Regular Meshes from Polygonal Patterns

PAPER AUTHORS:
AMIR VAXMAN,
CHRISTIAN MÜLLER,
OFIR WEBER

Motivation

- Regular Mesh Design and Generation

Method

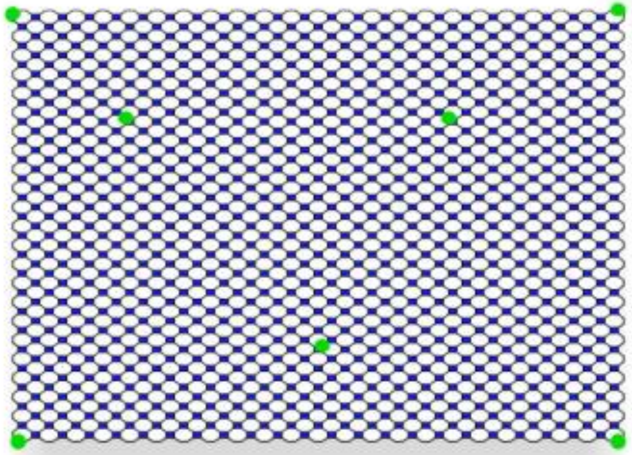
Deform a mesh to be "as regular as possible"

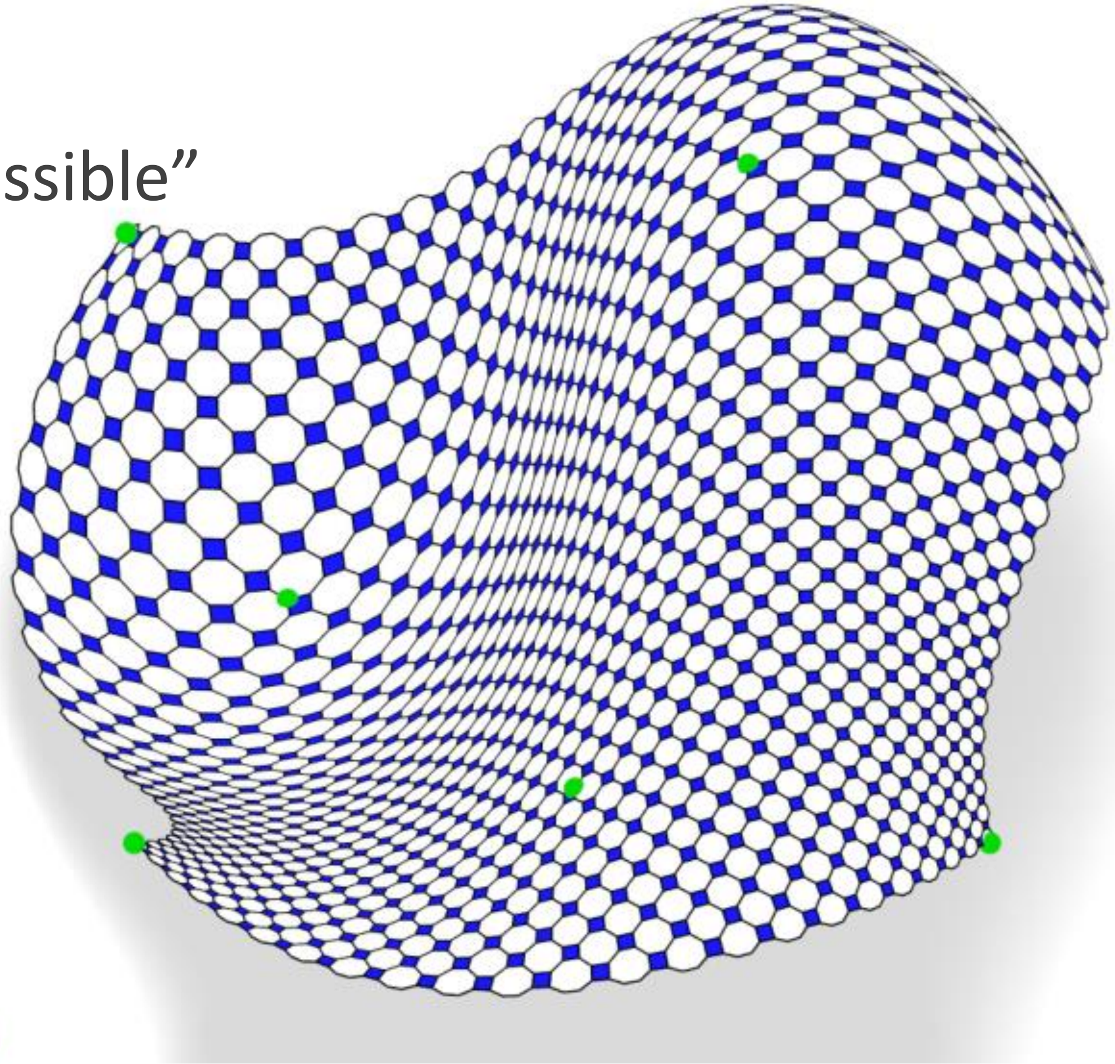
- Given perfect regular pattern
- Given imperfect regular mesh
- "Natural" boundary

Method

Deform a mesh to be "as regular as possible"

- Given perfect regular pattern
- Given imperfect regular mesh
- "Natural" boundary

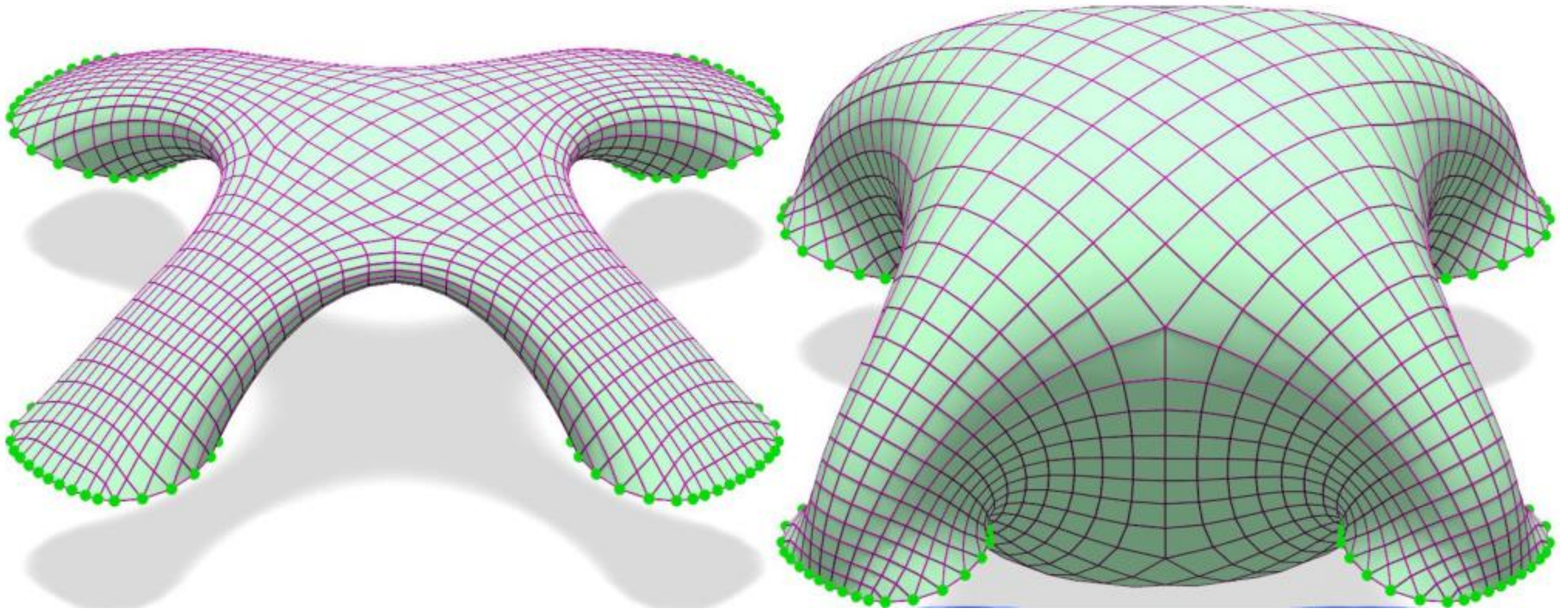




Method

Deform a mesh to be “as regular as possible”

