

CS159 Lecture 6: Planning under uncertainty and uncertain LMPC

Ugo Rosolia

Caltech

Spring 2021

Today's Class

- ▶ Planning under uncertainty
- ▶ Uncertain LMPC
- ▶ Discussion
- ▶ Key takeaways

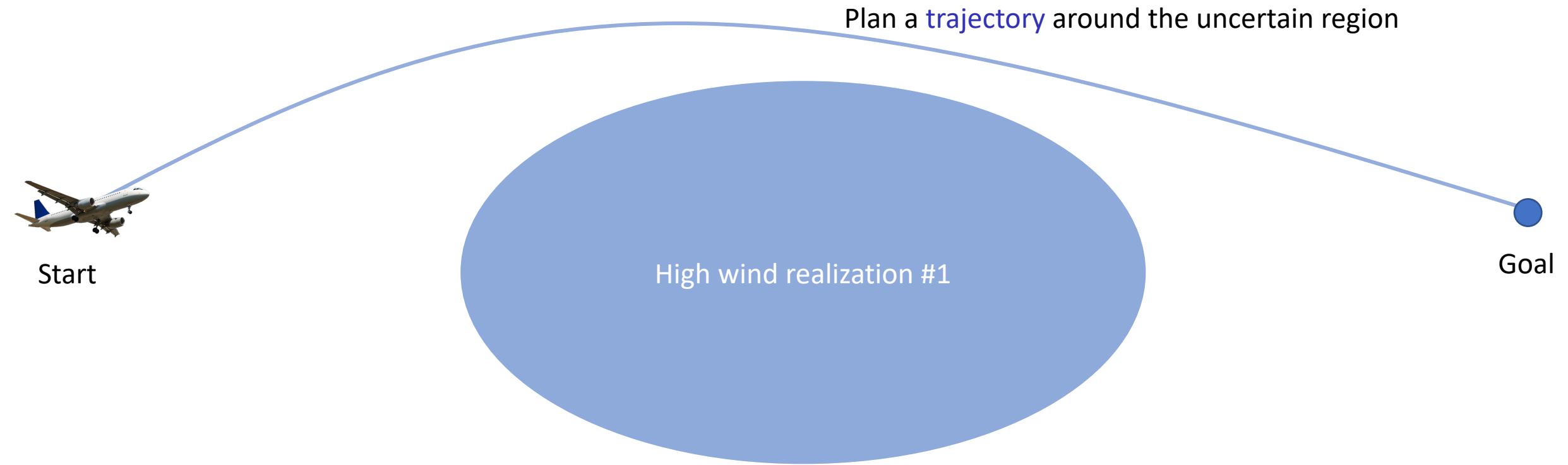
Planning under uncertainty

Optimizing over control policies

Why is planning in uncertain environments harder?



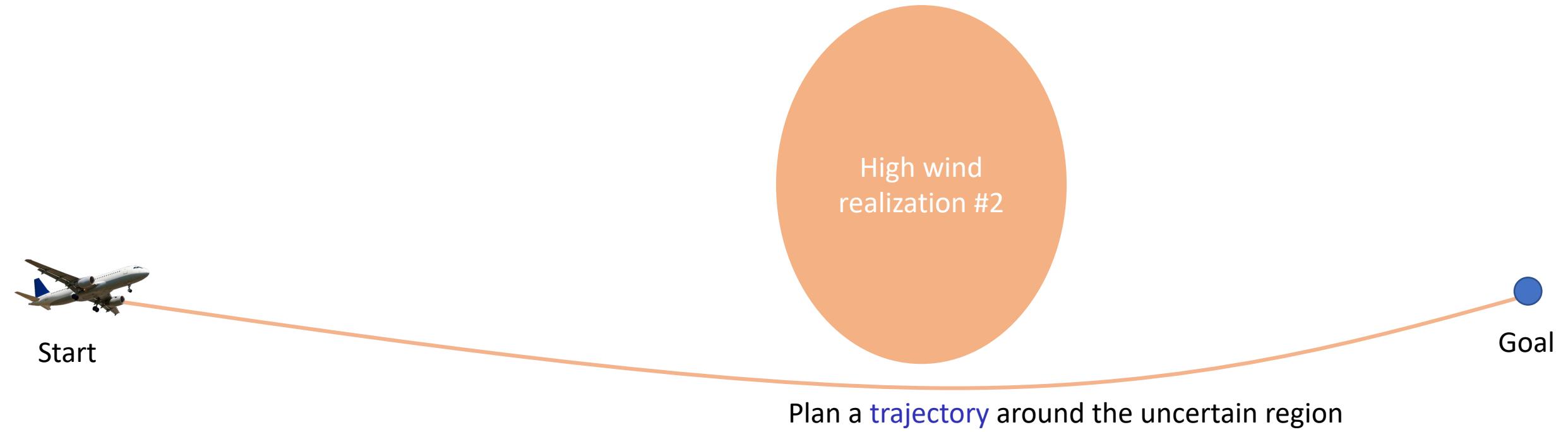
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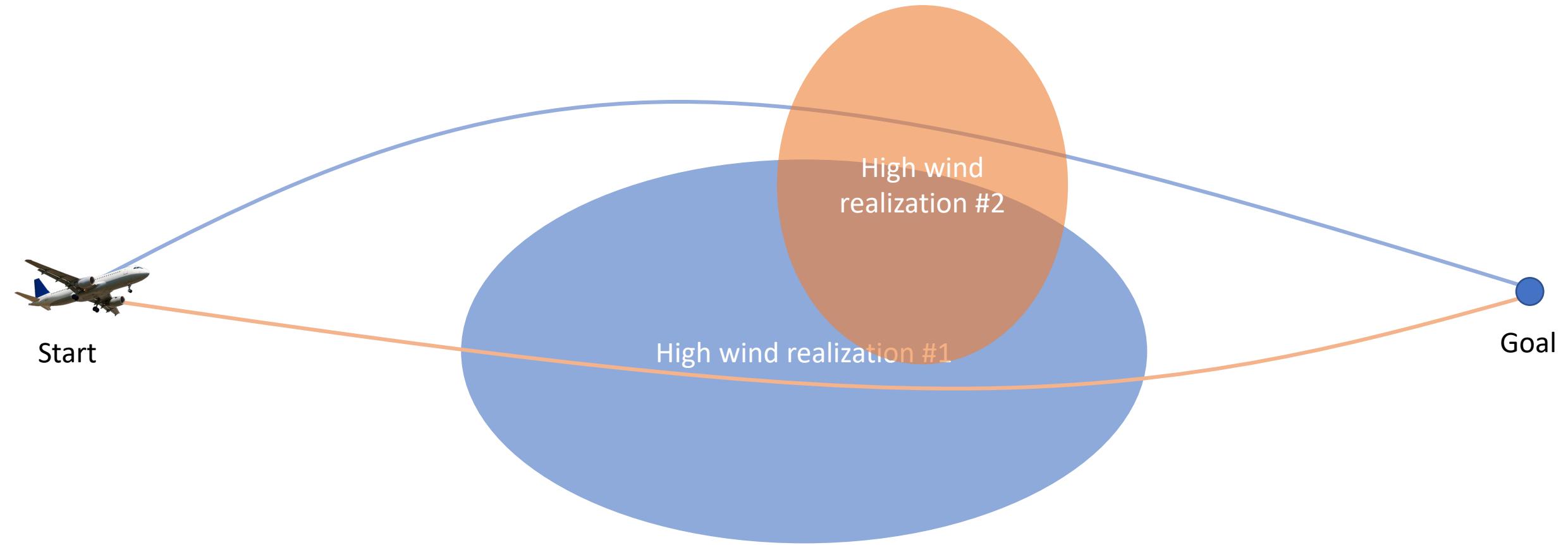
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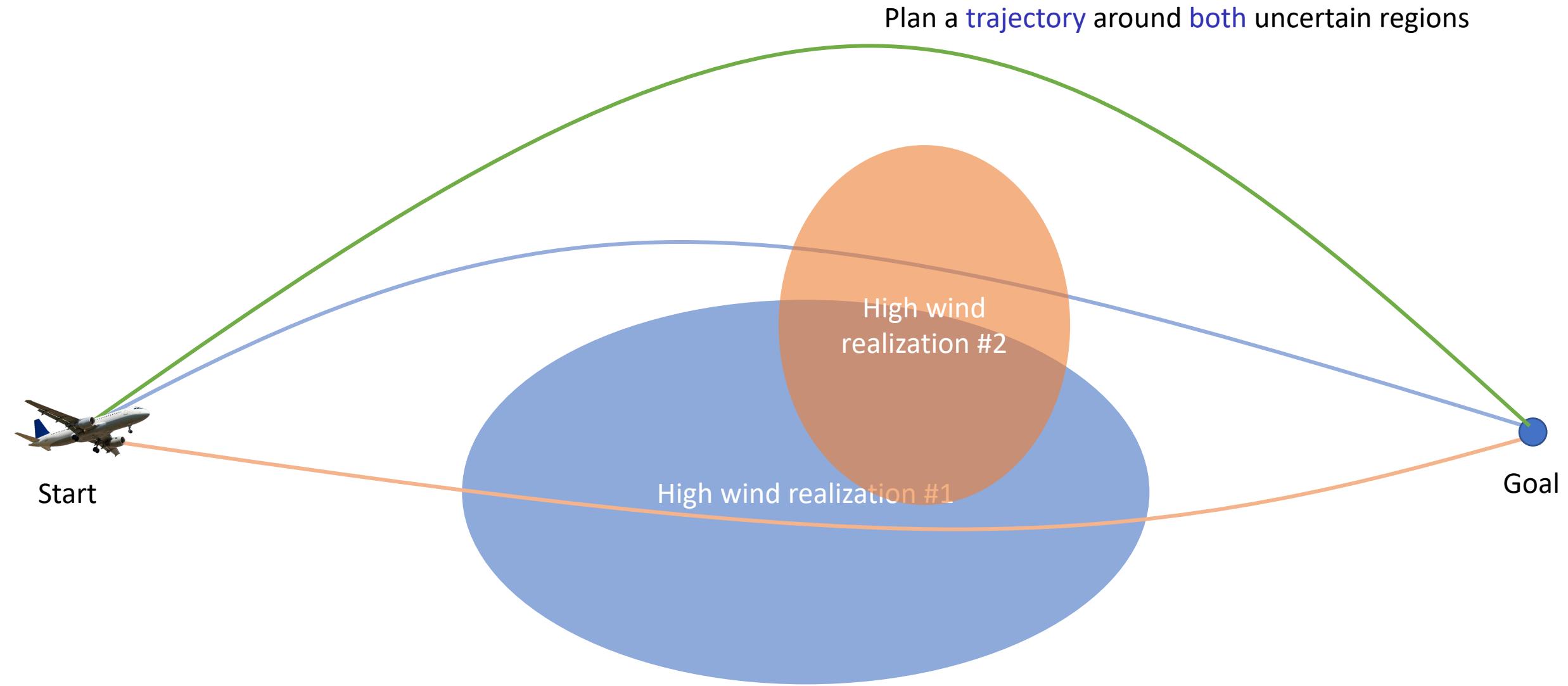
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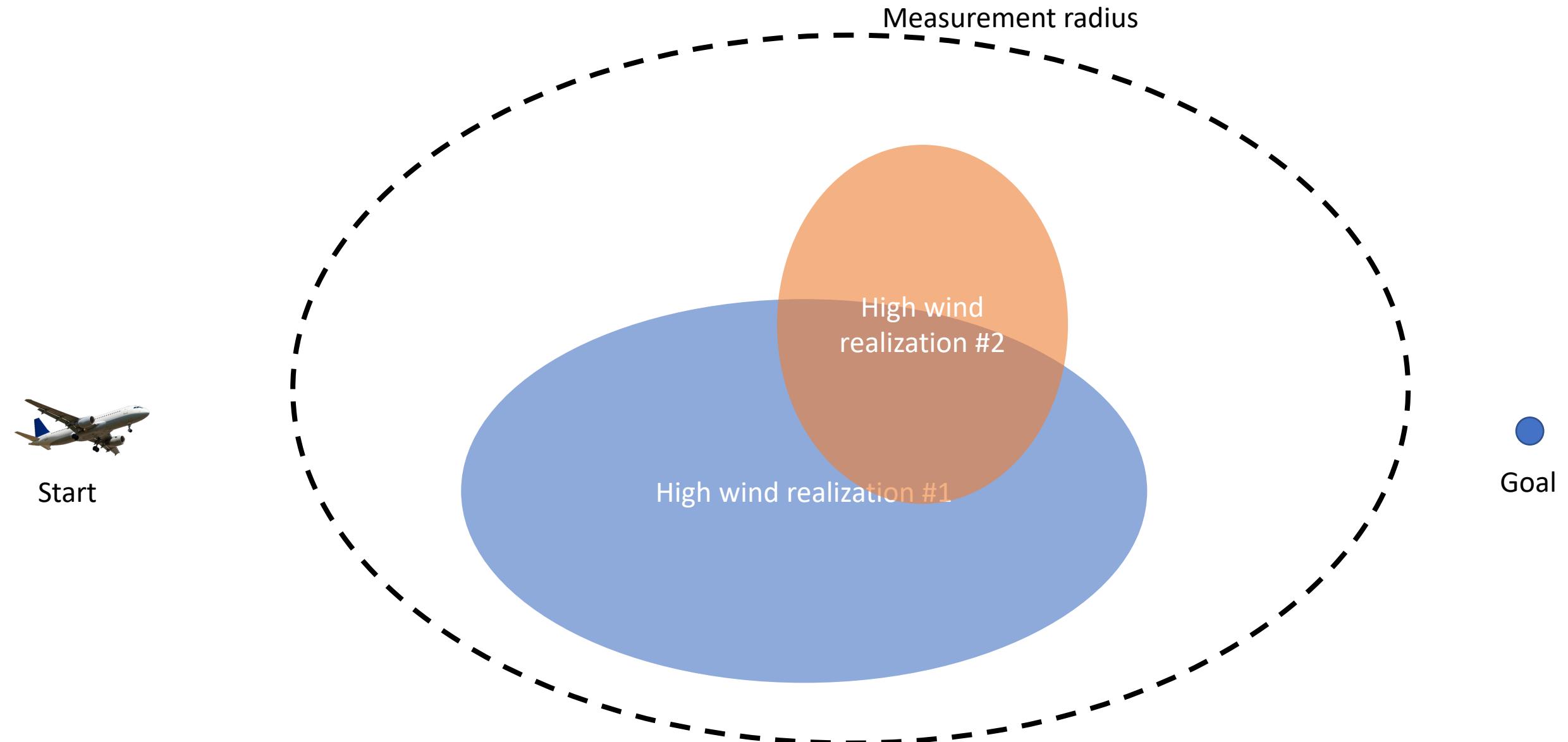
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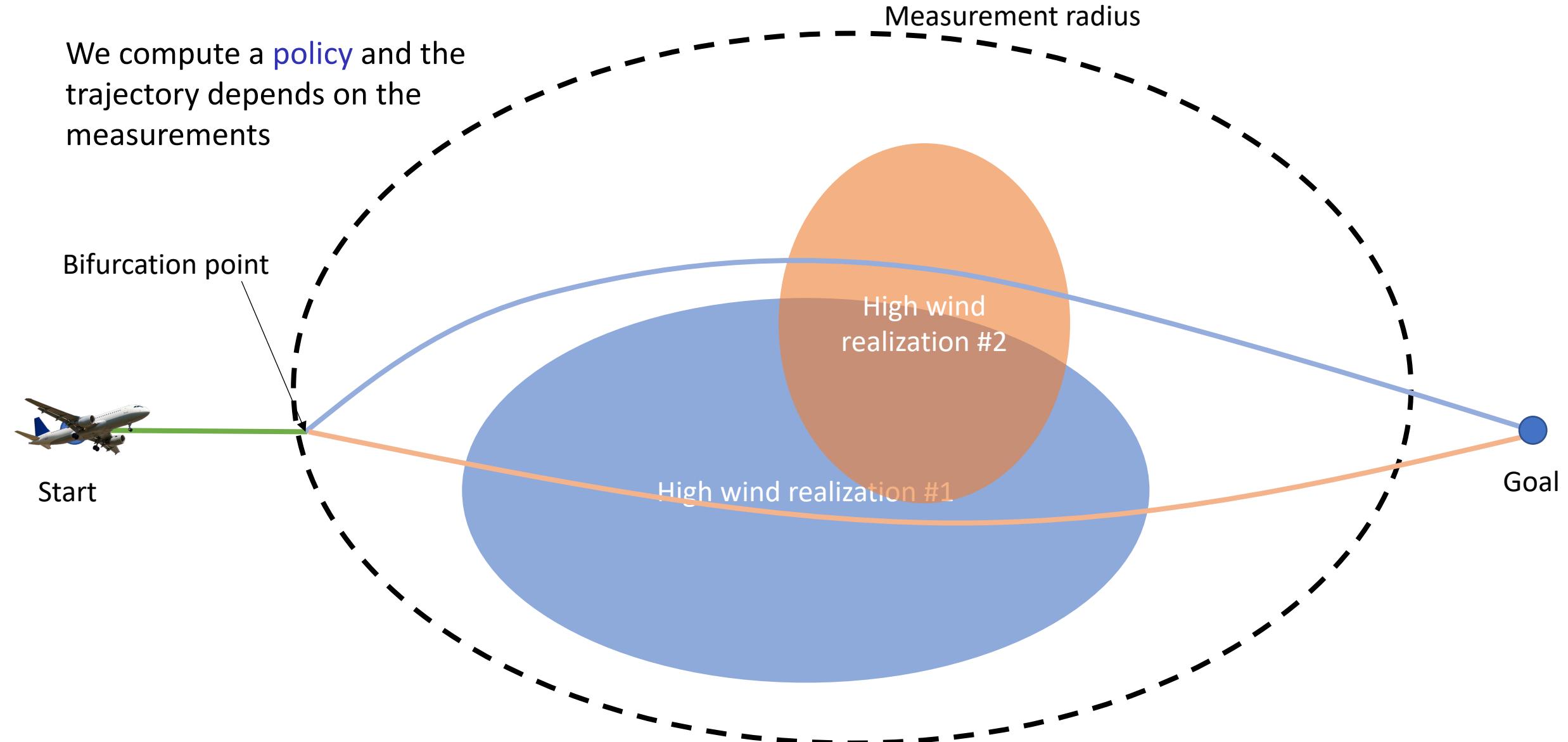


Why is planning in uncertain environments harder?



Why is planning in uncertain environments harder?

We compute a **policy** and the trajectory depends on the measurements



Problem Formulation

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

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Robust constraints:

Given a control policy $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^d$ designed using the nominal model, we want to guarantee that the closed-loop system

$$x_{k+1} = Ax_k + B\pi(x_k) + w_k$$

satisfies $x_k \in \mathcal{X}, \pi(x_k) \in \mathcal{U}$ for all $w_k \in \mathcal{W}$.

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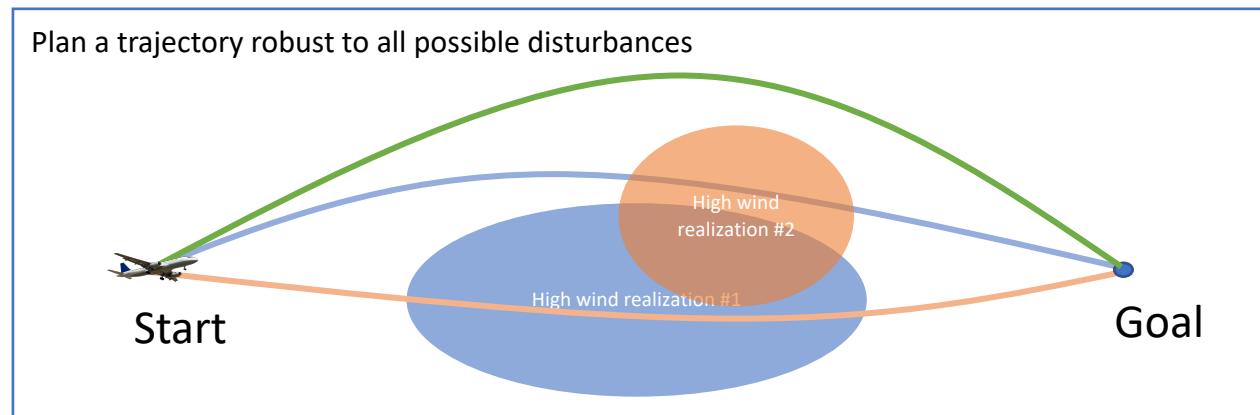
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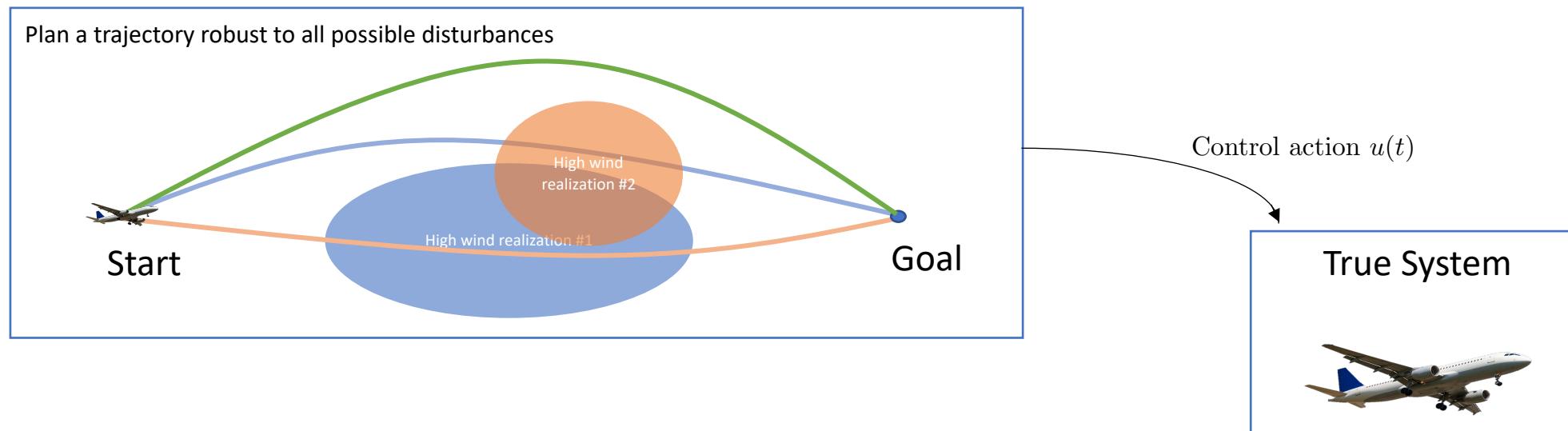
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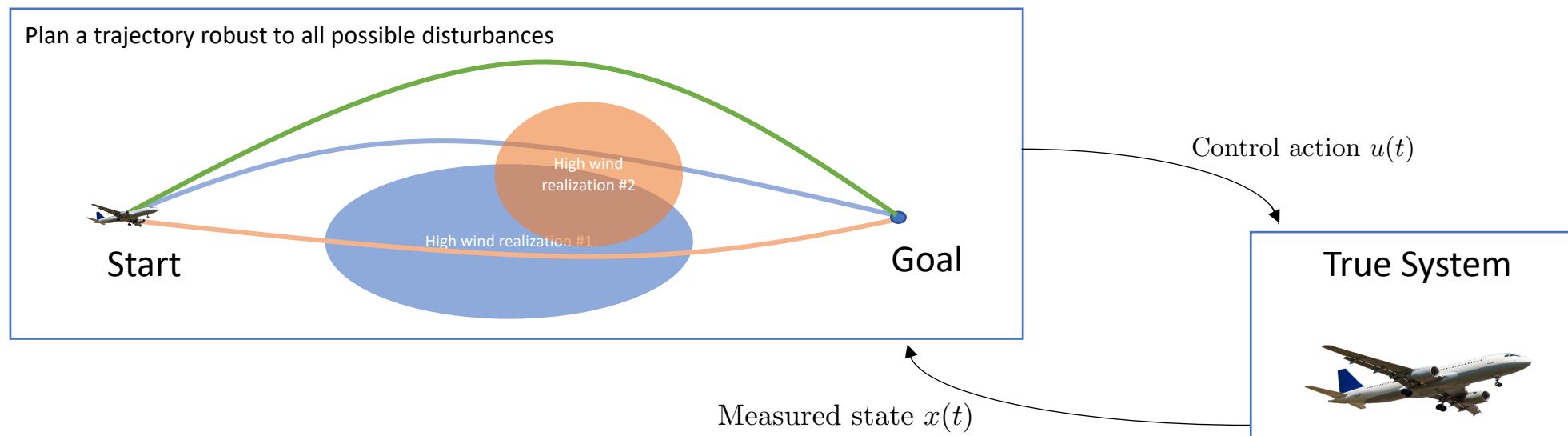
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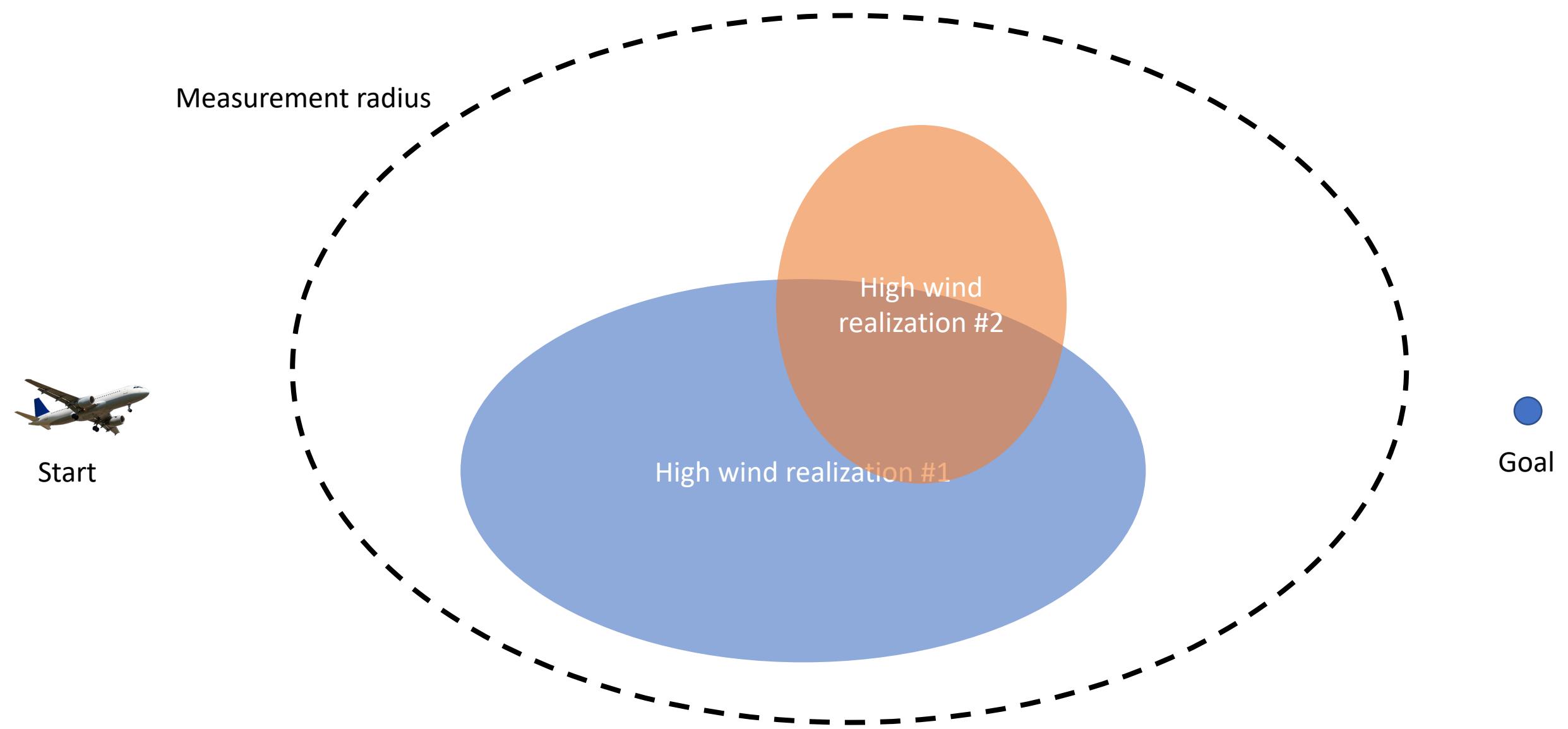
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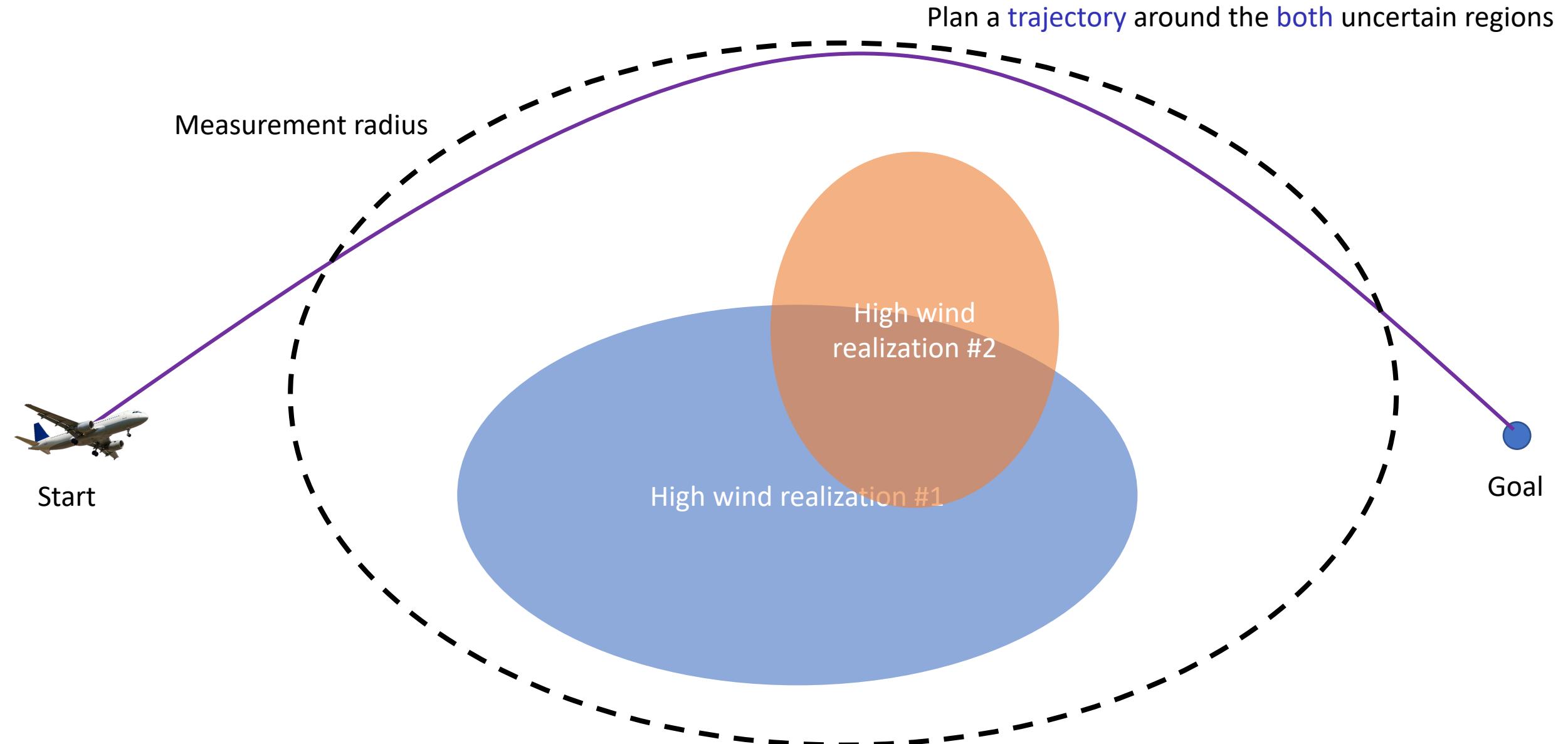
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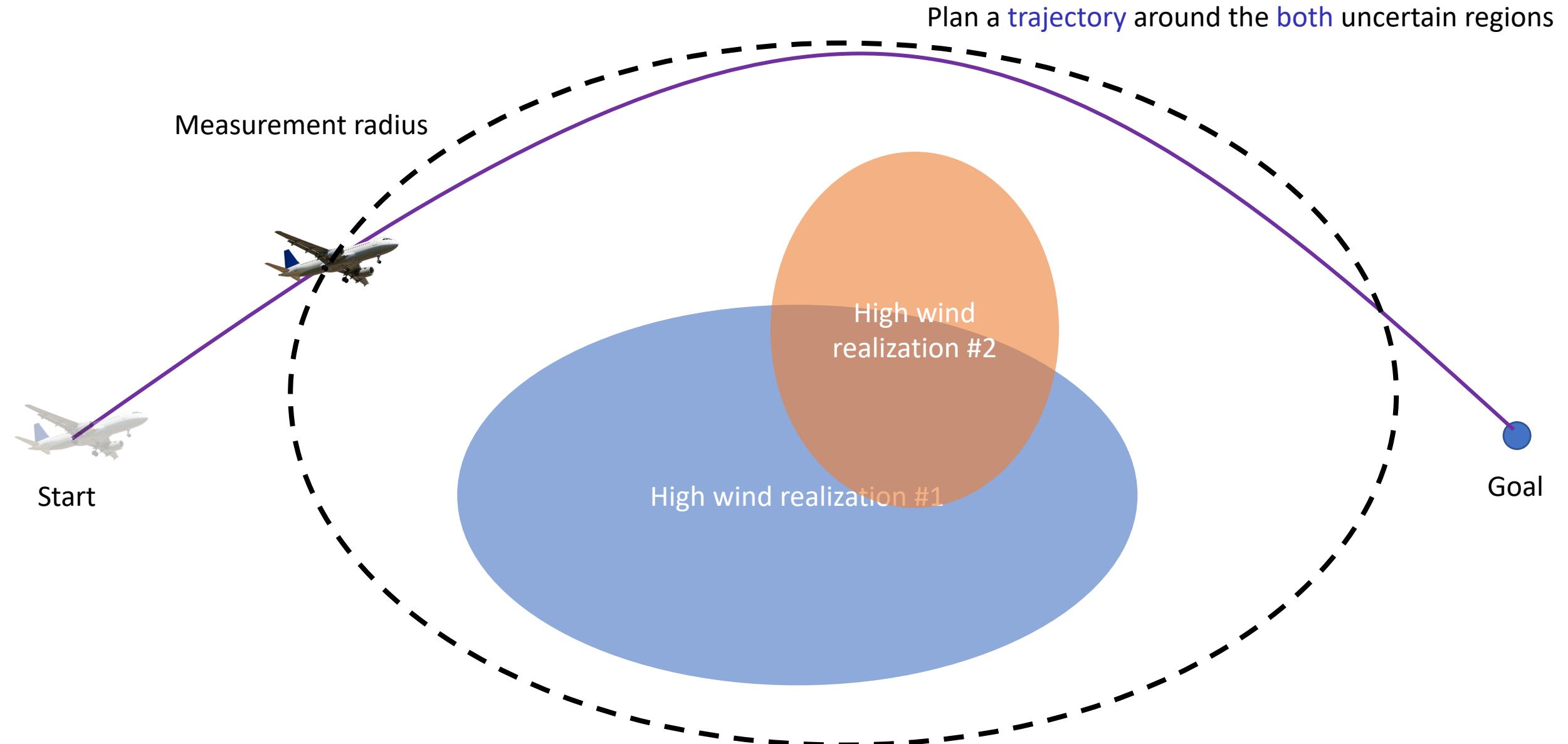
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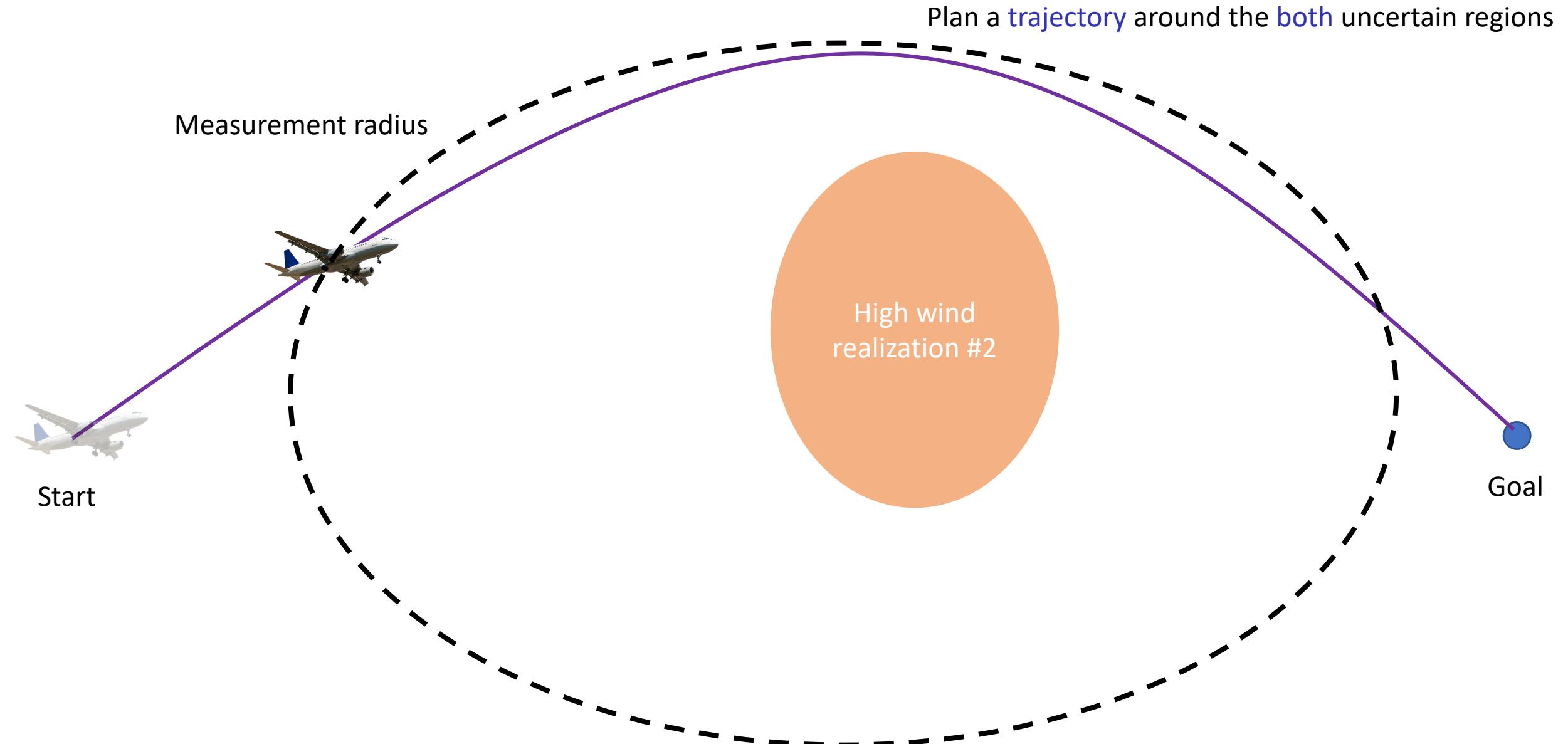
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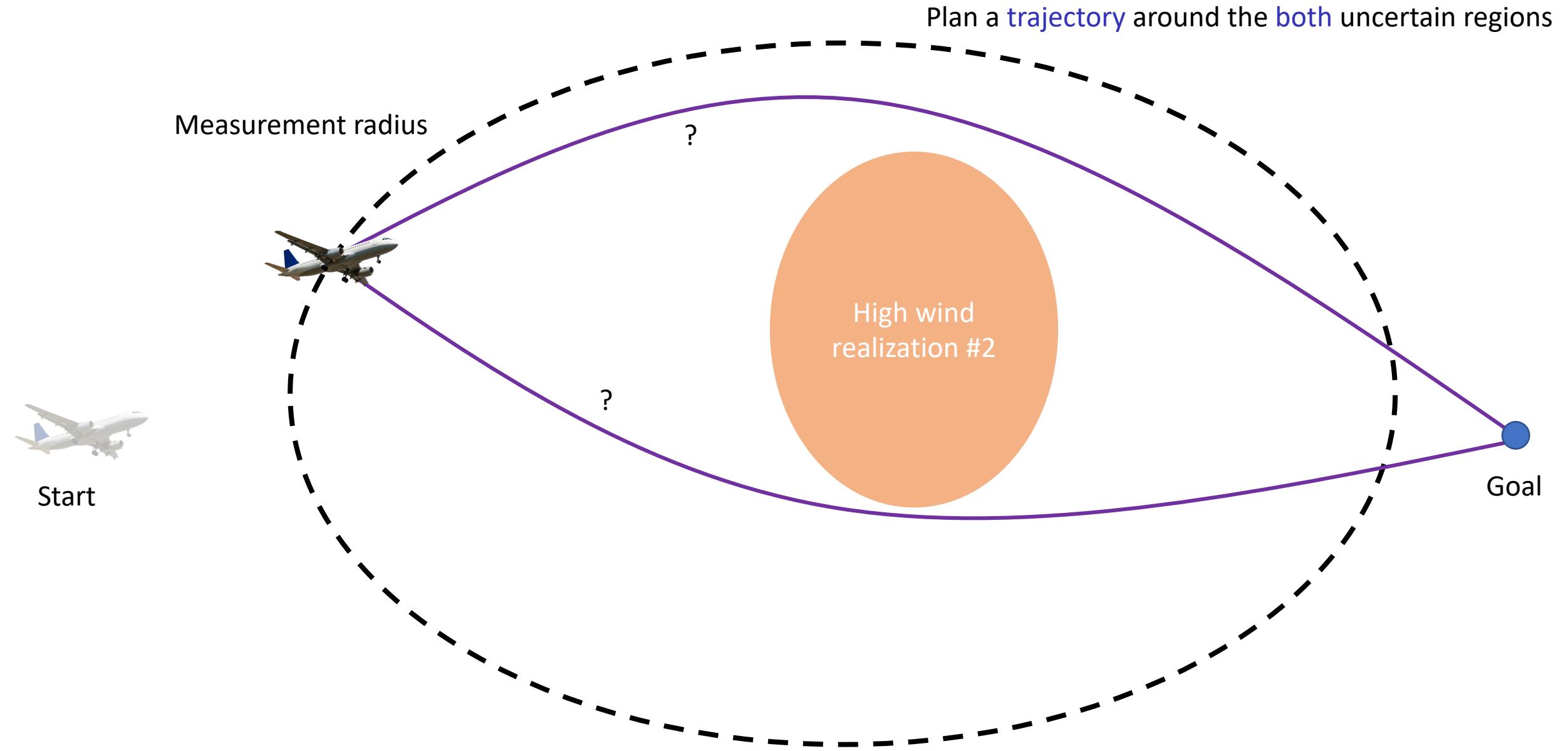
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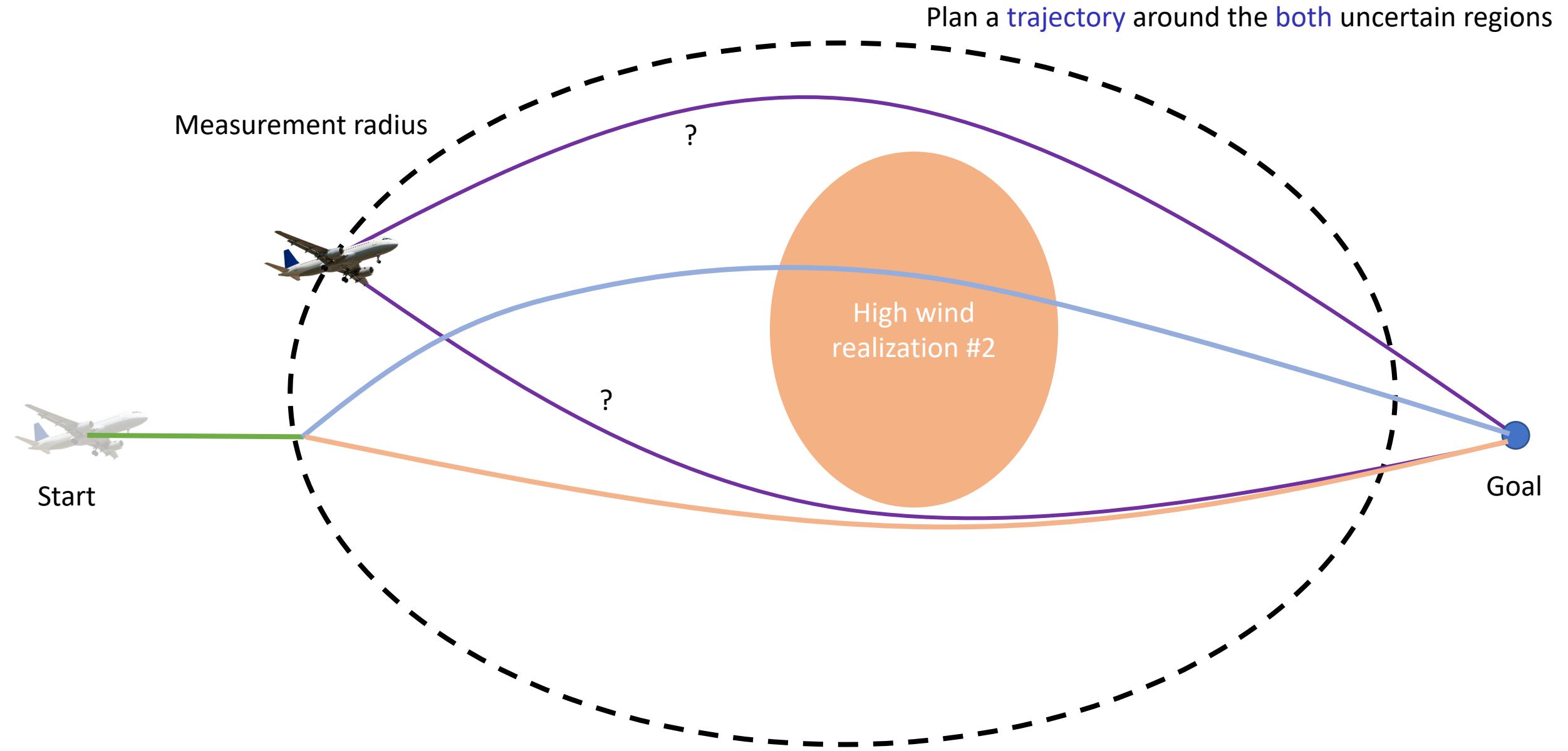
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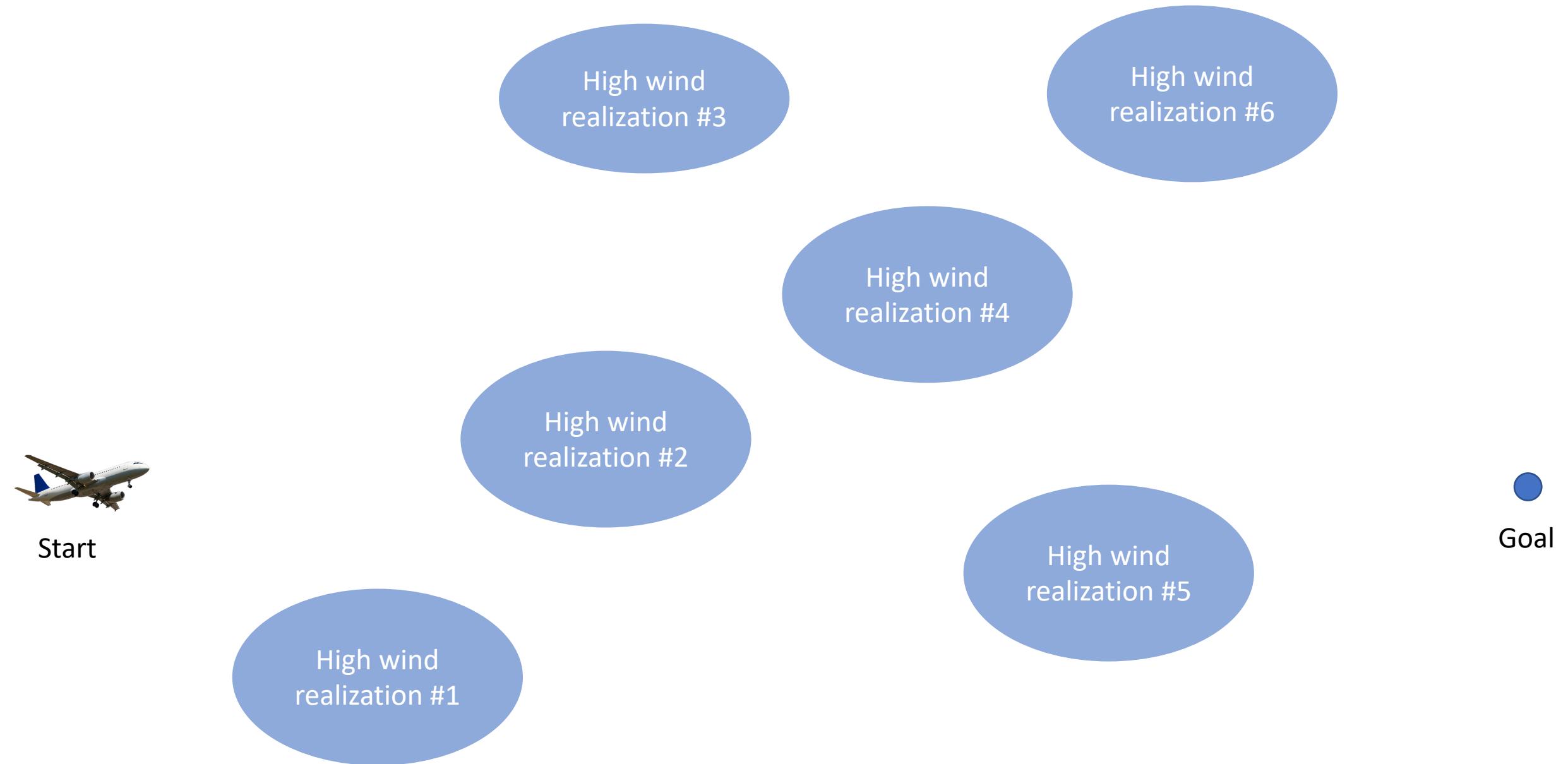
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The horizon dilemma



