CS159 Lecture 4 Supplementary Material: Synthesizing Terminal Components for MPC

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Adapted from Berkeley ME231A
Original slide set by F. Borrelli, M. Morari, C. Jones

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Polyhedra and polytopes

Polyhedra and polytopes

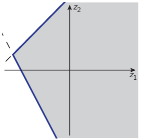
A polyhedron is the intersection of a *finite* number of closed halfspaces:

$$
\begin{aligned}
Z &= \{z \mid a_1^\top z \le b_1,\ a_2^\top z \le b_2, \dots, a_m^\top z \le b_m\} \\
&= \{z \mid Az \le b\}
\end{aligned}
$$

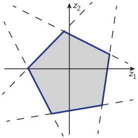
where $A := [a_1, a_2, \dots, a_m]^\top$ and $b := [b_1, b_2, \dots, b_m]^\top$.

A polytope is a *bounded* polyhedron.

Polyhedra and polytopes are always convex.



An (unbounded) polyhedron



A polytope

Polyhedra Representations

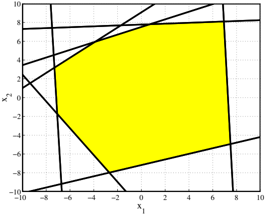
- An $\mathcal{H}$*-polyhedron* $\mathcal{P}$ in $\mathbb{R}^n$ denotes an intersection of a finite set of closed halfspaces in $\mathbb{R}^n$:

$$\mathcal{P} = \{x \in \mathbb{R}^n : \; Ax \leq b\}$$

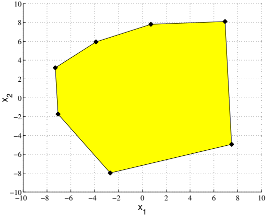
- A $\mathcal{V}$*-polytope* $\mathcal{P}$ in $\mathbb{R}^n$ is defined as

$$\mathcal{P} = \mathrm{conv}(V) = \{v \in \mathbb{R}^n | \exists \lambda \in \mathbb{R}^k, v = V\lambda, 1_k^\top \lambda = 1, \lambda \geq 0\}$$

for some $V = [V_1, \ldots, V_k] \in \mathbb{R}^{n \times k}$ and the vector of ones $1_k \in \mathbb{R}^k$.



$\mathcal{H}$-representation



$\mathcal{V}$-representation

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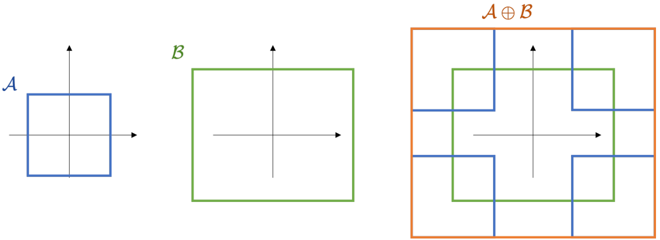
Basic Operations on Polytopes

- Given two sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{B} \subset \mathbb{R}^n$, the Minkowski sum of $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathcal{A} \oplus \mathcal{B} = \{x + y \in \mathbb{R}^n \mid x \in \mathcal{A}, y \in \mathcal{B}\}$$

Furthermore, given the V-representations $\mathcal{A} = \text{conv}([v_1^a, \ldots, v_a^a])$ and $\mathcal{B} = \text{conv}([v_1^b, \ldots, v_b^b])$ the Minkowski sum

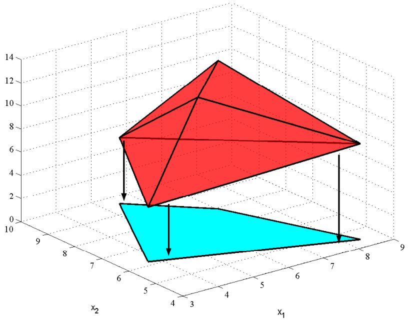
$$\mathcal{A} \oplus \mathcal{B} = \mathcal{A} = \text{conv}([v_{1,1}^{ab}, \ldots, v_{a,b}^{ab}]), \quad \text{where } v_{ij}^{ab} = v_i^a + v_j^b.$$