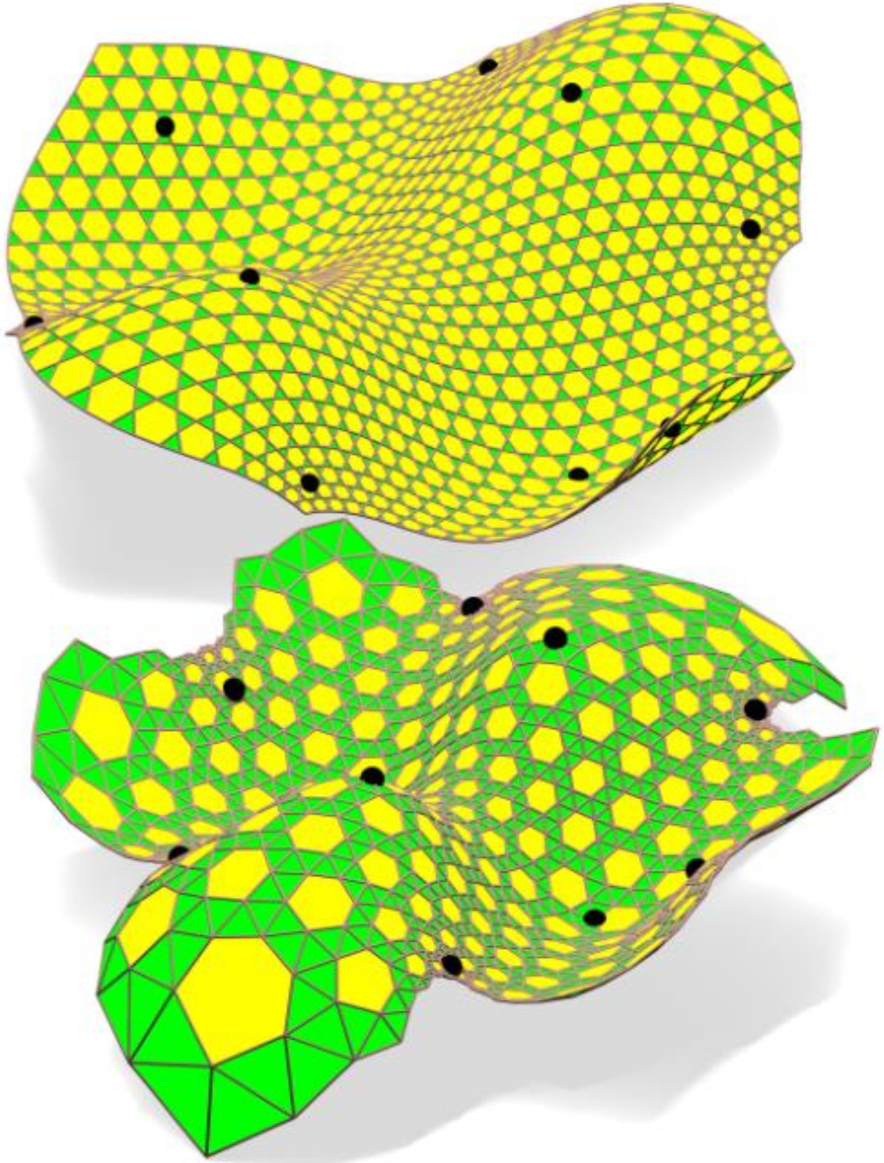
- Given perfect regular pattern
- Given imperfect regular mesh
- “Natural” boundary



Method

Deform a mesh to be "as regular as possible"

- Given perfect regular pattern
- Given imperfect regular mesh
- "Natural" boundary