The horizon dilemma

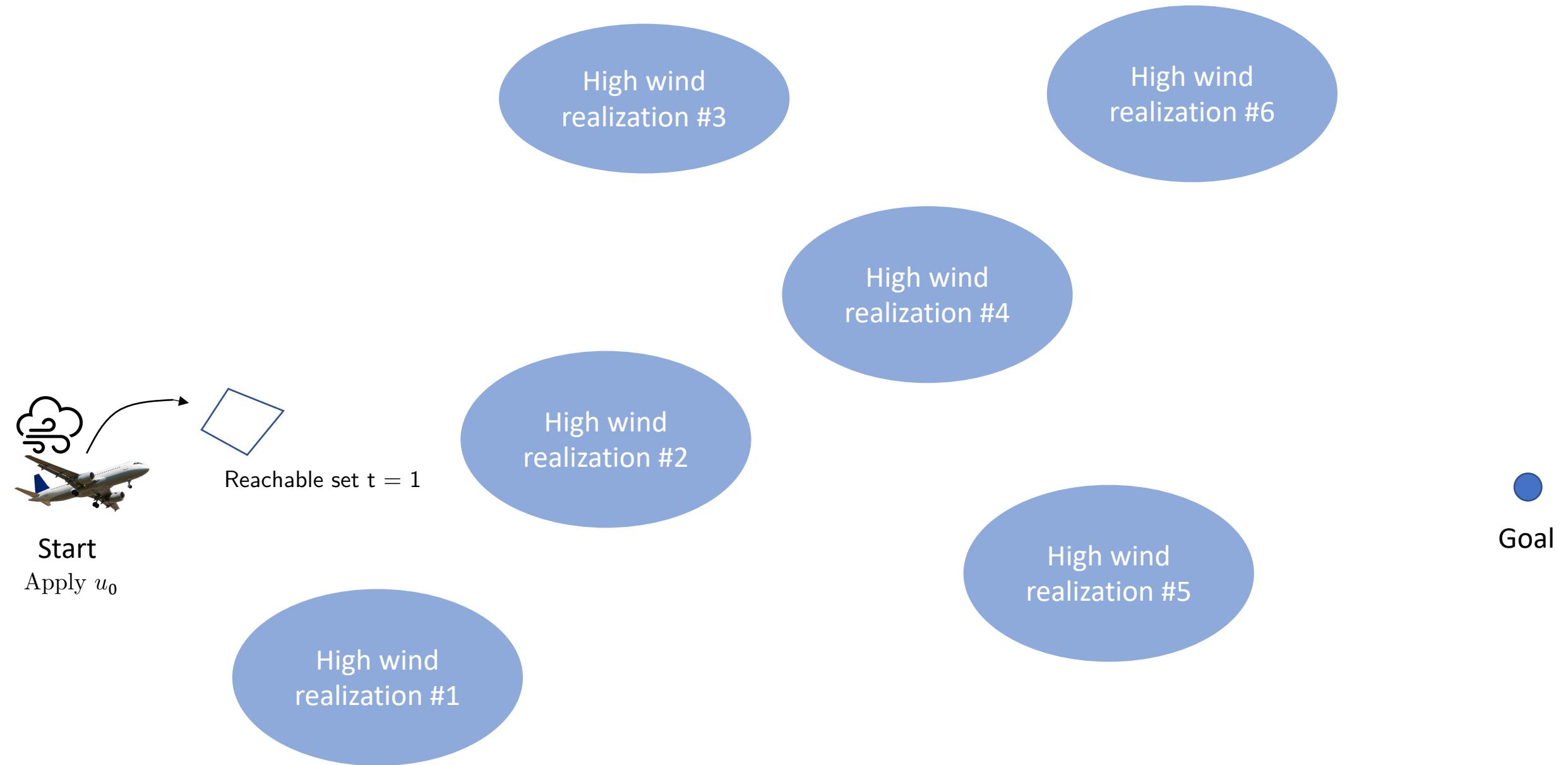


Start

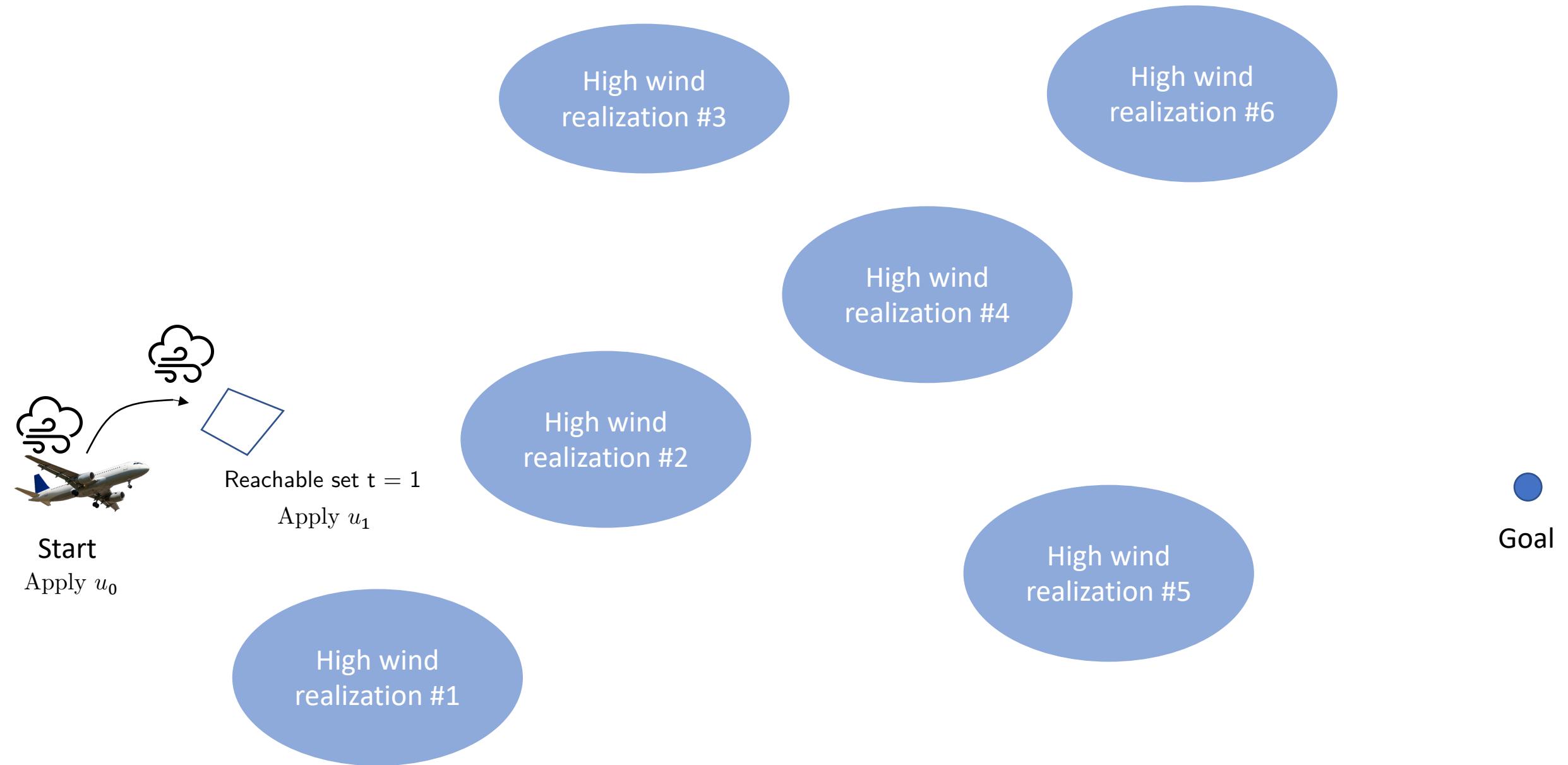


Goal

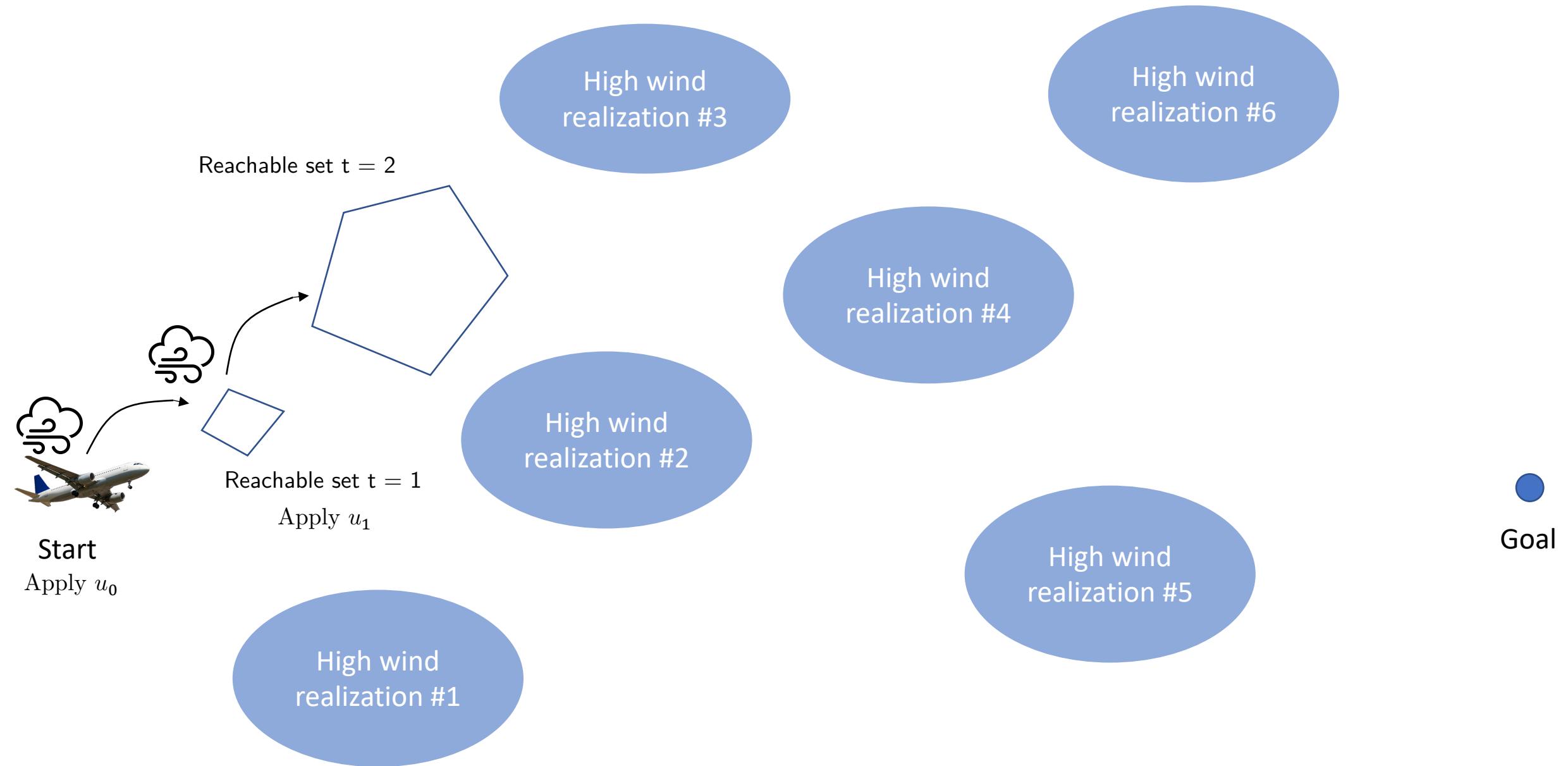
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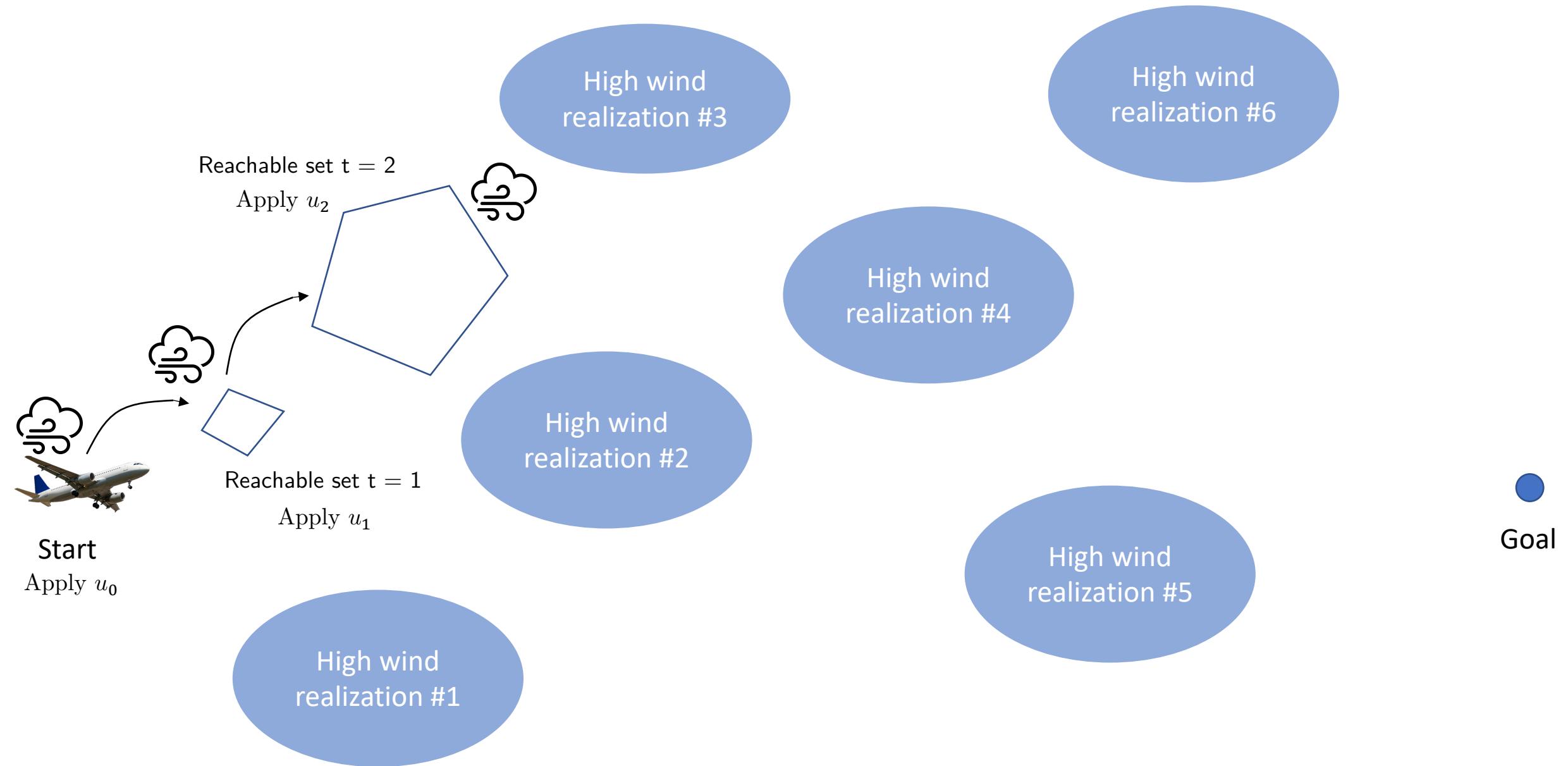
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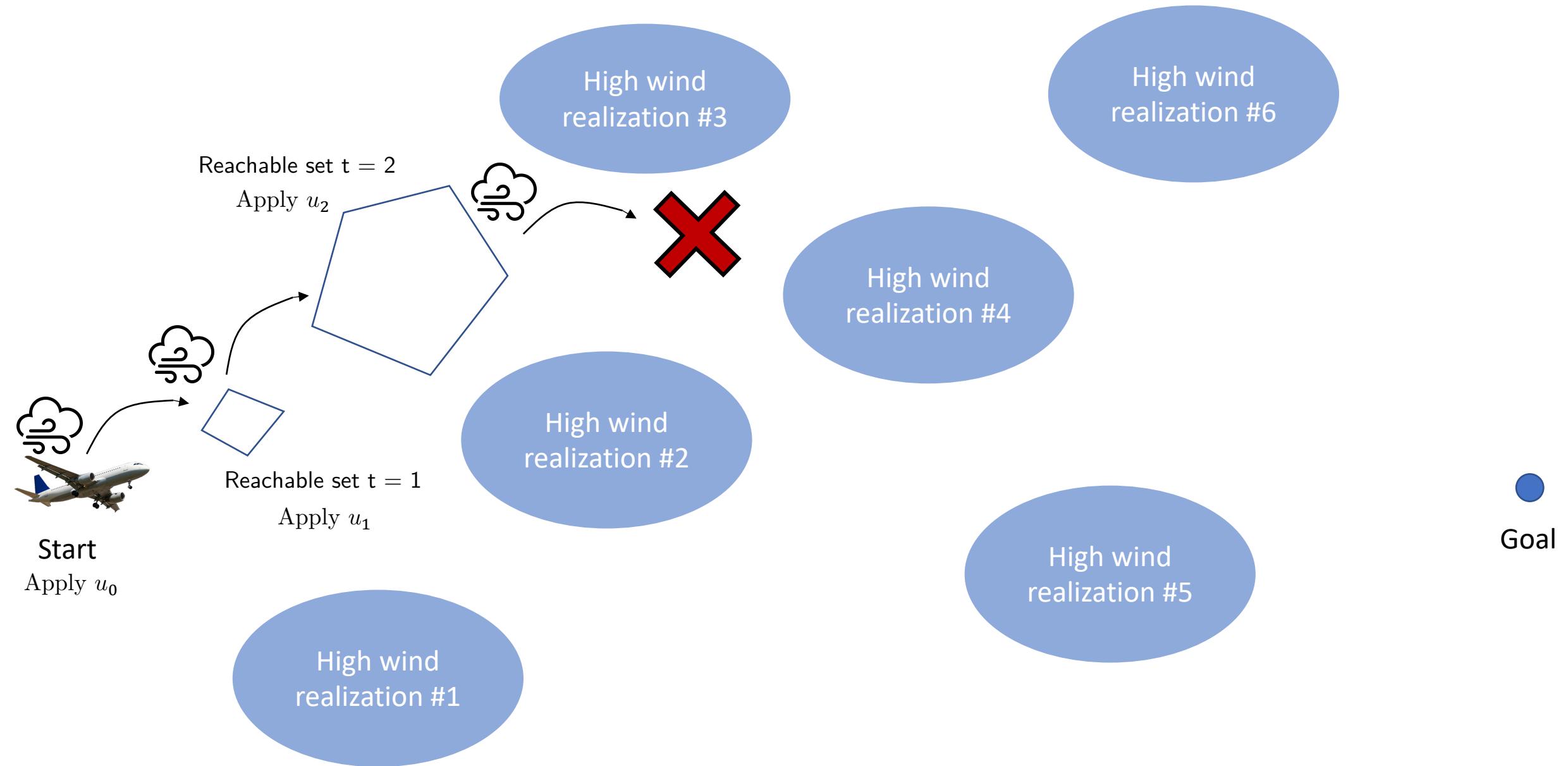
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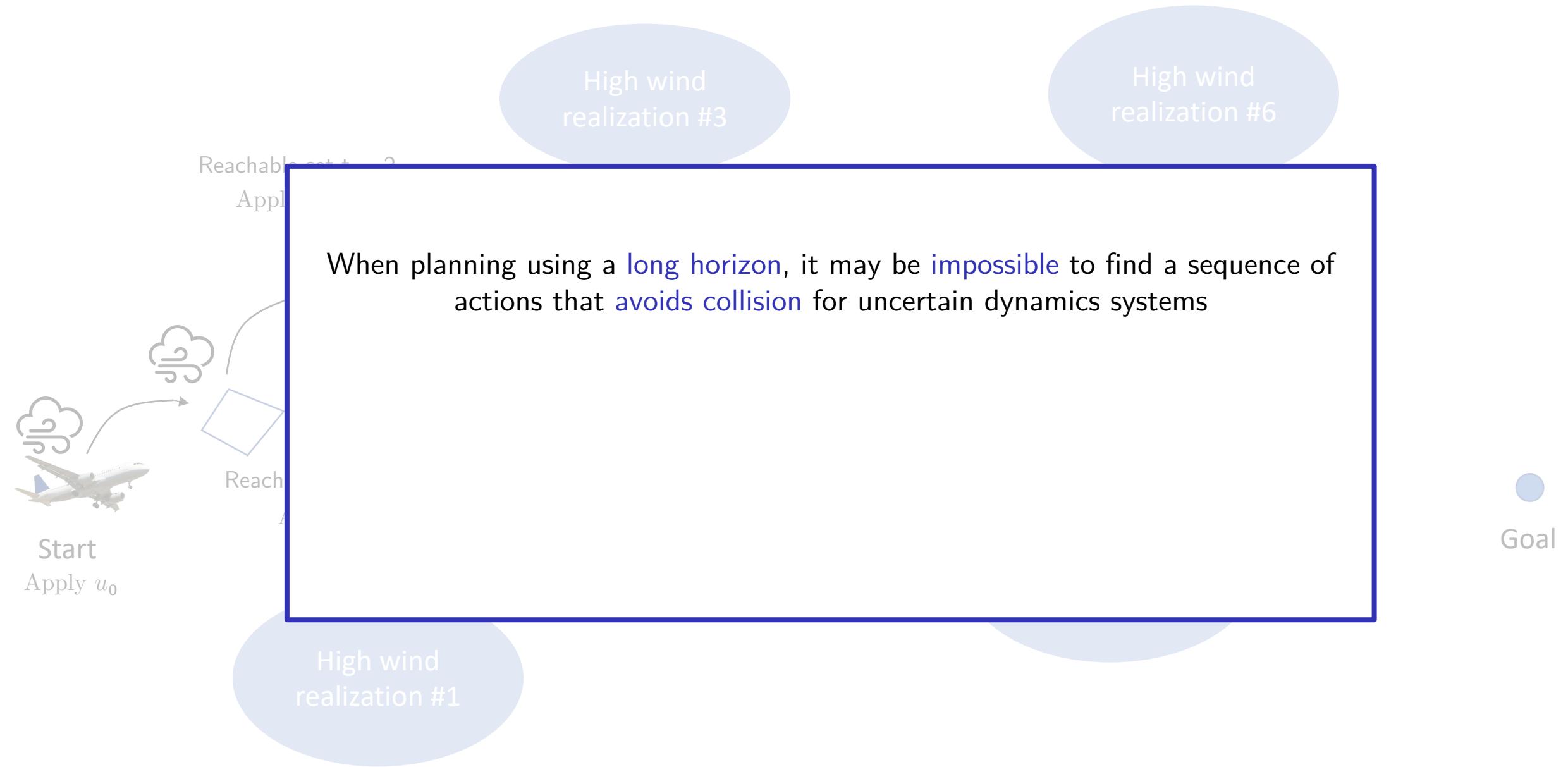
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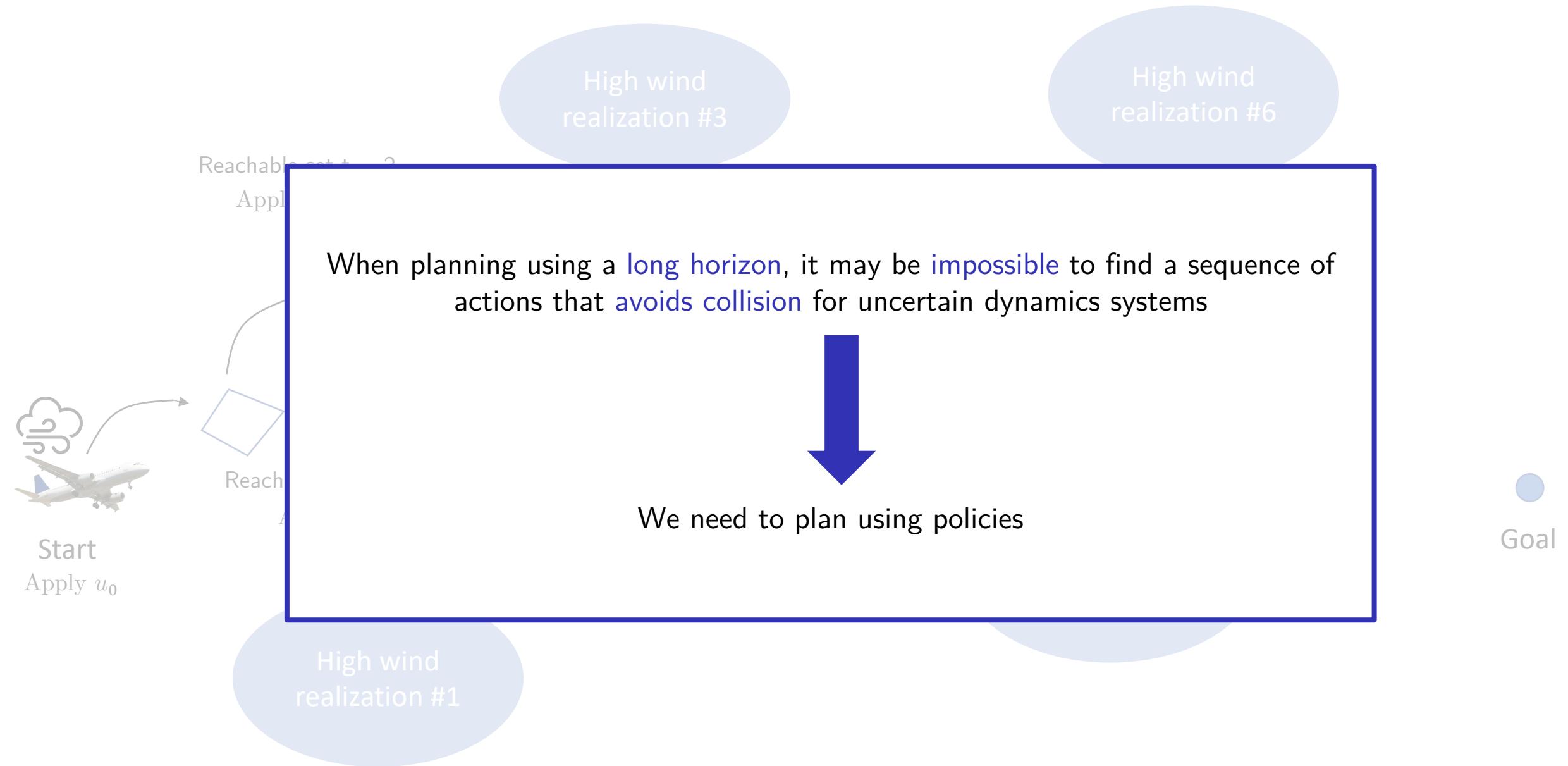
The horizon dilemma



The horizon dilemma in MBRL



The horizon dilemma in MBRL



How do we fix this problem?

1136

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 43, NO. 8, AUGUST 1998

Statement 2: Under the assumptions of this theorem for any $0 \leq t_0 \leq t$

$$\begin{aligned}\Phi_{\tilde{u}}(t) - \Phi_{\tilde{u}^*}(t_0) &\geq \Phi_{\tilde{u}^*}(t) - \Phi_{\tilde{u}^*}(t_0) \\ &= -\frac{1}{h} \Delta \vartheta^T(t) L^{-1} \Delta \vartheta(t) + o_\omega(h)\end{aligned}$$

where

$$\Phi_{\tilde{u}}(t) := \int_0^t (2\tilde{u}^T(\tau)d\vartheta(\tau) + \tilde{u}^T(\tau)L\tilde{u}(\tau)d\tau) \quad (20)$$

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and this minimum is reachable for

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As a result, we have: $d\Phi_{\tilde{u}^*}(t) \leq 0$ (in symbolic form).

Proof of Statement 2: Using the Euler–Maruyama's formula [3], [9] we obtain the following relation: $(h := t - t_0 \rightarrow 0), \Phi_{\tilde{u}}(t) - \Phi_{\tilde{u}^*}(t_0) = 2\tilde{u}^T(t)\Delta \vartheta(t) + \tilde{u}^T(t)L\tilde{u}(t)h + o_\omega(h)$.

Minimizing then the right side for each fixed t , we derive

$$\begin{aligned}\min_{\tilde{u}(t)} [\Phi_{\tilde{u}}(t) - \Phi_{\tilde{u}^*}(t_0)] &= \Phi_{\tilde{u}^*}(t) - \Phi_{\tilde{u}^*}(t_0) \\ &= -\frac{1}{h} \Delta \vartheta^T(t) L^{-1} \Delta \vartheta(t) + o_\omega(h) \leq o_\omega(h).\end{aligned}\quad (21)$$

Hence, taking into account the definition (20), we have $d\Phi_{\tilde{u}^*}(t) \leq 0$. \square

Rewriting (18) in differential form and taking into account that P is the solution of the Riccati equation (see Assumption 4 of this theorem) we finally derive

$$dV(e(t)) \stackrel{a.s.}{\leq} [\varphi(t) + I(t)] dt + S^T(t) dw(t) - e^T(t)Q_e e(t) dt.$$

According to the Kronecker lemma, the Large Number law for martingale, and Skorohod lemma (see [2] and [4]), we obtain the result of the theorem. Details of this proof can be found in [13].

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Brief paper

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Optimization over state feedback policies for robust control with constraints

Paul J. Goulart^{a,*}, Eric C. Kerrigan^b, Jan M. Maciejowski^a^aDepartment of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1FW, UK^bDepartment of Aeronautics and Department of Electrical and Electronic Engineering, Imperial College London, Exhibition Road, London SW7 2BY, UK

Received 15 March 2005; received in revised form 8 June 2005; accepted 25 August 2005

Abstract

This paper is concerned with the optimal control of linear discrete-time systems subject to unknown but bounded state dist mixed polytopic constraints on the state and input. It is shown that the class of admissible affine state feedback control policies with prior states is equivalent to the class of admissible feedback policies that are affine functions of the past disturbance sequence. that a broad class of constrained finite horizon robust and optimal control problems, where the optimization is over affine state policies, can be solved in a computationally efficient fashion using convex optimization methods. This equivalence result is used robust receding horizon control (RHC) state feedback policy such that the closed-loop system is input-to-state stable (ISS) and the are satisfied for all time and all allowable disturbance sequences. The cost to be minimized in the associated finite horizon op problem is quadratic in the disturbance-free state and input sequences. The value of the receding horizon control law can be calc sample instant using a single, tractable and convex quadratic program (QP) if the disturbance set is polytopic, or a tractable secon program (SOCP) if the disturbance set is given by a 2-norm bound. © 2006 Elsevier Ltd. All rights reserved.

Keywords: Robust control; Constraint satisfaction; Robust optimization; Predictive control; Optimal control

1. Introduction

This paper is concerned with the control of constrained discrete-time linear systems that are subject to additive, but bounded disturbances on the state. The main aim is to provide results that allow for the efficient computation of an optimal and stabilizing state feedback control policy that ensures a given set of state and input constraints are satisfied for all time, despite the presence of the disturbances. This is a problem that has been studied for some time now in the optimal control literature (Bertsekas & Rhodes, 1973) and a number of different

& Diaz-Bobillo, 1995) or predictive control (Mayne et al., 2000; Mayne, Rawlings, Rao, & Scokaert, 2000).

It is generally accepted that if disturbances accounted for in the formulation of a constraint control problem, then the optimization has to be admissible state feedback policies, rather than open sequences, otherwise infeasibility and instability may occur (Mayne et al., 2000). However, optimization contrary (nonlinear) feedback policies is particularly constraints have to be satisfied. Current proposals for this using finite dimensional optimization, such as

Abstract

This paper provides a novel solution to the problem of robust model predictive control of constrained, linear, discrete-time systems in the presence of bounded disturbances. The optimal control problem that is solved online includes, uniquely, the initial state of the model employed in the problem as a decision variable. The associated value function is zero in a disturbance invariant set that serves as the 'origin' when bounded disturbances are present, and permits a strong stability result, namely robust exponential stability of the disturbance invariant set for the controlled system with bounded disturbances, to be obtained. The resultant online algorithm is a quadratic program of similar complexity to that required in conventional model predictive control. © 2004 Elsevier Ltd. All rights reserved.

Keywords: Robust model predictive control; Robustness; Bounded disturbances

1. Introduction

Model predictive control is widely employed for the control of constrained systems and an extensive literature on the subject exists some of which is reviewed in Bemporad and Morari (1999); Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scokaert (2000). Several methods for achieving robustness have been considered. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scokaert & Rawlings, 1995; Marruedo, Álamo, & Camacho, 2002). *Open-loop* model predictive control that determines the

$\{u_0, u_1, \dots, u_{N-1}\}$ of control actions was proposed in Zheng and Morari (1993); this method cannot contain the 'spread' of predicted trajectories resulting from disturbances. Hence *feedback* model predictive control in which the decision variable is a *policy* π , which is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ of control laws, was advocated in, for example, Mayne (1995); Kothare, Balakrishnan, and Morari (1996); Lee and Yu (1997); Scokaert and Mayne (1998); De Nicalo, Magni, and Scattolini (2000); Magni, Nijmeijer, and van der Schaft (2001); Magni, De Nicolao, Scattolini, and Allgöwer (2003). Determination of a control policy is usually prohibitively difficult so research has concentrated on various simplifications and relaxations (Mayne,

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The problem is NP-hard!

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Optimization over state feedback policies for robust control with constraints

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Model predictive control is widely employed for the control of constrained systems and an extensive literature on the subject exists some of which is reviewed in Bemporad and Morari (1999); Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scokaert (2000). Several methods for achieving robustness have been considered. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scokaert & Rawlings, 1995; Marruedo, Álamo, & Camacho, 2002). *Open-loop* model predictive control that determines the

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How do we fix this problem?

1136

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Statement 2: Under the assumptions of this theorem for any $0 \leq t_0 \leq t$

$$\begin{aligned}\Phi_{\tilde{u}}(t) - \Phi_{\tilde{u}}(t_0) &\geq \Phi_{\tilde{u}^*}(t) - \Phi_{\tilde{u}^*}(t_0) \\ &= -\frac{1}{h} \Delta \vartheta^T(t) L^{-1} \Delta \vartheta(t) + o_\omega(h)\end{aligned}$$

where

$$\Phi_{\tilde{u}}(t) := \int_0^t (2\tilde{u}^T(\tau)d\vartheta(\tau) + \tilde{u}^T(\tau)L\tilde{u}(\tau)d\tau) \quad (20)$$

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As a result, we have: $d\Phi_{\tilde{u}^*}(t) \leq 0$ (in symbolic form).

Proof of Statement 2: Using the Euler–Maruyama's formula [3], [9] we obtain the following relation: $(h := t - t_0 \rightarrow 0), \Phi_{\tilde{u}}(t) - \Phi_{\tilde{u}}(t_0) = 2\tilde{u}^T(t)\Delta \vartheta(t) + \tilde{u}^T(t)L\tilde{u}(t)h + o_\omega(h)$.

Minimizing then the right side for each fixed t , we derive

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Rewriting (18) in differential form and taking into account that P is the solution of the Riccati equation (see Assumption 4 of this theorem) we finally derive

$$dV(e(t)) \stackrel{a.s.}{\leq} [\varphi(t) + I(t)]dt + S^T(t)dw(t) - e^T(t)Q_e(t)dt.$$

According to the Kronecker lemma, the Large Number law for martingale, and Skorohod lemma (see [2] and [4]), we obtain the result of the theorem. Details of this proof can be found in [13].

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Automatica 41 (2005) 219–224

Brief paper

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Min-Max Feedback Model Predictive Control for Constrained Linear Systems

P. O. M. Scokaert and D. Q. Mayne

Abstract—Min-max feedback formulations of model predictive control are discussed, both in the fixed and variable horizon contexts. The schemes the authors discuss introduce, in the control optimization notion that feedback is present in the receding-horizon implementation of the control. This leads to improved performance, compared to standard model predictive control, and resolves the feasibility difficulties that with the min-max techniques that are documented in the literature. The stabilizing properties of the methods are discussed as well as practical implementation details.

Index Terms—Feedback, min-max optimization, model predictive control.

I. INTRODUCTION

Model predictive control (MPC) is a control methodology that is becoming mature. The basic aspects of the method are well understood, and stabilizing formulations of the control law are documented in the literature, both for linear and nonlinear problems [1], [2].

The MPC strategy optimizes an open-loop control sequence each sample, to minimize a nominal cost function, subject to state and input constraints. The optimization is usually based on the assumption that the process model is exact and that disturbances are constant. Because the control law ignores the possibility of future changes in disturbance and model mismatch, closed-loop performance can be poor [3], with likely violations of constraints, when disturbances or model mismatch are present.

Some formulations of MPC have been proposed that address issues [4], [5]. These methods rely on a min–max optimization predicted performance. However, these formulations optimize a control profile over all possible disturbance (or model mismatch)

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Received 15 March 2005; received in revised form 8 June 2005; accepted 25 August 2005

Abstract

This paper is concerned with the optimal control of linear discrete-time systems subject to unknown but bounded state distibuted polytopic constraints on the state and input. It is shown that the class of admissible affine state feedback control policies with prior states is equivalent to the class of admissible feedback policies that are affine functions of the past disturbance sequence. that a broad class of constrained finite horizon robust and optimal control problems, where the optimization is over affine state policies, can be solved in a computationally efficient fashion using convex optimization methods. This equivalence result is used robust receding horizon control (RHC) state feedback policy such that the closed-loop system is input-to-state stable (ISS) and they are satisfied for all time and all allowable disturbance sequences. The cost to be minimized in the associated finite horizon optimization problem is quadratic in the disturbance-free state and input sequences. The value of the receding horizon control law can be calculated using a single, tractable and convex quadratic program (QP) if the disturbance set is polytopic, or a tractable second-order cone program (SOCP) if the disturbance set is given by a 2-norm bound. © 2006 Elsevier Ltd. All rights reserved.

Keywords: Robust control; Constraint satisfaction; Robust optimization; Predictive control; Optimal control

1. Introduction

This paper is concerned with the control of constrained discrete-time linear systems that are subject to additive, but bounded disturbances on the state. The main aim is to provide results that allow for the efficient computation of an optimal and stabilizing state feedback control policy that ensures a given set of state and input constraints are satisfied for all time, despite the presence of the disturbances. This is a problem that has been studied for some time now in the optimal control literature (Bertsekas & Rhodes, 1973) and a number of different

& Diaz-Bobillo, 1995) or predictive control (Mayne et al., 2000; Mayne, Rawlings, Rao, & Scokaert, 2000).