Basic Operations on Polytopes

- *Projection* Given a polytope $\mathcal{P} = \{[x'y']' \in \mathbb{R}^{n+m} \;:\; A^x x + A^y y \leq b\} \subset \mathbb{R}^{n+m}$ the projection onto the x-space $\mathbb{R}^n$ is defined as

$$\mathrm{proj}_x(\mathcal{P}) := \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m : A^x x + A^y y \leq b\}.$$



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Reachable Set

Reachable Set for a policy π

Consider the discrete-time system $x_{k+1} = f(x_k, u_k)$ and the state constraint set $\mathcal{X}$. The *Reachable Set for a policy π* from the set $\mathcal{S}$ is defined as

$$\mathrm{Reach}_\pi(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n \mid \exists\ x_0 \in \mathcal{S} \text{ s.t. } x = f(x_0, \pi(x_0))\}$$

Reachable Set

Consider the discrete-time system $x_{k+1} = f(x_k, u_k)$, the state constraint set $\mathcal{X}$ and input constraint set $\mathcal{U}$. The *Reachable Set* from the set $\mathcal{S}$ is defined as

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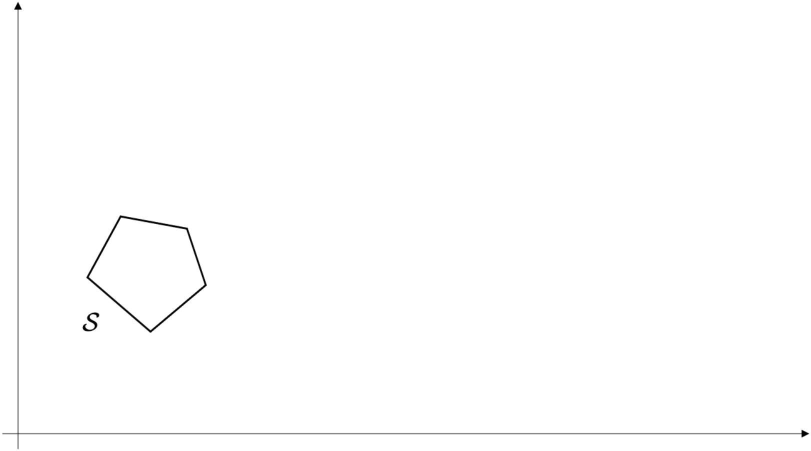
Reachable Set – Example



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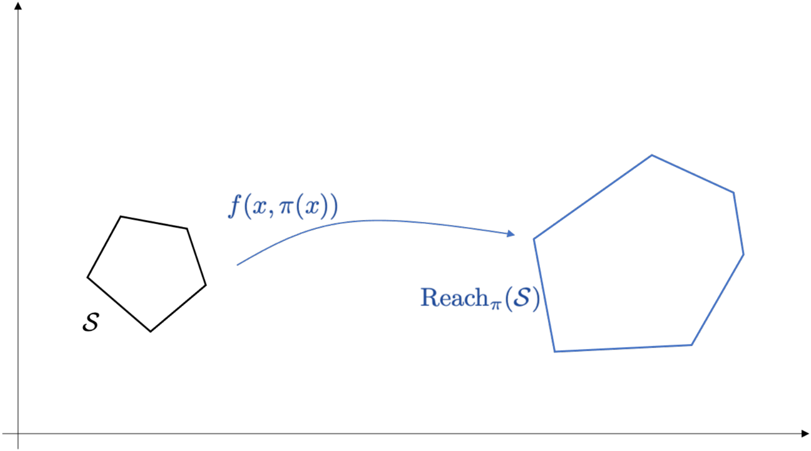

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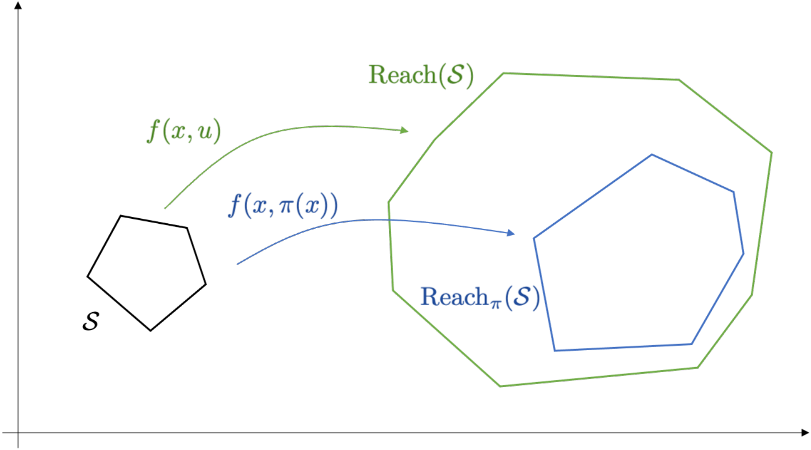

