CS159 Lecture 5: LMPC and Model Learning

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Caltech

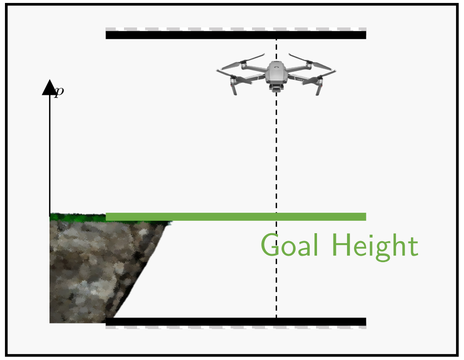
Spring 2021

Today's Class

- Recap of Lecture #4
- LMPC implementation
- Autonomous Racing Experiments
- Model learning in MPC (a brief summary)
- Planning under uncertainty

Recap of Lecture #4

Why did we focus on feasibility?



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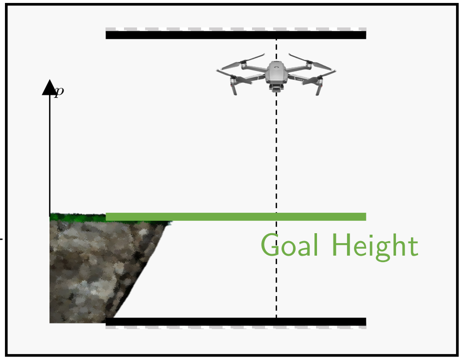
Solve a Finite Time Optimal Control Problem (FTOCP)

Measured state $x(t)$



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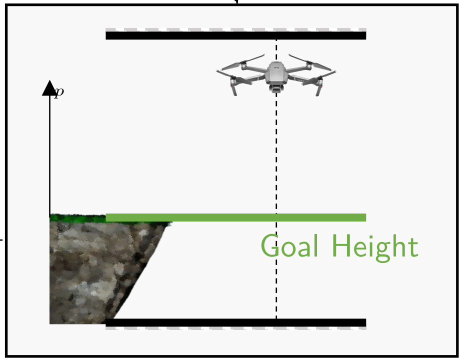

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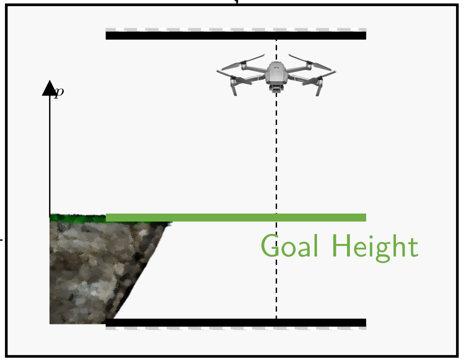

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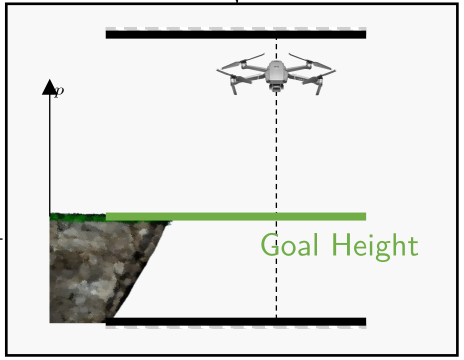

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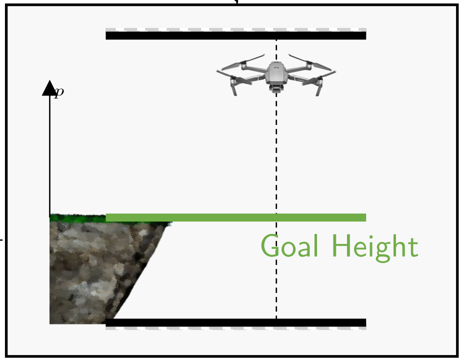

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Feasibility of the FTCOP at all times

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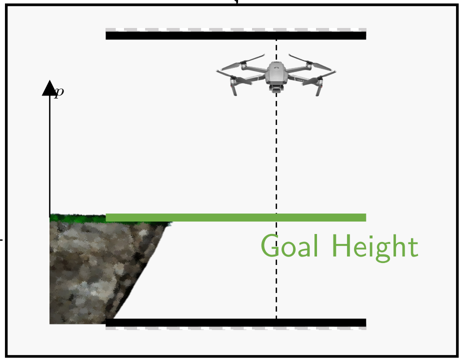

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Feasibility of the FTCOP at all times

⬇

Safety

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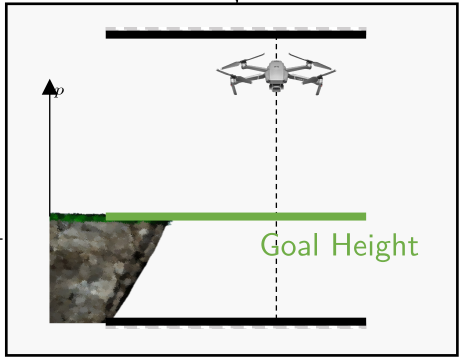

Measured state $x(t)$

Feasibility of the FTCOP at all times

~~$\Downarrow$~~ Not true when constraints are relaxed with slack variables!

Safety

Recap of Lecture #4

Stability Proof:

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We have shown that

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$$J_t^*(x) > 0, \forall x \neq 0$$

$$J_t^*(0) = 0$$

Recap of Lecture #4

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$$\left.\begin{aligned} &J_t^*(x(t)) < J_{t+1}^*(x(t+1)),\ \forall x(t) \neq 0 \\ &J_t^*(x) > 0, \forall x \neq 0 \\ &J_t^*(0) = 0 \end{aligned}\right\} \lim_{t\to\infty} J_t^*(x(t)) = 0$$

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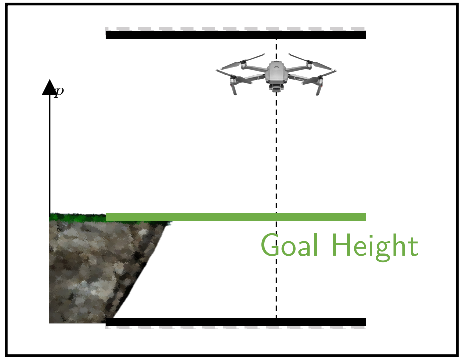
To show stability we need to that $J_t^*(x)$ is Lipchitz continuous (can be relaxed to continuity around the origin).

Lars Grüne, and Christopher M. Kellett. "ISS-Lyapunov functions for discontinuous discrete-time systems." IEEE Transactions on Automatic Control 59, no. 11 (2014): 3098-3103.

LMPC Recap #4

Time Loop: Perform the control task



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Time Loop: Perform the control task

Solve a Finite Time Optimal Control Problem (FTOCP) with:

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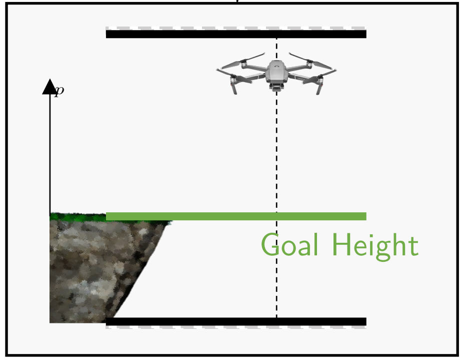

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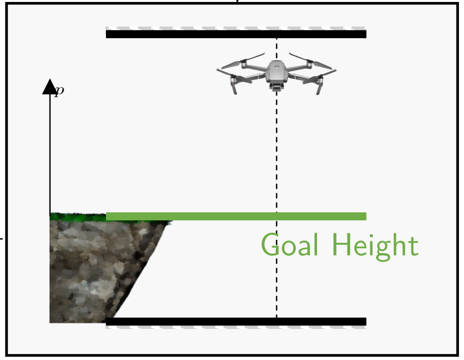

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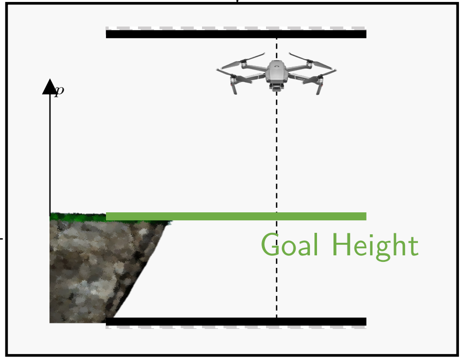

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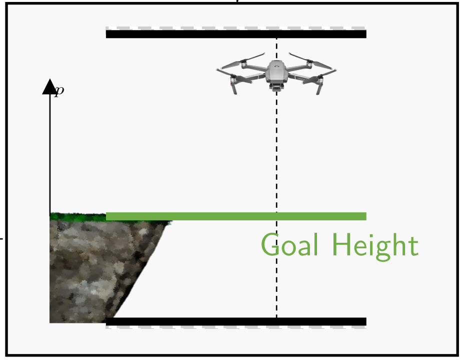


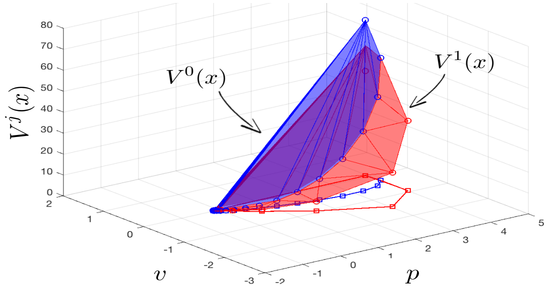
Measured state $x(t)$

Control task completed
Store closed-loop

$$\mathbf{x}^j = [x_0^j, x_1^j, \dots]$$

Update terminal constraint and terminal cost

$$\mathcal{CS}^{j+1} \text{ and } V^{j+1}(x)$$



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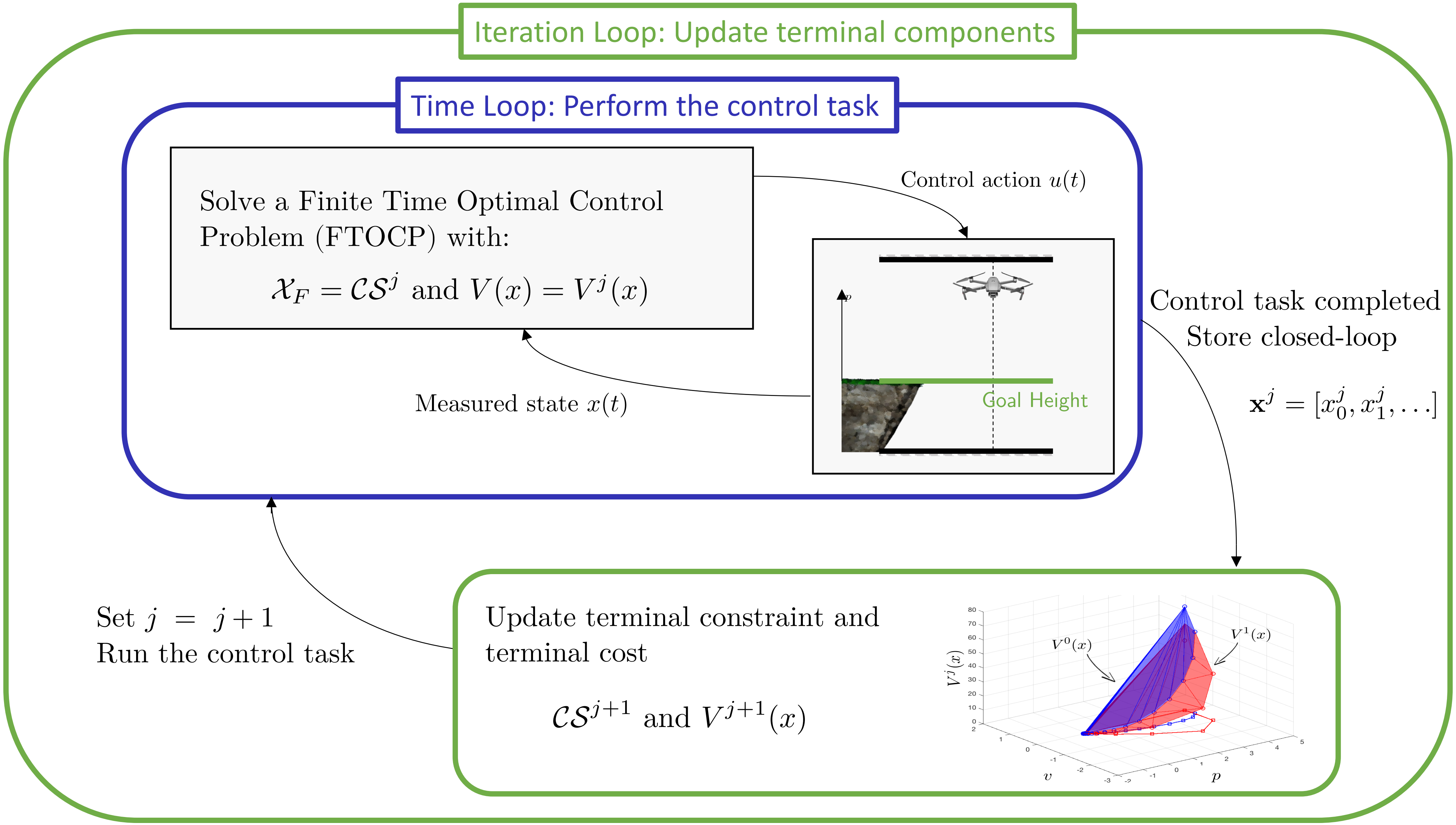

Iteration Loop: Update terminal components
Time Loop: Perform the control task
Solve a Finite Time Optimal Control Problem (FTOCP) with:
$\mathcal{X}_F = \mathcal{CS}^j$ and $V(x) = V^j(x)$
Control action $u(t)$
Goal Height
Measured state $x(t)$
Control task completed
Store closed-loop
$\mathbf{x}^j = [x_0^j, x_1^j, \ldots]$
Set $j = j+1$
Run the control task
Update terminal constraint and terminal cost
$\mathcal{CS}^{j+1}$ and $V^{j+1}(x)$
$V^0(x)$
$V^1(x)$
$V^j(x)$
v
p

Recap of Lecture #4

Theorem

Let $\mathbf{x}^j = [x_0^j, x_1^j, \ldots]$ be the closed-loop trajectory from the starting state x_S at iteration j. Consider sequence $\{\mathbf{x}^j\}$ of closed-loop trajectories and assume that for $c < \infty$ we have that

$$\mathbf{x}^c = \mathbf{x}^{c+1}$$

Then we have that

- At each iteration state and input constraints are satisfied.
- The closed-loop cost $J_{0\to\infty}^j(x_S)$ is non-increasing, i.e.,

$$J_{0\to\infty}^{j+1}(x_S) = \sum_{t=0}^{\infty} h(x_t^{j+1}, u_t^{j+1}) \leq \sum_{t=0}^{\infty} h(x_t^j, u_t^j) = J_{0\to\infty}^j(x_S)$$

- $\mathbf{x}^c = \mathbf{x}^*$, under mild conditions (LICQ holds at each time t)

Optimality Proof Sketch

Define the trajectory at convergence $\mathbf{x}^{\infty} = [x_0^{\infty}, x_1^{\infty}, \ldots]$

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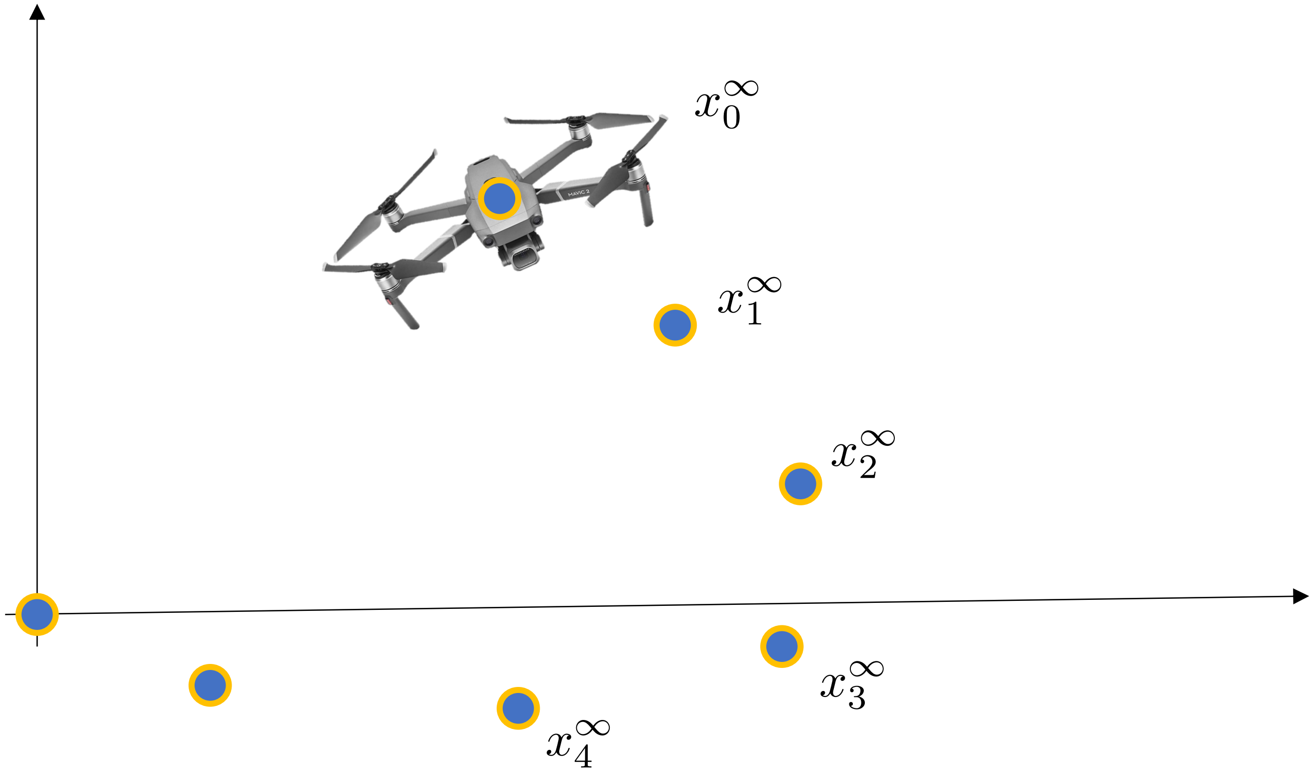

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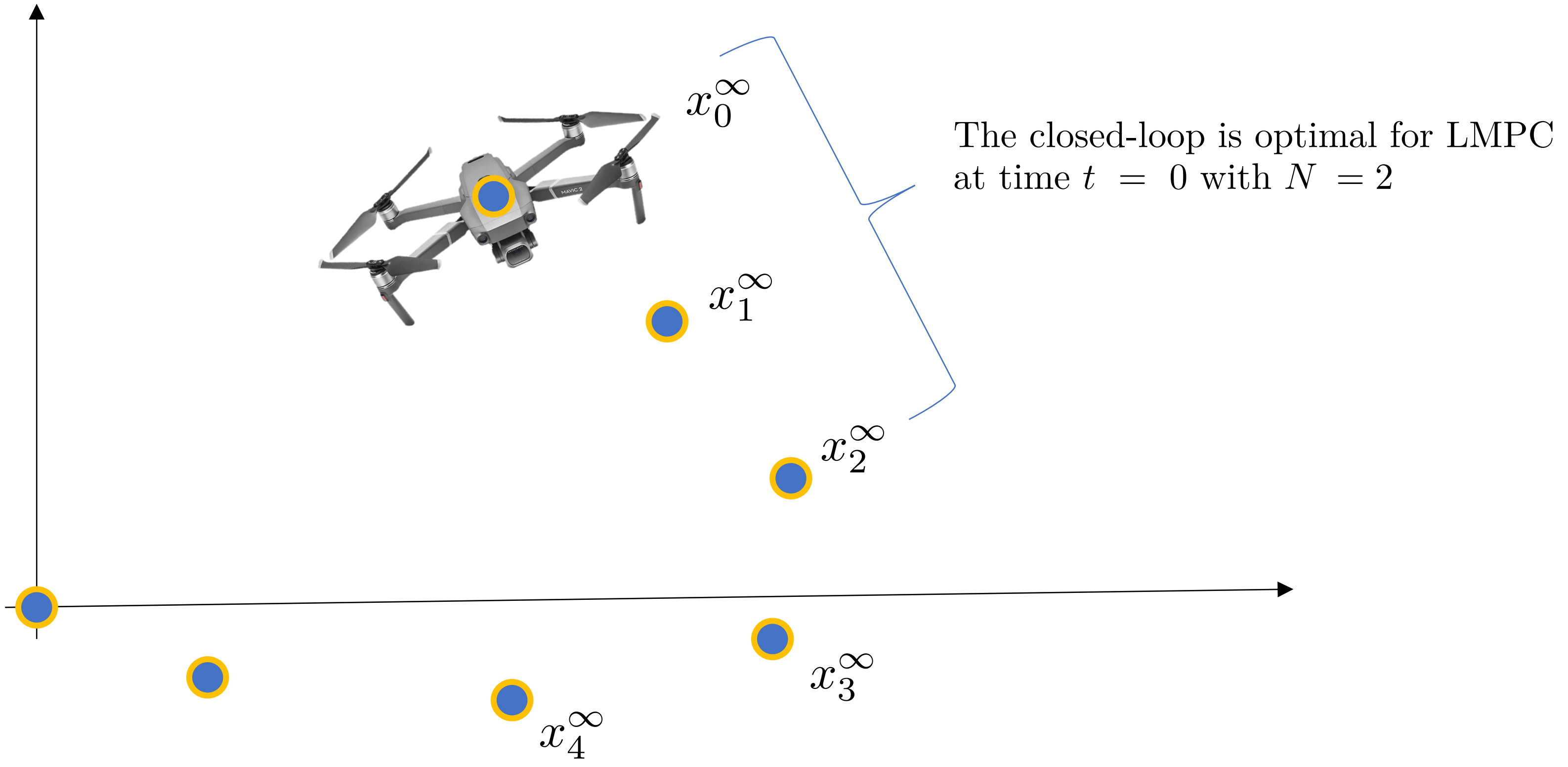

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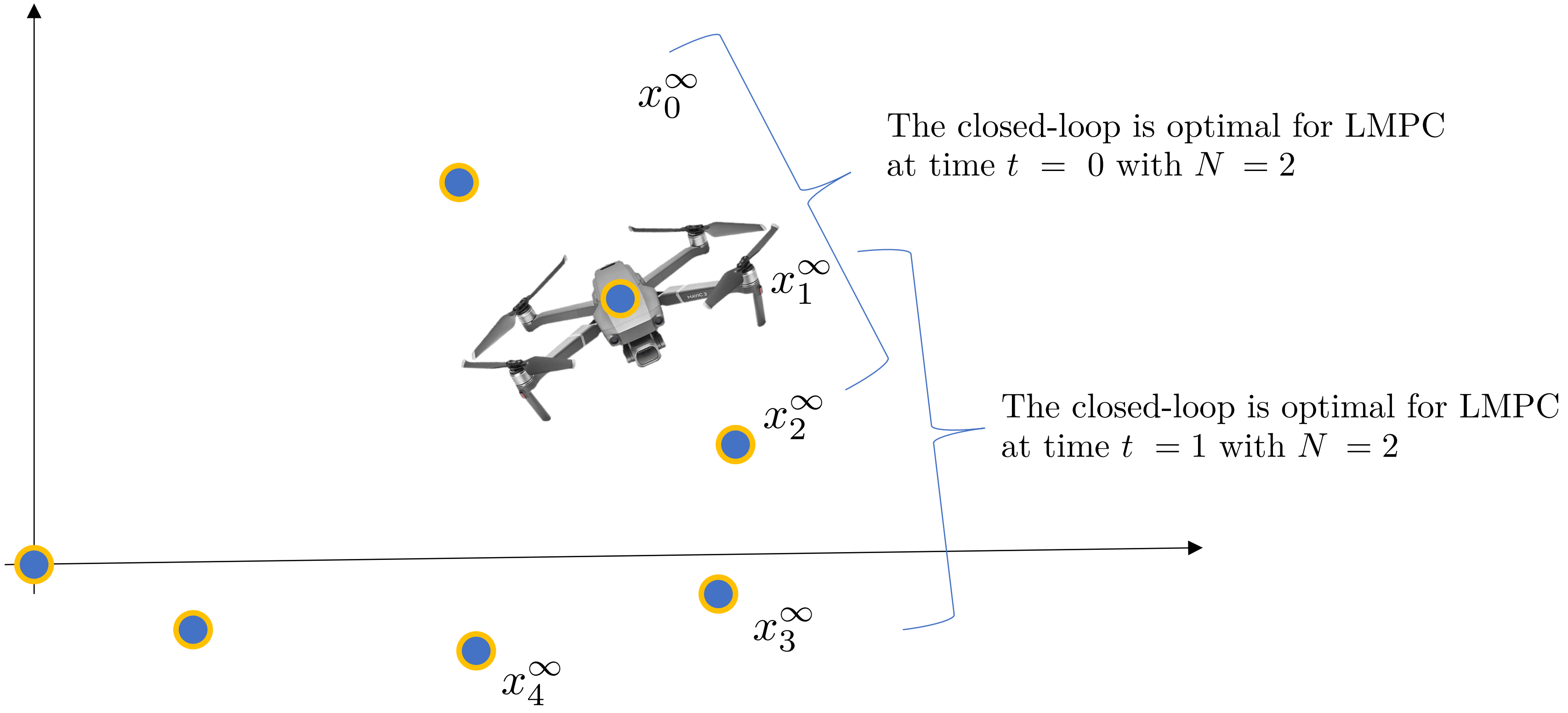

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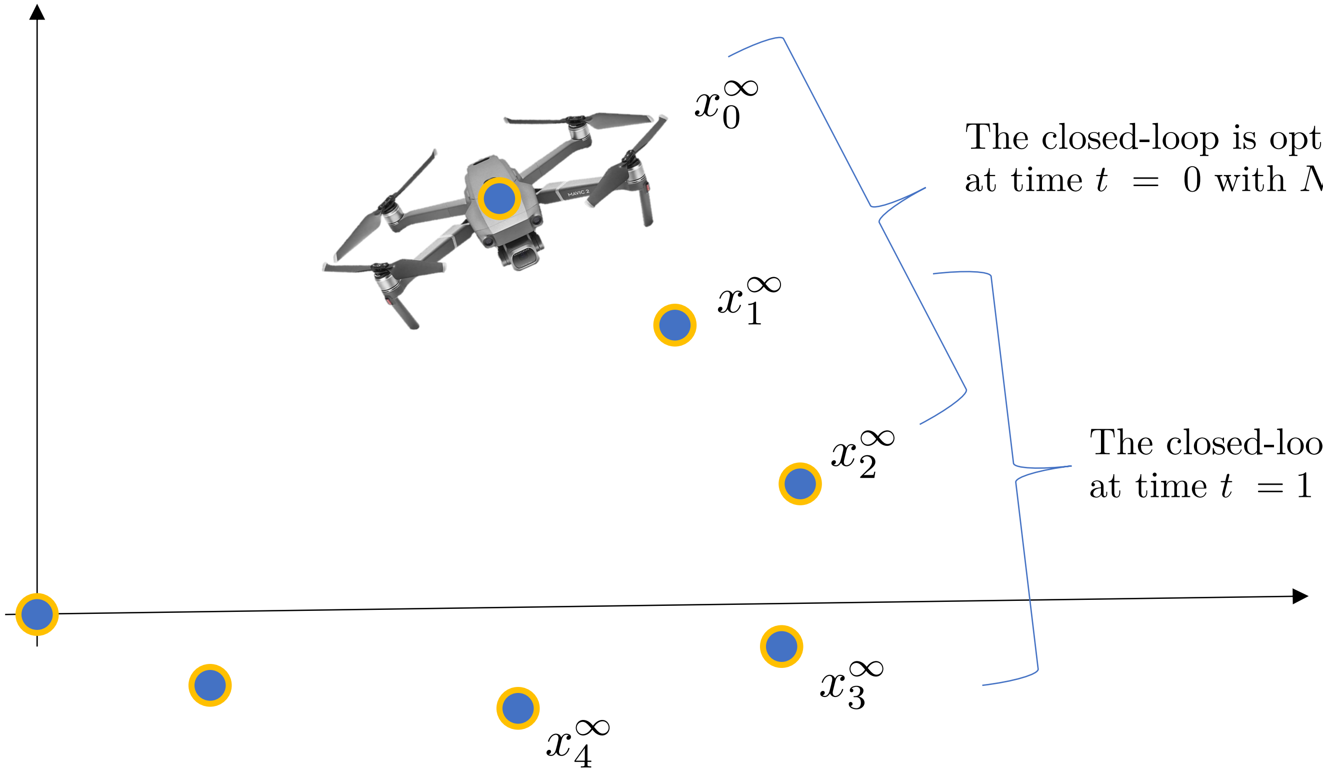


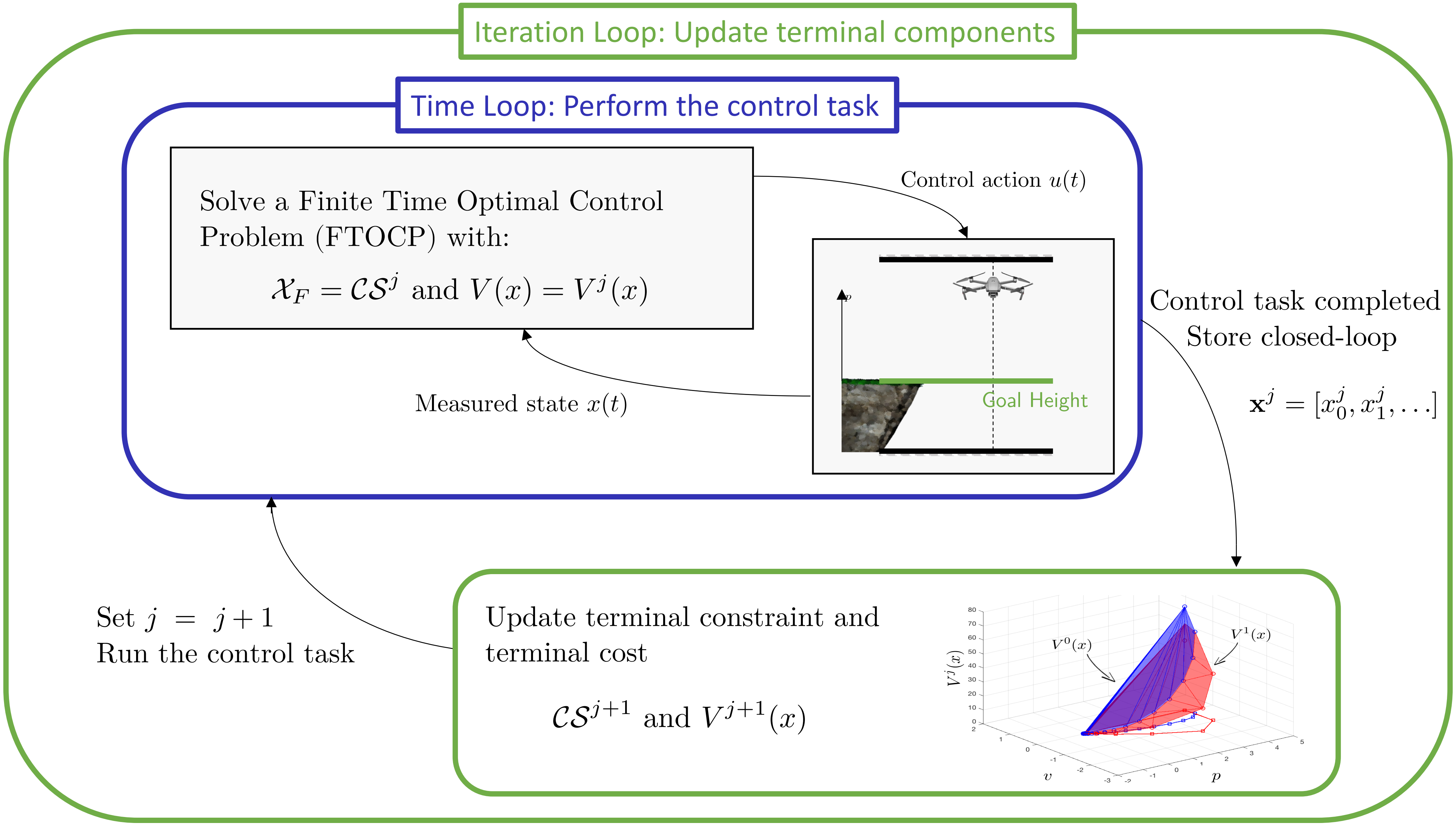
The closed-loop is optimal for LMPC at time $t = 0$ with $N = 2$

The closed-loop is optimal for LMPC at time $t = 1$ with $N = 2$

The Closed-loop is optimal for LMPC at time $t = 0$ with $N = 3$

Under some conditions associated the horizon length!

LMPC Recap #4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Linear(ized) LMPC

At time t of iteration j solve the following Constrained Finite Time Optimal Control Problem (FTOCP)

$$J_{t\to t+N}^{\mathrm{LMPC},j}(x_t^j) = \min_{u_{t|t},\dots,u_{t+N-1|t}} \sum_{k=t}^{t+N-1} h(x_{k|t}, u_{k|t}) + V^{j-1}(x_{t+N|t})$$

s.t.

$$x_{k+1|t} = Ax_{k|t} + Bu_{k|t},\ \forall k \in [t,\cdots,t+N-1]$$

$$x_{t|t} = x_t^j,$$

$$x_{k|t} \in \mathcal{X},\ u_{k|t} \in \mathcal{U},\ \forall k \in [t,\cdots,t+N-1]$$

$$x_{t+N|t} \in \mathcal{CS}^{j-1}$$

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$$\boxed{x_{t+N|t} = \sum_{i=0}^{j-1}\sum_{k} x^i_k \lambda^i_k,\ \ \sum_{i=0}^{j-1}\sum_{k} \lambda^i_k = 1,\ \lambda^i_k \geq 0.} \Longleftrightarrow \boxed{x_{t+N|t} \in \mathcal{CS}^{j-1}}$$

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$$\boxed{V^{j-1}(x_{t+N|t})}$$

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$$\Updownarrow$$

$$\boxed{V^{j-1}(x_{t+N|t})}$$

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- Convex optimization problem over inputs and lambdas
- Safety and performance improvement guarantees still hold (simple proofs as before)
- Converges to global optimal solution (LICQ required)

Learning MPC for Autonomous Racing

Real-time implementation on the Berkeley Autonomous Race Car (BARC)

Problem Formulation

Minimum Time Control Problem

$$\min_{T,\mathbf{u}} \quad \boxed{T}$$ Control objective

$$\boxed{x_0 = x_s, \;\; x_T = \mathcal{X}_F}$$ Start & end position

System dynamics
System constraints

$$\boxed{x_{k+1} = f(x_k, u_k), \qquad \forall k \in \{0, \ldots, T-1\}}$$

Safety constraints

$$\boxed{x_k \in \mathcal{X}, \;\; u_k \in \mathcal{U}, \qquad \forall k \in \{0, \ldots, T-1\}}$$

Learning Model Predictive Controller full-size vehicle experiments

Credits: Siddharth Nair, Nitin Kapania and Ugo Rosolia

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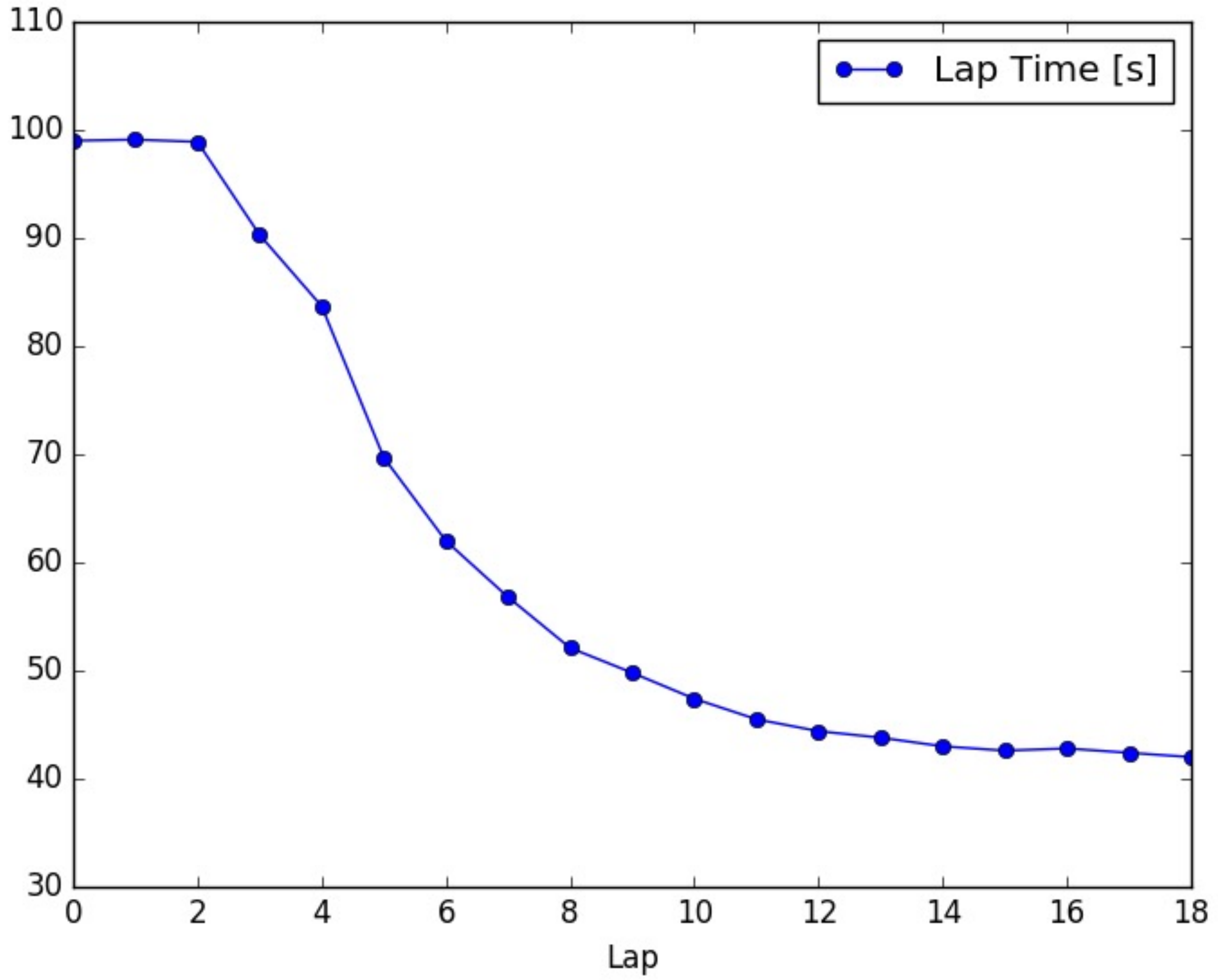
Lap Time



Learning Model Predictive Controller full-size vehicle experiments

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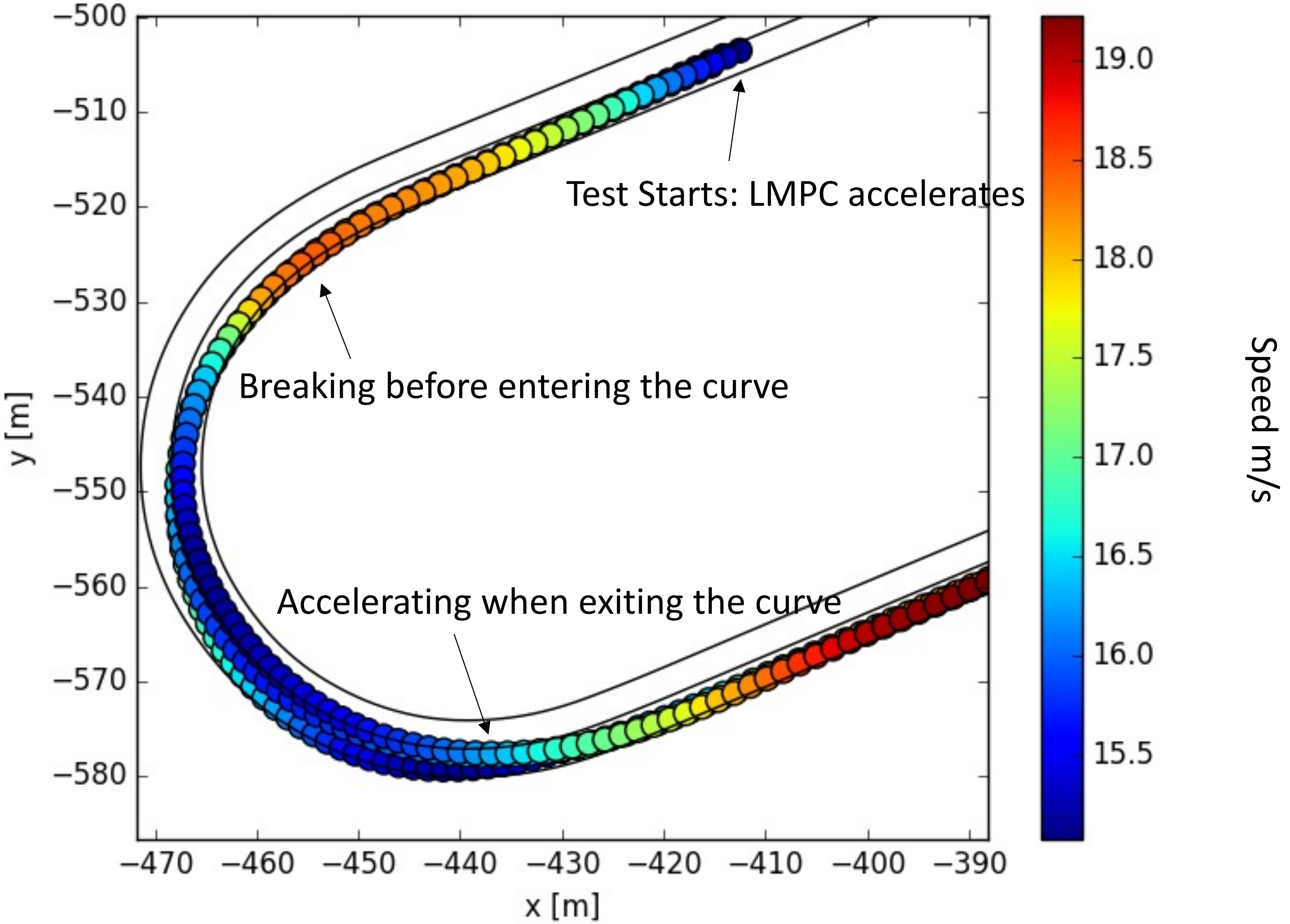
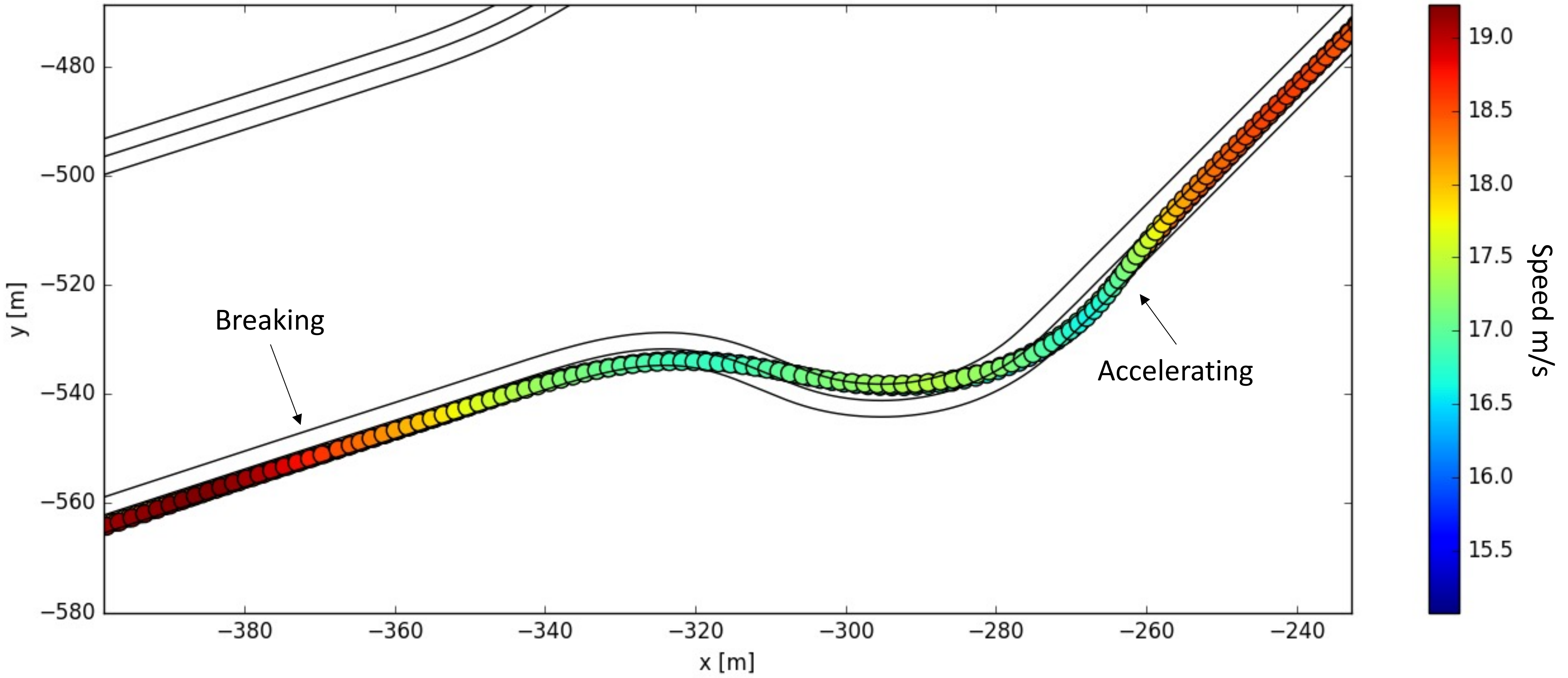
Velocity Profile at Convergence (Curve 1)



Velocity Profile at Convergence (Chicane)



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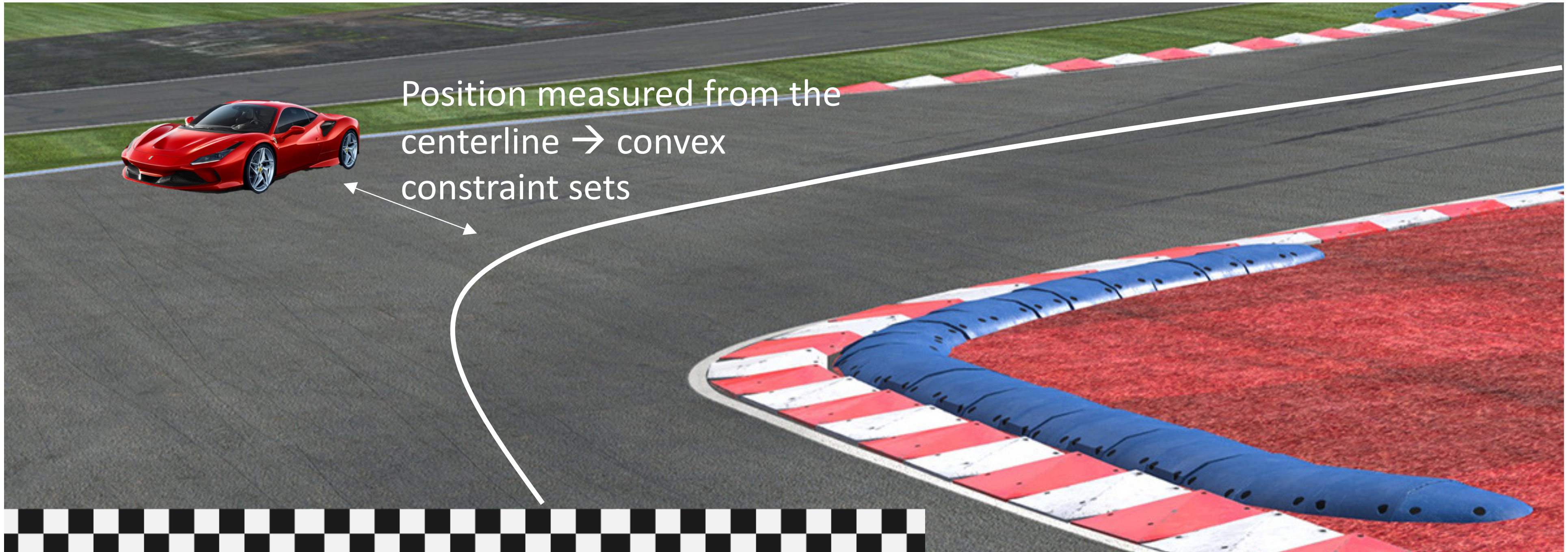
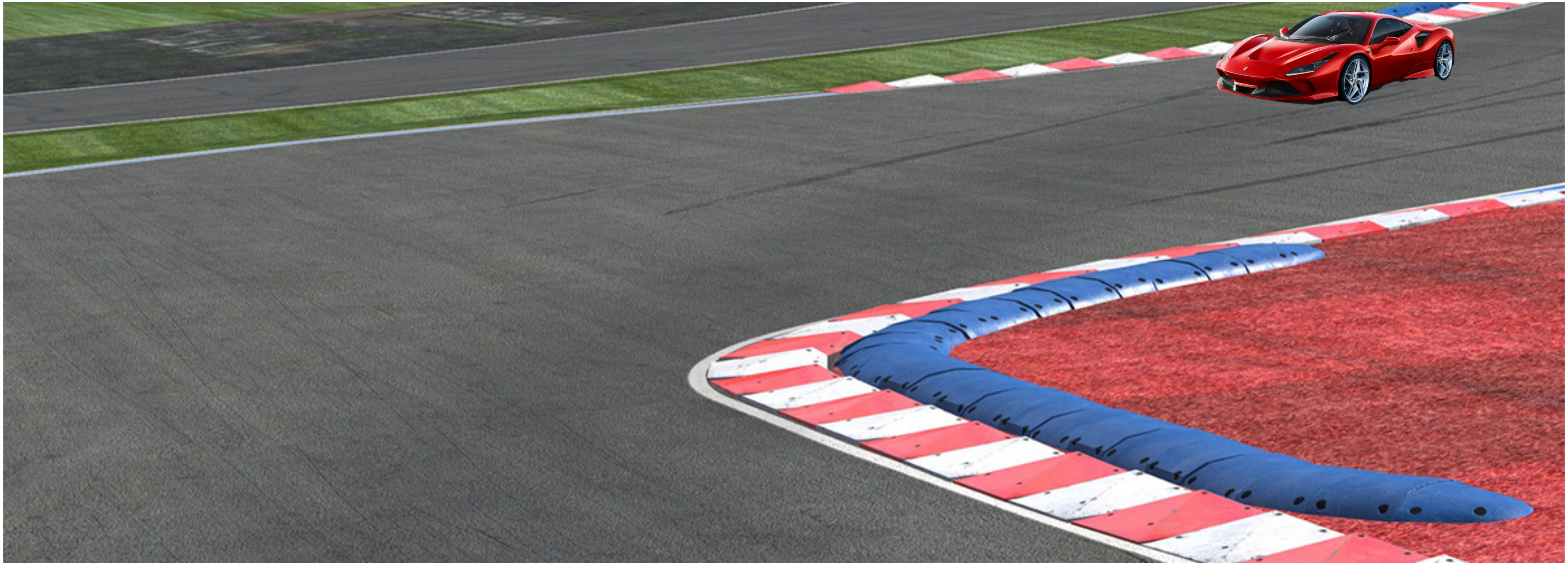