It is generally accepted that if disturbances accounted for in the formulation of a constrained control problem, then the optimization has to be admissible state feedback policies, rather than open sequences, otherwise infeasibility and instability may occur (Mayne et al., 2000). However, optimization of nonlinear feedback policies is particularly challenging because they have to be satisfied. Current proposals for this use finite dimensional optimization, such as

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An elegant approximation out of many

Optimizing over policies

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

Optimizing over policies

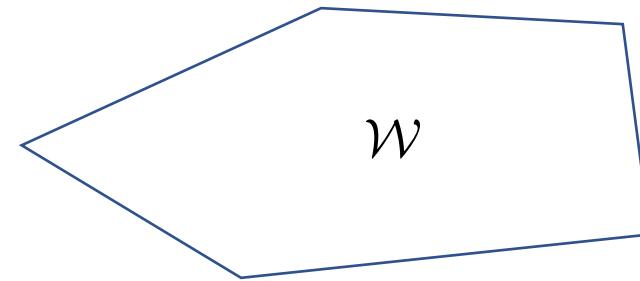
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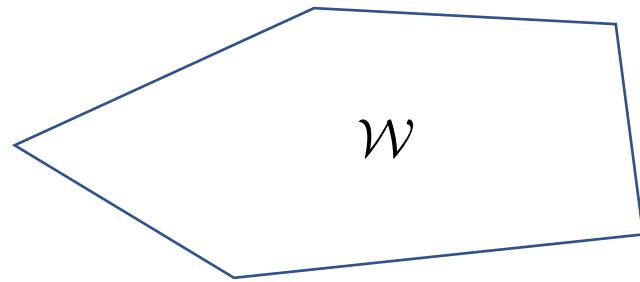
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The problem that we want to solve

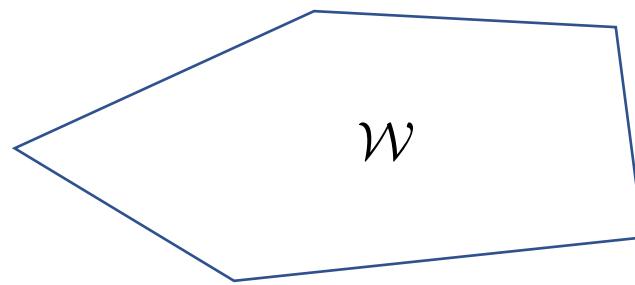
$$\begin{array}{ll}\min_{\pi} & f(x(t), \pi, w_{t|t}, \dots, w_{t+N|t}) \\ \text{subject to} & x_{k+1|t} = Ax_{k|t} + Bu_{k|t} + w_{k|t} \\ & \text{robust constraint satisfaction} \\ & x_{t|t} = x(t) \\ & u_{k|t} = \pi(x_{k|t}),\end{array}$$

Optimization over policies

Optimizing over policies

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Optimization over policies

Finite dimensional reformulation

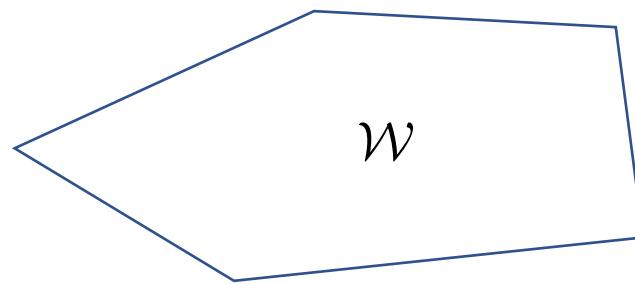
Optimize over a tree

Initial state
 u_0

Optimizing over policies

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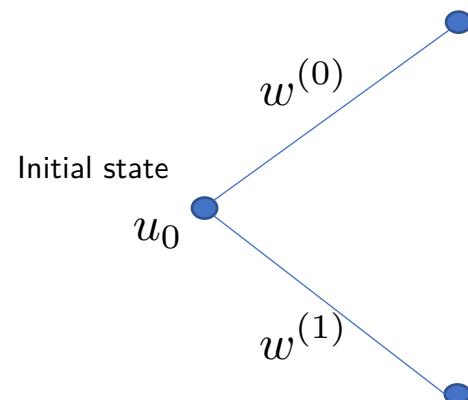
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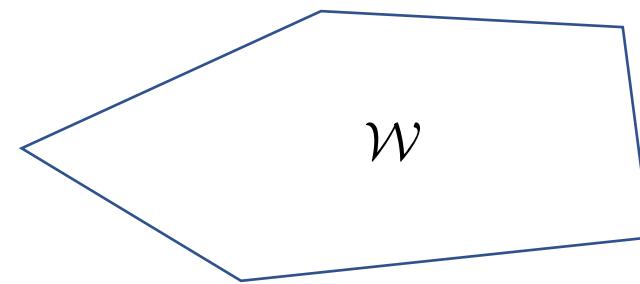
Optimize over a tree



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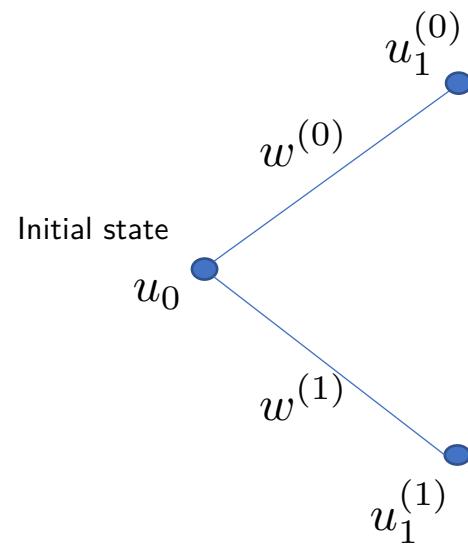
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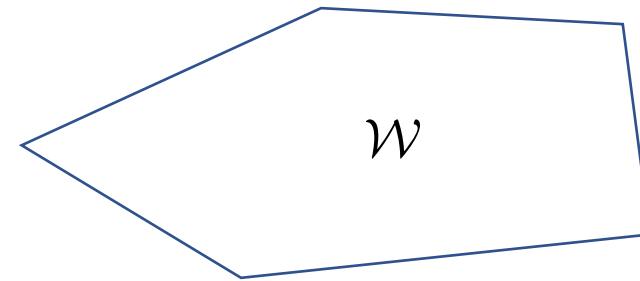
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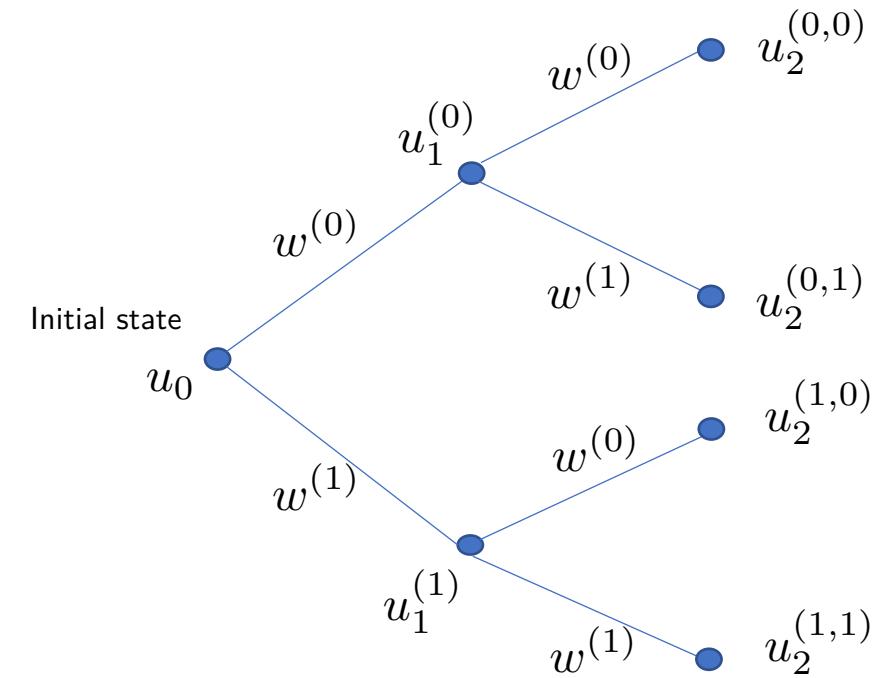
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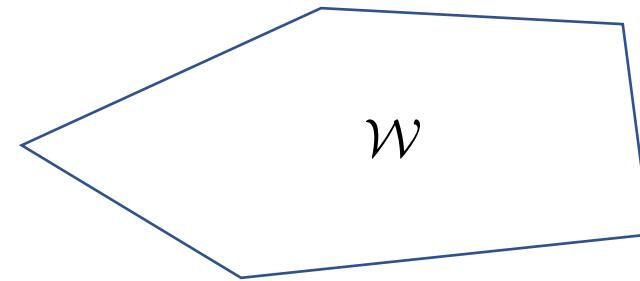
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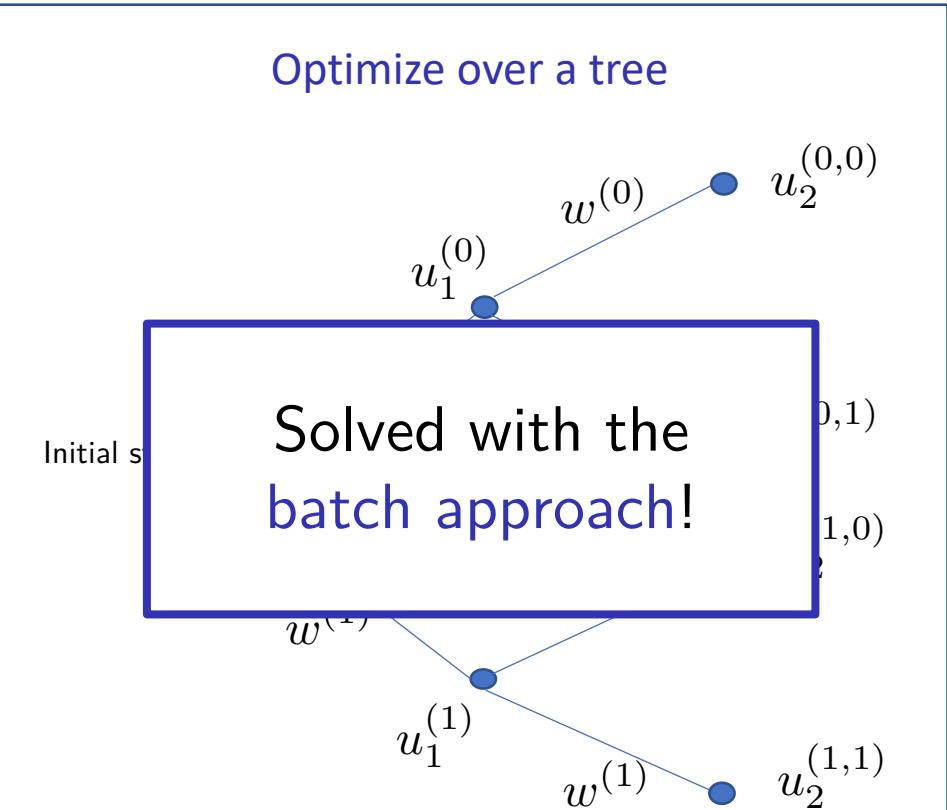
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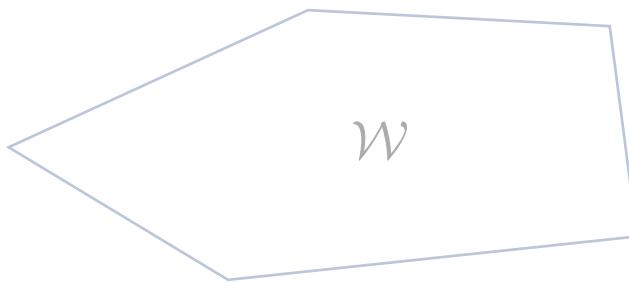
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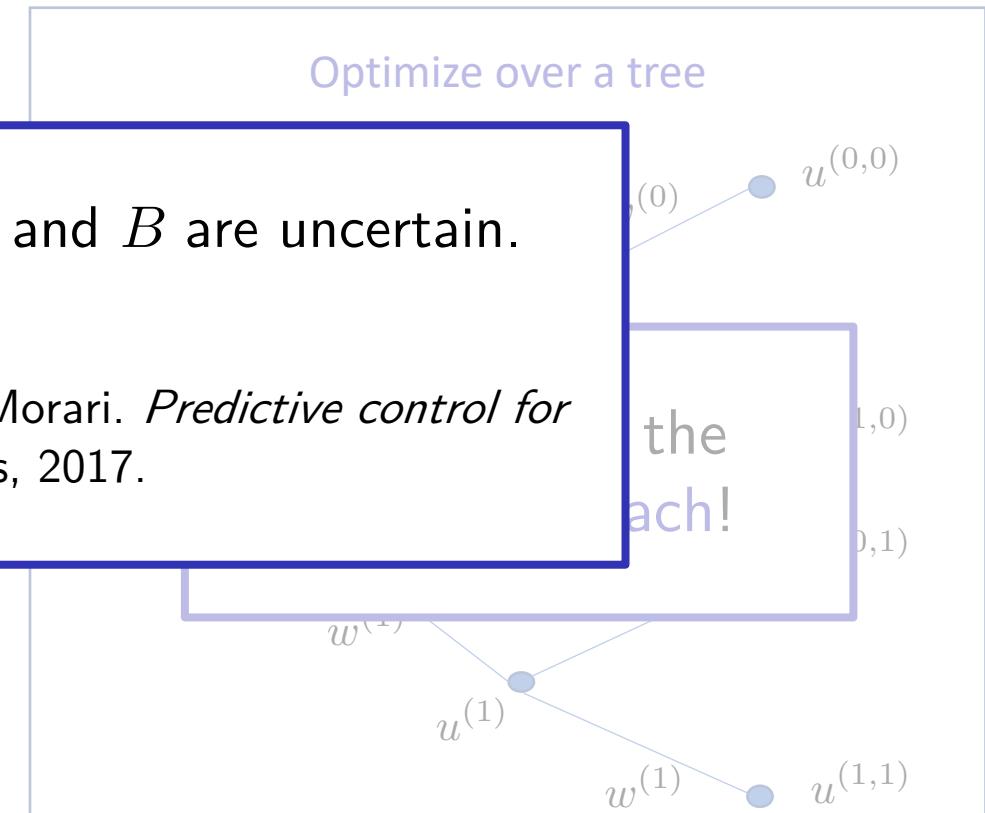
This approach can be used also when A and B are uncertain.

For further details check:

Francesco Borrelli, Alberto Bemporad, and Manfred Morari. *Predictive control for linear and hybrid systems*. Cambridge University Press, 2017.

Optimization over policies

Optimize over a tree



How do we fix this problem?

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Automatica 42 (2006) 523-533

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Automatica 41 (2005) 219-224

Brief paper



Automatica 41 (2005) 219-224

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Optimization over state feedback policies for robust control with constraints

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Received 15 March 2005; received in revised form 8 June 2005; accepted 25 August 2005

Abstract

This paper discusses min-max feedback formulations of model predictive control, both in the fixed and variable horizon contexts. The schemes the authors introduce, in the control optimization notion that feedback is present in the receding-horizon implementation of the control. This leads to improved performance, compared to standard model predictive control, and resolves the feasibility difficulties with the min-max techniques that are documented in the literature. The stabilizing properties of the methods are discussed as well as practical implementation details.

Index Terms—Feedback, min-max optimization, model predictive control.

I. INTRODUCTION

Model predictive control (MPC) is a control methodology that is becoming mature. The basic aspects of the method are well understood, and stabilizing formulations of the control are documented in the literature, both for linear and nonlinear problems [1], [2].

The MPC strategy optimizes an open-loop control sequence each sample, to minimize a nominal cost function, subject to state and input constraints. The optimization is usually based on the assumption that the process model is exact and that disturbances are constant. Because the control law ignores the possibility of future changes in disturbance and model mismatch, closed-loop performance can be poor [3], with likely violations of the constraints, when disturbances or model mismatch are present.

Some formulations of MPC have been proposed that address issues [4], [5]. These methods rely on a min-max optimization to predict performance. However, these formulations optimize a control profile over all possible disturbance (or model mismatch)

& Diaz-Bobillo, 1995) or predictive control (Mayne et al., 2000; Mayne, Rawlings, Rao, & Scokaert, 2000).

It is generally accepted that if disturbances are accounted for in the formulation of a constrained control problem, then the optimization has to be based on admissible state feedback policies, rather than open-loop sequences, otherwise infeasibility and instability may occur (Mayne et al., 2000). However, optimization of (nonlinear) feedback policies is particularly difficult because constraints have to be satisfied. Current proposals for this use finite dimensional optimization, such as quadratic programming (QCP) or semidefinite programming (SDP) (Goulart et al., 2005).

Abstract

This paper provides a novel solution to the problem of robust model predictive control of constrained, linear, discrete-time systems in the presence of bounded disturbances. The optimal control problem that is solved online includes, uniquely, the initial state of the model employed in the problem as a decision variable. The associated value function is zero in a disturbance invariant set that serves as the 'origin' when bounded disturbances are present, and permits a strong stability result, namely robust exponential stability of the disturbance invariant set for the controlled system with bounded disturbances, to be obtained. The resultant online algorithm is a quadratic program of similar complexity to that required in conventional model predictive control. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Robust model predictive control; Robustness; Bounded disturbances

1. Introduction

This paper is concerned with the control of constrained discrete-time linear systems that are subject to additive, but bounded disturbances on the state. The main aim is to provide results that allow for the efficient computation of an optimal and stabilizing state feedback control policy that ensures a given set of state and input constraints are satisfied for all time, despite the presence of the disturbances. This is a problem that has been studied for some time now in the optimal control literature (Bertsekas & Rhodes, 1973) and a number of different

An elegant approximation out of many

1. Introduction

Model predictive control is widely employed for the control of constrained systems and an extensive literature on the subject exists some of which is reviewed in Bemporad and Morari (1999); Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scokaert (2000). Several methods for achieving robustness have been considered. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scokaert & Rawlings, 1995; Marrucho, Álamo, & Camacho, 2002). *Open-loop* model predictive control that determines the

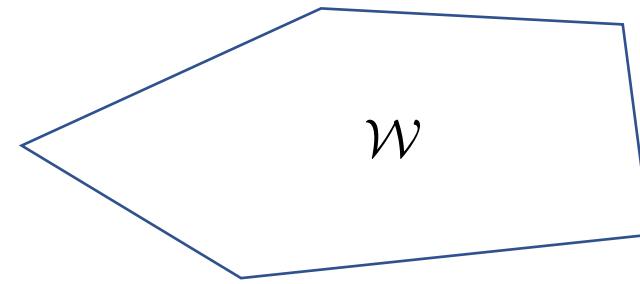
$\{u_0, u_1, \dots, u_{N-1}\}$ of control actions was proposed in Zheng and Morari (1993); this method cannot contain the 'spread' of predicted trajectories resulting from disturbances. Hence *feedback* model predictive control in which the decision variable is a *policy* π , which is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ of control laws, was advocated in, for example, Mayne (1995); Kothare, Balakrishnan, and Morari (1996); Lee and Yu (1997); Scokaert and Mayne (1998); De Nicolao, Magni, and Scattolini (2000); Magni, Nijmeijer, and van der Schaft (2001); Magni, De Nicolao, Scattolini, and Allgöwer (2003). Determination of a control policy is usually prohibitively difficult so research has concentrated on *constrained* (and *uncertain*) approximations (Mayne

What people do in practice

Optimizing over disturbance feedback policies

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

Assumption: the disturbance support $\mathcal{W} = \text{conv}(w^{(0)}, \dots, w^{(n_d)})$.



The problem that we want to solve

$$\min_{\pi} f(x(t), \pi, w_{t|t}, \dots, w_{t+N|t})$$

subject to $x_{k+1|t} = Ax_{k|t} + Bu_{k|t} + w_{k|t}$
robust constraint satisfaction

$$x_{t|t} = x(t)$$

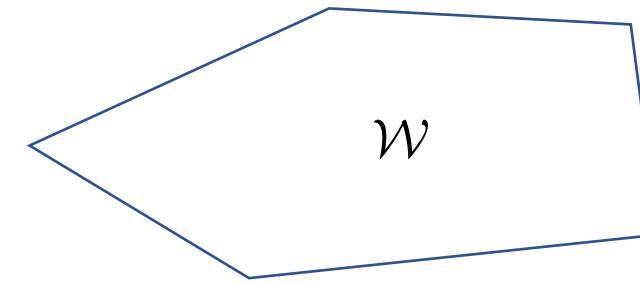
$$u_{k|t} = \pi(x_{k|t}),$$

Optimization over policies

Optimizing over disturbance feedback policies

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Optimization over policies

Tractable
approximation

Optimize over disturbance feedback policies

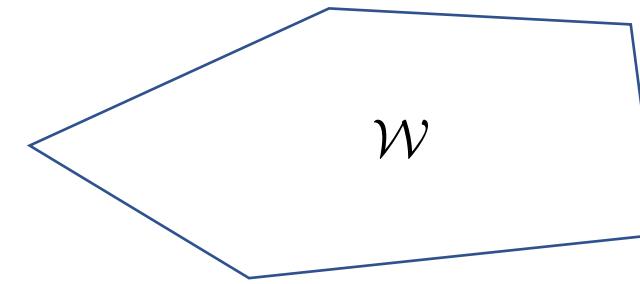
$$u_{k|t} = \pi(x(t), w_{t|t}, \dots, w_{k-1|t})$$

Function of disturbances!

Optimizing over disturbance feedback policies

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

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The problem that we want to solve

$$\begin{aligned} & \min_{\pi} \quad f(x(t), \pi, w_{t|t}, \dots, w_{t+N|t}) \\ \text{subject to} \quad & x_{k+1|t} = Ax_{k|t} + Bu_{k|t} + w_{k|t} \\ & \text{robust constraint satisfaction} \\ & x_{t|t} = x(t) \\ & u_{k|t} = \pi(x_{k|t}), \end{aligned}$$

Optimization over policies

Tractable
approximation

Optimize over disturbance feedback policies

$$u_{k|t} = \pi(x(t), w_{t|t}, \dots, w_{k-1|t})$$

Function of disturbances!

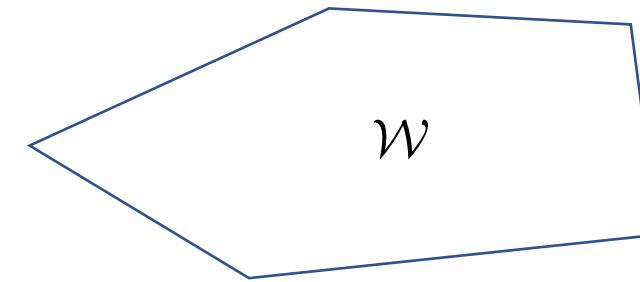
The problem is **tractable** when we pick:

$$u_{k|t} = M_{k|t} \begin{bmatrix} x(t) \\ w_{t|t} \\ \vdots \\ w_{k-1|t} \end{bmatrix}$$

Optimizing over disturbance feedback policies

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

Assumption: the disturbance support $\mathcal{W} = \text{conv}(w^{(0)}, \dots, w^{(n_d)})$.



The problem that we want to solve

$$\begin{array}{ll}\min_{\pi} & f(x(t), \pi, w_{t|t}, \dots, w_{t+N|t}) \\ \text{subject to} & x_{k+1|t} = Ax_{k|t} + Bu_{k|t} + w_{k|t} \\ & \text{robust constraint satisfaction} \\ & x_{t|t} = x(t) \\ & u_{k|t} = \pi(x_{k|t}),\end{array}$$

Optimization over policies

Tractable approximation

Optimize over disturbance feedback policies

$$u_{k|t} = \pi(x(t), w_{t|t}, \dots, w_{k-1|t})$$

Function of disturbances!

The problem is **tractable** when we pick:

$$u_{k|t} = M_{k|t} \begin{bmatrix} x(t) \\ w_{t|t} \\ \vdots \\ w_{k-1|t} \end{bmatrix} = K_{k|t} \begin{bmatrix} x(t) \\ x_{t+1|t} \\ \vdots \\ x_{k|t} \end{bmatrix}$$

Optimizing over disturbance feedback policies

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

Assumption:

This approach is **not convex** when A and B are uncertain → we need to overapproximate the uncertainty to convexify the problem!

The pro

For more details check:

Aharon Ben-Tal, Alexander Goryashko, Elana Guslitzer, and Arkadi Nemirovski. "Adjustable robust solutions of uncertain linear programs." *Mathematical programming* 99, no. 2 (2004): 351-376.

\min_{π}
subject to

Optimization

Optimizing over disturbance feedback policies

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Strategies to handle uncertainty in A and B (conservative, but tractable):

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- Sarah Dean, Stephen Tu, Nikolai Matni, and Benjamin Recht. "Safely learning to control the constrained linear quadratic regulator." In *2019 American Control Conference (ACC)*, pp. 5582-5588. IEEE, 2019.
- Wilbur Langson, Ioannis Chryssochoos, S. V. Raković, and David Q. Mayne. "Robust model predictive control using tubes." *Automatica* 40, no. 1 (2004): 125-133.
- Monimoy Bujarbaruah, Ugo Rosolia, Yvonne R. Stürz, Xiaojing Zhang, and Francesco Borrelli. "Robust MPC for LTI Systems with Parametric and Additive Uncertainty: A Novel Constraint Tightening Approach." *To appear in American Control Conference (ACC) (2020)*.

How do we fix this problem?

1136

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 43, NO. 8, AUGUST 1998

Statement 2: Under the assumptions of this theorem for any $0 \leq t_0 \leq t$

$$\begin{aligned}\Phi_{\omega}(t) - \Phi_{\omega}(t_0) &\geq \Phi_{\omega^*}(t) - \Phi_{\omega^*}(t_0) \\ &= -\frac{1}{h} \Delta \vartheta^T(t) L^{-1} \Delta \vartheta(t) + o_\omega(h)\end{aligned}$$

where

$$\Phi_{\omega}(t) := \int_{t_0}^t (2\bar{u}^T(\tau) d\vartheta(\tau) + \bar{u}^T(\tau) L \bar{u}(\tau) d\tau) \quad (20)$$

$$\begin{aligned}\Delta \vartheta(t) &:= B_{\omega}^T P[B_{\omega}(a_{\text{comp}}(t) - x^*(t)) h \\ &+ (K + A_3 C_b^+) \Delta y(t)], \quad \Delta y(t) = y(t) - y(t-h)\end{aligned}$$

and this minimum is reachable for

$$\begin{aligned}\dot{u}(t) dt &= \bar{u}^*(t) dt := -L^{-1} B_{\omega}^T P[B_{\omega}(a_{\text{comp}}(t) + x^*(t)) dt \\ &+ (K + A_3 C_b^+) dy(t)].\end{aligned}$$

As a result, we have: $d\Phi_{\omega^*}(t) \leq 0$ (in symbolic form).

Proof of Statement 2: Using the Euler-Maruyama's formula [3], [9] we obtain the following relation: $(h := t - t_0 \rightarrow 0), \Phi_{\omega}(t) - \Phi_{\omega}(t_0) = 2\bar{u}^T(t) \Delta \vartheta(t) + \bar{u}^T(t) L \bar{u}(t) h + o_\omega(h)$.

Minimizing then the right side for each fixed t , we derive

$$\begin{aligned}\min_{\omega} [\Phi_{\omega}(t) - \Phi_{\omega}(t_0)] &= \Phi_{\omega^*}(t) - \Phi_{\omega^*}(t_0) \\ &= -\frac{1}{h} \Delta \vartheta^T(t) L^{-1} \Delta \vartheta(t) + o_\omega(h) \leq o_\omega(h).\end{aligned}\quad (21)$$

Hence, taking into account the definition (20), we have $d\Phi_{\omega^*}(t) \leq 0$. \square

Rewriting (18) in differential form and taking into account that P is the solution of the Riccati equation (see Assumption 4 of this theorem) we finally derive

$$dV(e(t)) \leq [\varphi(t) + I(t)] dt + S^T(t) dw(t) - e^T(t) Q e(t) dt.$$

According to the Kronecker lemma, the Large Number law for martingale, and Skorohod lemma (see [2] and [4]), we obtain the result of the theorem. Details of this proof can be found in [13].

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The problem is NP-hard!

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Min-Max Feedback Model Predictive Control for Constrained Linear Systems

P. O. M. Scokaert and D. Q. Mayne

Abstract—Min-max feedback formulations of model predictive control are discussed, both in the fixed and variable horizon contexts. The schemes the authors discuss introduce, in the control optimization notion that feedback is present in the receding-horizon implementation of the control. This leads to improved performance, compared to standard model predictive control, and resolves the feasibility difficulties that with the min-max techniques that are documented in the literature. The stabilizing properties of the methods are discussed as well as practical implementation details.

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Model predictive control (MPC) is a control methodology that is becoming mature. The basic aspects of the method are well understood, and stabilizing formulations of the control law are documented in the literature, both for linear and nonlinear problems [1], [2].

The MPC strategy optimizes an open-loop control sequence each sample, to minimize a nominal cost function, subject to state and input constraints. The optimization is usually based on the assumption that the process model is exact and that disturbances are constant. Because the control law ignores the possibility of future changes in disturbance and model mismatch, closed-loop performance can be poor [3], with likely violations of the constraints, when disturbances or model mismatch are present.

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Optimization over state feedback policies for robust control with constrained disturbances

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Received 15 March 2005; received in revised form 8 June 2005; accepted 25 August 2005

Abstract

This paper is concerned with the optimal control of linear discrete-time systems subject to unknown but bounded state disjunctive polytopic constraints on the state and input. It is shown that the class of admissible affine state feedback control policies with prior states is equivalent to the class of admissible feedback policies that are affine functions of the past disturbance sequence. That a broad class of constrained finite horizon robust and optimal control problems, where the optimization is over affine policies, can be solved in a computationally efficient fashion using convex optimization methods. This equivalence result is used to robust receding horizon control (RHC) state feedback policy such that the closed-loop system is input-to-state stable (ISS) and it is satisfied for all time and all allowable disturbance sequences. The cost to be minimized in the associated finite horizon optimization problem is quadratic in the disturbance-free state and input sequences. The value of the receding horizon control law can be calculated using a single, tractable and convex quadratic program (QP) if the disturbance set is polytopic, or a tractable second-order cone program (SOCP) if the disturbance set is given by a 2-norm bound. © 2006 Elsevier Ltd. All rights reserved.

Keywords: Robust control; Constraint satisfaction; Robust optimization; Predictive control; Optimal control

1. Introduction

This paper is concerned with the control of constrained discrete-time linear systems that are subject to additive, but bounded disturbances on the state. The main aim is to provide results that allow for the efficient computation of an optimal and stabilizing state feedback control policy that ensures a given set of state and input constraints are satisfied for all time, despite the presence of the disturbances. This is a problem that has been studied for some time now in the optimal control literature (Bertsekas & Rhodes, 1973) and a number of different

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It is generally accepted that if disturbances accounted for in the formulation of a constraint control problem, then the optimization has to be able to handle state feedback policies, rather than open sequences, otherwise infeasibility and instability may occur (Mayne et al., 2000). However, optimization of trinary (nonlinear) feedback policies is particularly problematic because they have to be satisfied. Current proposals for this use finite dimensional optimization, such

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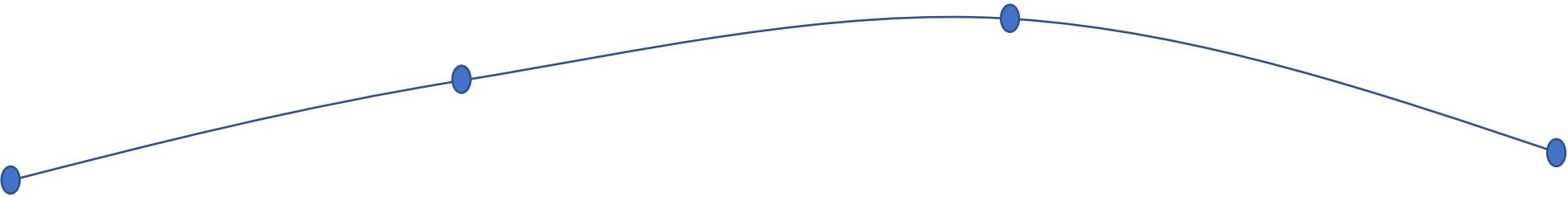
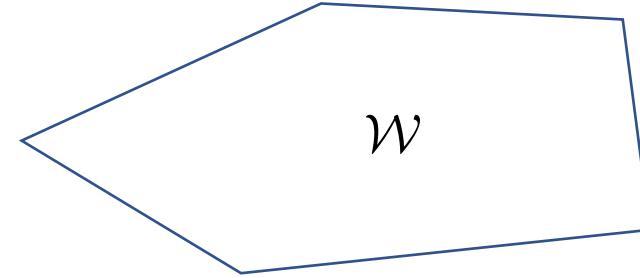
An elegant approximation out of many

What people do in practice

Fixed tube robust MPC

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

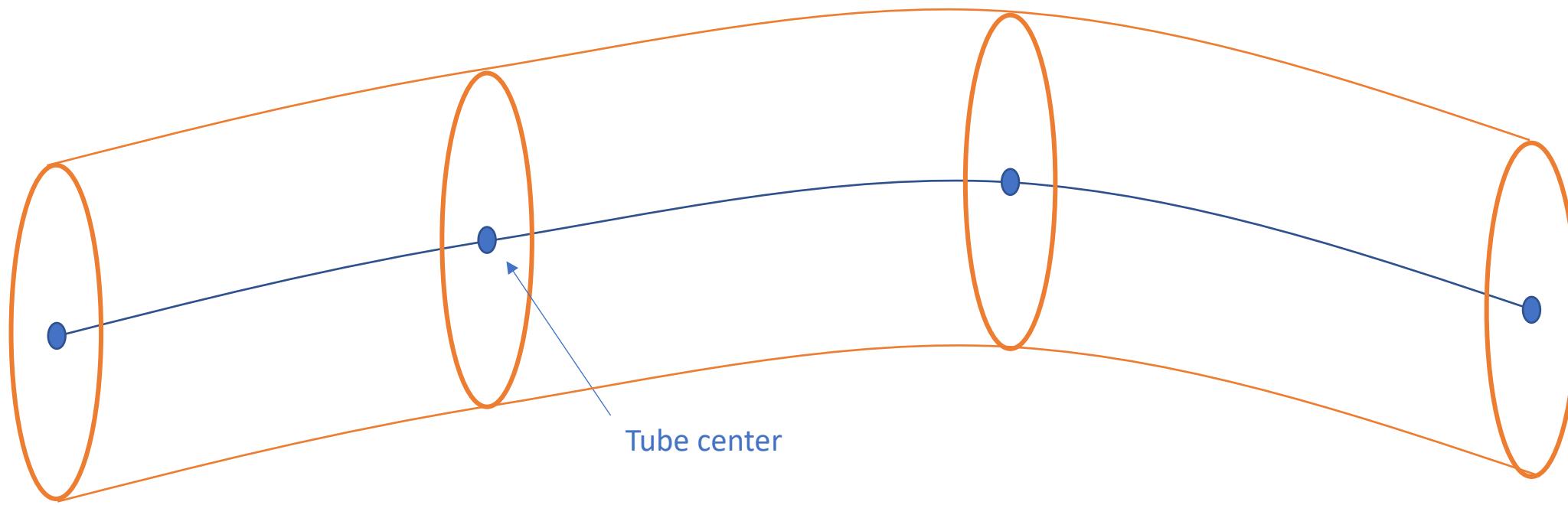
Assumption: the disturbance support $\mathcal{W} = \text{conv}(w^{(0)}, \dots, w^{(n_d)})$.



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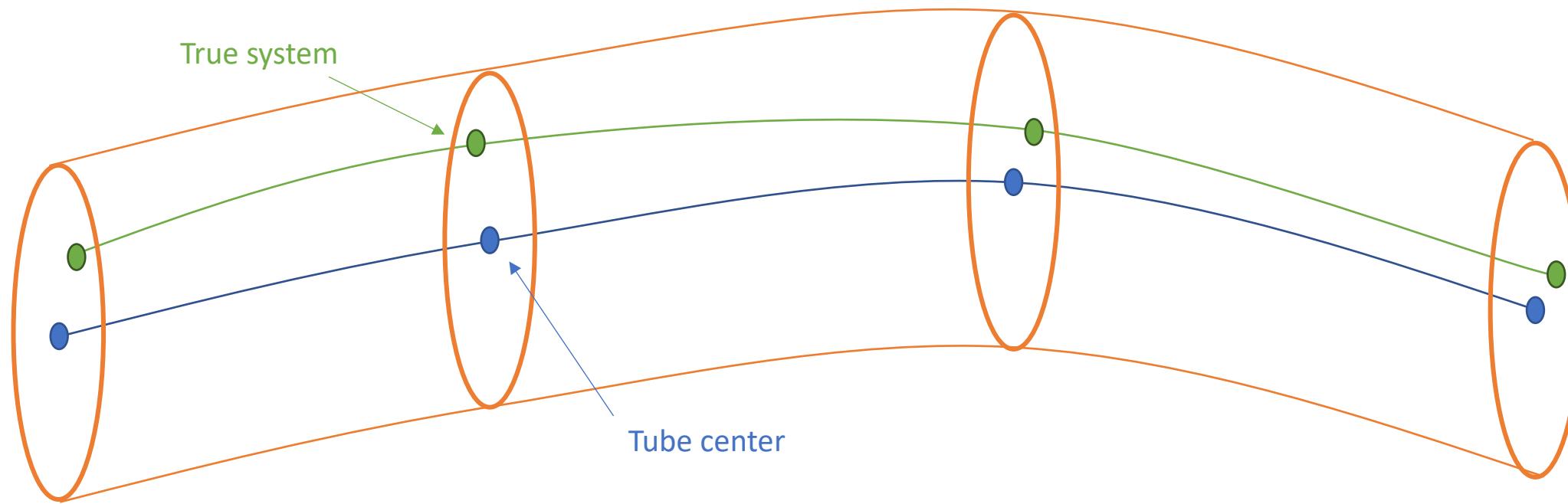
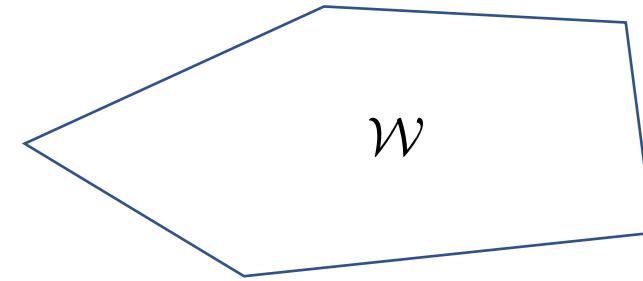
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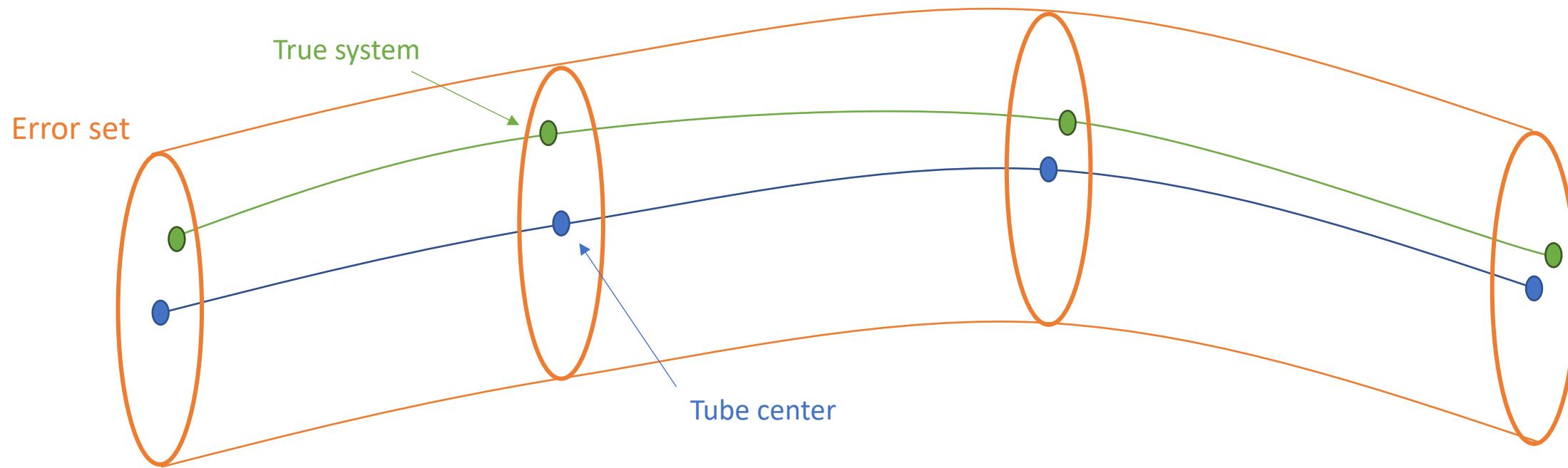
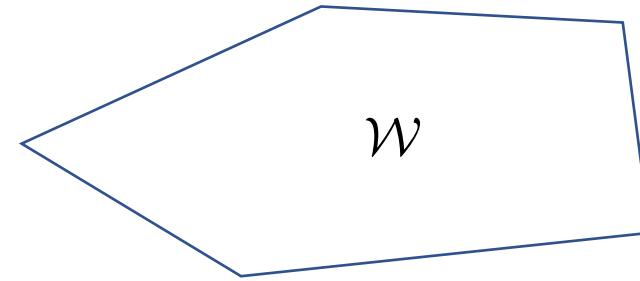
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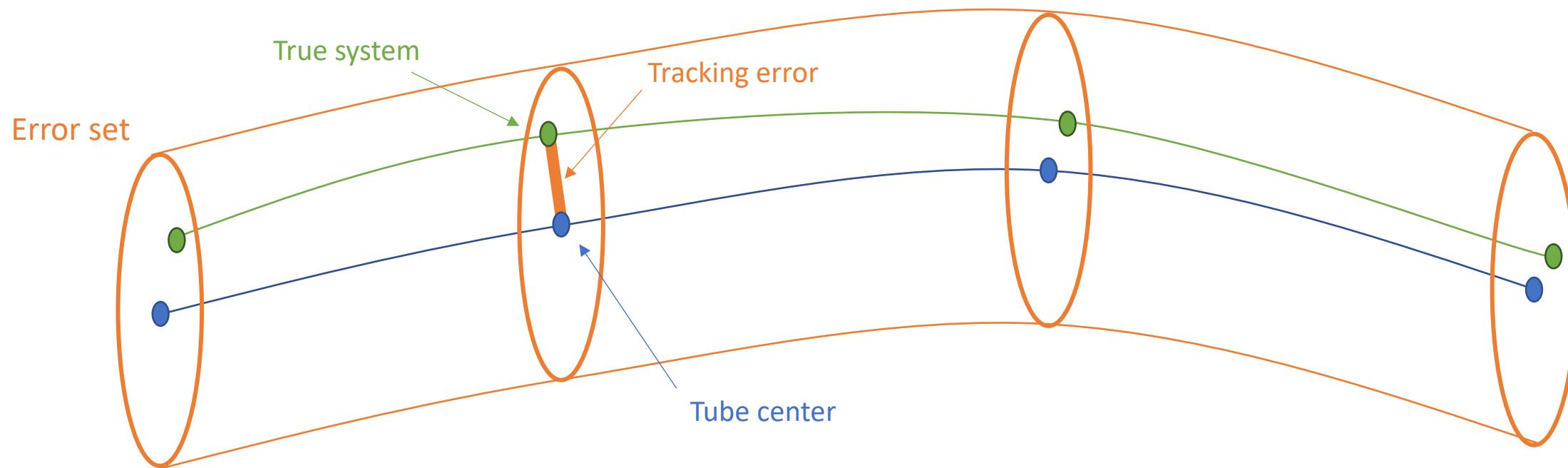
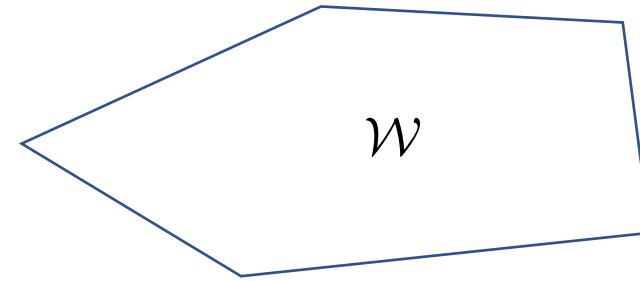
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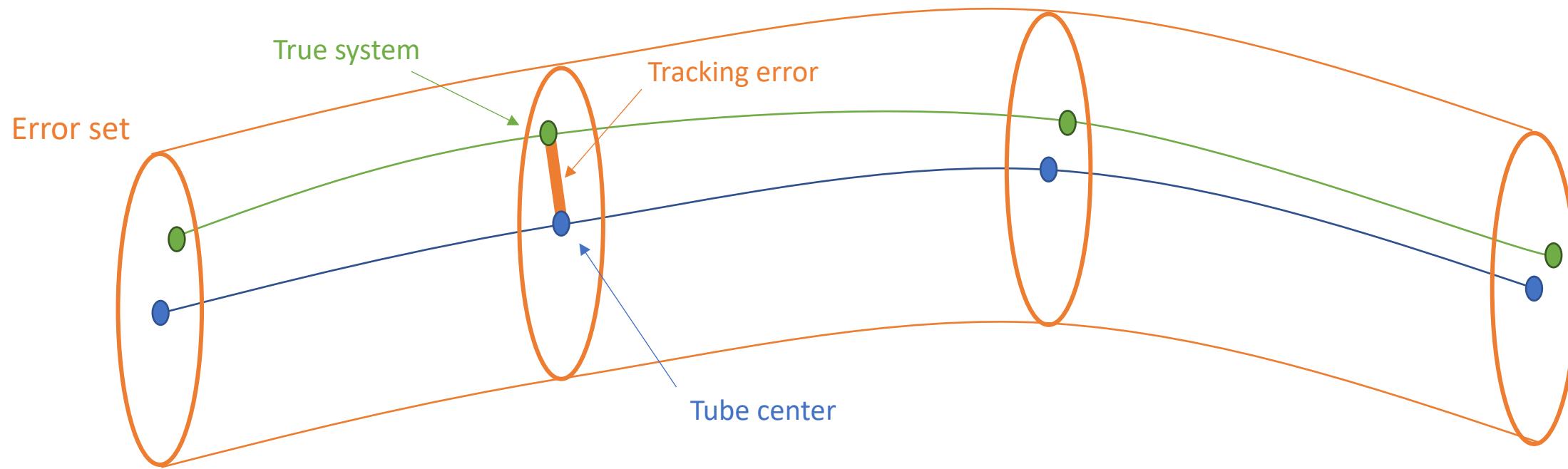
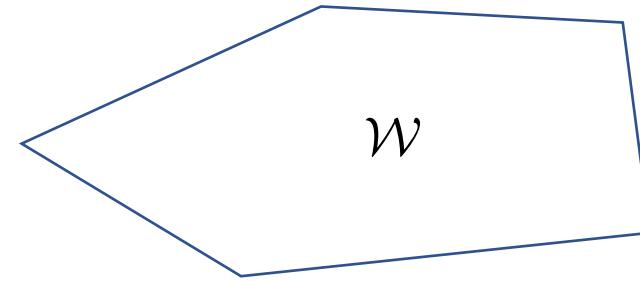
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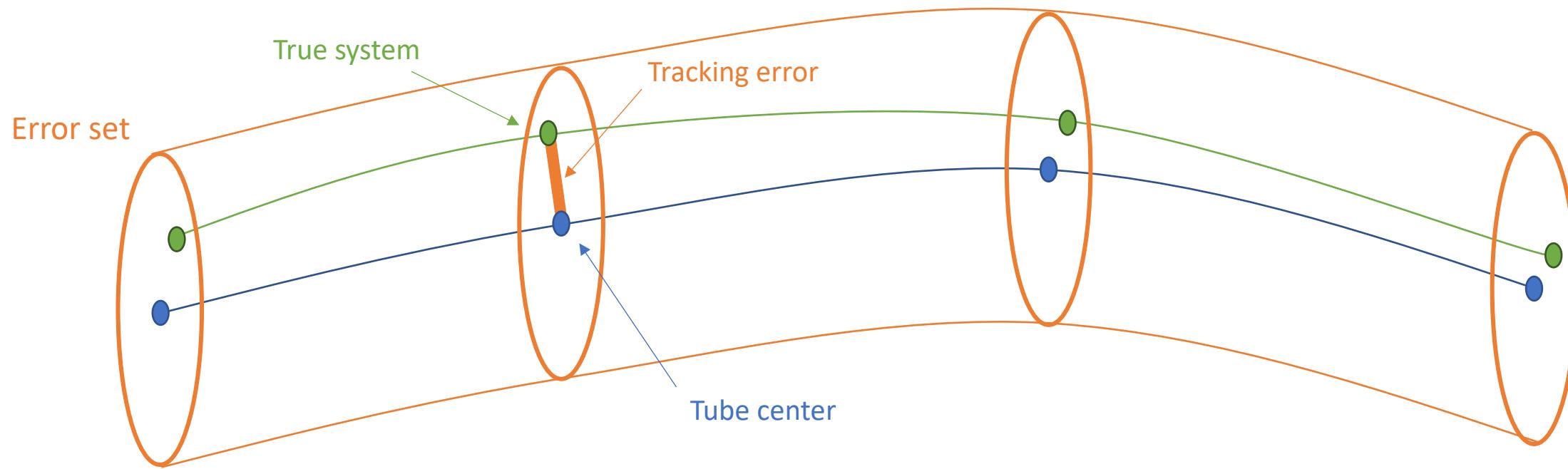
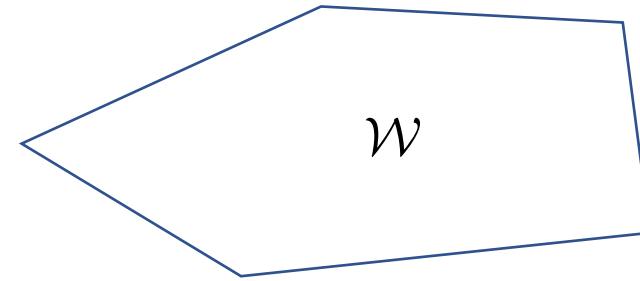
Key Ideas:

- ▶ Tube center planned via MPC
- ▶ Feedback to reduce tracking error

Fixed tube robust MPC

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Key Ideas:

- ▶ Tube center planned via MPC
- ▶ Feedback to reduce tracking error



$$\pi(x) = \pi^{\text{MPC}}(x) + \pi^{\text{tracking}}(x, \text{tube center})$$

LMPC for uncertain systems

A sample-based approach

LMPC for uncertain systems – Problem Formulation

LMPC for uncertain systems – Problem Formulation

- ▶ Model Choice: Linear system subject to bounded additive uncertainty,

$$x_{k+1} = Ax_k + Bu_k + w_k, \text{ with } w_k \in \mathcal{W} \quad \forall k \geq 0$$

LMPC for uncertain systems – Problem Formulation

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- ▶ Constraint Satisfaction:
 - Robust, $x_k \in \mathcal{X}, \forall w_k \in \mathcal{W} \quad k \geq 0.$

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- ▶ Cost: Worst Case.

LMPC for uncertain systems – Problem Formulation

- ▶ Model Choice: Linear system subject to bounded additive uncertainty,

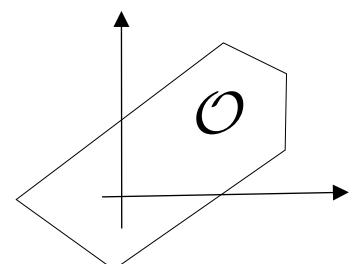
$$x_{k+1} = Ax_k + Bu_k + w_k, \text{ with } w_k \in \mathcal{W} \quad \forall k \geq 0$$

- ▶ Constraint Satisfaction:

- Robust, $x_k \in \mathcal{X}, \forall w_k \in \mathcal{W} \quad k \geq 0.$

- ▶ Cost: Worst Case.