Reach Set Computation

- Consider the polyhedron $\mathcal{X} = \text{conv}(V_x)$ and the linear discrete time system
$$x(t+1) = Ax(t) + Bu(t)$$
where the input $u \in \mathcal{U} = \text{conv}(V_u)$ and define
$$A \circ \mathcal{X} = \mathrm{conv}(AV_x).$$

- Then for the policy $\pi(x) = Kx$
$$\text{Reach}_\pi(\mathcal{X}) = (A - BK) \circ \mathcal{X}$$
and
$$\begin{aligned} \text{Reach}(\mathcal{X}) &= \{\bar{x} + \bar{u} \mid \bar{x} \in A \circ V_x,\ \bar{u} \in B \circ \mathcal{U}\} \\ &= (A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}). \end{aligned}$$

N-Step Reachable Sets

Definition (N-Step Reachable Set $\mathcal{R}_N(\mathcal{S})$)

Consider the discrete-time system $x_{k+1} = f(x_k, u_k)$, the state constraint set $\mathcal{X}$ and input constraint set $\mathcal{U}$. For a given initial set $\mathcal{S} \subseteq \mathcal{X}$, the N-step reachable set $\mathcal{R}_N(\mathcal{S})$ is

$$\mathcal{R}_{i+1}(\mathcal{S}) \triangleq \text{Reach}(\mathcal{R}_i(\mathcal{S})), \quad \mathcal{R}_0(\mathcal{S}) = \mathcal{S}, \qquad i = 0, \ldots, N-1$$

By definition all states $x_0 \in \mathcal{S}$ will evolve to the N-step reachable set $\mathcal{R}_N(\mathcal{S})$ in N time steps.

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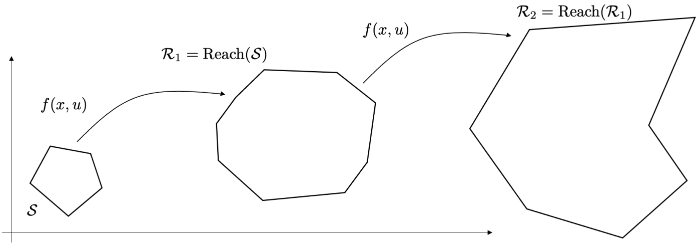

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Pre Set Definition

Pre Set for the policy π

Consider the discrete-time system $x_{k+1} = f(x_k, u_k)$ and the state constraint set $\mathcal{X}$. The *Pre Set for a policy π* from the set $\mathcal{S}$ is defined as

$$\text{Pre}_\pi(\mathcal{S}) \triangleq \{x \in \mathbb{R}^n \mid f(x, \pi(x)) \in \mathcal{S}\}$$

Pre Set

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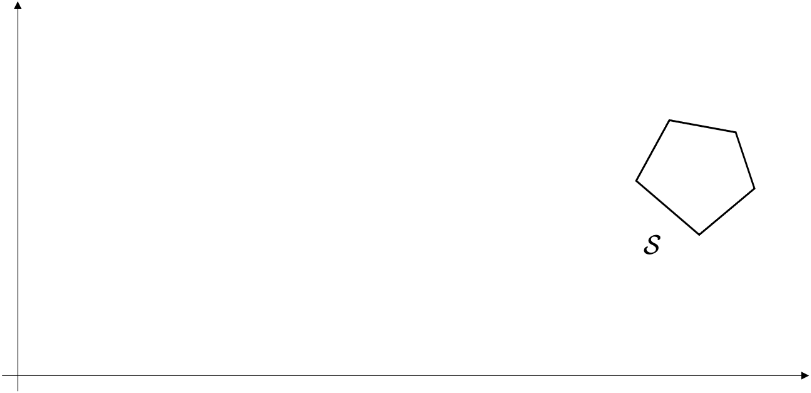
Pre Set – Example