Local Approximations



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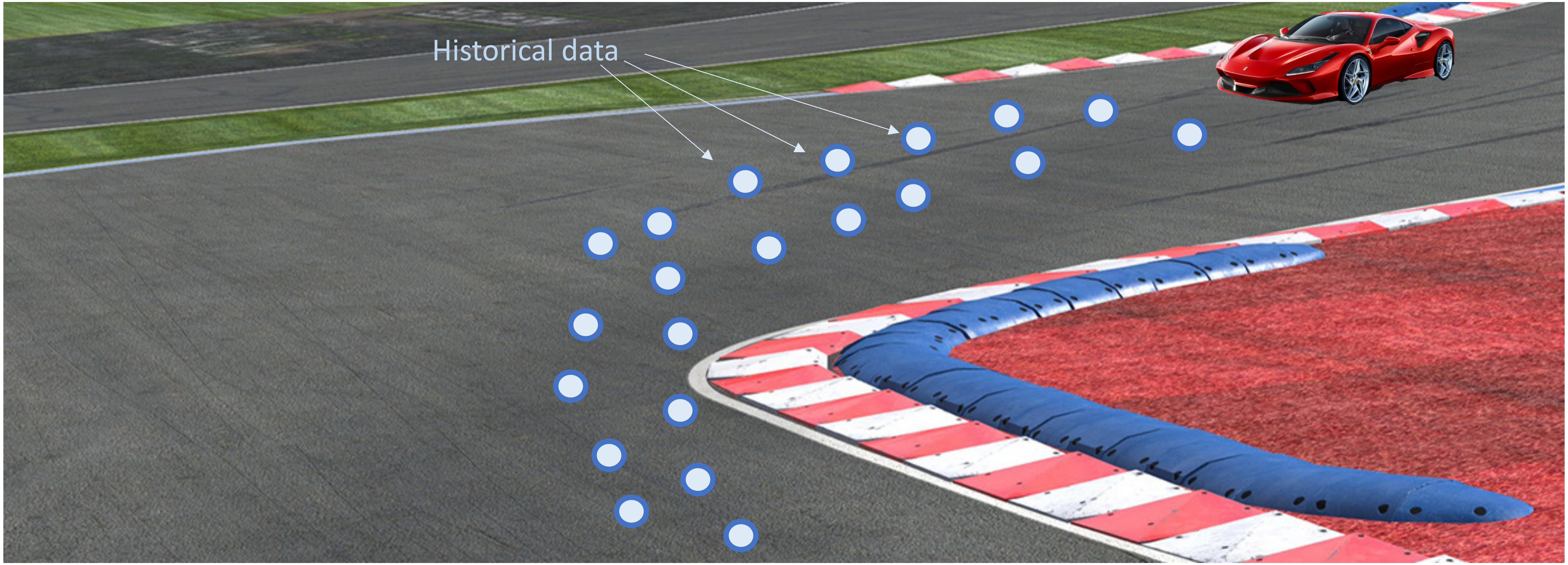

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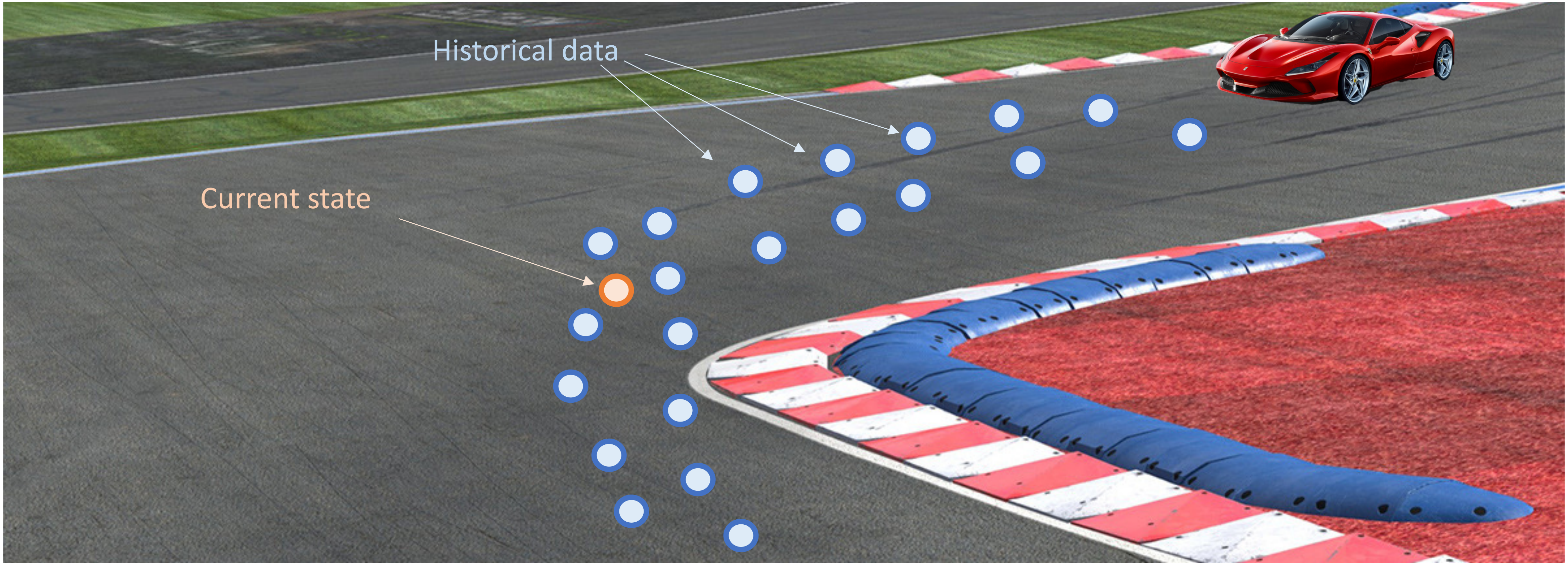

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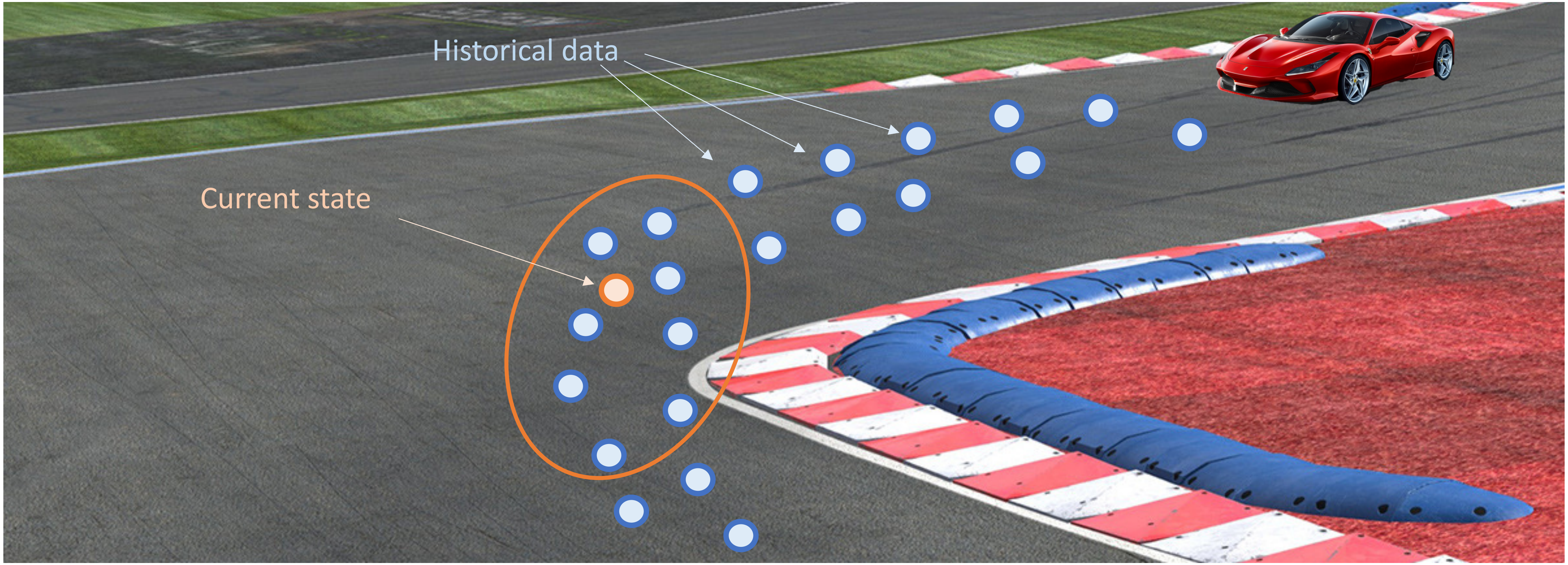

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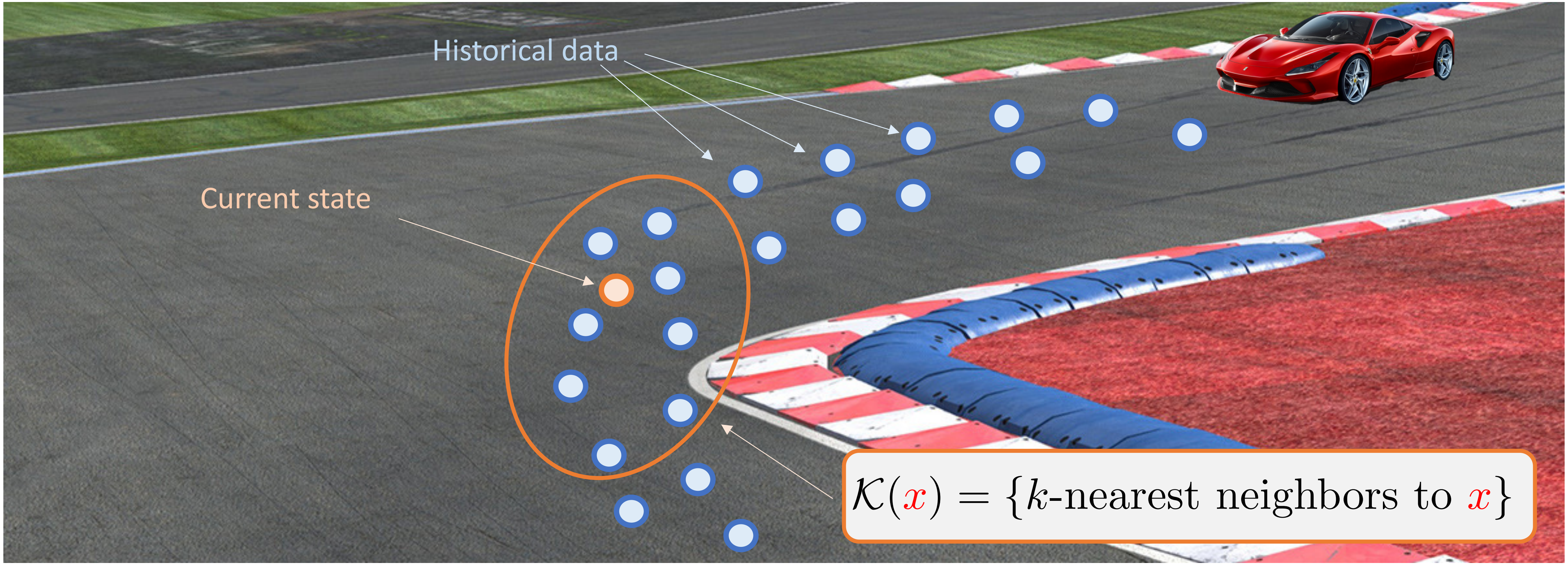

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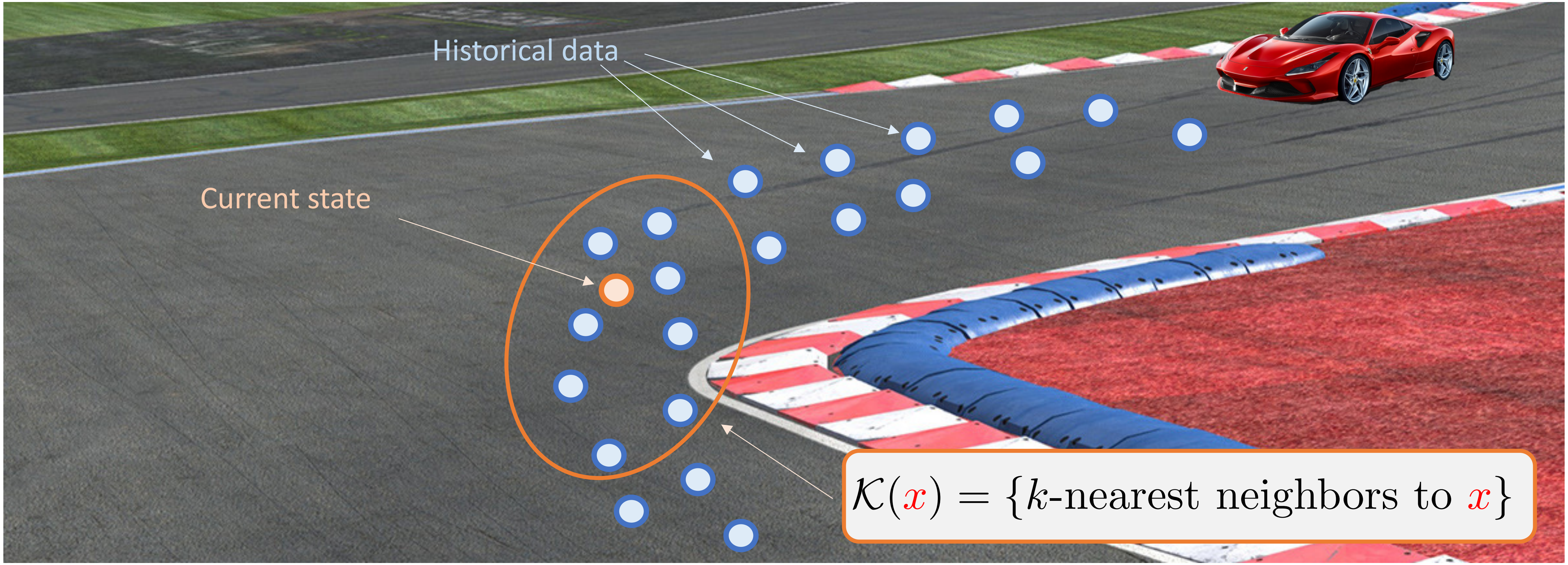
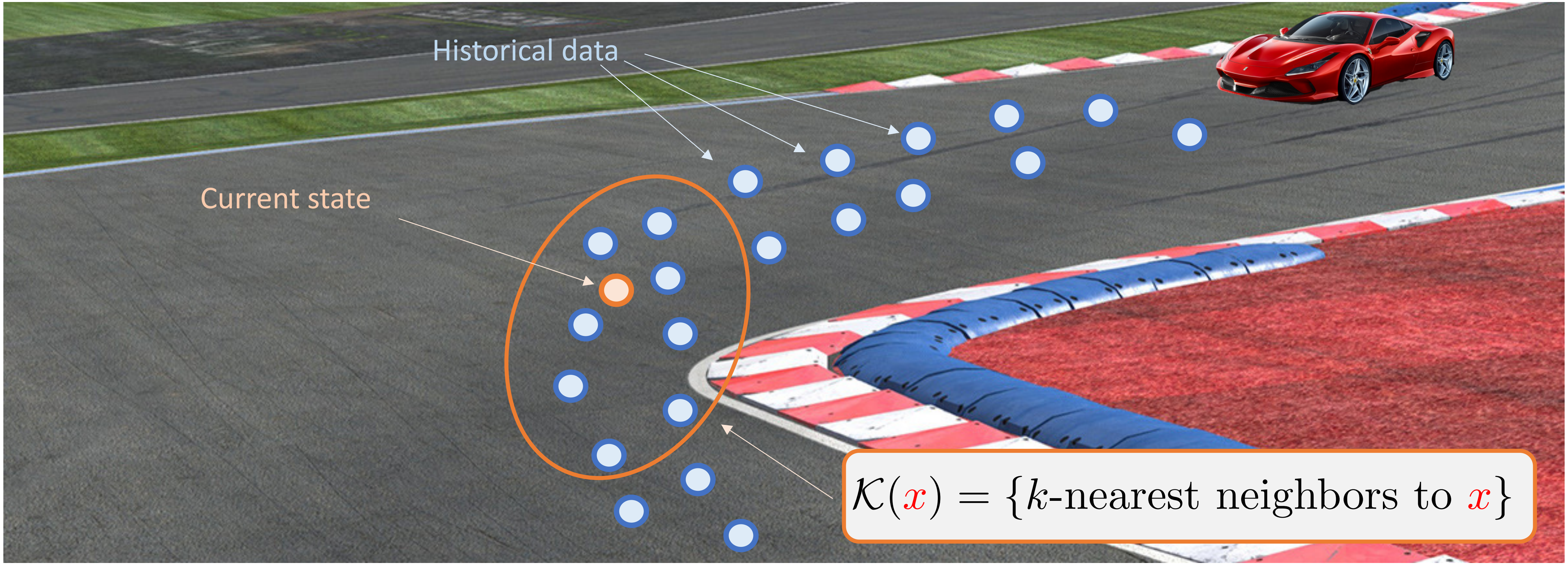


Local convex safe set approximation:

$$\mathcal{CS}^j(x) = \mathrm{conv}\left(\cup_{x_t^j \in \mathcal{K}(x)} x_t^j \right)$$

Local Approximations



Local value function approximation:

$$V^j(\bar{x}, x) = \min_{\lambda_t^i \geq 0} \quad \sum_{x_t^i \in \mathcal{K}^j(x)} J_t^i(x_t^i)\lambda_t^i$$

$$\text{subject to} \quad \sum_{x_t^i \in \mathcal{K}^j(x)} x_t^i \lambda_t^i = \bar{x}, \sum_i \sum_t \lambda_t^i = 1$$

Learning Model Predictive Controller

At time t of iteration j solve the following Constrained Finite Time Optimal Control Problem (CFTOCP)

$$J^{\mathrm{LMPC},j}_{t\rightarrow t+N}(x^j_t) = \min_{u^j_{t|t},\dots,u^j_{t+N-1|t}} \sum_{k=t}^{t+N-1} h(x^j_{k|t}, u^j_{k|t}) + V^{j-1}(x^j_{t+N|t}, x^+_{t+N})$$

s.t.

$$x^j_{k+1|t} = A^j_{k|t} x^j_{k|t} + B^j_{k|t} u^j_{k|t} + C^j_{k|t}$$

$$x^j_{t|t} = x^j_t,$$

$$x^j_{k|t} \in \mathcal{X},\ u^j_{k|t} \in \mathcal{U},\ \forall k \in [t, \cdots, t+N-1]$$

$$x^j_{t+N|t} \in \ \mathcal{CS}^{j-1}(x^+_{t+N}),$$

Learning Model Predictive Controller

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$$x^j_{t+N|t} \in \boxed{\mathcal{CS}^{j-1}(x^+_{t+N})},$$

Computed online
using LMPC strategy

where $x^+_{t+N} = x^*_{t+N-1|t-1}$

Learning Model Predictive Controller

At time t of iteration j solve the following Constrained Finite Time Optimal Control Problem (CFTOCP)

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Computed online
using LMPC strategy

where $x^+_{t+N} = x^*_{t+N-1|t-1}$ or $x^+_{t+N} = \hat{f}(x^*_{t+N-1|t-1}, u_f)$

Learning Model Predictive Controller

At time t of iteration j solve the following Constrained Finite Time Optimal Control Problem (CFTOCP)

$$J^{\mathrm{LMPC},j}_{t\rightarrow t+N}(x^j_t) = \min_{u^j_{t|t},\ldots,u^j_{t+N-1|t}} \sum_{k=t}^{t+N-1} h(x^j_{k|t}, u^j_{k|t}) + \boxed{V^{j-1}(x^j_{t+N|t}, x^+_{t+N})}$$

s.t.

$$\boxed{x^j_{k+1|t} = A^j_{k|t} x^j_{k|t} + B^j_{k|t} u^j_{k|t} + C^j_{k|t}}$$

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$$x^j_{k|t} \in \mathcal{X},\ u^j_{k|t} \in \mathcal{U},\ \forall k \in [t, \cdots, t+N-1]$$

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Computed online

Computed online using LMPC strategy

Learning Model Predictive Controller

At time t of iteration j solve the following Constrained Finite Time Optimal Control Problem (CFTOCP)

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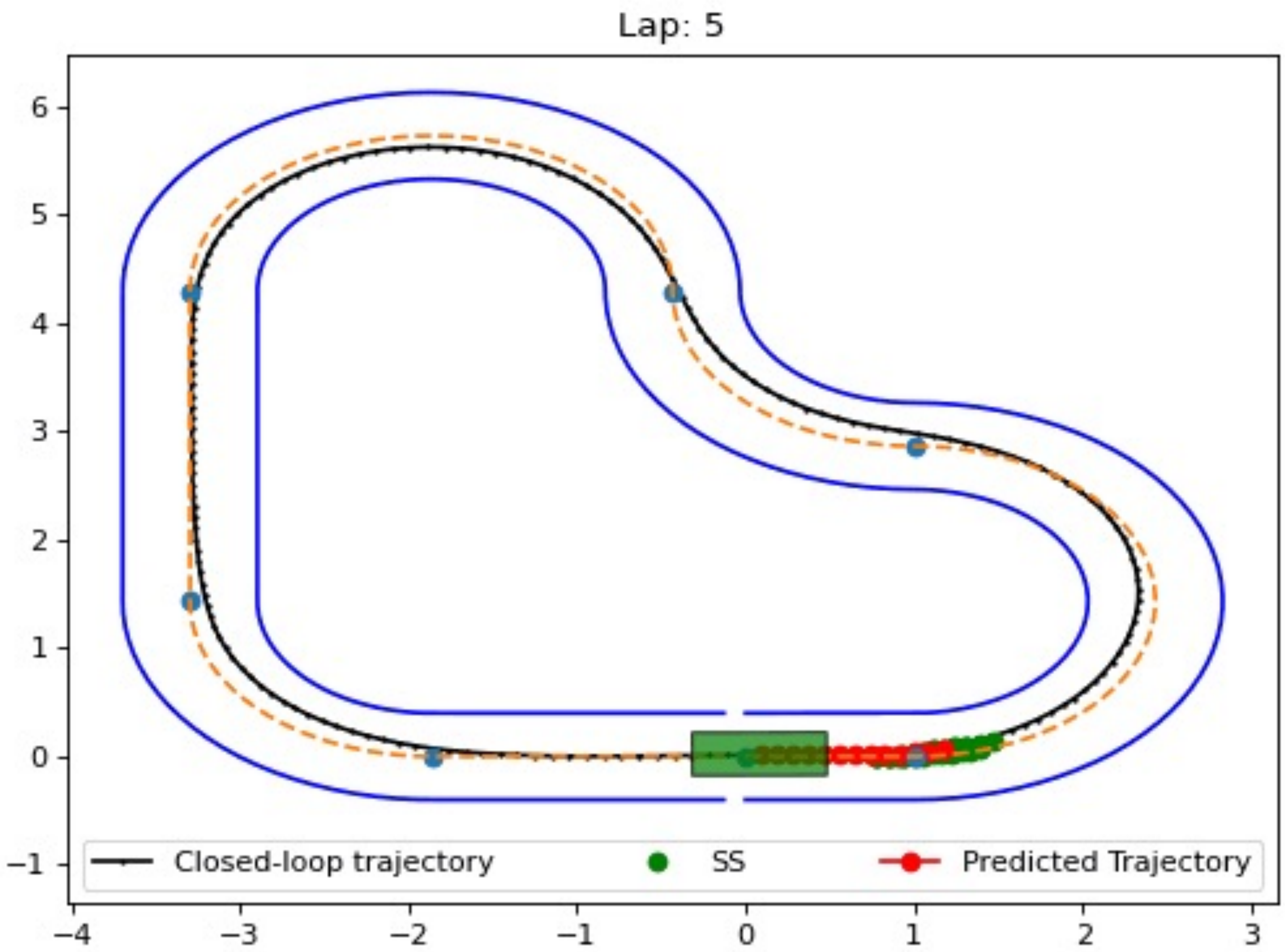
$$x^j_{t+N|t} \in \boxed{\mathcal{CS}^{j-1}(x^+_{t+N})},$$

Computed online

Computed online using LMPC strategy

Then apply to the system the control input $u^j_t = u^{*,j}_{t|t}$

Safe Set and Value Function Approximations



System ID in Autonomous Racing

- Nonlinear Dynamical System,

$$
\begin{aligned}
\ddot{x} &= \dot{y}\dot{\psi} + \tfrac{1}{m}\textstyle\sum_i F_{x_i} \\
\ddot{y} &= -\dot{x}\dot{\psi} + \tfrac{1}{m}\textstyle\sum_i F_{y_i} \\
\ddot{\psi} &= \tfrac{1}{I_z}(a(F_{y_{1,2}}) - b(F_{y_{2,3}}) + c(-F_{x_{1,3}} + F_{x_{2,4}}) \\
\dot{X} &= \dot{x}\cos\psi - \dot{y}\sin\psi, \quad \dot{Y} = \dot{x}\sin\psi + \dot{y}\cos\psi
\end{aligned}
$$

System ID in Autonomous Racing

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\begin{aligned}
\ddot{x} &= \dot{y}\dot{\psi} + \frac{1}{m}\sum_i F_{x_i} \\
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\end{aligned}
$$

Kinematic Equations

System ID in Autonomous Racing

- Nonlinear Dynamical System,

$$
\begin{aligned}
\ddot{x} &= \dot{y}\dot{\psi} + \frac{1}{m}\sum_i F_{x_i} \\
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&\boxed{\dot{X} = \dot{x}\cos\psi - \dot{y}\sin\psi, \quad \dot{Y} = \dot{x}\sin\psi + \dot{y}\cos\psi}
\end{aligned}
$$

Kinematic Equations

- Identifying the Dynamical System

$$
x^j_{k+1|t} = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} x^j_{k|t} + \begin{bmatrix} \\ \\ \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} \begin{bmatrix} u^j_{k|t} \\ 1 \end{bmatrix}
$$

Linearization around predicted trajectory

System ID in Autonomous Racing

- Nonlinear Dynamical System,

$$\boxed{\begin{aligned}\ddot{x} &= \dot{y}\dot{\psi} + \frac{1}{m}\textstyle\sum_i F_{x_i} \\ \ddot{y} &= -\dot{x}\dot{\psi} + \frac{1}{m}\textstyle\sum_i F_{y_i} \\ \ddot{\psi} &= \frac{1}{I_z}(a(F_{y_{1,2}}) - b(F_{y_{2,3}}) + c(-F_{x_{1,3}} + F_{x_{2,4}})\end{aligned}}$$

Dynamic Equations

$$\boxed{\dot{X} = \dot{x}\cos\psi - \dot{y}\sin\psi, \quad \dot{Y} = \dot{x}\sin\psi + \dot{y}\cos\psi}$$

Kinematic Equations

- Identifying the Dynamical System

$$x^j_{k+1|t} = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} x^j_{k|t} + \begin{bmatrix} \\ \\ \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{bmatrix} \begin{bmatrix} u^j_{k|t} \\ 1 \end{bmatrix}$$

Linearization around predicted trajectory

System ID in Autonomous Racing

- Nonlinear Dynamical System,

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Dynamic Equations

$$\boxed{\dot{X} = \dot{x}\cos\psi - \dot{y}\sin\psi, \quad \dot{Y} = \dot{x}\sin\psi + \dot{y}\cos\psi}$$

Kinematic Equations

- Identifying the Dynamical System

Local Linear Regression

$$x^j_{k+1|t} = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \begin{bmatrix} \multicolumn{3}{c}{\boxed{\arg\min \sum_{i,s} K(x^j_{k|t} - x^i_s)||\Lambda_y \begin{bmatrix} x^j_{k|t} \\ u^j_{k|t} \\ 1 \end{bmatrix} - y^i_{s+1}||, \forall y \in \{\dot{x}, \dot{y}, \ddot{\psi}\}}} \\ \boxed{\begin{matrix} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{matrix}} & x^j_{k|t} + & \boxed{\begin{matrix} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{matrix}} \end{bmatrix} \begin{bmatrix} u^j_{k|t} \\ 1 \end{bmatrix}$$

Linearization around predicted trajectory

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

$$x^j_{k+1|t} = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \left[\begin{array}{c} \boxed{\arg\min \sum_{i,s} K(x^j_{k|t} - x^i_s) || \Lambda_y \begin{bmatrix} x^j_{k|t} \\ u^j_{k|t} \\ 1 \end{bmatrix} - y^i_{s+1} ||, \forall y \in \{\dot{x}, \dot{y}, \ddot{\psi}\}} \\ \begin{array}{ccc} \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} & x^j_{k|t} + & \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} \end{array} \end{array} \right] \begin{bmatrix} u^j_{k|t} \\ 1 \end{bmatrix}$$

Linearization around predicted trajectory

Implementation Details

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

$$x^j_{k+1|t} = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \left[\begin{array}{c} \boxed{\arg\min \sum_{i,s} K(x^j_{k|t} - x^i_s) ||\Lambda_y \begin{bmatrix} x^j_{k|t} \\ u^j_{k|t} \\ 1 \end{bmatrix} - y^i_{s+1}||, \forall y \in \{\dot{x}, \dot{y}, \ddot{\psi}\}} \\ \begin{bmatrix} \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} \end{bmatrix} x^j_{k|t} + \begin{bmatrix} \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} \end{bmatrix} \end{array} \right] \begin{bmatrix} u^j_{k|t} \\ 1 \end{bmatrix}$$

Linearization around predicted trajectory

Implementation Details

- Compute trajectory to linearize around from previous optimal inputs and stored data

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

$$x^j_{k+1|t} = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \left[\begin{array}{c} \boxed{\arg\min \sum_{i,s} K(x^j_{k|t} - x^i_s)||\Lambda_y \begin{bmatrix} x^j_{k|t} \\ u^j_{k|t} \\ 1 \end{bmatrix} - y^i_{s+1}||, \forall y \in \{\dot{x}, \dot{y}, \ddot{\psi}\}} \\ \begin{bmatrix} \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} \end{bmatrix} x^j_{k|t} + \begin{bmatrix} \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} \end{bmatrix} \end{array} \right] \begin{bmatrix} u^j_{k|t} \\ 1 \end{bmatrix}$$

Linearization around predicted trajectory

Implementation Details

- Compute trajectory to linearize around from previous optimal inputs and stored data
- Enforce model-based sparsity in local linear regression

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

$$x^j_{k+1|t} = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \left[\begin{array}{c} \boxed{\arg\min \sum_{i,s} K(x^j_{k|t} - x^i_s) || \Lambda_y \begin{bmatrix} x^j_{k|t} \\ u^j_{k|t} \\ 1 \end{bmatrix} - y^i_{s+1} ||, \forall y \in \{\dot{x}, \dot{y}, \dot{\psi}\}} \\ \begin{bmatrix} \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} \end{bmatrix} x^j_{k|t} + \begin{bmatrix} \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} \end{bmatrix} \end{array} \right] \begin{bmatrix} u^j_{k|t} \\ 1 \end{bmatrix}$$

Linearization around predicted trajectory

Implementation Details

- Compute trajectory to linearize around from previous optimal inputs and stored data
- Enforce model-based sparsity in local linear regression
- Perform local linear regression at the linearization points using a subset of the stored data

System Identification – Design Steps

Identifying the Dynamical System

Local Linear Regression

$$x^j_{k+1|t} = \begin{bmatrix} \dot{x}_{k+1|t} \\ \dot{y}_{k+1|t} \\ \ddot{\psi} \\ \psi_{k+1|t} \\ X_{k+1|t} \\ Y_{k+1|t} \end{bmatrix} = \left[\begin{array}{c} \boxed{\arg\min \sum_{i,s} K(x^j_{k|t} - x^i_s)||\Lambda_y \begin{bmatrix} x^j_{k|t} \\ u^j_{k|t} \\ 1 \end{bmatrix} - y^i_{s+1}||, \forall y \in \{\dot{x}, \dot{y}, \ddot{\psi}\}} \\ \begin{array}{ccc} \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} & x^j_{k|t} + & \boxed{\begin{array}{c} \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \\ \text{Linearized Kinematics} \end{array}} \end{array} \end{array} \right] \begin{bmatrix} u^j_{k|t} \\ 1 \end{bmatrix}$$

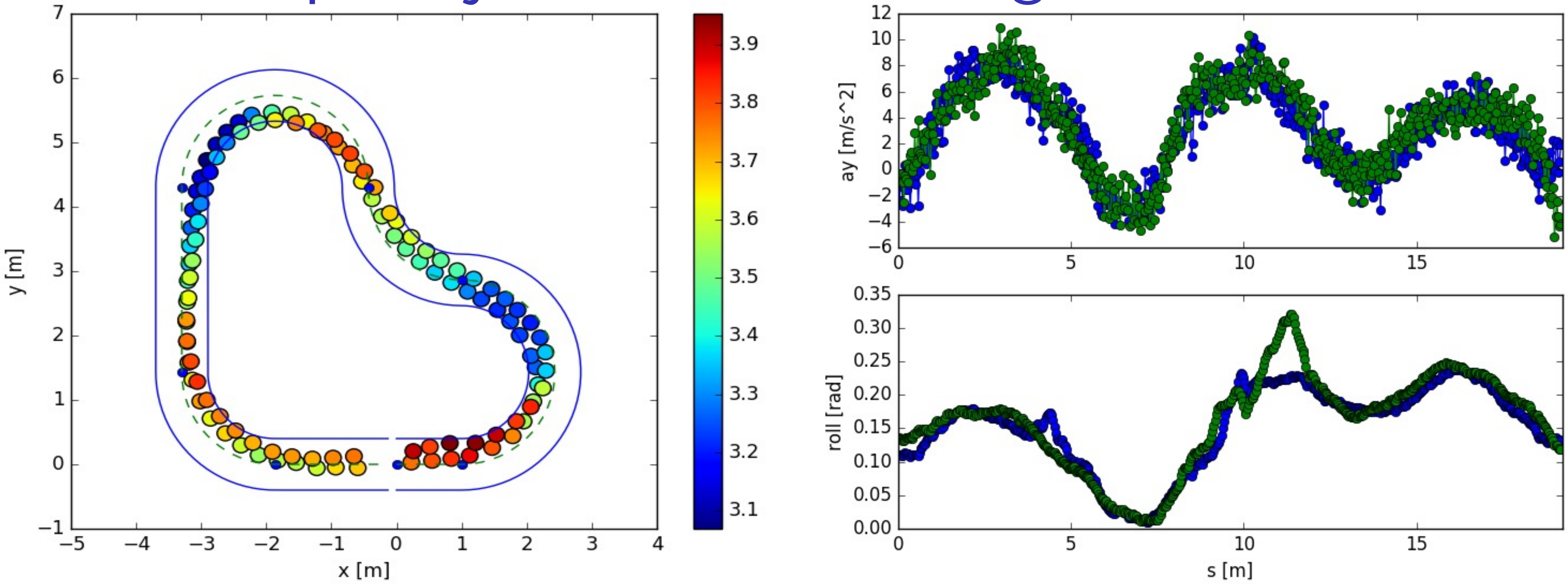
Linearization around predicted trajectory

Implementation Details

- Compute trajectory to linearize around from previous optimal inputs and stored data
- Enforce model-based sparsity in local linear regression
- Perform local linear regression at the linearization points using a subset of the stored data
- Use kernel *K()* to weight differently data as a function of distance to the linearization trajectory

Learning Model Predictictive Control for Autonomous Racing

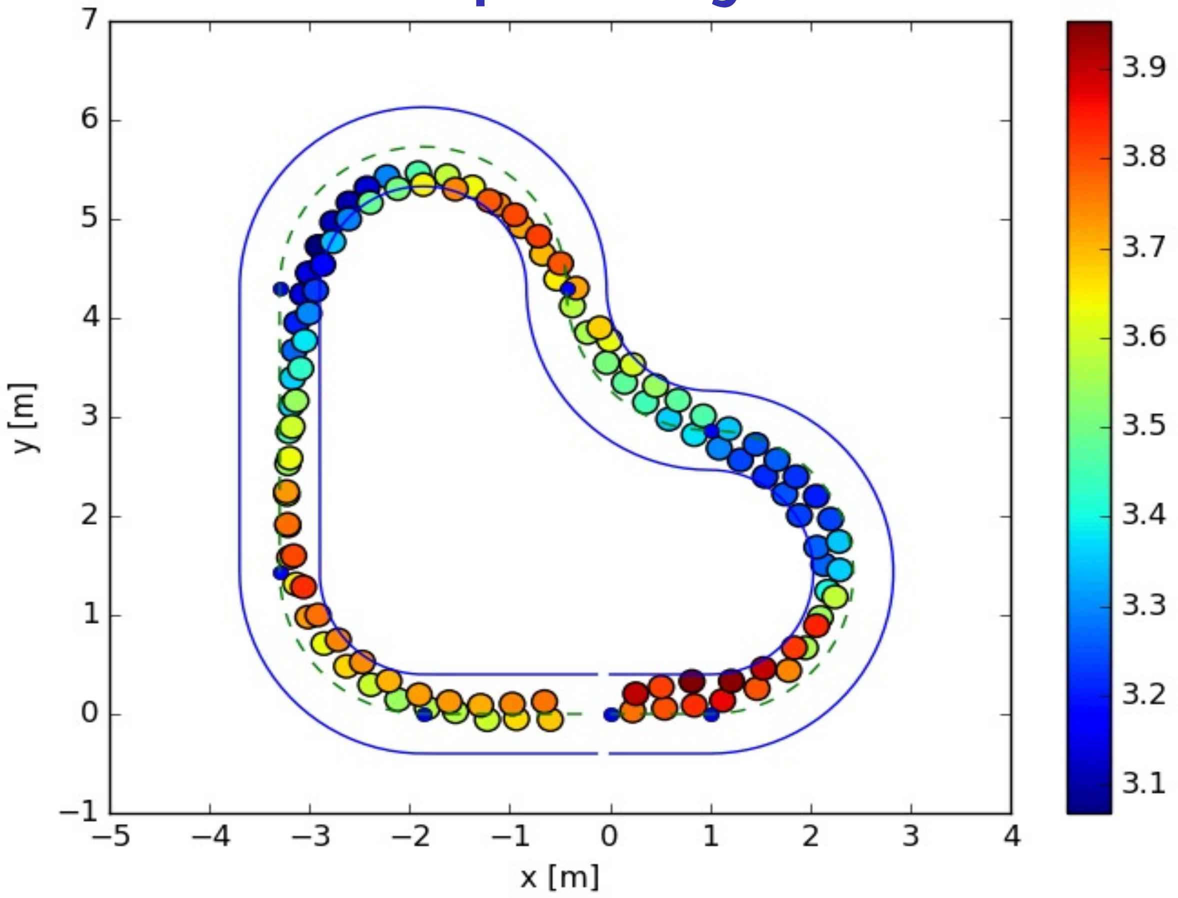
Closed-loop Trajectories at Convergence

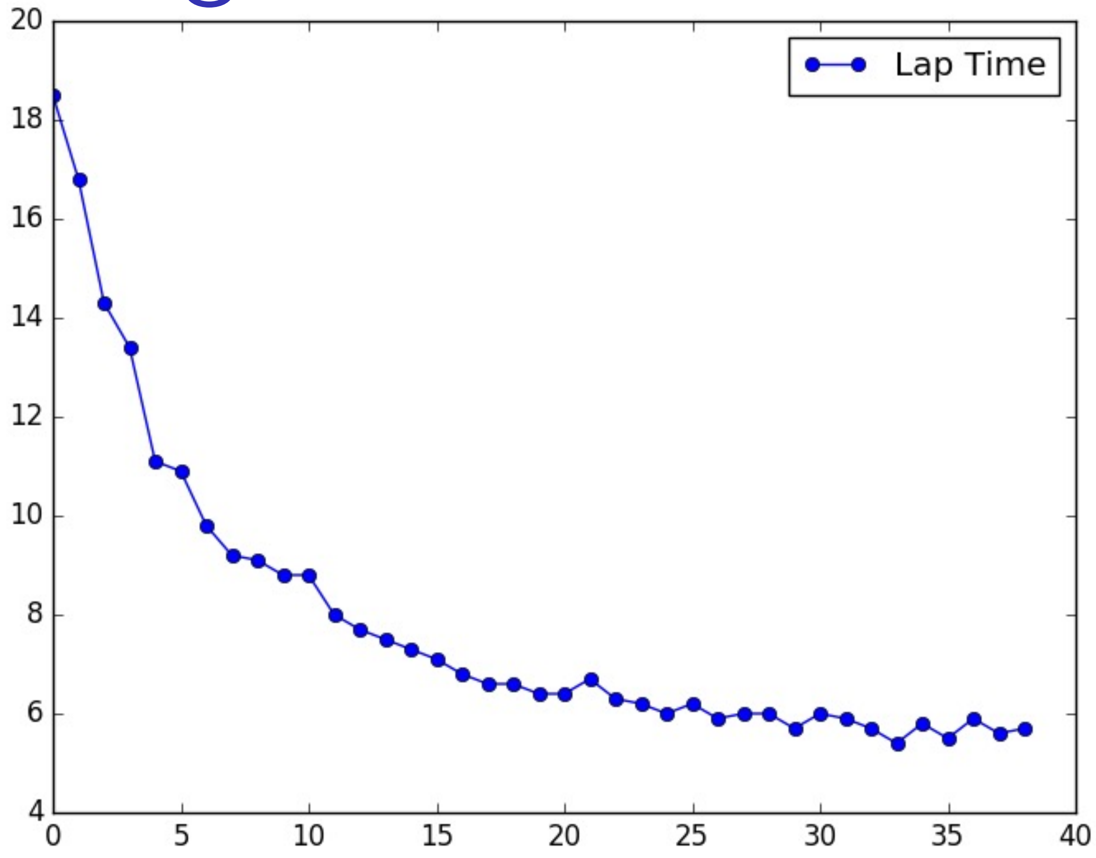


Remarks

- A **reference trajectory is not needed** for the controller implementation
- The controller converges after few laps: the learning process is data efficient
- The **controller safely explores the state space** iteratively improving the lap time

Closed-loop Trajectories at Convergence

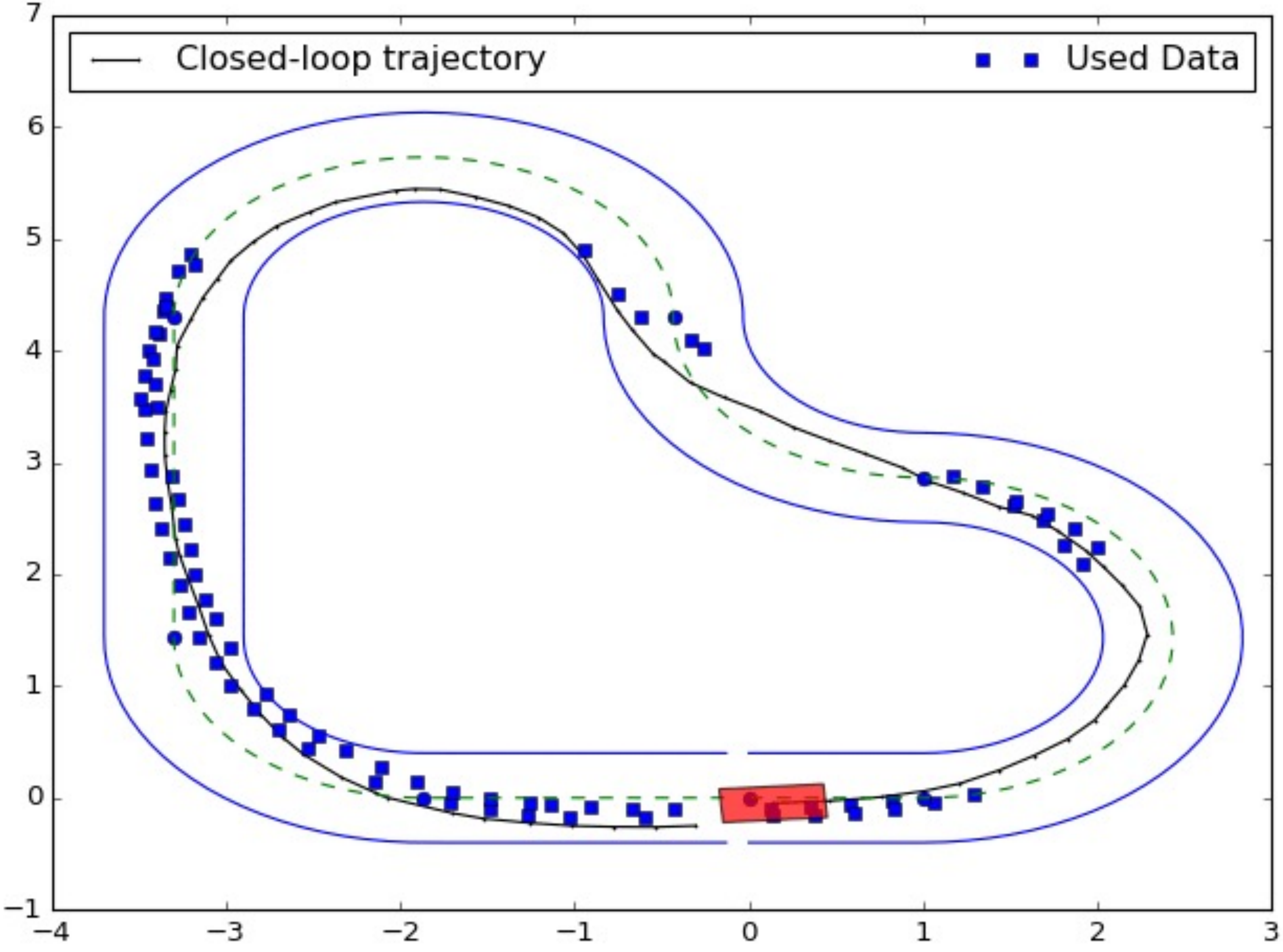




Remarks

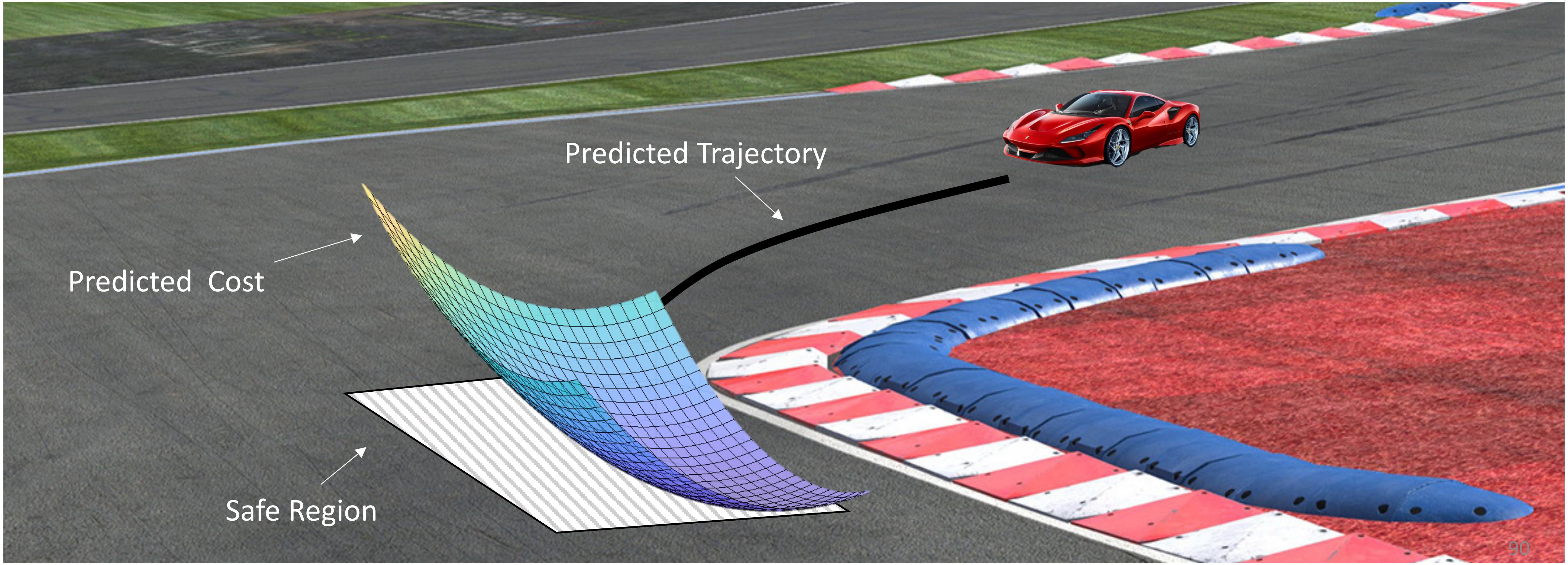
- A **reference trajectory is not needed** for the controller implementation
- The controller converges after few laps: the learning process is data efficient
- The **controller safely explores the state space** iteratively improving the lap time

Data Point Selection



The key components

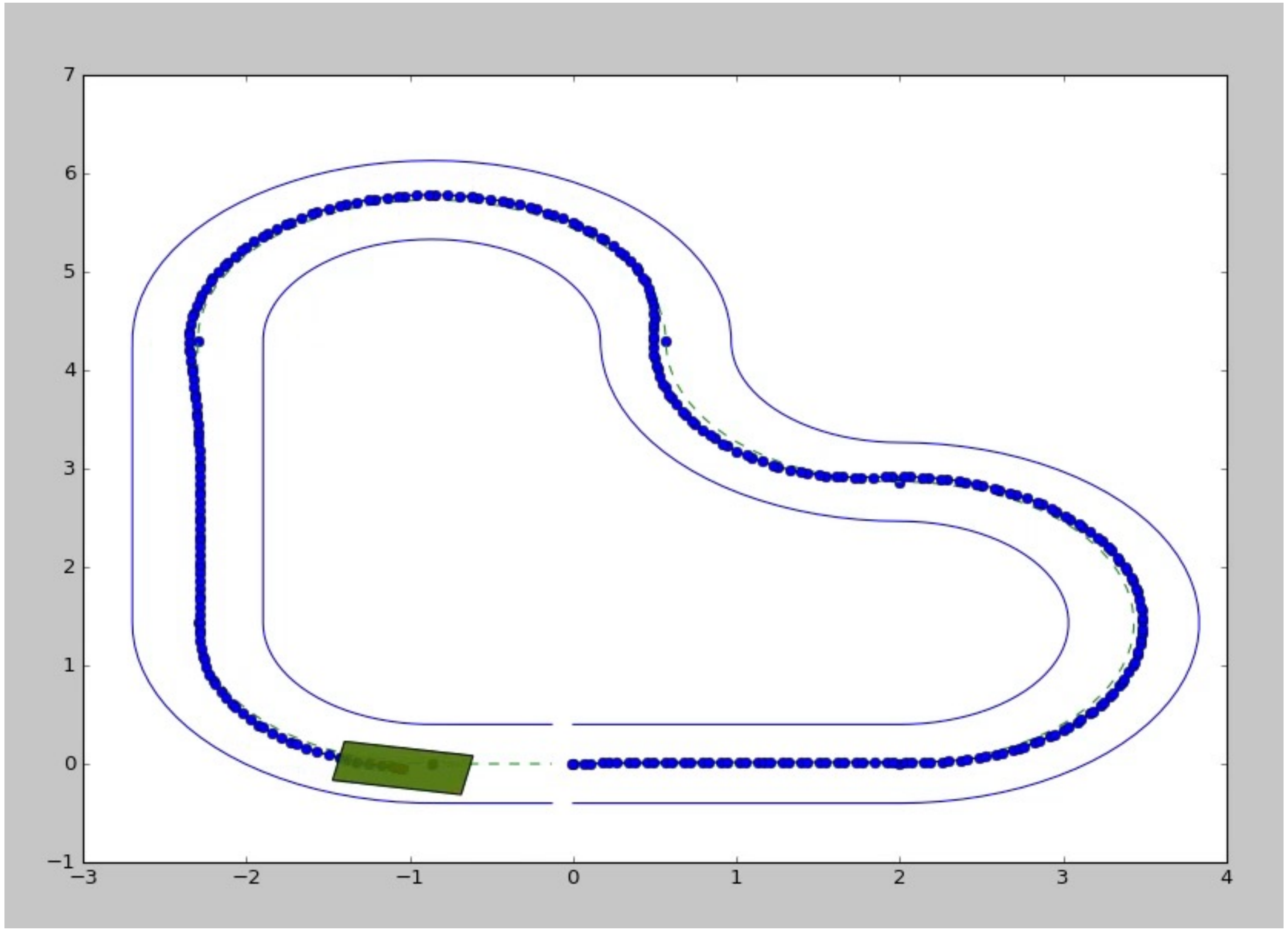
- Predicted trajectory given by prediction model
- Predicted cost estimated by value function approximation
- Safe region estimated by the safe set