- ▶ Convergence: Cannot regulate to a point but to a robust positive invariant $\mathcal{O}.$

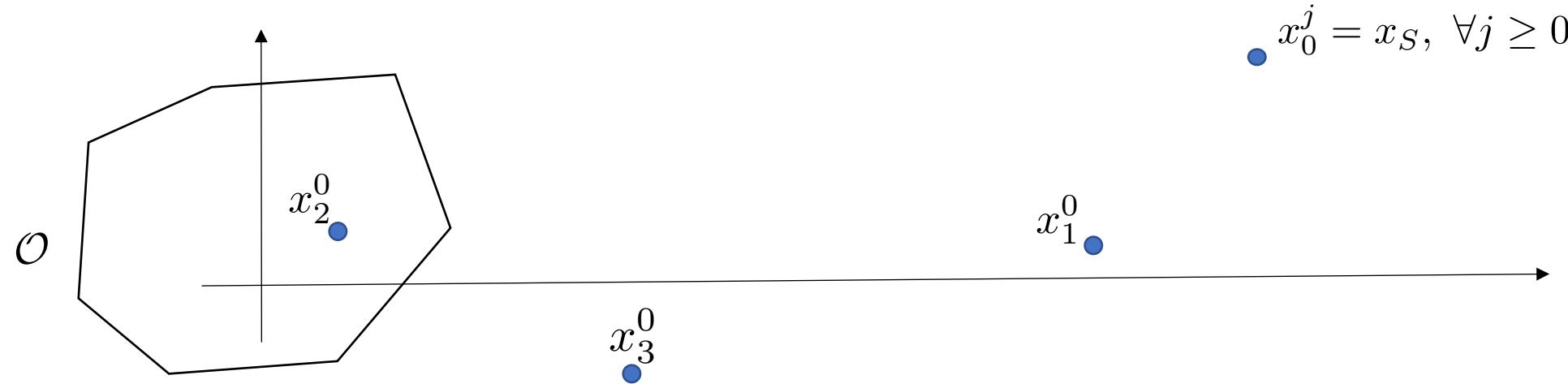


$$\forall x \in \mathcal{O} \rightarrow (A - KB)x + w \in \mathcal{O}, \quad \forall w \in \mathcal{W}$$

LMPC for uncertain systems – Differences w/ Nominal

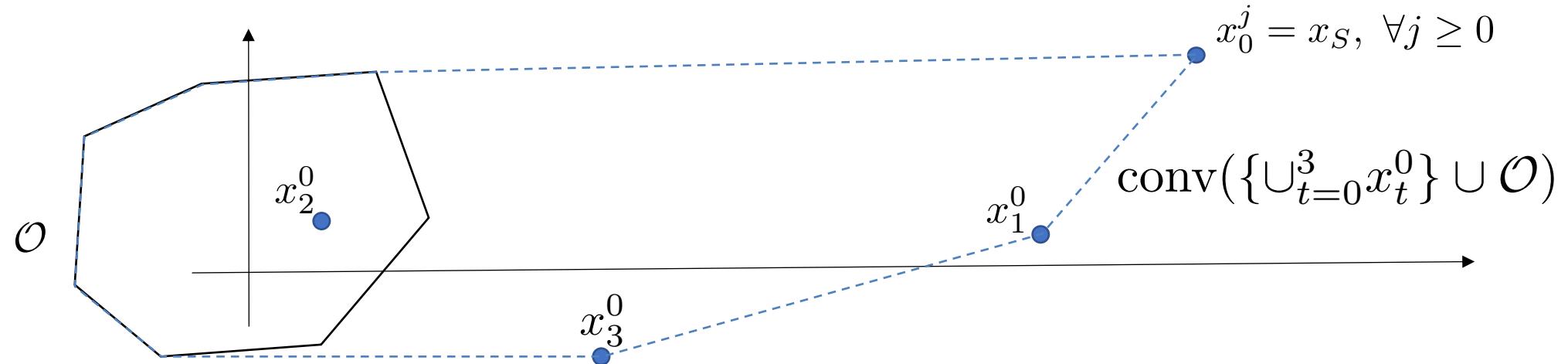
LMPC for uncertain systems – Differences w/ Nominal

Given a feasible trajectory which drives the system to the terminal robust invariant



LMPC for uncertain systems – Differences w/ Nominal

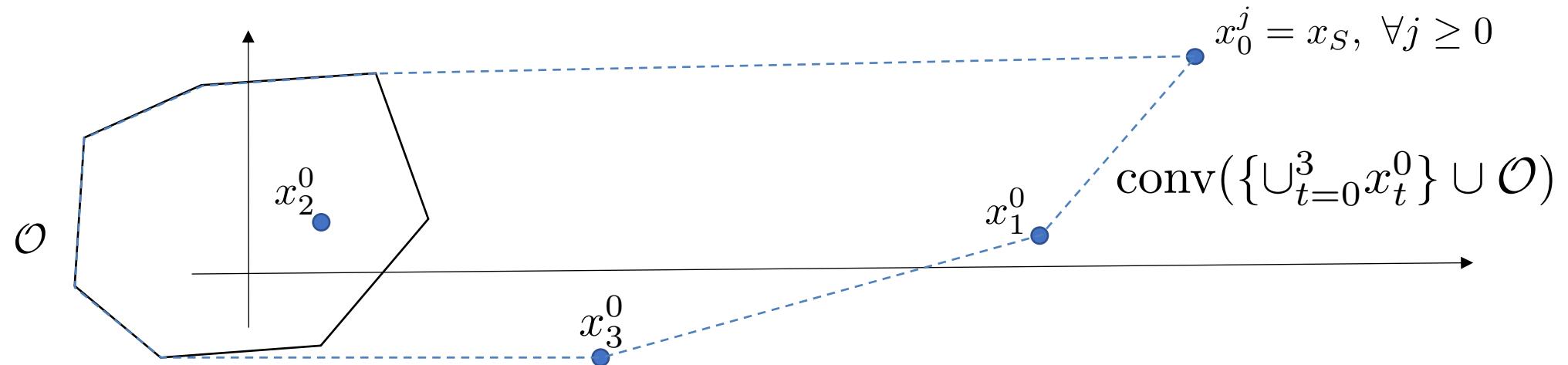
Given a feasible trajectory which drives the system to the terminal robust invariant



The convex hull of the stored data is not a robust control invariant.

LMPC for uncertain systems – Differences w/ Nominal

Given a feasible trajectory which drives the system to the terminal robust invariant



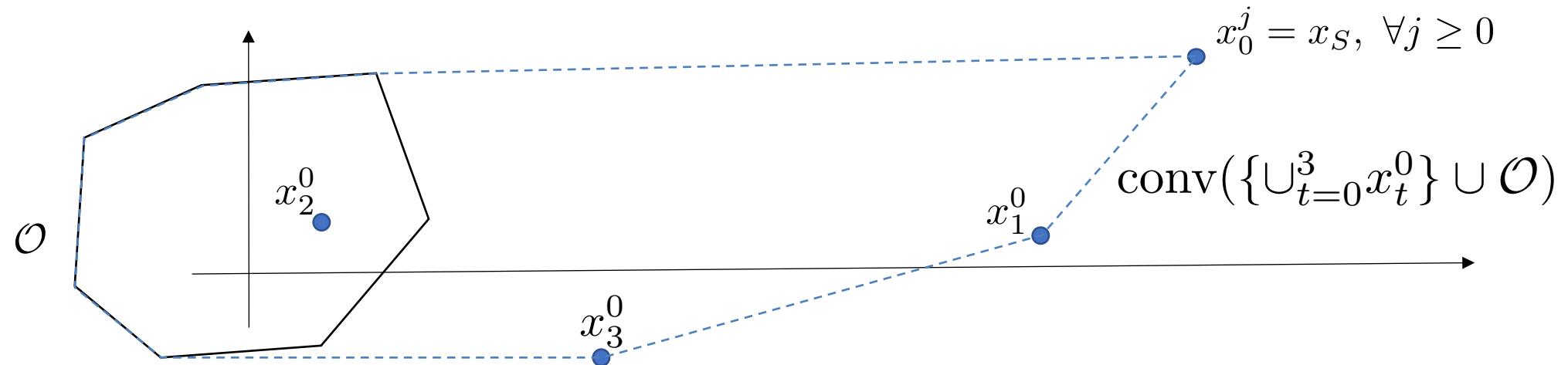
The convex hull of the stored data is not a robust control invariant.

For this reason,

- A. the safe set

LMPC for uncertain systems – Differences w/ Nominal

Given a feasible trajectory which drives the system to the terminal robust invariant



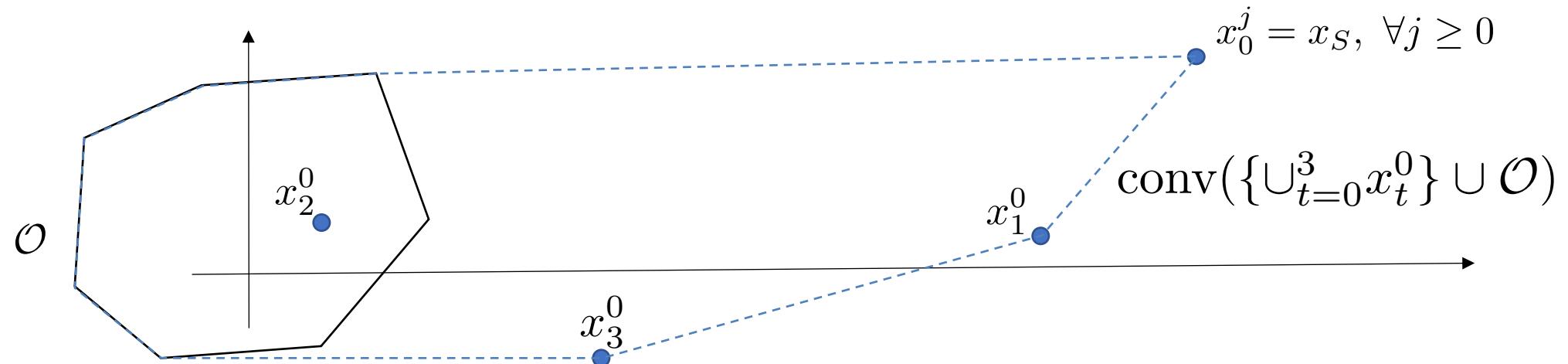
The convex hull of the stored data is not a robust control invariant.

For this reason,

- A. the safe set
- B. the terminal cost function

LMPC for uncertain systems – Differences w/ Nominal

Given a feasible trajectory which drives the system to the terminal robust invariant



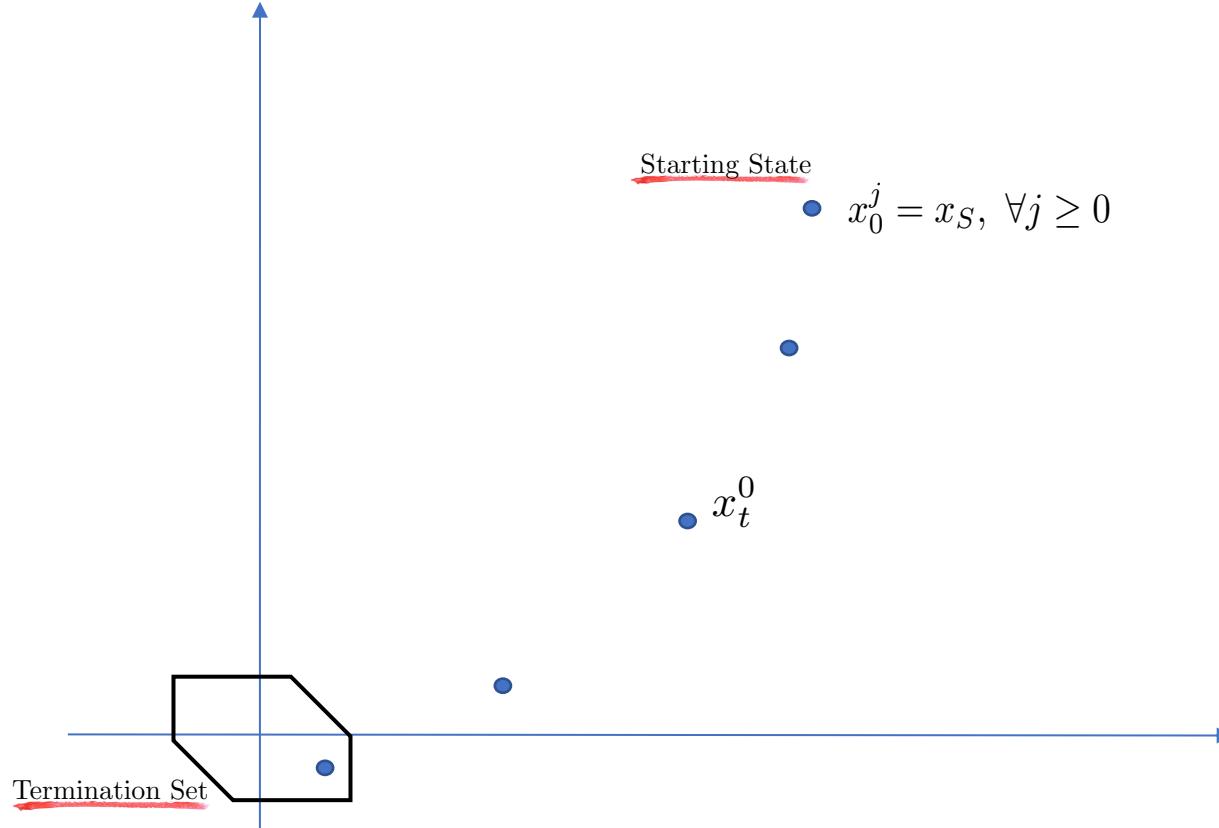
The convex hull of the stored data is not a robust control invariant.

For this reason,

- A. the safe set
- B. the terminal cost function

need to be designed differently in Robust LMPC.

Example: Constrained LQR Uncertain System



Control Problem

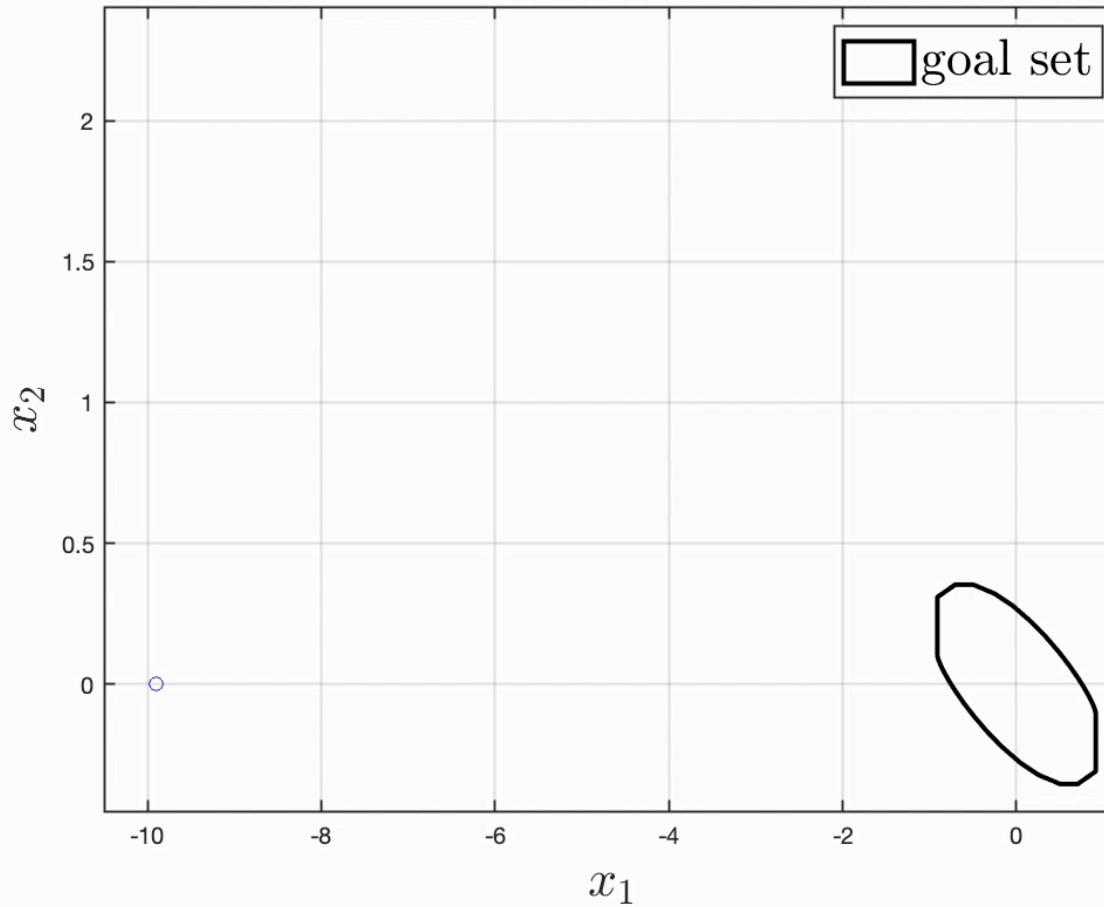
Consider the uncertain system

$$x_{k+1} = Ax_k + Bu_k + w_k, \text{ with } w_k \in \mathcal{W} \quad \forall k \geq 0$$

Our goal is to design a safe policy which steers the system to a terminal set

Example: Constrained LQR Uncertain System

- Policy π^1
- Policy π^2



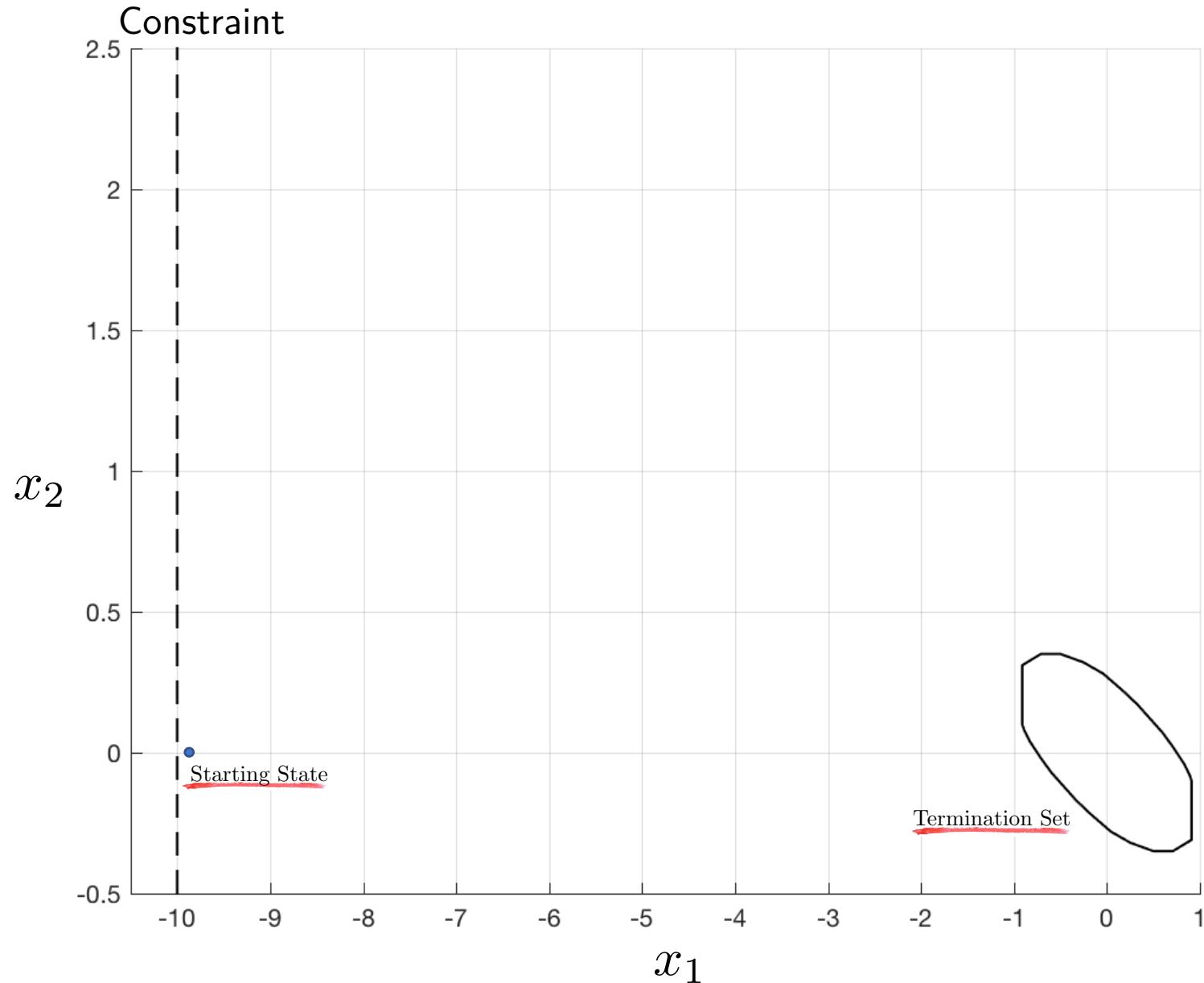
Control Problem

Consider the uncertain system

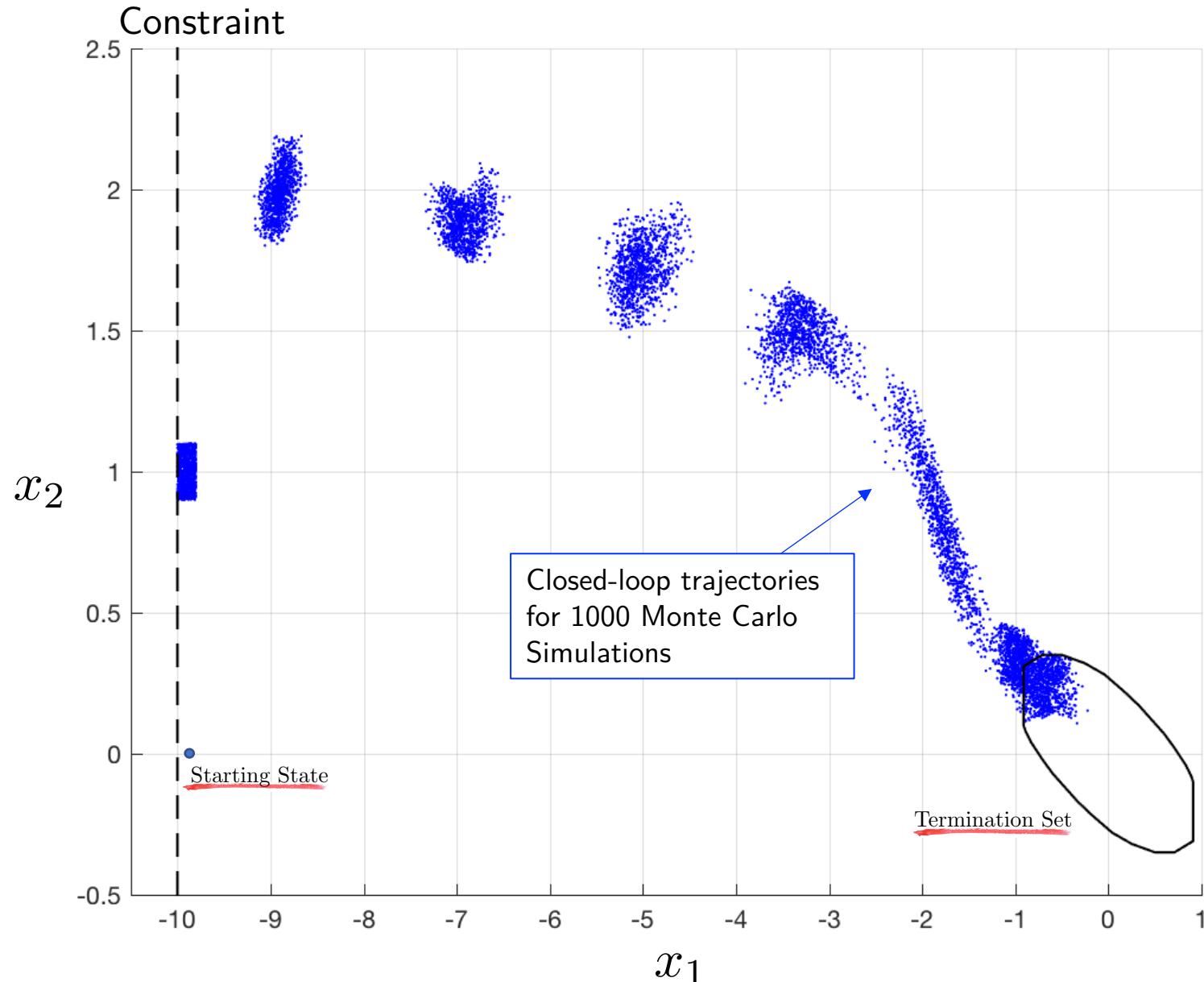
$$x_{k+1} = Ax_k + Bu_k + w_k, \text{ with } w_k \in \mathcal{W} \quad \forall k \geq 0$$

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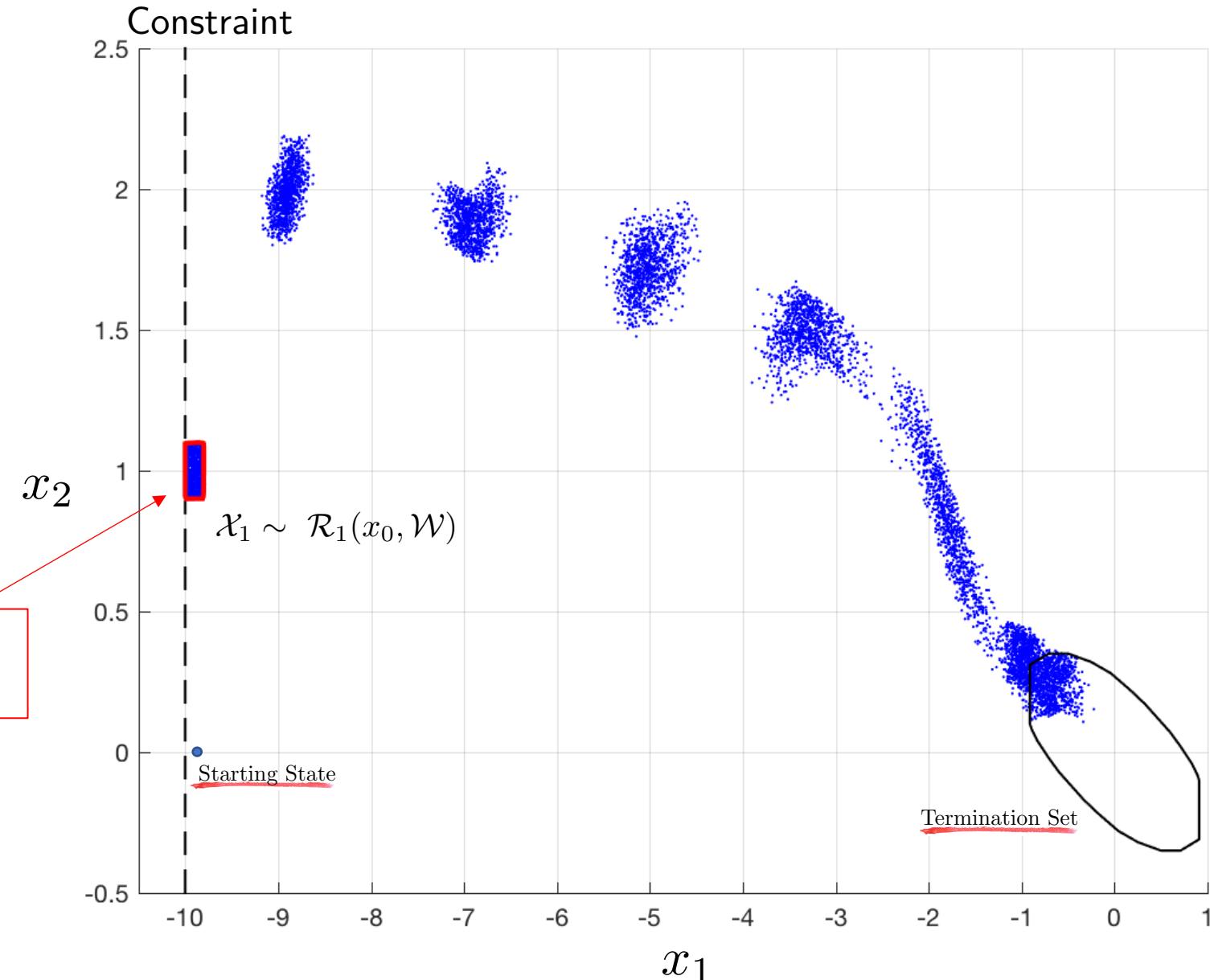
Example: Constrained LQR Uncertain System



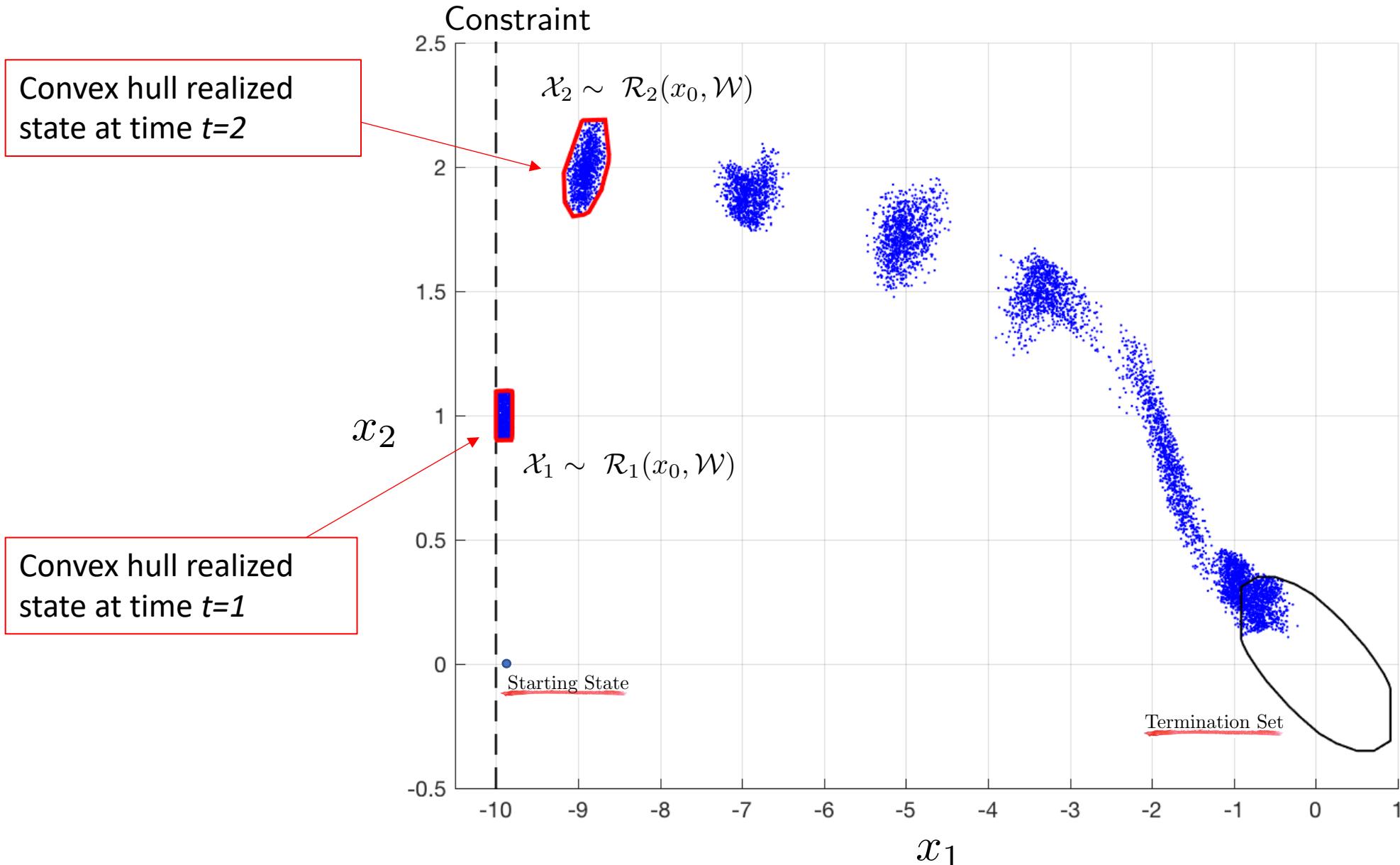
Example: Constrained LQR Uncertain System



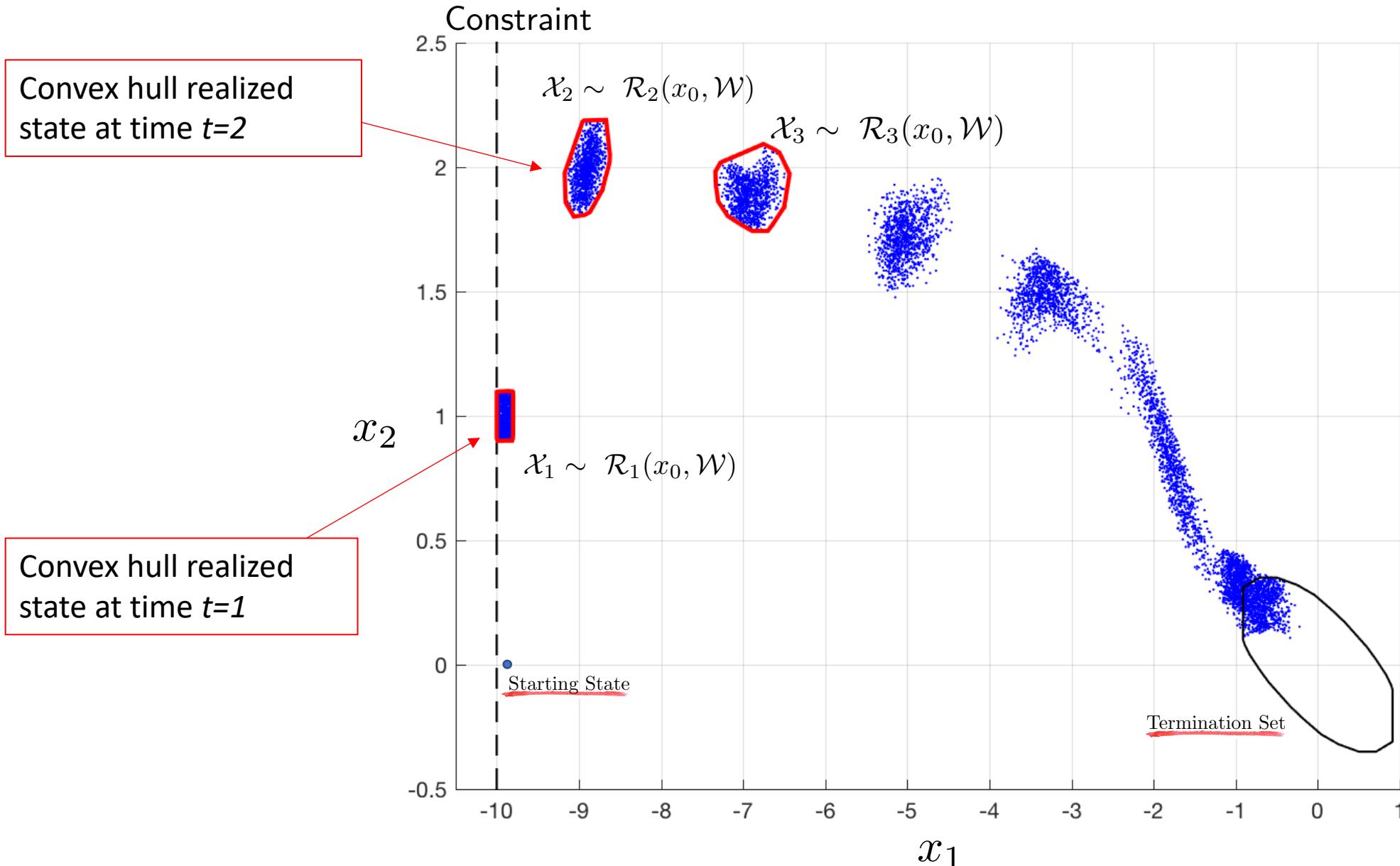
Example: Constrained LQR Uncertain System



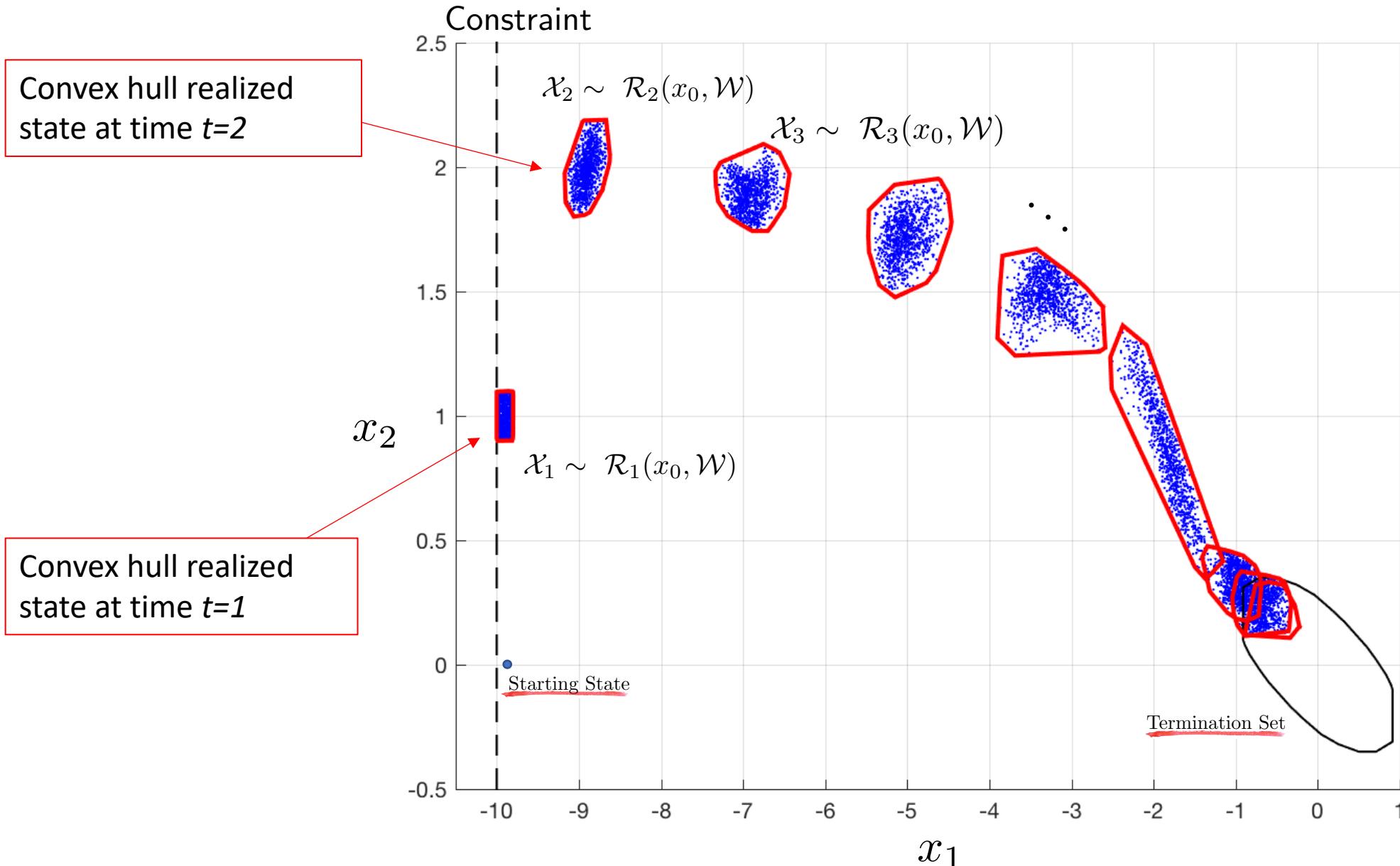
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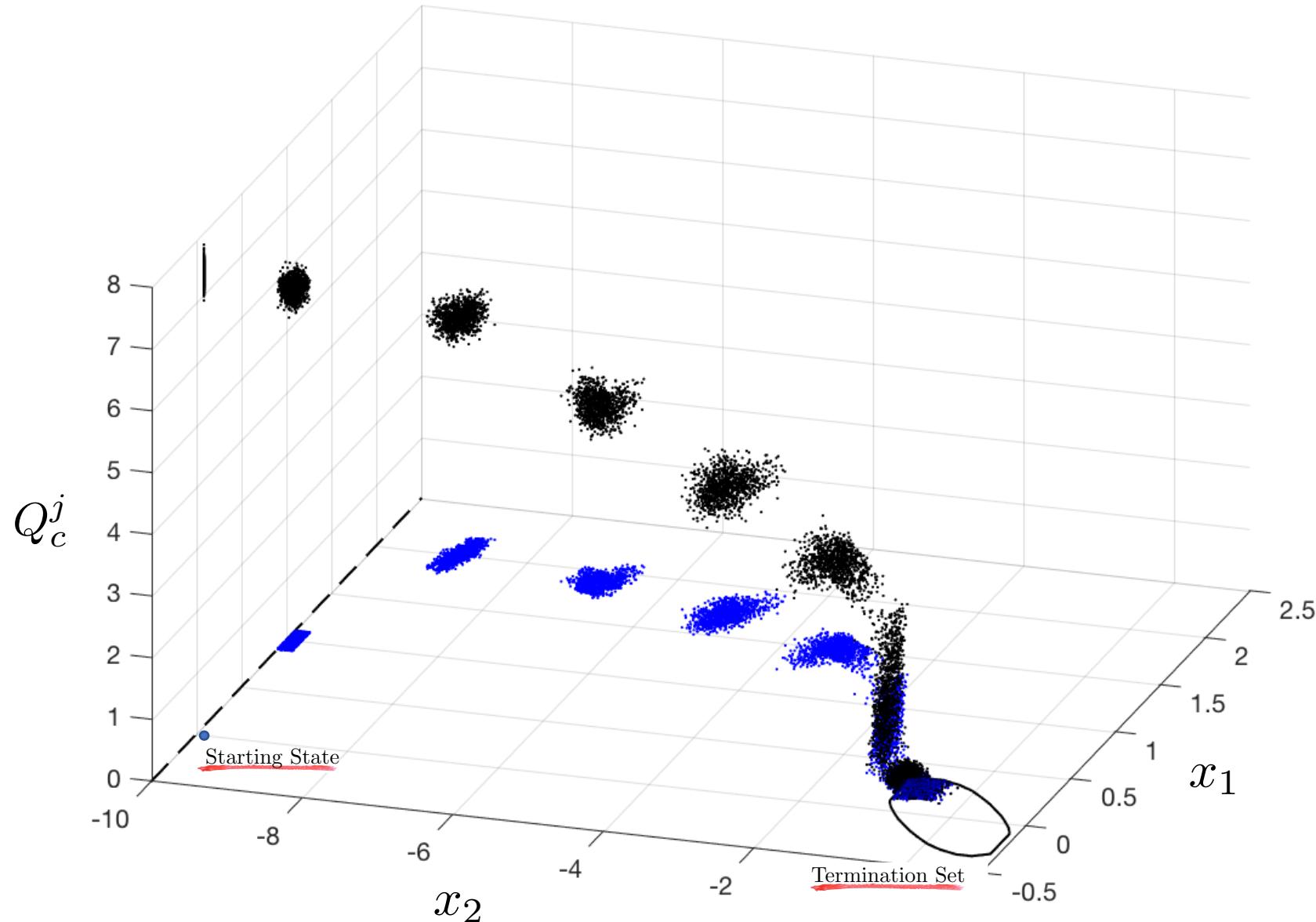
Example: Constrained LQR Uncertain System



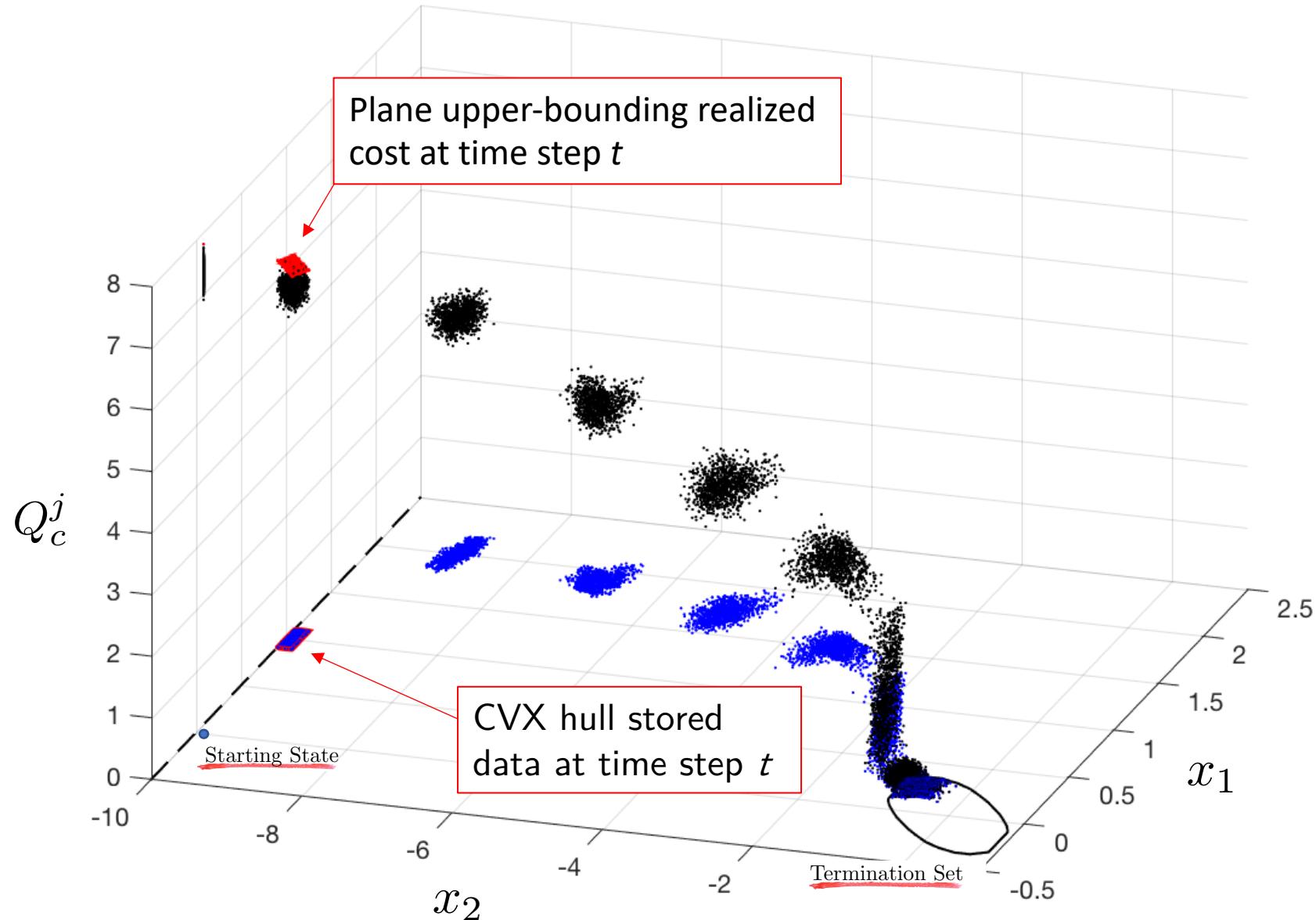
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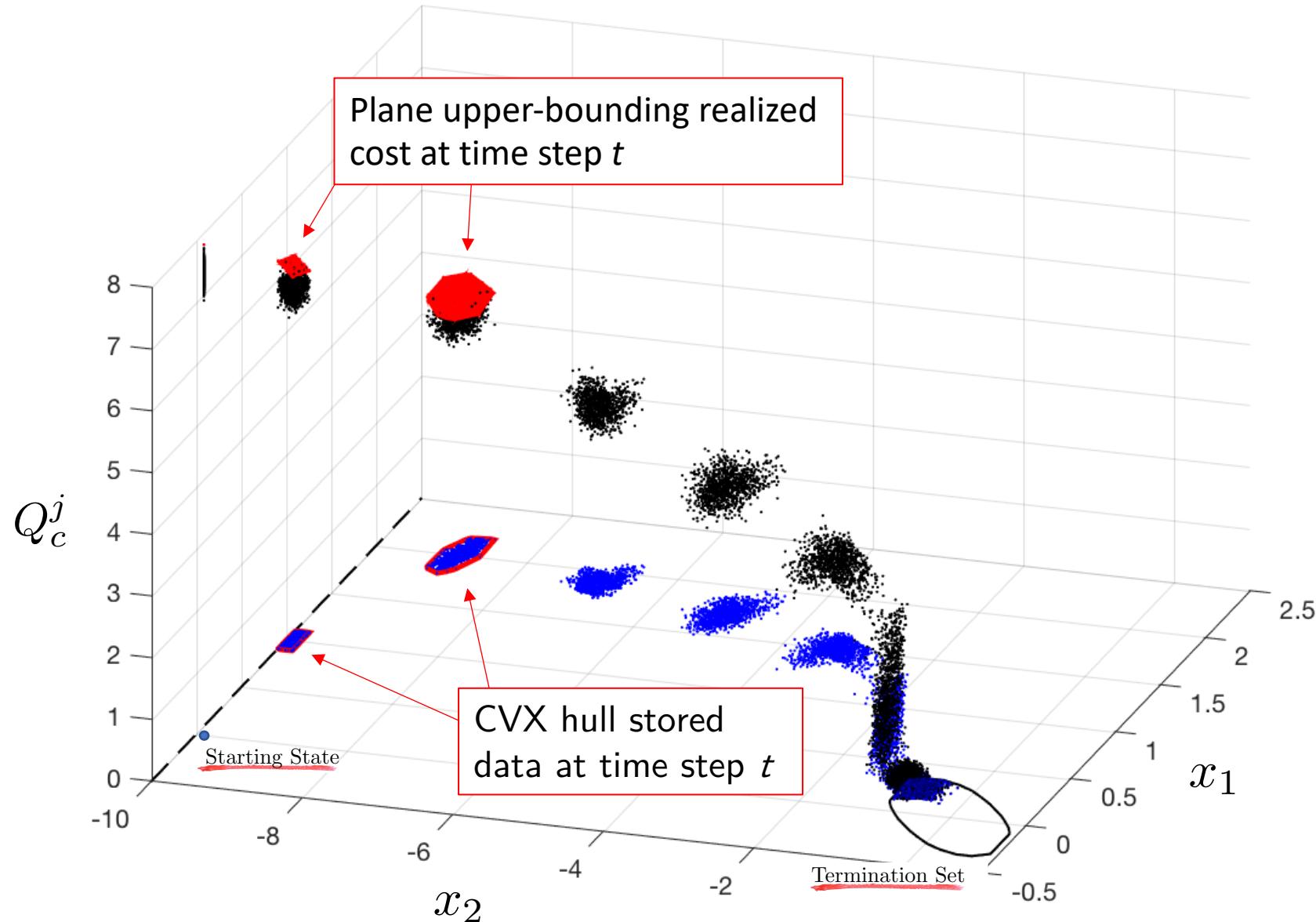
Example: Constrained LQR Uncertain System



