四元數
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1. 方向餘弦矩陣 DCM

先建立旋轉矩陣 R_x, R_y, R_z,
繞 X, Y, Z 軸旋轉角度分別為 θ, ϕ, ψ

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & Cos[\theta] & Sin[\theta] \\ 0 & -Sin[\theta] & Cos[\theta] \end{pmatrix}, \quad R_y = \begin{pmatrix} Cos[\phi] & 0 & -Sin[\phi] \\ 0 & 1 & 0 \\ Sin[\phi] & 0 & Cos[\phi] \end{pmatrix}, \quad R_z = \begin{pmatrix} Cos[\psi] & Sin[\psi] & 0 \\ -Sin[\psi] & Cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

設一向量 P(X, Y, Z), 並依 Z-Y-X 順序對向量旋轉
對 Z 軸旋轉, P'為 P 對 Z 軸旋轉後的向量

$$P' = R_z \cdot P$$

對 Y 軸旋轉, P''為 P'對 Y 軸旋轉後的向量

$$P'' = R_y \cdot P' = R_y \cdot R_z \cdot P$$

對 X 軸旋轉, P'''為 P''對 X 軸旋轉後的向量

$$P''' = R_x \cdot P'' = R_x \cdot R_y \cdot R_z \cdot P$$

令 $R_{xyz} = R_x \cdot R_y \cdot R_z$, 則

$$R_{xyz} = \begin{pmatrix} Cos[\phi]Cos[\psi] & Cos[\phi]Sin[\psi] & -Sin[\phi] \\ Cos[\psi]Sin[\theta]Sin[\phi] - Cos[\theta]Sin[\psi] & Cos[\theta]Cos[\psi] + Sin[\theta]Sin[\phi]Sin[\psi] & Cos[\phi]Sin[\theta] \\ Cos[\theta]Cos[\psi]Sin[\phi] + Sin[\theta]Sin[\psi] & -Cos[\psi]Sin[\theta] + Cos[\theta]Sin[\phi]Sin[\psi] & Cos[\theta]Cos[\phi] \end{pmatrix}$$

2. 四元數 Quaternions

四元數可以解決 DCM 旋轉所產生的萬向節鎖(Gimbal lock)問題，計算量也比 DCM 更為少
四元數由一個實數與三個虛數組成，可表示成

$$Q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k} = Cos[\frac{\theta}{2}] + \hat{n} \cdot Sin[\frac{\theta}{2}]$$

其中 n 為轉軸方向，$\theta/2$為轉角大小，i，j，k 關係為

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = -1$$
$$\hat{i} \cdot \hat{j} = -\hat{j} \cdot \hat{i} = \hat{k}, \quad \hat{j} \cdot \hat{k} = -\hat{k} \cdot \hat{j} = \hat{i}, \quad \hat{k} \cdot \hat{i} = -\hat{i} \cdot \hat{k} = \hat{j}$$

設有兩四元數Q_A，Q_B，其乘法關係為

$$Q = Q_A \cdot Q_B = (q_{A0} + q_{A1}\hat{i} + q_{A2}\hat{j} + q_{A3}\hat{k}) \cdot (q_{B0} + q_{B1}\hat{i} + q_{B2}\hat{j} + q_{B3}\hat{k})$$
$$= \begin{pmatrix} q_{A0} \cdot q_{B0} - q_{A1} \cdot q_{B1} - q_{A2} \cdot q_{B2} - q_{A3} \cdot q_{B3} \\ q_{A1} \cdot q_{B0} + q_{A0} \cdot q_{B1} - q_{A3} \cdot q_{B2} + q_{A2} \cdot q_{B3} \\ q_{A2} \cdot q_{B0} + q_{A3} \cdot q_{B1} + q_{A0} \cdot q_{B2} - q_{A1} \cdot q_{B3} \\ q_{A3} \cdot q_{B0} - q_{A2} \cdot q_{B1} - q_{A1} \cdot q_{B2} + q_{A0} \cdot q_{B3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ i \\ j \\ k \end{pmatrix}$$