Do you need the safe set? – Yes

LMPC without Invariant Set

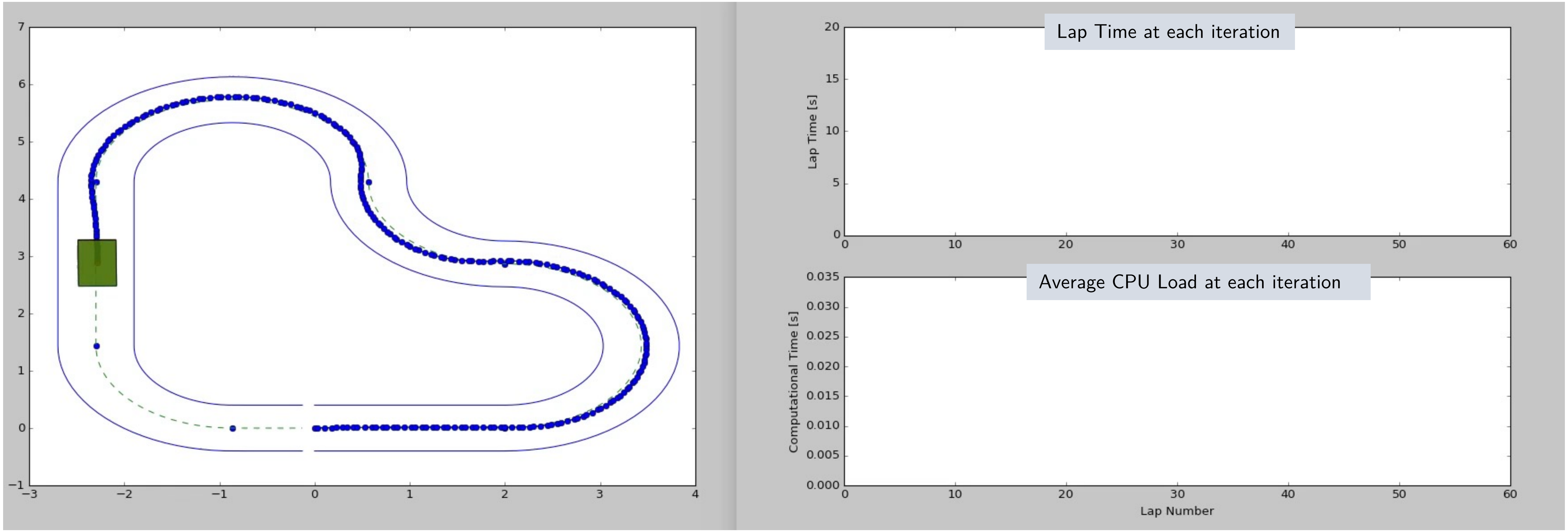
The controller extrapolates the value function on the Vx dimension



Do you need to Predict to Learn? Yes

When the LMPC horizon is N = 1 the controller

- solves the Bellman equation using the value function approximation
- does not explore the state space as it cannot plan outside the safe set



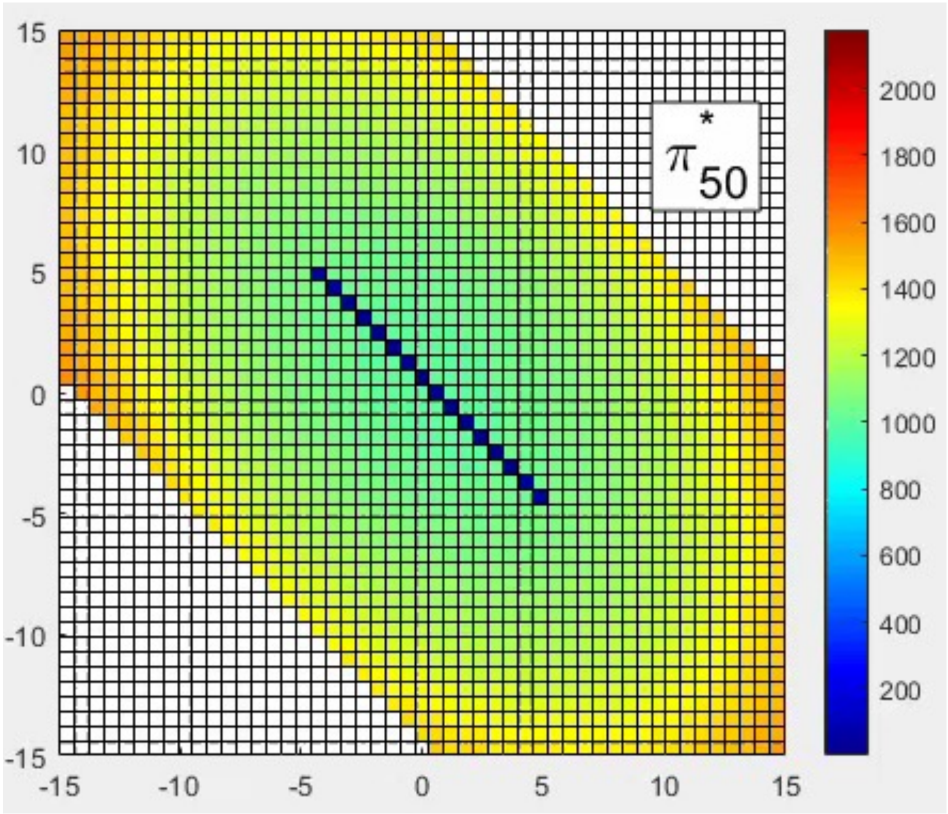
Comparison with Approximate DP (aka RL)

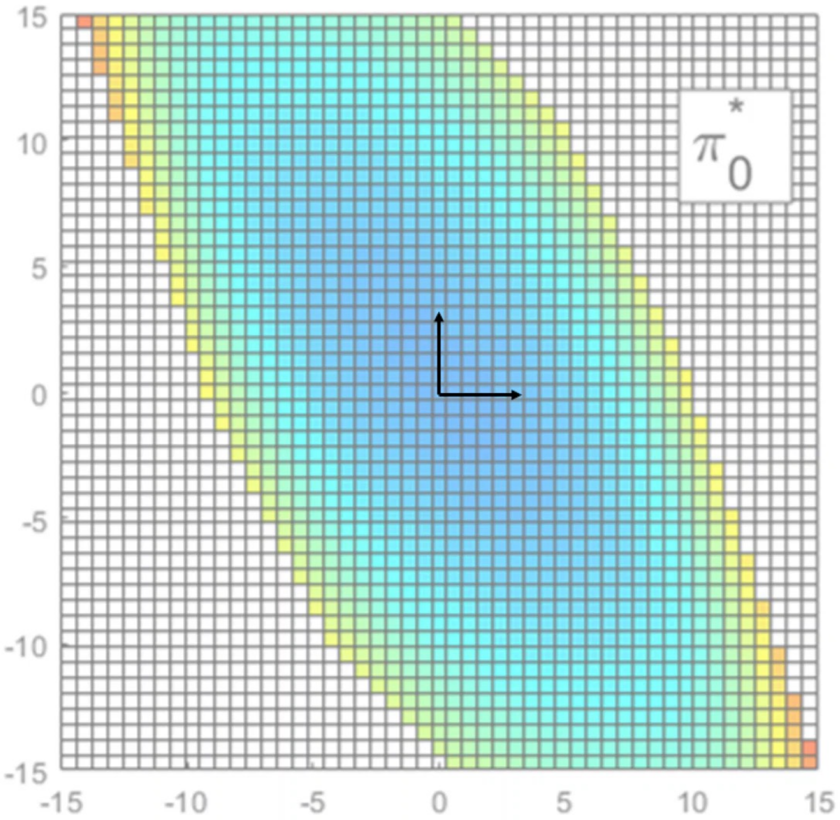
- Some references:
  - Bertsekas paper connecting MPC and ADP [1], books on RL and OC [2,3]
  - Lewis and Vrabie survey [4]
  - Recht survey [5]

- LMPC highlights
  - Continuous state and action formulation
  - Constraints satisfaction and Sampled Safe Sets
  - Q-function constructed locally based on cost/model driven exploration
  - Q-function at stored state is “exact” and upperbounds at intermediate points

[1] D. Bertsekas, “Dynamic programming and suboptimal control: A survey from ADP to MPC.” European Journal of Control 11.4-5 (2005)
[2] D. Bertsekas, “Reinforcement learning and optimal control.” Athena Scientific, 2019.
[3] D. Bertsekas, “Distributed Reinforcement Learning” http://web.mit.edu/dimitrib/www/RL_2_Rollout_&_PI.pdf
[4] F. Lewis, Frank, and D. Vrabie. "Reinforcement learning and adaptive dynamic programming for feedback control." IEEE circuits and systems magazine 9.3 (2009)
[5] R. Benjamin. "A tour of reinforcement learning: The view from continuous control." Annual Review of Control, Robotics, and Autonomous Systems 2 (2019)

Forward Value Iteration





Dynamic Programming:

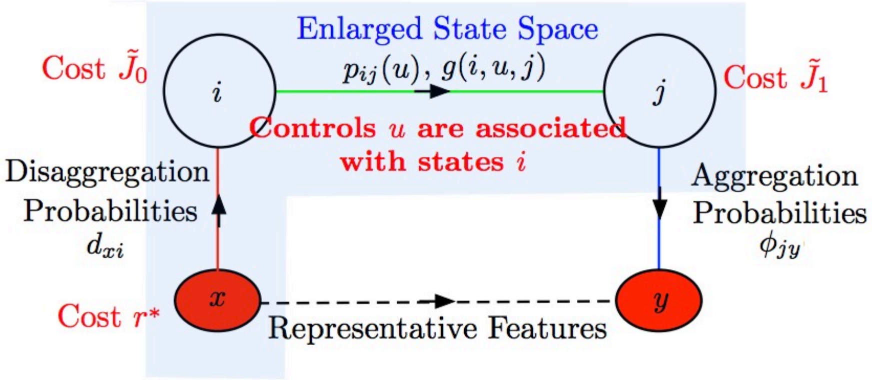
- Gridding, global properties
- Backward, one-step iteration

LMPC:

- No Gridding, local properties
- Forward, multi-step prediction
- LICQ required for optimality

LMPC as Efficient and Safe Aggregated ADP

More Accurate Version: The Enlarged Aggregate Problem



Bellman equations for the enlarged problem

$$r_x^* = \sum_{i=1}^{n} d_{xi}\tilde{J}_0(i), \qquad x \in \mathcal{A},$$

$$\tilde{J}_0(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha \tilde{J}_1(j)\big), \qquad i = 1,\dots,n,$$

$$\tilde{J}_1(j) = \sum_{y \in \mathcal{A}} \phi_{jy} r_y^*, \qquad j = 1,\dots,n$$

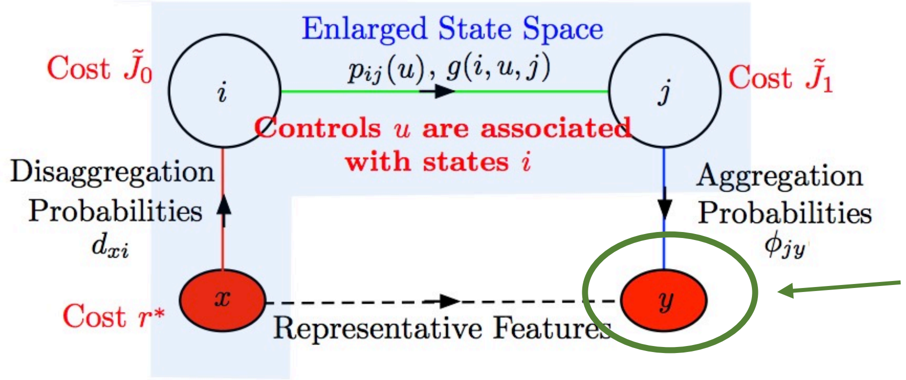
r^* solves uniquely the composite Bellman equation $r^* = Hr^*$:

$$r_x^* = (Hr^*)(x) = \sum_{i=1}^{n} d_{xi} \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\left(g(i,u,j) + \alpha \sum_{y \in \mathcal{A}} \phi_{jy}\, r_y^*\right), \qquad x \in \mathcal{A}$$

Slide from Prof. D. Bertsekas class on aggregation, available at http://web.mit.edu/dimitrib/www/Slides_Lecture12_RLOC.pdf
Details on aggregation in Chapter 6 of “Reinforcement Learning and Optimal Control” by Prof. D. Bertsekas

LMPC as Efficient and Safe Aggregated ADP

More Accurate Version: The Enlarged Aggregate Problem



1 - Let MPC explore and iteratively construct the set of aggregate states

Bellman equations for the enlarged problem

$$r_x^* = \sum_{i=1}^{n} d_{xi} \tilde{J}_0(i), \qquad x \in \mathcal{A},$$

$$\tilde{J}_0(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha \tilde{J}_1(j)\big), \qquad i = 1, \ldots, n,$$

$$\tilde{J}_1(j) = \sum_{y \in \mathcal{A}} \phi_{jy} r_y^*, \qquad j = 1, \ldots, n$$

r^* solves uniquely the composite Bellman equation $r^* = Hr^*$:

$$r_x^* = (Hr^*)(x) = \sum_{i=1}^{n} d_{xi} \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u) \left(g(i,u,j) + \alpha \sum_{y \in \mathcal{A}} \phi_{jy}\, r_y^* \right), \qquad x \in \mathcal{A}$$

Slide from Prof. D. Bertsekas class on aggregation, available at http://web.mit.edu/dimitrib/www/Slides_Lecture12_RLOC.pdf
Details on aggregation in Chapter 6 of "Reinforcement Learning and Optimal Control" by Prof. D. Bertsekas

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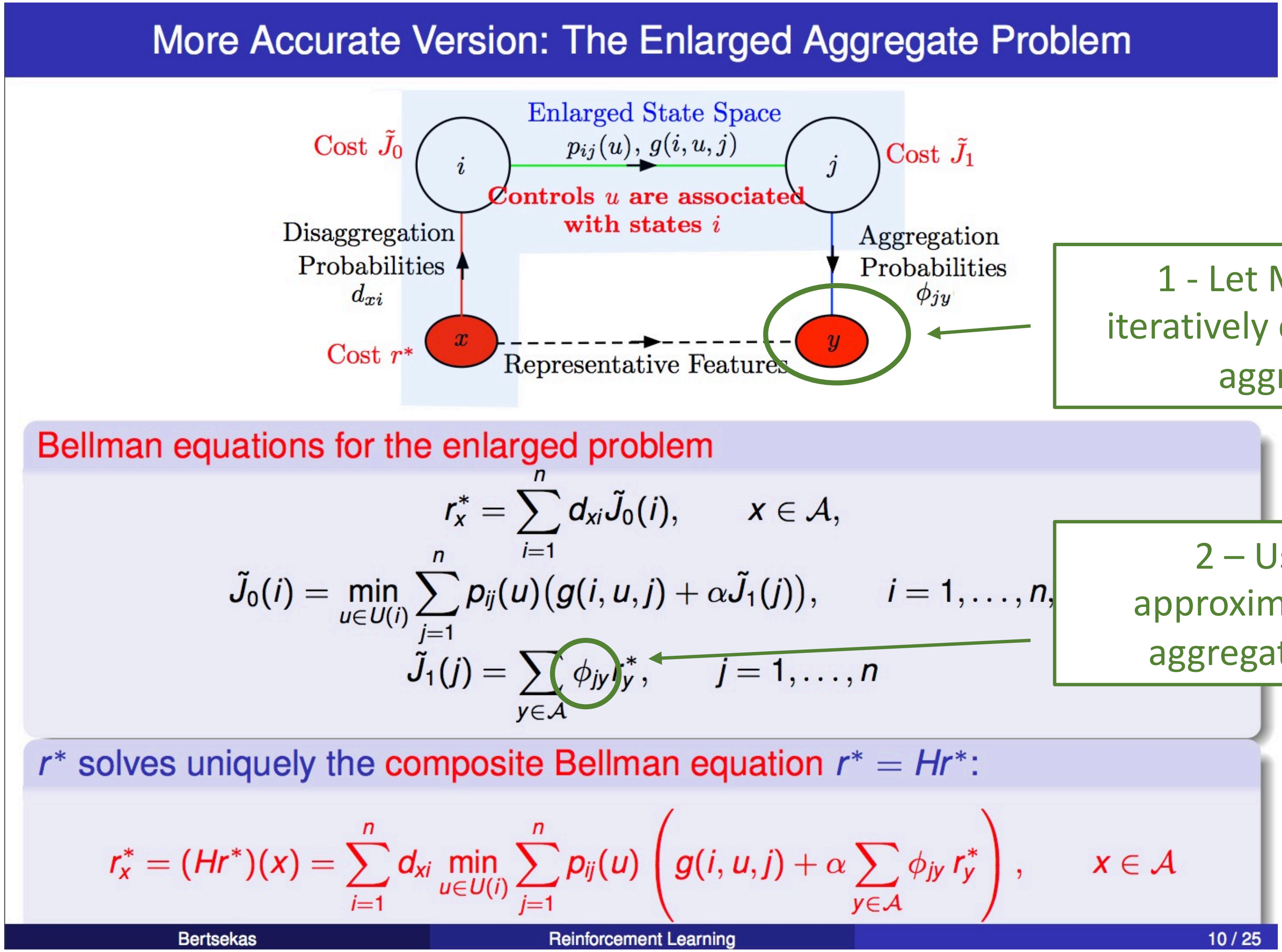

1 - Let MPC explore and iteratively construct the set of aggregate states

2 – Use barycentric approximation to compute aggregation probabilities

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Model Learning in MPC

A short and non-comprehensive summary

The complexity of the prediction model

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Linear

$$x_{k+1} = Ax_k + Bu_k + w_k$$

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$$x_{k+1} = Ax_k + Bu_k + w_k$$

Gray Box

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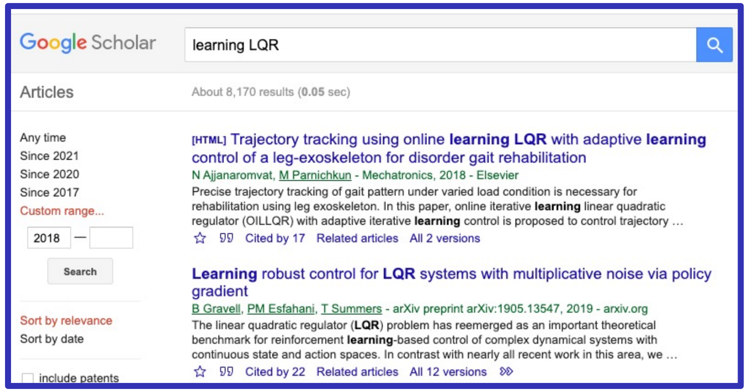

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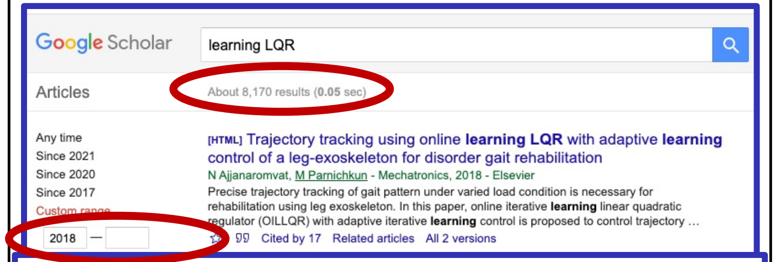

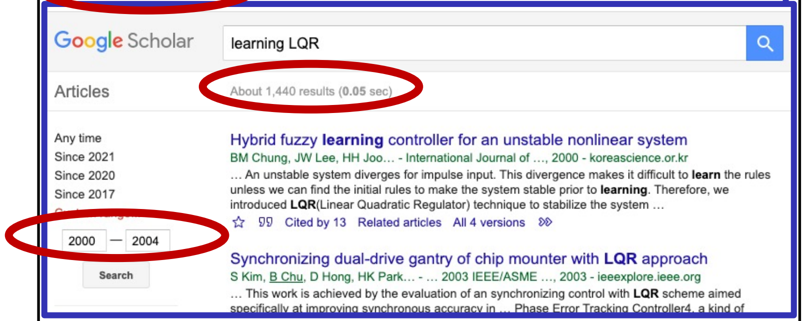

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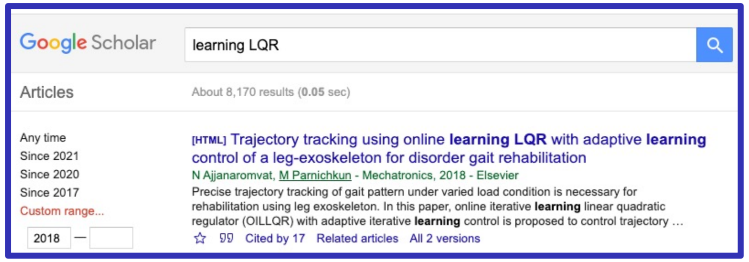

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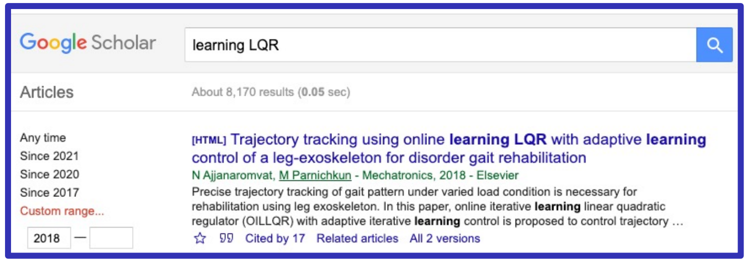

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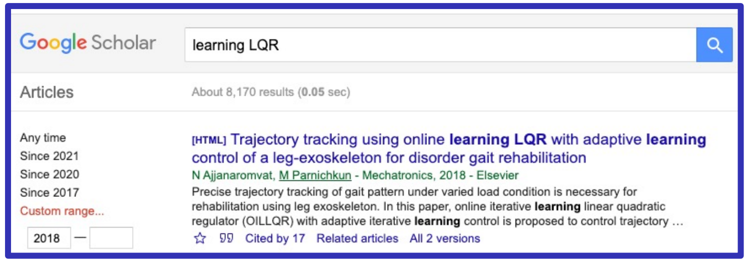

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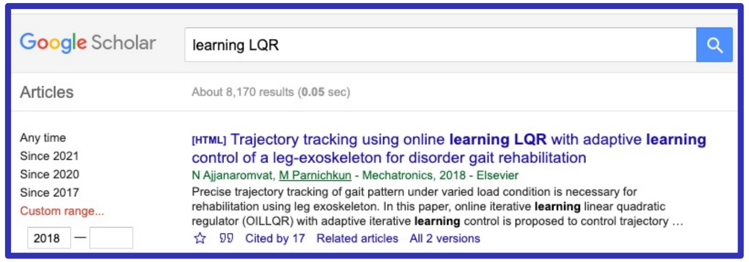

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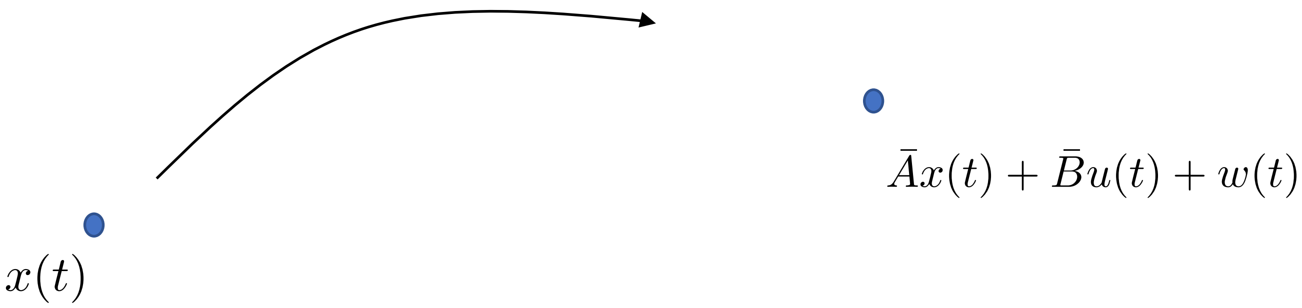

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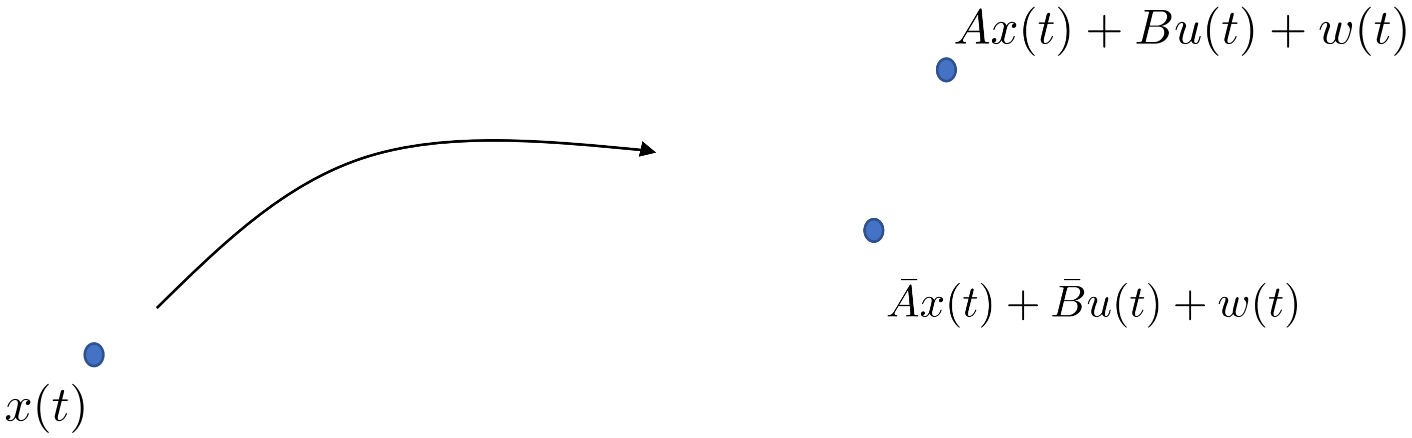

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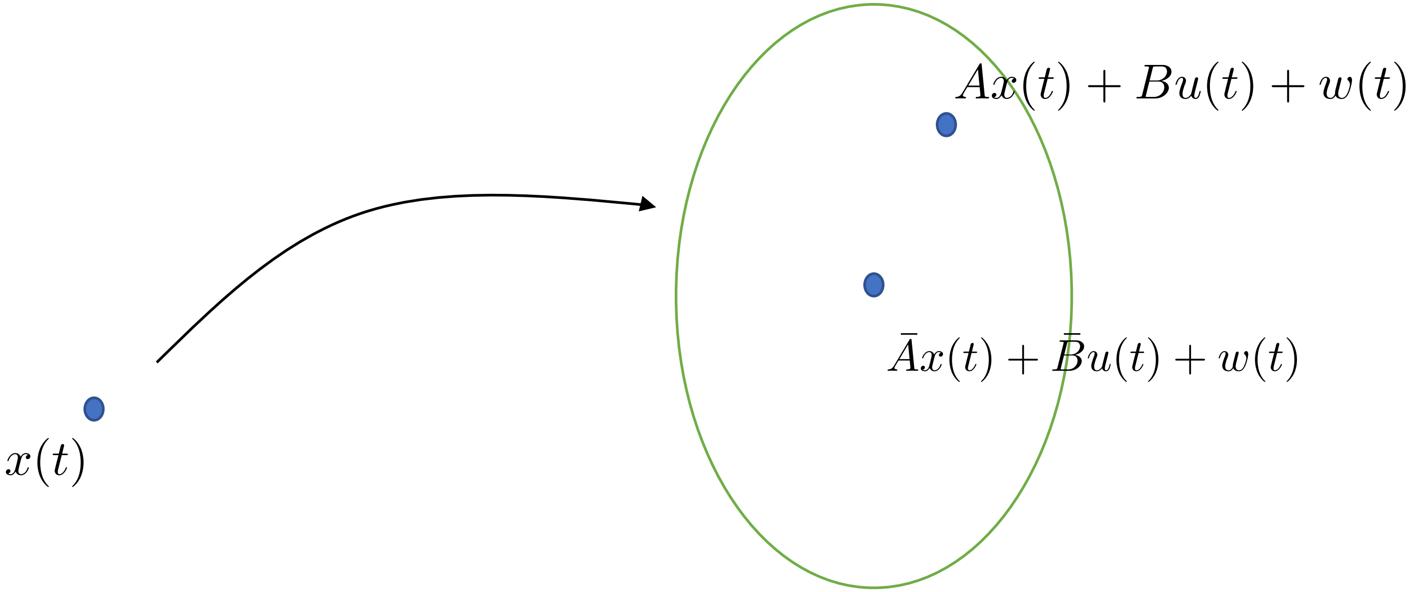

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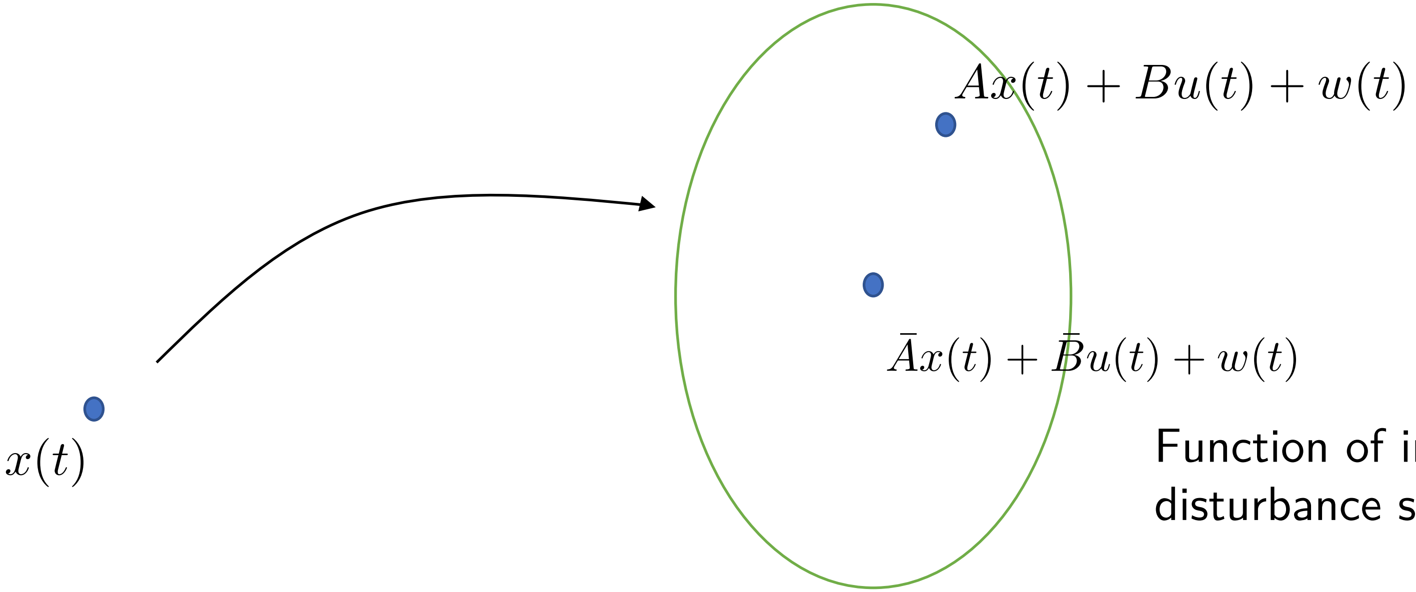

Function of initial condition, applied input, disturbance support and the identification error:

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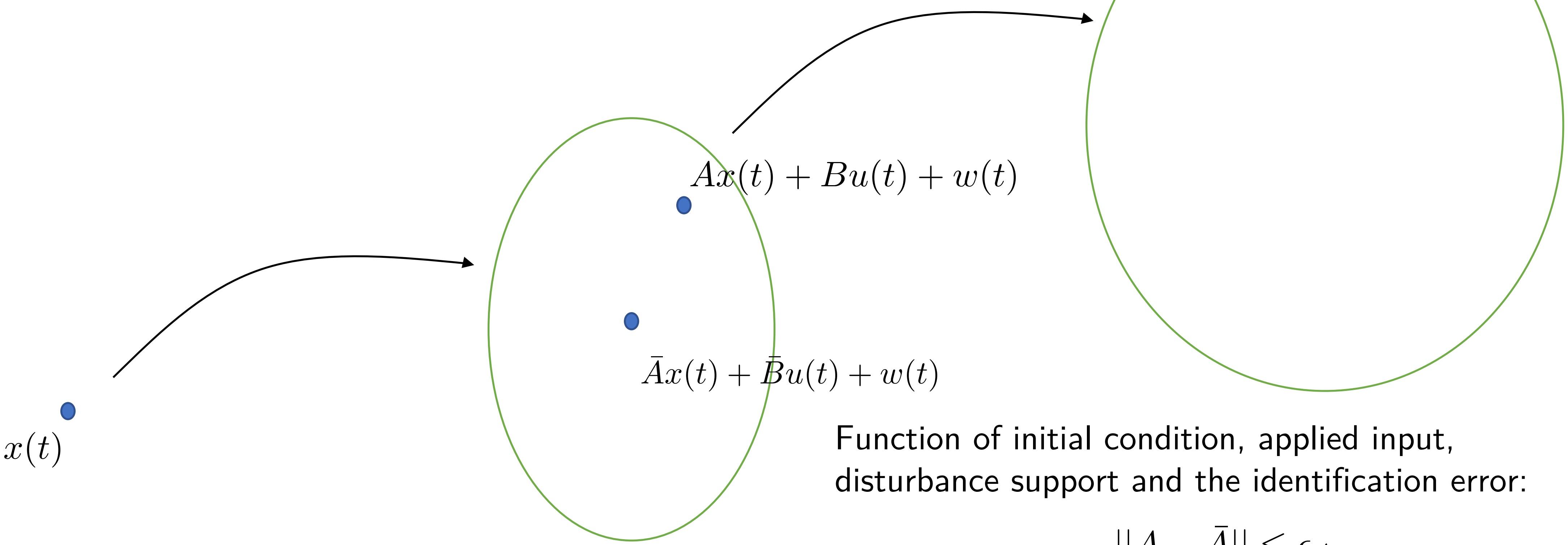

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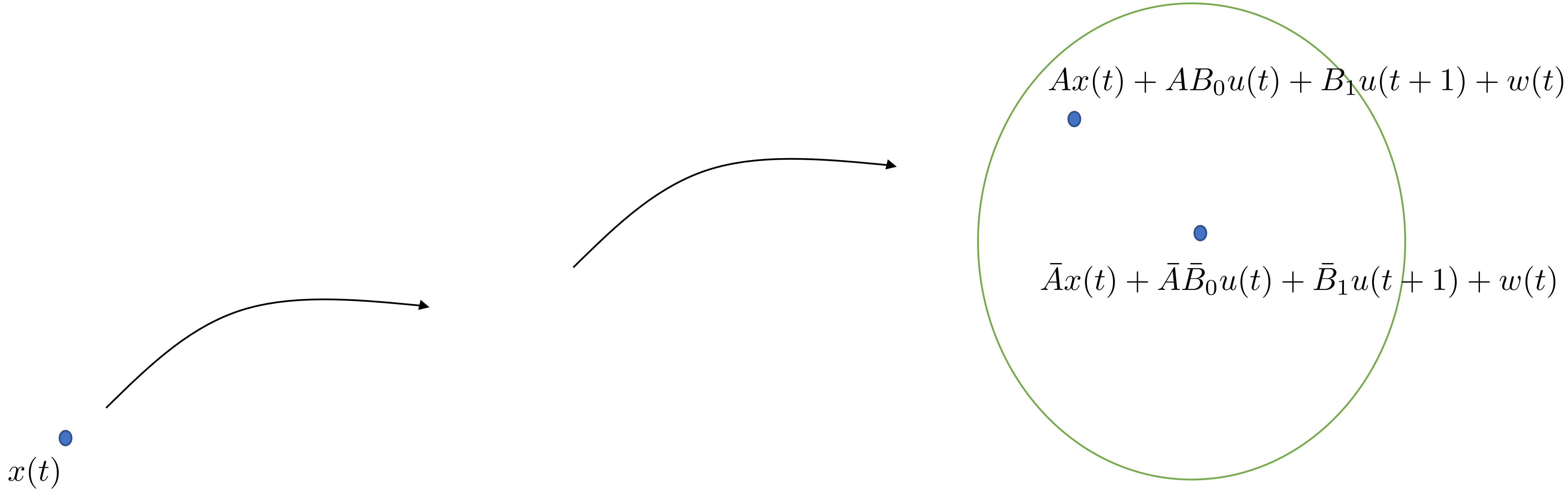

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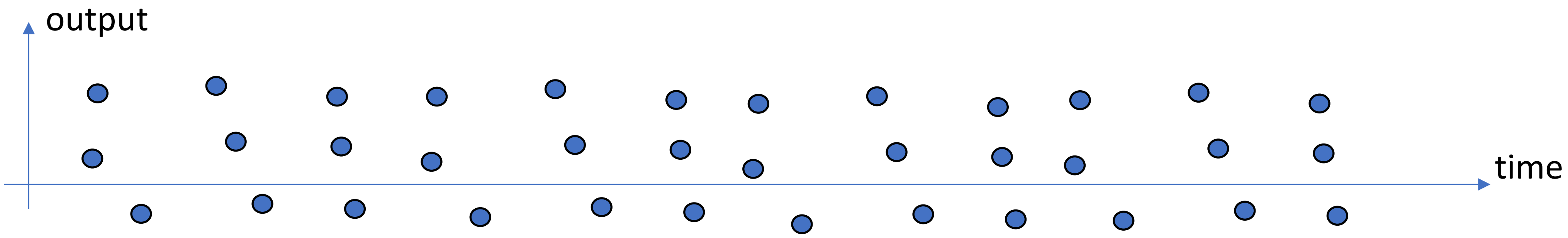

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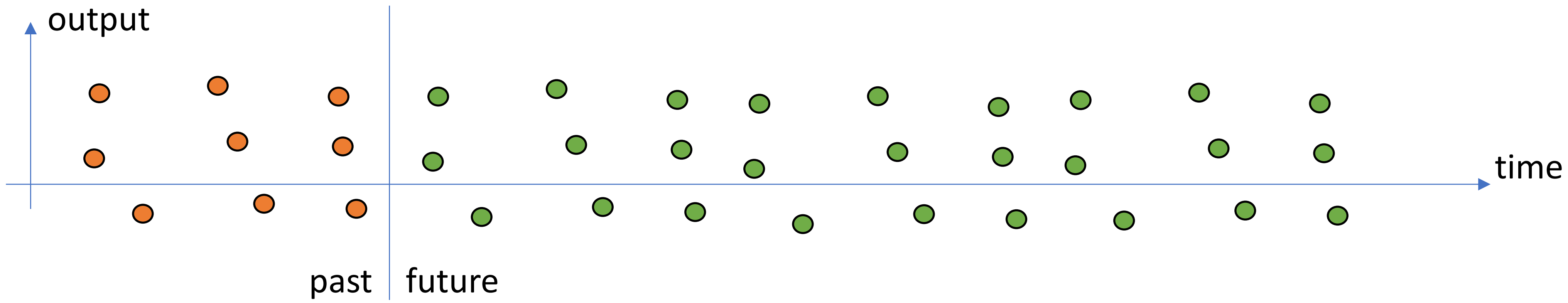

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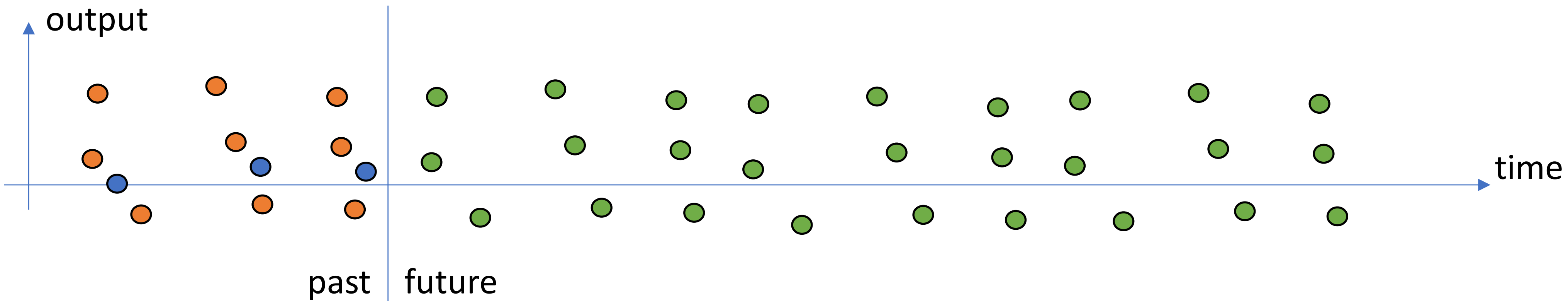

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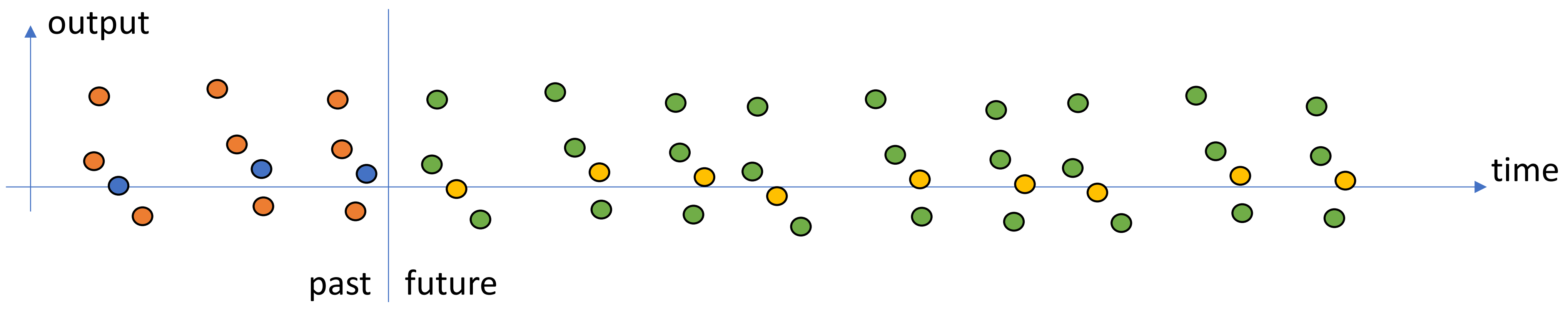

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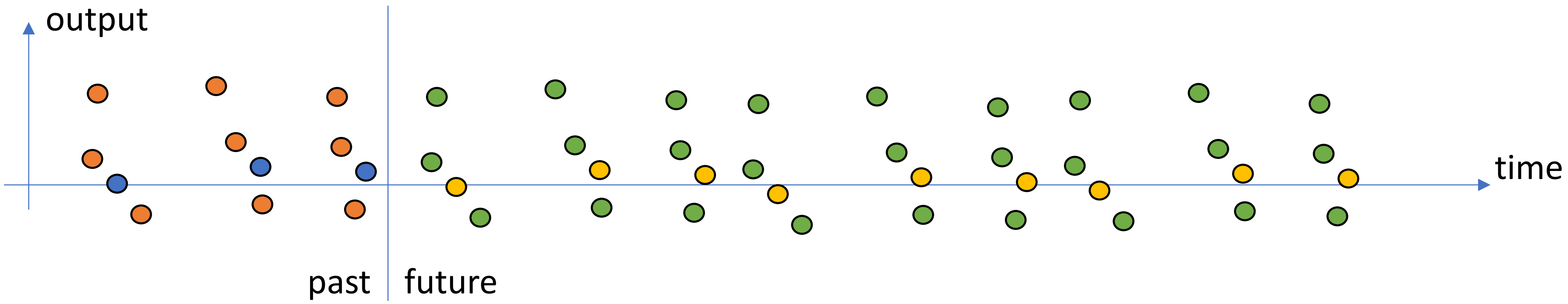

Construct the Hankel Matrix using Data

$$\mathscr{H}_L(u) := \begin{pmatrix} u_1 & \cdots & u_{T-L+1} \\ \vdots & \ddots & \vdots \\ u_L & \cdots & u_T \end{pmatrix}$$

$$\begin{pmatrix} U_\mathrm{p} \\ U_\mathrm{f} \end{pmatrix} := \mathscr{H}_{T_\mathrm{ini}+N}(u^\mathrm{d}), \quad \begin{pmatrix} Y_\mathrm{p} \\ Y_\mathrm{f} \end{pmatrix} := \mathscr{H}_{T_\mathrm{ini}+N}(y^\mathrm{d}),$$

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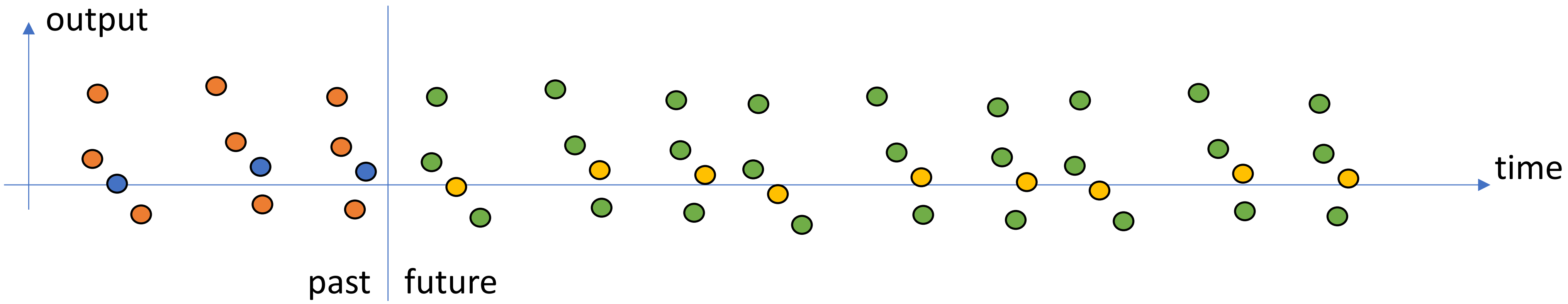

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Replace the model

$$\begin{aligned} \underset{g,u,y}{\text{minimize}} \quad & \sum_{k=0}^{N-1} \left(\|y_k - r_{t+k}\|_Q^2 + \|u_k\|_R^2 \right) \\ \text{subject to} \quad & \begin{pmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \\ U_\mathrm{f} \\ Y_\mathrm{f} \end{pmatrix} g = \begin{pmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} \\ u \\ y \end{pmatrix}, \\ & u_k \in \mathcal{U}, \ \forall k \in \{0, \dots, N-1\}, \\ & y_k \in \mathcal{Y}, \ \forall k \in \{0, \dots, N-1\}. \end{aligned}$$

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The complexity of the prediction model

Linear

$$x_{k+1} = Ax_k + Bu_k + w_k$$

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Gray Box

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unknown

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Most popular method is Gaussian Process

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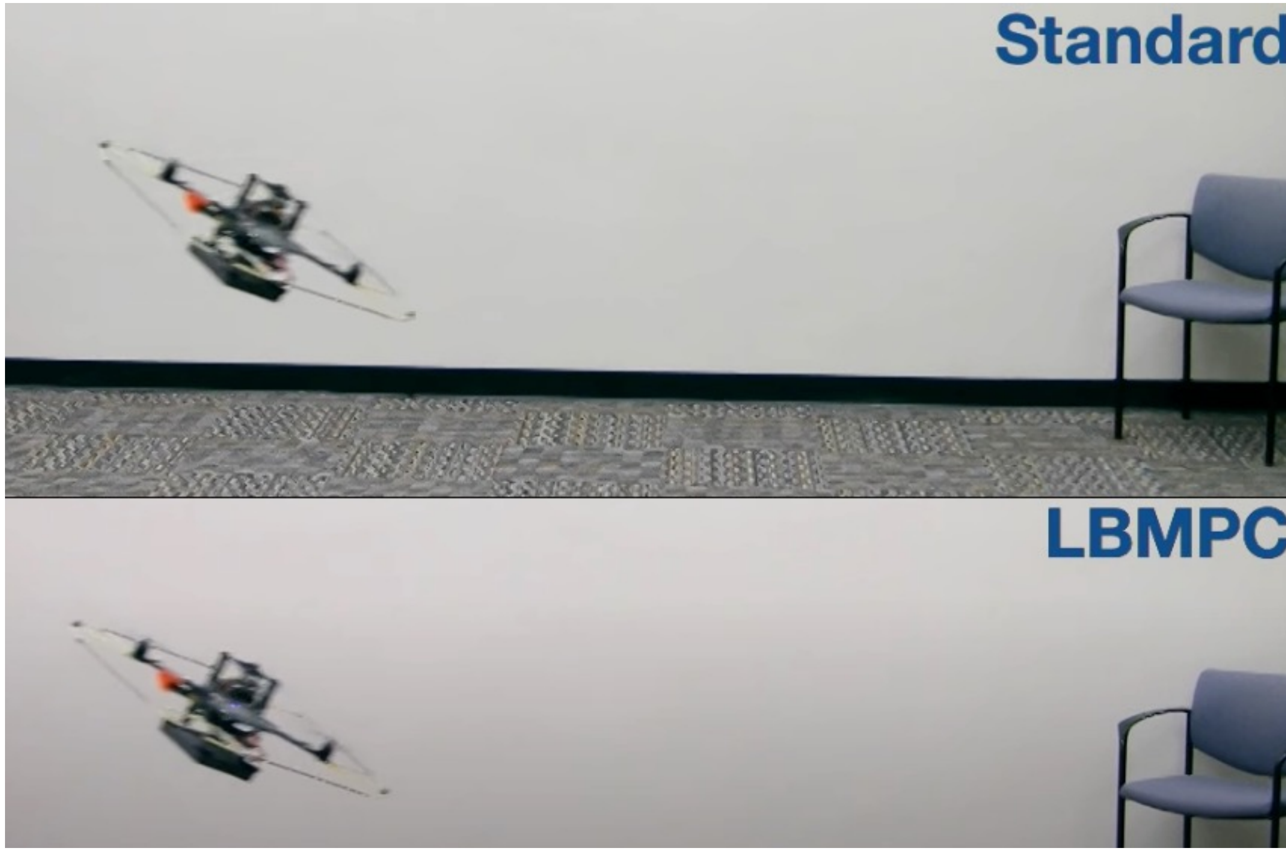

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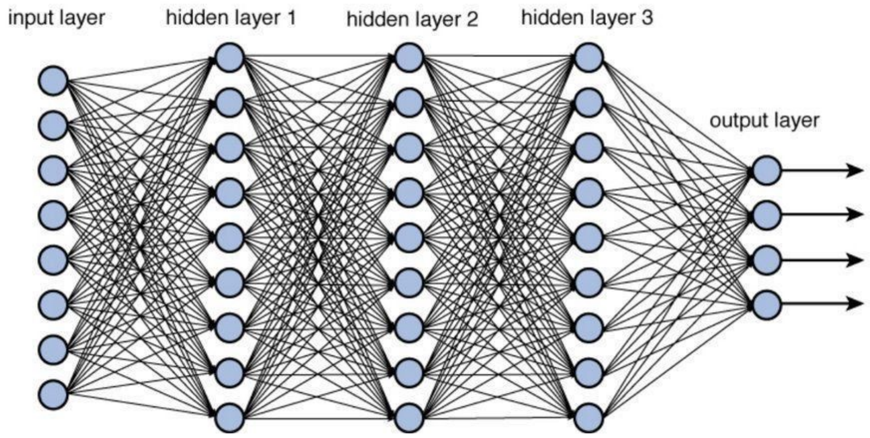

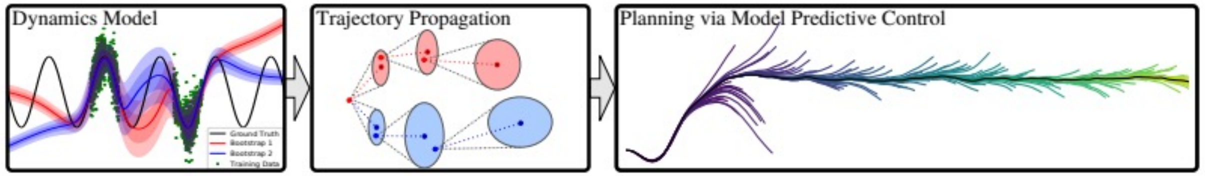

Figure 1: Our method (PE-TS): **Model**: Our probabilistic ensemble (PE) dynamics model is shown as an ensemble of two bootstraps (bootstrap disagreement far from data captures epistemic uncertainty: our subjective uncertainty due to a lack of data), each a probabilistic neural network that captures aleatoric uncertainty (inherent variance of the observed data). **Propagation**: Our trajectory sampling (TS) propagation technique uses our dynamics model to re-sample each particle (with associated bootstrap) according to its probabilistic prediction at each point in time, up until horizon T. **Planning**: At each time step, our MPC algorithm computes an optimal action sequence, applies the first action in the sequence, and repeats until the task-horizon.

