{\dfrac{\mathrm{U}}{\gamma^4-1}(\gamma^2-\mathrm{B})}} \\ U=\dfrac{1}{(X_0^2+AX_0Y_0+BY_0^2)-E} \\ \gamma=\dfrac{1-B}{A}\pm\sqrt{\left(\dfrac{1-B}{A}\right)^2+1} \end{cases}$$

數據A

X	Y
5.12	7.112
4.338	7.812
3.501	8.379
2.64	8.794
1.791	9.046
0.998	9.123
0.291	9.018
-0.296	8.741
-0.736	8.296
-1.017	7.71
-1.123	7.003
-1.049	6.21
-0.794	5.364
-0.373	4.497
0.189	3.66
0.879	2.878
1.657	2.192
2.502	1.622
3.36	1.203
4.209	0.951
5.002	0.879
5.709	0.982
6.293	1.262
6.737	1.707
7.019	2.293
7.125	2.999
7.045	3.794
6.792	4.635
6.378	5.499
5.808	6.339





結果：

$\theta = 45.02108°$

$X_0 = 2.99923$

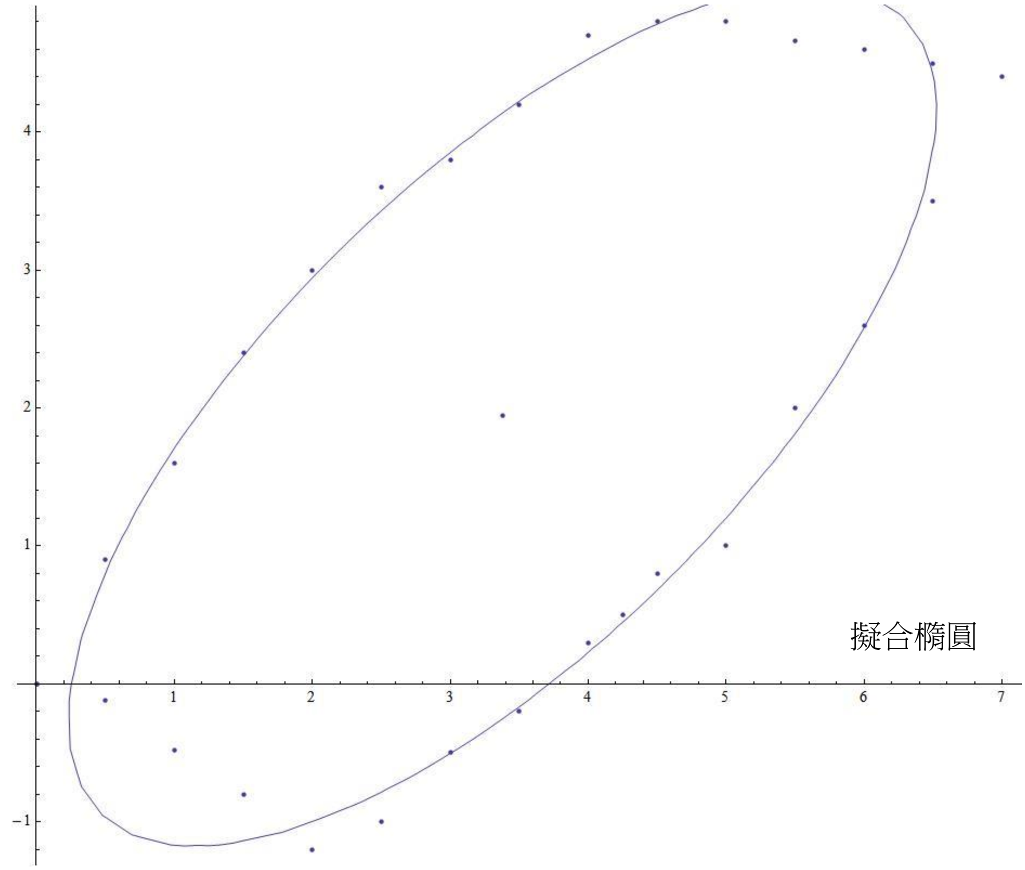
$Y_0 = 4.99968$

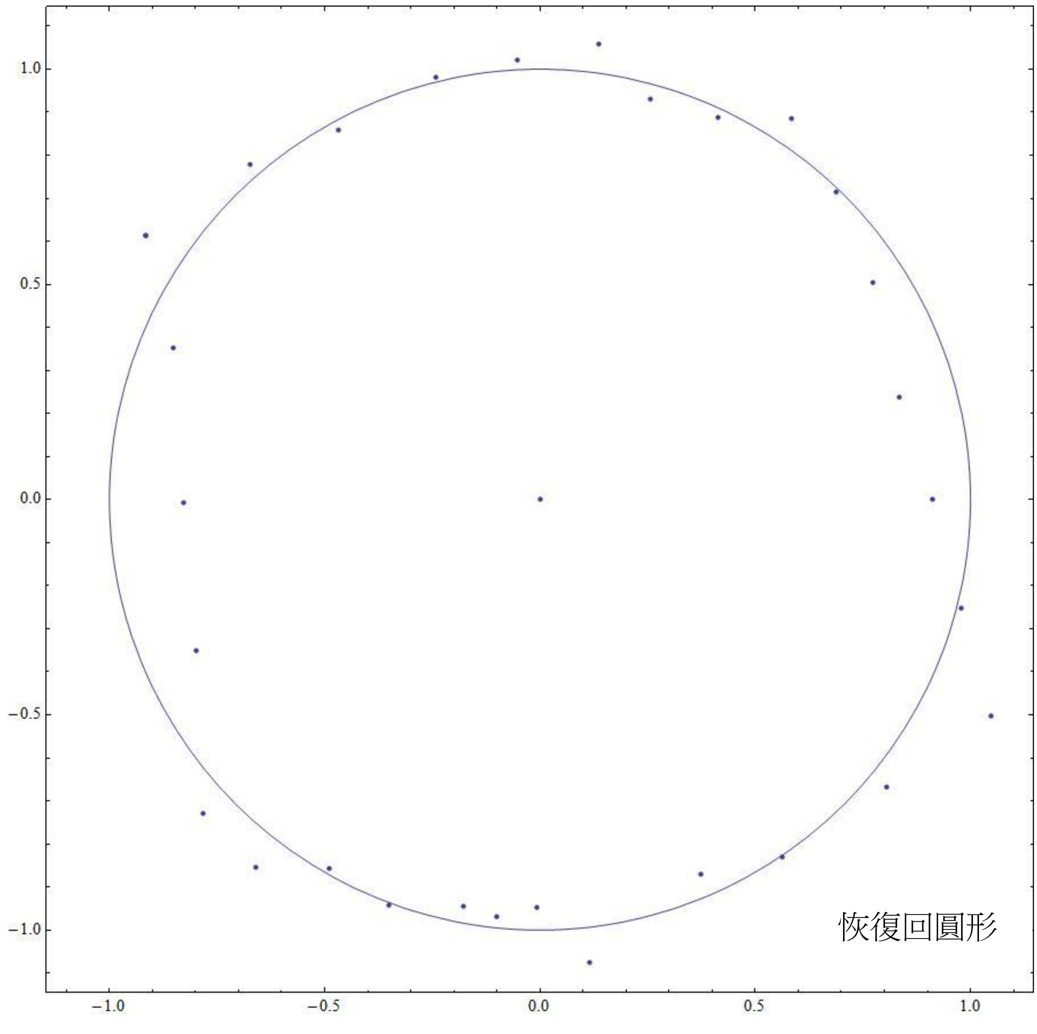
$a = 5.00079$

$b = 2.99949$

數據B

X	Y
0	0
0.5	0.9
1	1.6
1.5	2.4
2	3
2.5	3.6
3	3.8
3.5	4.2
4	4.7
4.5	4.8
5	4.8
5.5	4.66
6	4.6
6.5	4.5
7	4.4
6.5	3.5
6	2.6
5.5	2
5	1
4.5	0.8
4	0.3
3.5	-0.2
3	-0.5
2.5	-1
2	-1.2
1.5	-0.8
1	-0.48
0.5	-0.12
0	0
4.25	0.5