旋轉四元數

$$Q_{xyz} = Q_z \cdot Q_y \cdot Q_x = (Cos[\frac{\psi}{2}] + Sin[\frac{\psi}{2}]\hat{k}) \cdot (Cos[\frac{\phi}{2}] + Sin[\frac{\phi}{2}]\hat{j}) \cdot (Cos[\frac{\theta}{2}] + Sin[\frac{\theta}{2}]\hat{i})$$
$$\Rightarrow \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} Cos\left[\frac{\psi}{2}\right] Cos\left[\frac{\phi}{2}\right] Cos\left[\frac{\theta}{2}\right] + Sin\left[\frac{\psi}{2}\right] Sin\left[\frac{\phi}{2}\right] Sin\left[\frac{\theta}{2}\right] \\ Cos\left[\frac{\psi}{2}\right] Cos\left[\frac{\phi}{2}\right] Sin\left[\frac{\theta}{2}\right] - Sin\left[\frac{\psi}{2}\right] Sin\left[\frac{\phi}{2}\right] Cos\left[\frac{\theta}{2}\right] \\ Cos\left[\frac{\psi}{2}\right] Sin\left[\frac{\phi}{2}\right] Cos\left[\frac{\theta}{2}\right] + Sin\left[\frac{\psi}{2}\right] Cos\left[\frac{\phi}{2}\right] Sin\left[\frac{\theta}{2}\right] \\ Sin\left[\frac{\psi}{2}\right] Cos\left[\frac{\phi}{2}\right] Cos\left[\frac{\theta}{2}\right] - Cos\left[\frac{\psi}{2}\right] Sin\left[\frac{\phi}{2}\right] Sin\left[\frac{\theta}{2}\right] \end{pmatrix} \cdot \begin{pmatrix} 1 \\ i \\ j \\ k \end{pmatrix}$$

設一向量 $\mathrm{R} = \mathrm{x}\hat{\mathrm{i}} + \mathrm{y}\hat{\mathrm{j}} + \mathrm{z}\hat{\mathrm{k}}$，及旋轉後向量 $\mathrm{R}' = \mathrm{x}'\hat{\mathrm{i}} + \mathrm{y}'\hat{\mathrm{j}} + \mathrm{z}'\hat{\mathrm{k}}$，四元數滿足

$$R' = Q^{-1} \cdot R \cdot Q$$

四元數倒數

$$Q^{-1} = \frac{1}{Q} = \frac{Q^*}{Q \cdot Q^*} = \frac{Q^*}{|\, Q \,|}$$

當四元數被歸一化後 $|\, Q \,| = 1$

$$Q^{-1} = Q^*$$

可得到

$$R' = Q^* \cdot R \cdot Q$$

將其展開

$$x'\hat{i} + y'\hat{j} + z'\hat{k} = (q_0 - q_1\hat{i} - q_2\hat{j} - q_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k})$$
$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} {q_0}^2 + {q_1}^2 - {q_2}^2 - {q_3}^2 & 2(q_1 \cdot q_2 + q_0 \cdot q_3) & 2(q_1 \cdot q_3 - q_0 \cdot q_2) \\ 2(q_1 \cdot q_2 - q_0 \cdot q_3) & {q_0}^2 - {q_1}^2 + {q_2}^2 - {q_3}^2 & 2(q_2 \cdot q_3 + q_0 \cdot q_1) \\ 2(q_1 \cdot q_3 + q_0 \cdot q_2) & 2(q_2 \cdot q_3 - q_0 \cdot q_1) & {q_0}^2 - {q_1}^2 - {q_2}^2 + {q_3}^2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

令四元數旋轉矩陣為 M_q

$$M_q = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} {q_0}^2 + {q_1}^2 - {q_2}^2 - {q_3}^2 & 2(q_1 \cdot q_2 + q_0 \cdot q_3) & 2(q_1 \cdot q_3 - q_0 \cdot q_2) \\ 2(q_1 \cdot q_2 - q_0 \cdot q_3) & {q_0}^2 - {q_1}^2 + {q_2}^2 - {q_3}^2 & 2(q_2 \cdot q_3 + q_0 \cdot q_1) \\ 2(q_1 \cdot q_3 + q_0 \cdot q_2) & 2(q_2 \cdot q_3 - q_0 \cdot q_1) & {q_0}^2 - {q_1}^2 - {q_2}^2 + {q_3}^2 \end{pmatrix}$$