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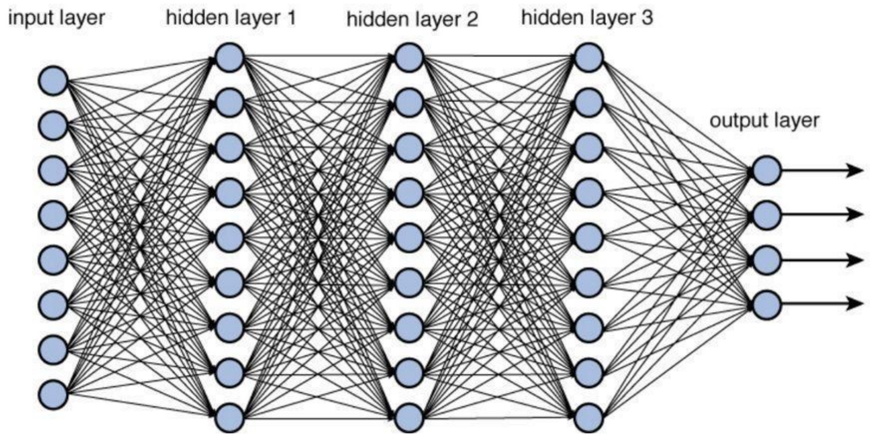

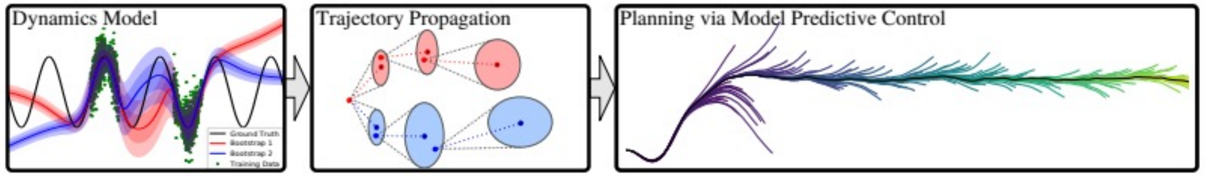

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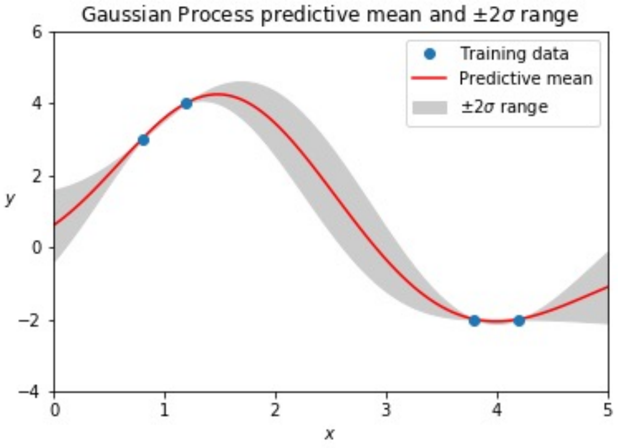

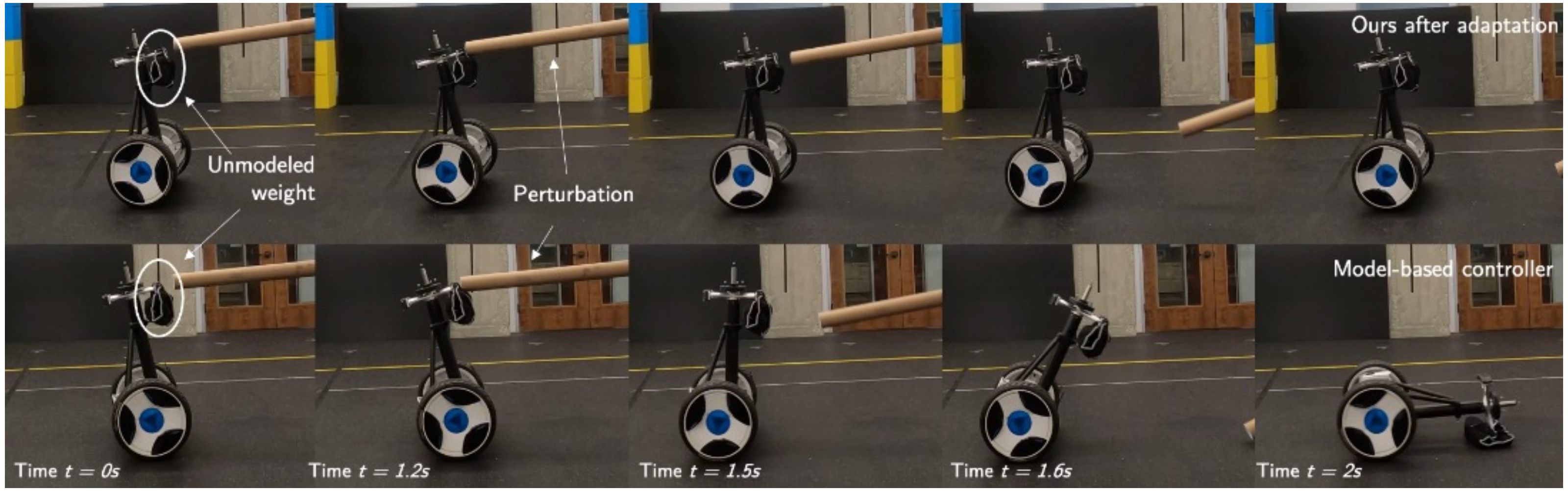

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Planning under uncertainty

Optimizing over control policies

Why is planning in uncertain environments harder?

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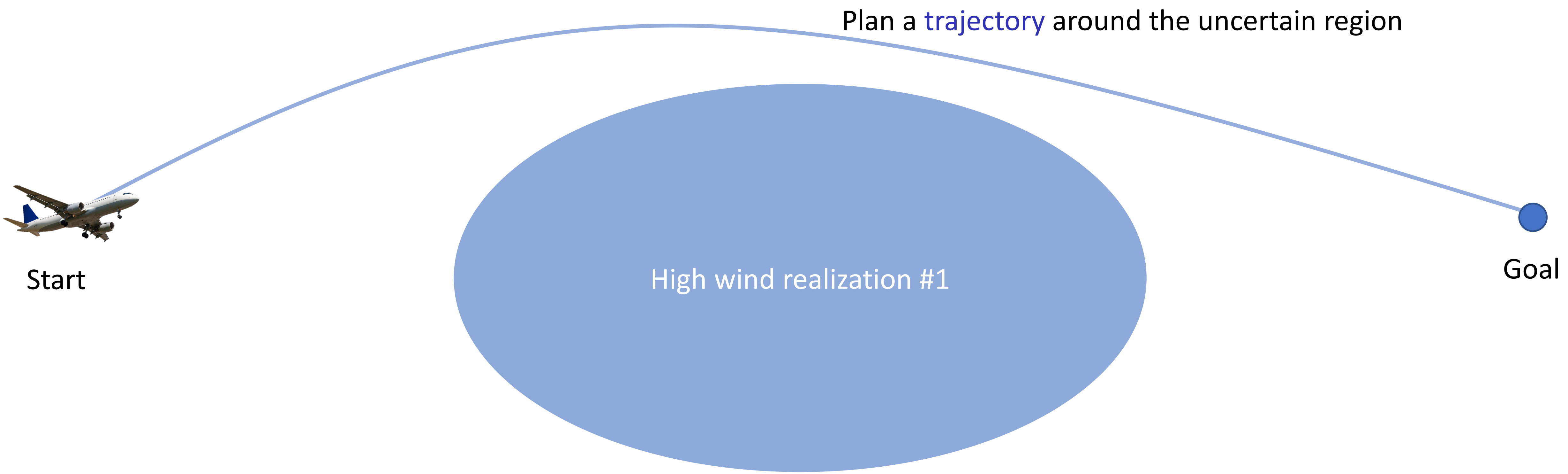

Why is planning in uncertain environments harder?

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Plan a trajectory around the uncertain region

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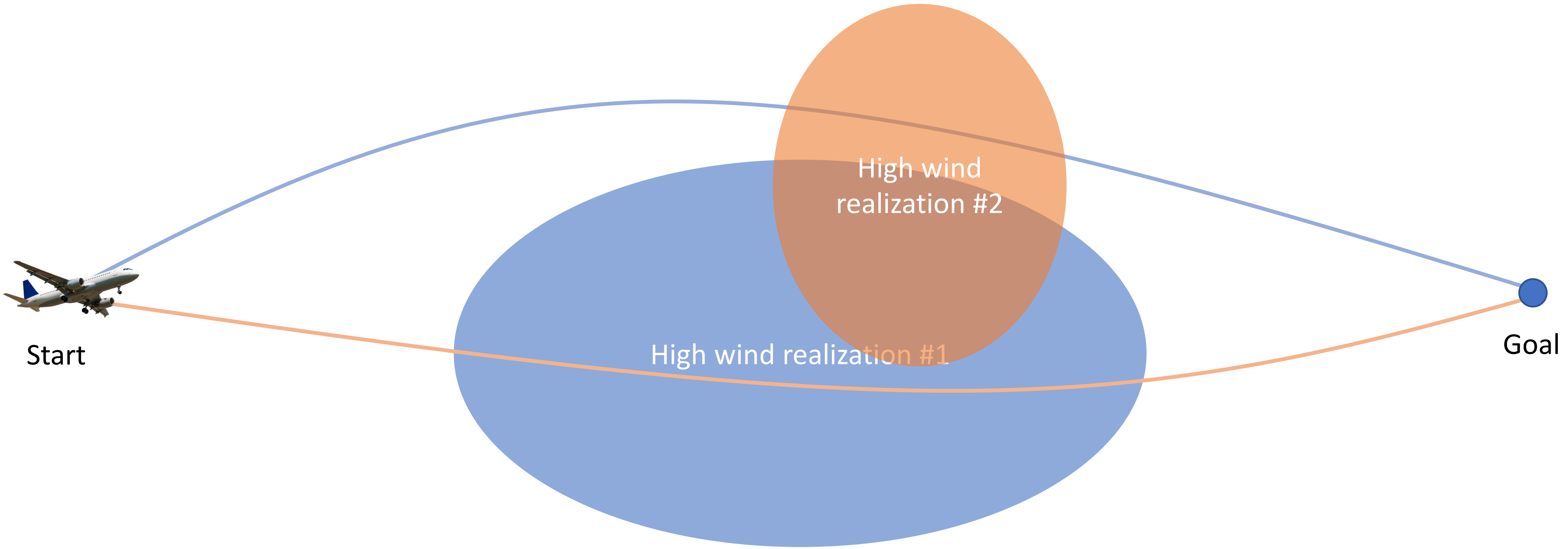

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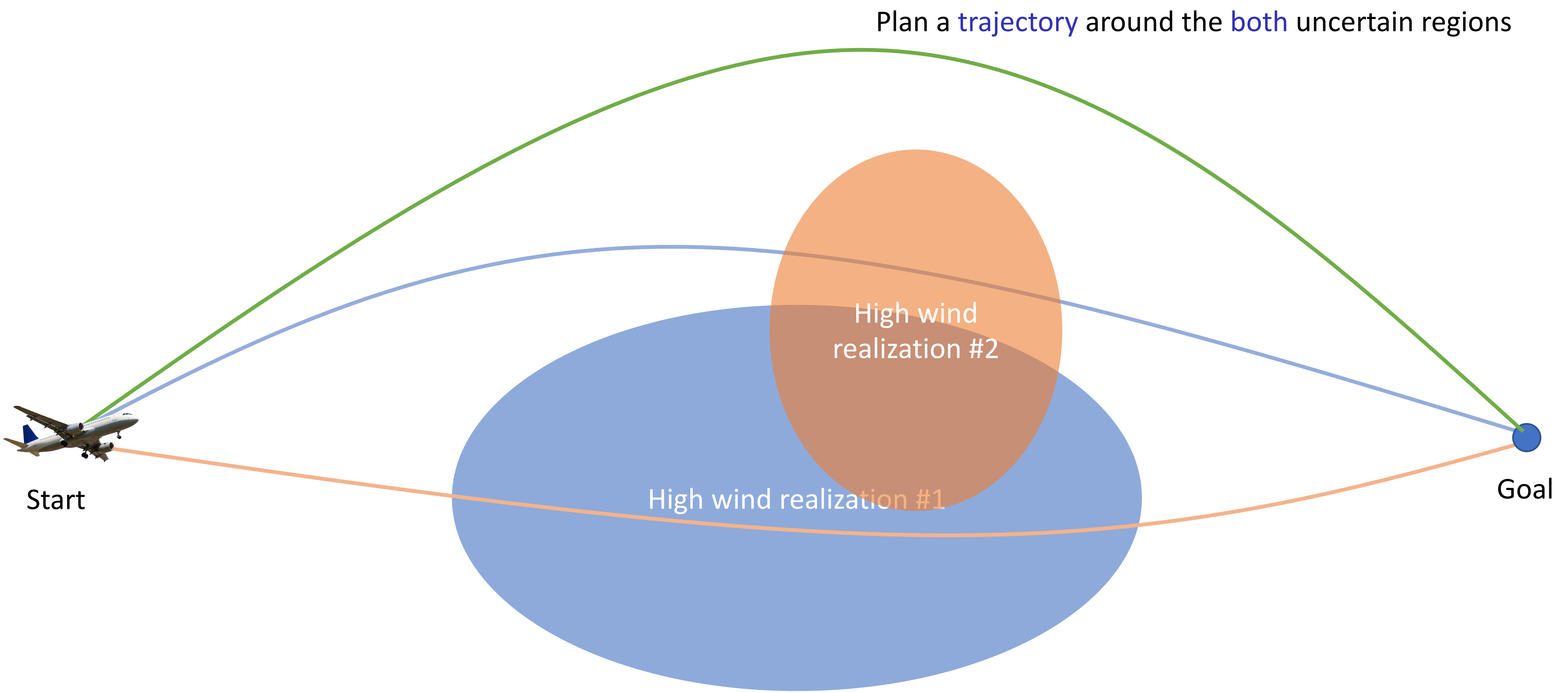

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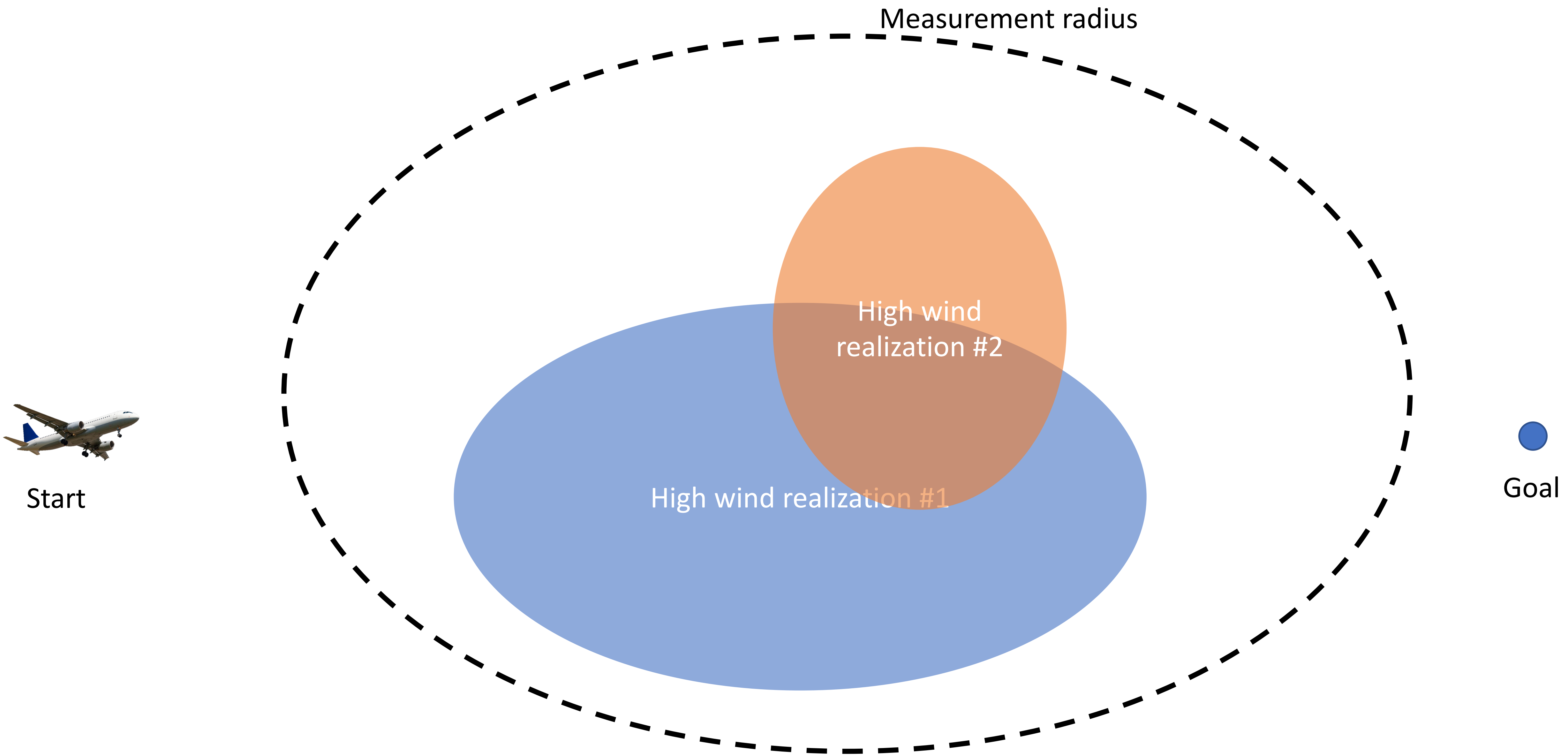
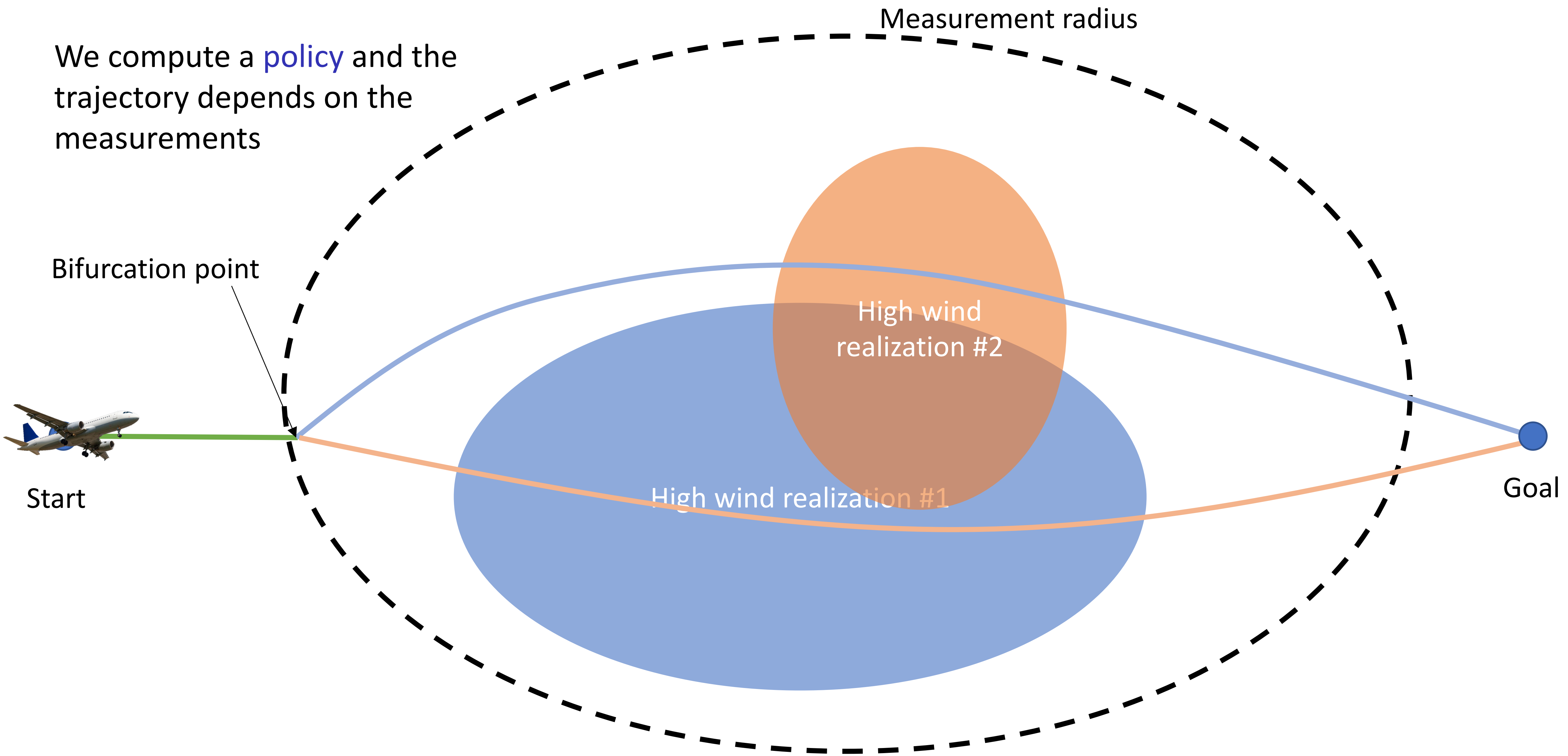


Why is planning in uncertain environments harder?

We compute a policy and the trajectory depends on the measurements



Problem Formulation

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

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Nominal model: $\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}\bar{u}_k + w_k$ where $||A - \bar{A}|| \leq \epsilon_A, ||B - \bar{B}|| \leq \epsilon_B$

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Robust constraints:

Given a control policy $\pi : \mathbb{R}^n \to \mathbb{R}^d$ designed using the nominal model, we want to guarantee that the closed-loop system

$$x_{k+1} = Ax_k + B\pi(x_k) + w_k$$

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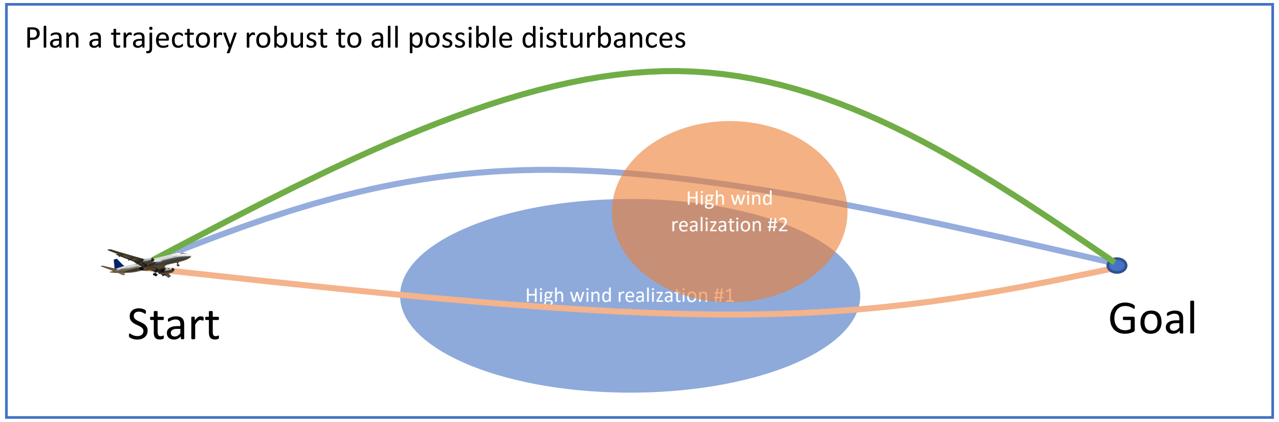

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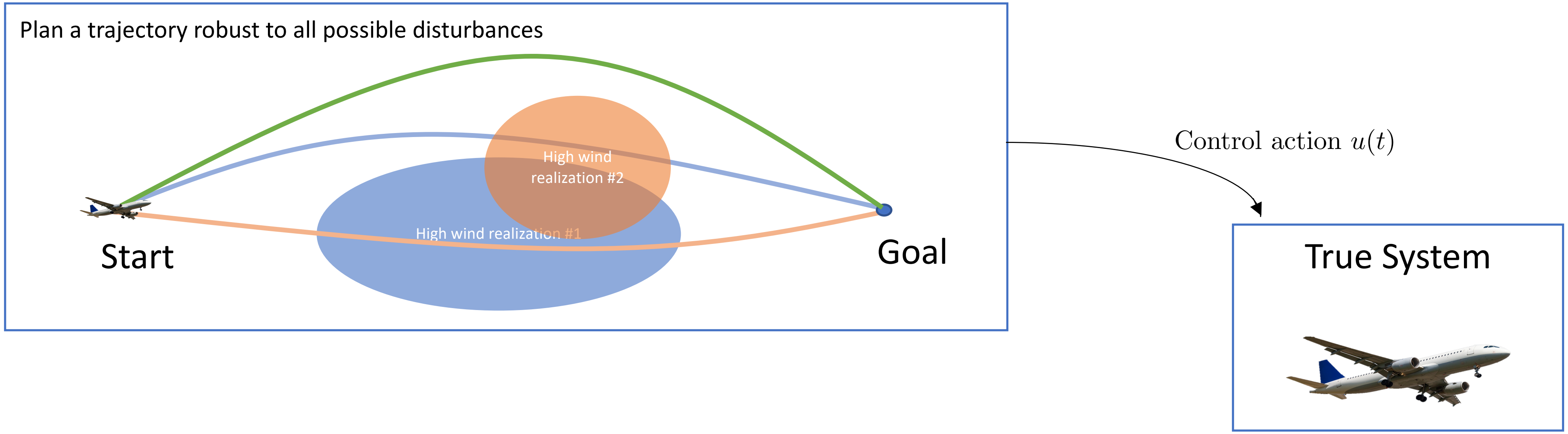

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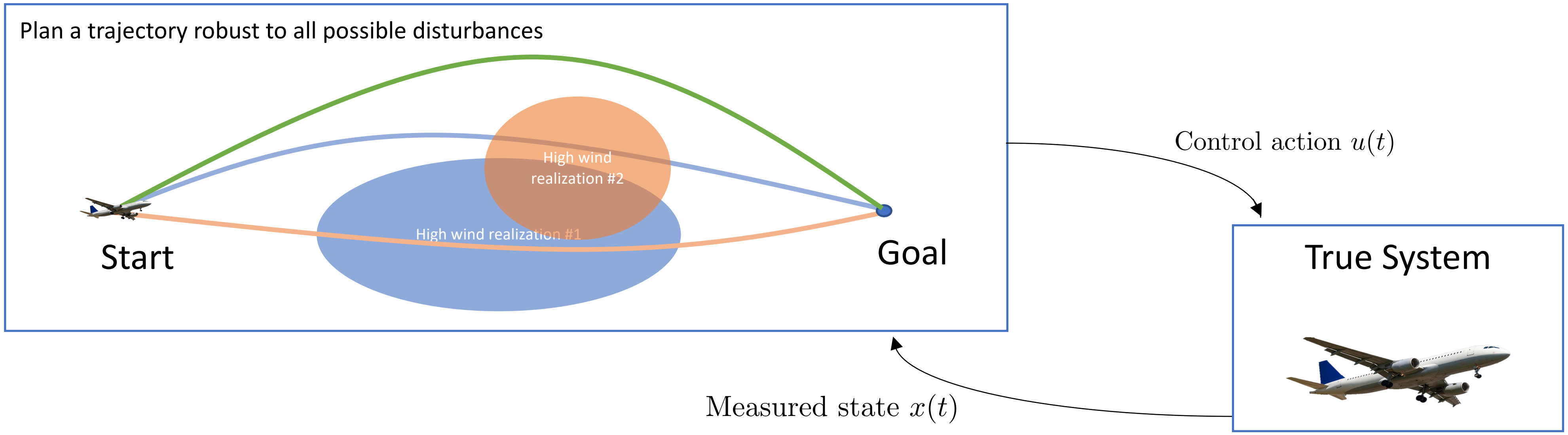

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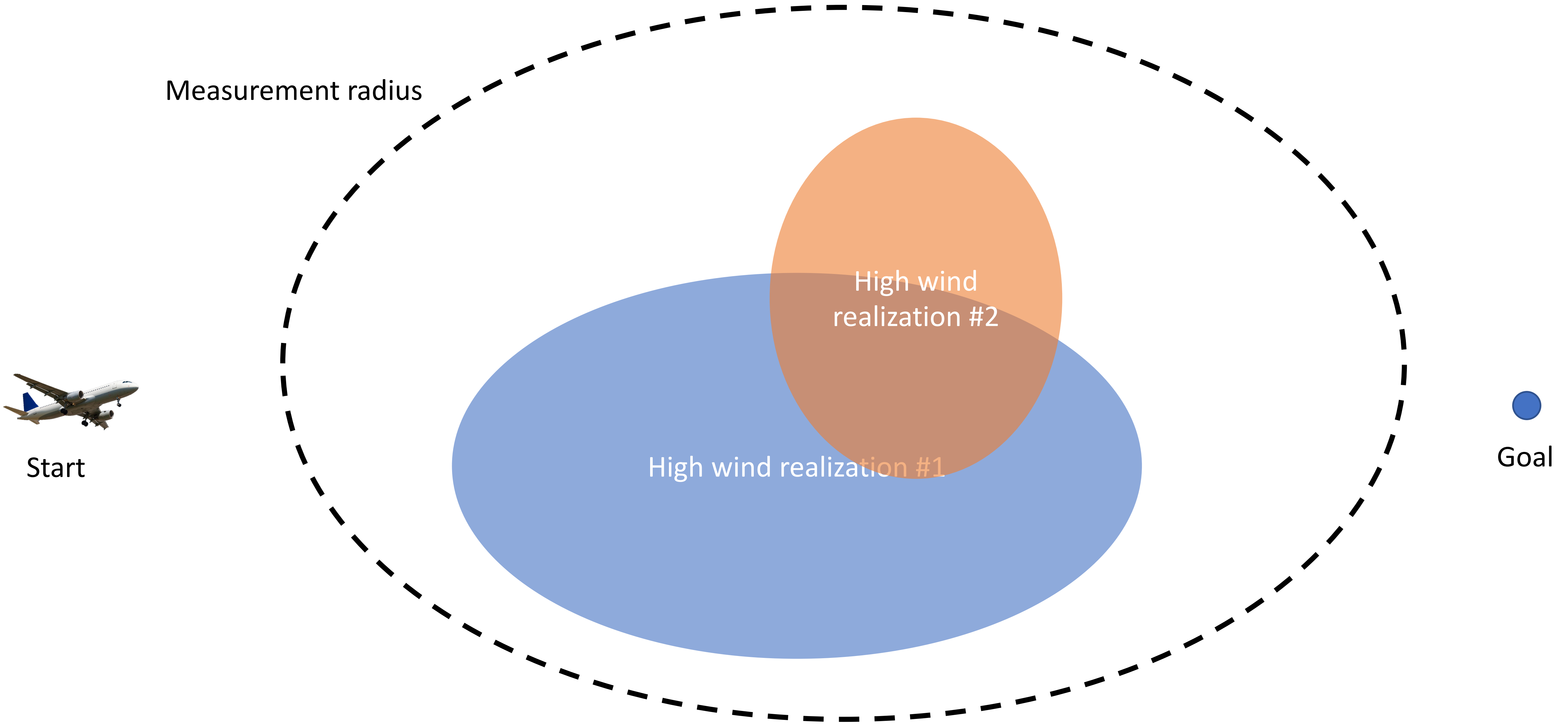

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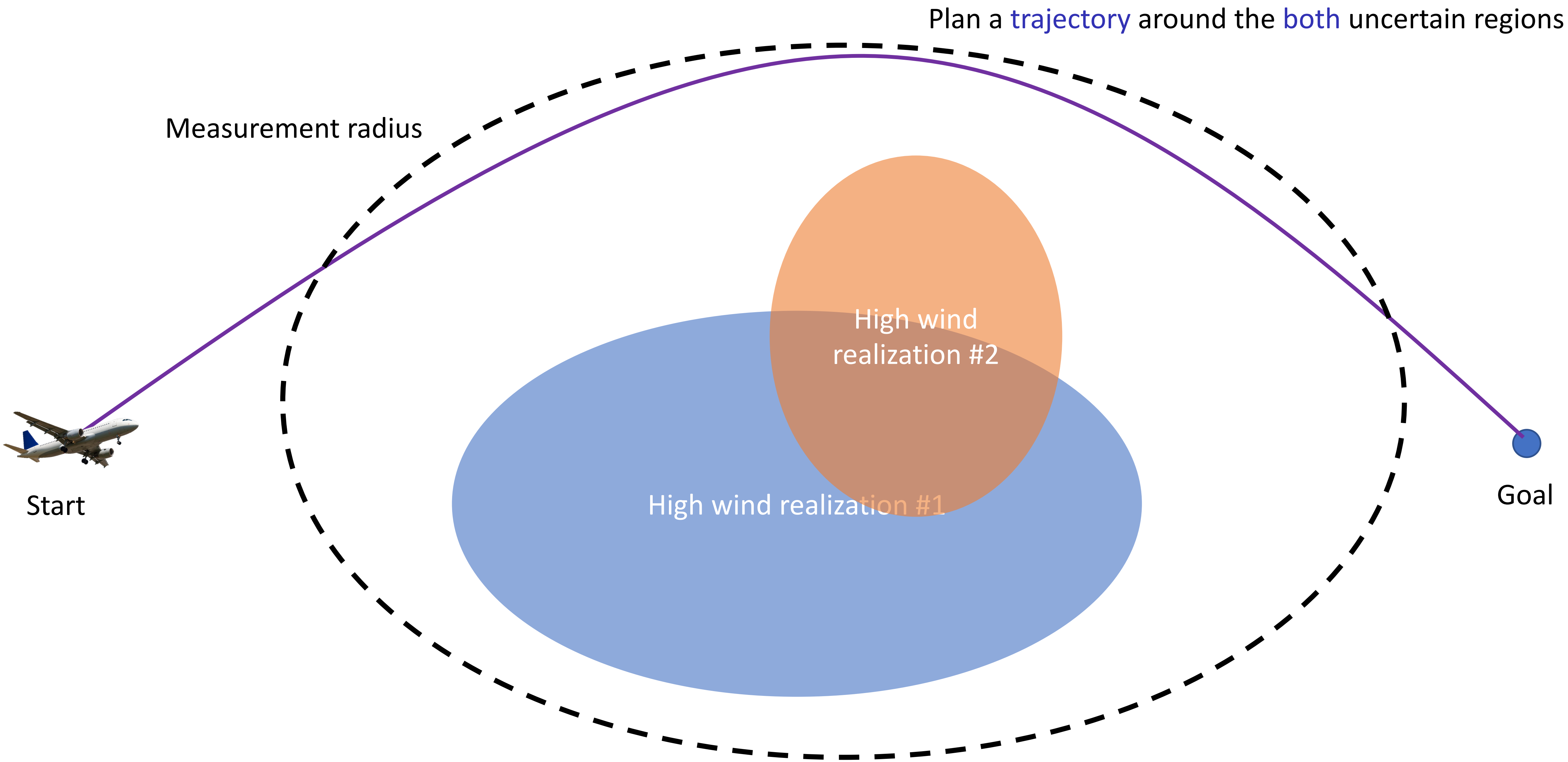

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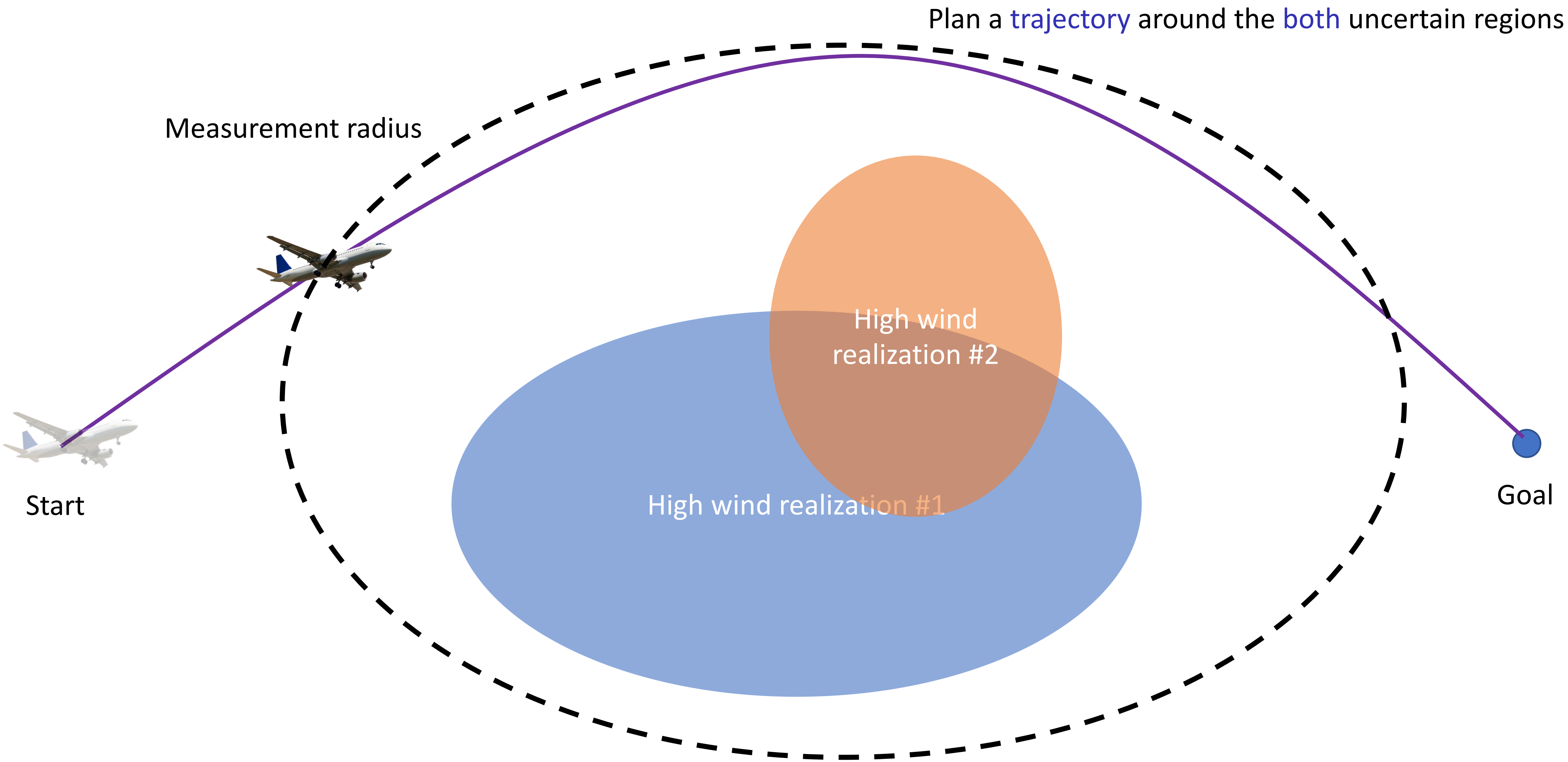

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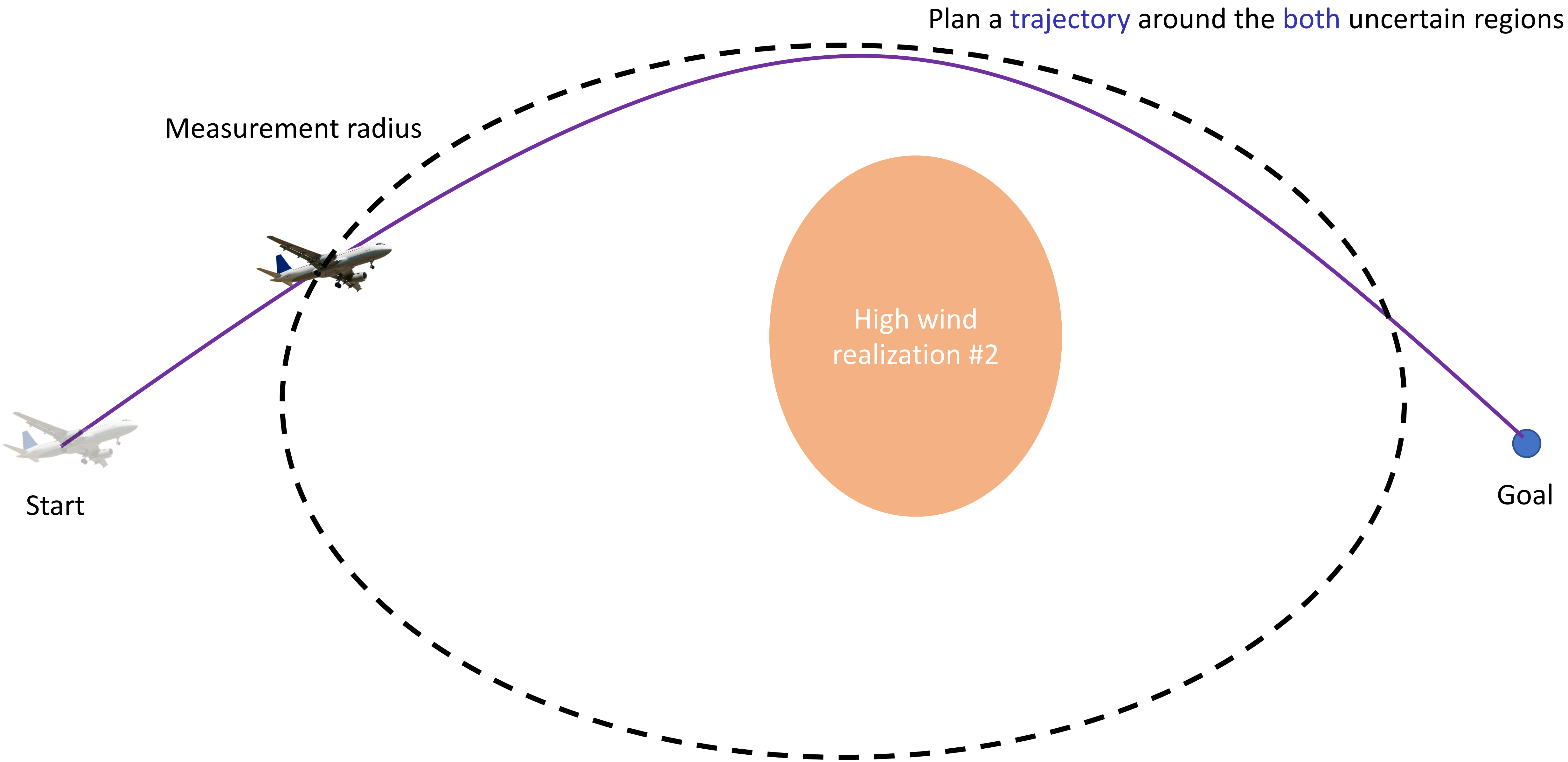

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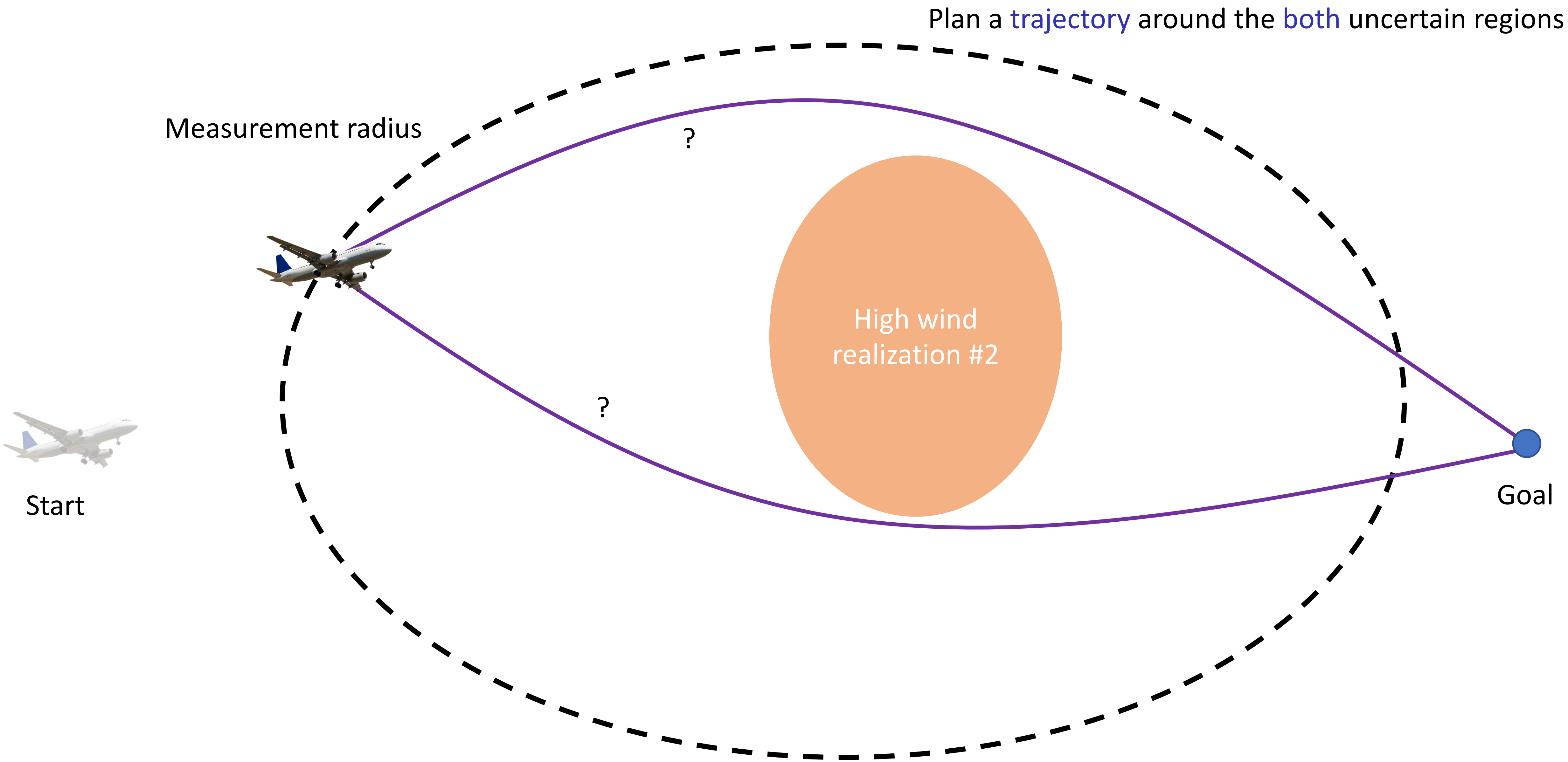

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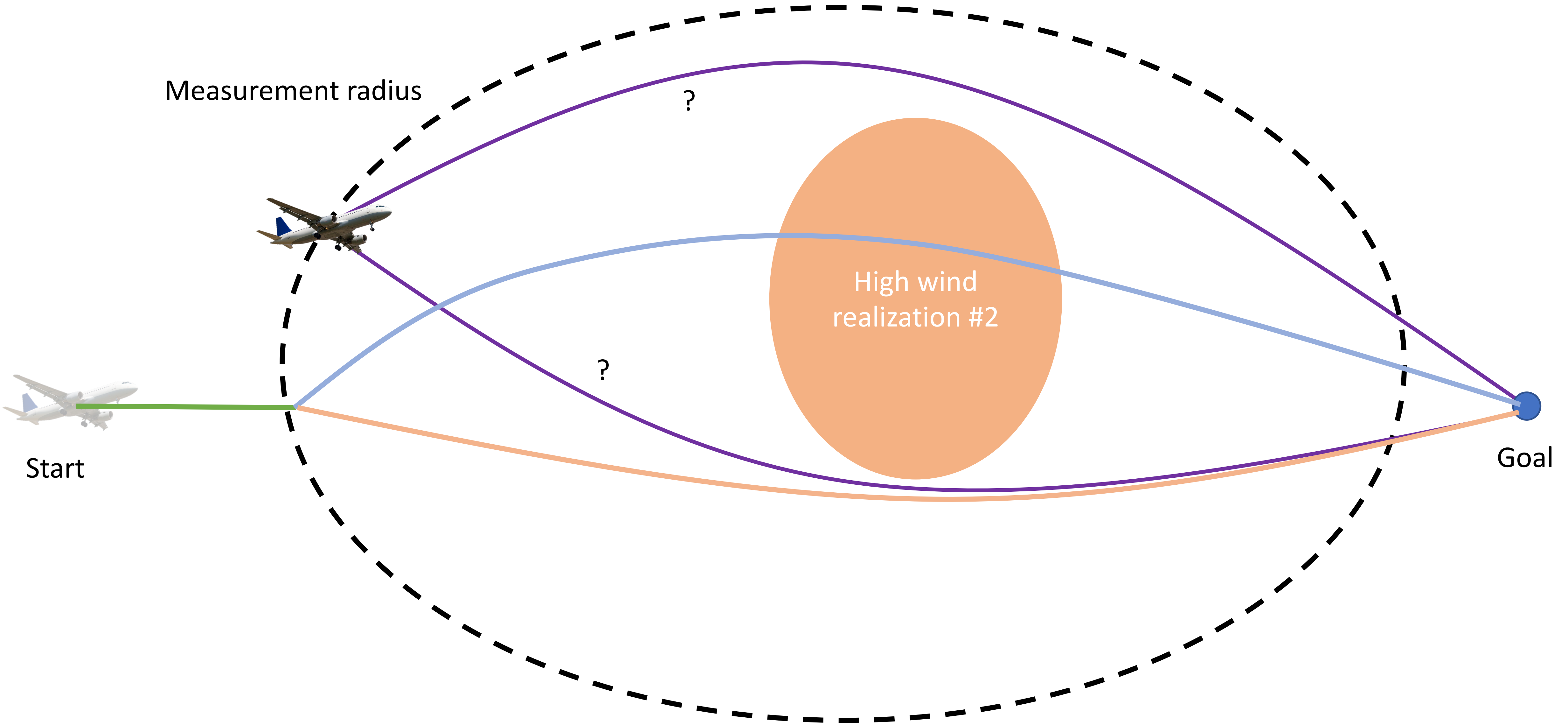
Plan a trajectory around the both uncertain regions



Why should I care?

Why should I care?

Model-based RL

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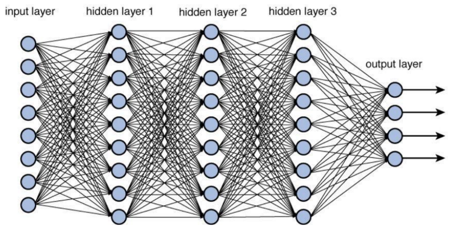
Model-based RL

Step 1: Learn a Model

(x_k, u_k)



x_{k+1}

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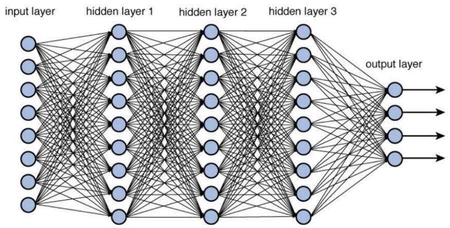
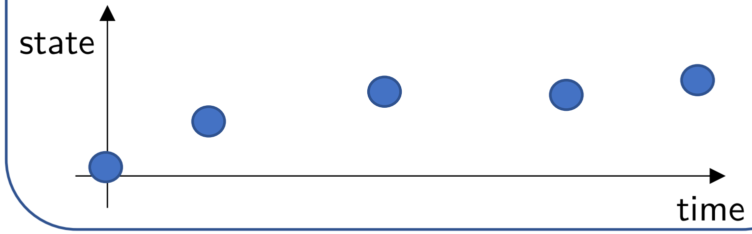


x_{k+1}

Step 2: Receding horizon planning

$$\min \mathbb{E}_{\substack{a_t \sim \pi \\ s_t \sim p}} [\textstyle\sum_{t=0}^{H} \gamma^t c(a_t, s_t)]$$



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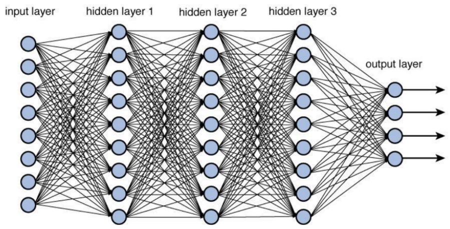

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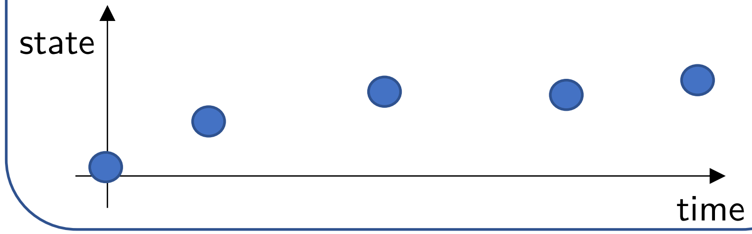
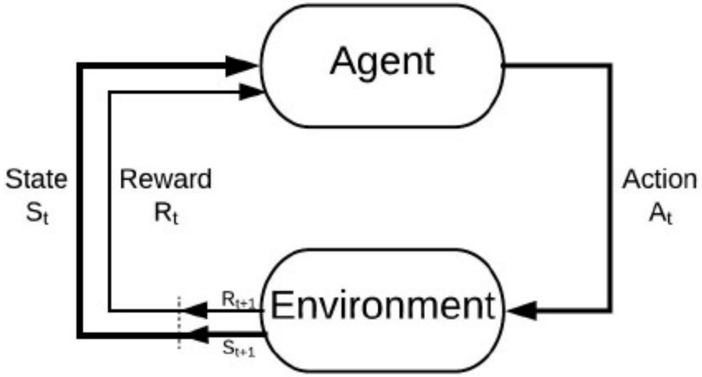
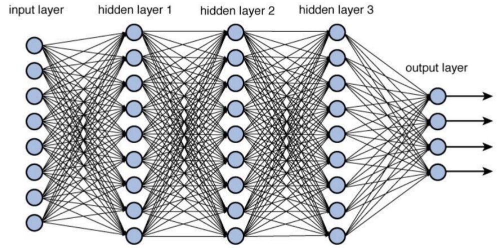


Model-free RL

Step 1: Learn a policy



x_k



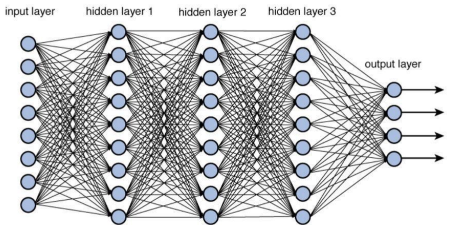
u_k

Why should I care?