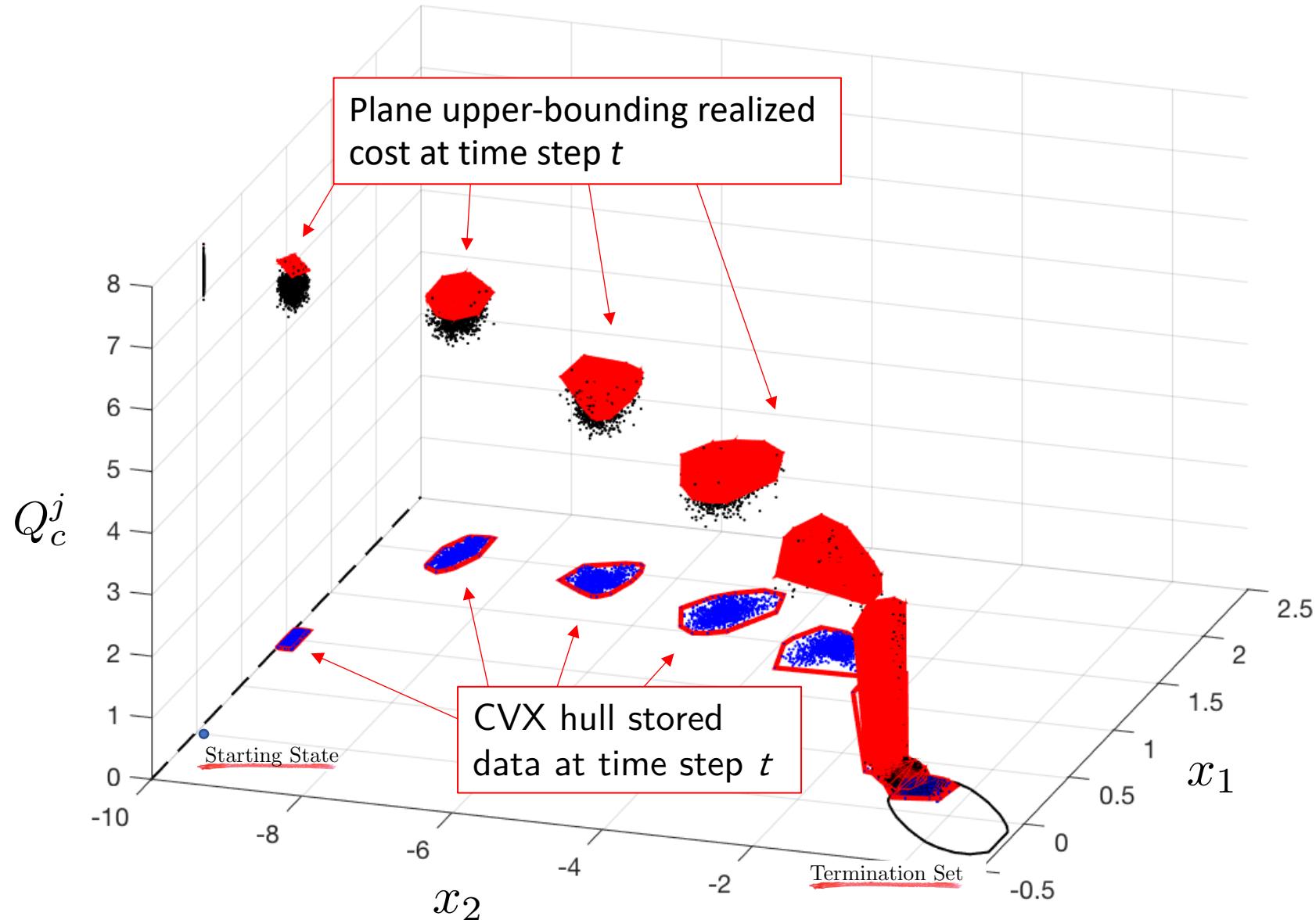
Example: Constrained LQR Uncertain System



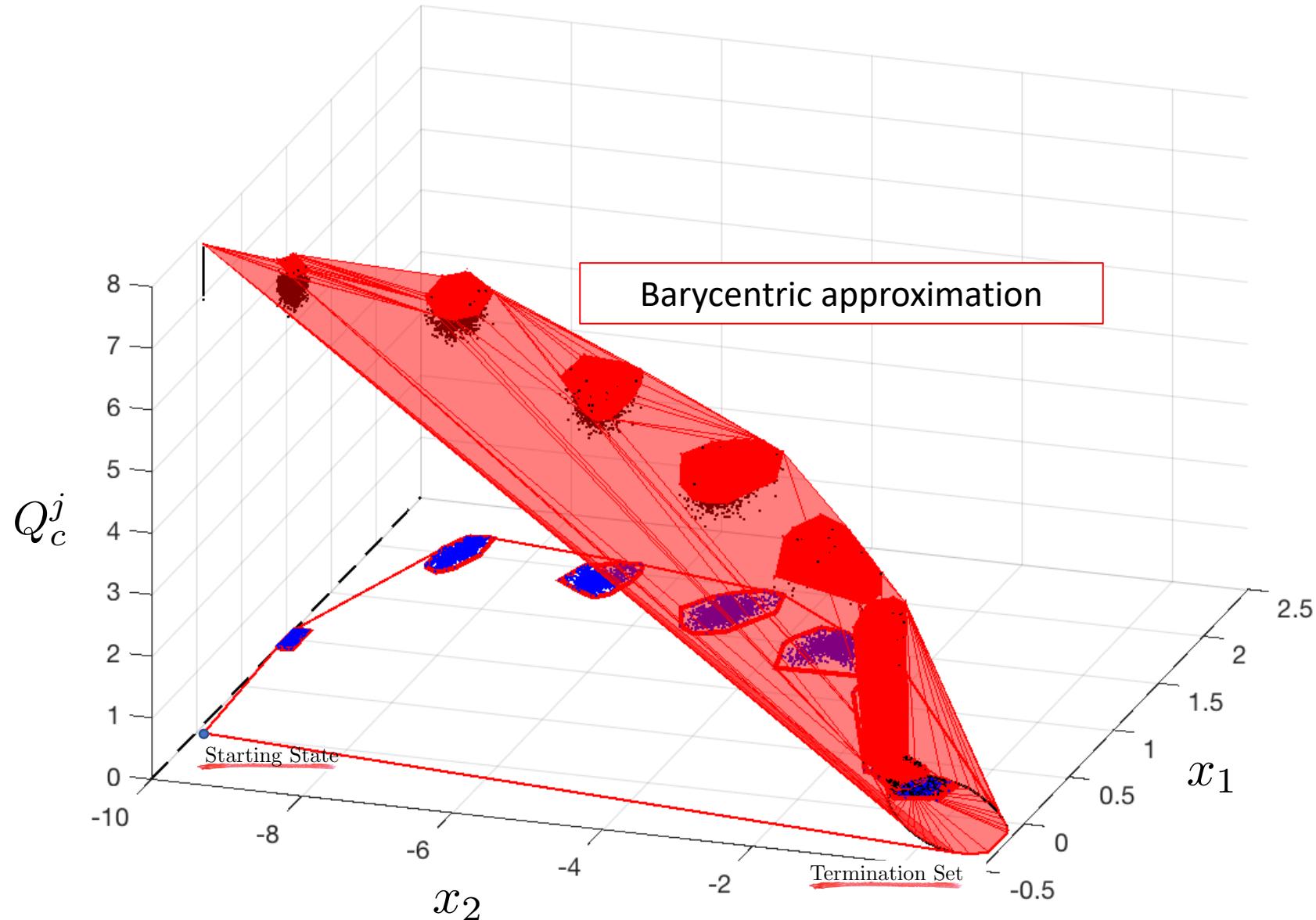
Example: Constrained LQR Uncertain System



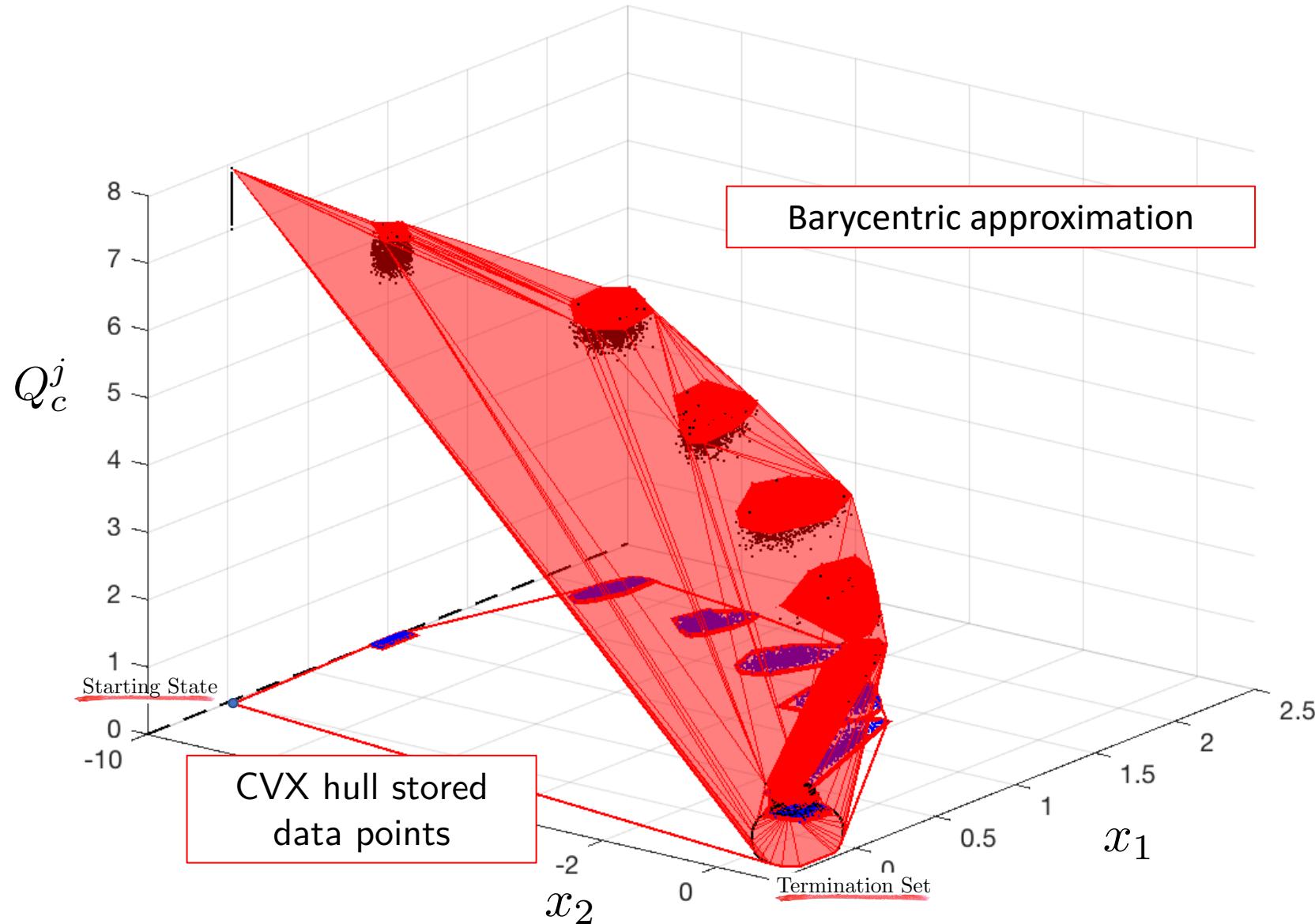
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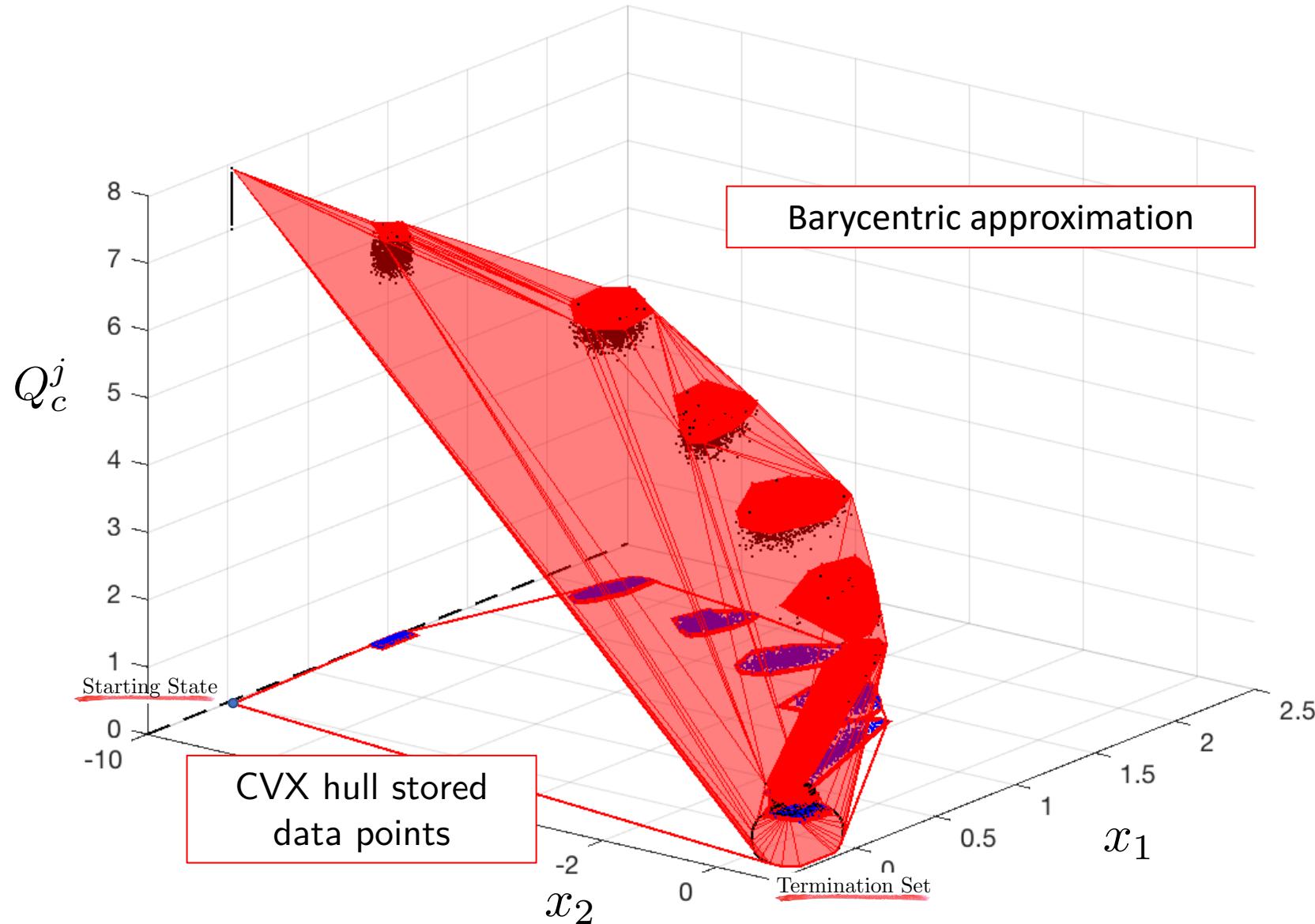
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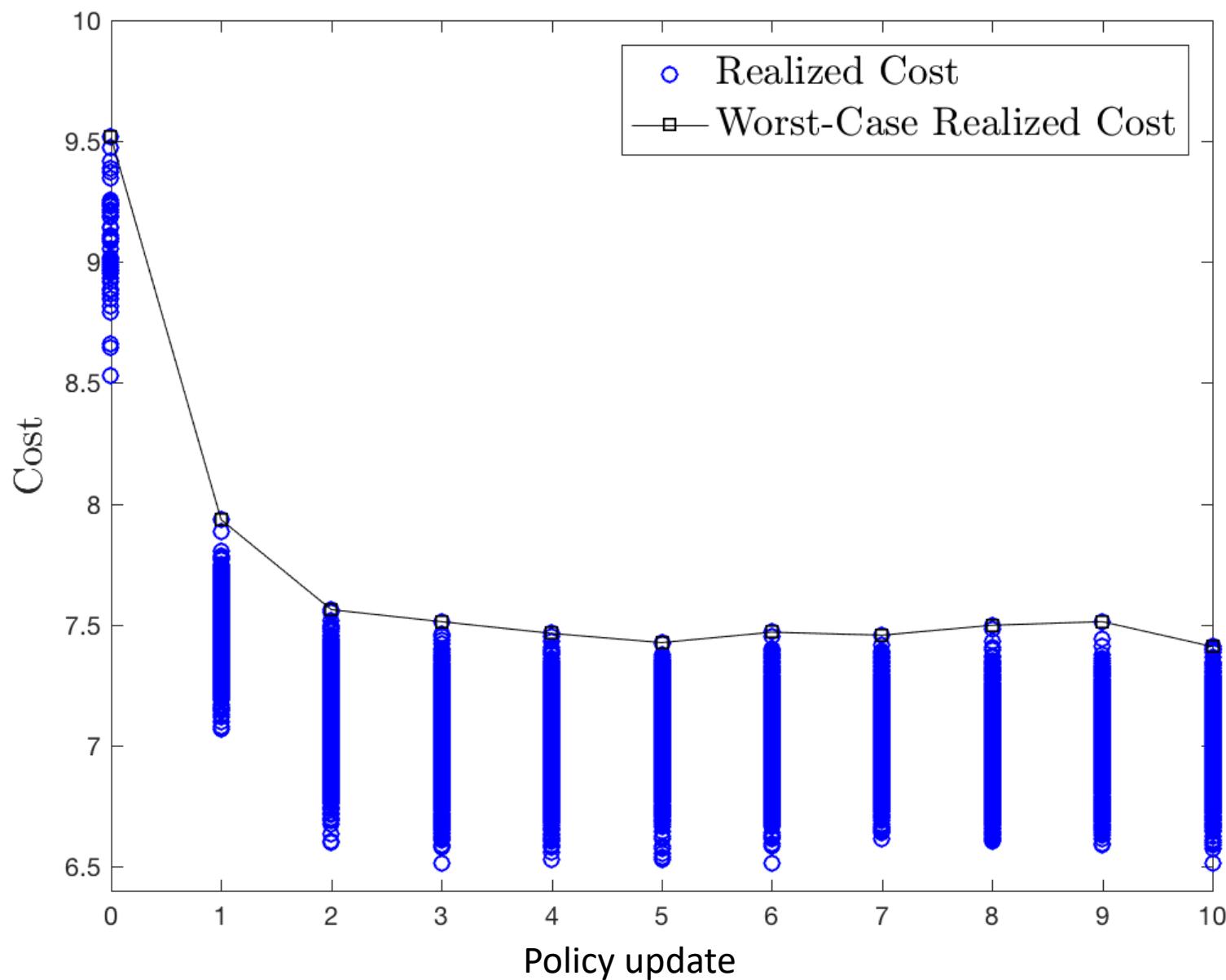
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Example: Constrained LQR Uncertain System