結果：

$\theta = 45.3439°$

$X_0 = 3.37720$

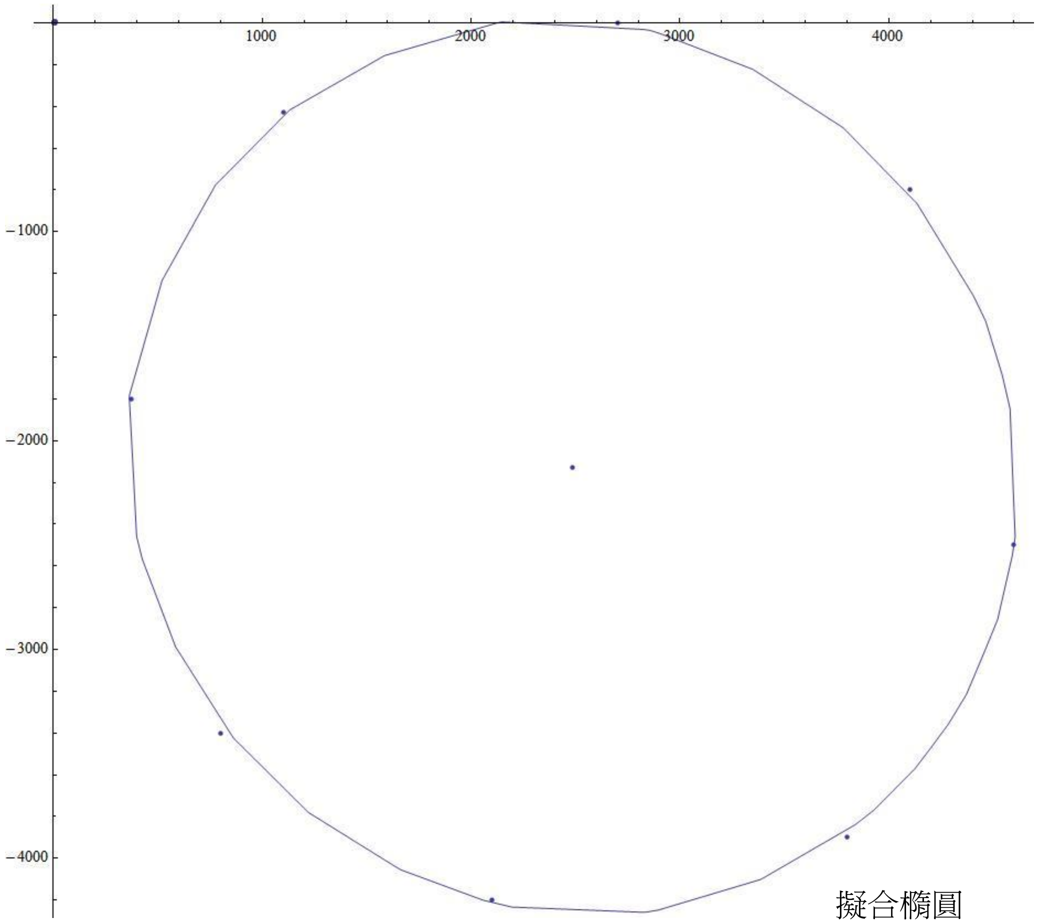
$Y_0 = 1.94285$

$a = 1.68913$

$b = 4.09959$

數據 C

X	Y
2700	0
1100	-430
370	-1800
800	-3400
2100	-4200