Model-based RL

Step 1: Learn a Model

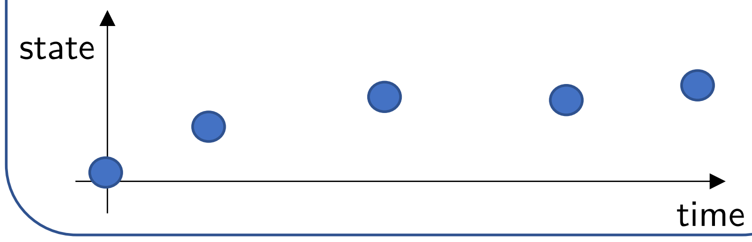
(x_k, u_k)



x_{k+1}

Step 2: Receding horizon planning

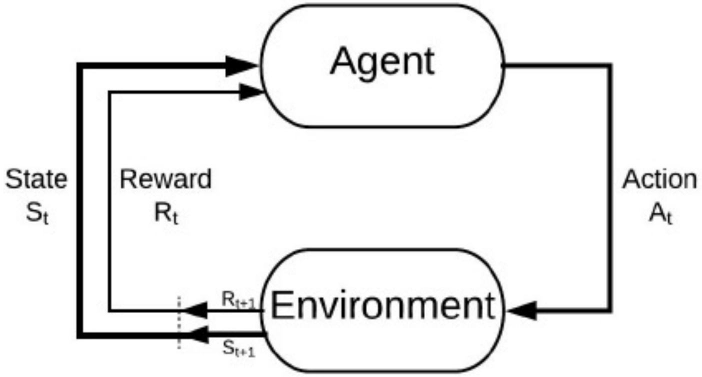
$$\min \mathbb{E}_{\substack{a_t \sim \pi \\ s_t \sim p}} [\textstyle\sum_{t=0}^{H} \gamma^t c(a_t, s_t)]$$



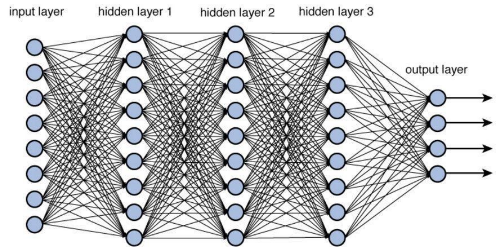
Which one has better asymptotic performance?

Model-free RL

Step 1: Learn a policy



x_k



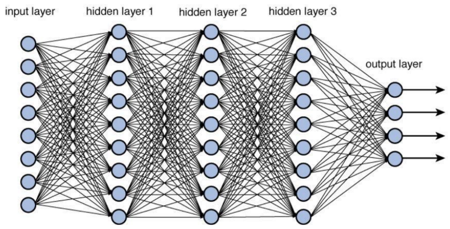
u_k

Why should I care?

Model-based RL

Step 1: Learn a Model

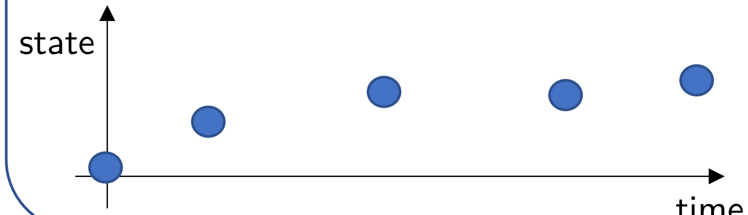
(x_k, u_k)



x_{k+1}

Step 2: Receding horizon planning

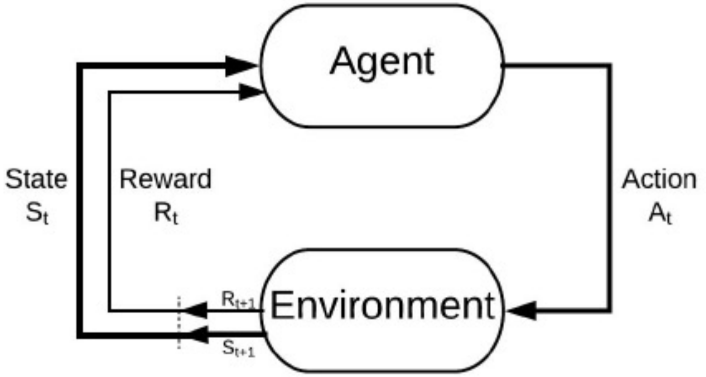
$$\min \mathbb{E}_{\substack{a_t \sim \pi \\ s_t \sim p}} [\textstyle\sum_{t=0}^{H} \gamma^t c(a_t, s_t)]$$



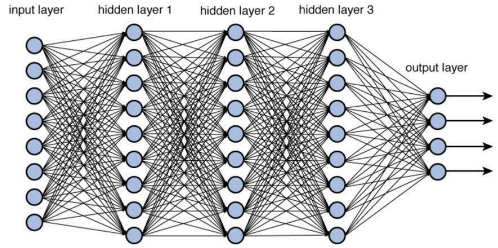
Which one has better asymptotic performance?

Model-free RL

Step 1: Learn a policy



x_k



u_k

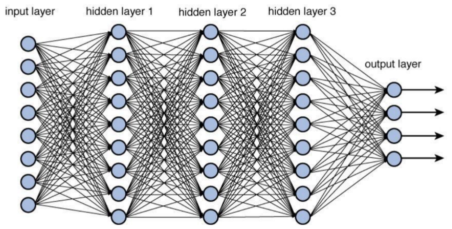
What can go wrong with model-based RL?

Why should I care?

Model-based RL

Step 1: Learn a Model

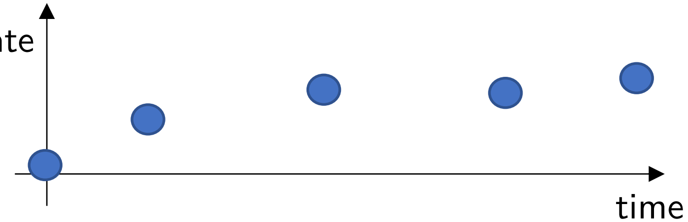
(x_k, u_k)



x_{k+1}

Step 2: Receding horizon planning

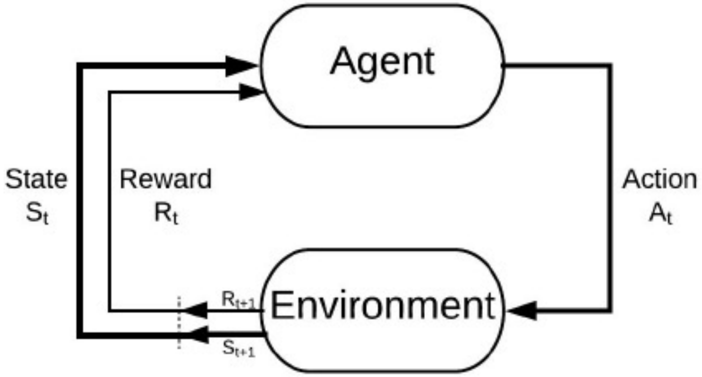
$$\min \mathbb{E}_{\substack{a_t \sim \pi \\ s_t \sim p}} [\sum_{t=0}^{H} \gamma^t c(a_t, s_t)]$$



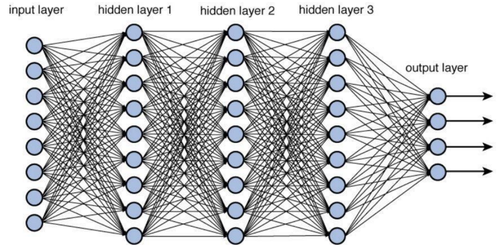
Which one has better asymptotic performance?

Model-free RL

Step 1: Learn a policy



x_k



u_k

What can go wrong with model-based RL?

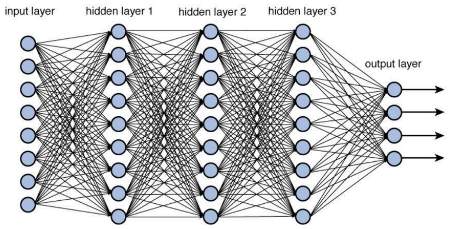
- The prediction horizon is short → need a value function and safe set

Why should I care?

Model-based RL

Step 1: Learn a Model

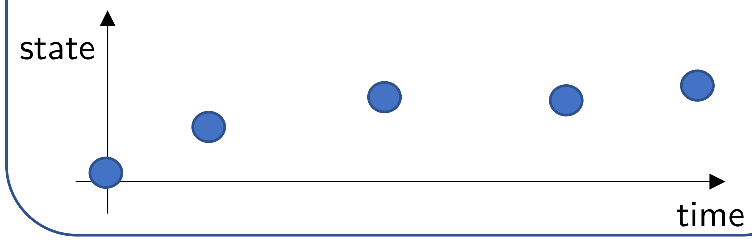
(x_k, u_k)



x_{k+1}

Step 2: Receding horizon planning

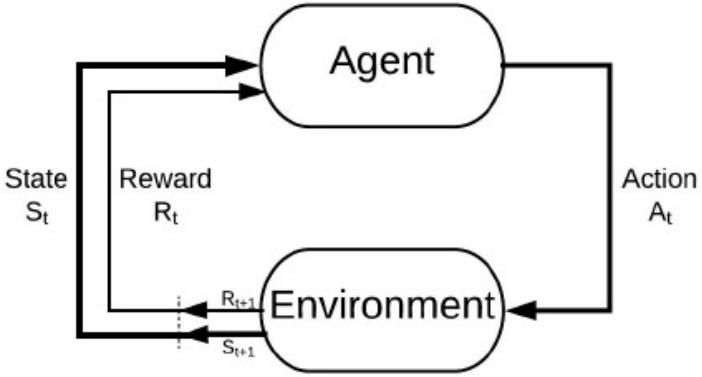
$$\min \mathbb{E}_{\substack{a_t \sim \pi \\ s_t \sim p}}\left[\sum_{t=0}^{H} \gamma^t c(a_t, s_t)\right]$$



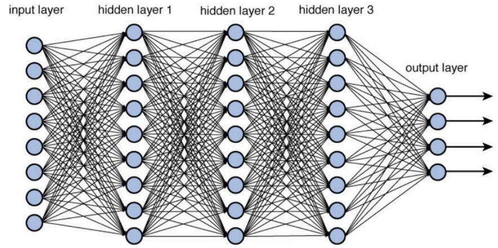
Which one has better asymptotic performance?

Model-free RL

Step 1: Learn a policy



x_k



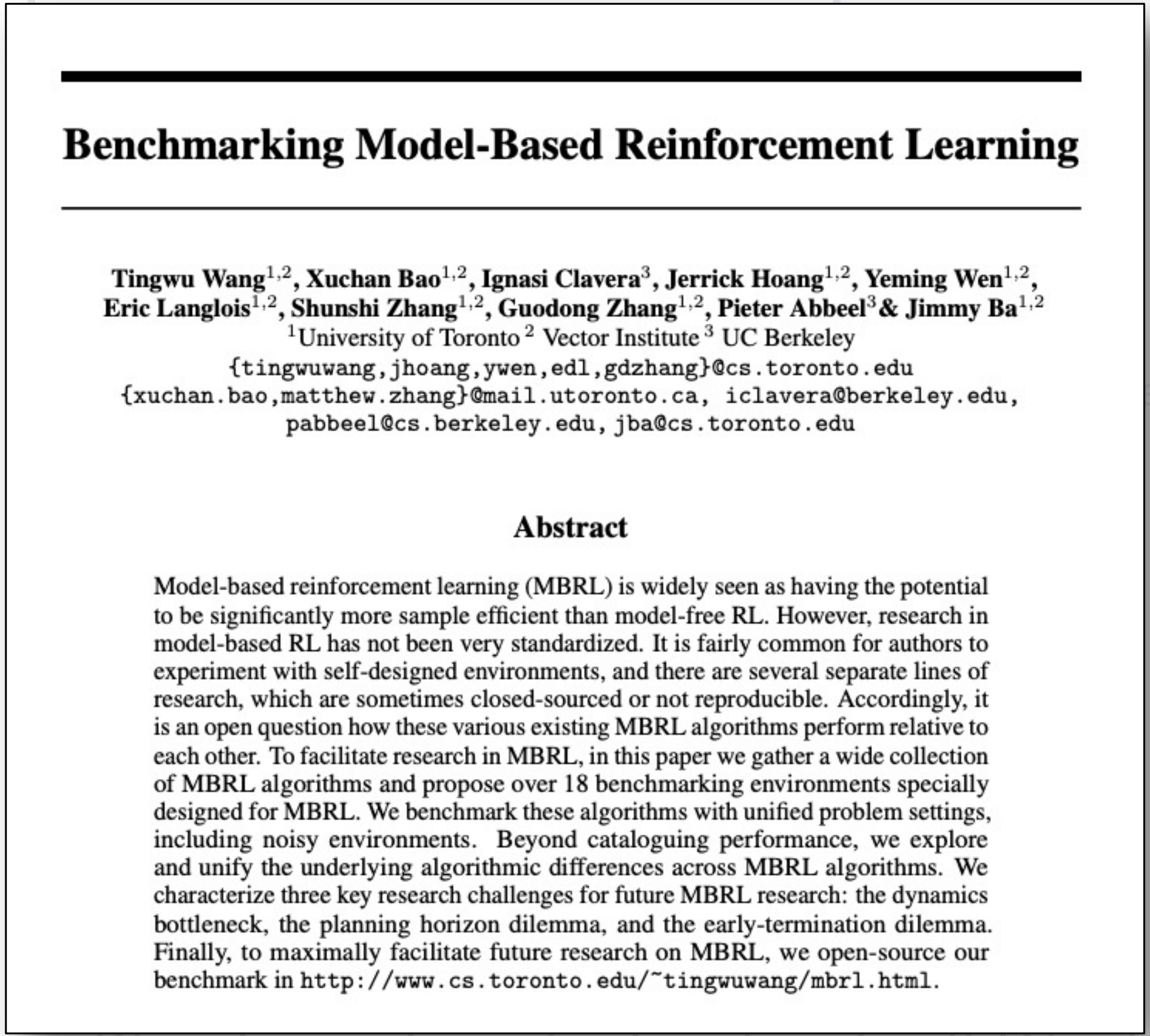
u_k

What can go wrong with model-based RL?

- The prediction horizon is short → need a value function and safe set
- Planning trajectories is suboptimal! We should plan over policies

Why should I care?

Model-based RL



Benchmarking Model-Based Reinforcement Learning

Tingwu Wang[1,2], Xuchan Bao[1,2], Ignasi Clavera[3], Jerrick Hoang[1,2], Yeming Wen[1,2], Eric Langlois[1,2], Shunshi Zhang[1,2], Guodong Zhang[1,2], Pieter Abbeel[3] & Jimmy Ba[1,2]
[1]University of Toronto [2] Vector Institute [3] UC Berkeley
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Abstract

Model-based reinforcement learning (MBRL) is widely seen as having the potential to be significantly more sample efficient than model-free RL. However, research in model-based RL has not been very standardized. It is fairly common for authors to experiment with self-designed environments, and there are several separate lines of research, which are sometimes closed-sourced or not reproducible. Accordingly, it is an open question how these various existing MBRL algorithms perform relative to each other. To facilitate research in MBRL, in this paper we gather a wide collection of MBRL algorithms and propose over 18 benchmarking environments specially designed for MBRL. We benchmark these algorithms with unified problem settings, including noisy environments. Beyond cataloguing performance, we explore and unify the underlying algorithmic differences across MBRL algorithms. We characterize three key research challenges for future MBRL research: the dynamics bottleneck, the planning horizon dilemma, and the early-termination dilemma. Finally, to maximally facilitate future research on MBRL, we open-source our benchmark in http://www.cs.toronto.edu/~tingwuwang/mbrl.html.

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Model-Based Reinforcement Learning via Meta-Policy Optimization

Ignasi Clavera* — UC Berkeley — iclavera@berkeley.edu
Jonas Rothfuss* — KIT, UC Berkeley — jonas.rothfuss@kit.edu
John Schulman — OpenAI
Yasuhiro Fujita — Preferred Networks
Tamim Asfour — Karlsruhe Inst. of Technology (KIT)
Pieter Abbeel — UC Berkeley, Covariant.AI

Abstract: Model-based reinforcement learning approaches carry the promise of being data efficient. However, due to challenges in learning dynamics models that sufficiently match the real-world dynamics, they struggle to achieve the same asymptotic performance as model-free methods. We propose Model-Based Meta-Policy-Optimization (MB-MPO), an approach that foregoes the strong reliance on accurate learned dynamics models. Using an ensemble of learned dynamic models, MB-MPO meta-learns a policy that can quickly adapt to any model in the ensemble with one policy gradient step. This steers the meta-policy towards internalizing consistent dynamics predictions among the ensemble while shifting the burden of behaving optimally w.r.t. the model discrepancies towards the adaptation step. Our experiments show that MB-MPO is more robust to model imperfections than previous model-based approaches. Finally, we demonstrate that our approach is able to match the asymptotic performance of model-free methods while requiring significantly less experience.

Keywords: Reinforcement Learning, Meta-Learning, Model-Based, Model-Free

► Planning trajectories is suboptimal! We should plan over policies

Why should I care?

Model-based RL

Model-free RL

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Keywords: Reinforcement Learning, Meta-Learning, Model-Based, Model-Free

- Planning trajectories is suboptimal! We should plan over policies

How do we fix this problem?

How do we fix this problem?

Statement 2: Under the assumptions of this theorem for any $0 \le t_0 \le t$

$$\Phi_{\tilde{u}}(t) - \Phi_{\tilde{u}}(t_0) \ge \Phi_{\tilde{u}^*}(t) - \Phi_{\tilde{u}^*}(t_0) = -\frac{1}{h}\Delta\vartheta^T(t)L^{-1}\Delta\vartheta(t) + o_\omega(h)$$

where

$$\Phi_{\tilde{u}}(t) := \int_0^t (2\tilde{u}^T(\tau)d\vartheta(\tau) + \tilde{u}^T(\tau)L\tilde{u}(\tau)d\tau) \quad (20)$$

$$\Delta\vartheta(t) := B_{u0}^T P[B_{u0}(u_{comp}(t) - x^*(t))h + (K + A_0 C_0^+)\Delta y(t)], \quad \Delta y(t) = y(t) - y(t-h)$$

and this minimum is reachable for

$$\tilde{u}(t)dt = \tilde{u}^*(t)dt := -L^{-1}B_{u0}^T P[B_{u0}(u_{comp}(t) + x^*(t))dt + (K + A_0 C_0^+)dy(t)].$$

As a result, we have: $d\Phi_{\tilde{u}^*}(t) \le 0$ (in symbolic form).

Proof of Statement 2: Using the Euler–Maruyama's formula [3], [9] we obtain the following relation: $(h := t - t_0 \to 0)$: $\Phi_{\tilde{u}}(t) - \Phi_{\tilde{u}}(t_0) = 2\tilde{u}^T(t)\Delta\vartheta(t) + \tilde{u}^T(t)L\tilde{u}(t)h + o_\omega(h)$.

Minimizing then the right side for each fixed t, we derive

$$\min_{\tilde{u}(t)}[\Phi_{\tilde{u}}(t) - \Phi_{\tilde{u}}(t_0)] = \Phi_{\tilde{u}^*}(t) - \Phi_{\tilde{u}^*}(t_0) = -\frac{1}{h}\Delta\vartheta^T(t)L^{-1}\Delta\vartheta(t) + o_\omega(h) \le o_\omega(h). \quad (21)$$

Hence, taking into account the definition (20), we have $d\Phi_{\tilde{u}^*}(t) \le 0$. □

Rewriting (18) in differential form and taking into account that P is the solution of the Riccati equation (see Assumption 4 of this theorem) we finally derive

$$dV(e(t)) \overset{a.s.}{\le} [\varphi(t) + I(t)]dt + S^T(t)dw(t) - e^T(t)Q_c e(t)dt.$$

According to the Kronecker lemma, the Large Number law for martingale, and Skorohod lemma (see [2] and [4]), we obtain the result of the theorem. Details of this proof can be found in [13].

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Min–Max Feedback Model Predictive Control for Constrained Linear Systems

P. O. M. Scokaert and D. Q. Mayne

Abstract—**Min–max feedback formulations of model predictive ... are discussed, both in the fixed and variable horizon contexts. The ... schemes the authors discuss introduce, in the control optimizati... notion that feedback is present in the receding-horizon implemen... of the control. This leads to improved performance, compared to sta... model predictive control, and resolves the feasibility difficulties tha... with the min–max techniques that are documented in the lite... The stabilizing properties of the methods are discussed as well as... practical implementation details.**

Index Terms—**Feedback, min–max optimization, model predictiv... trol.**

I. INTRODUCTION

Model predictive control (MPC) is a control methodolog... is becoming mature. The basic aspects of the method are... well understood, and stabilizing formulations of the control la... documented in the literature, both for linear and nonlinear pro... [1], [2].

The MPC strategy optimizes an open-loop control sequen... each sample, to minimize a nominal cost function, subject to... state and input constraints. The optimization is usually bas... the assumption that the process model is exact and that ... disturbances are constant. Because the control law ignores the ... of possible future changes in disturbance and model mis... closed-loop performance can be poor [3], with likely violatio... the constraints, when disturbances or model mismatch are pres...

Some formulations of MPC have been proposed that address... issues [4], [5]. These methods rely on a min–max optimizat... predicted performance. However, these formulations optimize a... control profile over all possible disturbance (or model mis...

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Automatica 42 (2006) 523–533

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Optimization over state feedback policies for robust control with constr...

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[b] Department of Aeronautics and Department of Electrical and Electronic Engineering, Imperial College London, Exhibition Road, London S...

Received 15 March 2005; received in revised form 8 June 2005; accepted 25 August 2005

Abstract

This paper is concerned with the optimal control of linear discrete-time systems subject to unknown but bounded state distu... mixed polytopic constraints on the state and input. It is shown that the class of admissible affine state feedback control policies wit... of prior states is equivalent to the class of admissible feedback policies that are affine functions of the past disturbance sequence. ... that a broad class of constrained finite horizon robust and optimal control problems, where the optimization is over affine st... policies, can be solved in a computationally efficient fashion using convex optimization methods. This equivalence result is used ... robust receding horizon control (RHC) state feedback policy such that the closed-loop system is input-to-state stable (ISS) and th... are satisfied for all time and all allowable disturbance sequences. The cost to be minimized in the associated finite horizon op... problem is quadratic in the disturbance-free state and input sequences. The value of the receding horizon control law can be calcu... sample instant using a single, tractable and convex quadratic program (QP) if the disturbance set is polytopic, or a tractable secon... program (SOCP) if the disturbance set is given by a 2-norm bound.

Keywords: Robust control; Constraint satisfaction; Robust optimization; Predictive control; Optimal control

1. Introduction

This paper is concerned with the control of constrained discrete-time linear systems that are subject to additive, but bounded disturbances on the state. The main aim is to provide results that allow for the efficient computation of an optimal and stabilizing state feedback control policy that ensures a given set of state and input constraints are satisfied for all time, despite the presence of the disturbances. This is a problem that has been studied for some time now in the optimal control literature (Bertsekas & Rhodes, 1973) and a number of different ...

& Diaz-Bobillo, 1995) or predictive control (M... 2002; Mayne, Rawlings, Rao, & Scokaert, 2000).

It is generally accepted that if disturbances ... accounted for in the formulation of a constrain... control problem, then the optimization has to be... admissible state feedback policies, rather than open... sequences, otherwise infeasibility and instability pr... occur (Mayne et al., 2000). However, optimizatio... trary (nonlinear) feedback policies is particularly... constraints have to be satisfied. Current proposals fo... this using finite dimensional optimization, such ...

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Brief paper

Robust model predictive control of constrained linear systems with bounded disturbances☆

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Received 9 February 2004; received in revised form 14 July 2004; accepted 23 August 2004

Available online 8 December 2004

Abstract

This paper provides a novel solution to the problem of robust model predictive control of constrained, linear, discrete-time systems in the presence of bounded disturbances. The optimal control problem that is solved online includes, uniquely, the initial state of the model employed in the problem as a decision variable. The associated value function is zero in a disturbance invariant set that serves as the 'origin' when bounded disturbances are present, and permits a strong stability result, namely robust exponential stability of the disturbance invariant set for the controlled system with bounded disturbances, to be obtained. The resultant online algorithm is a quadratic program of similar complexity to that required in conventional model predictive control.

Keywords: Robust model predictive control; Robustness; Bounded disturbances

1. Introduction

Model predictive control is widely employed for the control of constrained systems and an extensive literature on the subject exists some of which is reviewed in Bemporad and Morari (1999); Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scokaert (2000). Several methods for achieving robustness have been considered. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scokaert & Rawlings, 1995; Marruedo, Álamo, & Camacho, 2002). *Open-loop* model predictive control that determines the ...

$\{u_0, u_1, \ldots, u_{N-1}\}$ of control *actions* was proposed in Zheng and Morari (1993); this method cannot contain the 'spread' of predicted trajectories resulting from disturbances. Hence *feedback* model predictive control in which the decision variable is a *policy* π, which is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ of control *laws*, was advocated in, for example, Mayne (1995); Kothare, Balakrishnan, and Morari (1996); Lee and Yu (1997); Scokaert and Mayne (1998); De Nicolao, Magni, and Scattolini (2000); Magni, Nijmeijer, and van der Schaft (2001); Magni, De Nicolao, Scattolini, and Allgöwer (2003). Determination of a control policy is usually prohibitively difficult so research has con...

How do we fix this problem?

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where

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$$\Delta\vartheta(t) := B_{u0}^T P[B_{u0}(u_{\text{comp}}(t) - x^*(t))h + (K + A_0C_0^+)\Delta y(t)], \quad \Delta y(t) = y(t) - y(t-h)$$

and this minimum is reachable for

$$\bar{u}(t)\,dt = \bar{u}^*(t)\,dt := -L^{-1}B_{u0}^T P[B_{u0}(u_{\text{comp}}(t) + x^*(t))\,dt + (K + A_0C_0^+)\,dy(t)].$$

As a result, we have: $d\Phi_{\bar{u}^*}(t) \le 0$ (in symbolic form).

Proof of Statement 2: Using the Euler–Maruyama's formula [3], [9] we obtain the following relation: $(h := t - t_0 \to 0)$: $\Phi_{\bar{u}}(t) - \Phi_{\bar{u}}(t_0) = 2\bar{u}^T(t)\Delta\vartheta(t) + \bar{u}^T(t)L\bar{u}(t)h + o_\omega(h)$. Minimizing then the right side for each fixed t, we derive

$$\min_{\bar{u}(t)} [\Phi_{\bar{u}}(t) - \Phi_{\bar{u}}(t_0)] = \Phi_{\bar{u}^*}(t) - \Phi_{\bar{u}^*}(t_0) = -\frac{1}{h}\Delta\vartheta^T(t)L^{-1}\Delta\vartheta(t) + o_\omega(h) \le o_\omega(h). \quad (21)$$

Hence, taking into account the definition (20), we have $d\Phi_{\bar{u}^*}(t) \le 0$. □

Rewriting (18) in differential form and taking into account that P is the solution of the Riccati equation (see Assumption 4 of this theorem) we finally derive

$$dV(e(t)) \overset{a.s.}{\le} [\varphi(t) + I(t)]\,dt + S^T(t)\,dw(t) - e^T(t)Q_e e(t)\,dt.$$

According to the Kronecker lemma, the Large Number law for martingale, and Skorohod lemma (see [2] and [4]), we obtain the result of the theorem. Details of this proof can be found in [13].

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Min–Max Feedback Model Predictive Control for Constrained Linear Systems

P. O. M. Scokaert and D. Q. Mayne

Abstract—**Min–max feedback formulations of model predictive ... are discussed, both in the fixed and variable horizon contexts. The ... schemes the authors discuss introduce, in the control optimizati... notion that feedback is present in the receding-horizon implemen... of the control. This leads to improved performance, compared to sta... model predictive control, and resolves the feasibility difficulties tha... with the min–max techniques that are documented in the liter... The stabilizing properties of the methods are discussed as well a... practical implementation details.**

Index Terms—**Feedback, min–max optimization, model predictiv... trol.**

I. Introduction

Model predictive control (MPC) is a control methodolog... is becoming mature. The basic aspects of the method are well understood, and stabilizing formulations of the control la... documented in the literature, both for linear and nonlinear pro... [1], [2].

The MPC strategy optimizes an open-loop control sequen... each sample, to minimize a nominal cost function, subject to state and input constraints. The optimization is usually bas... the assumption that the process model is exact and that disturbances are constant. Because the control law ignores the ... of possible future changes in disturbance and model mis... closed-loop performance can be poor [3], with likely violatio... the constraints, when disturbances or model mismatch are pres...

Some formulations of MPC have been proposed that address... issues [4], [5]. These methods rely on a min–max optimizat... predicted performance. However, these formulations optimize a... control profile over all possible disturbance (or model mis...

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Optimization over state feedback policies for robust control with constr...

Paul J. Goulart[a,*], Eric C. Kerrigan[b], Jan M. Maciejowski[a]
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[b] Department of Aeronautics and Department of Electrical and Electronic Engineering, Imperial College London, Exhibition Road, London S...

Received 15 March 2005; received in revised form 8 June 2005; accepted 25 August 2005

Abstract

This paper is concerned with the optimal control of linear discrete-time systems subject to unknown but bounded state distu... mixed polytopic constraints on the state and input. It is shown that the class of admissible affine state feedback control policies wi... of prior states is equivalent to the class of admissible feedback policies that are affine functions of the past disturbance sequence. ... that a broad class of constrained finite horizon robust and optimal control problems, where the optimization is over affine st... policies, can be solved in a computationally efficient fashion using convex optimization methods. This equivalence result is used... robust receding horizon control (RHC) state feedback policy such that the closed-loop system is input-to-state stable (ISS) and th... are satisfied for all time and all allowable disturbance sequences. The cost to be minimized in the associated finite horizon op... problem is quadratic in the disturbance-free state and input sequences. The value of the receding horizon control law can be calcu... sample instant using a single, tractable and convex quadratic program (QP) if the disturbance set is polytopic, or a tractable secon... program (SOCP) if the disturbance set is given by a 2-norm bound.

Keywords: Robust control; Constraint satisfaction; Robust optimization; Predictive control; Optimal control

1. Introduction

This paper is concerned with the control of constrained discrete-time linear systems that are subject to additive, but bounded disturbances on the state. The main aim is to provide results that allow for the efficient computation of an optimal and stabilizing state feedback control policy that ensures a given set of state and input constraints are satisfied for all time, despite the presence of the disturbances. This is a problem that has been studied for some time now in the optimal control literature (Bertsekas & Rhodes, 1973) and a number of different ...

... & Diaz-Bobillo, 1995) or predictive control (M... 2002; Mayne, Rawlings, Rao, & Scokaert, 2000).

It is generally accepted that if disturbances ... accounted for in the formulation of a constrain... control problem, then the optimization has to be... admissible state feedback policies, rather than open... sequences, otherwise infeasibility and instability pr... occur (Mayne et al., 2000). However, optimization... trary (nonlinear) feedback policies is particularly... constraints have to be satisfied. Current proposals fo... this using finite dimensional optimization, such ...

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Brief paper

Robust model predictive control of constrained linear systems with bounded disturbances☆

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Abstract

This paper provides a novel solution to the problem of robust model predictive control of constrained, linear, discrete-time systems in the presence of bounded disturbances. The optimal control problem that is solved online includes, uniquely, the initial state of the model employed in the problem as a decision variable. The associated value function is zero in a disturbance invariant set that serves as the 'origin' when bounded disturbances are present, and permits a strong stability result, namely robust exponential stability of the disturbance invariant set for the controlled system with bounded disturbances, to be obtained. The resultant online algorithm is a quadratic program of similar complexity to that required in conventional model predictive control.

Keywords: Robust model predictive control; Robustness; Bounded disturbances

1. Introduction

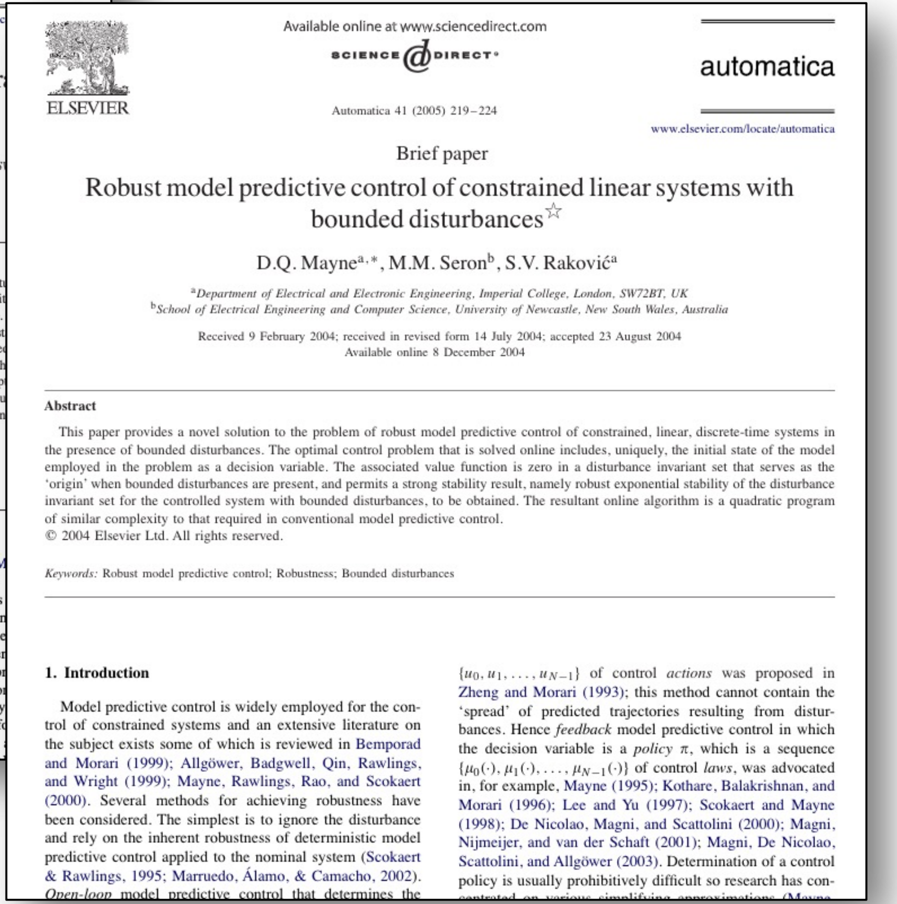
Model predictive control is widely employed for the control of constrained systems and an extensive literature on the subject exists some of which is reviewed in Bemporad and Morari (1999); Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scokaert (2000). Several methods for achieving robustness have been considered. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scokaert & Rawlings, 1995; Marruedo, Álamo, & Camacho, 2002). *Open-loop* model predictive control that determines the ...

$\{u_0, u_1, \ldots, u_{N-1}\}$ of control *actions* was proposed in Zheng and Morari (1993); this method cannot contain the 'spread' of predicted trajectories resulting from disturbances. Hence *feedback* model predictive control in which the decision variable is a *policy* π, which is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ of control *laws*, was advocated in, for example, Mayne (1995); Kothare, Balakrishnan, and Morari (1996); Lee and Yu (1997); Scokaert and Mayne (1998); De Nicolao, Magni, and Scattolini (2000); Magni, Nijmeijer, and van der Schaft (2001); Magni, De Nicolao, Scattolini, and Allgöwer (2003). Determination of a control policy is usually prohibitively difficult so research has con...

The problem is NP-hard!

How do we fix this problem?

The problem is NP-hard!



An elegant approximation

How do we fix this problem?

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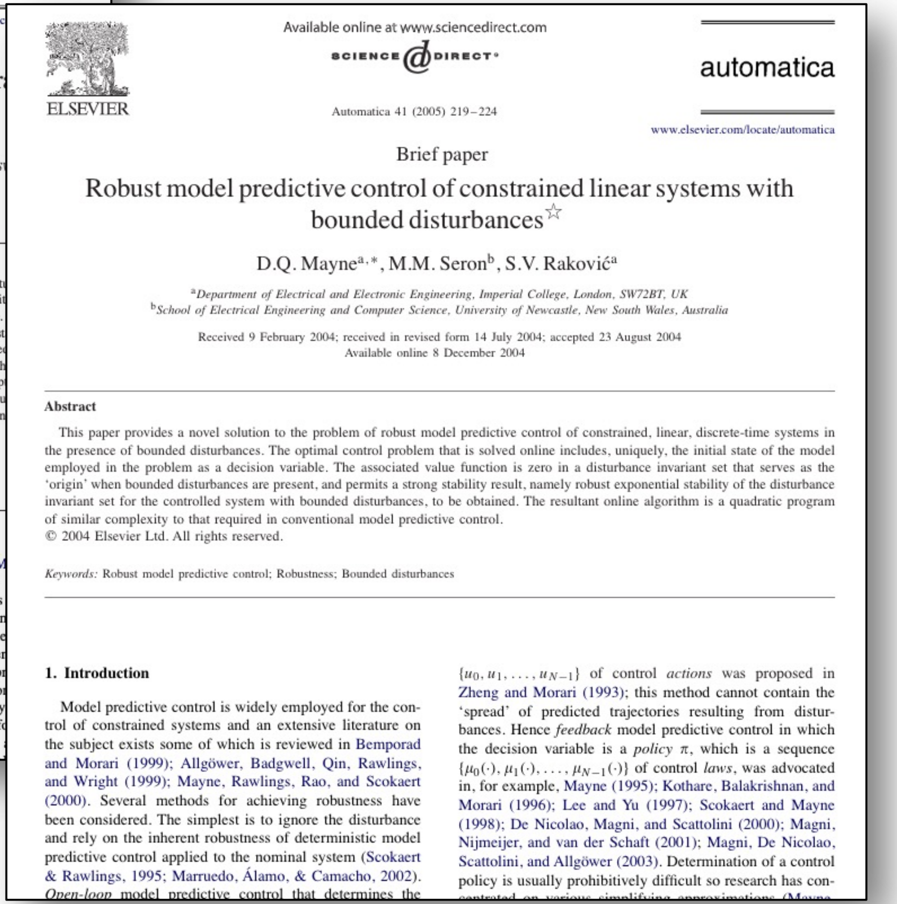
An elegant approximation



What people do in practice

How do we fix this problem?

The problem is NP-hard!

An elegant approximation

What people do in practice

Optimizing over policies

True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

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Assumption: the disturbance support $\mathcal{W} = \mathrm{conv}(w^{(0)}, \ldots, w^{(n_d)})$.

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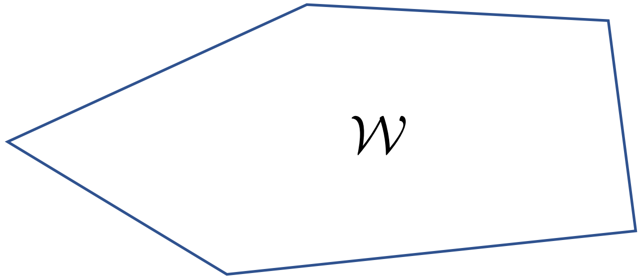

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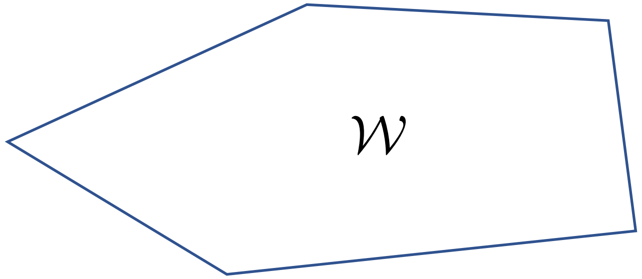

The problem that we want to solve

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\begin{aligned}
\min_{\pi} \quad & f(x(t), \pi, w_{t|t}, \ldots, w_{t+N|t}) \\
\text{subject to} \quad & x_{k+1|t} = Ax_{k|t} + Bu_{k|t} + w_{k|t} \\
& \text{robust constraint satisfaction} \\
& x_{t|t} = x(t) \\
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Optimization over policies

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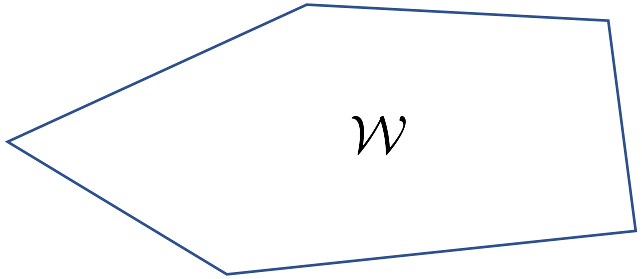

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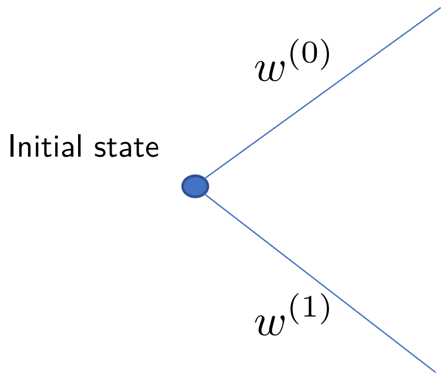
Optimization over policies

Finite dimensional reformulation

Optimize over a tree



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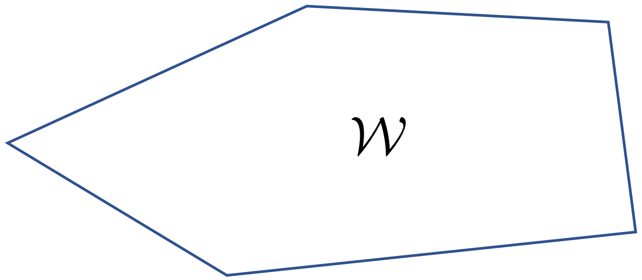

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Optimization over policies

Finite dimensional reformulation

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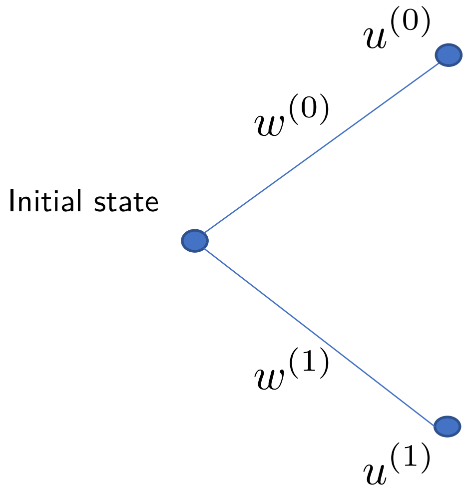

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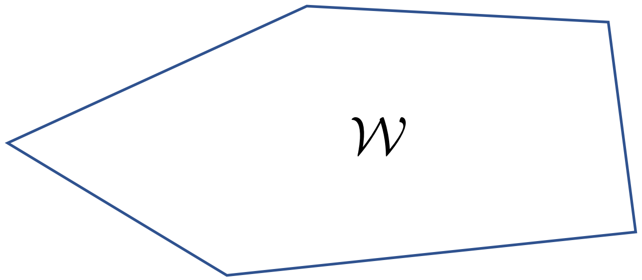

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Optimization over policies

Finite dimensional reformulation

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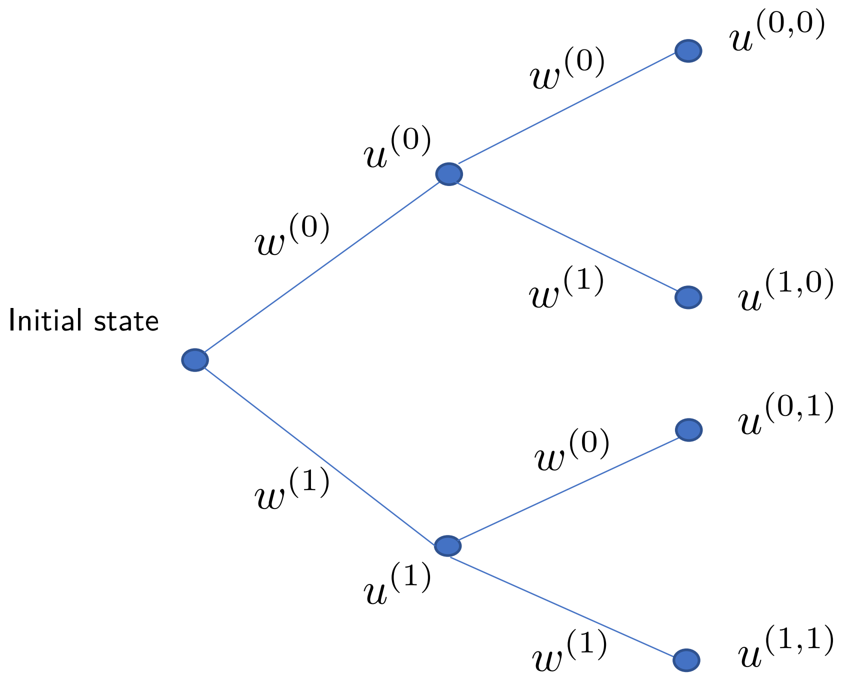

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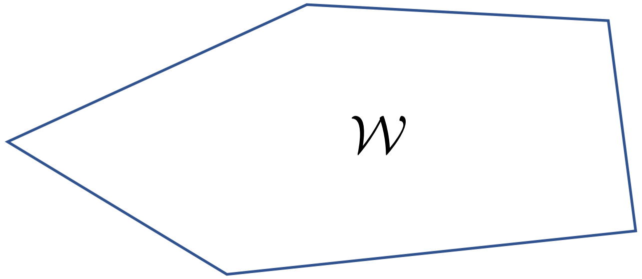

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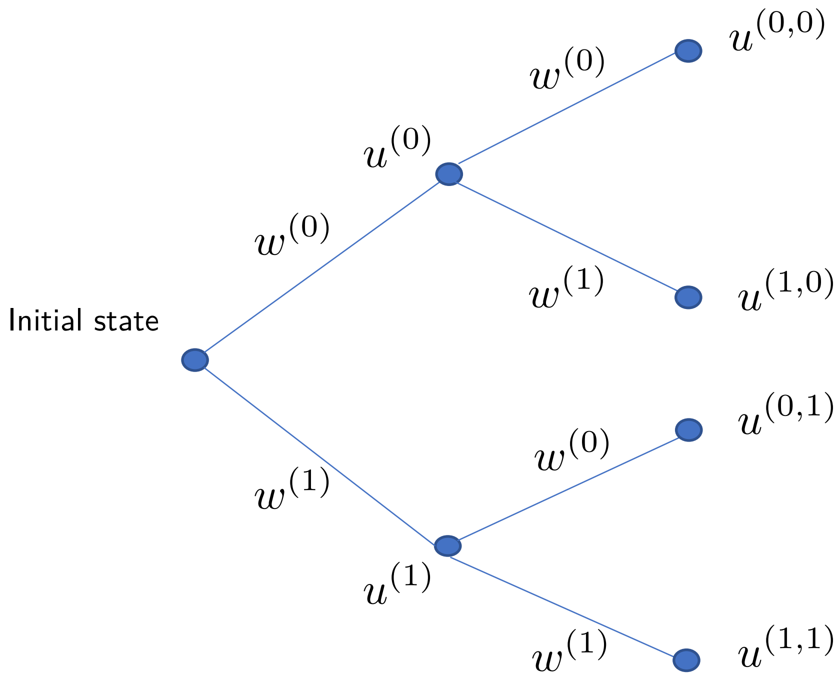
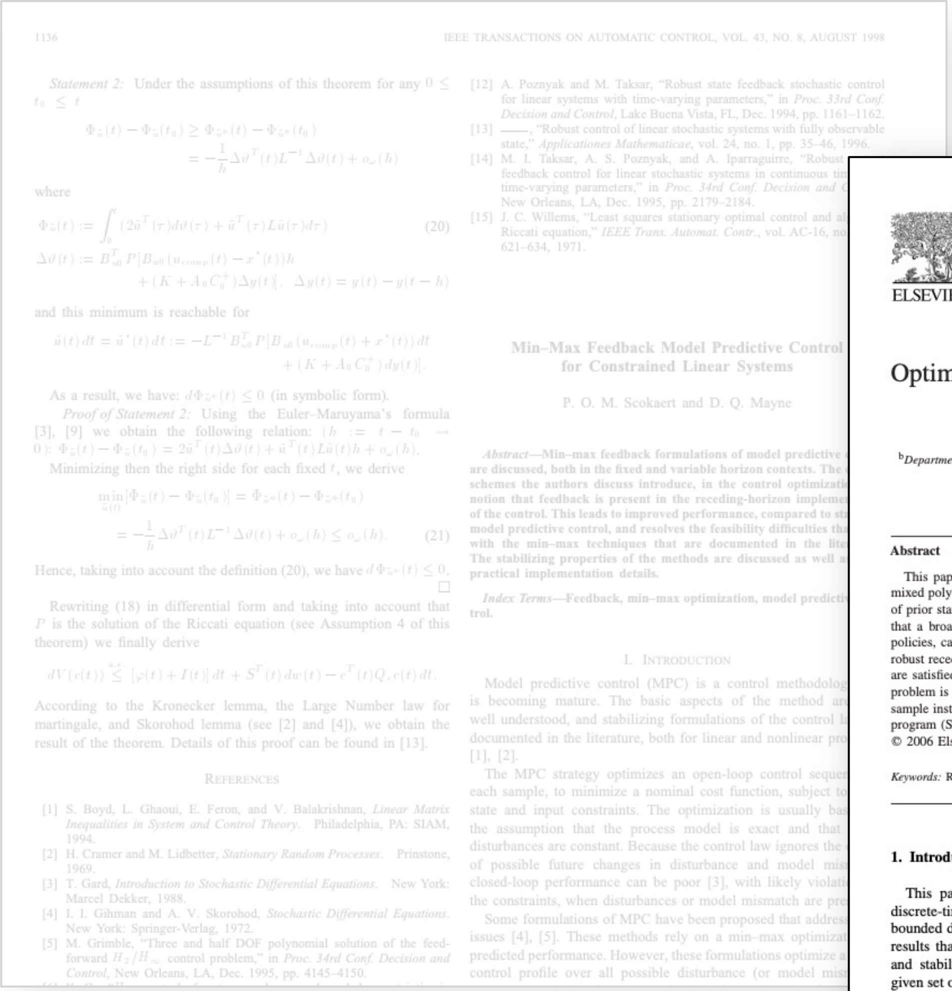


This approach can be used also when A and B are uncertain. For further details check:
Francesco Borrelli, Alberto Bemporad, and Manfred Morari. *Predictive control for linear and hybrid systems*. Cambridge University Press, 2017.

How do we fix this problem?



The problem is NP-hard!

An elegant approximation

What people do in practice

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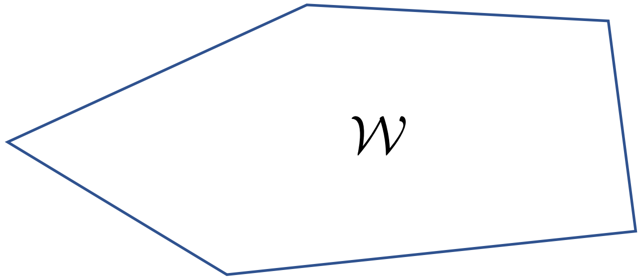

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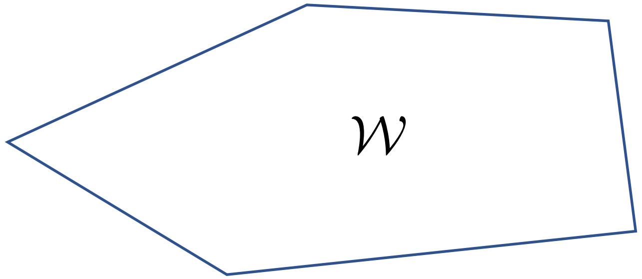

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Optimization over policies

Tractable approximation

Optimize over disturbance feedback policies

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Function of disturbances!