Example: Constrained LQR Uncertain System

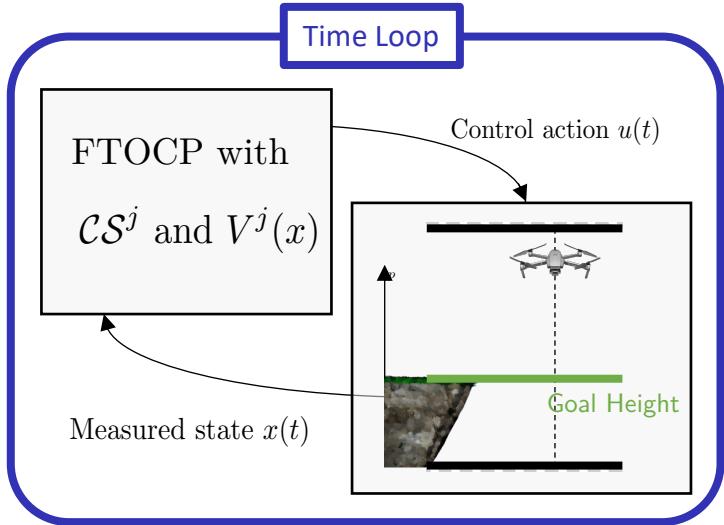


Summary Deterministic Vs Robust LMPC

LMPC for deterministic systems

LMPC for uncertain systems

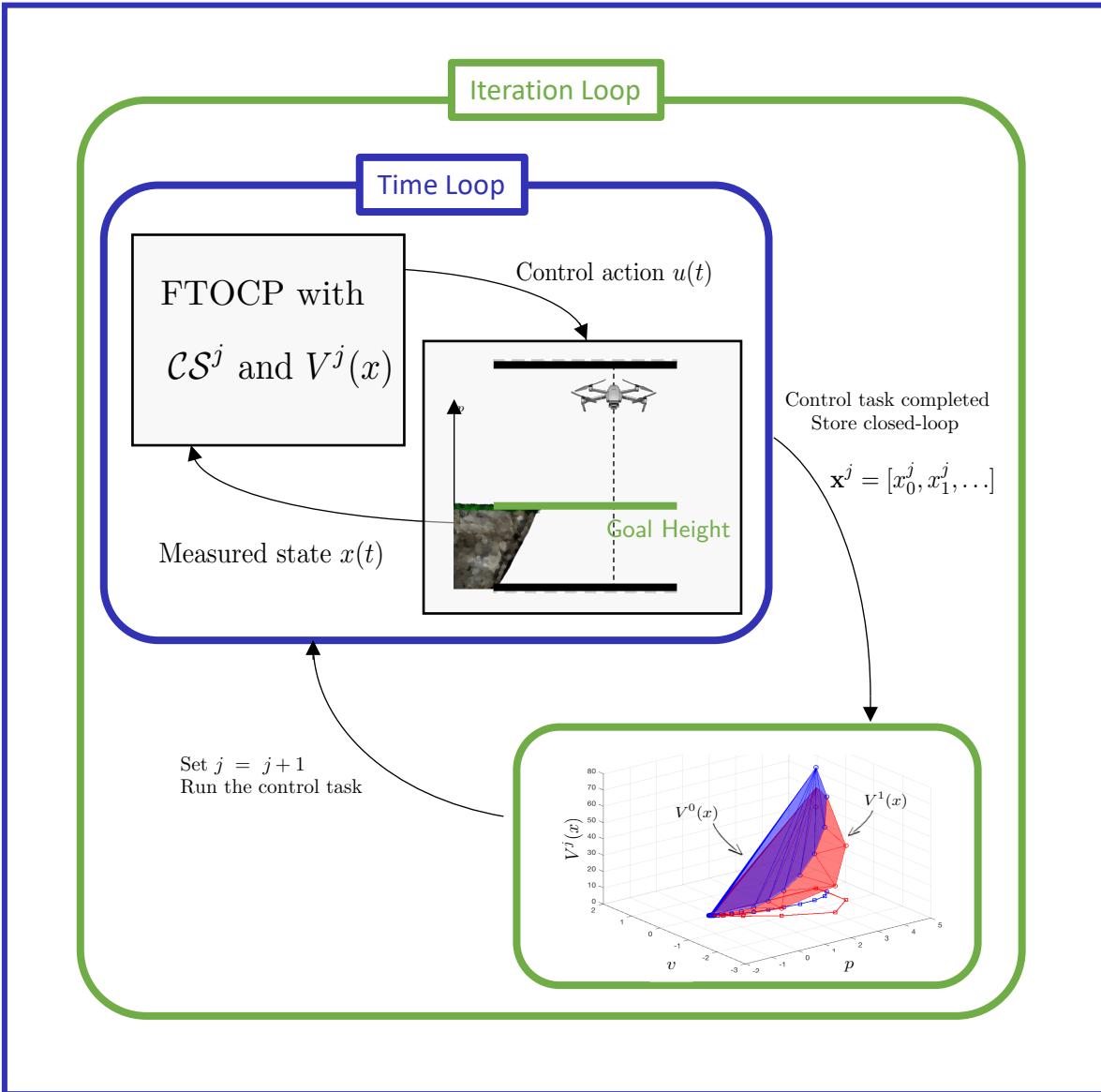
Summary Deterministic Vs Robust LMPC



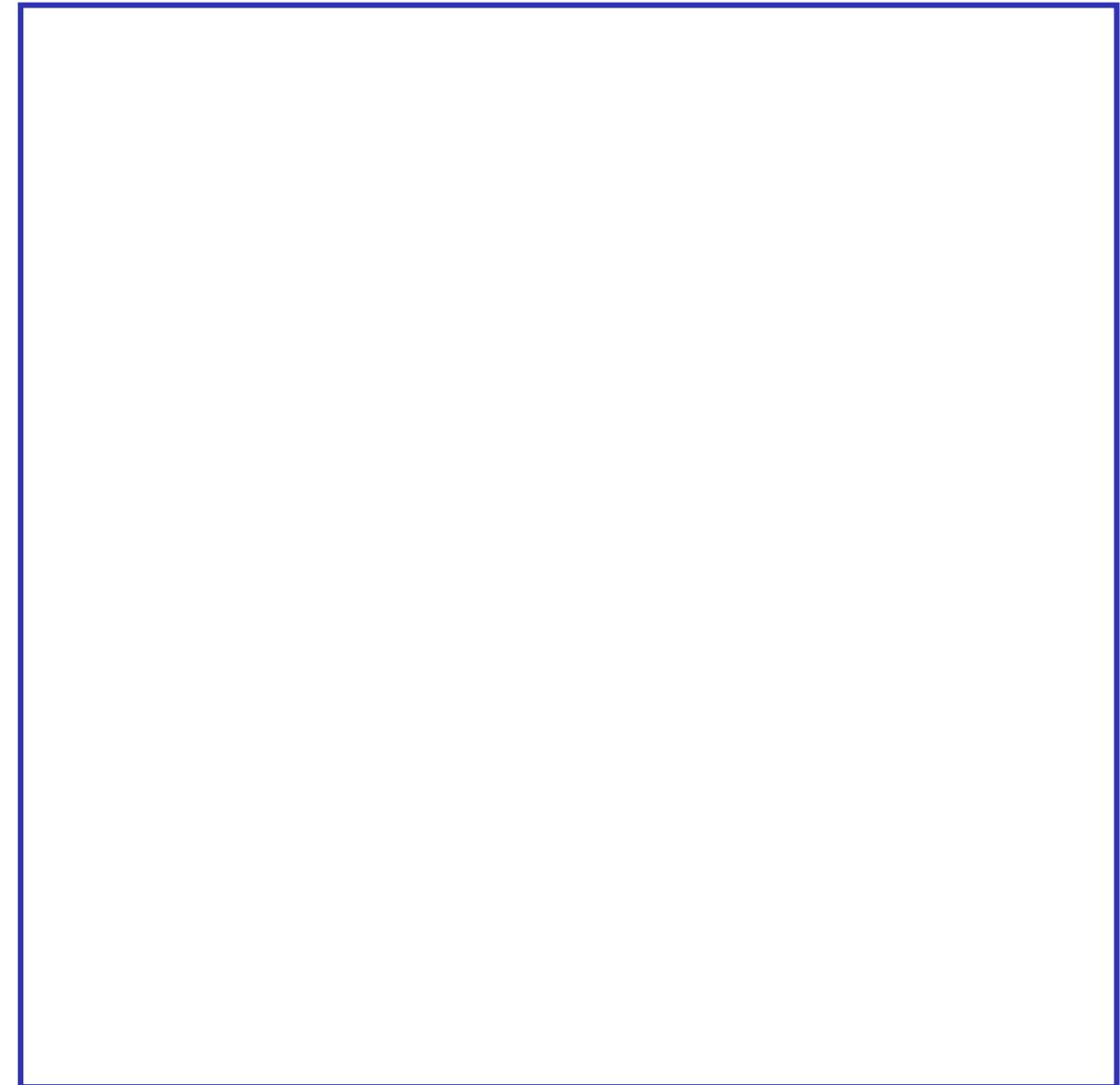
LMPC for deterministic systems

LMPC for uncertain systems

Summary Deterministic Vs Robust LMPC

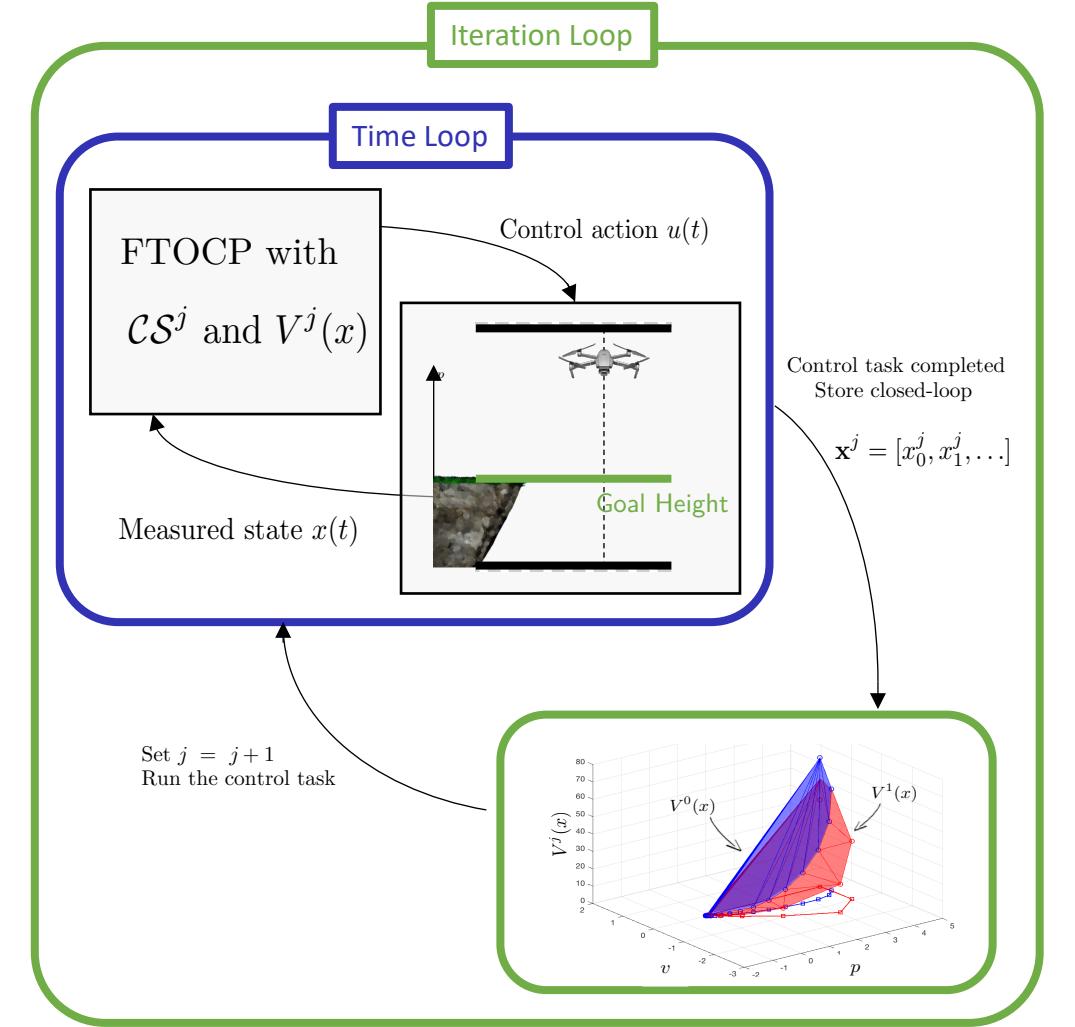


LMPC for deterministic systems

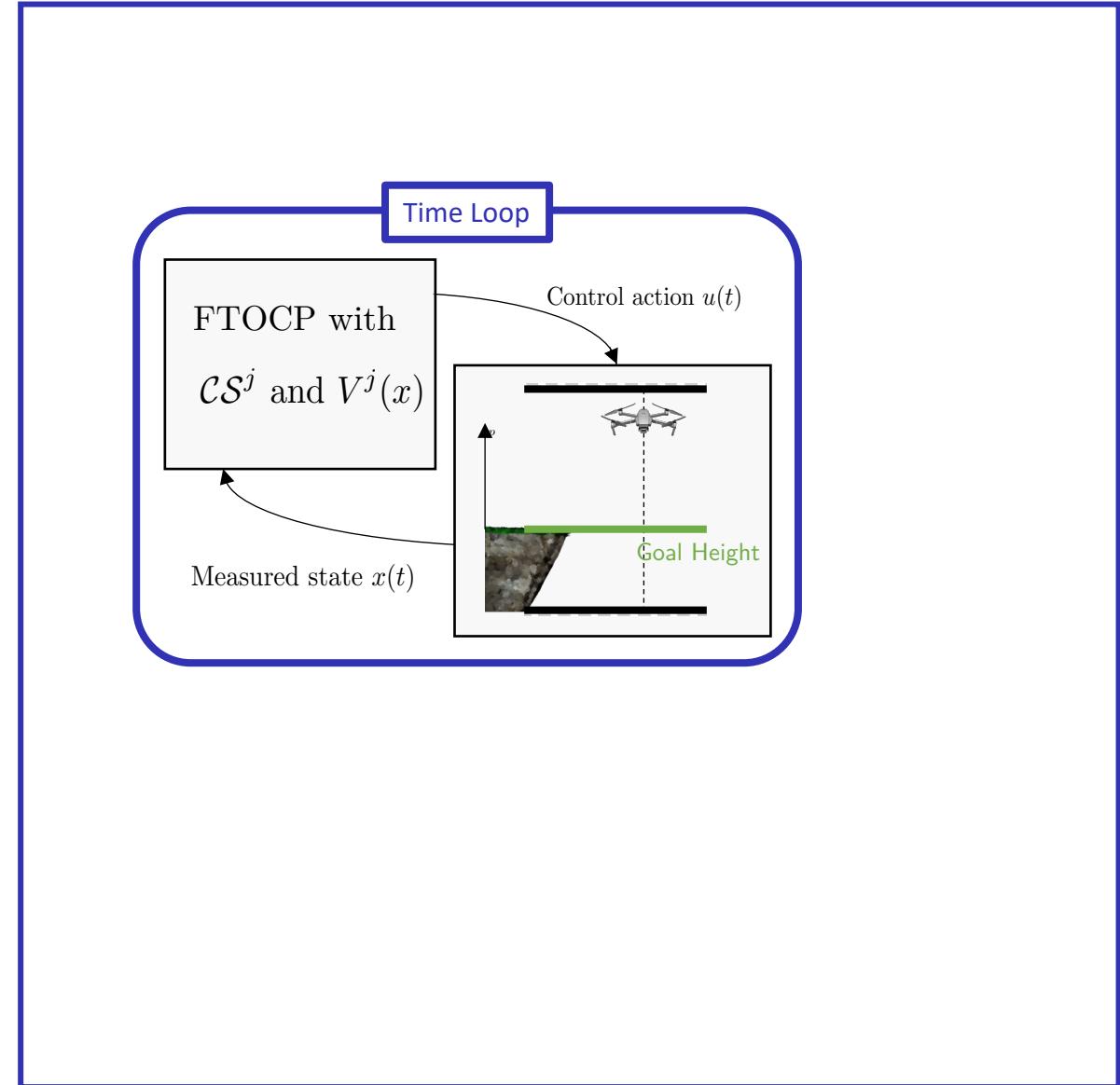


LMPC for uncertain systems

Summary Deterministic Vs Robust LMPC

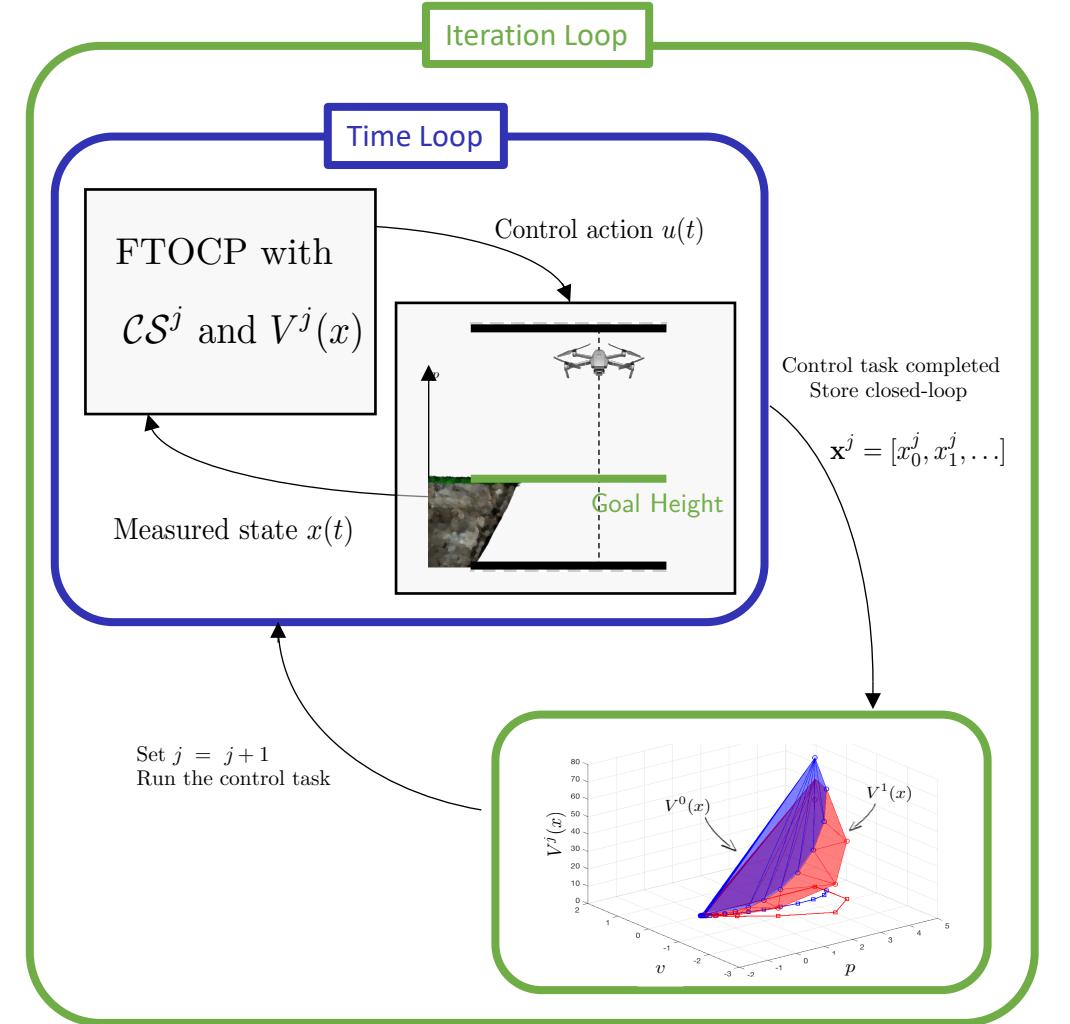


LMPC for deterministic systems

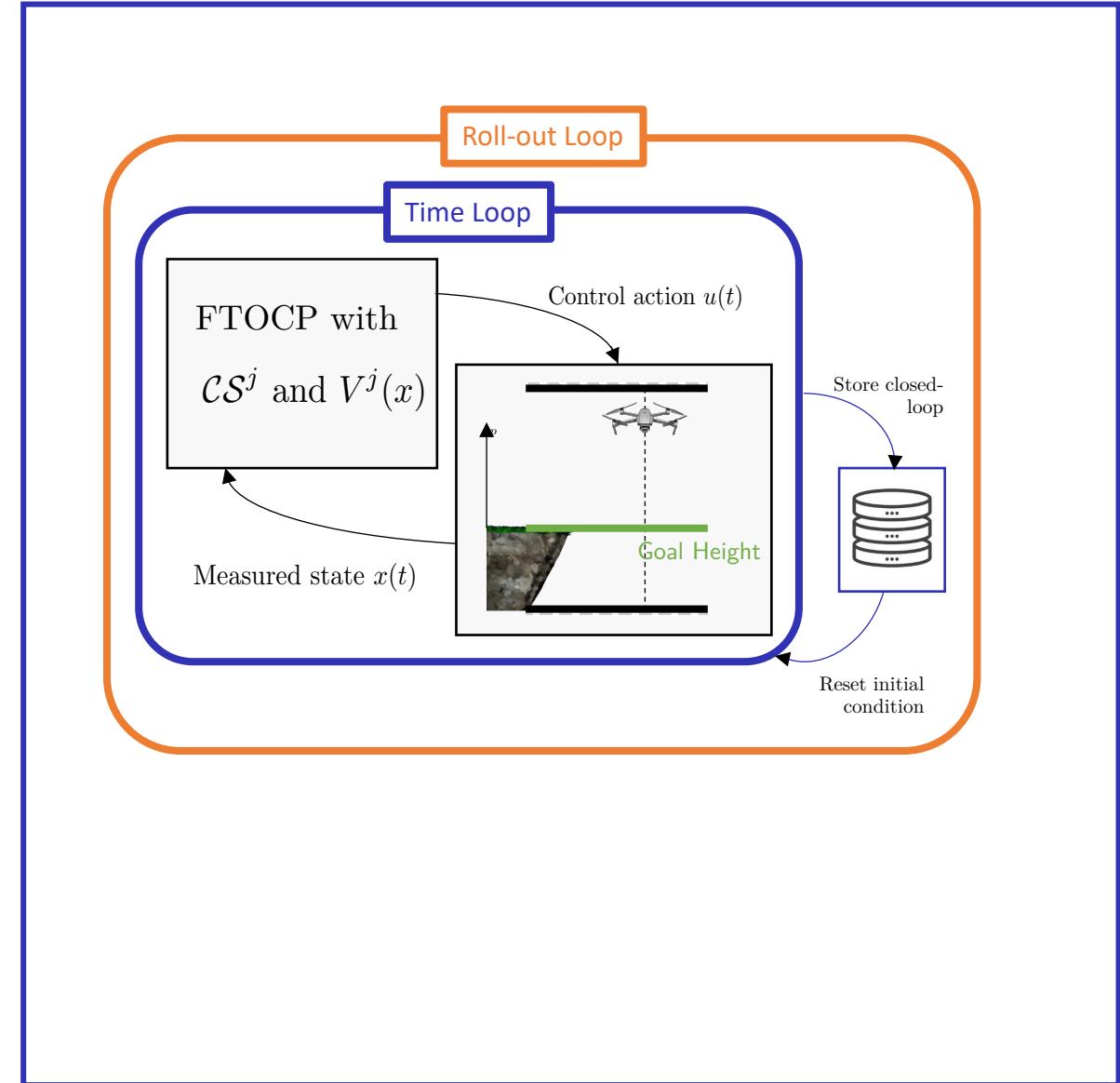


LMPC for uncertain systems

Summary Deterministic Vs Robust LMPC

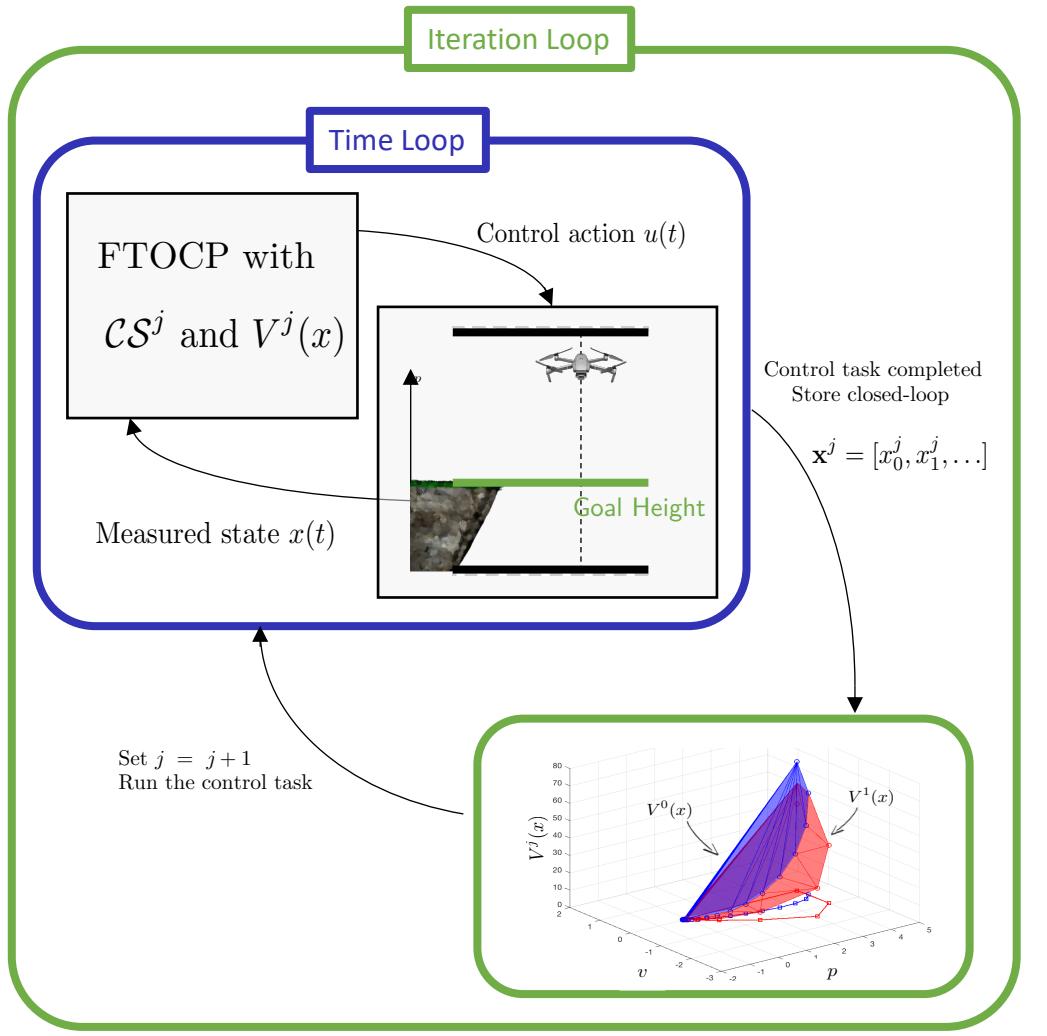


LMPC for deterministic systems

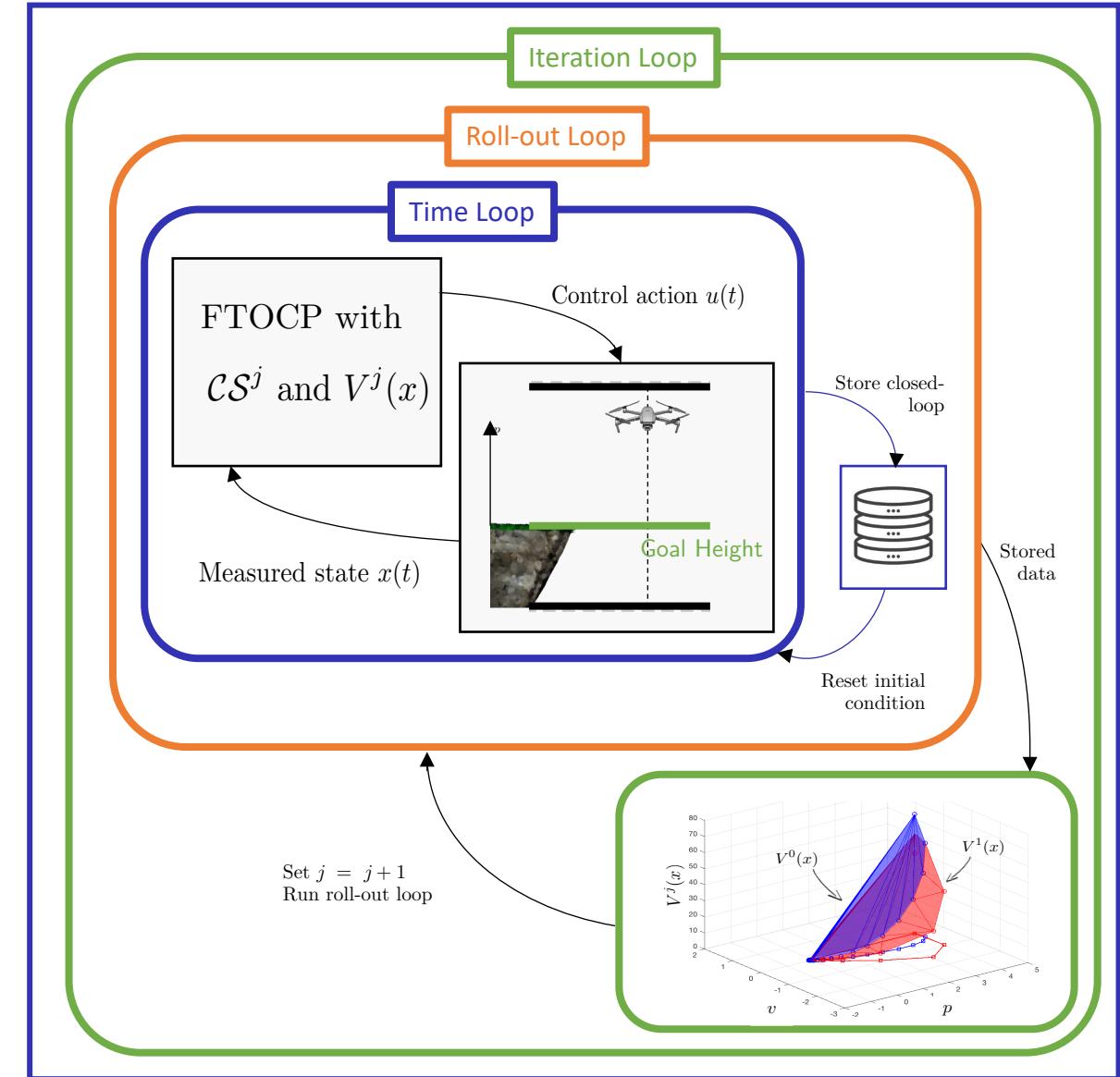


LMPC for uncertain systems

Summary Deterministic Vs Robust LMPC

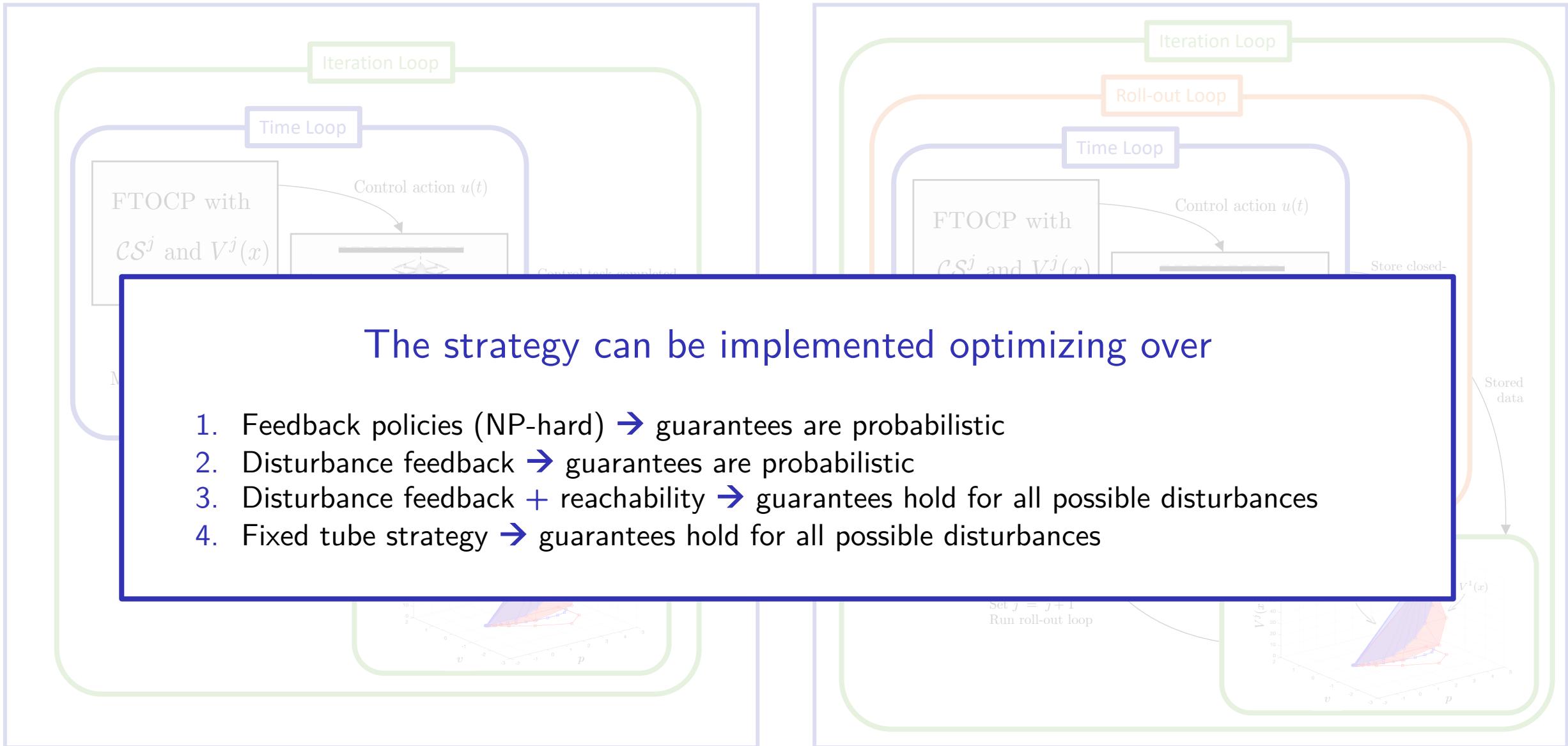


LMPC for deterministic systems



LMPC for uncertain systems

Summary Deterministic Vs Robust LMPC



LMPC for deterministic systems

LMPC for uncertain systems

Discussion

Some high-level ideas and on-going research

Three types of learning

Learning classification borrowed
from Prof. Borrelli

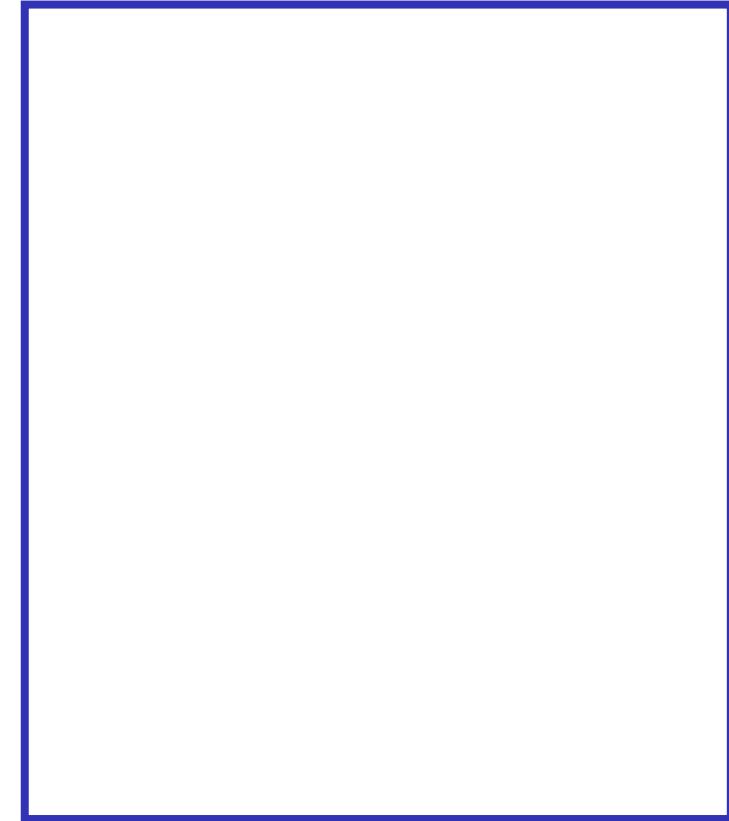


Three types of learning

Learning classification borrowed
from Prof. Borrelli



Skill acquisition

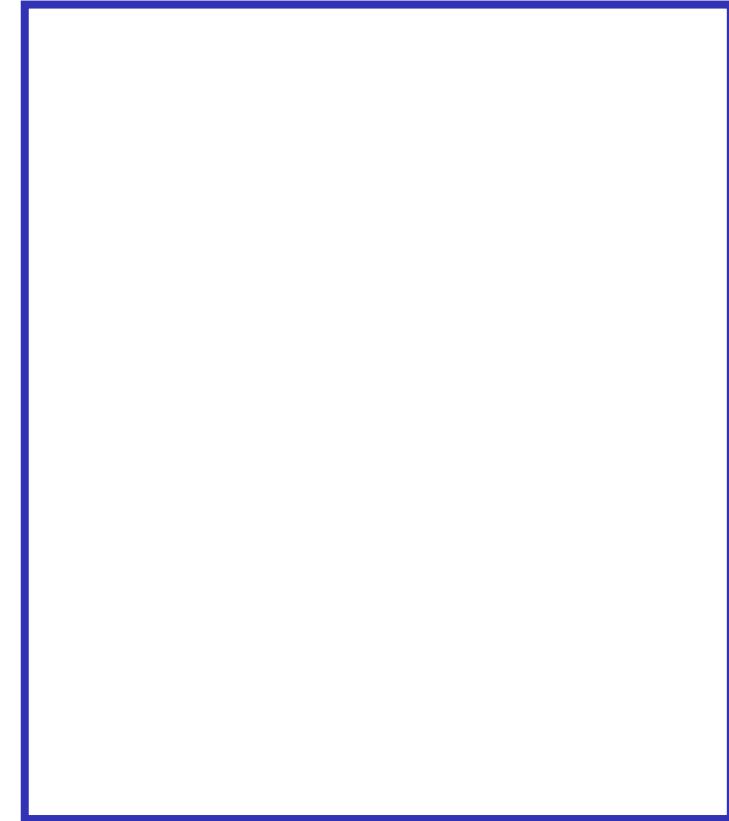


Three types of learning

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Skill acquisition



Three types of learning

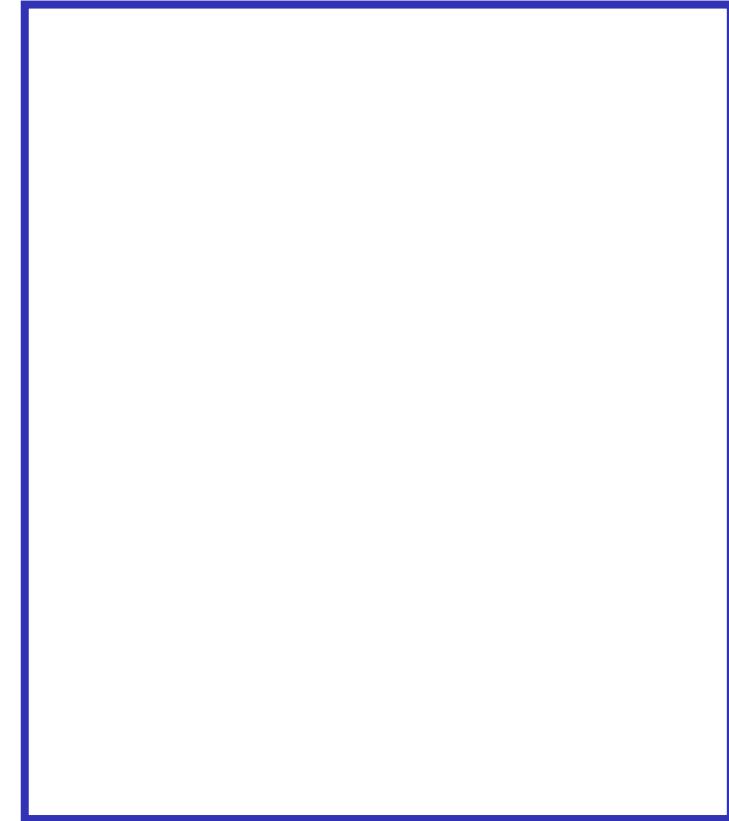
Learning classification borrowed
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Skill acquisition



Performance Improvement

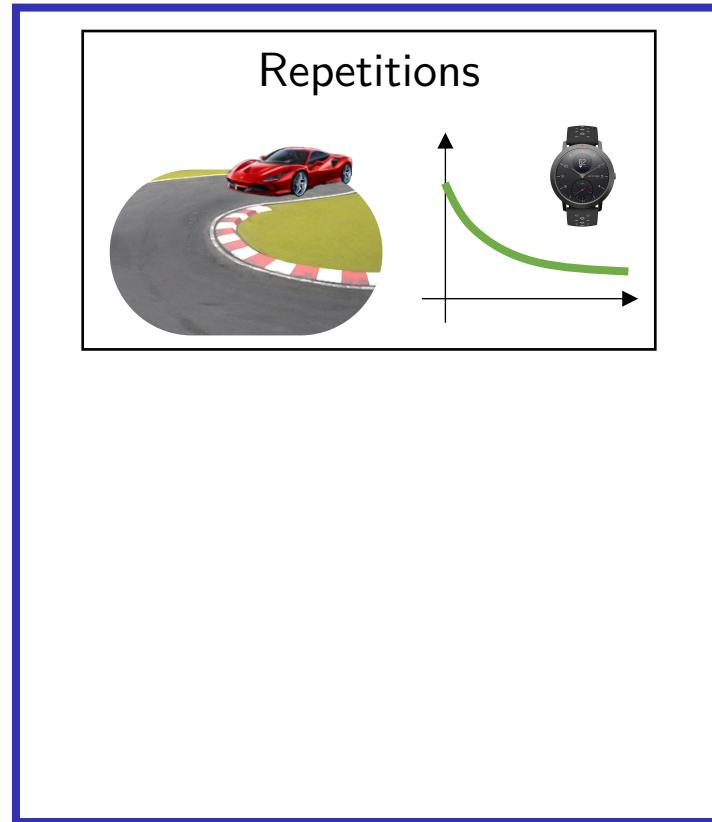


Three types of learning

Learning classification borrowed
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Skill acquisition



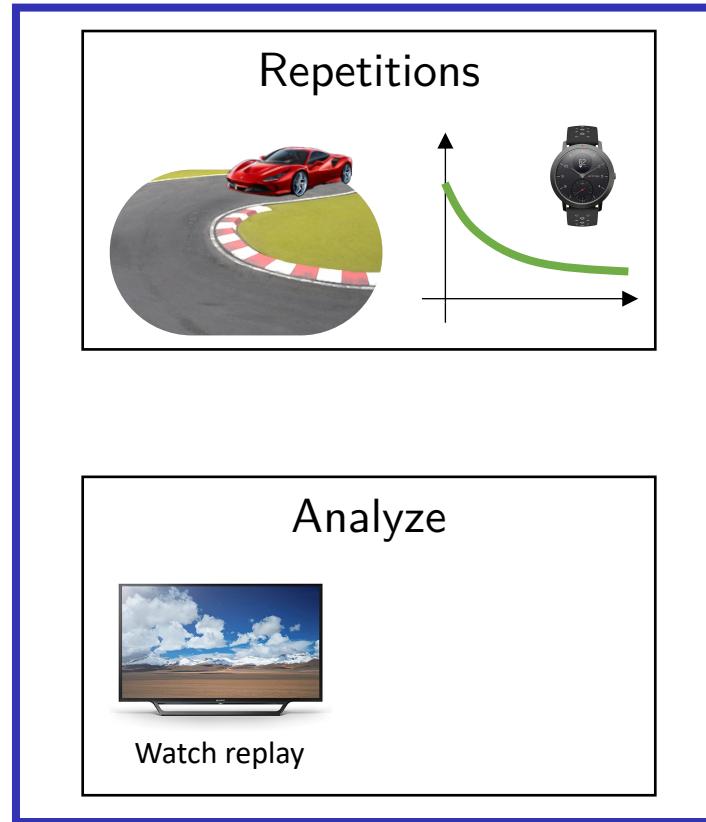
Performance Improvement

Three types of learning

Learning classification borrowed from Prof. Borrelli



Skill acquisition



Analyze



Watch replay

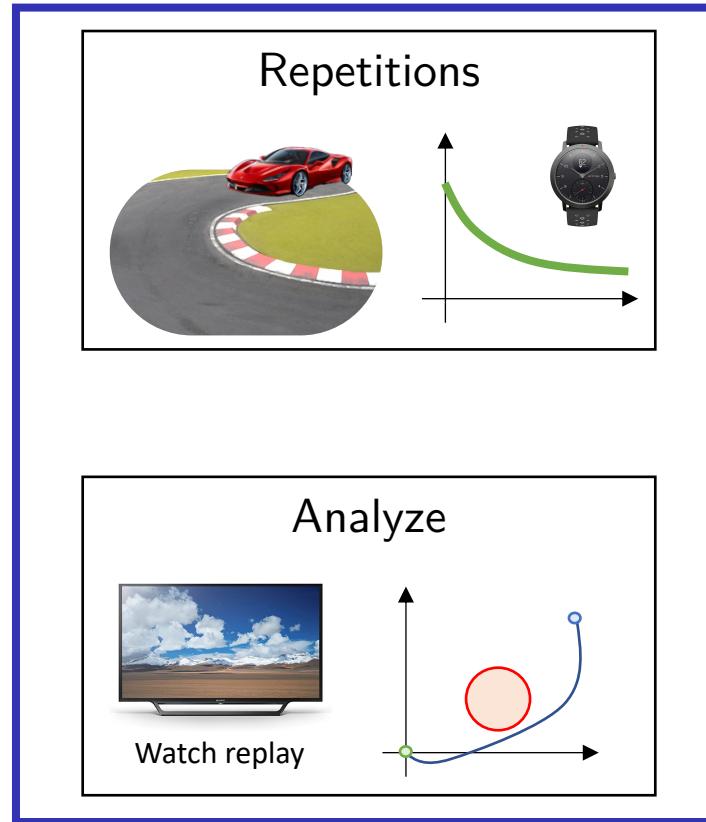


Three types of learning

Learning classification borrowed from Prof. Borrelli



Skill acquisition



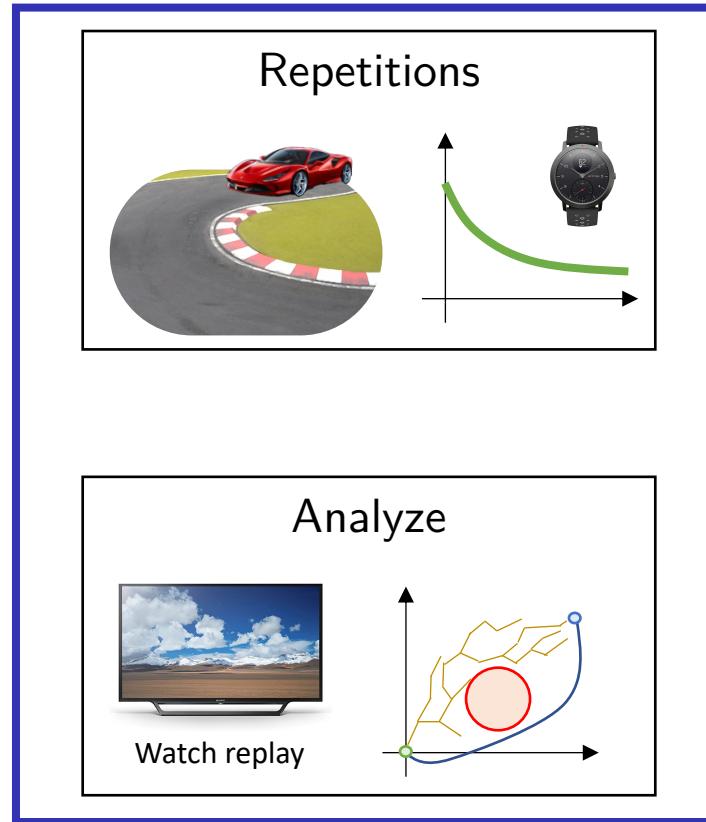
Performance Improvement

Three types of learning

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Skill acquisition



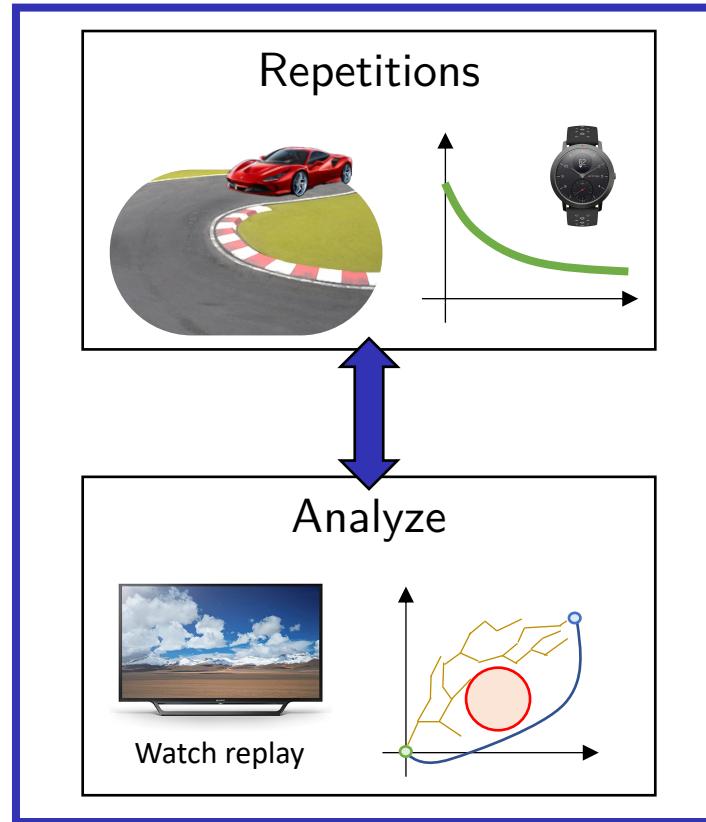
Performance Improvement

Three types of learning

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Skill acquisition



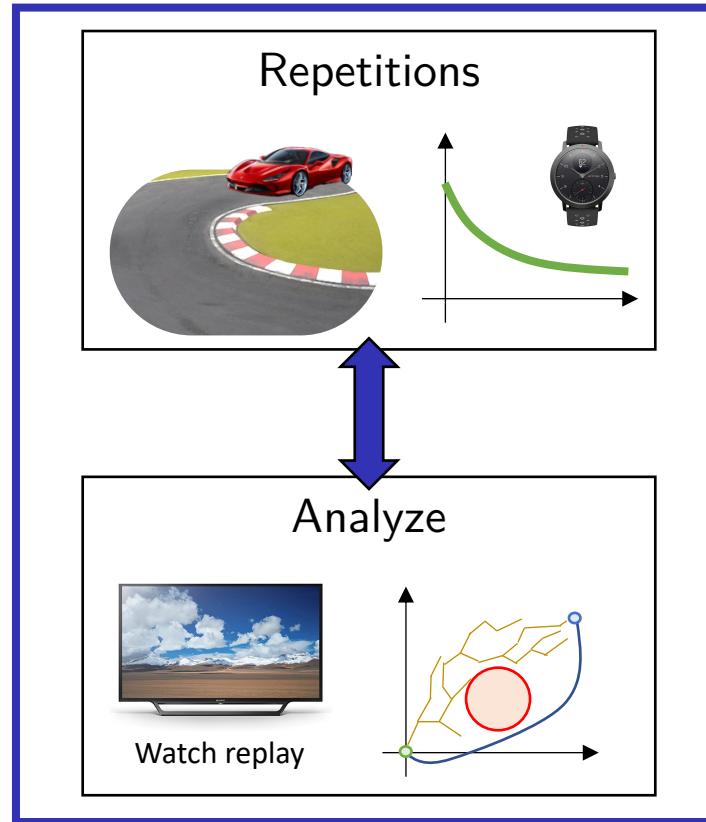
Performance Improvement

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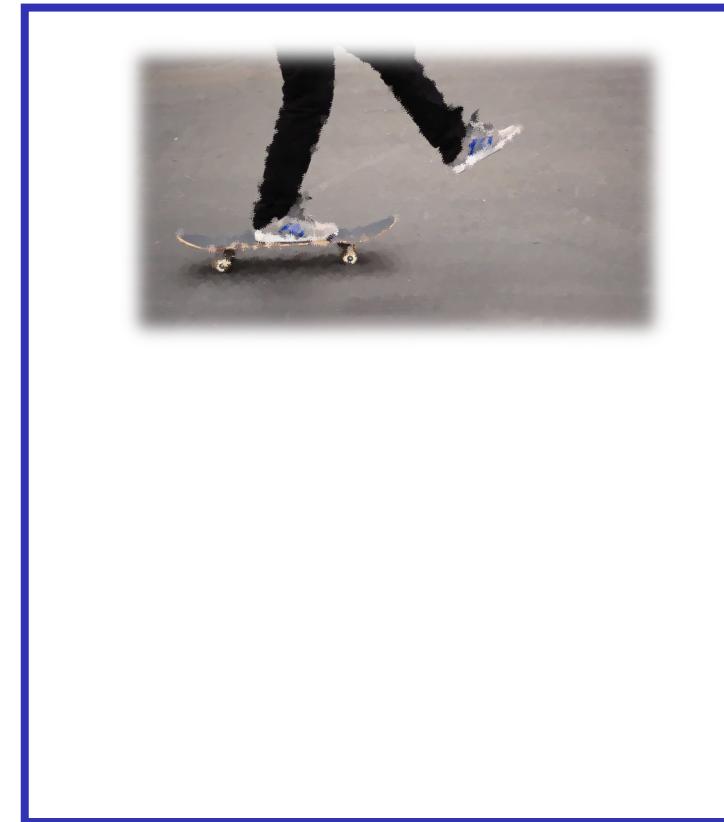
Learning classification borrowed from Prof. Borrelli



Skill acquisition



Performance Improvement



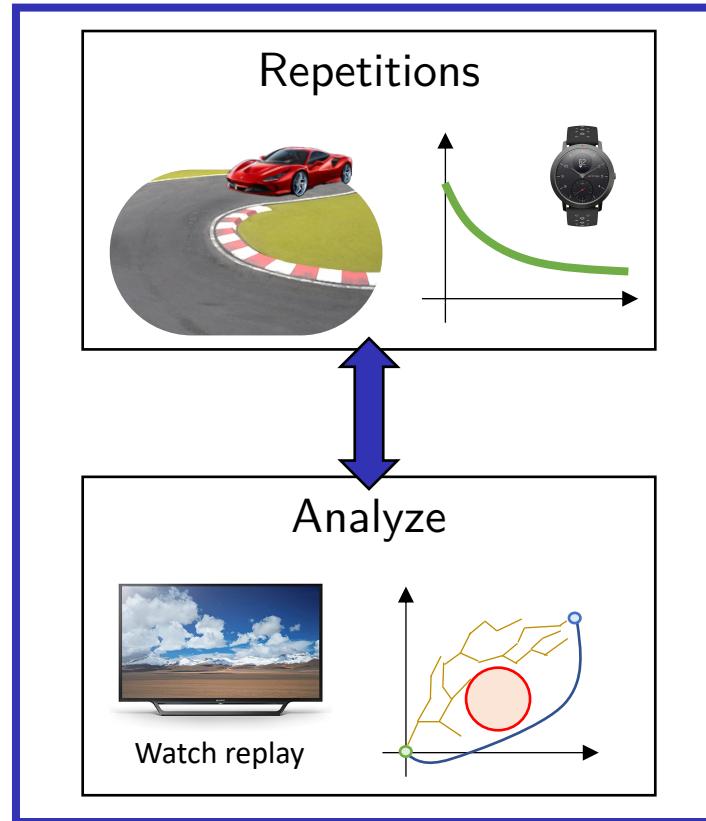
Building muscle memory

Three types of learning

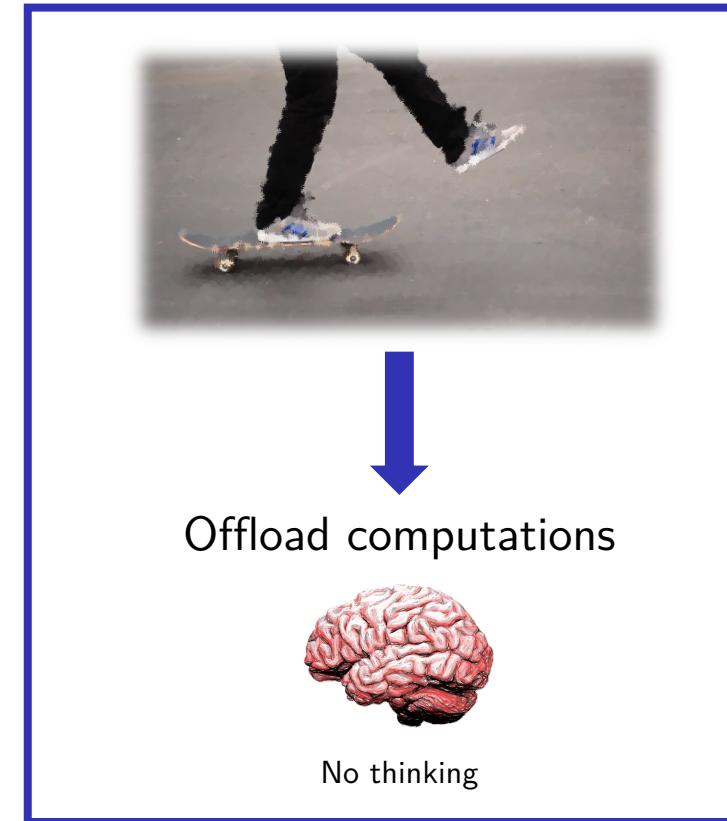
Learning classification borrowed
from Prof. Borrelli



Skill acquisition



Performance Improvement



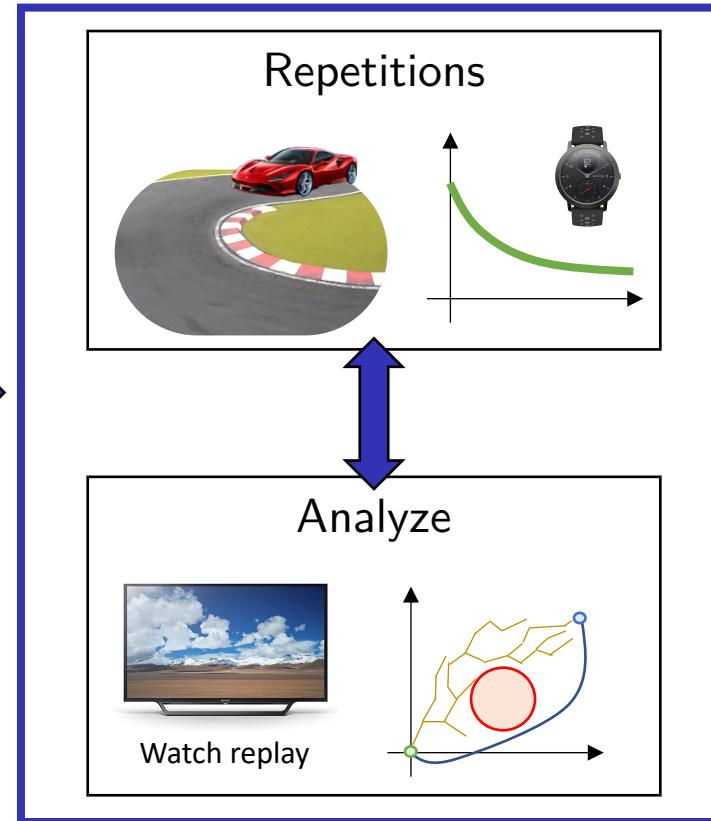
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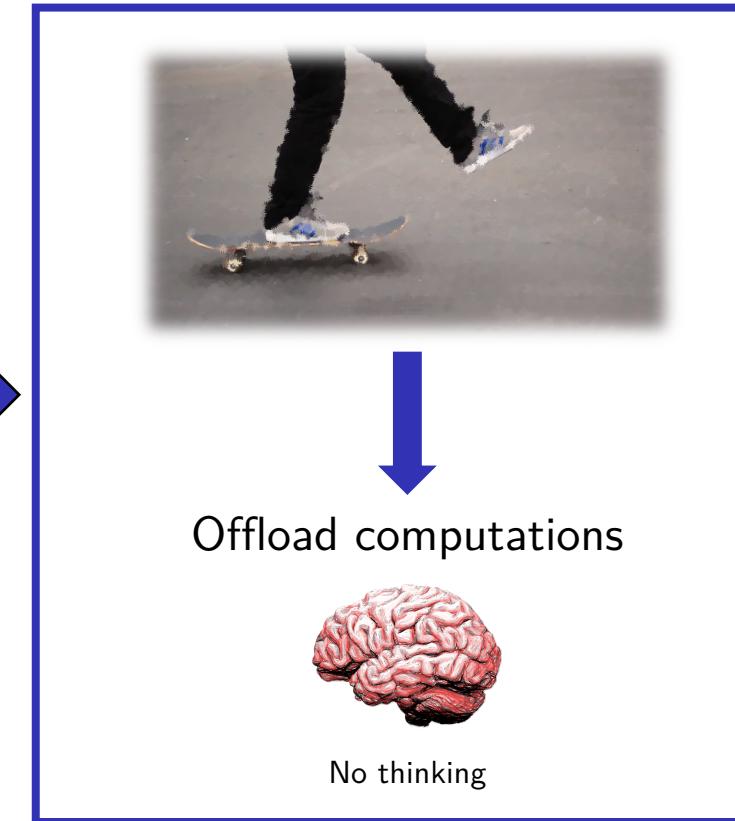
Learning classification borrowed
from Prof. Borrelli



Skill acquisition



Performance Improvement



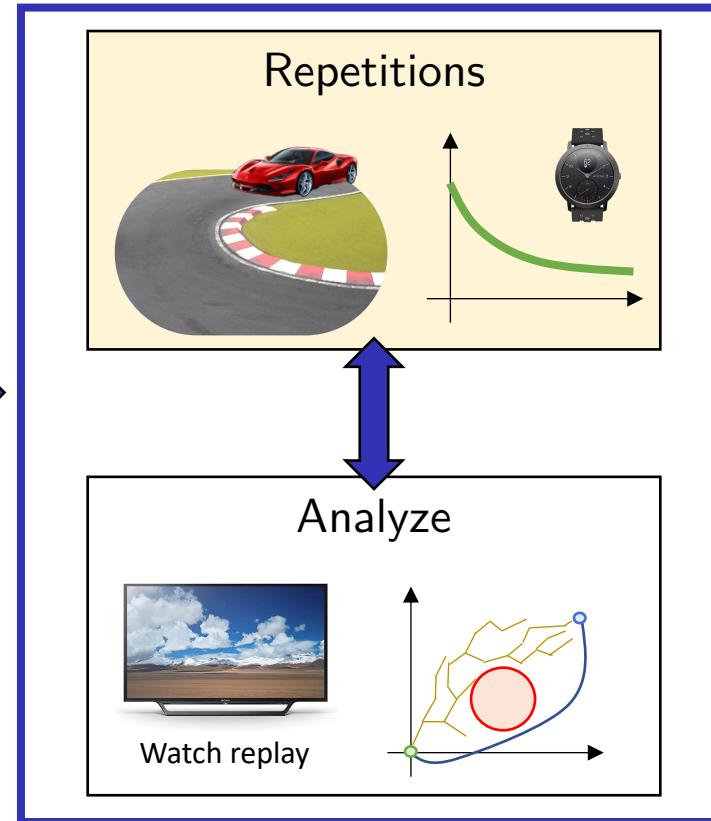
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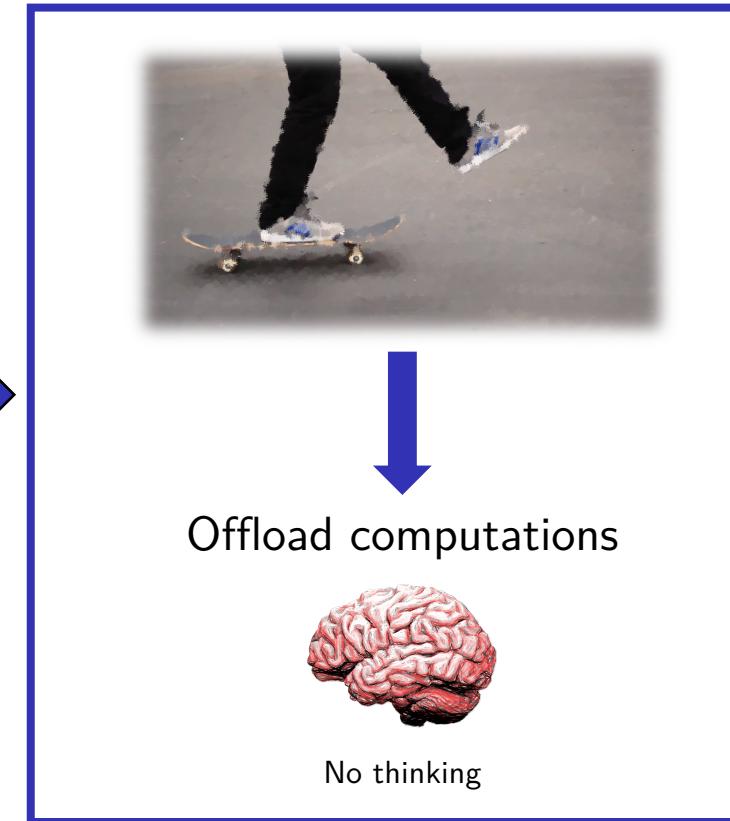
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Skill acquisition



Performance Improvement



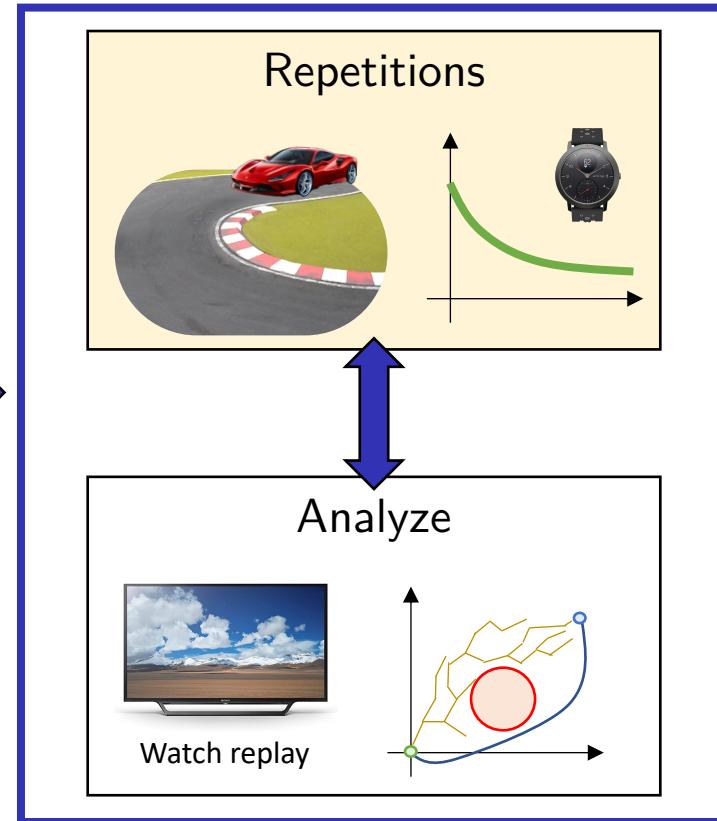
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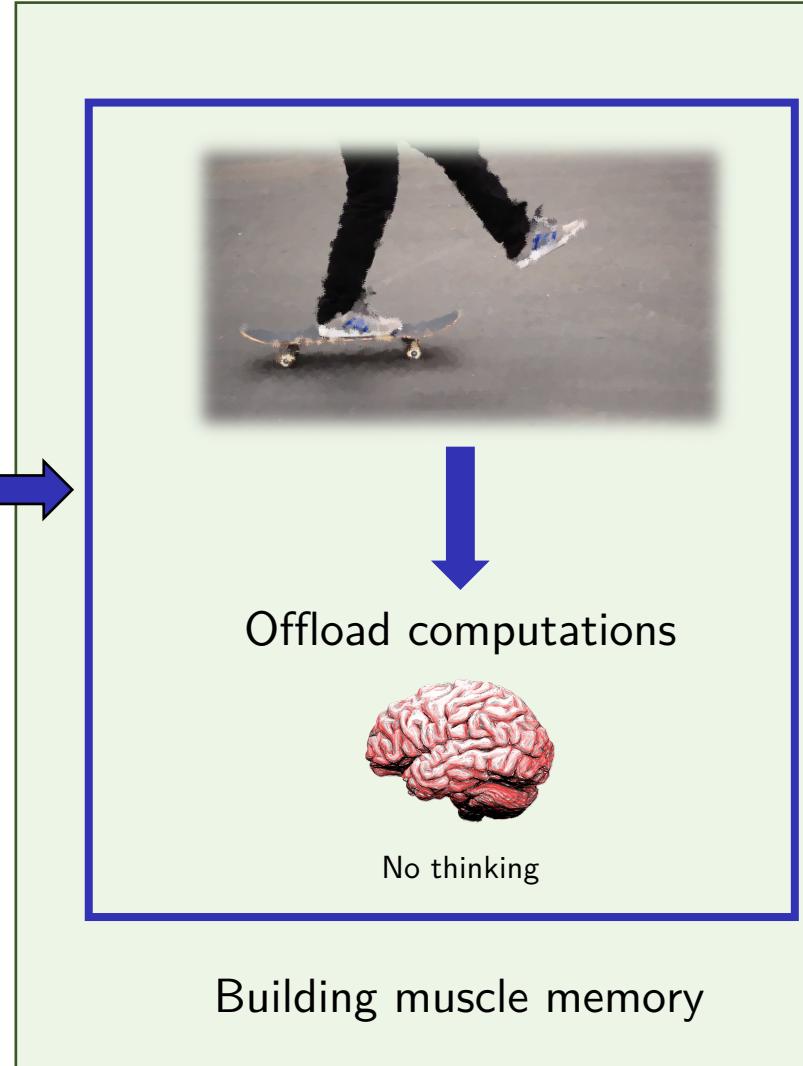
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Skill acquisition