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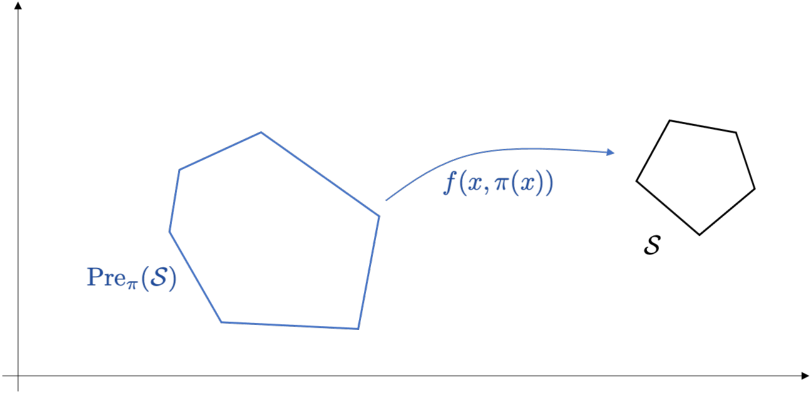
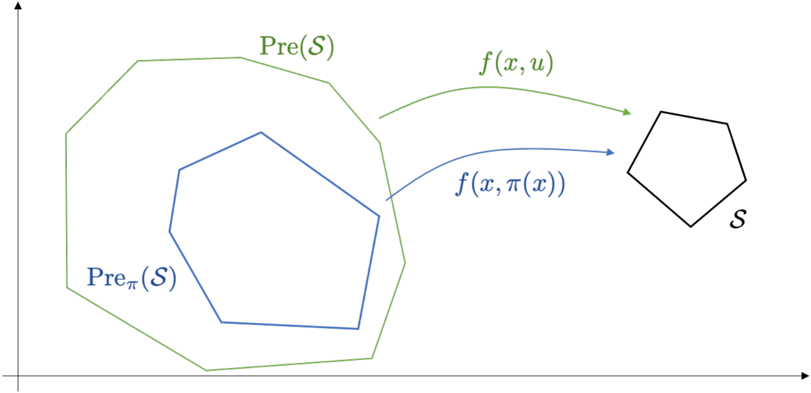


Pre Set – Example



Pre Set Computation - Autonomous Systems

- Consider the polyhedron $\mathcal{X} = \{x \mid H_x x \leq h_x\}$ and the linear discrete time autonomous system

$$x(t+1) = Ax(t) + Bu(t)$$

- Then

$$\mathrm{Pre}(\mathcal{S}) = \{x \mid HAx \leq h\}$$

Pre Set Computation - System with Inputs

- Consider the polyhedron $\mathcal{X} = \{x \mid H_x x \leq h_x\}$ and the linear discrete time system

$$x(t+1) = Ax(t) + Bu(t)$$

where the input $u \in \mathcal{U} = \{u \mid H_u u \leq h_u\}$ and define

$$A \circ \mathcal{X} = \mathrm{conv}(AV_x).$$

- Then

$$\mathrm{Pre}(\mathcal{S}) = \left\{ x \in \mathbb{R}^n \,|\, \exists u \in \mathbb{R} \mid \begin{bmatrix} H_x A & H_x B \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq \begin{bmatrix} h_x \\ h_u \end{bmatrix} \right\}$$

which is the projection onto the x-space (with dimension $\mathbb{R}^n$) of the polyhedron

$$\mathcal{T} := \left\{ \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq \begin{bmatrix} h \\ h_u \end{bmatrix} \right\}.$$

N-Step Controllable Sets

N-Step Controllable Set $\mathcal{K}_N(\mathcal{S})$

For a given target set $\mathcal{S} \subseteq \mathcal{X}$, the N-step controllable set $\mathcal{K}_N(\mathcal{S})$ is defined as:

$$\mathcal{K}_N(\mathcal{S}) \triangleq \text{Pre}(\mathcal{K}_{N-1}(\mathcal{S})) \cap \mathcal{X}, \quad \mathcal{K}_0(\mathcal{S}) = \mathcal{S}, \qquad N \in \mathbb{N}^+.$$

By definition all states $x_0 \in \mathcal{K}_N(\mathcal{S})$ can be driven, through a time-varying control law, to the target set $\mathcal{O}$ in N steps, while satisfying input and state constraints.