Optimizing over disturbance feedback policies

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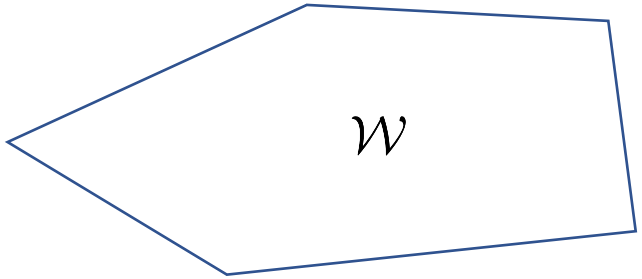

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Optimization over policies

Tractable approximation

Optimize over disturbance feedback policies

$$u_{k|t} = \pi(x(t), w_{t|t}, \ldots, w_{k-1|t})$$

Function of disturbances!

The problem is tractable when we pick:

$$u_{k|t} = M_{k|t} \begin{bmatrix} x(t) \\ w_{t|t} \\ \vdots \\ w_{k-1|t} \end{bmatrix}$$

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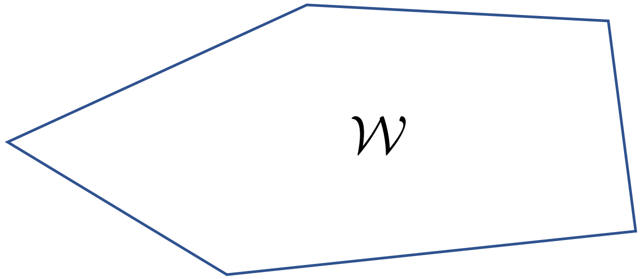

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Aharon Ben-Tal, Alexander Goryashko, Elana Guslitzer, and Arkadi Nemirovski. "Adjustable robust solutions of uncertain linear programs." *Mathematical programming* 99, no. 2 (2004): 351-376.

How do we fix this problem?

1136 IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 43, NO. 8, AUGUST 1998

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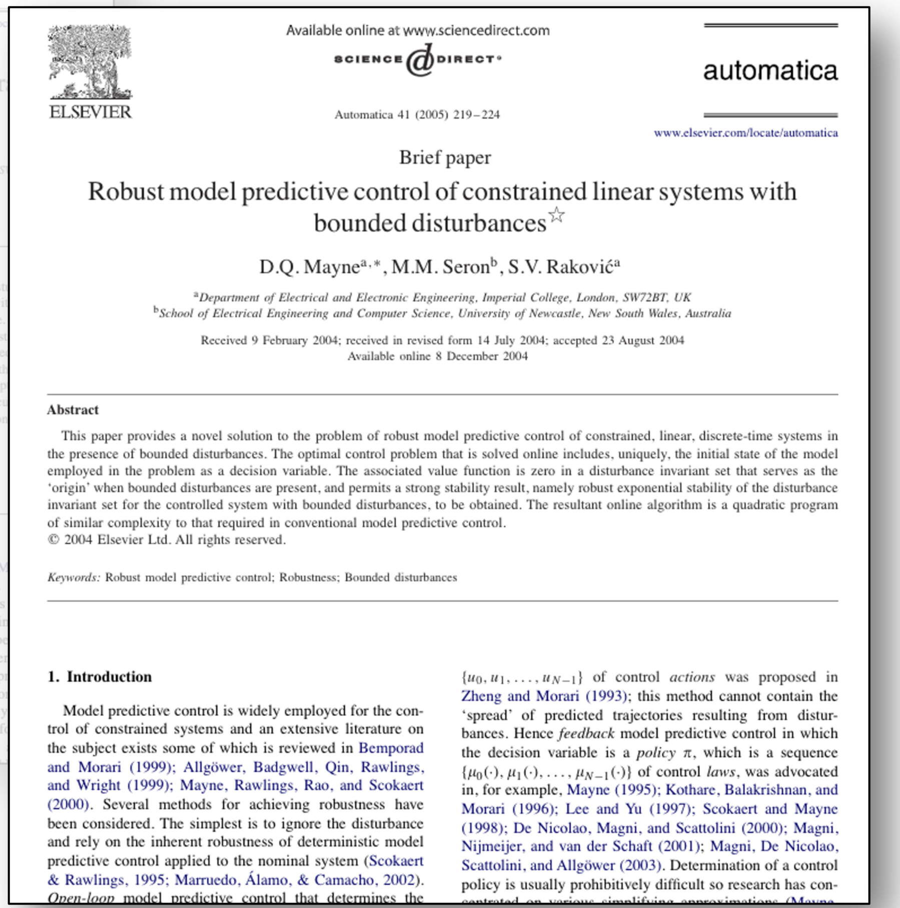
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Automatica 41 (2005) 219–224

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Brief paper

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D.Q. Mayne[a,*], M.M. Seron[b], S.V. Raković[a]

[a] Department of Electrical and Electronic Engineering, Imperial College, London, SW72BT, UK

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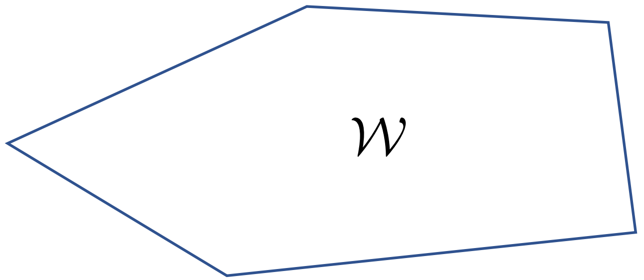
Keywords: Robust model predictive control; Robustness; Bounded disturbances

1. Introduction

Model predictive control is widely employed for the control of constrained systems and an extensive literature on the subject exists some of which is reviewed in Bemporad and Morari (1999); Allgöwer, Badgwell, Qin, Rawlings, and Wright (1999); Mayne, Rawlings, Rao, and Scokaert (2000). Several methods for achieving robustness have been considered. The simplest is to ignore the disturbance and rely on the inherent robustness of deterministic model predictive control applied to the nominal system (Scokaert & Rawlings, 1995; Marruedo, Álamo, & Camacho, 2002). *Open-loop* model predictive control that determines the $\{u_0, u_1, \ldots, u_{N-1}\}$ of control *actions* was proposed in Zheng and Morari (1993); this method cannot contain the 'spread' of predicted trajectories resulting from disturbances. Hence *feedback* model predictive control in which the decision variable is a *policy* π, which is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$ of control *laws*, was advocated in, for example, Mayne (1995); Kothare, Balakrishnan, and Morari (1996); Lee and Yu (1997); Scokaert and Mayne (1998); De Nicolao, Magni, and Scattolini (2000); Magni, Nijmeijer, and van der Schaft (2001); Magni, De Nicolao, Scattolini, and Allgöwer (2003). Determination of a control policy is usually prohibitively difficult so research has con…

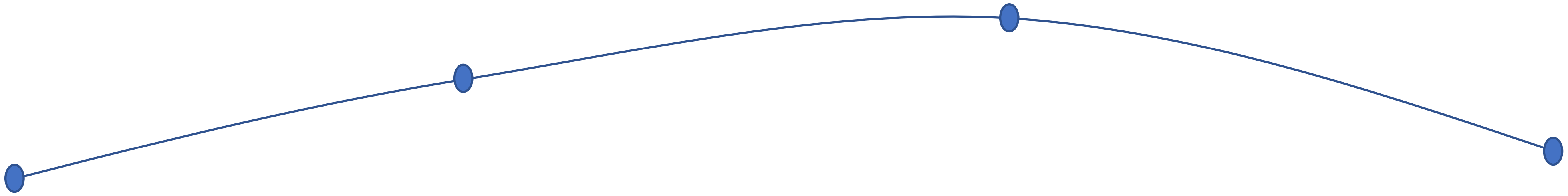
What people do in practice

Fixed tube robust MPC



True system dynamics: $x_{k+1} = Ax_k + Bu_k + w_k$

Assumption: the disturbance support $\mathcal{W} = \mathrm{conv}(w^{(0)}, \dots, w^{(n_d)})$.



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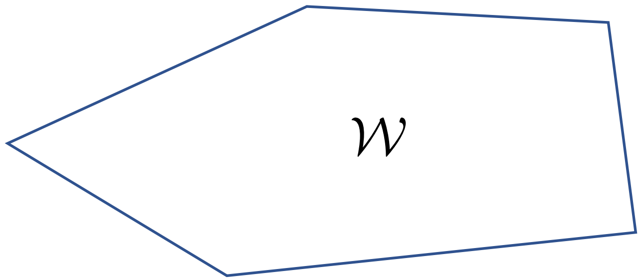

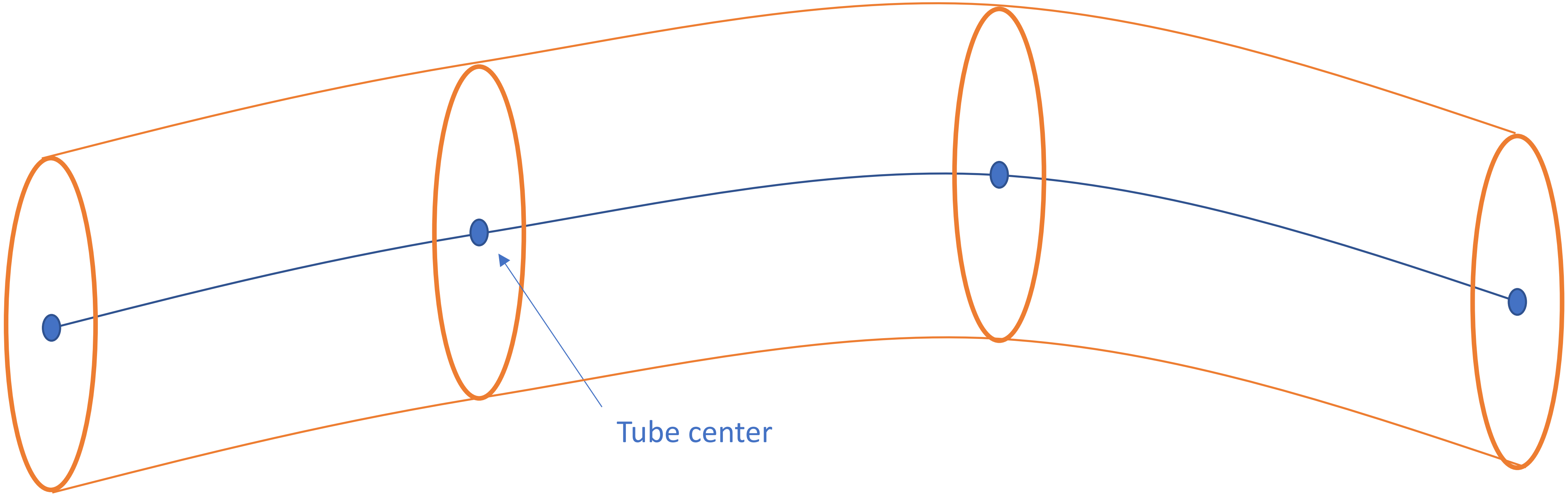

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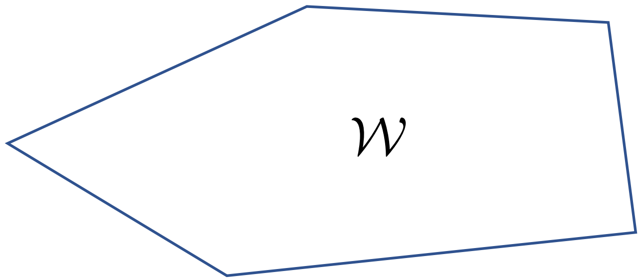

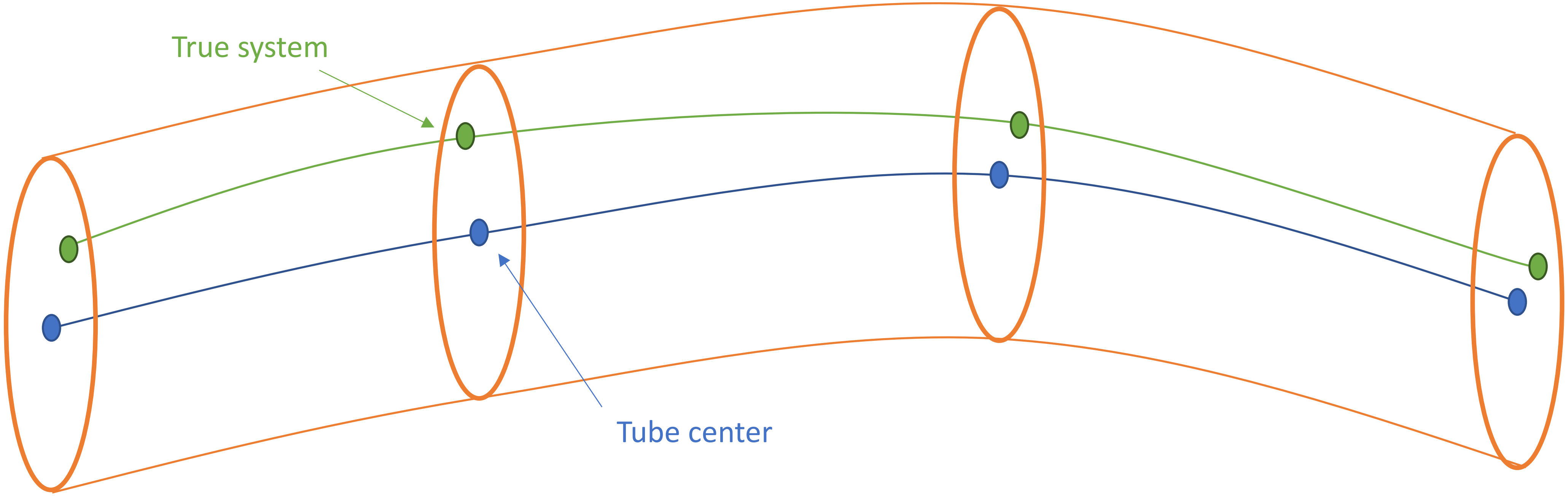

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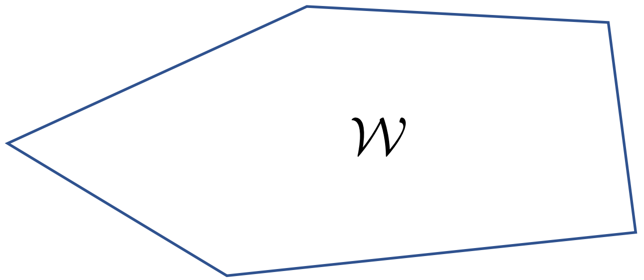

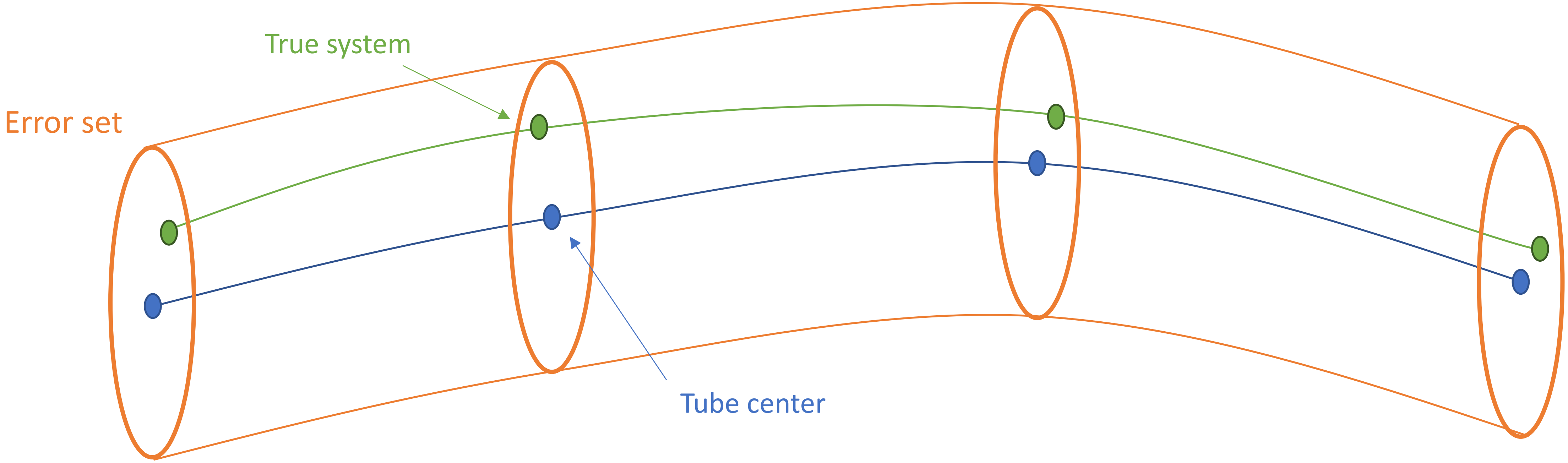

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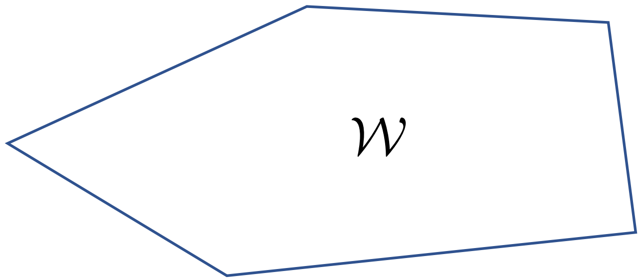

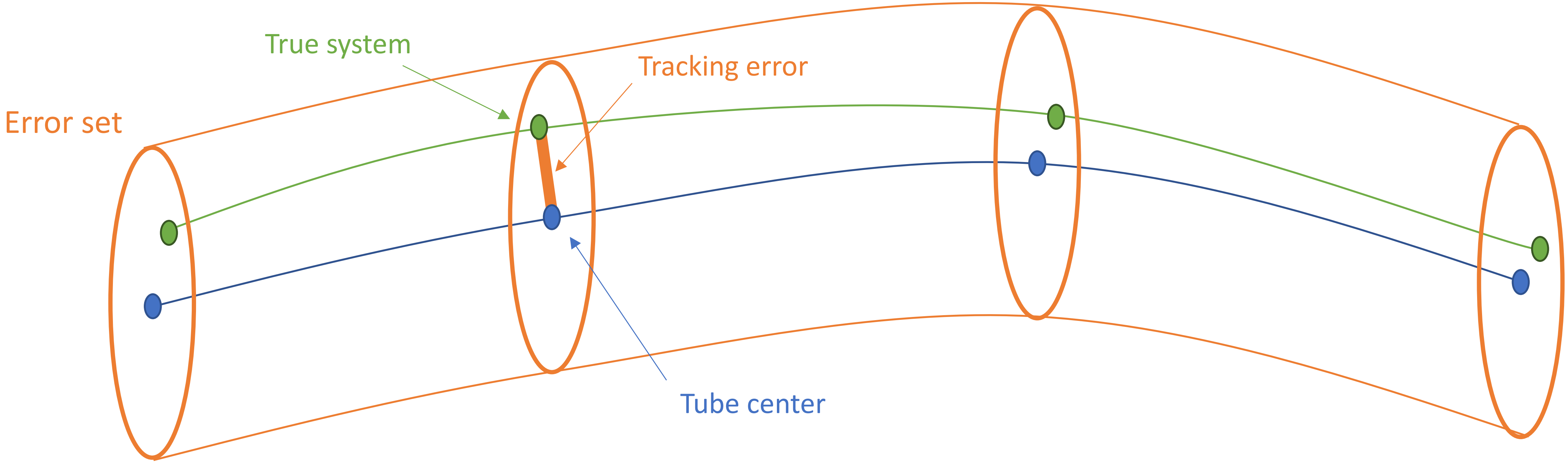

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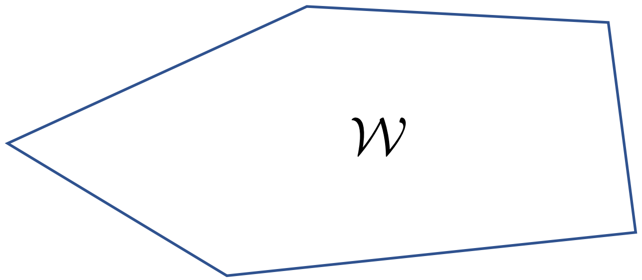

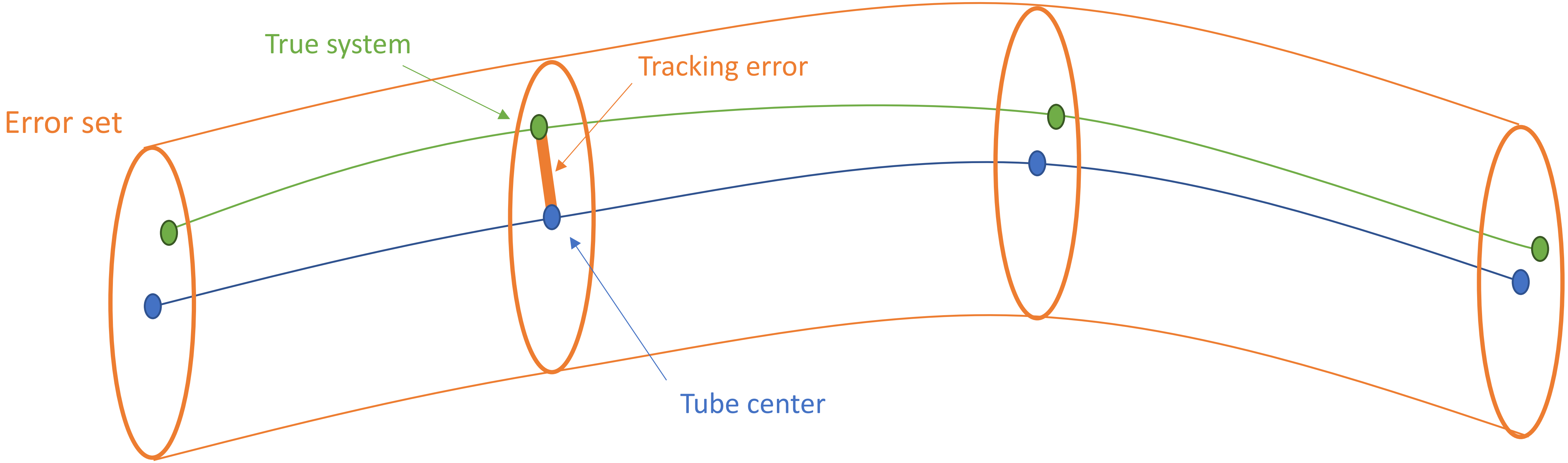

Key Ideas:

- Tube center planned via MPC
- Feedback to reduce for tracking error

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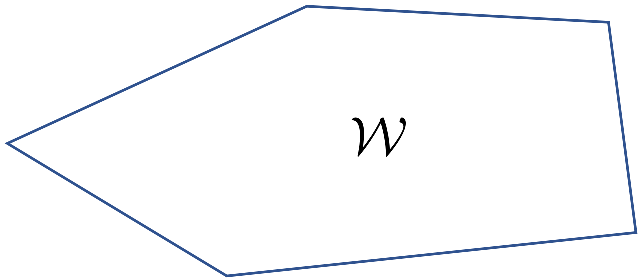
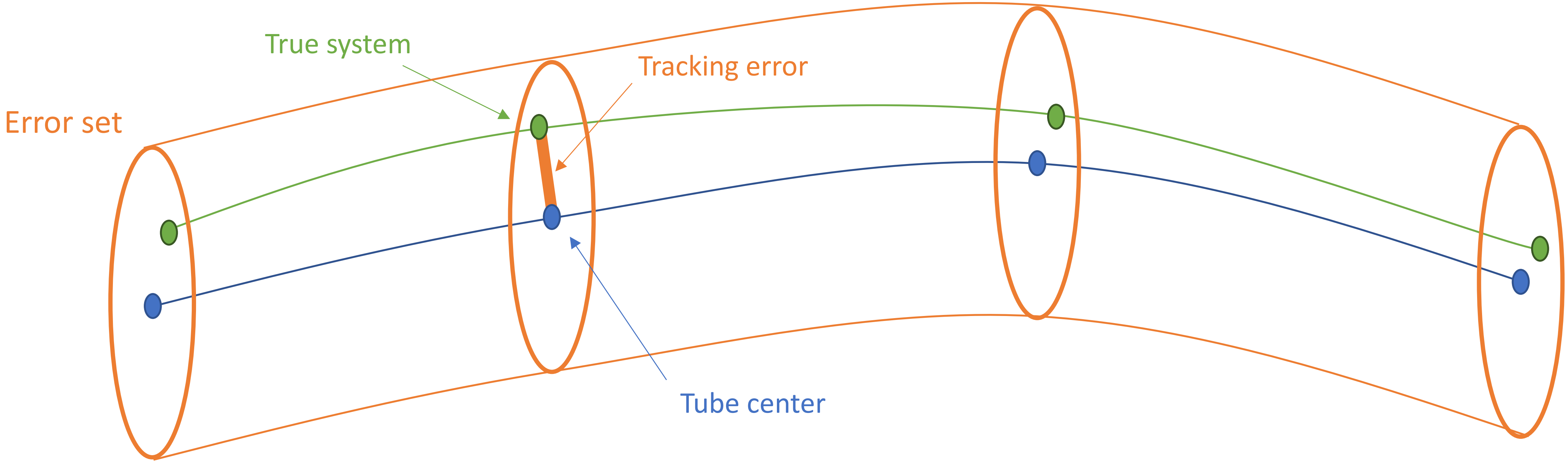




Key Ideas:

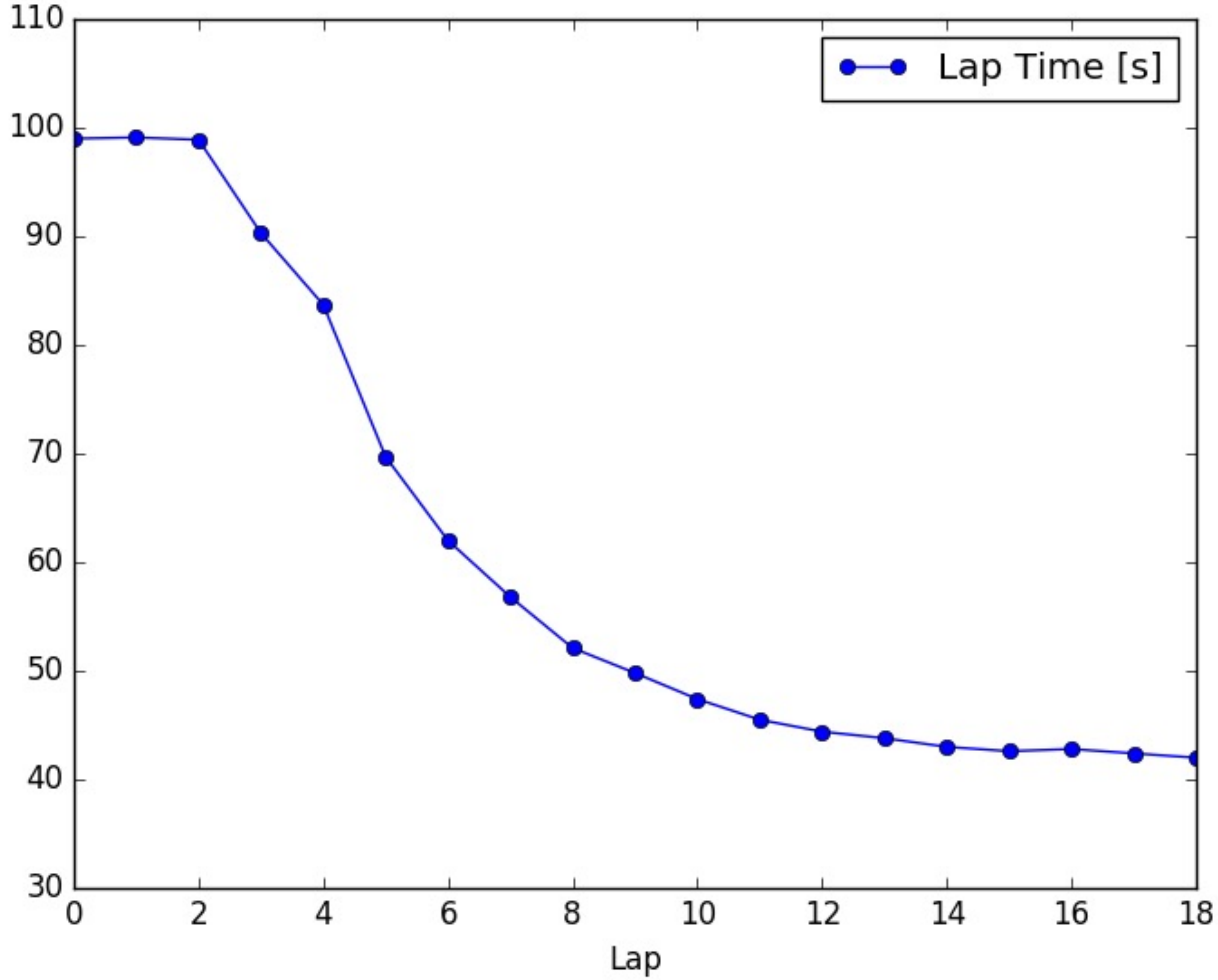
- Tube center planned via MPC
- Feedback to reduce for tracking error

$$\pi(x) = \pi^{\mathrm{MPC}}(x) + \pi^{\mathrm{tracking}}(x, \text{tube center})$$

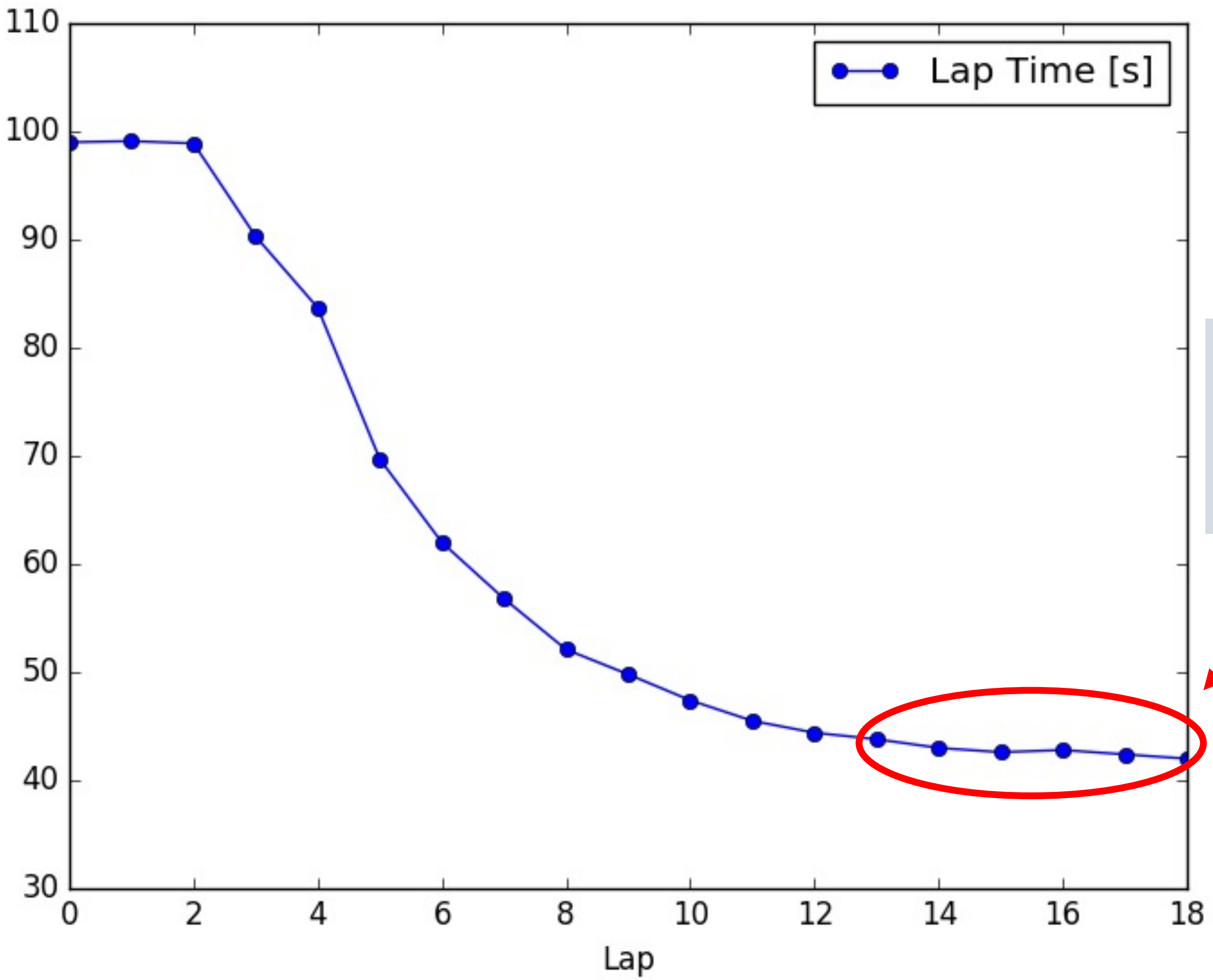
Model-free Control

How to reduce the computation load associated with MPC policies

Motivation: Lap Time Improvement

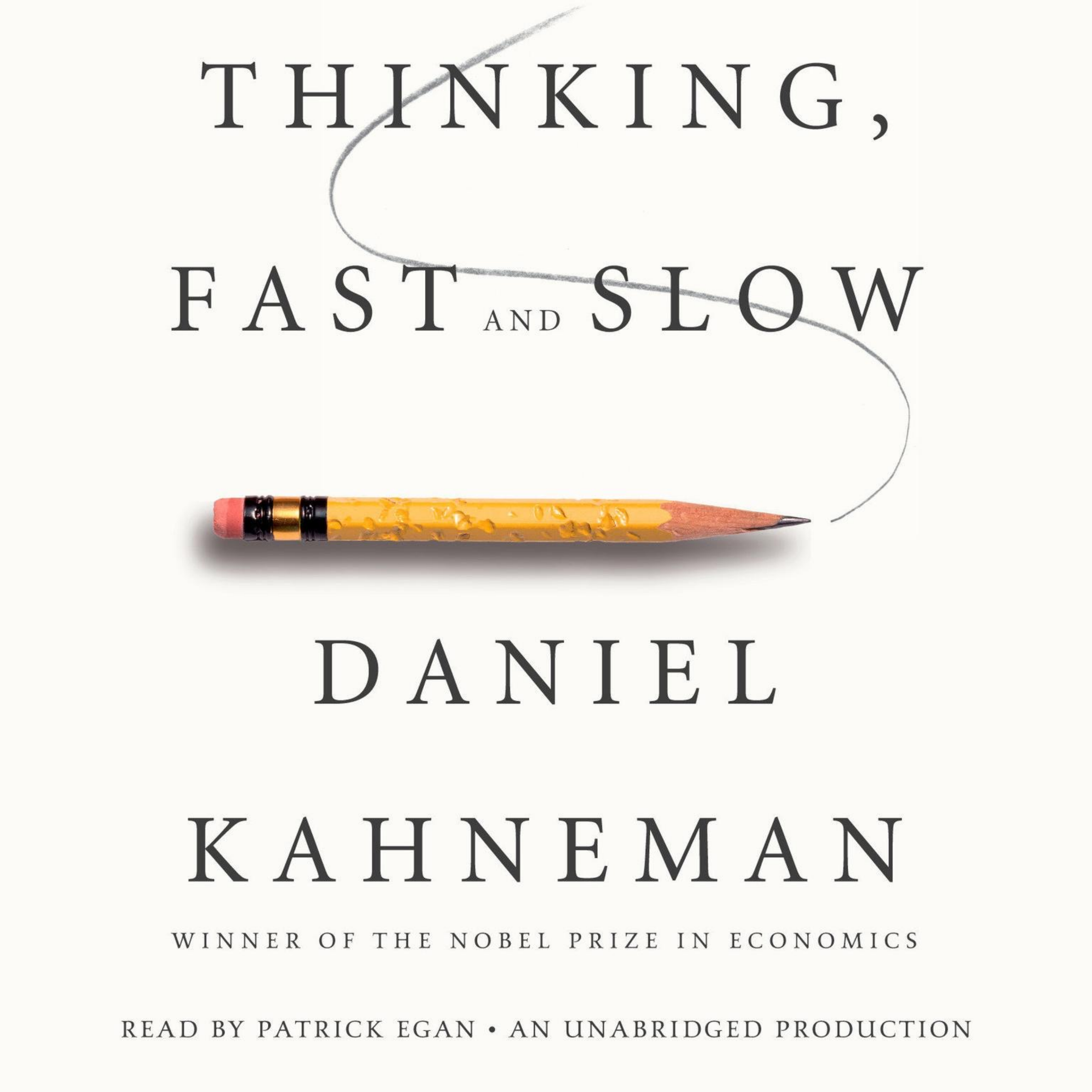


Goal: Reduce Computational Complexity

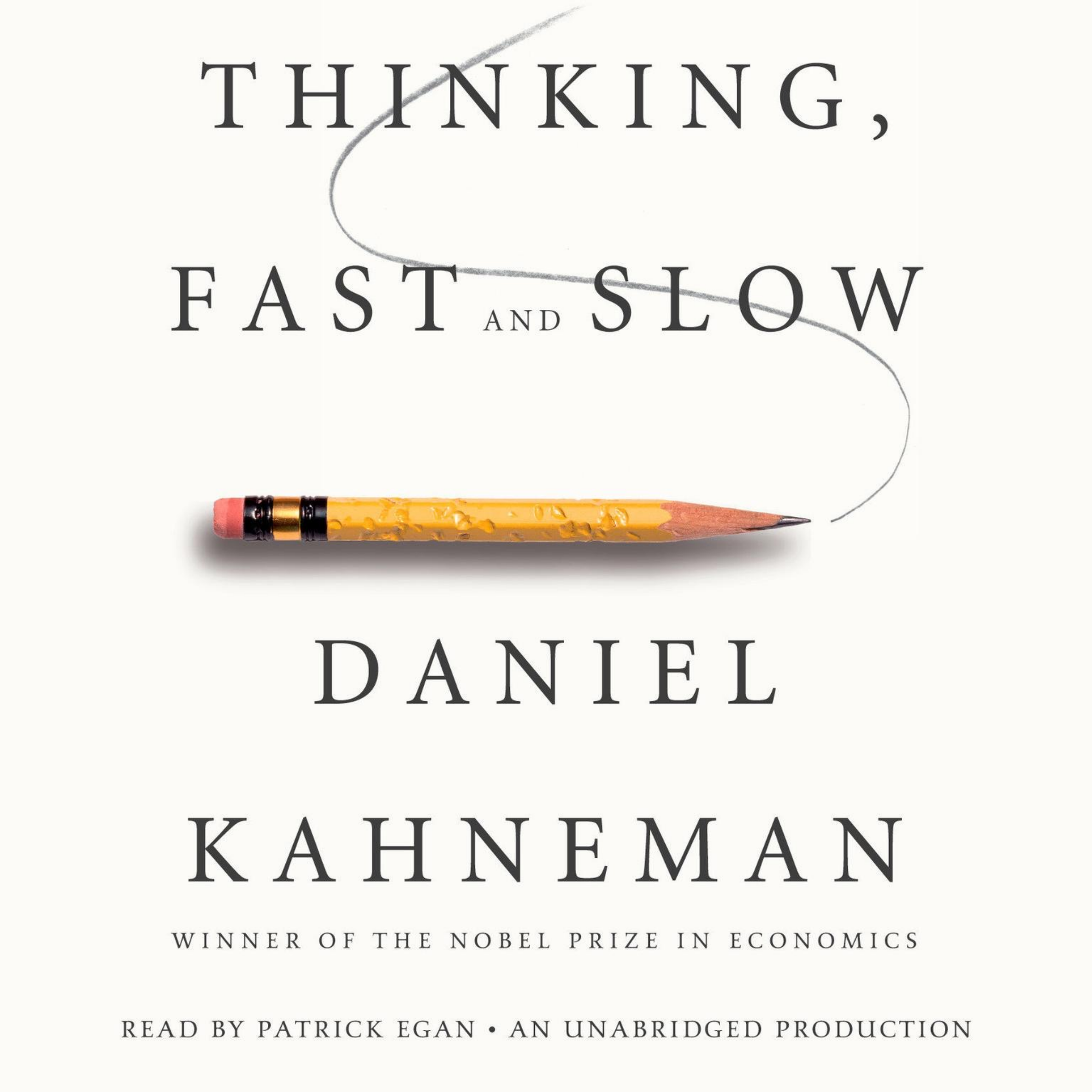


At convergence the **control policy** does not change

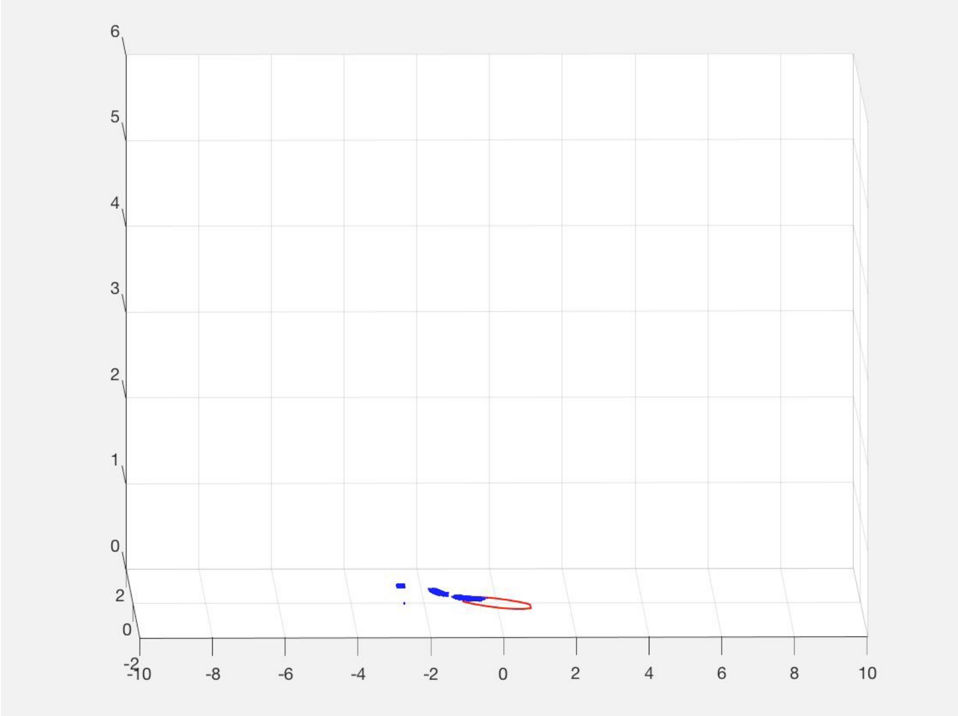
Learning as Computational Load Reduction



Learning as Computational Load Reduction

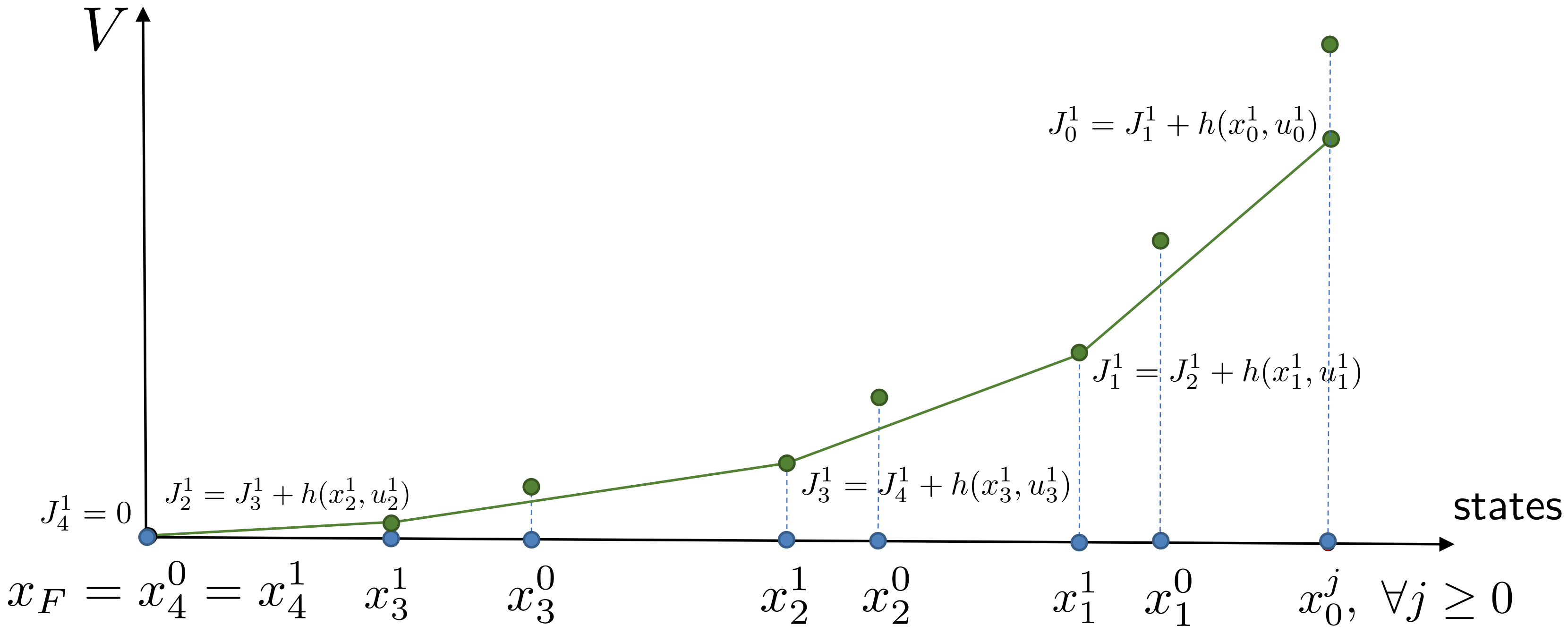


At convergence: the value function approximation does not change as we collect more data



Can we design a policy which used data and **"plays back"** the inputs that we computed with the expensive policy

Data-Based Policy



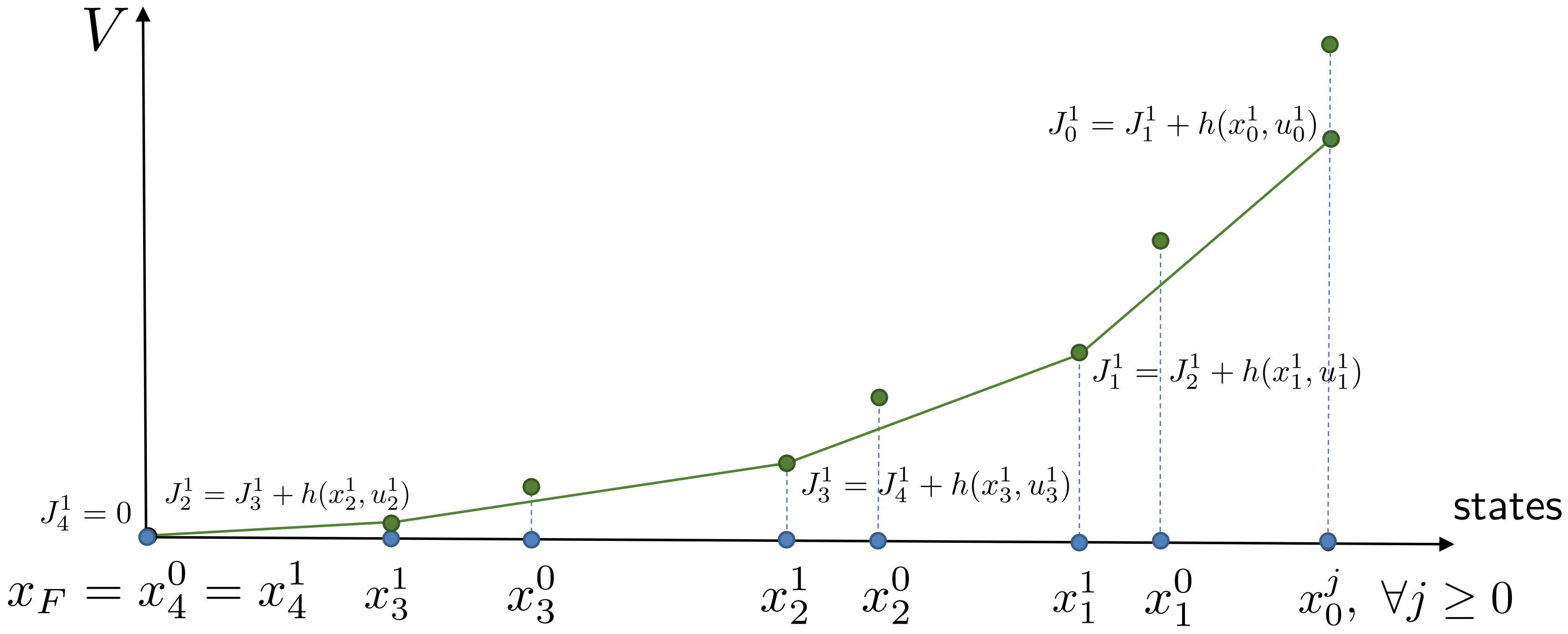
Multipliers Defining the Value Function Approximation

$$[\lambda_0^{0,*}, \ldots, \lambda_{T^j}^{T^j,*}] = \arg \min_{\lambda_i^j \in [0,1]} \quad \sum_i \sum_j J_i^j \lambda_i^j$$

$$\text{s.t} \quad \sum_i \sum_j x_i^j \lambda_i^j = x,$$

$$\sum_i \sum_j \lambda_i^j = 1$$

Data-Based Policy



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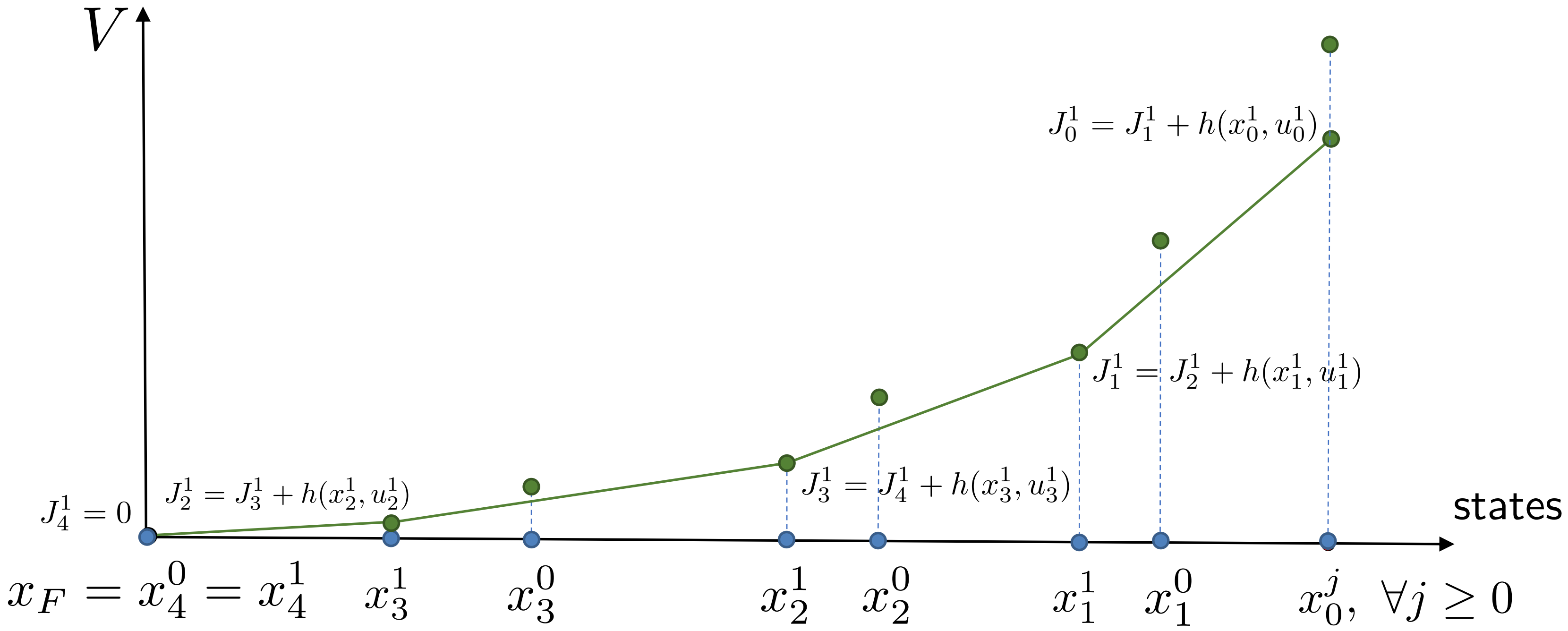