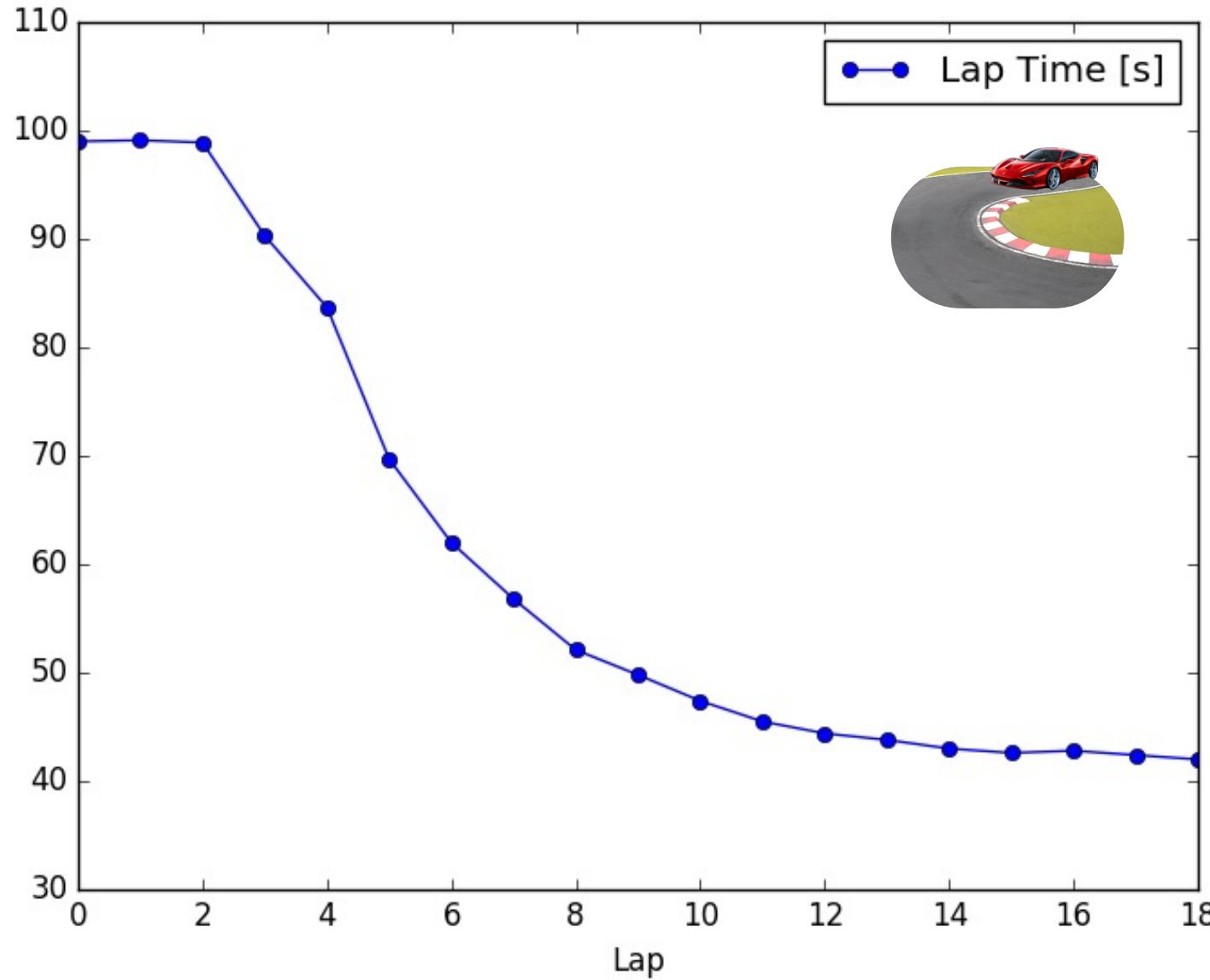


Performance Improvement

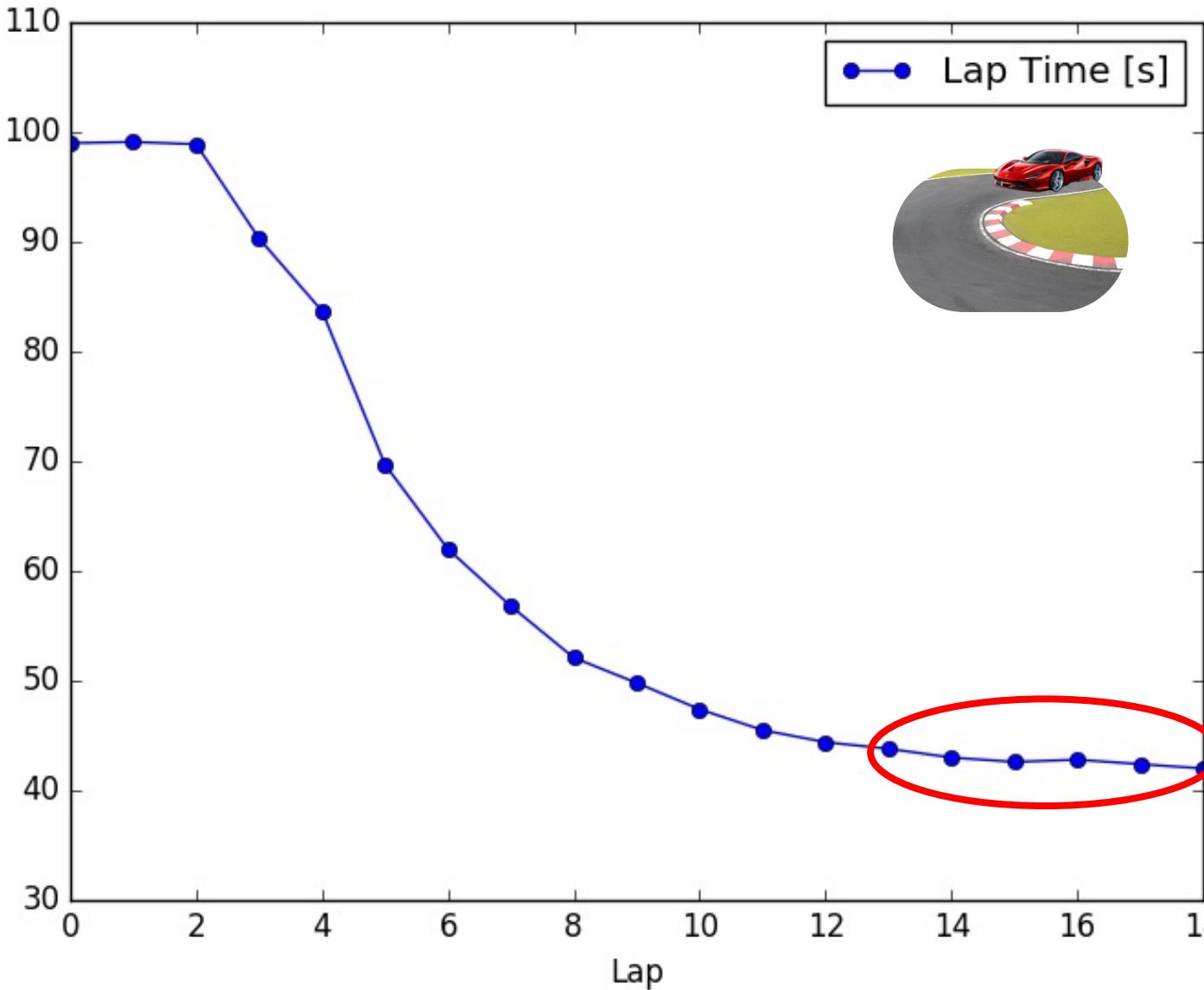


Building muscle memory

Motivation: Lap Time Improvement



Goal: Reduce Computational Complexity



At convergence the control policy does not change

Learning as Computational Load Reduction

THINKING,
FAST AND SLOW

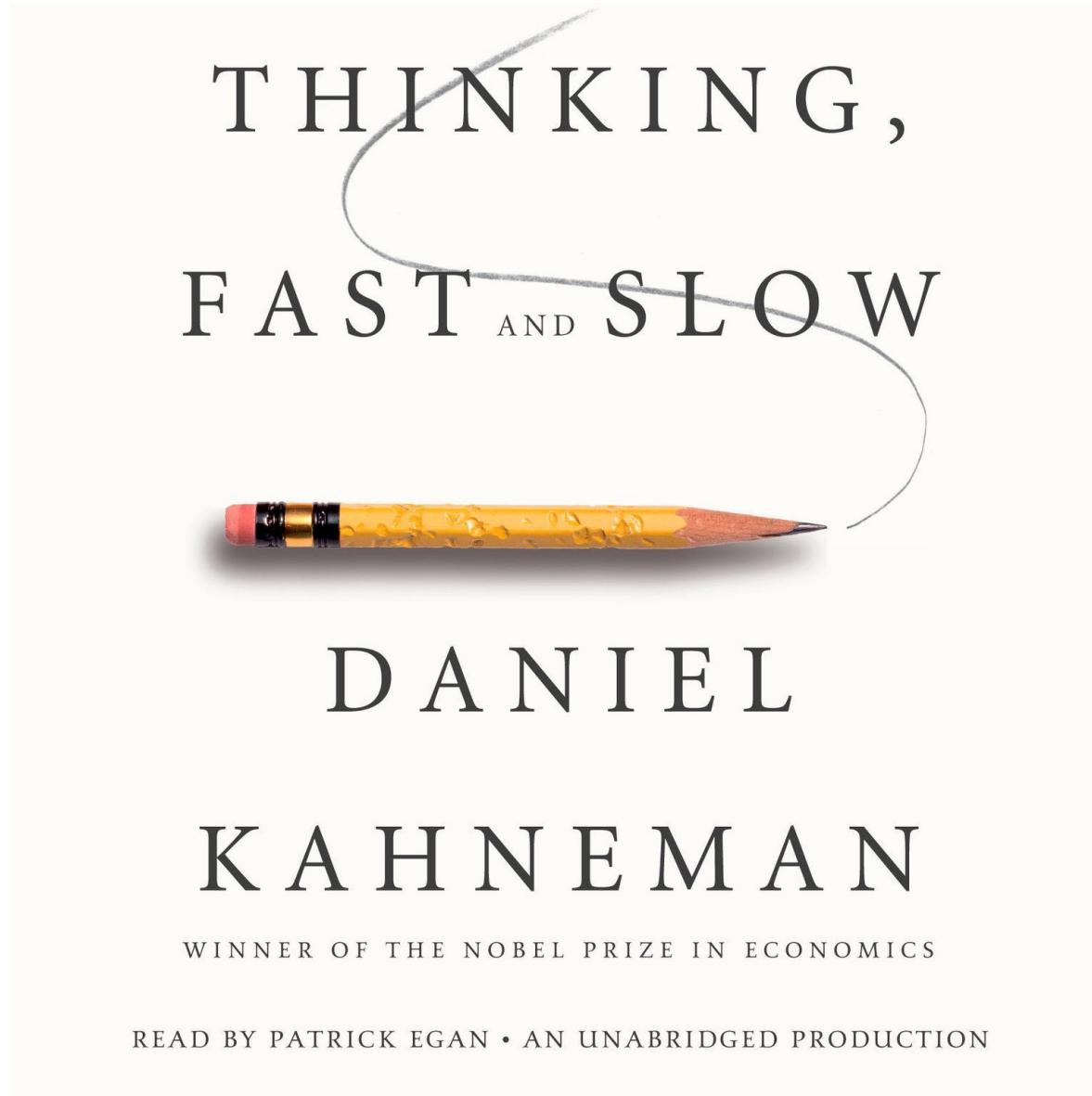


DANIEL
KAHNEMAN

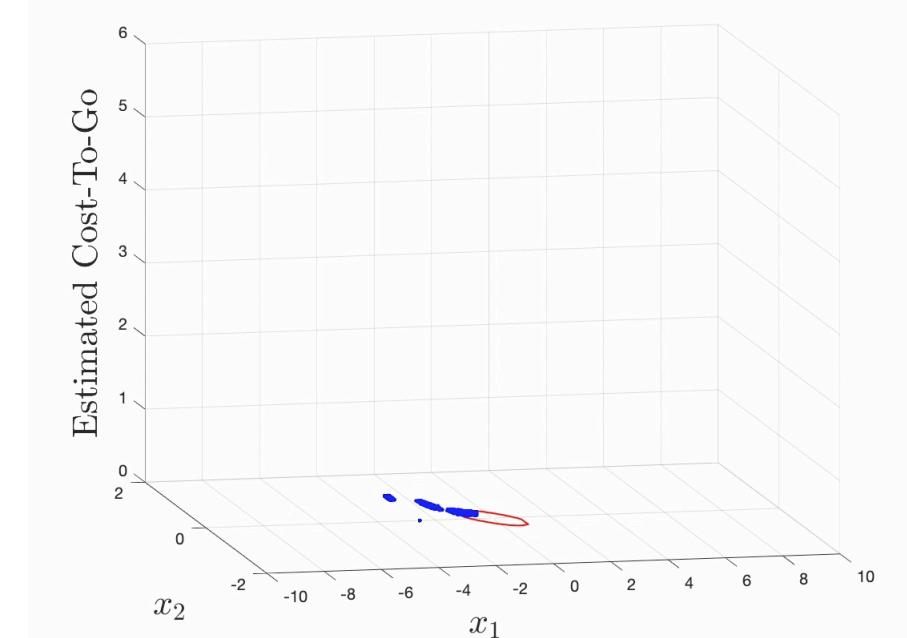
WINNER OF THE NOBEL PRIZE IN ECONOMICS

READ BY PATRICK EGAN • AN UNABRIDGED PRODUCTION

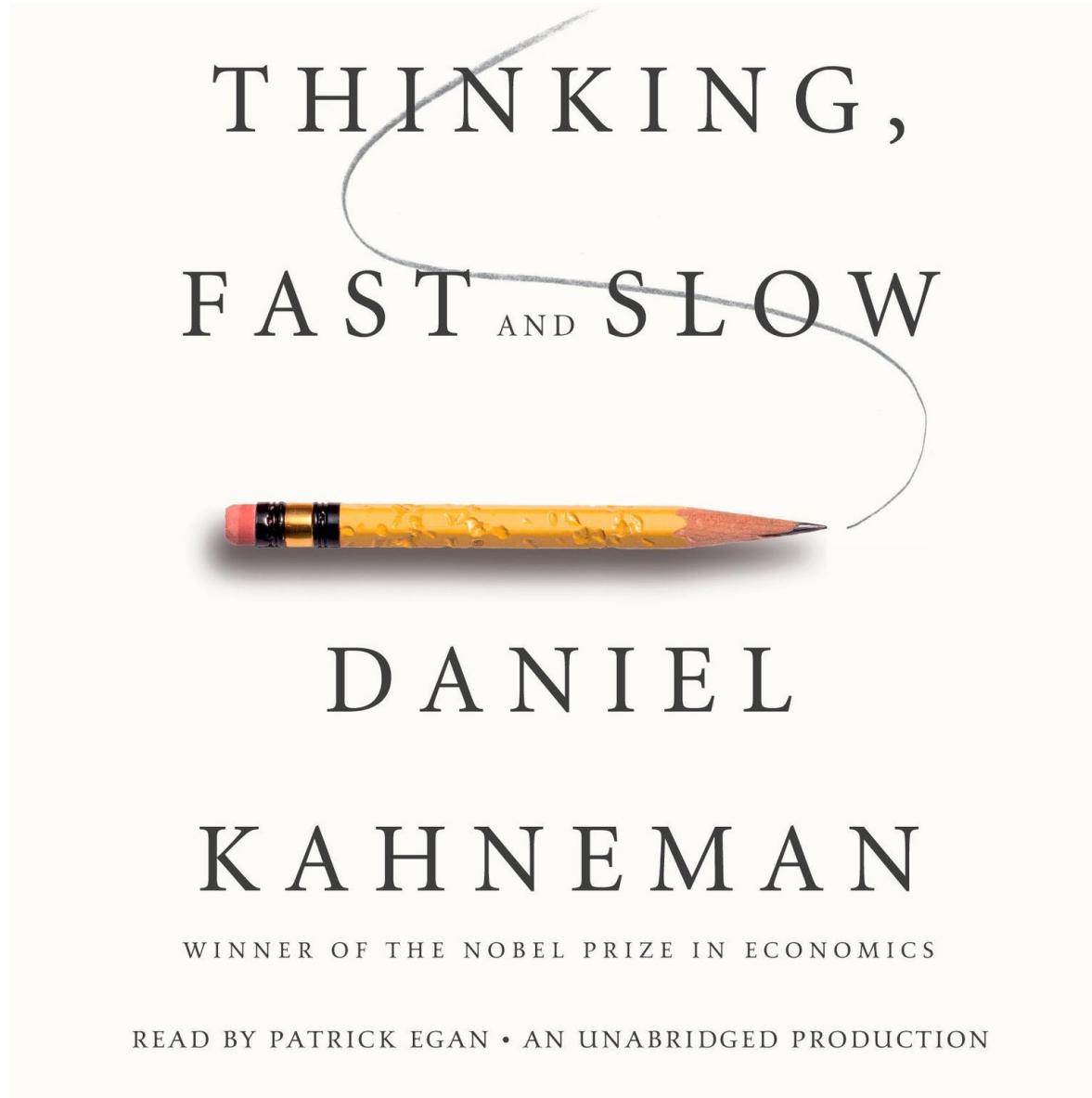
Learning as Computational Load Reduction



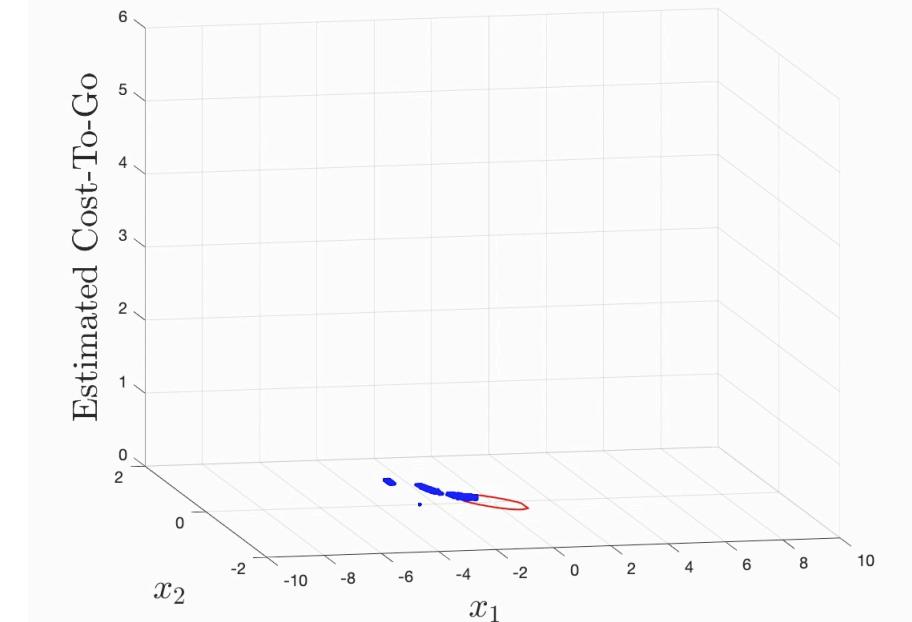
At convergence: the value function approximation does not change as we collect more data



Learning as Computational Load Reduction



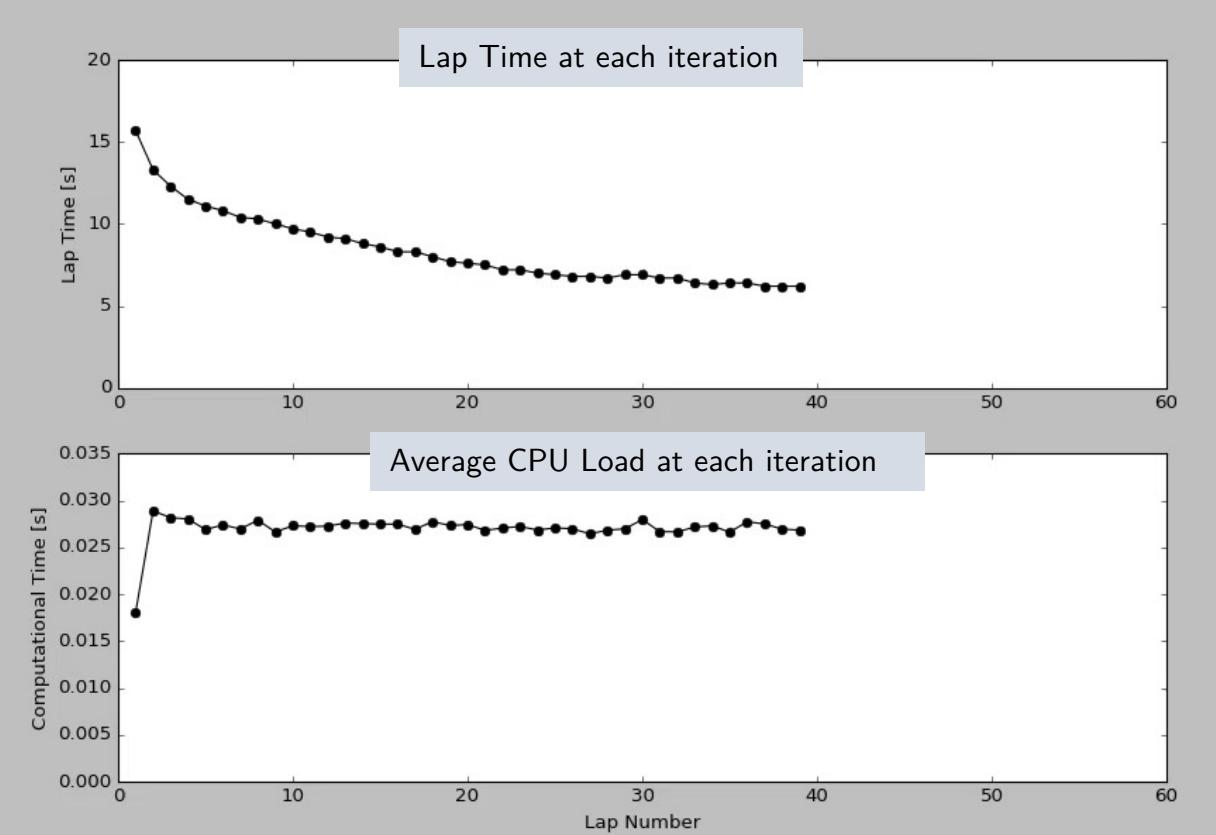
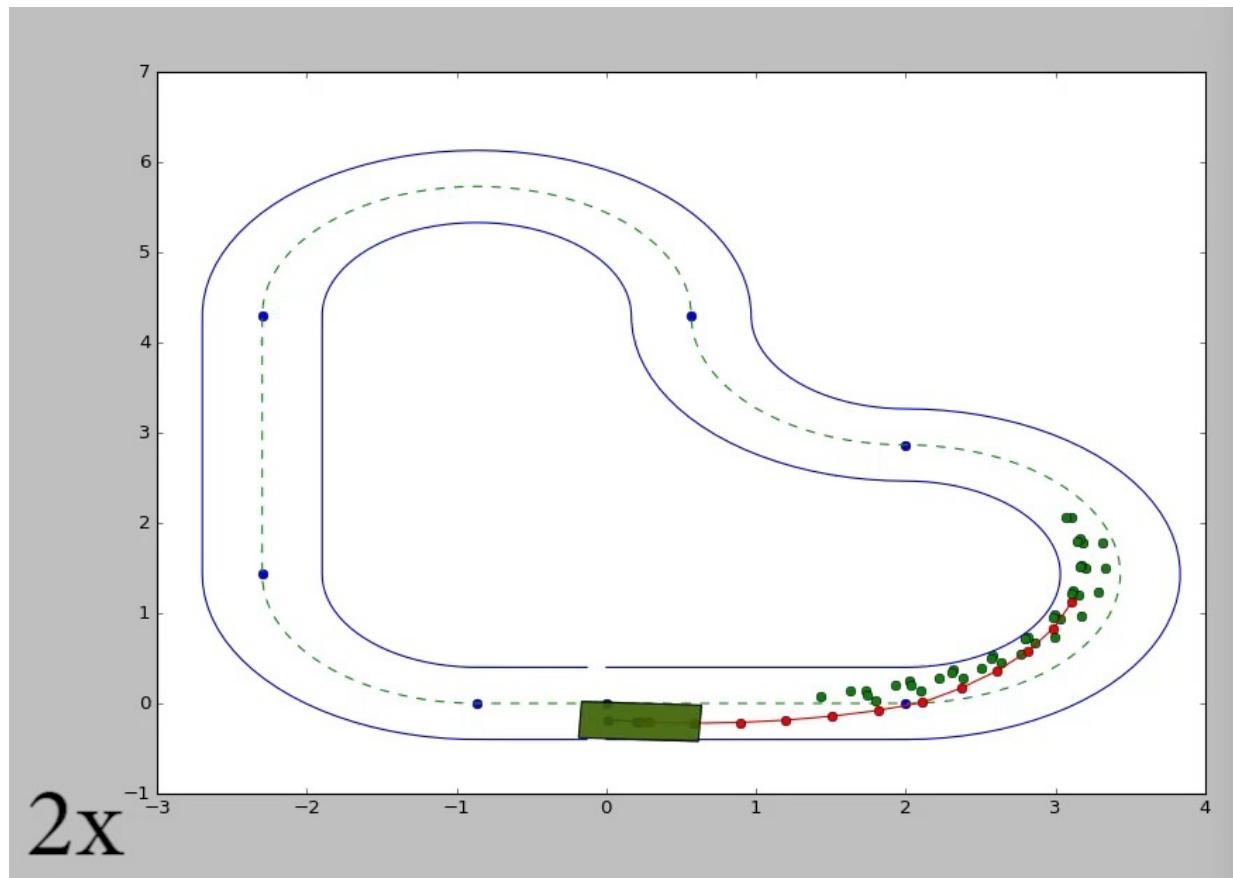
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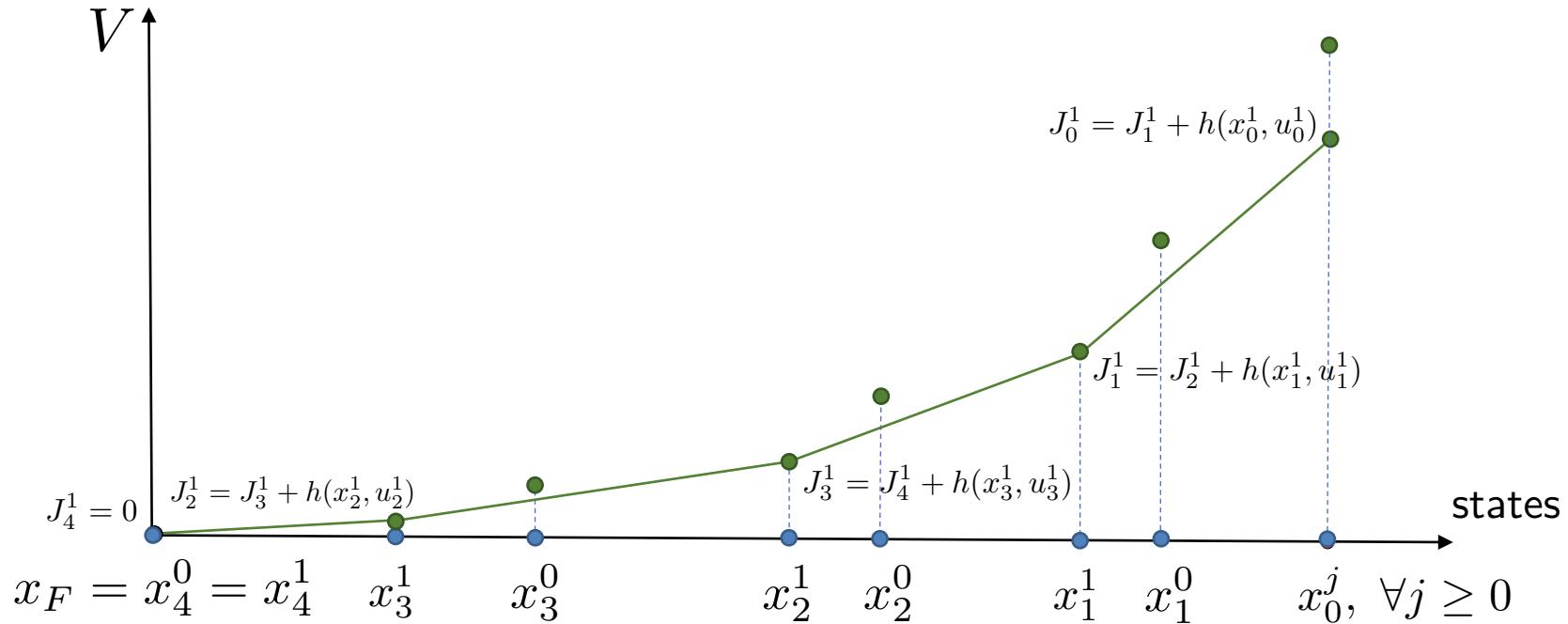
Can we design a policy which used data and “plays back” the inputs that we computed with the expensive policy

Do you need to Predict at Convergence?

Do you need to Predict at Convergence? No



Data-Based Policy

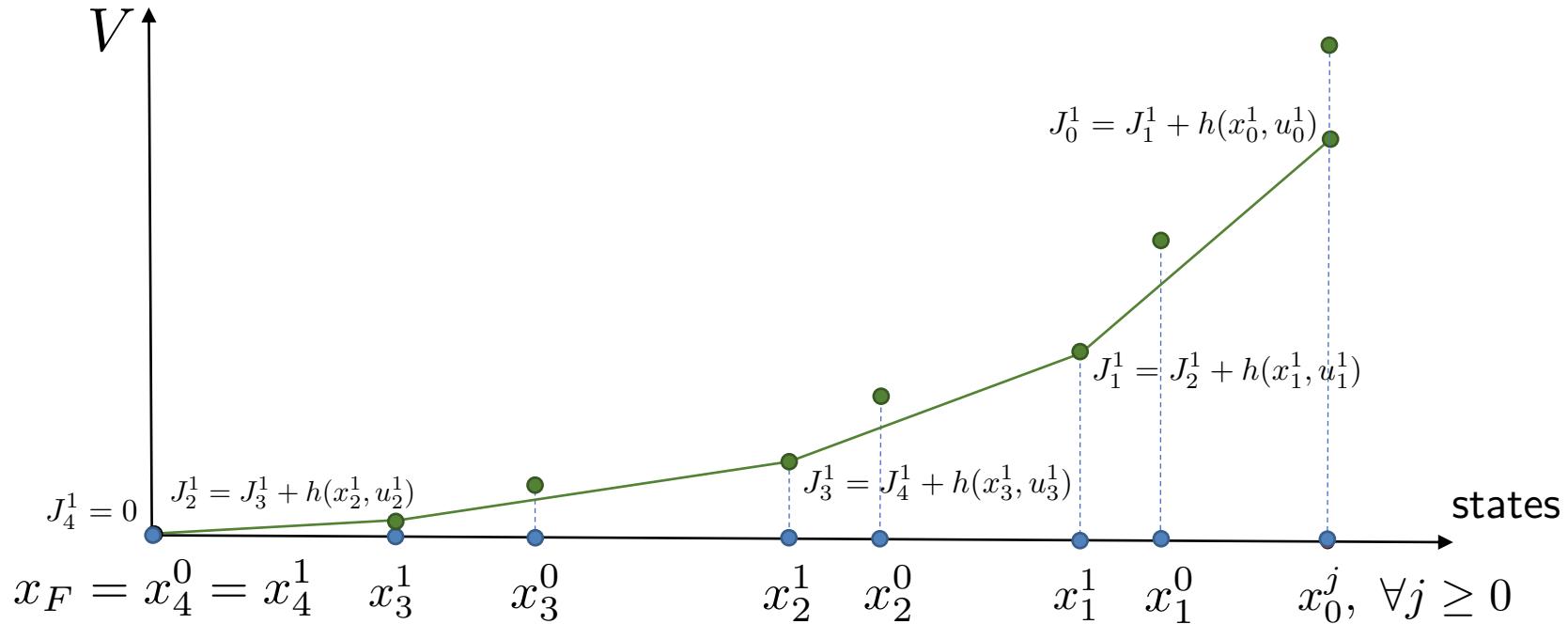


Multipliers Defining the Value Function Approximation

$$[\lambda_0^{0,*}, \dots, \lambda_{T^j}^{T^j,*}] = \arg \min_{\lambda_i^j \in [0,1]} \sum_i \sum_j J_i^j \lambda_i^j$$

$$\text{s.t} \quad \sum_i \sum_j x_i^j \lambda_i^j = \textcolor{red}{x}, \\ \sum_i \sum_j \lambda_i^j = 1$$

Data-Based Policy



Multipliers Defining the Value Function Approximation

$$[\lambda_0^{0,*}, \dots, \lambda_{T^j}^{T^j,*}] = \arg \min_{\lambda_i^j \in [0,1]} \sum_i \sum_j J_i^j \lambda_i^j$$

s.t

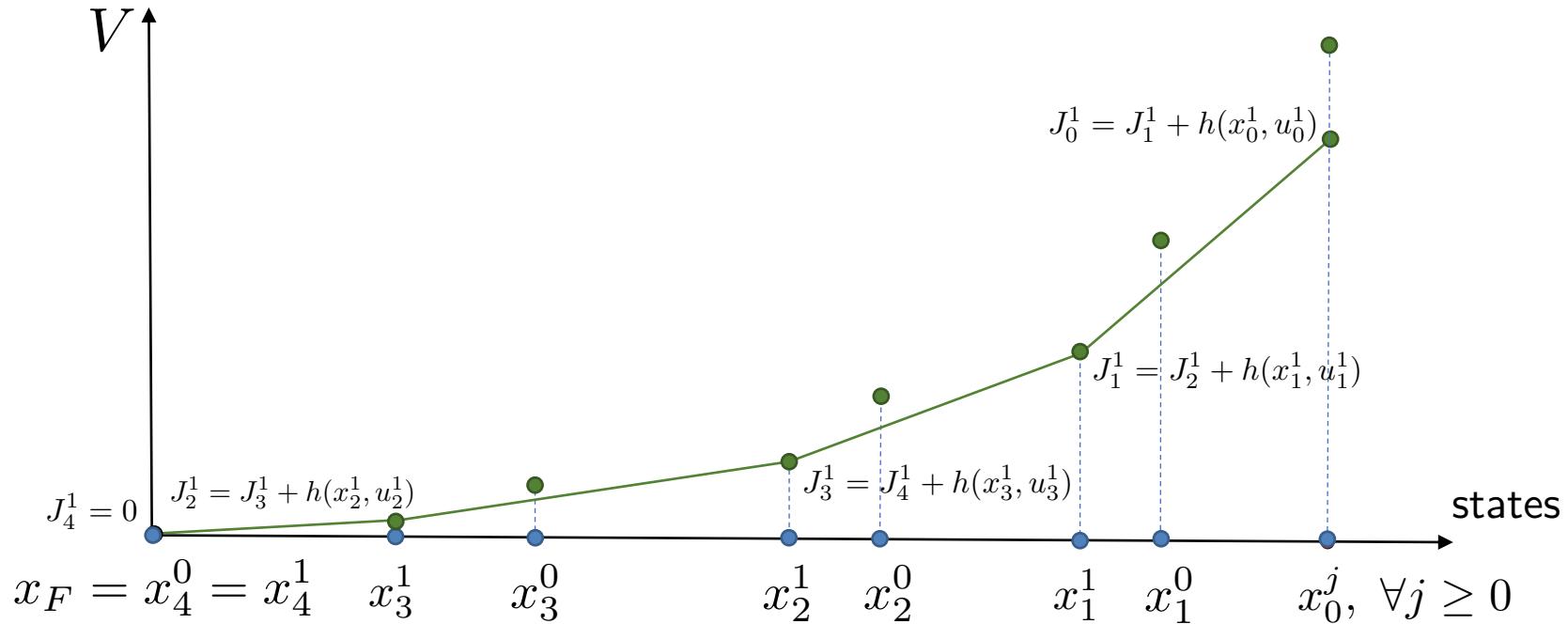
$$\sum_i \sum_j x_i^j \lambda_i^j = \textcolor{red}{x},$$

$$\sum_i \sum_j \lambda_i^j = 1$$

Data-Based Policy

$$\kappa^j(\textcolor{red}{x}) = \sum_i \sum_j u_i^j \lambda_i^{j,*}$$

Data-Based Policy



Multipliers Defining the Value Function Approximation

$$[\lambda_0^{0,*}, \dots, \lambda_{T^j}^{T^j,*}] = \arg \min_{\lambda_i^j \in [0,1]} \sum_i \sum_j J_i^j \lambda_i^j$$

s.t

$$\sum_i \sum_j x_i^j \lambda_i^j = \textcolor{red}{x},$$

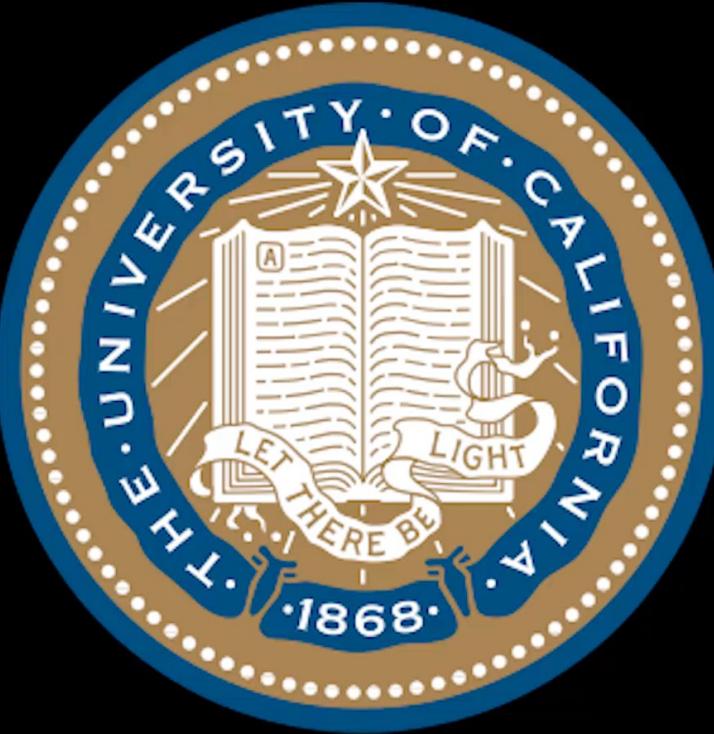
$$\sum_i \sum_j \lambda_i^j = 1$$

Data-Based Policy

$$\kappa^j(\textcolor{red}{x}) = \sum_i \sum_j u_i^j \lambda_i^{j,*}$$

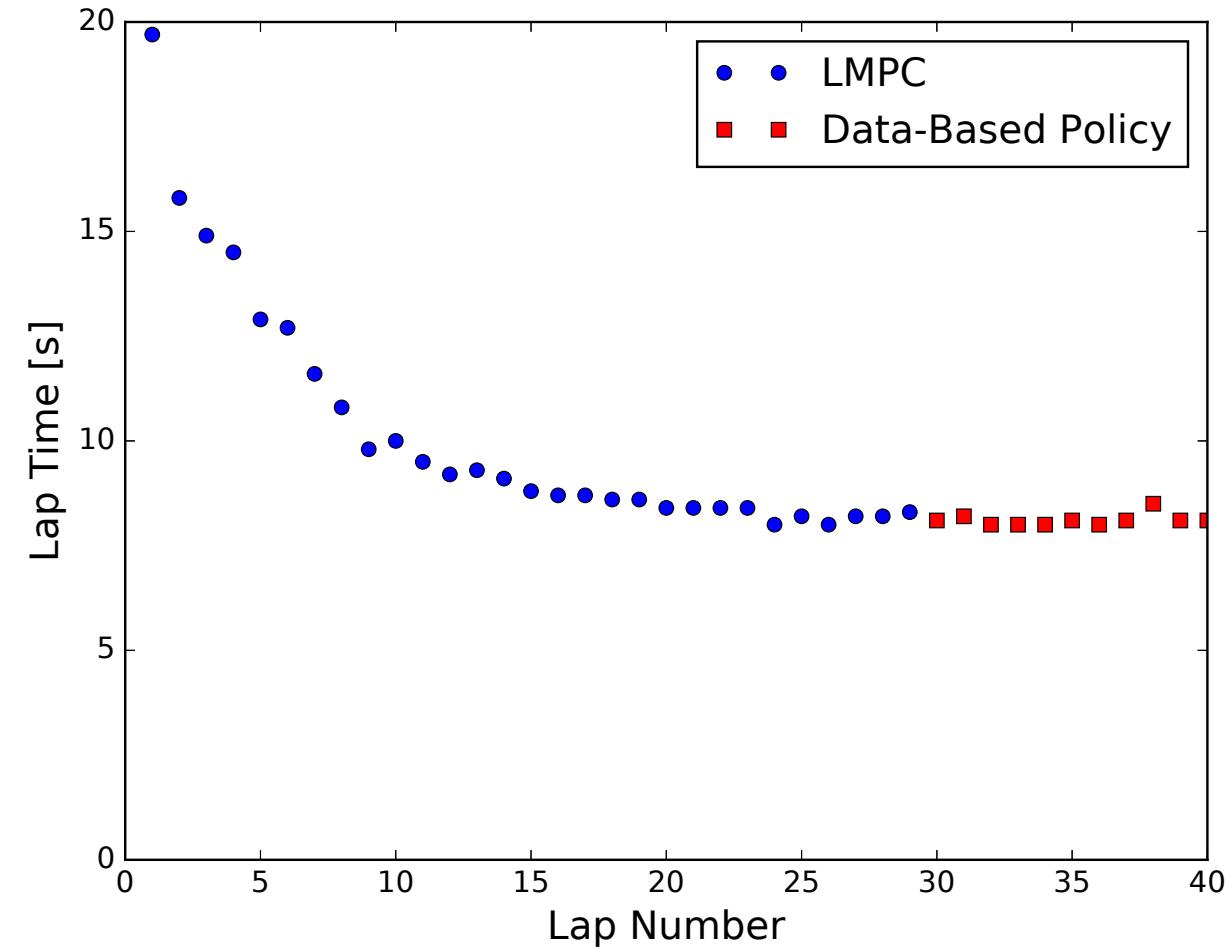
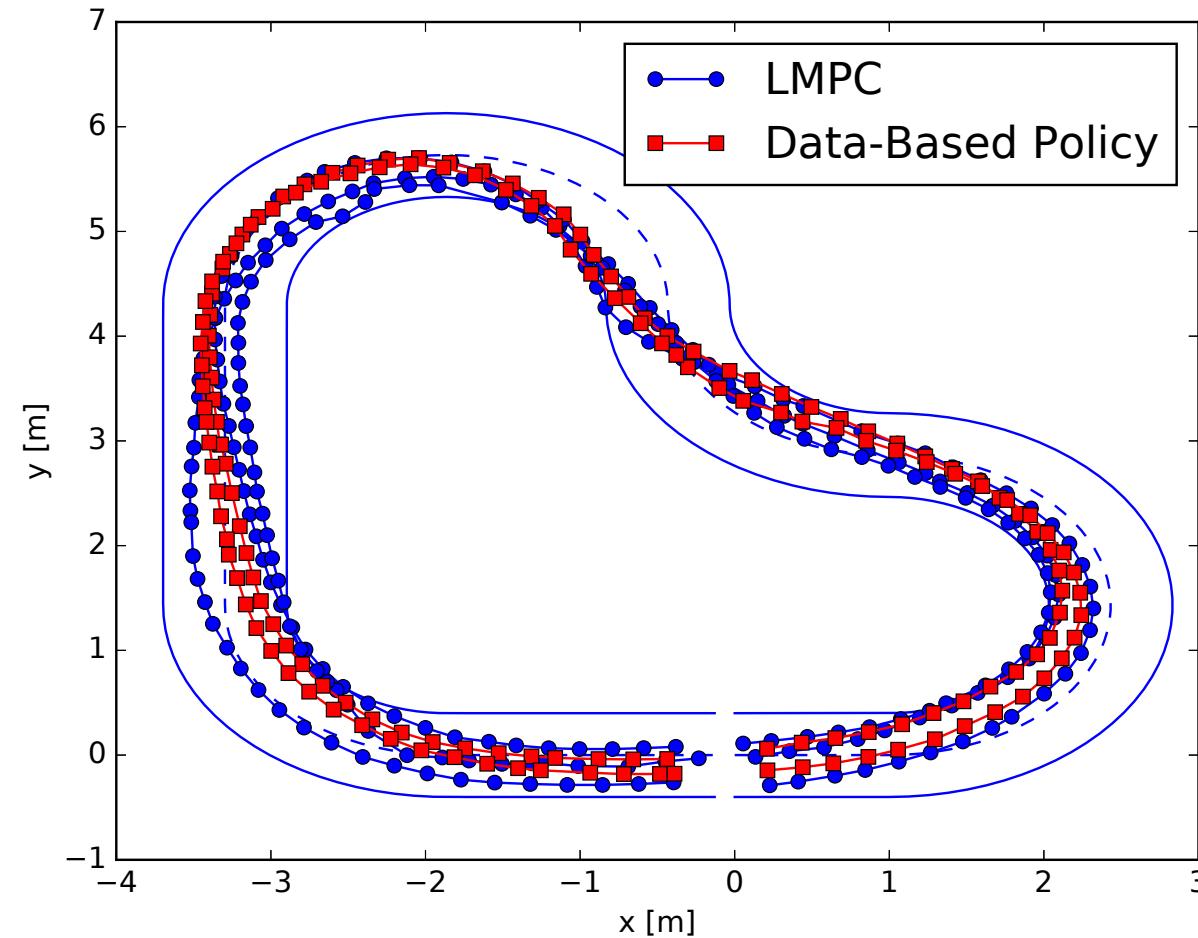
Historical Data

Berkeley Autonomous Race Car Implementation



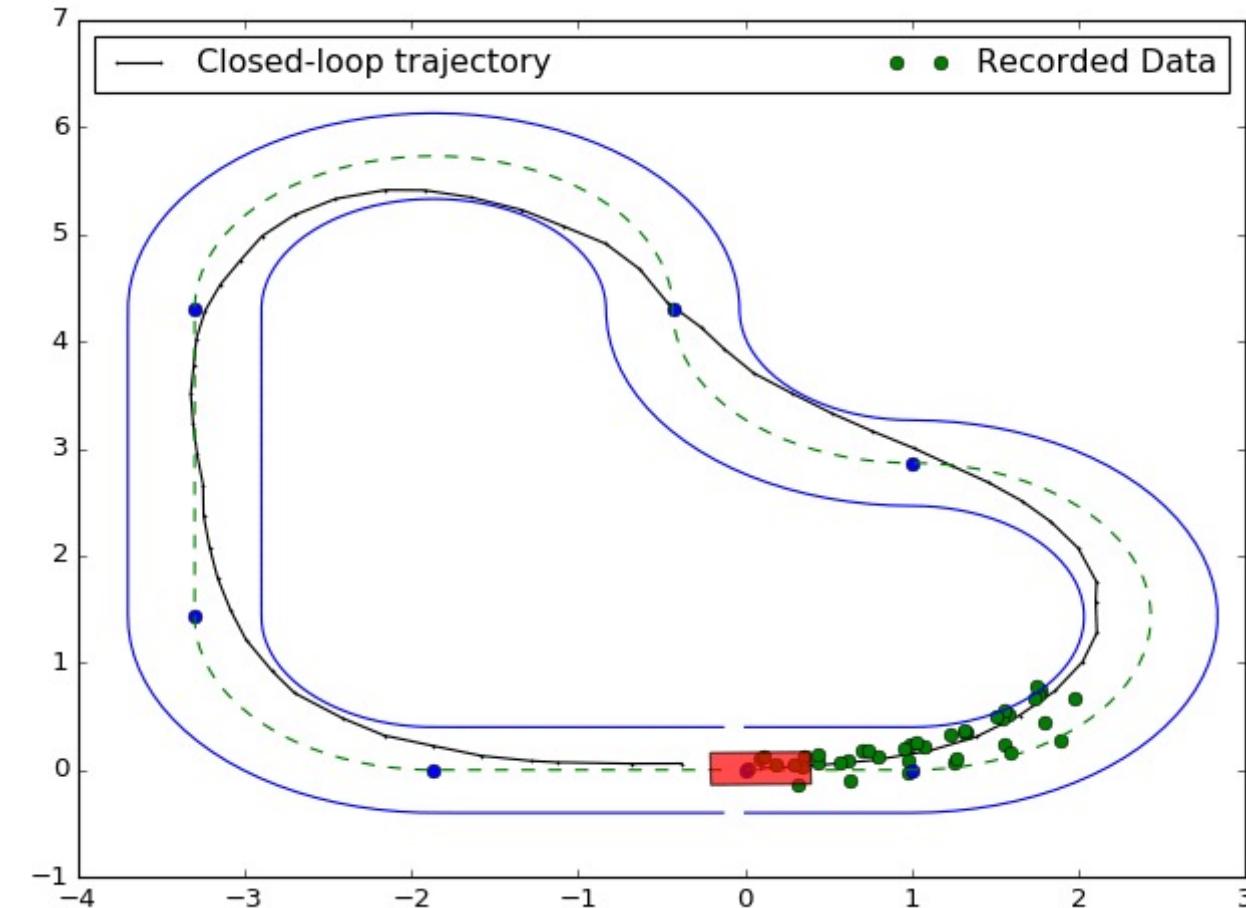
Learning Model Predictive Control
for Autonomous Racing