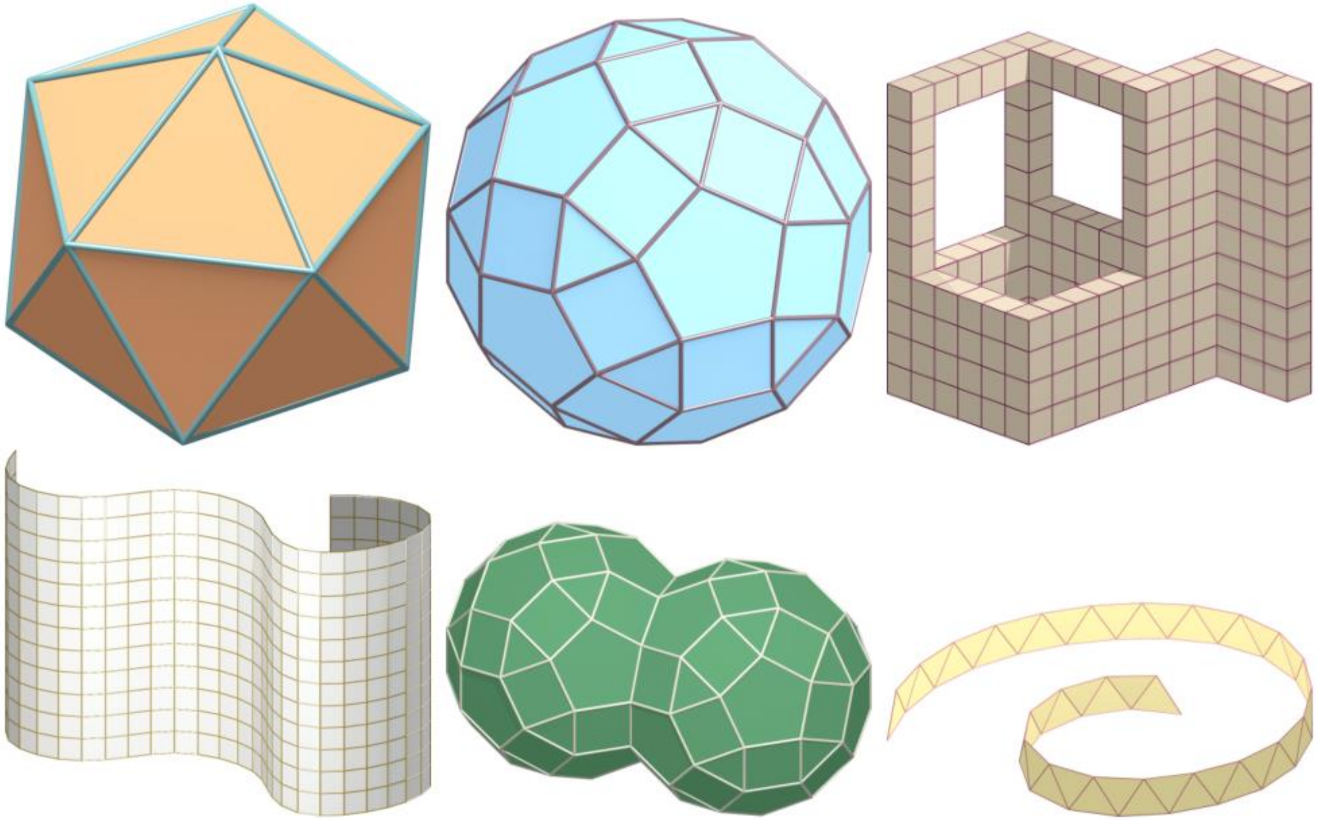


Regular Meshes

- Euclidean-regular meshes
- Möbius-regular meshes

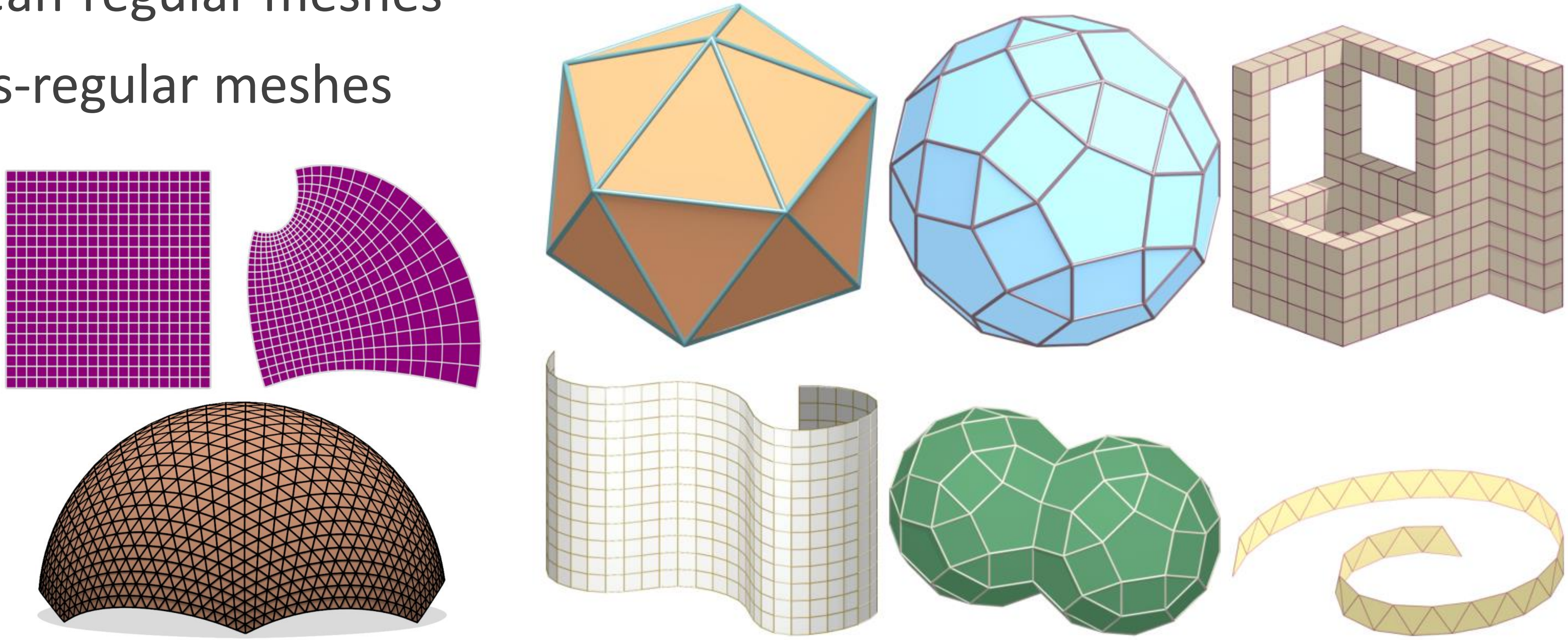
Regular Meshes

- Euclidean-regular meshes
- Möbius-regular meshes

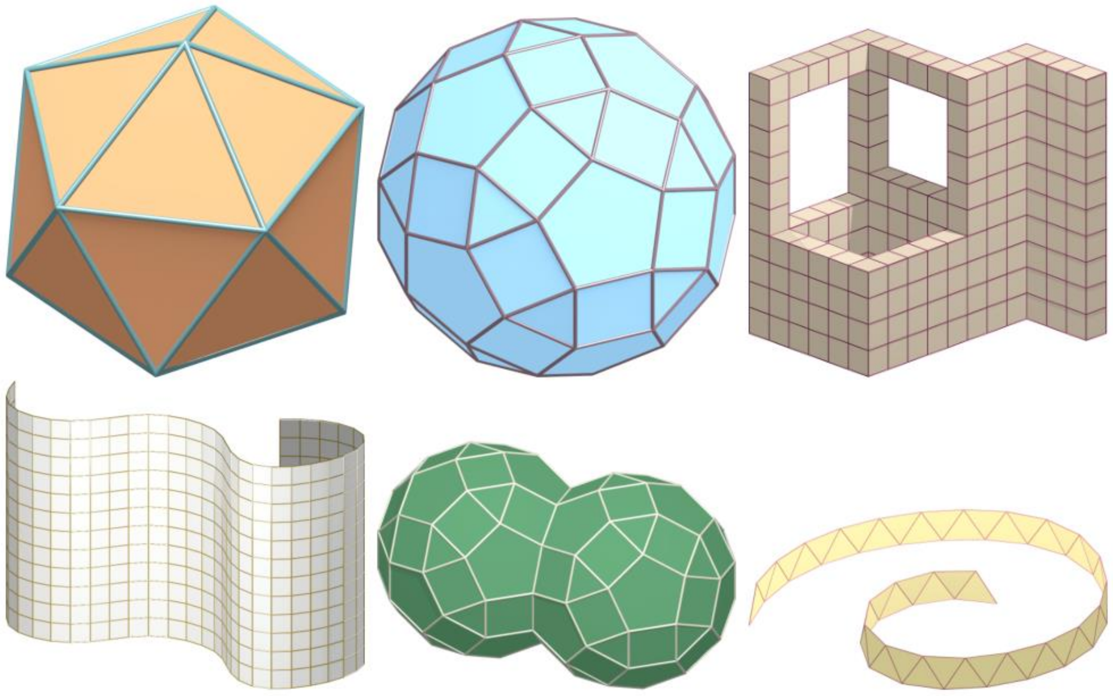


Regular Meshes

- Euclidean-regular meshes
- Möbius-regular meshes



Perfectly Euclidean-regular meshes



- Each face is rotationally symmetric
  - All edge length equivalence
  - Planarity of all vertices in the face

Quaternion Introduction

- Quaternion q
- Non commutative multiplication
- Re(q) = 0 for all vertices in 3D Mesh

Quaternion Introduction

- Quaternion q
- Non commutative multiplication
- Re(q) = 0 for all vertices in 3D Mesh

$$q = [r, x, y, z] = [r, v]$$

$$v = (x, y, z) = \mathrm{Im}(q)$$

Quaternion Introduction

- Quaternion q
- Non commutative multiplication
- Re(q) = 0 for all vertices in 3D Mesh

$$q = [r, x, y, z] = [r, v]$$

$$v = (x, y, z) = \mathrm{Im}(q)$$

$$q_1 q_2 = [r_1 r_2 - \langle v_1, v_2 \rangle, r_1 v_2 + r_2 v_1 + \underline{v_1 \times v_2}]$$