3800	-3900
4600	-2500
4100	-800



擬合橢圓

結果：

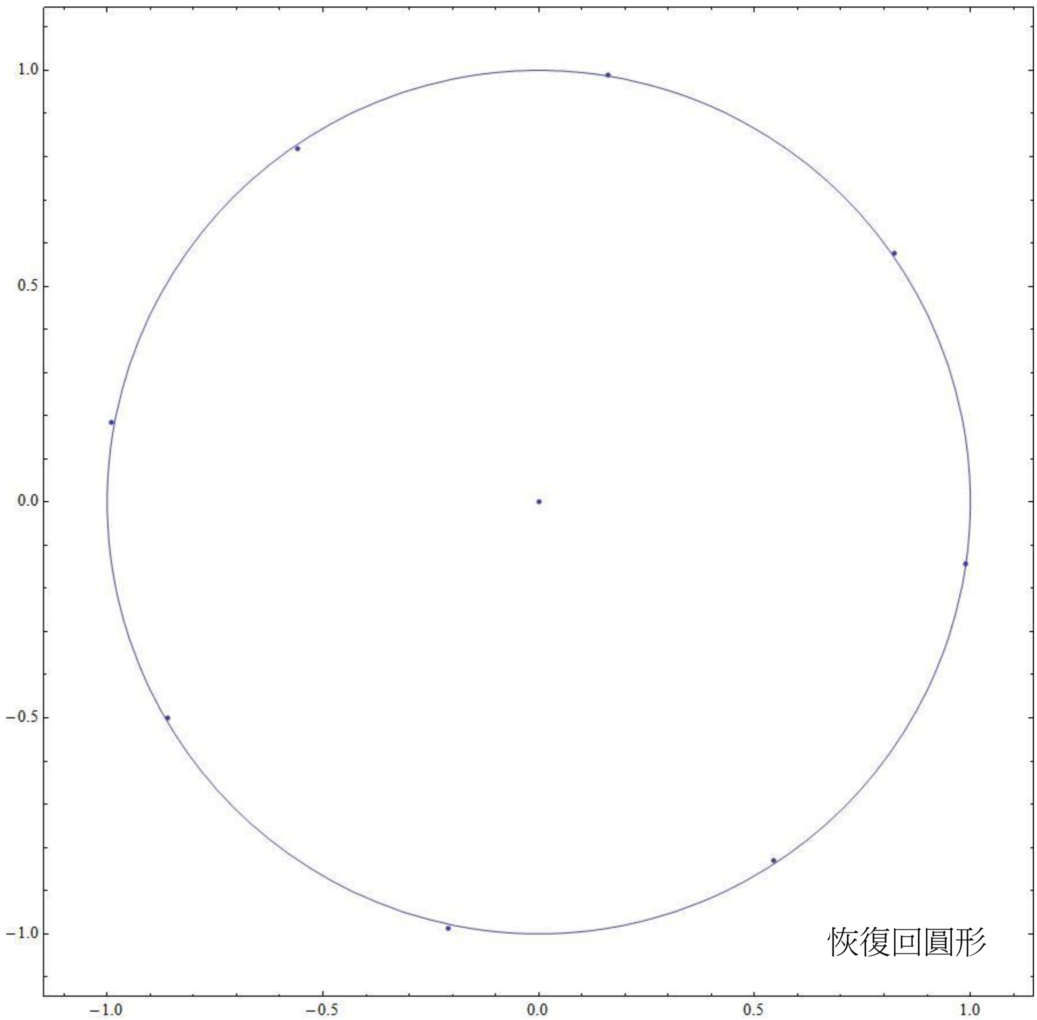
$\theta = 48.0413°$

$X_0 = 2485.70$

$Y_0 = -2128.80$

$a = 2192.14$

$b = 2092.12$



恢復回圓形