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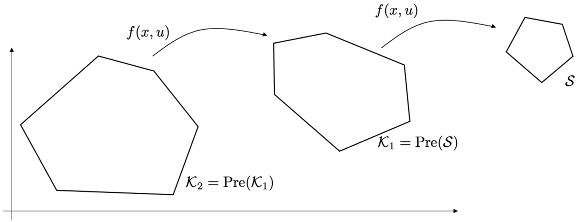

Maximal Controllable Set

Maximal Controllable Set $\mathcal{K}_\infty(\mathcal{S})$

For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the maximal controllable set $\mathcal{K}_\infty(\mathcal{S})$ for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints $x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$ is the union of all N-step controllable sets contained in $\mathcal{X}$ ($N \in \mathbb{N}$).

As we will be discussing, Maximal Controllable Set characterize the MPC region of attraction. However, computing these set may be challenging as these sets are computed using projections.

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Invariant Sets

Invariant sets

- are computed for ***autonomous systems***
- for a *given* feedback controller $u = \pi(x)$, will contain the evolution of the system for all times.

Positive Invariant Set

A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a positive invariant set for the autonomous system $x(t+1) = f(x(t), \pi(x(t))$ subject to the constraints $x(t) \in \mathcal{X}$, if

$$x(0) \in \mathcal{O} \quad \Rightarrow \quad x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}^+$$

Maximal Positive Invariant Set $\mathcal{O}_\infty$

The set $\mathcal{O}_\infty$ is the maximal invariant set if $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all the invariant sets contained in $\mathcal{X}$.

Invariant Sets

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Maximal Positive Invariant Set $\mathcal{O}_\infty$

The set $\mathcal{O}_\infty$ is the maximal invariant set if $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all the invariant sets contained in $\mathcal{X}$.

Invariant Sets

Theorem (Geometric condition for invariance)

A set $\mathcal{O}$ is a positive invariant set if and only if $\mathcal{O} \subseteq Pre_{\pi}(\mathcal{O})$

NOTE: $\mathcal{O} \subseteq \text{Pre}_{\pi}(\mathcal{O}) \Leftrightarrow \text{Pre}_{\pi}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$

Algorithm

Input: System model f, control policy π, constraint set $\mathcal{X}$
Output: $\mathcal{O}_\infty$
1. Let $\Omega_0 = \mathcal{X}$
2. Let $\Omega_{k+1} = \text{Pre}_{\pi}(\Omega_k) \cap \Omega_k$
3. If $\Omega_{k+1} = \Omega_k$ **then** $\Omega_\infty \leftarrow \Omega_{k+1}$
4. If else go to 2

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in \mathbb{N}$ and it terminates when $\Omega_{k+1} = \Omega_k$ so that Ω_k is the maximal positive invariant set $\mathcal{O}_\infty$ for $x(t+1) = f_a(x(t))$.

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Control Invariant Sets

Control invariant sets

- are computed for systems ***subject to external inputs***
- provide the set of initial states for which *there exists* a controller such that the system constraints are never violated.

Control Invariant Set

A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set if

$$x(t) \in \mathcal{C} \quad \Rightarrow \quad \exists u(t) \in \mathcal{U} \text{ such that } f(x(t), u(t)) \in \mathcal{C}, \quad \forall t \in \mathbb{N}^+$$

Maximal Control Invariant Set

The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints in $x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$, if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$.

Control Invariant Sets

Control invariant sets

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A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set if

$$x(t) \in \mathcal{C} \quad \Rightarrow \quad \exists u(t) \in \mathcal{U} \text{ such that } f(x(t), u(t)) \in \mathcal{C}, \quad \forall t \in \mathbb{N}^+$$

Maximal Control Invariant Set

The set $\mathcal{C}_\infty$ is said to be the maximal control invariant set for the system $x(t+1) = f(x(t), u(t))$ subject to the constraints in $x(t) \in \mathcal{X},\ u(t) \in \mathcal{U}$, if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$.