Quaternion Introduction

- Quaternion q
- Non commutative multiplication
- Re(q) = 0 for all vertices in 3D Mesh

$$q = [r, x, y, z] = [r, v]$$

$$v = (x, y, z) = \mathrm{Im}(q)$$

$$q_1 q_2 = [r_1 r_2 - \langle v_1, v_2 \rangle, r_1 v_2 + r_2 v_1 + \underline{v_1 \times v_2}]$$

$$q_1 q_2 = [-\langle \mathrm{Im}(q_1), \mathrm{Im}(q_2) \rangle, \mathrm{Im}(q_1) \times \mathrm{Im}(q_2)]$$

$$q_{ij}^{-1} = -q_{ij} / |q_{ij}|^2$$

Euclidean Regularity

- Edge representation
  - Re(q) = 0

$$q_{ij} = q_j - q_i$$

$$N[i,j,k] = q_{ij} q_{jk}^{-1}$$

- Normal ratio
  - Modulus
  - Imaginary part

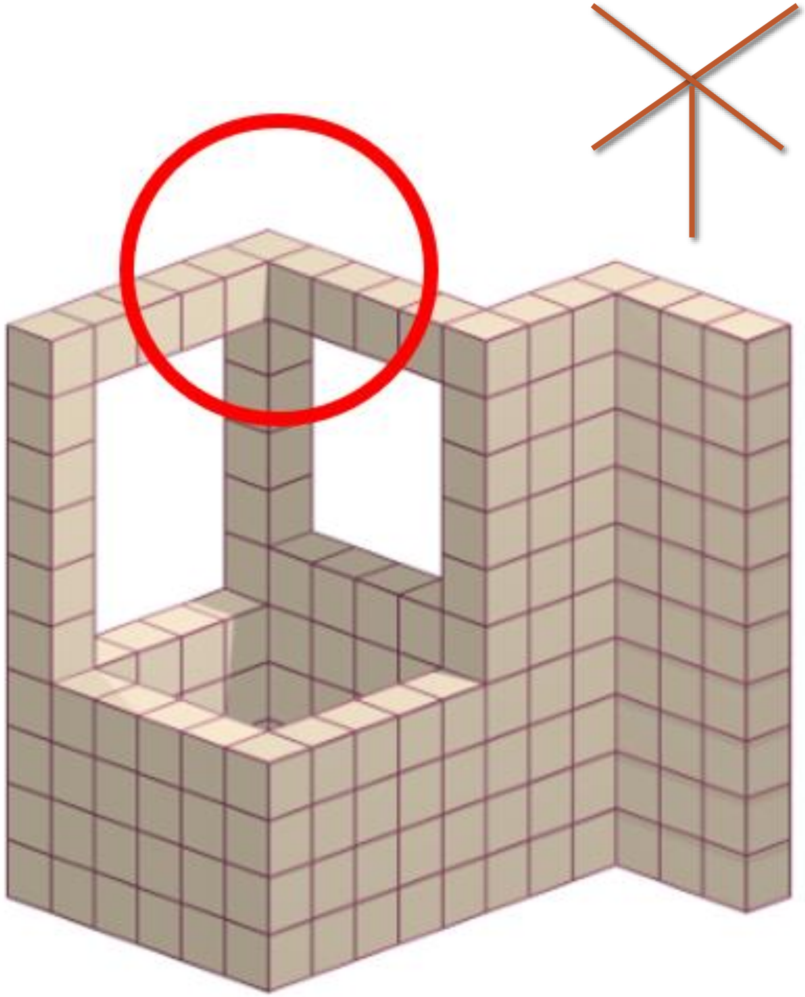
$$|q_{ij}|/|q_{jk}|$$

$$q_1 q_2^{-1} = [\langle Im(q_1), Im(q_2) \rangle, Im(q_1) \times Im(q_2)]$$

Euclidean Regularity Energy

- d is valence, $\chi_d = 2\pi/d$
- Imaginary $\boldsymbol{n}_f$ is unit normal to the face
- $\boldsymbol{n}_f, w$ are optimization variable

$$E_{\mathrm{ER}} = \sum_{f \in \mathcal{F}} \sum_{\substack{(ki),(ij) \\ \text{adjacent edges} \in f}} \left| w_{ij} w_{ki}^{-1} - [\cos(\chi_n), -\sin(\chi_n)\mathbf{n}_f] \right|^2$$



Möbius-regular meshes

Every vertex's 1-ring is individually Möbius Transformed and:

- Face: regular polygon
- Vertex 1-ring star:
  - Cospherical/planar after transform

➢ Möbius-regular ⊉ Euclidean-regular

Should we Optimize **all** vertex's individual **transform** and **Vertices**?

Möbius-regular meshes

Every vertex's 1-ring is individually Möbius Transformed and:

- Face: regular polygon
- Vertex 1-ring star:
  - Cospherical/planar after transform

➢ Möbius-regular ⊉ Euclidean-regular

Möbius-Transform

- Imaginary-preserving möbius-transform
  - $q \mapsto w = (aq + b)(cq + d)^{-1}$
  - Re(w) = 0

Transform-invariants

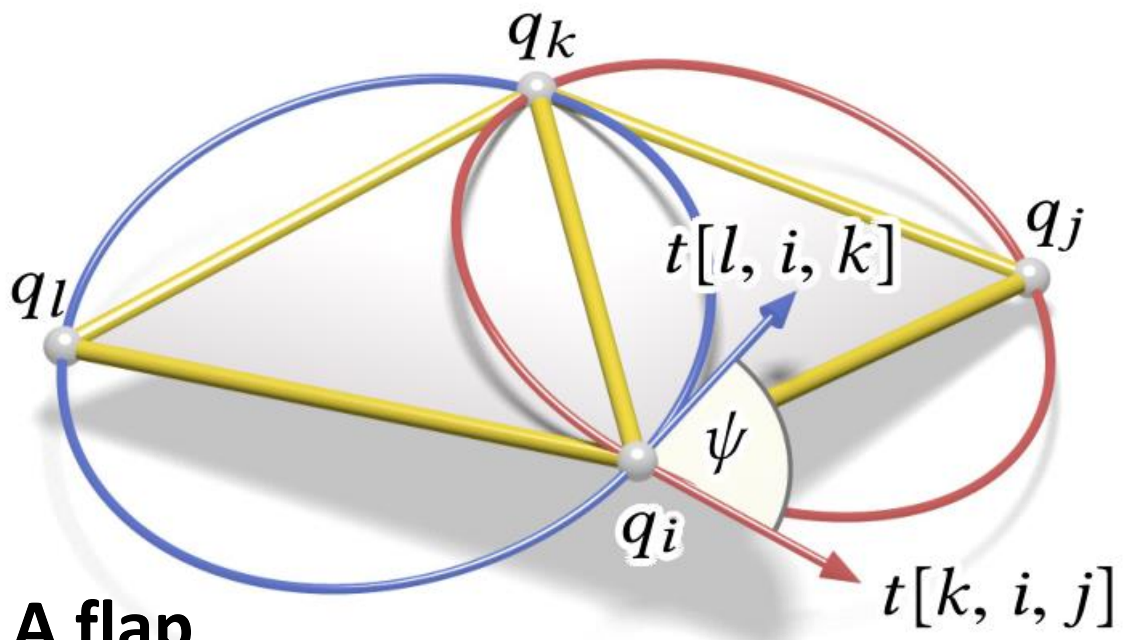
- Corner tangent $$t[k,i,j] := -q_{ki}^{-1} q_{jk} q_{ij}^{-1} = q_{ki}^{-1} + q_{ij}^{-1}$$
- Cross ratio $$\mathrm{cr}[i,j,k,l] = t[k,i,j]^{-1} \cdot t[k,i,l]$$
$$= q_{ij}\, q_{jk}^{-1}\, q_{kl}\, q_{li}^{-1}$$