將 Q_{xyz} 帶入 M_q 會得到與 R_{xyz} 相同結果

$$\begin{aligned} M_q &= \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \\ &= \begin{pmatrix} Cos[\phi]Cos[\psi] & Cos[\phi]Sin[\psi] & -Sin[\phi] \\ Cos[\psi]Sin[\theta]Sin[\phi] - Cos[\theta]Sin[\psi] & Cos[\theta]Cos[\psi] + Sin[\theta]Sin[\phi]Sin[\psi] & Cos[\phi]Sin[\theta] \\ Cos[\theta]Cos[\psi]Sin[\phi] + Sin[\theta]Sin[\psi] & -Cos[\psi]Sin[\theta] + Cos[\theta]Sin[\phi]Sin[\psi] & Cos[\theta]Cos[\phi] \end{pmatrix} \end{aligned}$$

將尤拉角轉成四元數，由 M_q 可得出

$$\theta = ArcTan\left[\frac{M_{23}}{M_{33}}\right], \qquad \phi = -ArcSin[M_{13}], \qquad \psi = ArcTan[\frac{M_{12}}{M_{11}}]$$

3. 四元數微分

四元數微分，已知一四元數 $Q=Cos[\frac{\theta}{2}]+\hat{n}\cdot Sin[\frac{\theta}{2}]$，對時間微分

$$\frac{dQ}{dt}=-\frac{1}{2}Sin[\frac{\theta}{2}]\cdot\frac{d\theta}{dt}+\frac{d\hat{n}}{dt}\cdot Sin[\frac{\theta}{2}]+\hat{n}\cdot\frac{1}{2}Cos[\frac{\theta}{2}]\cdot\frac{d\theta}{dt}$$

已知 $\hat{\mathrm{n}}\cdot\hat{\mathrm{n}}=-1$，$\frac{d\hat{n}}{dt}=0$，$\frac{\mathrm{d}\theta}{\mathrm{dt}}=\omega_{\mathrm{Eb}}^{\mathrm{E}}$, E 為地理座標系, b 為飛行器坐標系

$$\frac{dQ}{dt}=\frac{1}{2}\hat{n}\cdot\omega_{Eb}^{E}\cdot(Cos\left[\frac{\theta}{2}\right]+\hat{n}\cdot Sin[\frac{\theta}{2}])=\frac{1}{2}\vec{\omega}_{Eb}^{E}\cdot Q$$

因為陀螺儀在飛行器上測到的角速度為 $\vec{\omega}_{Eb}^{b}=\omega_x\hat{i}+\omega_y\hat{j}+\omega_z\hat{k}$，故將 $\vec{\omega}_{Eb}^{E}$ 轉換成 $\vec{\omega}_{Eb}^{b}$ 會較為方便

$$\vec{\omega}_{Eb}^{b}=Q^{*}\vec{\omega}_{Eb}^{E}\cdot Q\Rightarrow Q\cdot\vec{\omega}_{Eb}^{b}=Q\cdot Q^{*}\cdot\vec{\omega}_{Eb}^{E}\cdot Q=\vec{\omega}_{Eb}^{E}\cdot Q$$
$$\Rightarrow\frac{dQ}{dt}=\frac{1}{2}\vec{\omega}_{Eb}^{E}\cdot Q=\frac{1}{2}Q\cdot\vec{\omega}_{Eb}^{b}$$

將 $\frac{\mathrm{dQ}}{\mathrm{dt}}=\frac{1}{2}\mathrm{Q}\cdot\vec{\omega}_{\mathrm{Eb}}^{\mathrm{b}}$ 展開

$$\frac{dQ}{dt}=\frac{1}{2}(q_0+q_1\hat{i}+q_2\hat{j}+q_3\hat{k})\cdot(\omega_x\hat{i}+\omega_y\hat{j}+\omega_z\hat{k})=\frac{1}{2}\begin{pmatrix}0&-\omega_x&-\omega_y&-\omega_z\\\omega_x&0&\omega_z&-\omega_y\\\omega_y&-\omega_z&0&\omega_x\\\omega_z&\omega_y&-\omega_x&0\end{pmatrix}\cdot\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix}$$