Control Invariant Sets

Same geometric condition for control invariants holds: $\mathcal{C}$ is a control invariant set if and only if

$$\mathcal{C} \subseteq \mathrm{Pre}(\mathcal{C})$$

Algorithm

Input: System model f, constraint sets $\mathcal{X}$ and $\mathcal{U}$
Output: $\mathcal{O}_\infty$
1. Let $\Omega_0 = \mathcal{X}$
2. Let $\Omega_{k+1} = \mathrm{Pre}(\Omega_k) \cap \Omega_k$
3. If $\Omega_{k+1} = \Omega_k$ **then** $\mathcal{C}_\infty \leftarrow \Omega_{k+1}$
4. If else go to 2

The algorithm generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k, \forall k \in \mathbb{N}$ and it terminates if $\Omega_{k+1} = \Omega_k$ so that Ω_k is the maximal control invariant set $\mathcal{C}_\infty$ for the constrained system.

Invariant Sets and Control Invariant Sets

TO DO: Add figure

- The set $\mathcal{O}_\infty$ ($\mathcal{C}_\infty$) is ***finitely determined*** if and only if $\exists\ i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$.
- The smallest element $i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$ is called the ***determinedness index***.
- For all states contained in the maximal control invariant set $\mathcal{C}_\infty$ there exists a control law, such that the system constraints are never violated.

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Loss of Feasibility

MPC policies compute control actions by solving finite time optimal control problems over shifted time windows:

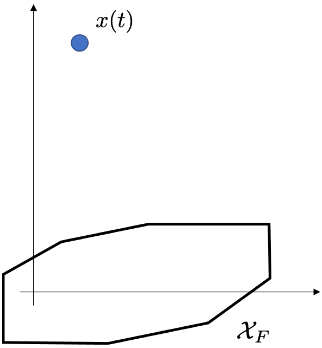
$$
\begin{aligned}
J_t^*(x(0)) = \min_{u_{t|t},\ldots,u_{t+N-1|t}} \quad & \sum_{k=0}^{T-1} h(x_{k|t}, u_{k|t}) + V(x_{t+T|t}) \\
\text{such that} \quad & x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \forall k \in \{t, \ldots, t+N-1\} \\
& x_{k|t} \in \mathcal{X}, u_{k|t} \in \mathcal{U}, \forall k \in \{t, \ldots, t+N-1\} \\
& x_{t|t} = x(0), x_N \in \mathcal{X}_F
\end{aligned}
$$

Solution: The terminal cost $V(x_{t+T|t})$ and terminal constraint $\mathcal{X}_F$, often referred to as *terminal components*, should approximate the tail of cost and constraints beyond the prediction horizon.

Recursive Feasibility

Let the terminal set $\mathcal{X}_F$ be *control invariant*. Next, we show by induction that the terminal set $\mathcal{X}_F$ guarantees that the controller is recursively feasible.

- At time step t assume that the MPC problem is feasible and let $\{u^*_{t|t}, \ldots, u^*_{t+N-1|t}\}$ and $\{x^*_{t|t}, \ldots, x^*_{t+N|t}\}$ be the optimal sequences of states and actions.
- At the next time step $t+1$, we have that $x(t+1) = x^*_{t+1|t}$.
- Therefore, at the next time step $t+1$ the sequences $\{u^*_{t+1|t}, \ldots, u^*_{t+N-1|t}, 0\}$ and $\{x^*_{t+1|t}, \ldots, x^*_{t+N|t}, 0\}$ are feasible, as $x^*_{t+N|t} = 0 \in \mathcal{X}_F$ and the origin is an unforced equilibrium point.



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- At the next time step $t+1$, we have that $x(t+1) = x^*_{t+1|t}$.
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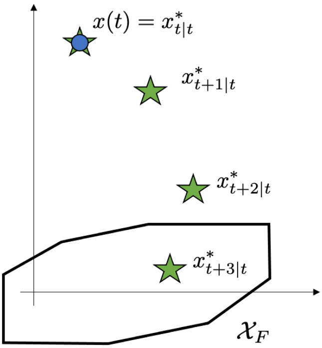