Transform-invariants

- Corner tangent $$t[k,i,j] := -q_{ki}^{-1} q_{jk} q_{ij}^{-1} = q_{ki}^{-1} + q_{ij}^{-1}$$

- Cross ratio $$\mathrm{cr}[i,j,k,l] = t[k,i,j]^{-1} \cdot t[k,i,l]$$

$$= \quad q_{ij}\, q_{jk}^{-1}\, q_{kl}\, q_{li}^{-1}$$



A flap

Transform-invariants

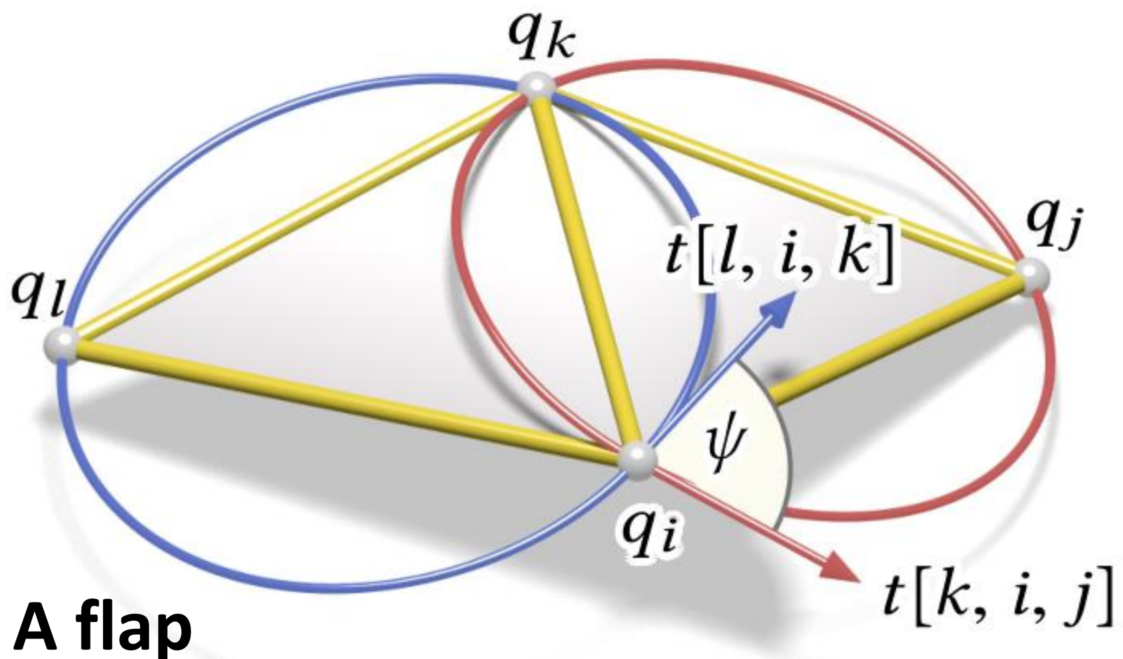
- Corner tangent

$$t[k,i,j] := -q_{ki}^{-1} q_{jk} q_{ij}^{-1} = q_{ki}^{-1} + q_{ij}^{-1}$$

- Cross ratio

$$\mathrm{cr}[i,j,k,l] = t[k,i,j]^{-1} \cdot t[k,i,l]$$

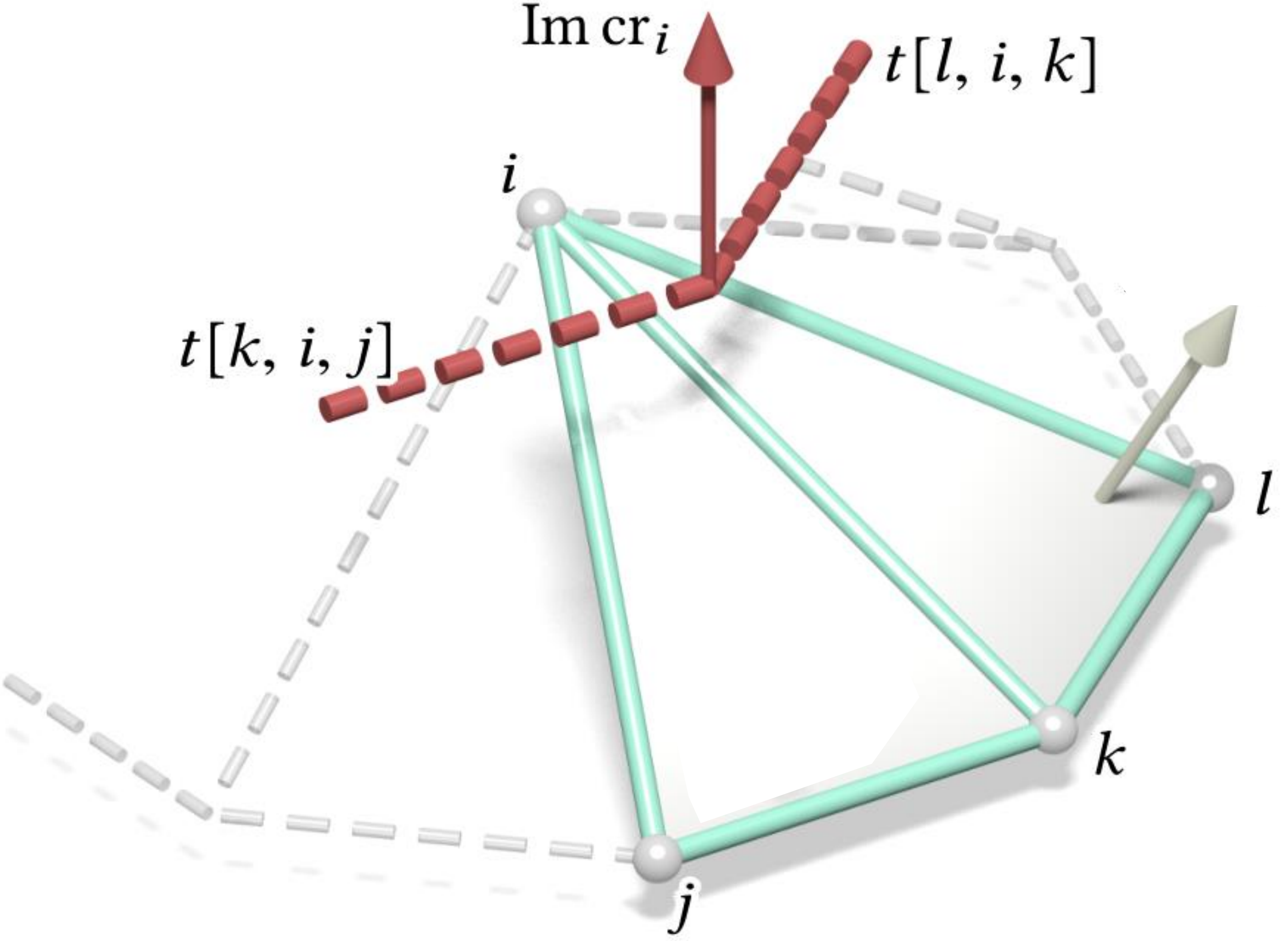
$$= q_{ij}\, q_{jk}^{-1}\, q_{kl}\, q_{li}^{-1}$$