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Historical Data

Berkeley Autonomous Race Car Implementation

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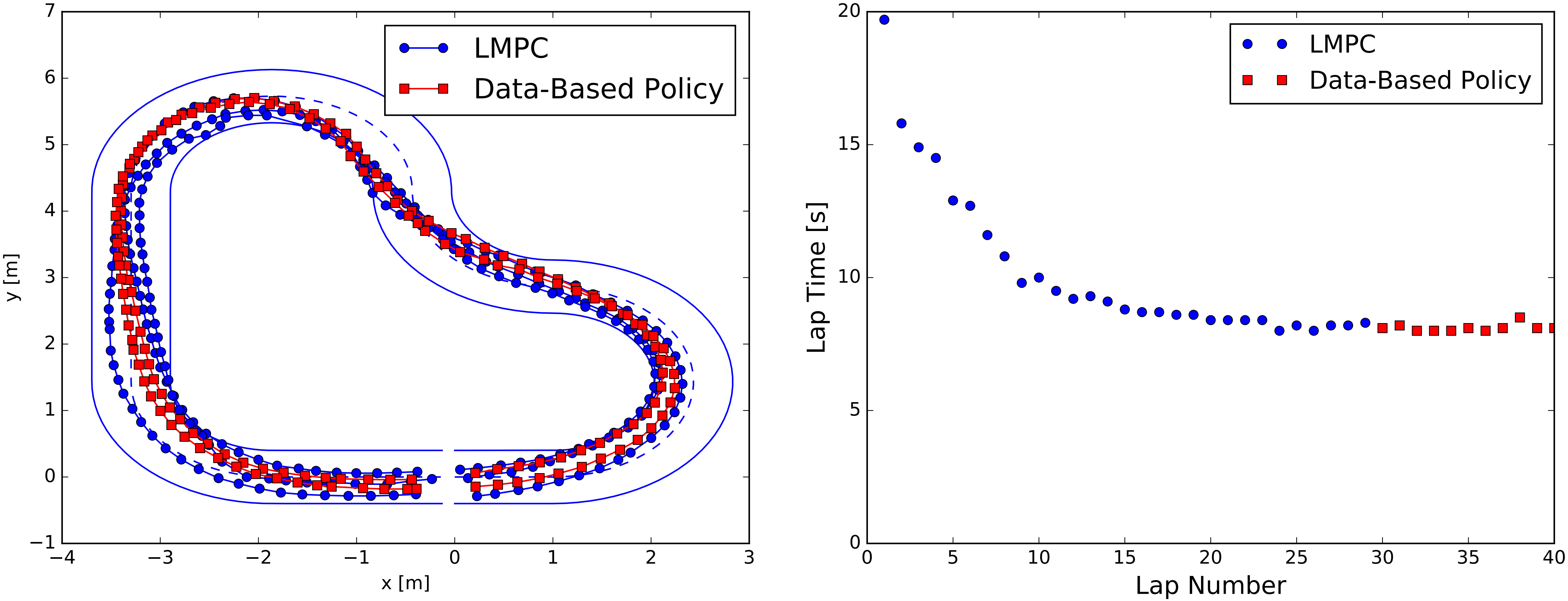
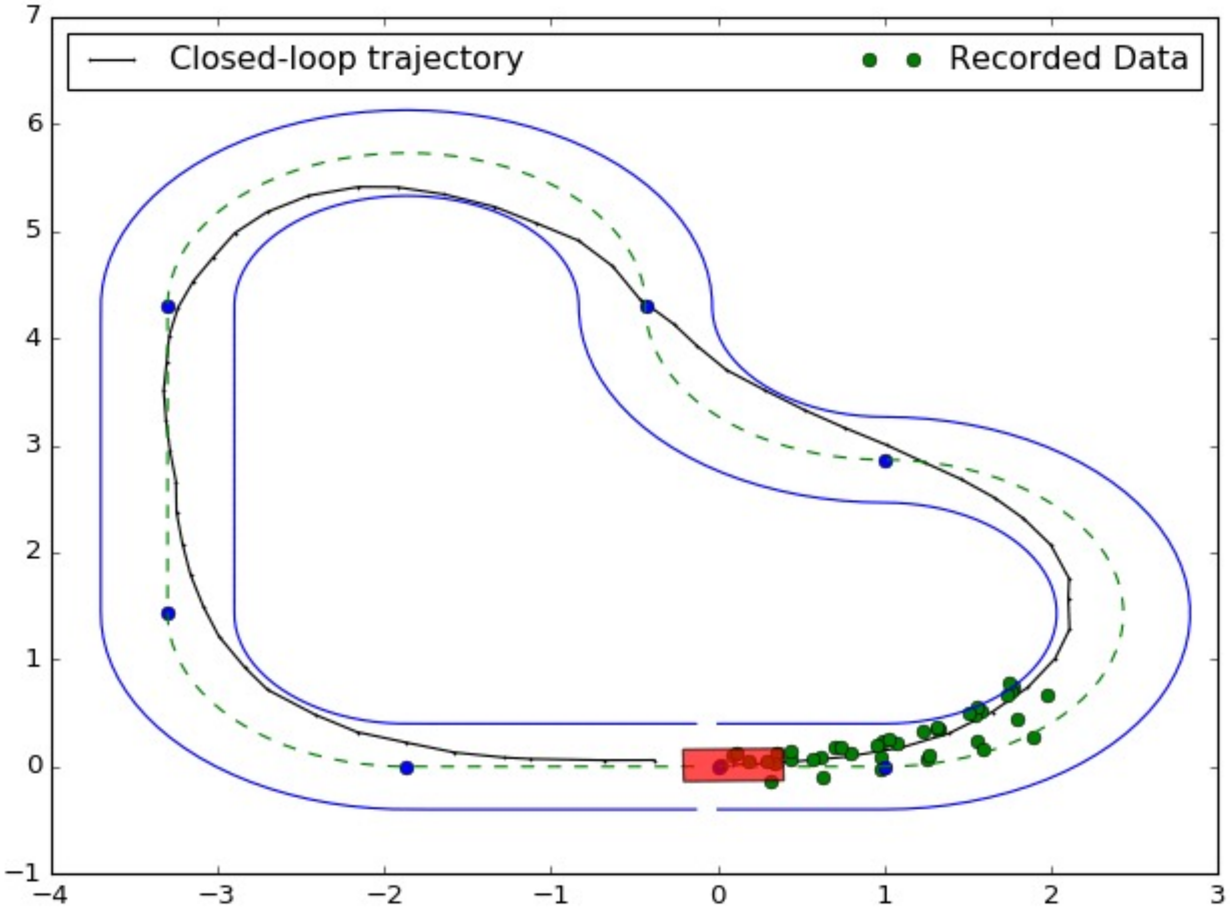


- A **reference trajectory** is **not needed** for the controller implementation
- The controller converges after few laps: the learning process is **data efficient**
- The controller **safely explores** the state space **iteratively improving** the **lap time**

Berkeley Autonomous Race Car Implementation



At each time t the data-based policy is computed with the K nearest neighbors (in green) to the current state. This allows us to account for the nonlinearity of the vehicle

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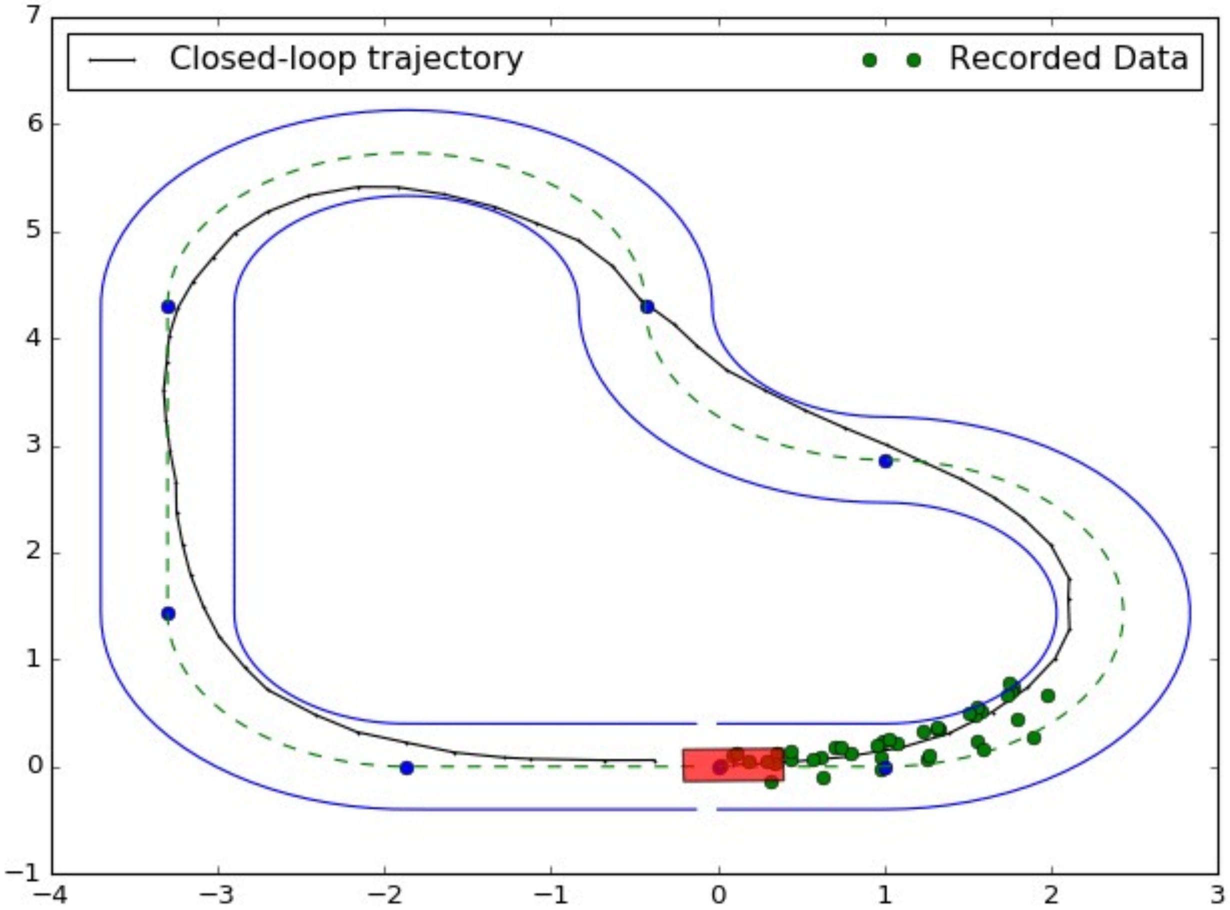

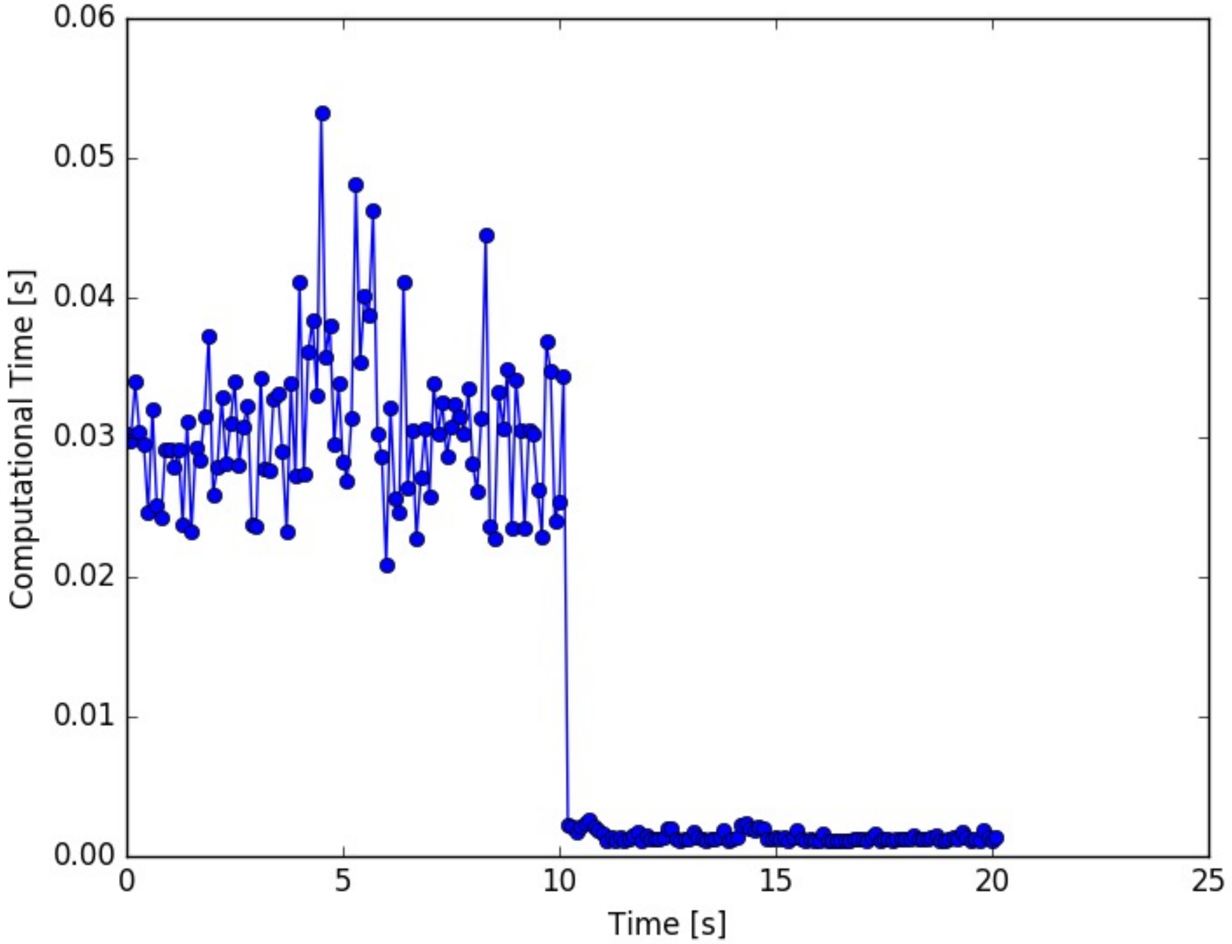

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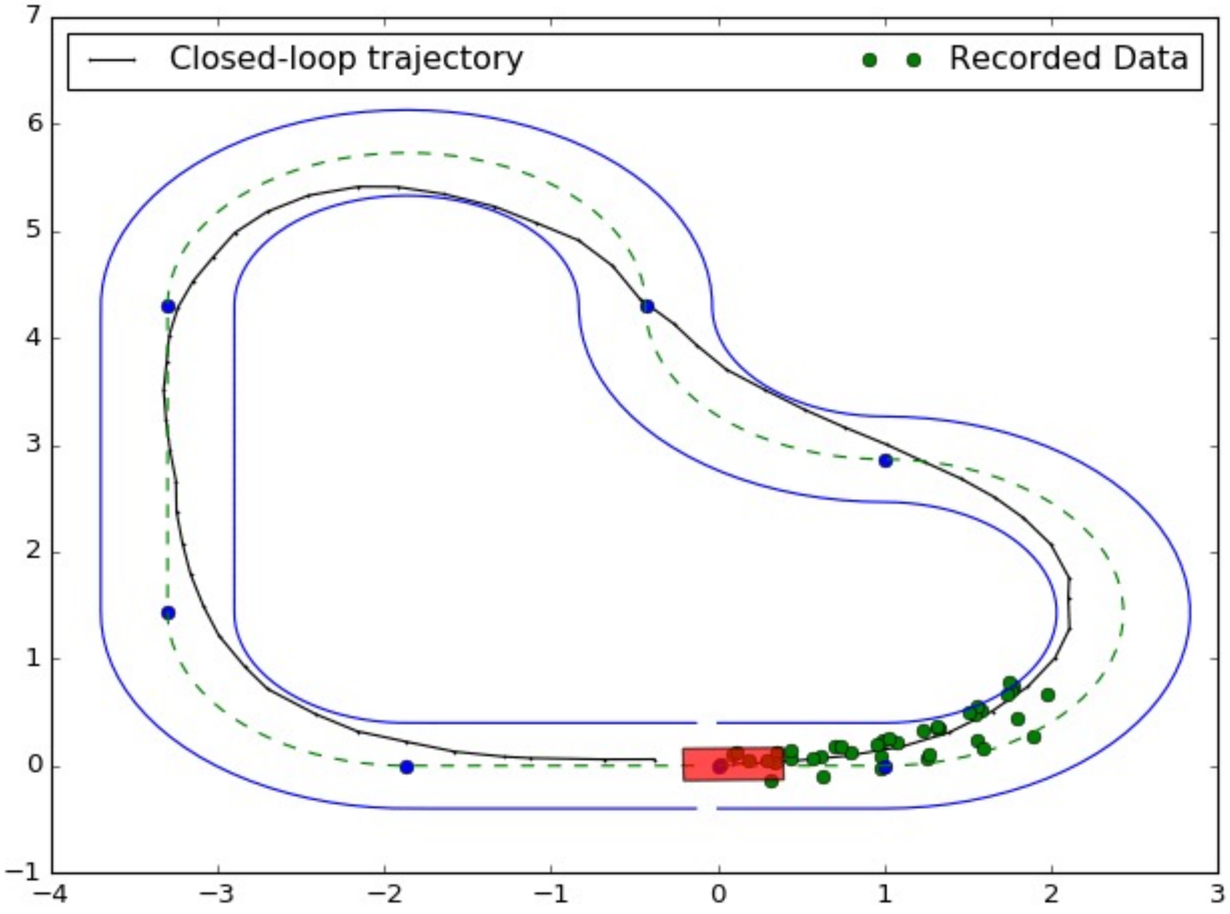

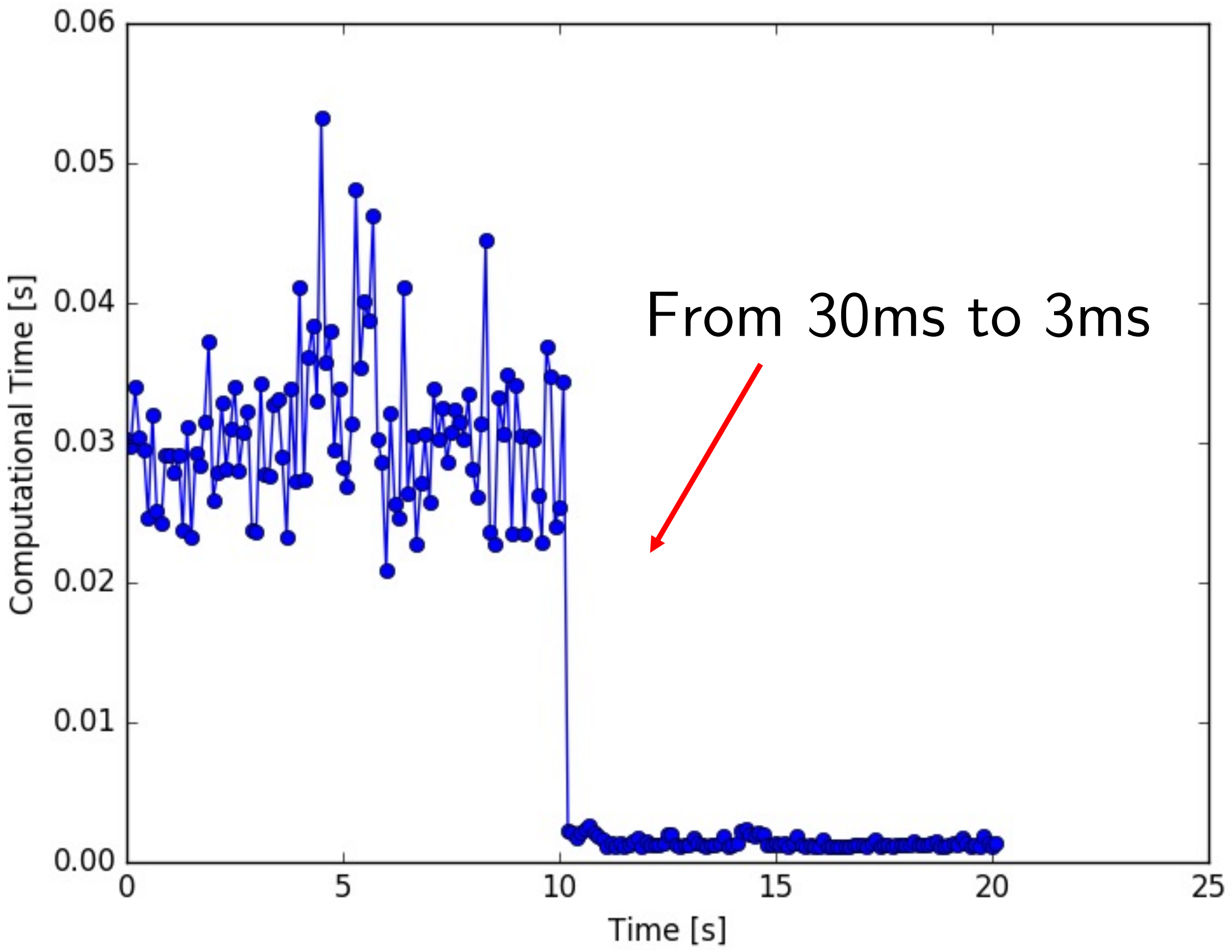

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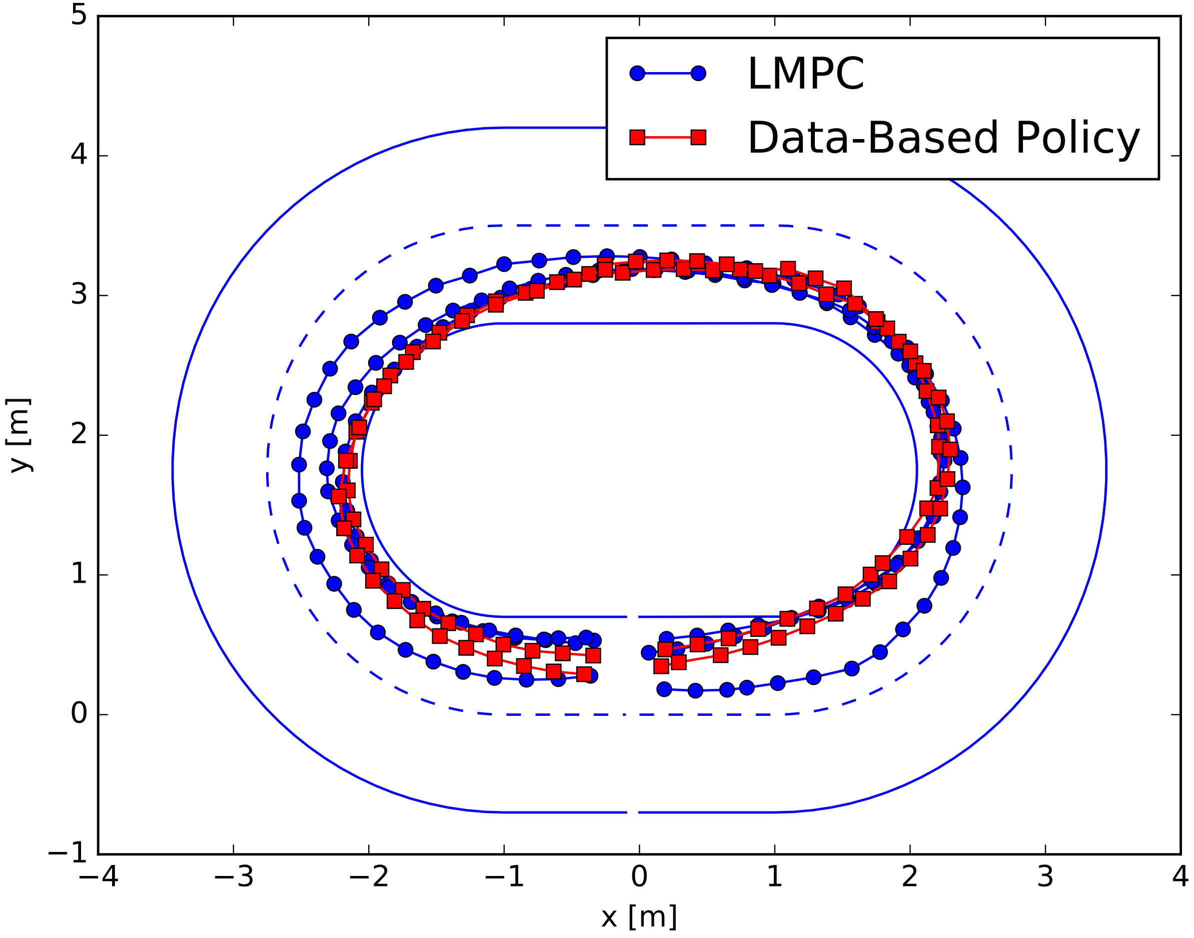

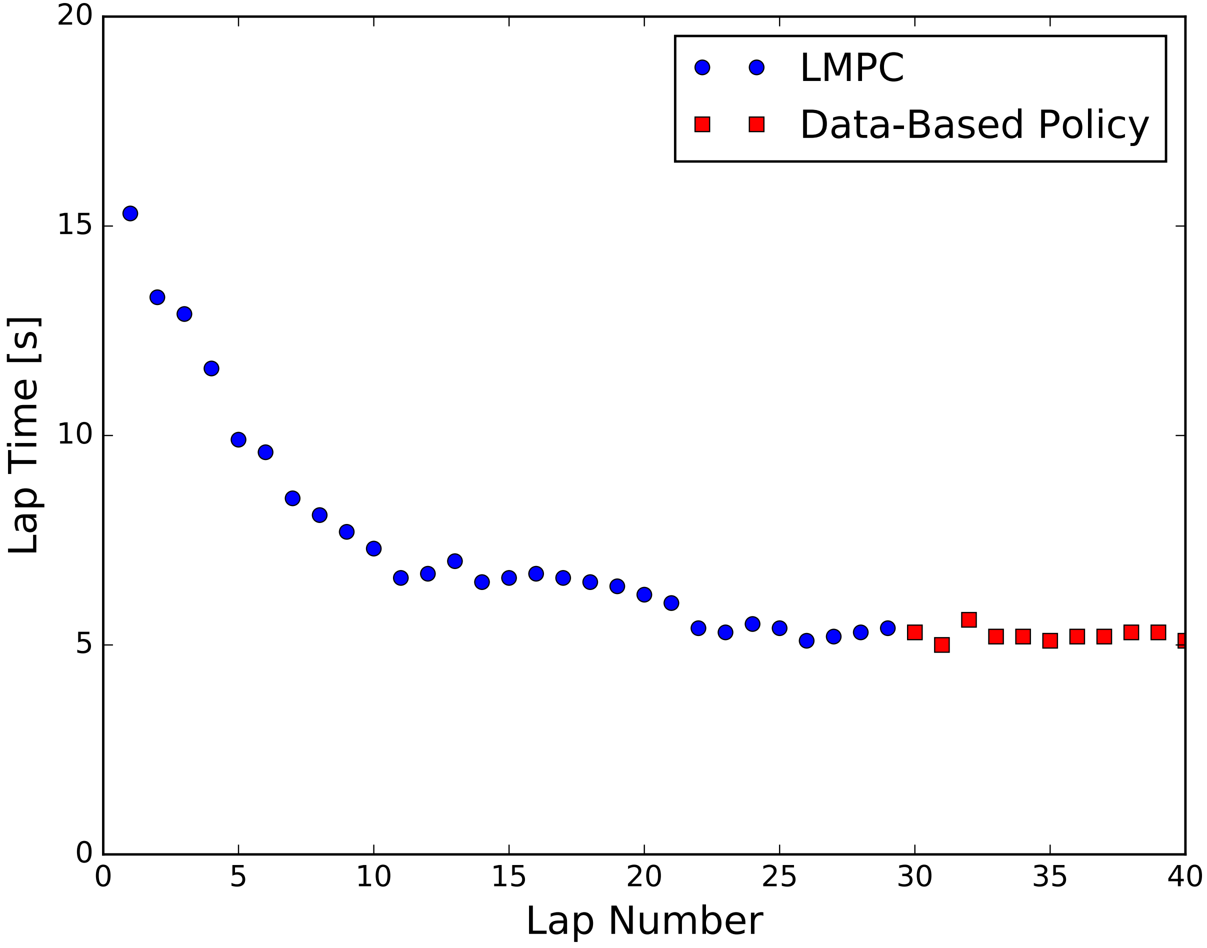

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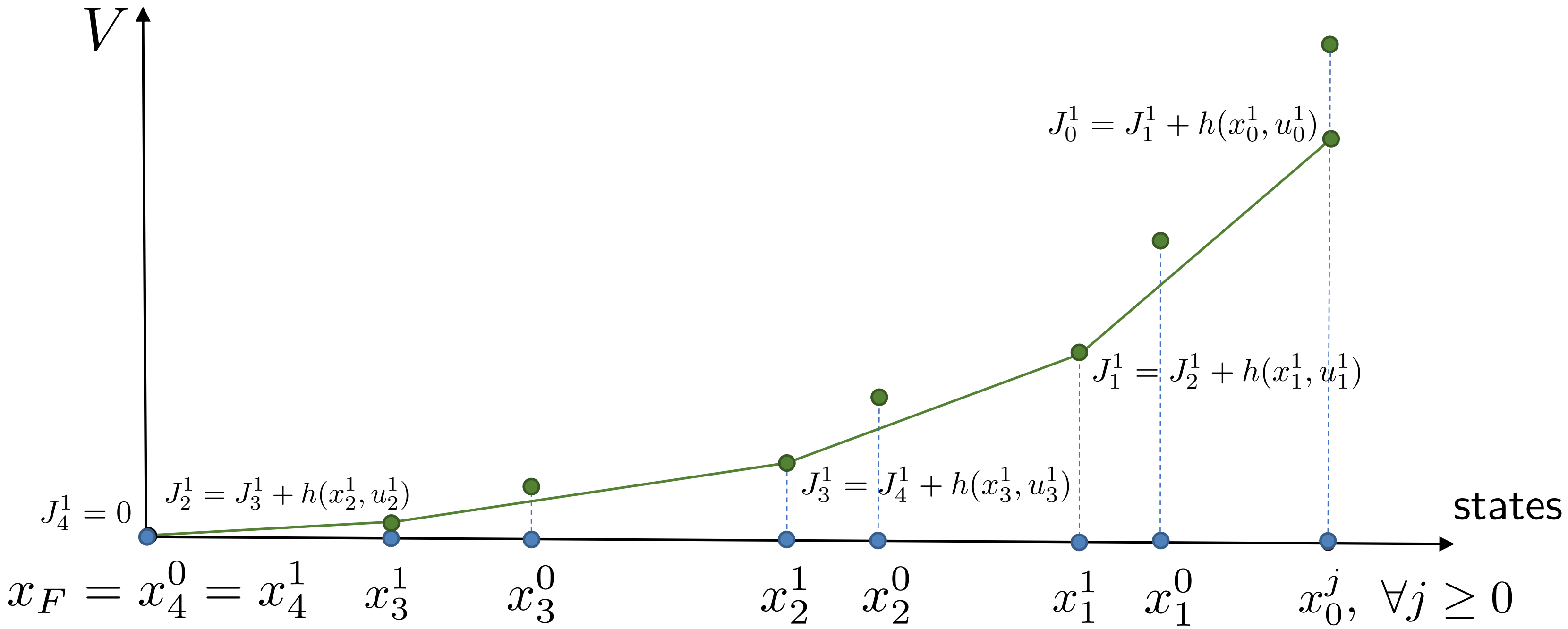

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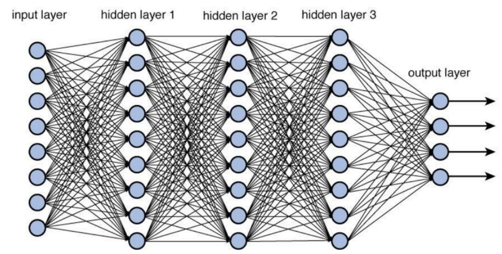
Historical Data

Speeding-up the solver

Offline

Primal Neural-Network



Learns input policy $\pi^p(x)$

Speeding-up the solver

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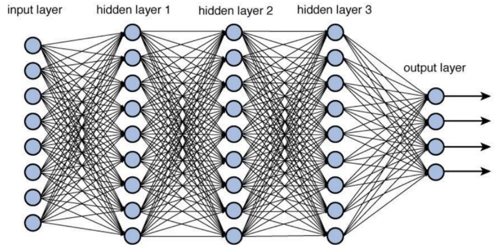

Learns input policy $\pi^p(x)$

Online

Apply the control policy $\pi^p(x)$

Steven Chen, Kelsey Saulnier, Nikolay Atanasov, Daniel D. Lee, Vijay Kumar, George J. Pappas, and Manfred Morari. "Approximating explicit model predictive control using constrained neural networks." In *2018 Annual American control conference (ACC)*

Primal-Dual Neural Network

Primal-Dual Neural Network

Offline

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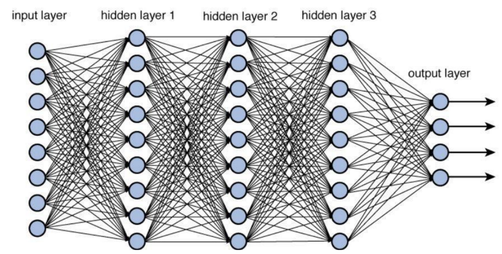

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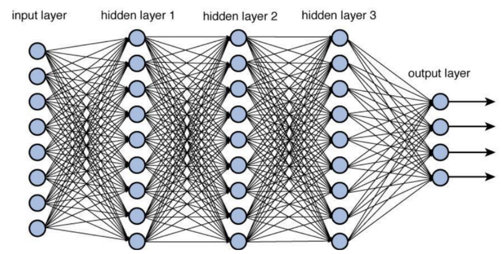

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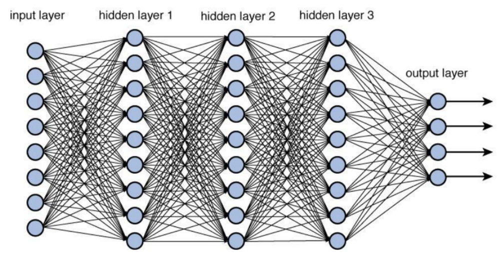
Dual Neural-Network



Learns optimality *certificate* $\lambda^d(x)$

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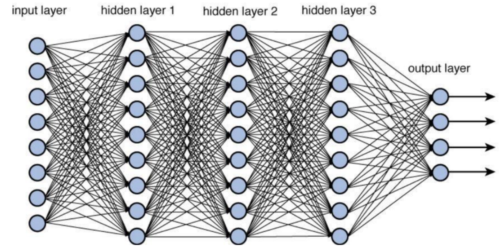

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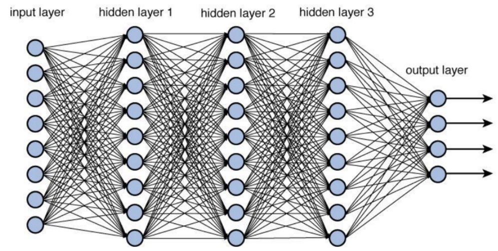

Learns optimality *certificate* $\lambda^d(x)$

Offline Verification

Sample at random N states and check that $|J_N(\pi^p(x) \; - J_N(\pi^p(x))| \leq \gamma$

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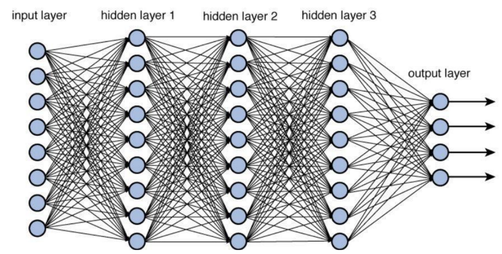

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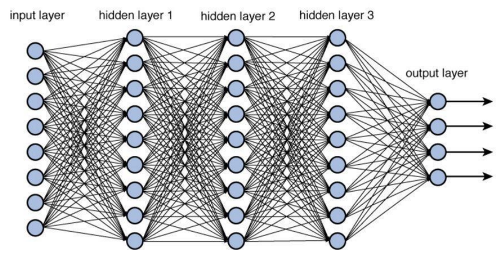

Learns optimality *certificate* $\lambda^d(x)$

Offline Verification

Sample at random N states and check that $|J_N(\pi^p(x)) - J_N^*| > \gamma/2$ w.p $\epsilon/2$ and $|J_N(\pi^p(x) - J_N^*| > \gamma/2$ w.p $\epsilon/2$

If $N \geq \frac{\ln(\frac{1}{\beta/2})}{\ln(\frac{1}{1-\epsilon/2})}$ $\Longrightarrow$ With probability at least $1-\epsilon$, learned controllers will be at most γ-suboptimal

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Tempo, Roberto, Er-Wei Bai, and Fabrizio Dabbene. "Probabilistic robustness analysis: Explicit bounds for the minimum number of samples." In *Proceedings of 35th IEEE Conference on Decision and Control*, (1996)

Primal-Dual Neural Network

Offline

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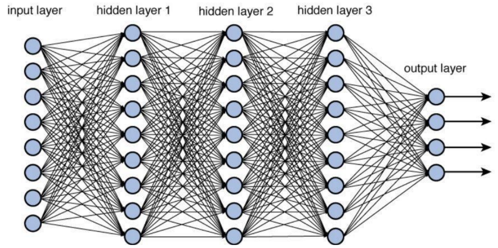

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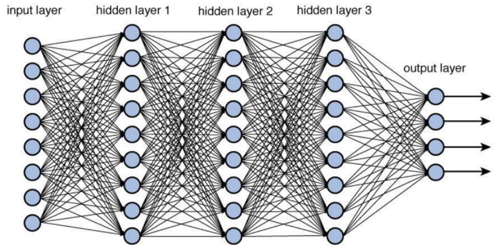

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LMPC for uncertain systems

A sample-based approach

LMPC for uncertain systems – Problem Formulation

LMPC for uncertain systems – Problem Formulation

- Model Choice: Linear system subject to bounded additive uncertainty,

$$x_{k+1} = Ax_k + Bu_k + w_k, \ \text{with } w_k \in \mathcal{W} \ \forall k \geq 0$$

LMPC for uncertain systems – Problem Formulation

- **Model Choice:** Linear system subject to bounded additive uncertainty,

$$x_{k+1} = Ax_k + Bu_k + w_k, \text{ with } w_k \in \mathcal{W} \ \forall k \geq 0$$

- **Constraint Satisfaction:**
  - Robust, $x_k \in \mathcal{X}, \ \forall w_k \in \mathcal{W} \ k \geq 0.$

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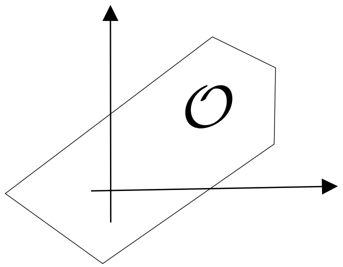
- Cost: Worst Case.

- Convergence: Cannot regulate to a point but to a robust positive invariant . $\mathcal{O}$

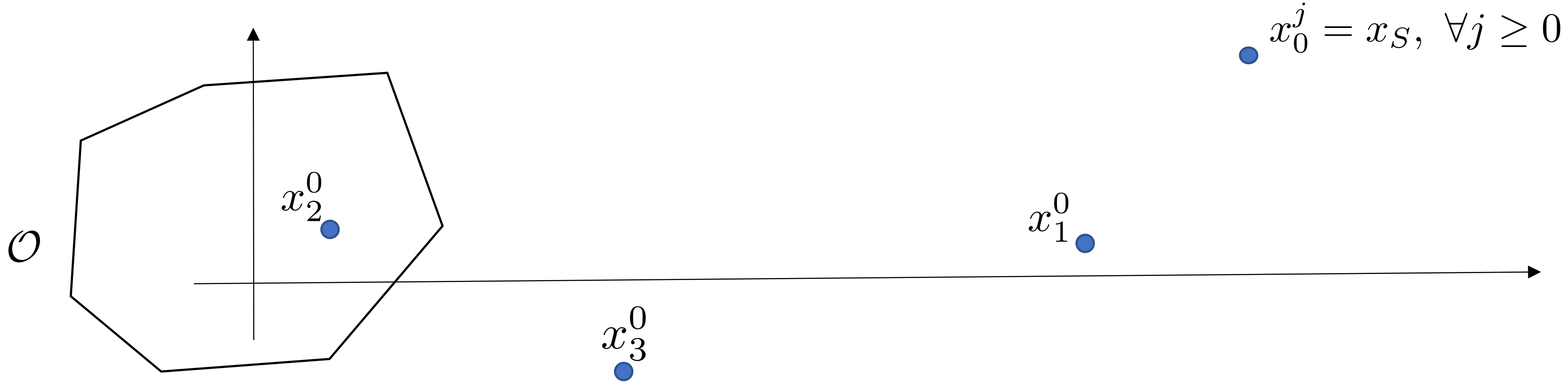


$$\forall x \in \mathcal{O} \rightarrow (A - KB)x + w \in \mathcal{O}, \; \forall w \in \mathcal{W}$$

LMPC for uncertain systems – Differences w/ Nominal

LMPC for uncertain systems – Differences w/ Nominal

Given a feasible trajectory which drives the system to the terminal robust invariant



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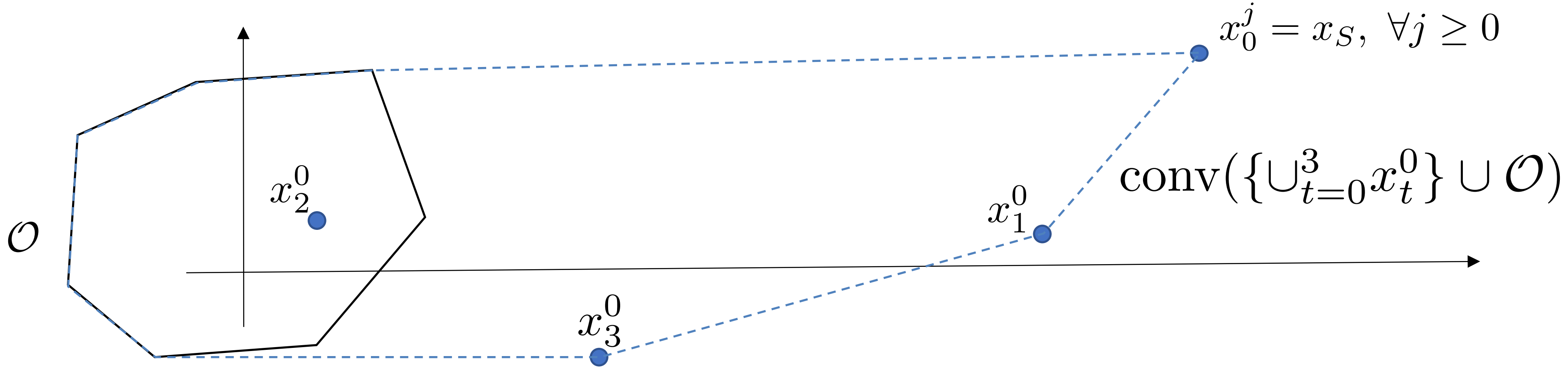

The convex hull of the stored data is **not a robust control invariant**.

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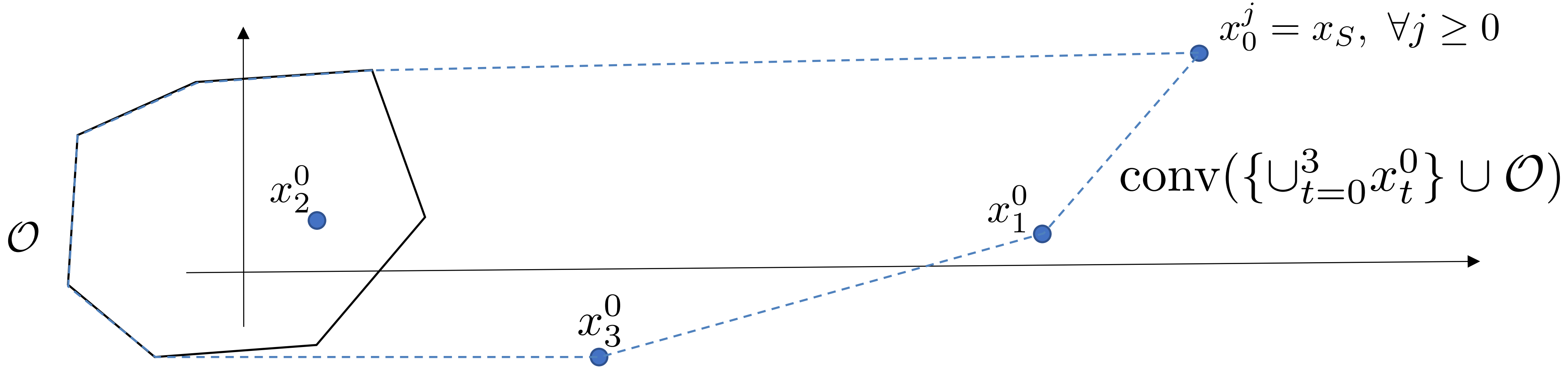

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For this reason,

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LMPC for uncertain systems – Differences w/ Nominal

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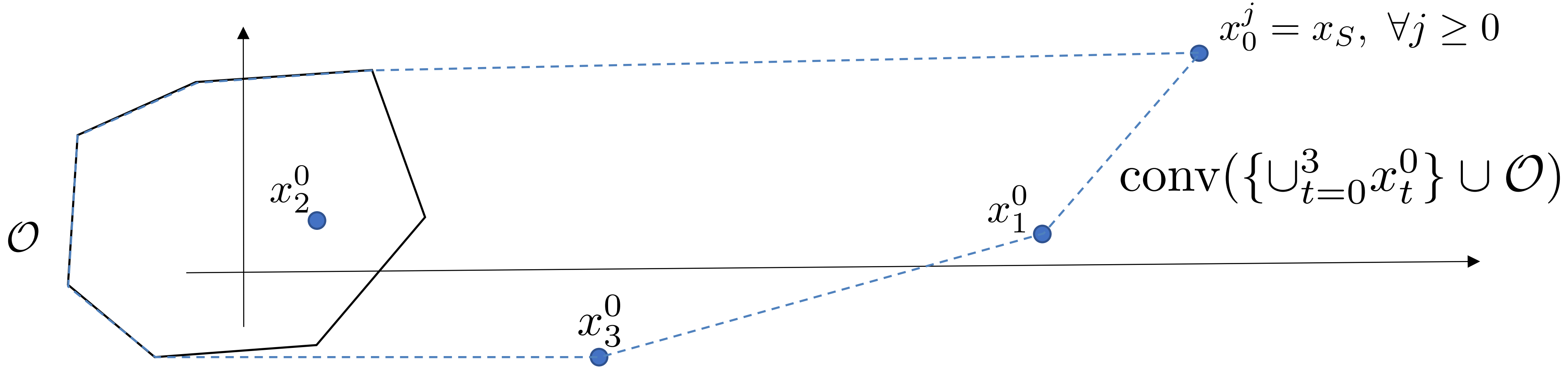

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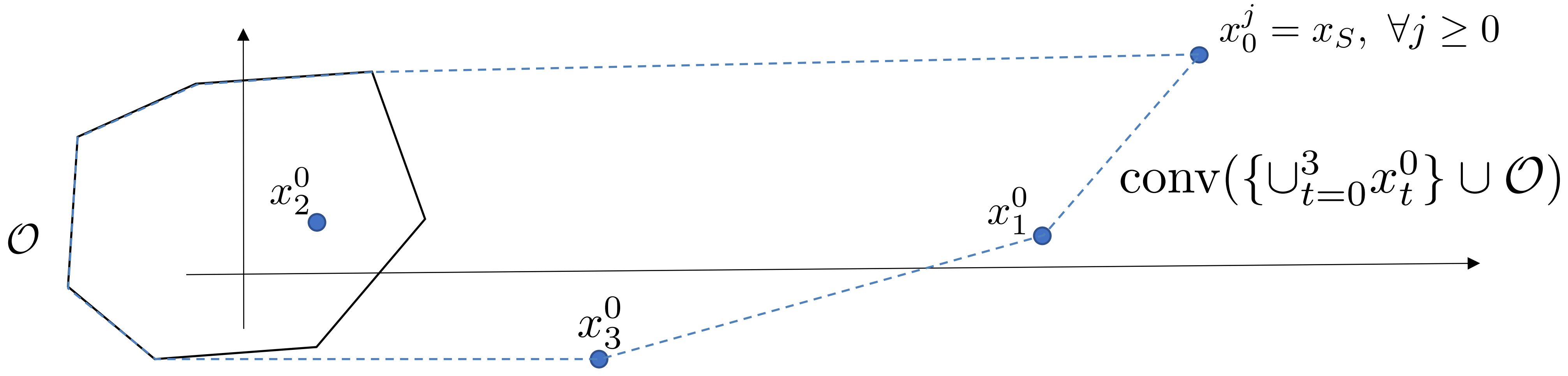

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need to be designed differently in Robust LMPC.

A. Robust Safe Set

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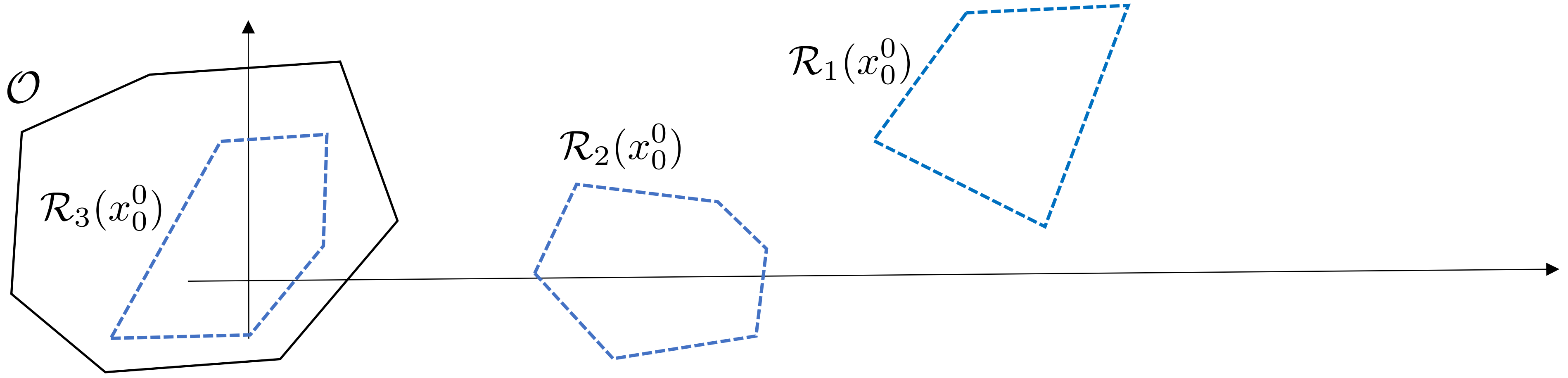
Assume that given control policy $\pi^0 : \mathbb{R}^n \to \mathbb{R}^d$ robustly steers, the closed-loop system $x^0_{k+1} = Ax^0_k + B\pi^0(x^0_k) + w^0_k$ converges to the terminal set $\mathcal{O}$.