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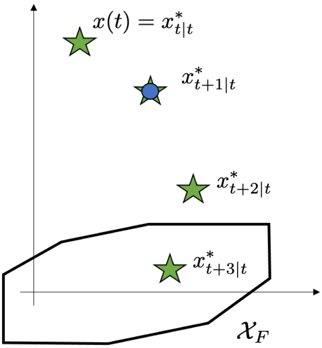
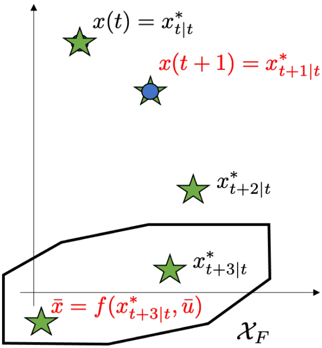
- At the next time step $t+1$, we have that $x(t+1) = x^*_{t+1|t}$.
- Therefore, at the next time step $t+1$ the sequences $\{u^*_{t+1|t}, \dots, u^*_{t+N-1|t}, 0\}$ and $\{x^*_{t+1|t}, \dots, x^*_{t+N|t}, 0\}$ are feasible, as $x^*_{t+N|t} = 0 \in \mathcal{X}_F$ and the origin is an unforced equilibrium point.



Recursive Feasibility

Let the terminal set $\mathcal{X}_F$ be <u>*control invariant*</u>. Next, we show by induction that the terminal set $\mathcal{X}_F$ guarantees that the controller is recursively feasible.

- At time step t assume that the MPC problem is feasible and let $\{u^*_{t|t}, \ldots, u^*_{t+N-1|t}\}$ and $\{x^*_{t|t}, \ldots, x^*_{t+N|t}\}$ be the optimal sequences of states and actions.
- At the next time step $t+1$, we have that $x(t+1) = x^*_{t+1|t}$.
- As $x^*_{t+N|t} \in \mathcal{X}_F$ there exists $\bar{u} \in \mathcal{U}$ such that $\bar{x} = f(x^*_{t+N|t}, \bar{u}) \in \mathcal{X}_F$. Thus, at the next time step $t+1$ the sequences $\{u^*_{t+1|t}, \ldots, u^*_{t+N-1|t}, \bar{u}\}$ and $\{x^*_{t+1|t}, \ldots, x^*_{t+N|t}, \bar{x}\}$ are feasible.



Stability – Assumptions

Let the following assumptions hold

- The stage cost satisfies

$$h(x,u) = 0 \forall x \in \mathcal{X} \setminus \{0\}, \forall u \in \mathcal{U} \setminus \{0\}$$

  and $h(0,0) = 0$.
- The terminal set $\mathcal{X}_F$ is a *control invariant set*
- The terminal cost function $V : \mathbb{R}^n \to \mathbb{R}$ is a control Lyapunov function for the set $\mathcal{X}_F$, i.e.,

$$\forall x \in \mathcal{X}_F, \exists u \in \mathcal{U} \text{ such that } V(f(x,u)) - V(x) \geq -h(x,u) \\ \text{and } f(x,u) \in \mathcal{X}_F.$$

Next, we show by induction that the open-loop cost $J_t^*(x(t))$ is a Lyapunov function for the closed-loop system, i.e.,

$$J_{t+1}^*(x(t+1)) < J_t^*(x(t)), \forall x(t) \in \mathcal{X} \setminus \{0\}.$$

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Stability – Proof (1/2)

- At time step t, assume that the MPC problem is feasible and let $\{u^*_{t|t}, \ldots, u^*_{t+N-1|t}\}$ and $\{x^*_{t|t}, \ldots, x^*_{t+N|t}\}$ be the optimal sequences of states and actions. Then the open-loop cost is

$$\begin{aligned} J^*_t(x(t)) &= \sum_{k=t}^{N-1} h(x^*_{k|t}, u^*_{k|t}) + V(x^*_{t+N|t}) \\ &\geq \sum_{k=t}^{N-1} h(x^*_{k|t}, u^*_{k|t}) + h(x^*_{t+N|t}, \bar{u}) + V(f(x^*_{t+N|t}, \bar{u})) \end{aligned}$$

for $\bar{u} \in \mathcal{U}$ such that $f(x^*_{t+N|t}, \bar{u})$.