A flap

$$q = s[cos\phi, sin\phi\mathbf{v}]$$

$$argument\ \phi[i,j,k,l] = \pi - \psi[i,j,k,l]$$



Transform-invariants

Tangent polygon

Transform-invariants

$$q \mapsto w = (aq + b)(cq + d)^{-1}$$

$$\mathrm{cr}_w[i,j,k,l] = (\overline{cq_i + d})^{-1}\, \mathrm{cr}_q[i,j,k,l]\, (\overline{cq_i + d})$$

$$t_w[k,i,j] = (cq_i + d)\, t_q[k,i,j]\, (\overline{cq_i + d})$$

Transform-invariants

$$q \mapsto w = (aq + b)(cq + d)^{-1}$$

$$\mathrm{cr}_w[i,j,k,l] = (\overline{cq_i + d})^{-1}\, \mathrm{cr}_q[i,j,k,l]\, (\overline{cq_i + d})$$

rotation: keep argument and modulus

$$t_w[k,i,j] = (cq_i + d)\, t_q[k,i,j]\, (\overline{cq_i + d})$$

Transform-invariants

$$q \mapsto w = (aq + b)(cq + d)^{-1}$$

$$\mathrm{cr}_w[i,j,k,l] = (\overline{cq_i + d})^{-1}\, \mathrm{cr}_q[i,j,k,l]\, (\overline{cq_i + d})$$

rotation: keep argument and modulus

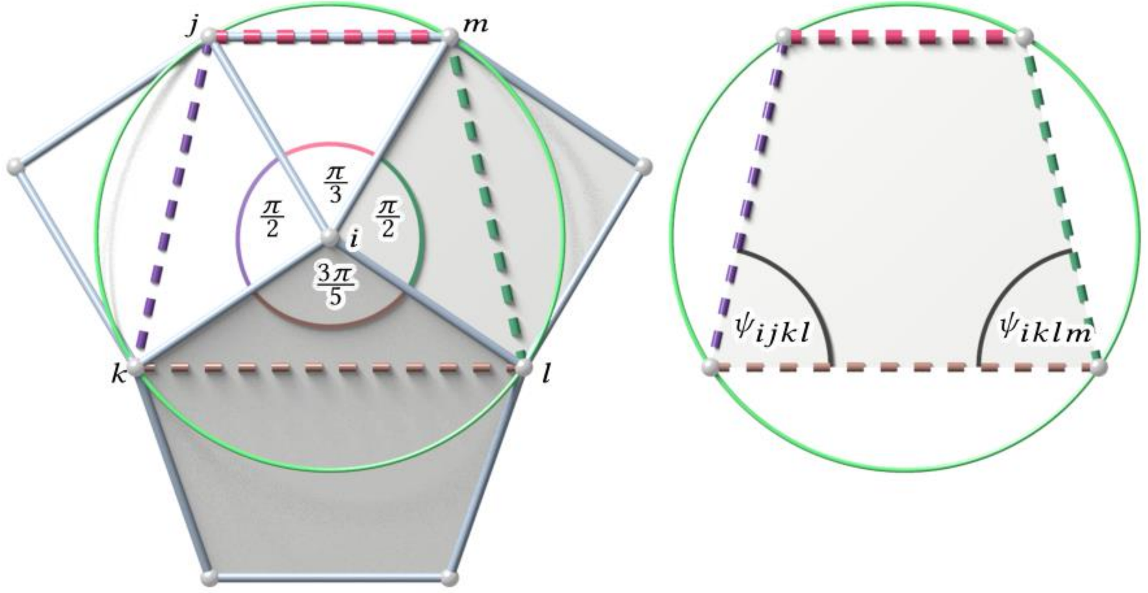
$$t_w[k,i,j] = (cq_i + d)\, t_q[k,i,j]\, (\overline{cq_i + d})$$

similarity

Möbius regularity (1)

- Every facet is regular polygon
- For u_i in facet

$$\mathrm{cr}[u_i, u_{i+1}, u_{i+2}, u_{i+3}] = [-(1 + 2\cos(2\pi/d))^{-1}, 0, 0, 0]$$



Möbius regularity (2)

- Vertex 1-ring star:
  - Cospherical/planar after transform
- Given regularity(1)
  - Tangent polygon = boundary polygon
  - Both of them are concyclic
- Get ideal tangent polygon according to face valence

Möbius Regularity Energy

- Optimize w and $\mathbf{n}_i$
- ϕ is argument of ideal cross ratio
- l_{ijkl} is modulus of ideal cross ratio

$$E_{\mathrm{MR}} = \sum_{f \in \mathcal{F}} \sum_{p=1}^{d} \left| \mathrm{cr}[w_p^f, w_{p+1}^f, w_{p+2}^f, w_{p+3}^f] - [\frac{-1}{(1+2\cos(2\pi/d)}, \mathbf{0}] \right|^2$$

$$+ \sum_{w_i \in \mathcal{V}} \sum_{\mathrm{flap}\,(ijkl)} \left| \mathrm{cr}[w_i, w_j, w_k, w_l] - l_{ijkl}[\cos\phi_{ijkl}, \sin\phi_{ijkl}\mathbf{n}_i] \right|^2$$

Other tricks

- $E_R = \lambda_{\mathrm{MR}} E_{\mathrm{MR}} + \lambda_{\mathrm{ER}} E_{\mathrm{ER}}$

Other tricks

- $E_R = \lambda_{\mathrm{MR}} E_{\mathrm{MR}} + \lambda_{\mathrm{ER}} E_{\mathrm{ER}}$
- Position constrains