Berkeley Autonomous Race Car Implementation



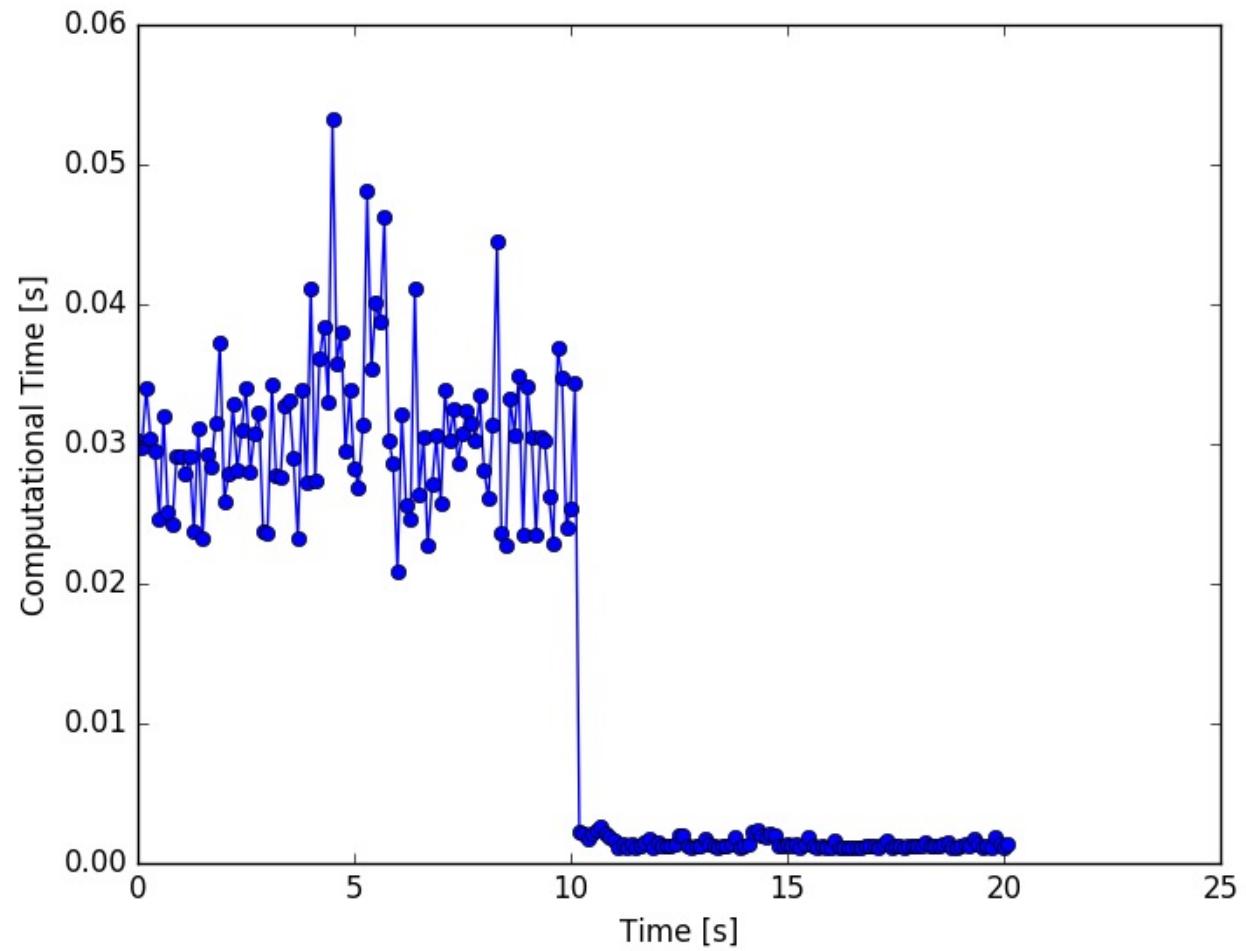
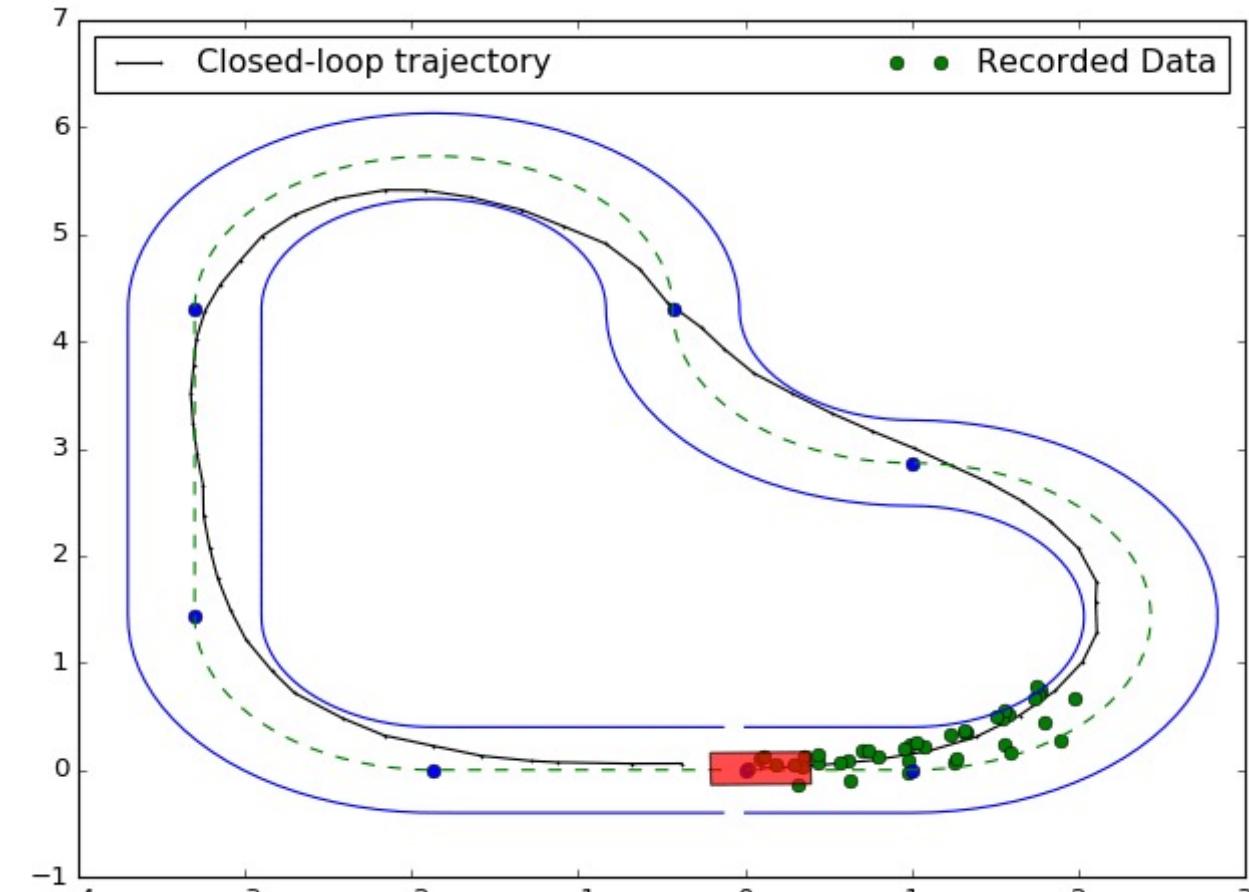
- ▶ A **reference trajectory** is **not needed** for the controller implementation
- ▶ The controller converges after few laps: the learning process is **data efficient**
- ▶ The controller **safely explores** the state space **iteratively improving** the lap time

Berkeley Autonomous Race Car Implementation



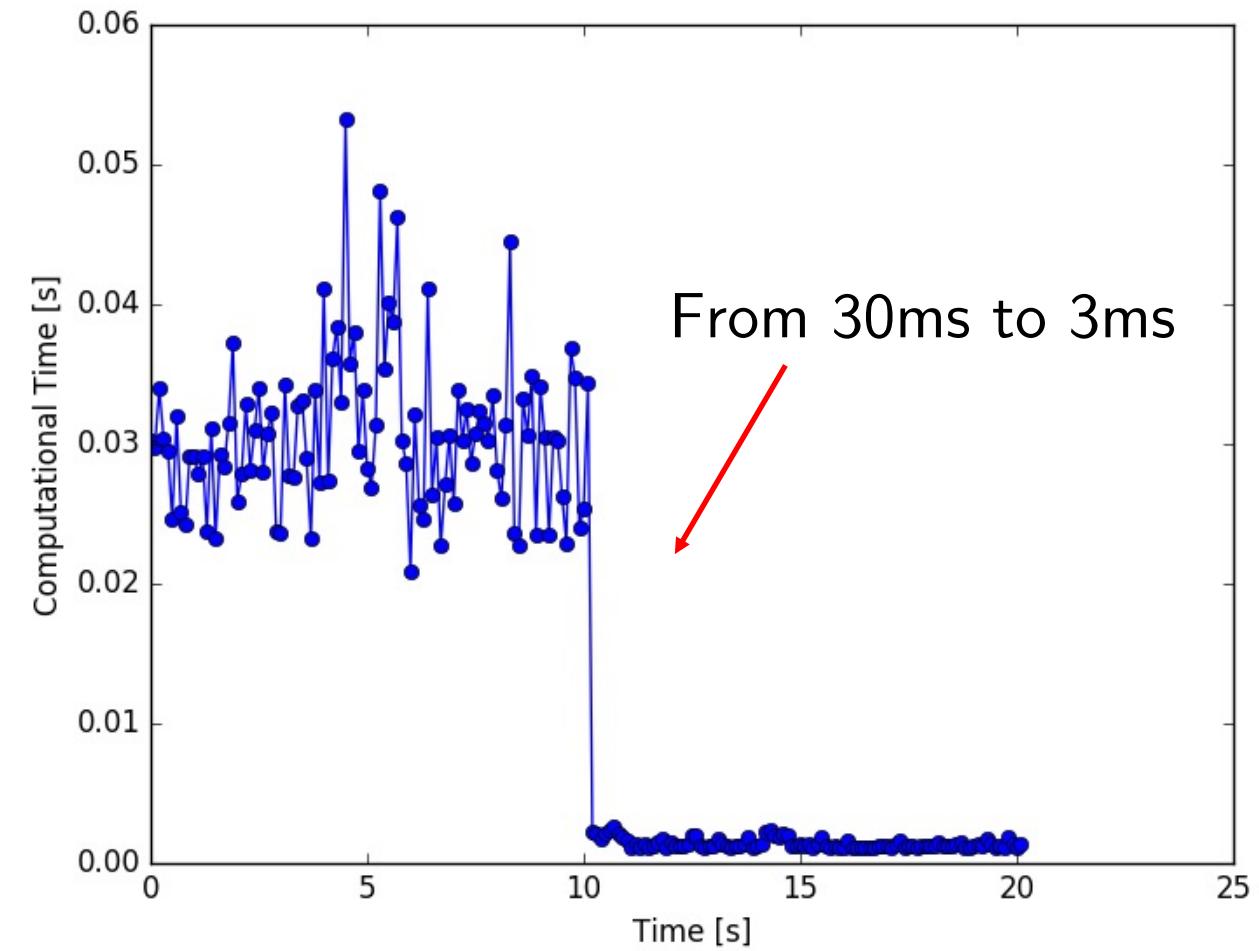
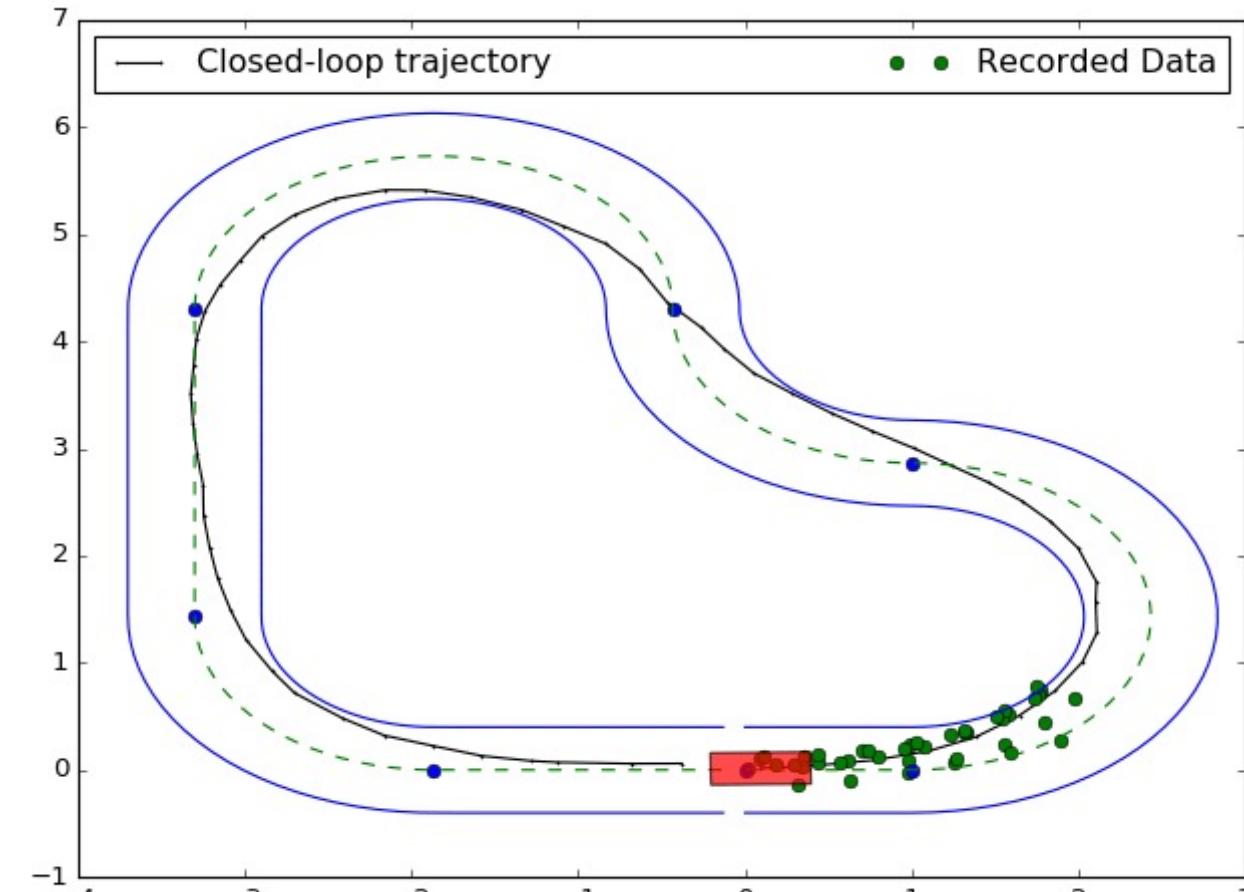
At each time t the data-based policy is computed with the K nearest neighbors (in green) to the current state. This allows us to account for the nonlinearity of the vehicle

Berkeley Autonomous Race Car Implementation



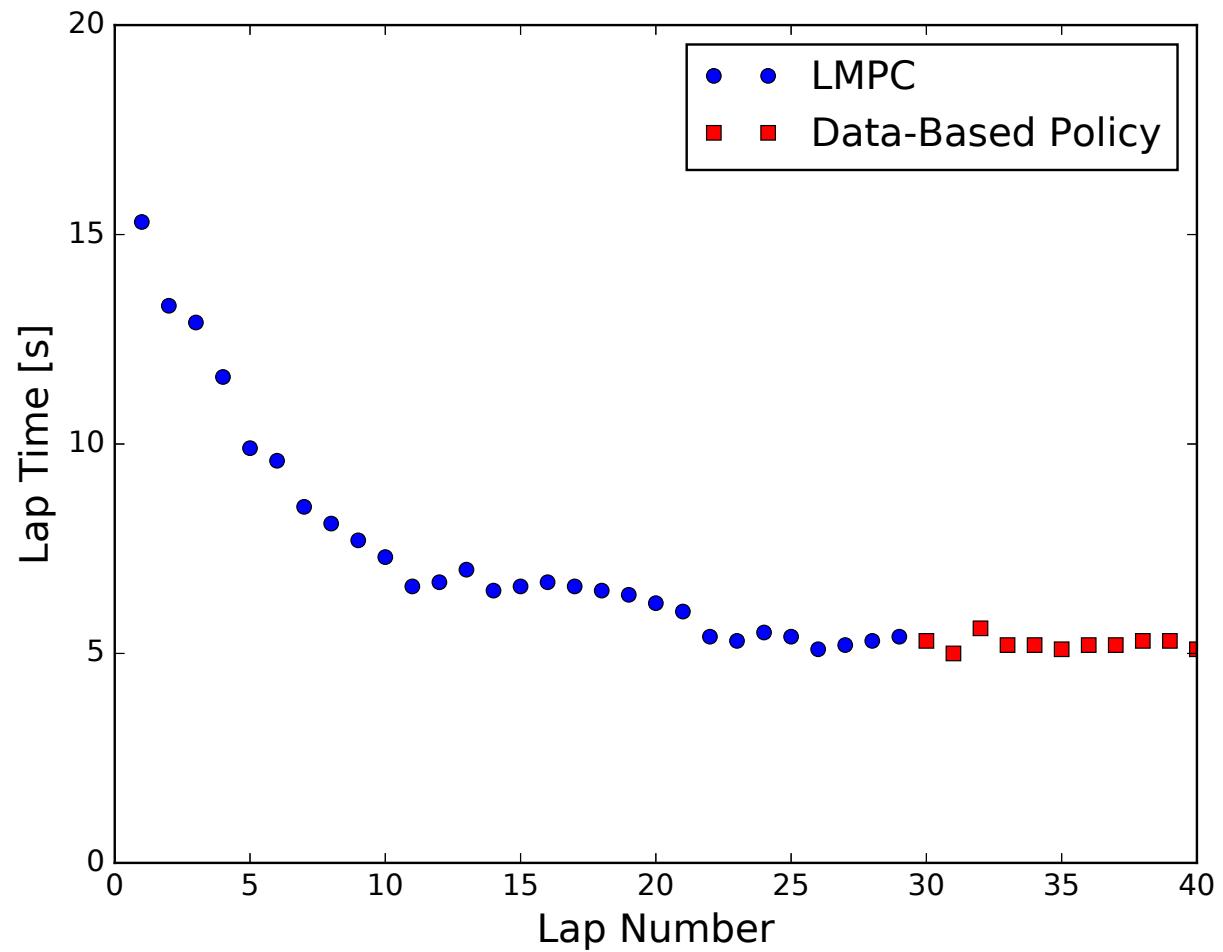
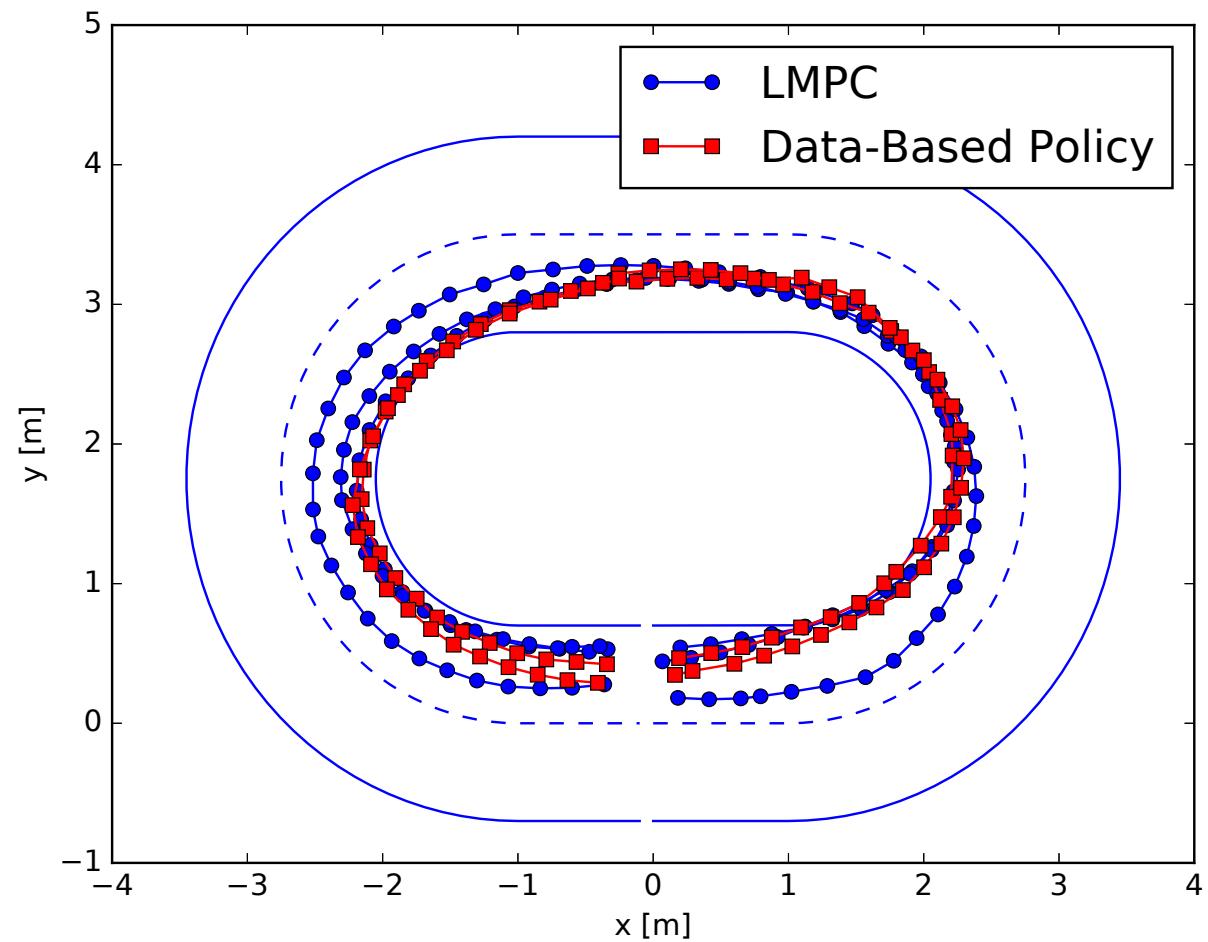
At each time t the data-based policy is computed with the K nearest neighbors (in green) to the current state. This allows us to account for the nonlinearity of the vehicle

Berkeley Autonomous Race Car Implementation

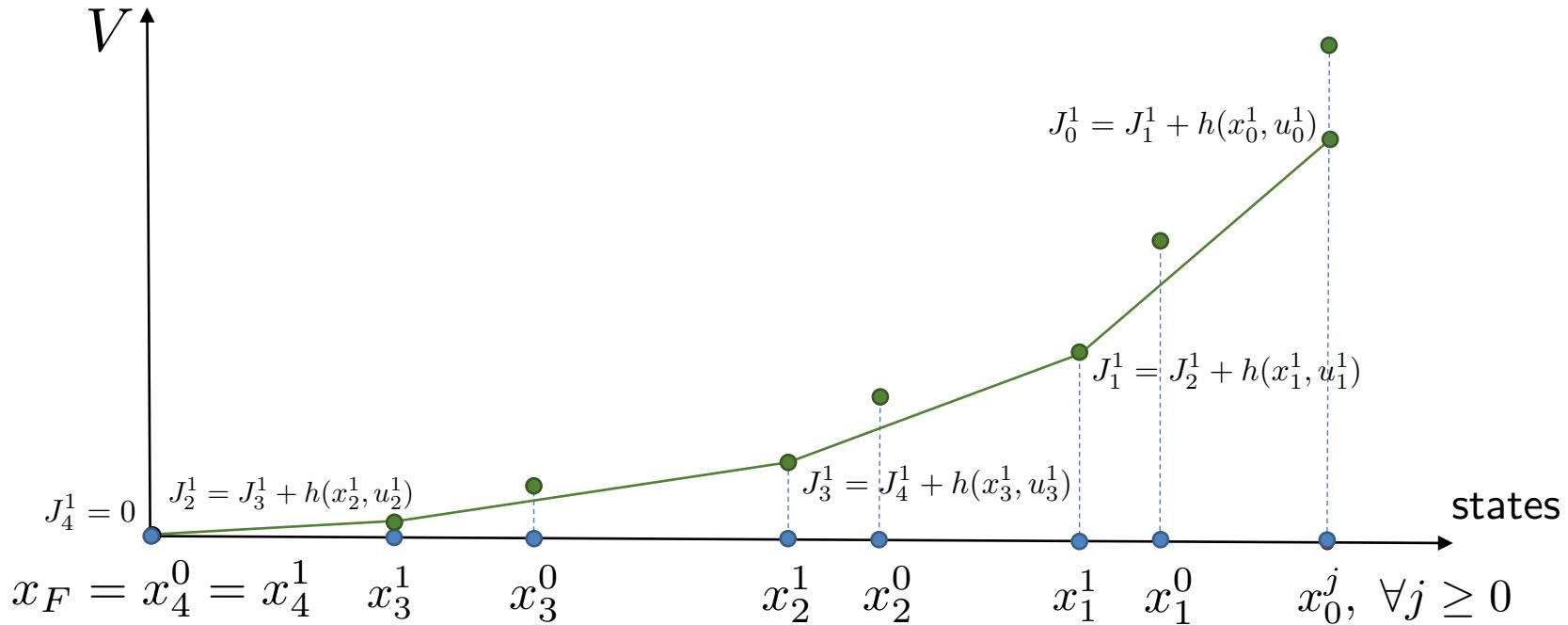


At each time t the data-based policy is computed with the K nearest neighbors (in green) to the current state. This allows us to account for the nonlinearity of the vehicle

Berkeley Autonomous Race Car Implementation



Data-Based Policy



Multipliers Defining the Value Function Approximation

$$[\lambda_0^{0,*}, \dots, \lambda_{T^j}^{T^j,*}] = \arg \min_{\lambda_i^j \in [0,1]} \sum_i \sum_j J_i^j \lambda_i^j$$

s.t

$$\sum_i \sum_j x_i^j \lambda_i^j = \textcolor{red}{x},$$

$$\sum_i \sum_j \lambda_i^j = 1$$

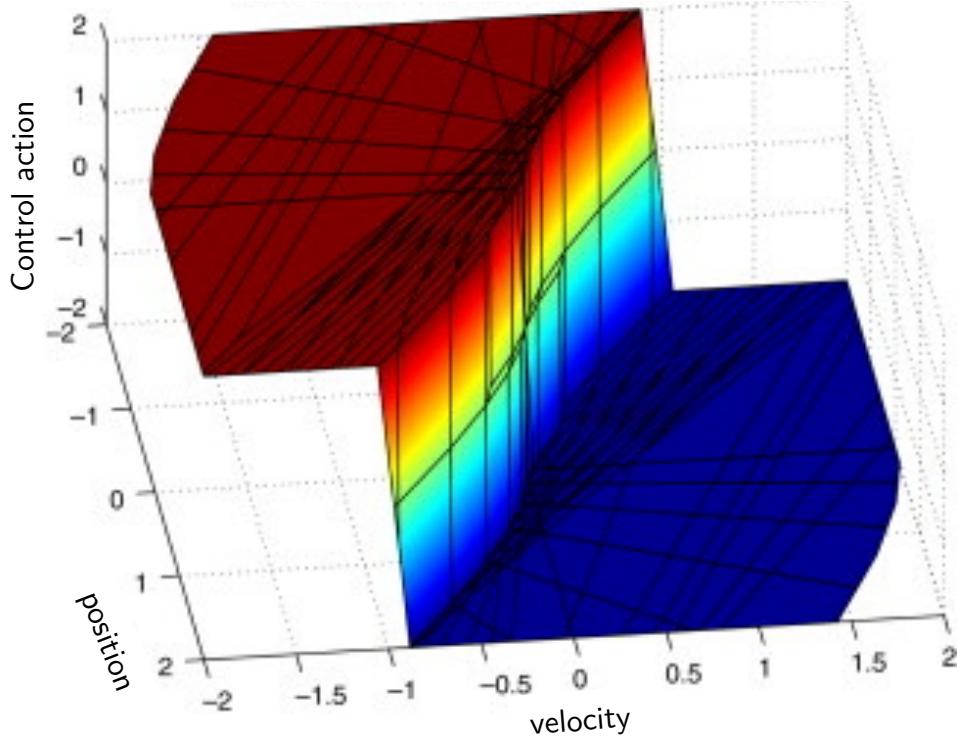
Data-Based Policy

$$\kappa^j(\textcolor{red}{x}) = \sum_i \sum_j u_i^j \lambda_i^{j,*}$$

Historical Data

Speeding-up the solver

Leverage the explicit solution

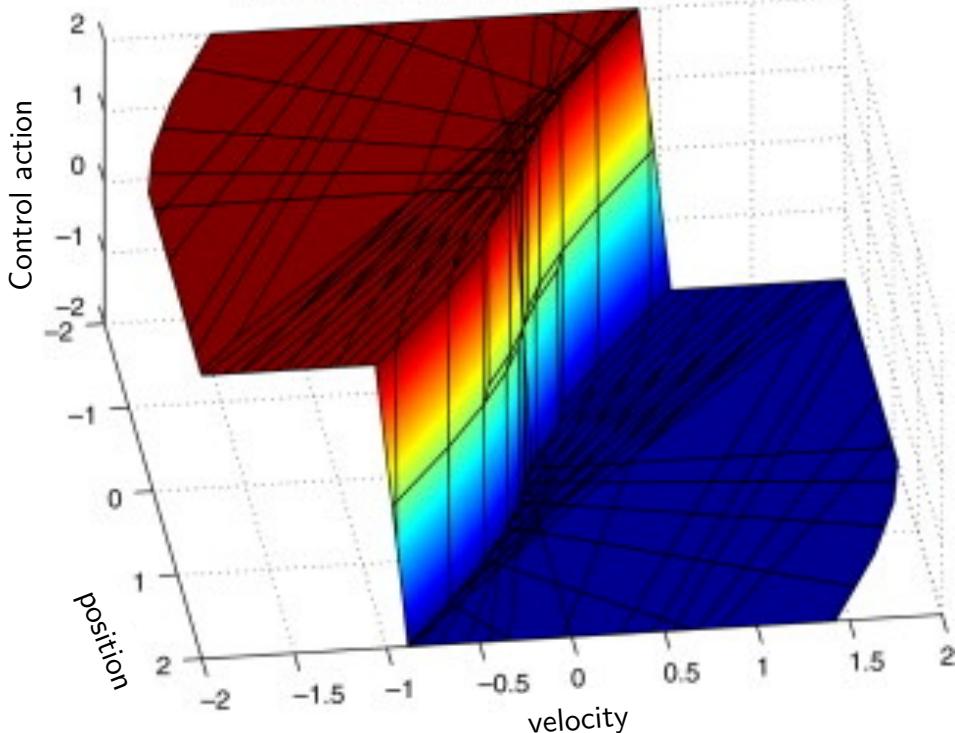


Alberto Bemporad, Manfred Morari, Vivek Dua, and Efstratios N. Pistikopoulos. "The explicit solution of model predictive control via multiparametric quadratic programming." In *Proceedings of the 2000 American Control Conference. ACC (IEEE Cat. No. 00CH36334)*, vol. 2, pp. 872-876. IEEE, 2000.

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Speeding-up the solver

Leverage the explicit solution

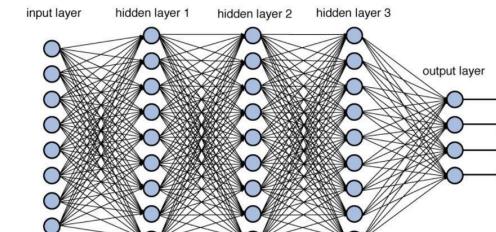


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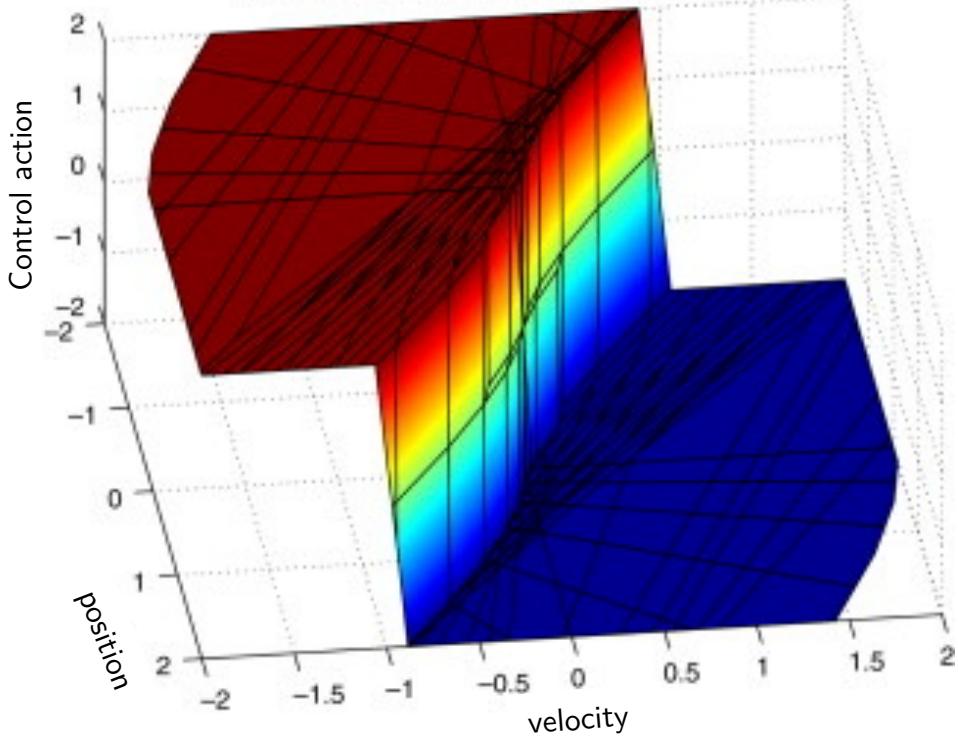
Primal Neural-Network



Learns input policy $\pi^p(x)$

Speeding-up the solver

Leverage the explicit solution

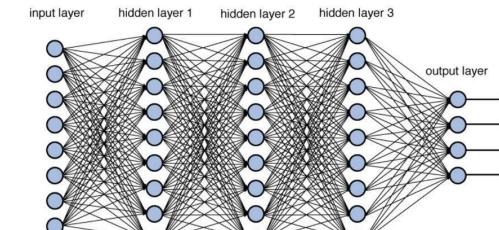


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Offline

Primal Neural-Network



Learns input policy $\pi^p(x)$

Online

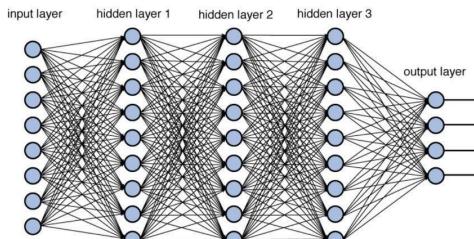
Apply the control policy $\pi^p(x)$

Primal-Dual Neural Network

Primal-Dual Neural Network

Offline

Primal Neural-Network

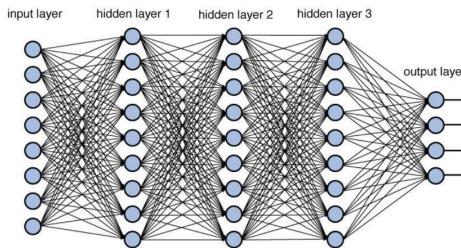


Learns input policy $\pi^p(x)$

Primal-Dual Neural Network

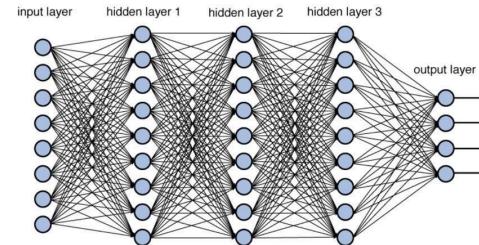
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Primal Neural-Network



Learns input policy $\pi^p(x)$

Dual Neural-Network



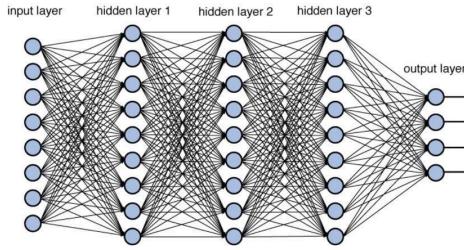
Learns optimality certificate $\pi^d(x)$



Primal-Dual Neural Network

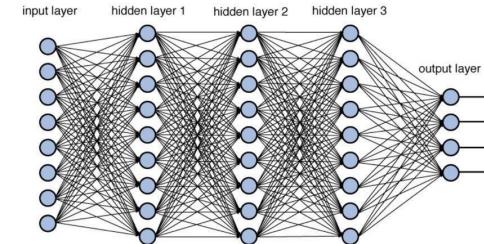
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Primal Neural-Network



Learns input policy $\pi^p(x)$

Dual Neural-Network



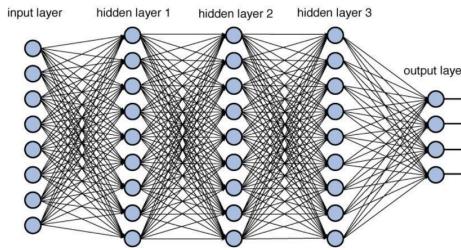
Learns optimality certificate $\pi^d(x)$

Online

Primal-Dual Neural Network

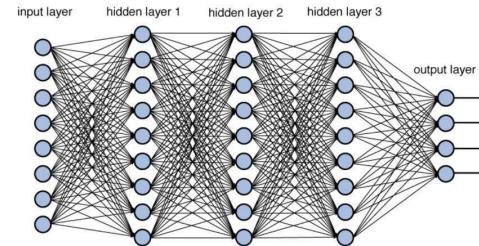
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Primal Neural-Network



Learns input policy $\pi^p(x)$

Dual Neural-Network



Learns optimality certificate $\pi^d(x)$

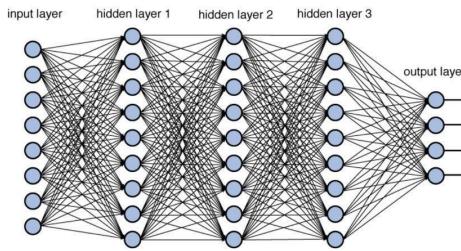
Online

Compute primal cost $J_N^p(x(t), \pi^p)$

Primal-Dual Neural Network

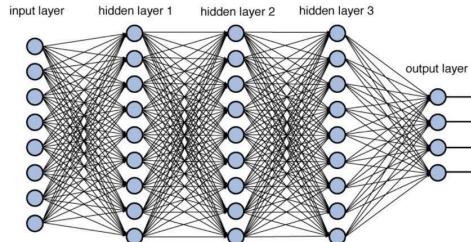
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Primal Neural-Network



Learns input policy $\pi^p(x)$

Dual Neural-Network



Learns optimality certificate $\pi^d(x)$

Online

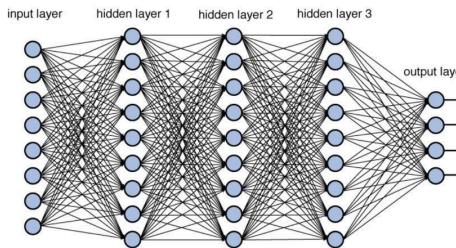
Compute primal cost $J_N^p(x(t), \pi^p)$

Compute dual cost $J_N^d(x(t), \pi^d)$

Primal-Dual Neural Network

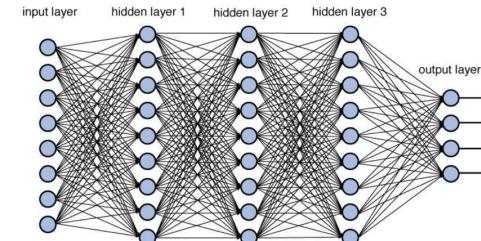
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Primal Neural-Network



Learns input policy $\pi^p(x)$

Dual Neural-Network



Learns optimality certificate $\pi^d(x)$

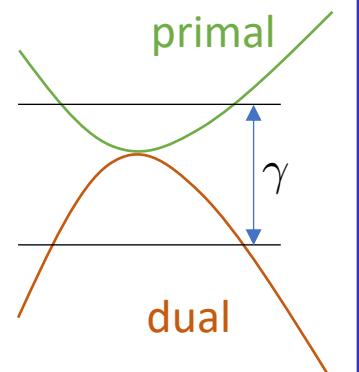
Online

Compute primal cost $J_N^p(x(t), \pi^p)$

Compute dual cost $J_N^d(x(t), \pi^d)$

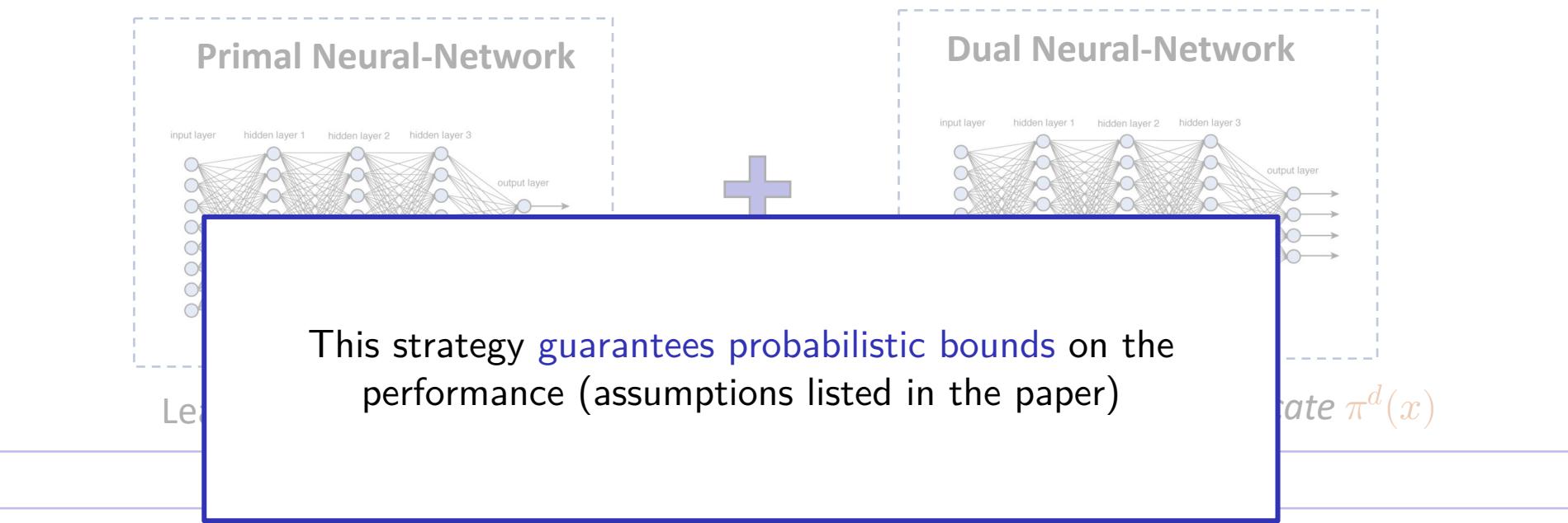
If (primal = feasible) and (dual = feasible) and $|J_N^d(x(t), \pi^d) - J_N^p(x(t), \pi^d)| \leq \gamma$ then

Apply $u_t = \pi^p(x)$



Primal-Dual Neural Network

Offline

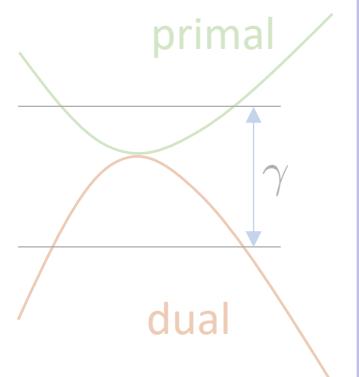


Online

Compute primal cost $J_N^p(x(t), \pi^p)$

Compute dual cost $J_N^d(x(t), \pi^d)$

If (primal = feasible) and (dual = feasible) and $|J_N^d(x(t), \pi^d) - J_N^p(x(t), \pi^p)| \leq \gamma$ then
Apply $u_t = \pi^p(x)$



Primal-Dual Neural Network



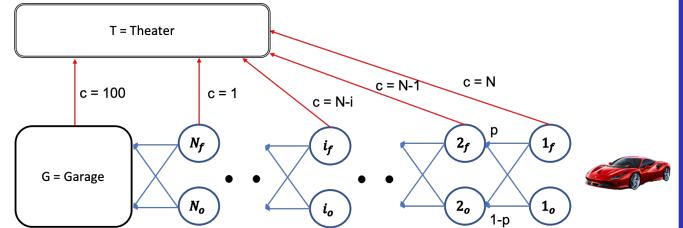
Xiaojing Zhang., Monimoy Bujarbarua, and Francesco Borrelli. "Safe and near-optimal policy learning for model predictive control using primal-dual neural networks". In *2019 American Control Conference (ACC) (2019)*
Xiaojing Zhang, Monimoy Bujarbarua, and Francesco Borrelli. "Near-optimal rapid MPC using neural networks: A primal-dual policy learning framework." *IEEE Transactions on Control Systems Technology* (2020).

Key takeaways

Two slides summary

Lecture #1

- ▶ Control problem with discrete state and action spaces can be solved using **value iteration** and **policy iteration**
- ▶ For large problem we can use **Approximate Dynamic Programming (ADP)**.
The engine behind alphaGo (more details: http://web.mit.edu/dimitrib/www/Slides_Lecture13_RLOC.pdf)



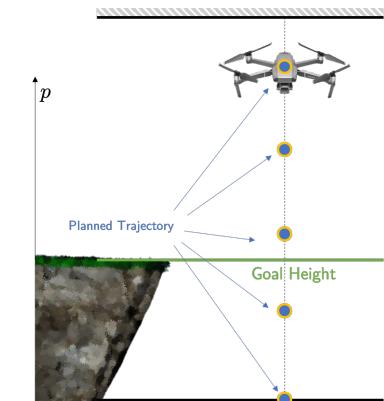
Lecture #2

- ▶ A control policy for the **Linear Quadratic Regulator (LQR)** problem can be computed exactly using dynamic programming
- ▶ For **constrained** systems is hard to compute a policy, but it is “easy” to compute a trajectory using the **batch approach**

$$\begin{aligned} & \min_z \quad z^\top Q z \\ \text{s.t.} \quad & Fz = x(t) \\ & Gz \leq Ex(t) + b \end{aligned}$$

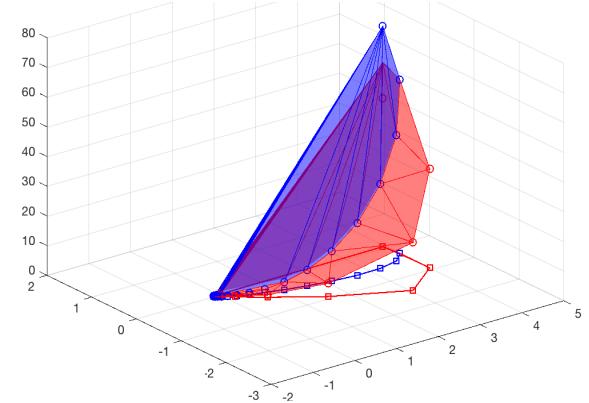
Lecture #3

- ▶ A control policy for constrained problems can be defined iteratively replanning a trajectory over a time horizon N
- ▶ Receding horizon strategies may lead to safety constraint violation and instabilities when the terminal components are not designed correctly



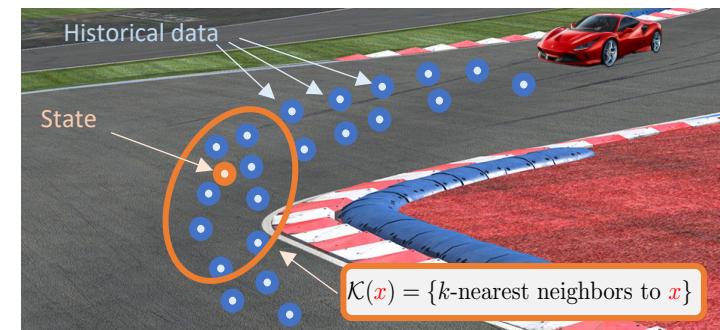
Lecture #4

- ▶ When terminal cost is a control Lyapunov function and the terminal set is an invariant set, the MPC policy guarantees safety and stability
- ▶ We can compute the terminal set and a terminal cost iteratively from historical data. This iterative strategy guarantees safety, stability and performance improvement



Lecture #5

- ▶ A control strategy tailored to uncertain Linear Time-Varying (LTV) systems could work surprisingly well for smooth systems
- ▶ There are several strategies for model learning! Pick the one that best fit your application keeping in mind the accuracy-computation trade off



Lecture #6

- ▶ When disturbances are acting on the system, we should plan of policies to reduce conservatism and the uncertainty along the planned trajectory
- ▶ For stochastic system, we should collect several closed-loop trajectories associated with a control policy to estimate the value function and safe set