Stability – Proof (2/2)

- At the next time step $t+1$,

$$\bar{J} = \sum_{k=t+1}^{N-1} h(x^*_{k|t}, u^*_{k|t}) + h(x^*_{t+N|t}, \bar{u}) + V(f(x^*_{t+N|t}, \bar{u}))$$

  is the cost associated with the feasible sequence of inputs $\{u^*_{t+1|t}, \dots, u^*_{t+N-1|t}, \bar{u}\}$, thus

$$J^*_t(x(t)) = h(x^*_{k|t}, u^*_{k|t}) + \bar{J} \geq h(x^*_{t|t}, u^*_{t|t}) + J^*_{t+1}(x(t+1)).$$

- Concluding, the open-loop cost satisfies

$$J^*_{t+1}(x(t+1)) - J^*_t(x(t)) \leq -h(x(t), u(t))$$

  as $x^*_{t|t} = x(t)$ and $u^*_{t|t} = x(t)$, and it is a Lyapunov function for the closed-loop system.

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Constrained Linear Quadratic Regulator

Consider the following finite time optimal control problem:

$$
\begin{aligned}
J_t^*(x(0)) = \min_{u_{t|t},\ldots,u_{t+N-1|t}} \quad & \sum_{k=0}^{T-1} h(x_{k|t}, u_{k|t}) + x_{t+T|t}^\top P x_{t+T|t} \\
\text{such that} \quad & x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \forall k \in \{t, \ldots, t+N-1\} \\
& x_{k|t} \in \mathcal{X}, u_{k|t} \in \mathcal{U}, \forall k \in \{t, \ldots, t+N-1\} \\
& x_{t|t} = x(0), x_N \in \mathcal{X}_F
\end{aligned}
$$

where $h(x, u) = x^\top Qx + u^\top Ru$.

Next, we discuss how to construct the terminal cost $V(x) = x^\top Px$ and the terminal set $\mathcal{X}_F$ to guarantee recursive feasibility and closed-loop stability.

Design Rules

1. Design unconstrained LQR control law

$$K_\infty = (B'P_\infty B + R)^{-1}B'P_\infty A$$

   where P_∞ is the solution to the discrete-time algebraic Riccati equation:

$$P_\infty = A'P_\infty A + Q - A'P_\infty B(B'P_\infty B + R)^{-1}B'P_\infty A$$

2. Choose the terminal weight $P = P_\infty$
3. Choose the terminal set $\mathcal{X}_F$ to be the maximum invariant set for the closed-loop system $x_{k+1} = (A - BK_\infty)x_k$:

$$x_{k+1} = (A - BK_\infty)x_k \in \mathcal{X}_F, \quad \text{for all } x_k \in \mathcal{X}_F$$

   All state and input **constraints are satisfied** in $\mathcal{X}_F$:

$$\mathcal{X}_F \subseteq \mathcal{X}, \; F_\infty x_k \in \mathcal{U}, \quad \text{for all } x_k \in \mathcal{X}_F$$

Stability and Feasibility Proof

By construction all the Assumptions of the required to guarantee recursive feasibility and stability are verified:

1. The stage cost is a positive definite function
2. By construction the terminal set is **invariant** under the local control law $v = -K_\infty x$
3. Terminal cost is a continuous **Lyapunov function** in the terminal set $\mathcal{X}_F$ and satisfies:

$$\begin{aligned}
&x_{k+1}^\top P x_{k+1} - x_k^\top P x_k \\
&\qquad = x_k'(-P_\infty + A'P_\infty A - A'P_\infty B(B'P_\infty B + R)^{-1}B'P_\infty A)x_k \\
&\qquad = -x_k' Q x_k
\end{aligned}$$

Summary of Safety and Stability Properties

Key Message: When the MPC terminal components are not designed correctly, the closed-loop system may violate safety constraints and convergence to the goal state/set is not guaranteed

Solution: We have shown that given a terminal set set $\mathcal{X}_F$ which is control invariant, and a terminal cost function $V(x)$ which is a control Lyapunov function.

- The MPC problem is feasible at all times
- The closed-loop system is stable as for the positive definite open-loop cost we have $J^*_{t+1}(x(t+1)) < J^*_t(x(t)), \forall x(t) \notin \mathcal{X}_F$

Main drawback: These terminal components are hard to compute even for linear constrained deterministic systems.