Other tricks

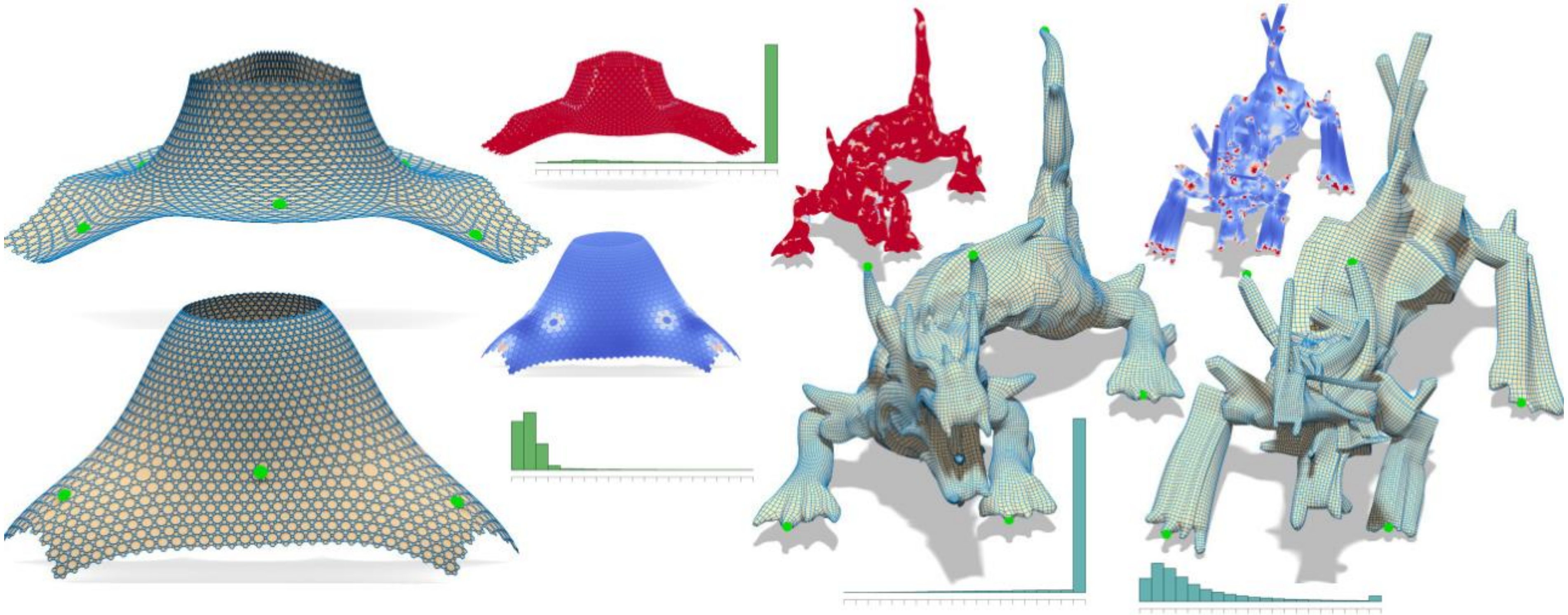
- $E_R = \lambda_{\mathrm{MR}} E_{\mathrm{MR}} + \lambda_{\mathrm{ER}} E_{\mathrm{ER}}$
- Position constrains
- For Möbius regularity
  - ambiguity of Möbius transform
  - $\lambda_{\mathrm{MR}} = 1$ and $\lambda_{\mathrm{ER}} = 0.01$

Other tricks

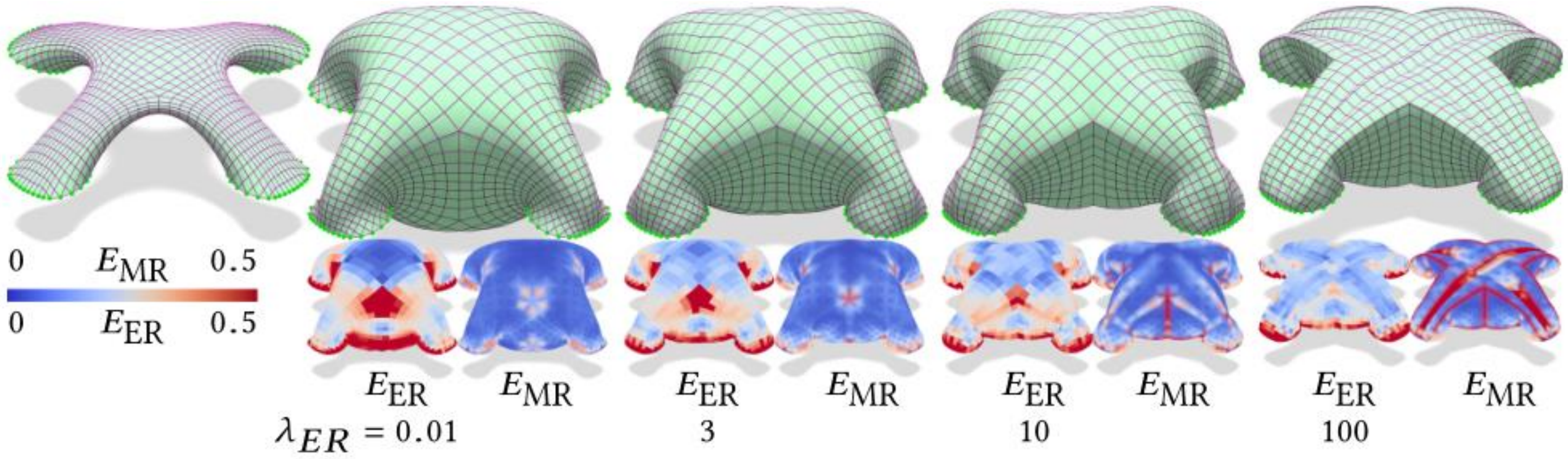
- $E_R = \lambda_{\mathrm{MR}} E_{\mathrm{MR}} + \lambda_{\mathrm{ER}} E_{\mathrm{ER}}$
- Position constrains
- For Möbius regularity
  - ambiguity of Möbius transform
  - $\lambda_{\mathrm{MR}} = 1$ and $\lambda_{\mathrm{ER}} = 0.01$
- E_{ER} *for initial* $\mathbf{n}_f$ *or* $\mathbf{n}_r$

Other tricks

- $E_R = \lambda_{\mathrm{MR}} E_{\mathrm{MR}} + \lambda_{\mathrm{ER}} E_{\mathrm{ER}}$
- Position constrains
- For Möbius regularity
  - ambiguity of Möbius transform
  - $\lambda_{\mathrm{MR}} = 1$ and $\lambda_{\mathrm{ER}} = 0.01$
- E_{ER} *for initial* $\mathbf{n}_f$ *or* $\mathbf{n}_r$
- Levenberg-Marquadt algorithm for nonlinear optimization
  - Google Ceres solver



Result


0 E_{MR} 0.5
0 E_{ER} 0.5
E_{ER} E_{MR} E_{ER} E_{MR} E_{ER} E_{MR} E_{ER} E_{MR}
$\lambda_{ER} = 0.01$ 3 10 100

Result

Local-global based approach

$$E'_R = \lambda_{\mathrm{MR}} \sum_{i=1}^{|\mathcal{V}|+|\mathcal{F}|} |A_i W_i - P_{\mathrm{MR}}(A_i W_i)|^2$$

$$+ \lambda_{\mathrm{ER}} \sum_{i=1}^{|\mathcal{F}|} |A_i W_i - P_{\mathrm{ER}}(A_i W_i)|^2,$$

- W is a subset of vertex positions participating in a projection
- A is averaging operator
- P is projection operator