整理為

$$\frac{dQ}{dt}=\Omega_b\cdot Q$$

其中

$$\Omega_b=\frac{1}{2}\begin{pmatrix}0&-\omega_x&-\omega_y&-\omega_z\\\omega_x&0&\omega_z&-\omega_y\\\omega_y&-\omega_z&0&\omega_x\\\omega_z&\omega_y&-\omega_x&0\end{pmatrix}$$

4. 更新四元數

使用一階 Runge-Kutta 更新四元數，假設有一微分方程

$$\frac{dX}{dt} = f[X[t], \omega[t]]$$

則其解為

$$X[t + \Delta t] = X[t] + \Delta t \cdot f[X[t], \omega[t]]$$

其中 Δt 為取樣週期，將套用至四元數

$$Q[t + \Delta t] = Q[t] + \Delta t \cdot \Omega_b[t] \cdot Q[t]$$

展開上式

$$\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}_{t+\Delta t} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}_{t} + \frac{\Delta t}{2} \begin{pmatrix} -\omega_x \cdot q_1 - \omega_y \cdot q_2 - \omega_z \cdot q_3 \\ +\omega_x \cdot q_0 - \omega_y \cdot q_3 + \omega_z \cdot q_2 \\ +\omega_x \cdot q_3 + \omega_y \cdot q_0 - \omega_z \cdot q_1 \\ -\omega_x \cdot q_2 + \omega_y \cdot q_1 + \omega_z \cdot q_0 \end{pmatrix}$$

只需利用角速度即可更新四元數

5. 使用四元數融合加速度計

利用四元數將地理的重力加速度旋轉至飛行器上面,
再與加速度計讀出的值(已歸一化的)做外積, 得出誤差,
用此誤差對角速度做校正融合

重力加速度

$$\vec{g} = g\hat{z}$$

做歸一化

$$\vec{g} \to \hat{g} = \hat{z}$$

設旋轉至飛行器上的重力加速度為 $\hat{g}_b$

$$\hat{g}_b = M_q \cdot \hat{g} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$\Rightarrow \hat{g}_b = \begin{pmatrix} g_{\mathrm{bx}} \\ g_{\mathrm{by}} \\ g_{\mathrm{bz}} \end{pmatrix} = \begin{pmatrix} M_{13} \\ M_{23} \\ M_{33} \end{pmatrix}$$

飛行器上的加速度計測得的加速度 $\vec{a}_b$ 做歸一化

$$\vec{a}_b \to \hat{a}_b$$

做外積, 計算誤差 $\vec{e}$

$$\vec{e} = \hat{a}_b \times \hat{g}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_{\mathrm{bx}} & a_{\mathrm{by}} & a_{\mathrm{bz}} \\ g_{\mathrm{bx}} & g_{\mathrm{by}} & g_{\mathrm{bz}} \end{vmatrix} = \begin{vmatrix} a_{\mathrm{by}} & a_{\mathrm{bz}} \\ g_{\mathrm{by}} & g_{\mathrm{bz}} \end{vmatrix} \hat{x} + \begin{vmatrix} a_{\mathrm{bz}} & a_{\mathrm{bx}} \\ g_{\mathrm{bz}} & g_{\mathrm{bx}} \end{vmatrix} \hat{y} + \begin{vmatrix} a_{\mathrm{bx}} & a_{\mathrm{by}} \\ g_{\mathrm{bx}} & g_{\mathrm{by}} \end{vmatrix} \hat{z}$$
$$\Rightarrow \vec{e} = \begin{pmatrix} e_{\mathrm{x}} \\ e_{\mathrm{y}} \\ e_{\mathrm{z}} \end{pmatrix} = \begin{pmatrix} a_{\mathrm{by}} \cdot g_{\mathrm{bz}} - a_{\mathrm{bz}} \cdot g_{\mathrm{by}} \\ a_{\mathrm{bz}} \cdot g_{\mathrm{bx}} - a_{\mathrm{bx}} \cdot g_{\mathrm{bz}} \\ a_{\mathrm{bx}} \cdot g_{\mathrm{by}} - a_{\mathrm{by}} \cdot g_{\mathrm{bx}} \end{pmatrix}$$

將 e 融合至角速度上

參考書籍：
鄭正隆, ”慣性技術 INERTIAL TECHNOLOGY”, 崧博出版事業有限公司, 2011