Euclidean Regularity Projection

The closest regular polygon

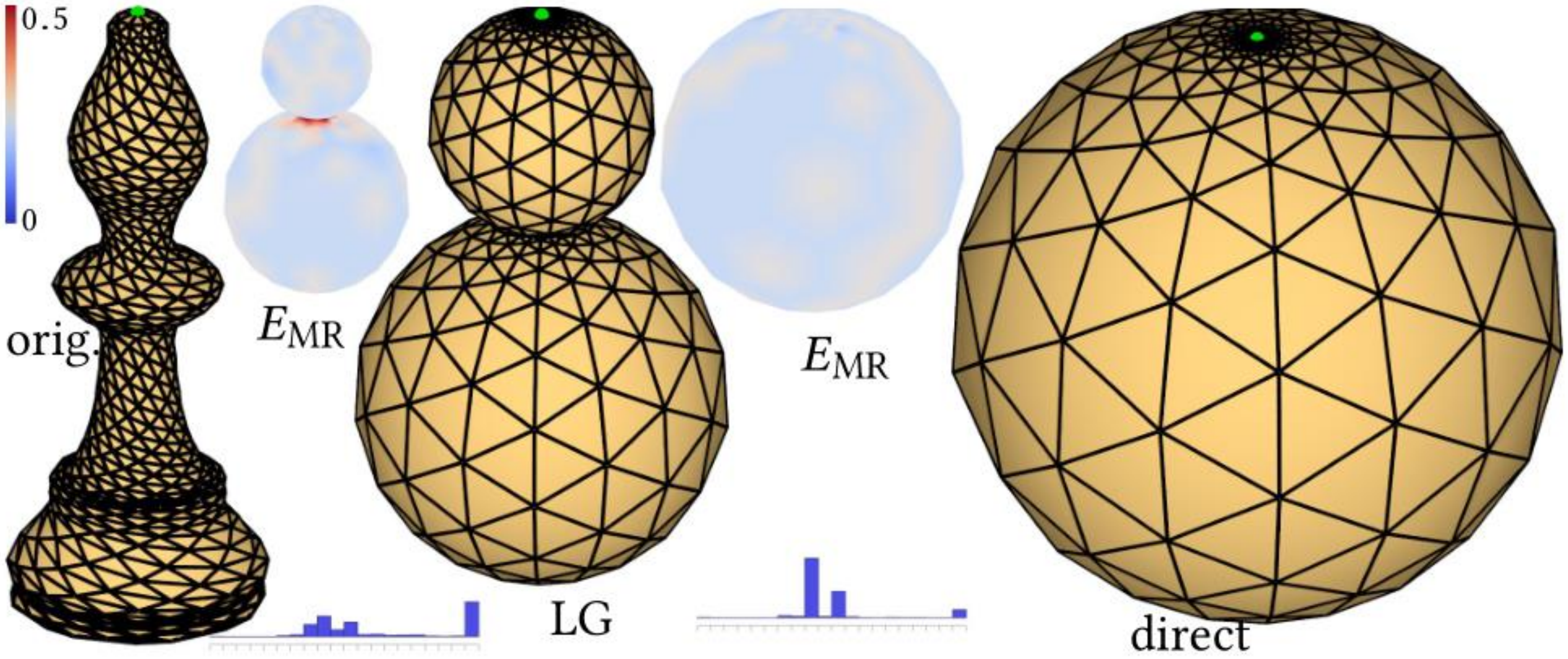
Möbius Regularity Projection

For facets

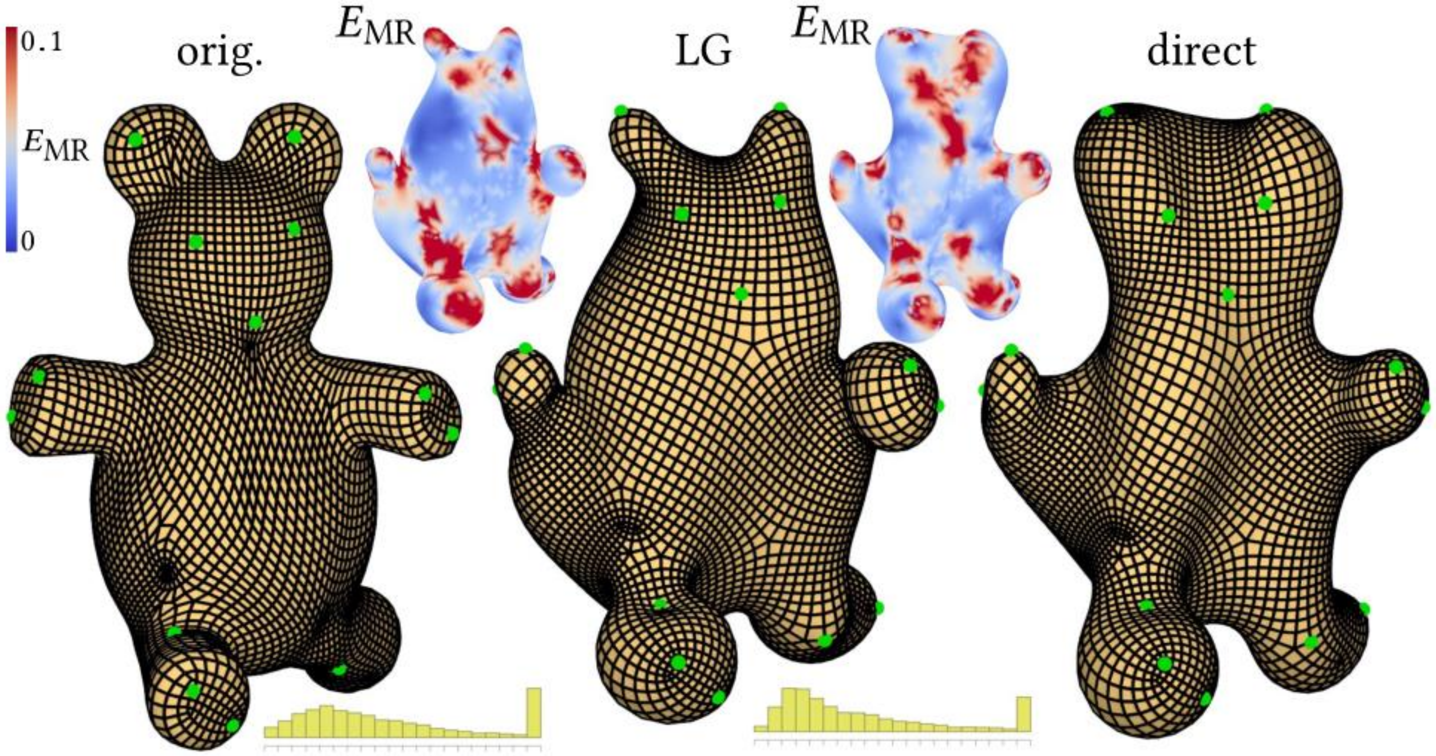
- Find rotation
- Find Möbius transform
- Compose their inverse

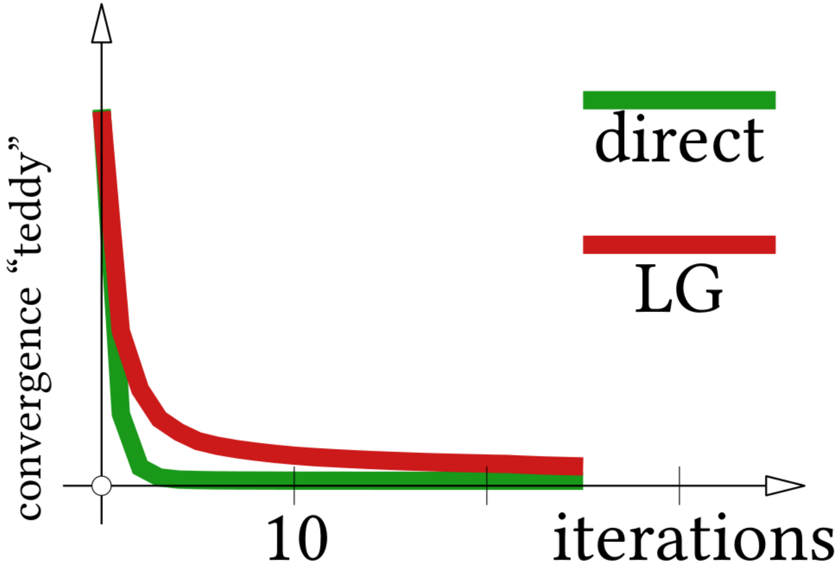
For vertex star

- Find canonical embedding
- Find similarity
- Find Möbius transform, this is the projection



Comparision





Comparision

Thanks