$x^j_0 = x_S, \ \forall j \geq 0$



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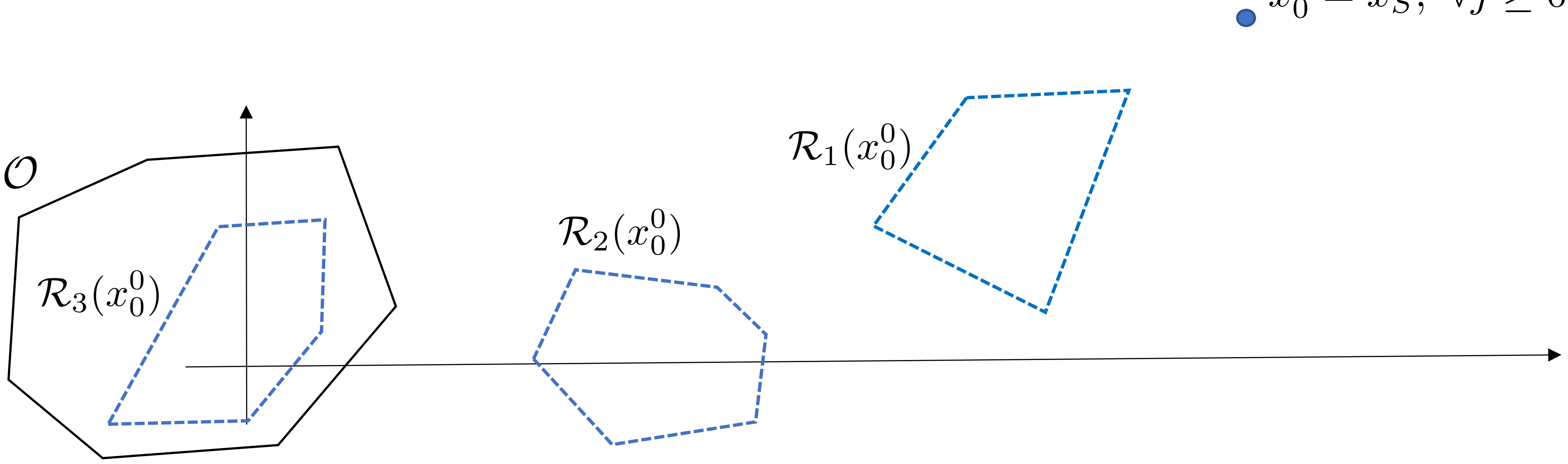
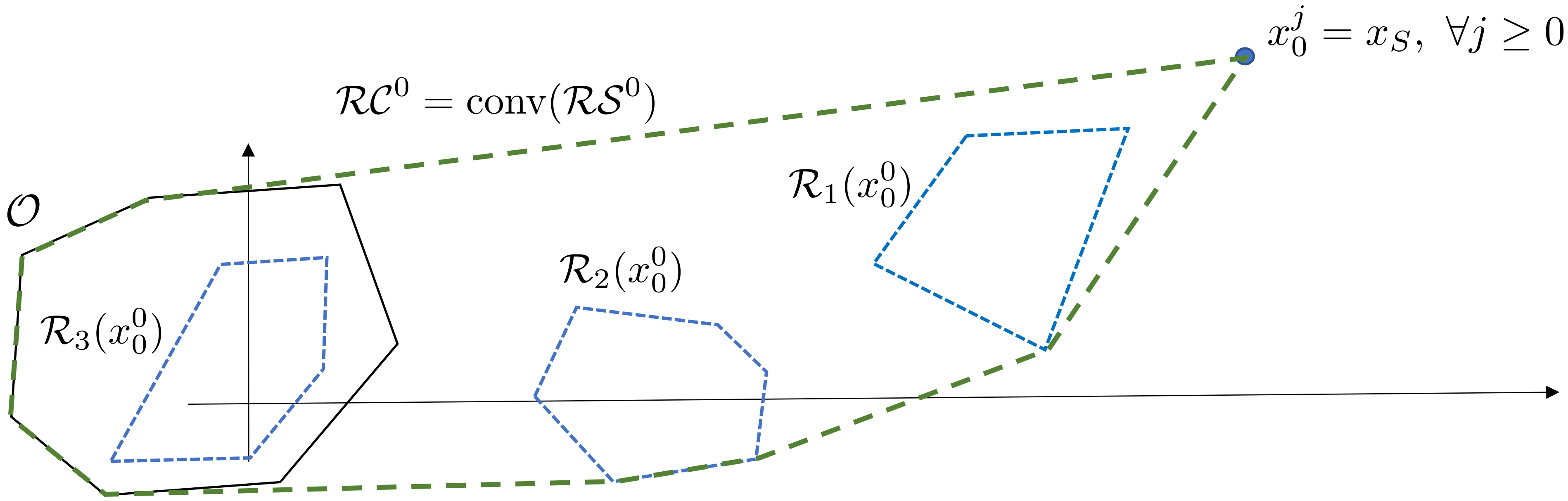


Definition: Robust Safe Set

The robust safe set $\mathcal{RS}^j = \{\cup_k \mathcal{R}_k(x_0^j)\} \cup \mathcal{O}$ is a **robust control invariant set** for constrained uncertain linear systems.

A. Robust Convex Safe Set

Assume that given control policy $\pi^0 : \mathbb{R}^n \to \mathbb{R}^d$ robustly steers, the closed-loop system $x^0_{k+1} = Ax^0_k + B\pi^0(x^0_k) + w^0_k$ converges to the terminal set $\mathcal{O}$.



Definition: Robust Convex Safe Set

The robust convex safe set $\mathcal{RC}^j = \text{conv}\left(\mathcal{RS}^j\right)$ is a **robust control invariant** set for constrained uncertain linear systems.

B. Robust Q-Function

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Consider the Bellman recursion for the control policy , π^j

$$J^j_{\pi^j}(x) = \max_{w \in \mathcal{W}} [h(x, \pi^j(x)) + J^j_{\pi^j}(Ax + B\pi^j(x) + w)]$$

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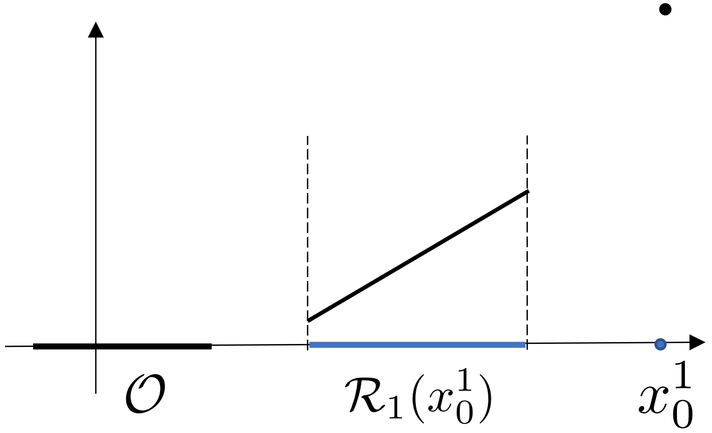
The robust Q-function, which represents the worst case cost-to-go over the robust safe set, is defined as

$$Q^j_r(x) = \begin{cases} \max_{w \in \mathcal{W}} [h(x, \pi^j(x)) + Q^j_r(Ax^j + Bu^j + w)] & \text{If } x \in \mathcal{RS}^j \\ +\infty & \text{If } x \notin \mathcal{RS}^j \end{cases}$$



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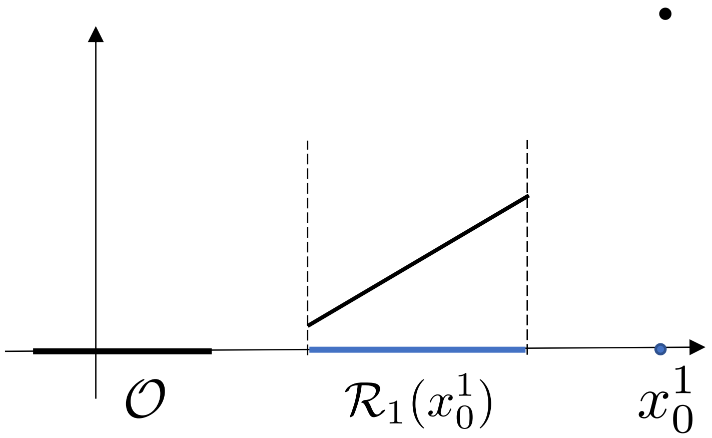
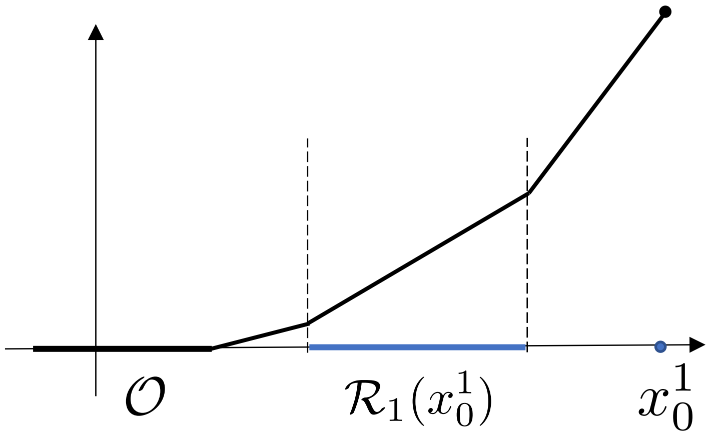


Definition: Robust Convex Q-function

The robust convex Q-function defined as

$$Q^j_{rc}(x) = \min_{\mu} \mu \mid (x,\mu) \in \mathrm{conv}\Big(\bigcup_{k=0}^{j} \mathrm{epi}(Q^j_r(x))\Big),$$

is a **robust control Lyapunov function**.



Robust Learning Model Predictive Control

At time *t* of iteration *j* solve the following Constrained Finite Time Optimal Control Problem (CFTOCP)

$$
\begin{aligned}
J^{\mathrm{LMPC},j}_{t\rightarrow t+N}(x_t^j) = \min_{\boldsymbol{\pi}_t^j(\cdot)} \max_{\bar{\mathbf{w}}_t^j} \; & \Big[\sum_{k=t}^{t+N-1} h(x^j_{k|t}, u^j_{k|t}) + Q^{j-1}_{rc}(x^j_{t+N|t}) \Big] \\
& x^j_{k+1|t} = A x^j_{k|t} + B u^j_{k|t} + \bar{w}^j_{k|t} \\
& u^j_{k|t} = \pi^j_{k|t}(x^j_{k|t}) \\
& x^j_{k|t} \in \mathcal{X}, u^j_{k|t} \in \mathcal{U} \\
& x^j_{t+N|t} \in \mathcal{RC}^{j-1}, \;\; \forall \bar{w}^j_{k|t} \in \mathcal{W}, k \in \{t, \dots, t+N\}
\end{aligned}
$$

for $\boldsymbol{\pi}_t^j(\cdot) = [\pi^j_{t|t}(\cdot), \dots, \pi^j_{t+N|t}(\cdot)]$ and $\bar{\mathbf{w}}_t^j = [\bar{w}^j_{t|t}, \dots, \bar{w}^j_{t+N|t}]$

Then apply to the system $\quad u_t^j = \pi^{j,*}_{t|t}(x_t^j)$

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Iteratively constructed using the system model

for $\boldsymbol{\pi}^j_t(\cdot) = [\pi^j_{t|t}(\cdot), \dots, \pi^j_{t+N|t}(\cdot)]$ and $\bar{\mathbf{w}}^j_t = [\bar{w}^j_{t|t}, \dots, \bar{w}^j_{t+N|t}]$

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Robust Learning Model Predictive Control – Remarks

At iteration *j* the robust convex safe set $\mathcal{RC}^j$ and the robust Q-function $Q^j_{rc}(\cdot)$ can be:

- Computed analytically (hard even for linear systems)

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- Approximated for a large number of rollout N_{roll}. Then, we could guarantee

$$\mathbb{P}(Ax + Bu + w \in \mathcal{CS}^j | x \in \mathcal{CS}^j) \geq 1 - \epsilon(N_{roll})$$

$$\mathbb{P}(Q^j_{rc}(Ax + B\pi^j(u) + w) - Q^j_{rc}(x) + h(x, \pi^j(u)) \leq 0 | x \in \mathcal{RS}^j) \geq 1 - \gamma(N_{roll})$$

where $\epsilon(N_{roll})$ and $\gamma(N_{roll})$ increase as the number of rollouts N_{roll} increases.

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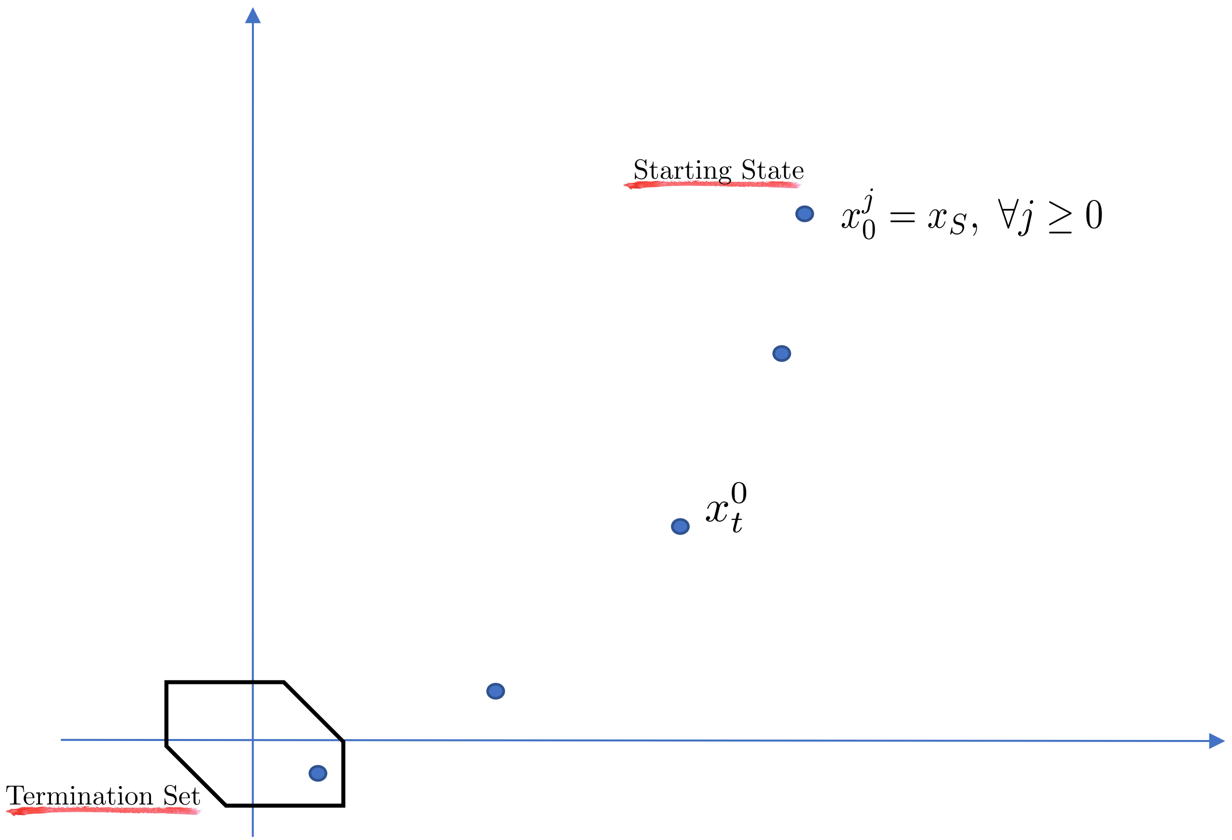
Iterative approximation strategy based on historical data

Example: Constrained LQR Uncertain System



Control Problem

Let

$$J^j_{\pi^j}(x) = \max_{w \in \mathcal{W}} [h(x, \pi^j(x)) + J^j_{\pi^j}(Ax + B\pi^j(x) + w)]$$

be the solution to the Bellman recursion for the control policy $\pi^j(\cdot)$. We want to solve the following finite time robust optimal control problem

$$J^{j,*}_{0 \to T^j}(x_0^j) = \min_{\pi^j(\cdot)} \quad J^j_{\pi^j}(x_0^j)$$

$$x^j_{k+1} = Ax^j_k + B\pi^j(x^j_k) + w^j_k$$

$$u^j_k = \pi^j(x^j_k)$$

$$x^j_k \in \mathcal{X}, u^j_k \in \mathcal{U}, x^j_{T^j} \in \mathcal{O}$$

$$\forall w^j_k \in \mathcal{W}, k \in \{0, \ldots, T^j\}$$

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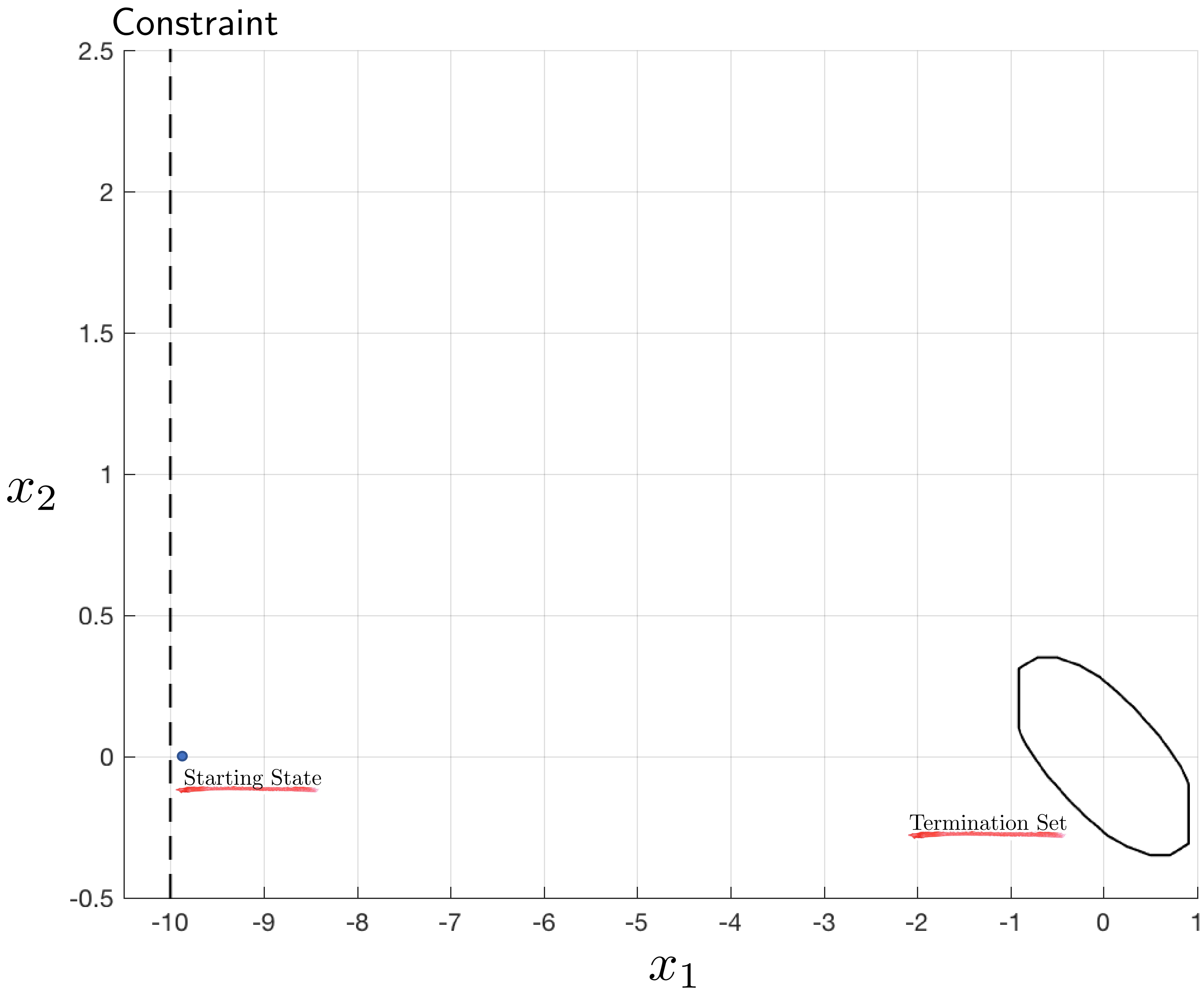

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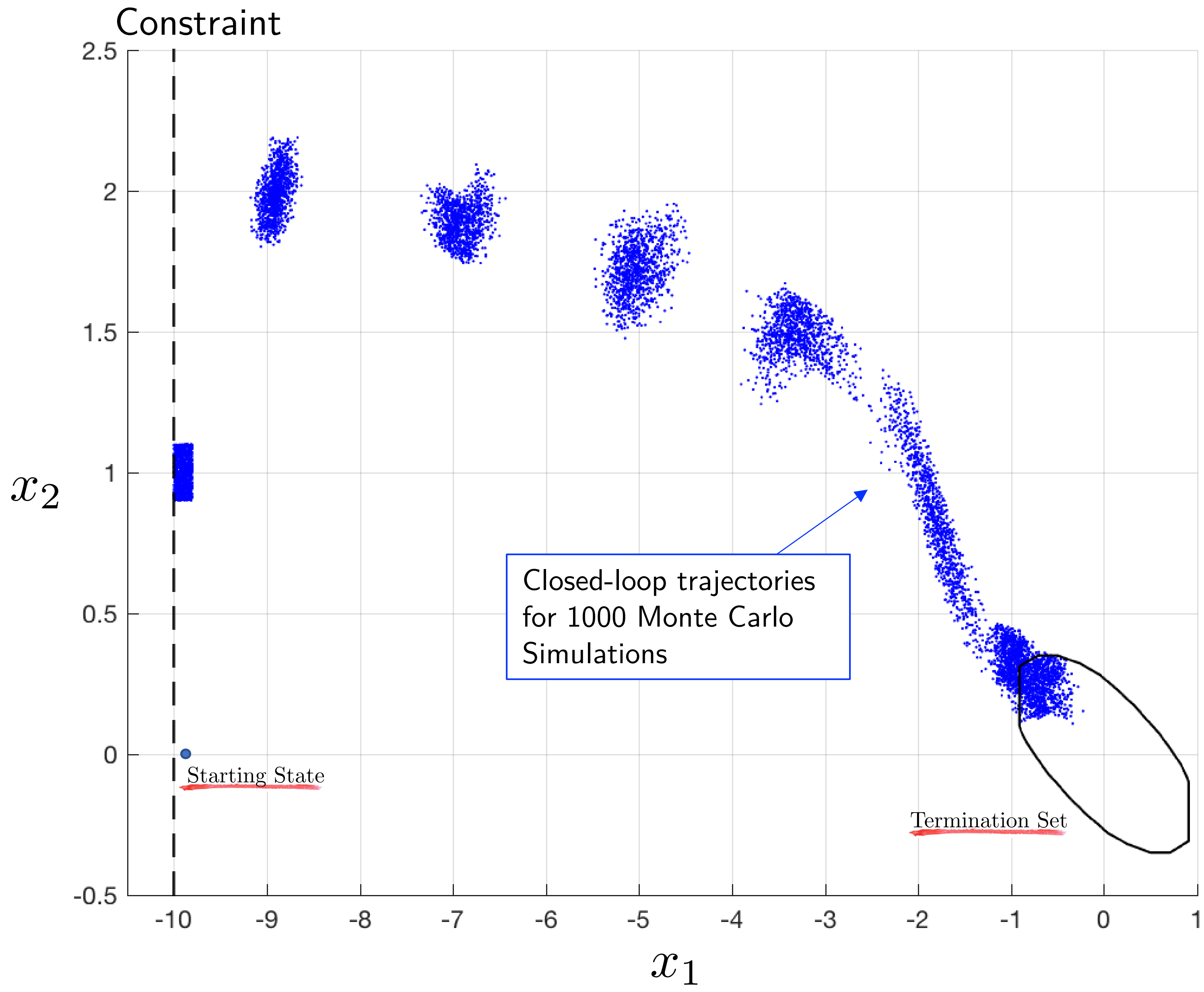

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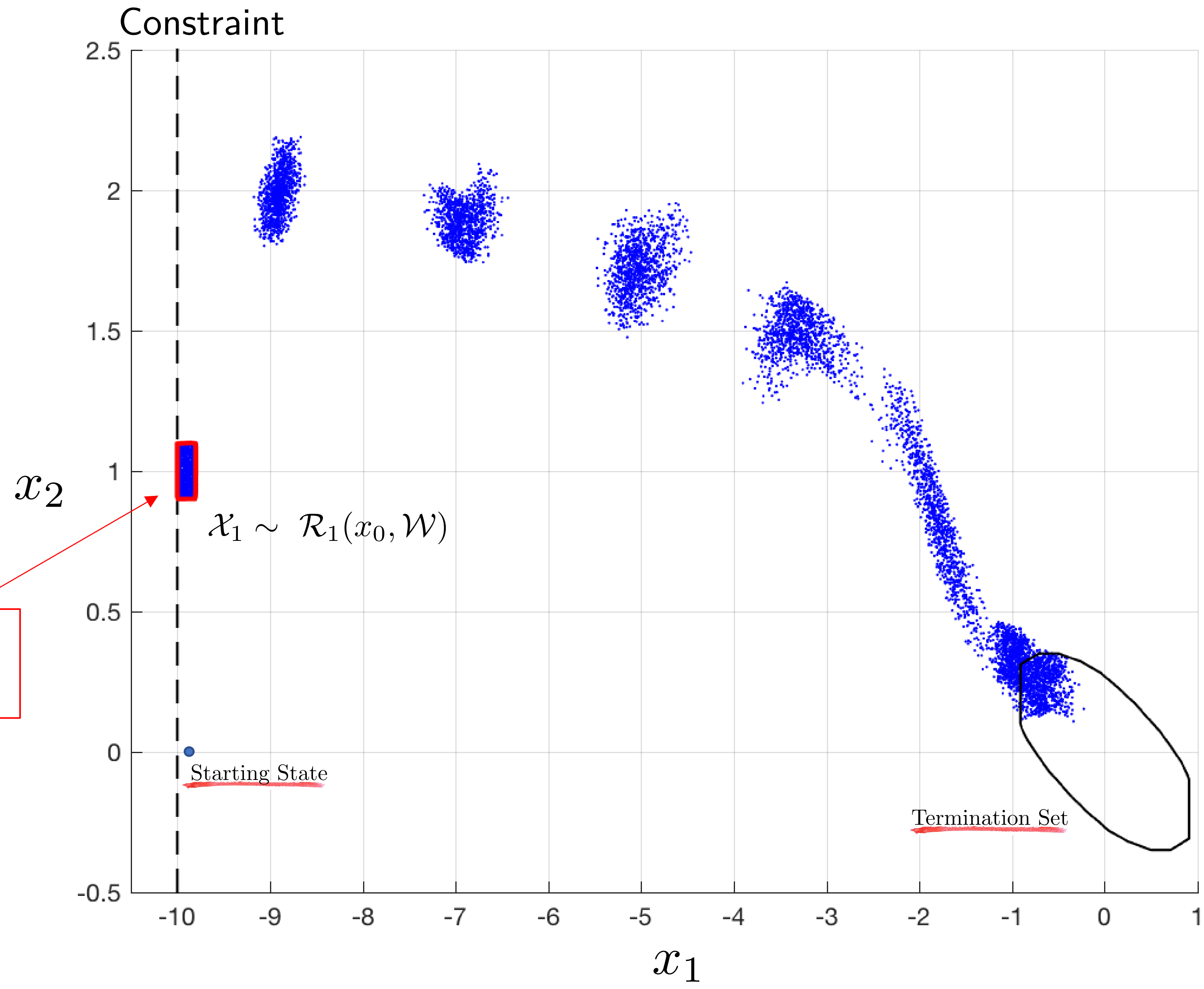
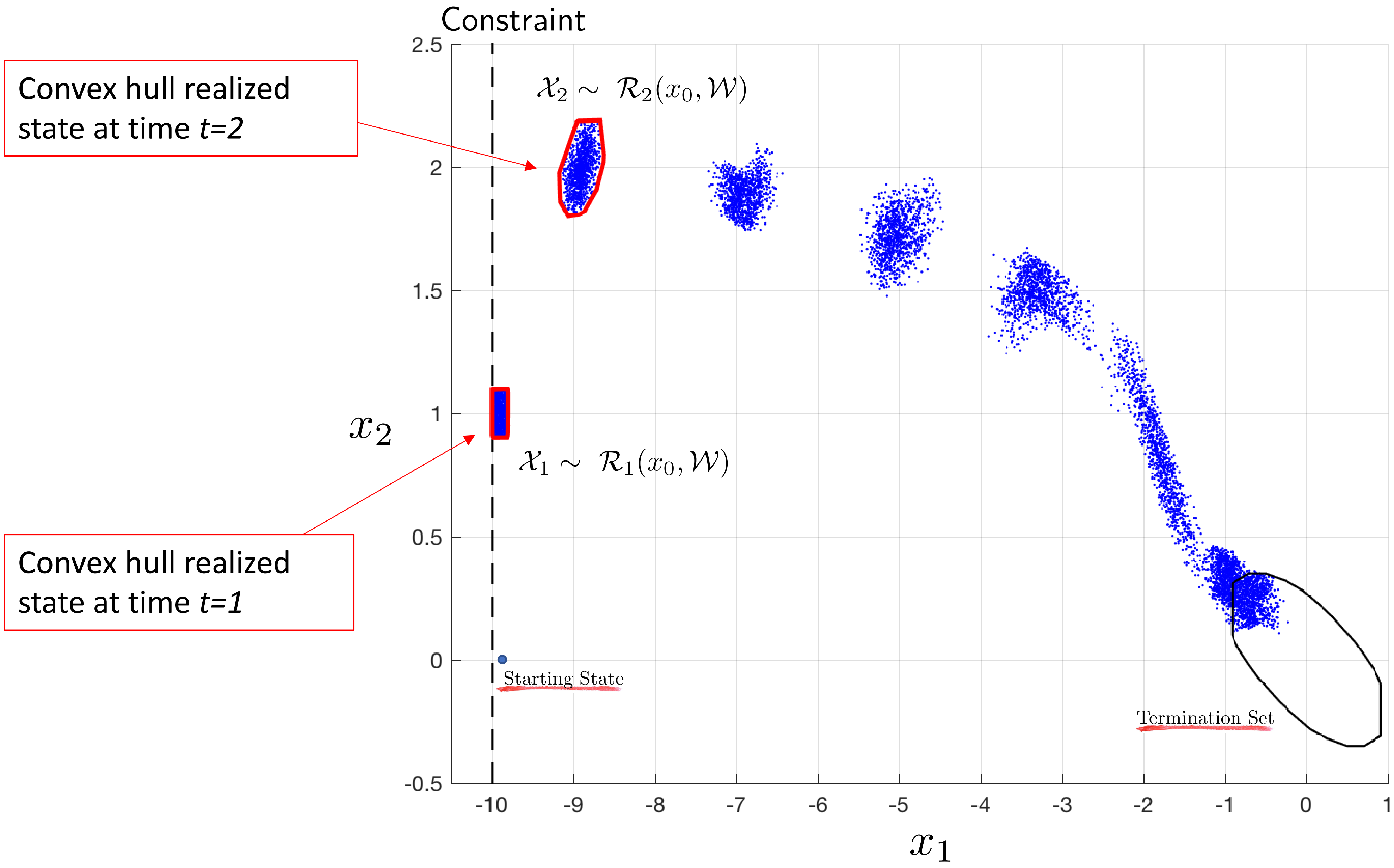


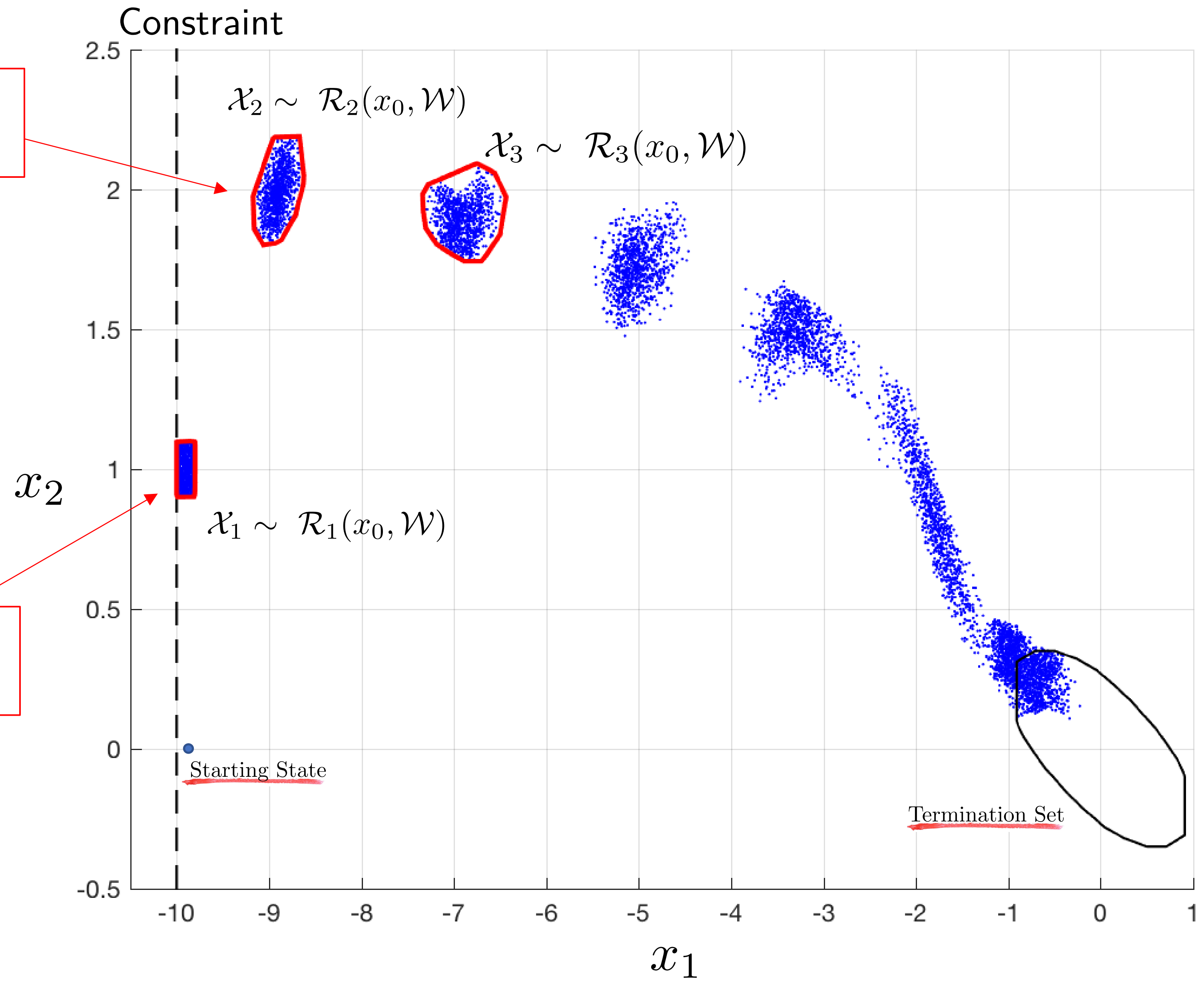
Convex hull realized state at time *t=1*

Example: Constrained LQR Uncertain System



Example: Constrained LQR Uncertain System

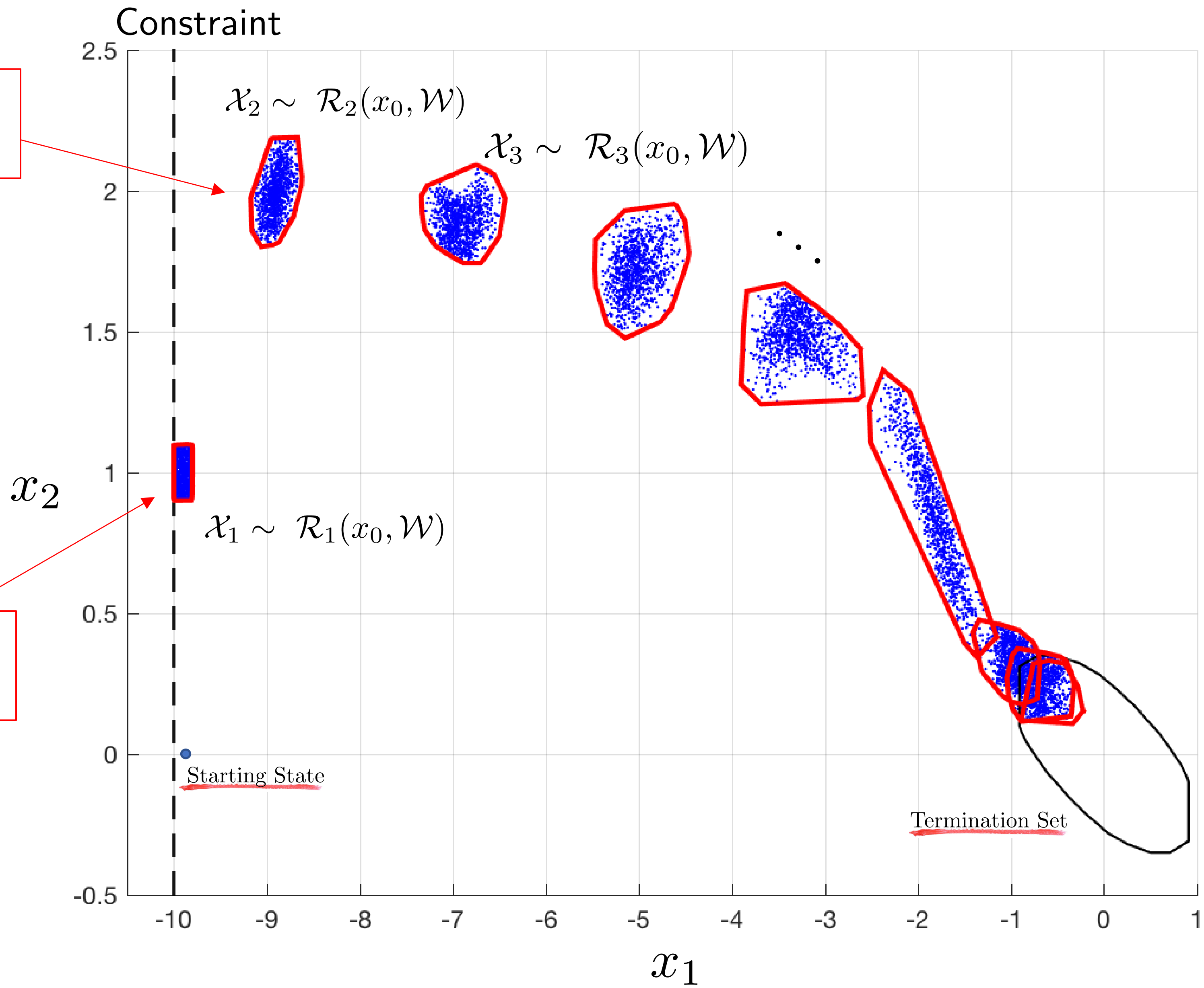
Convex hull realized state at time *t=2*



Convex hull realized state at time *t=1*

Example: Constrained LQR Uncertain System

Convex hull realized state at time *t=2*



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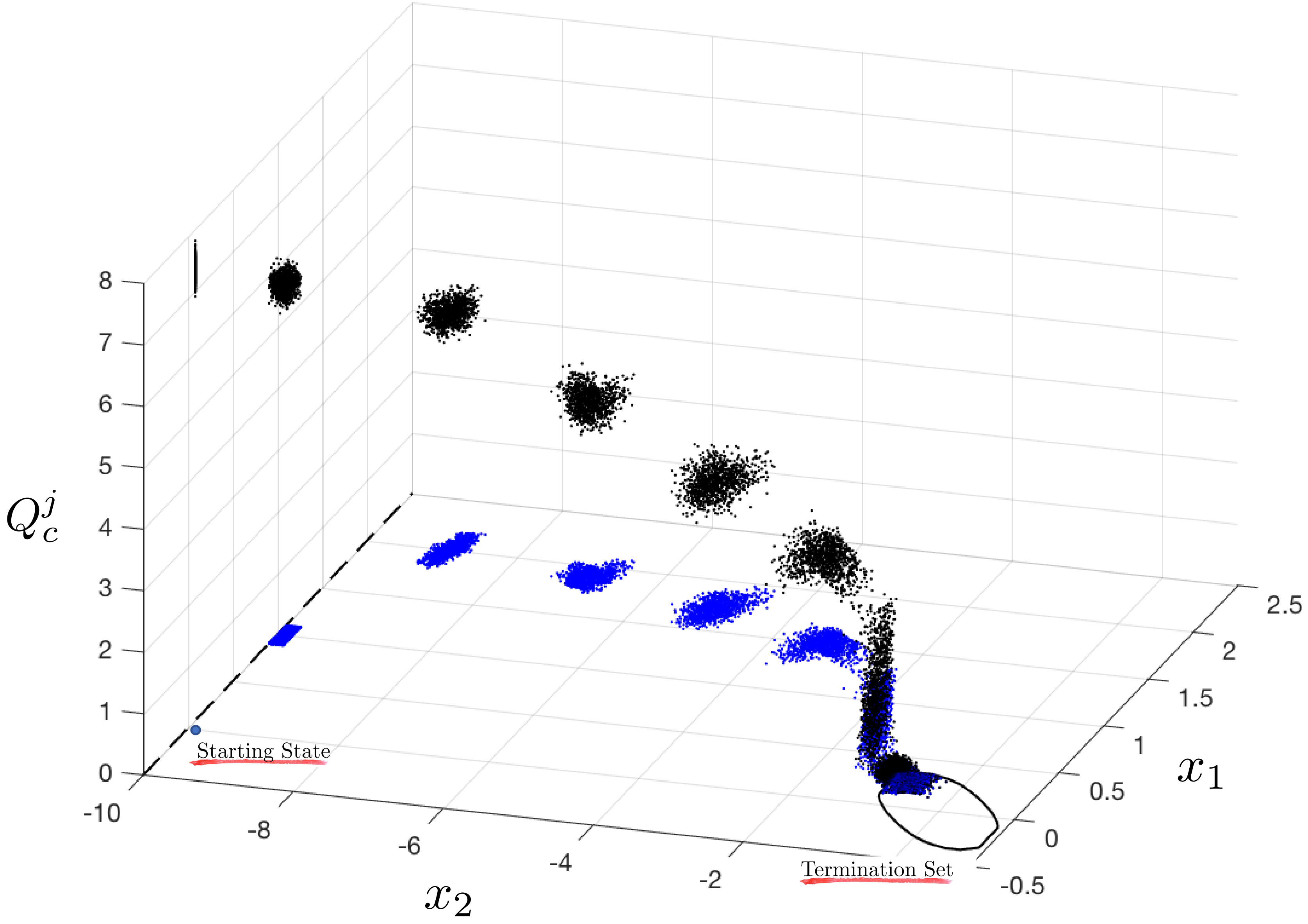

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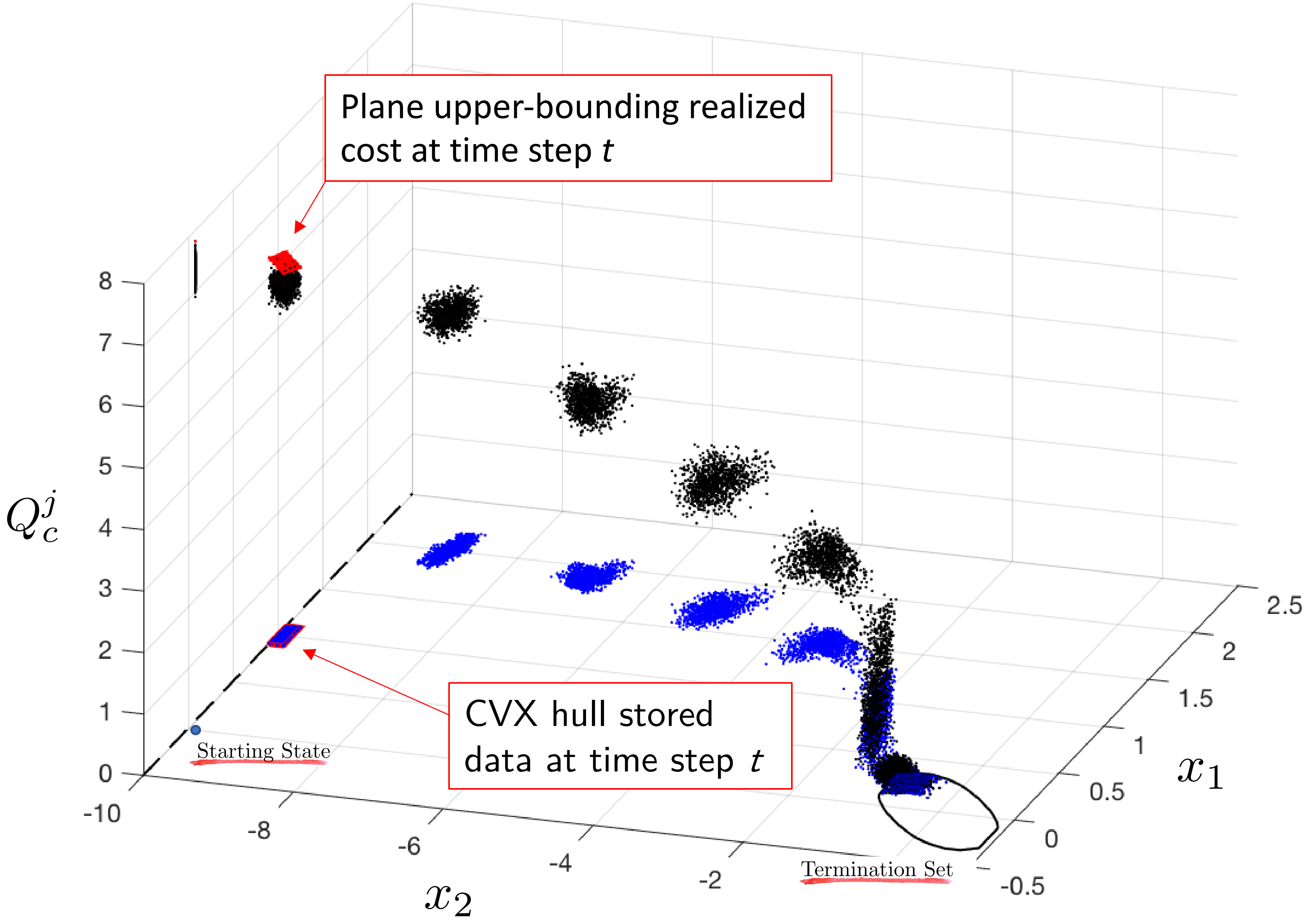

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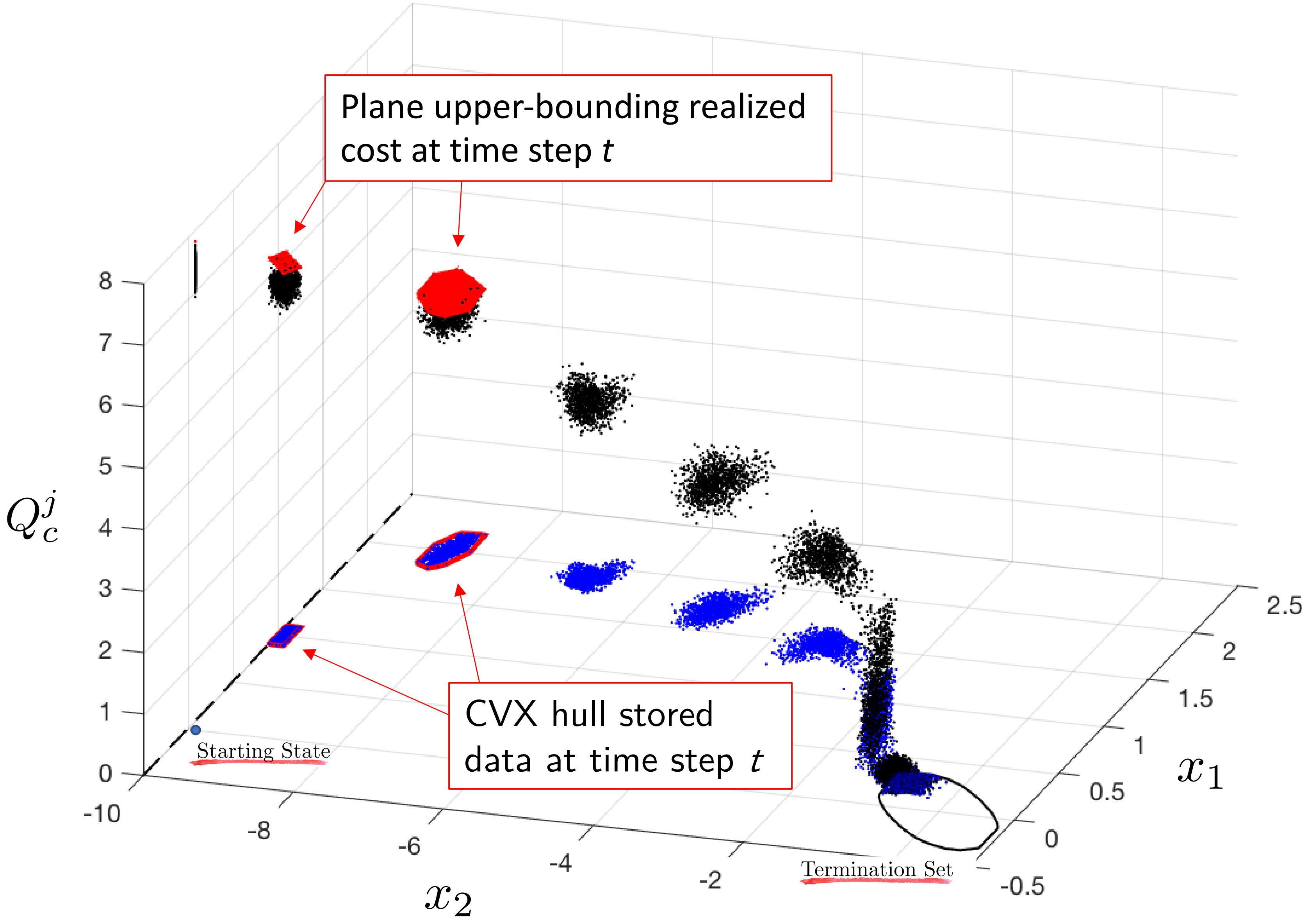

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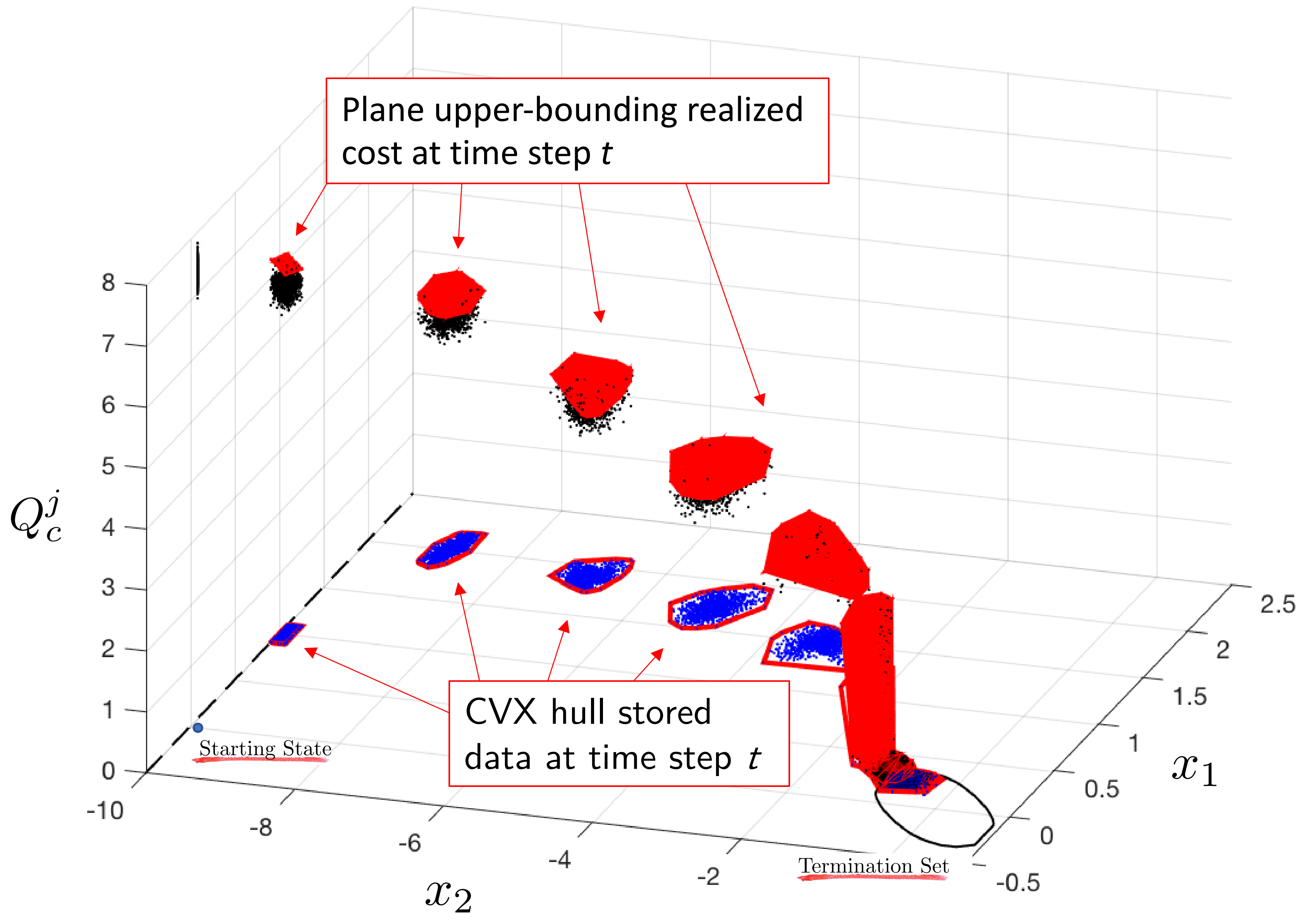

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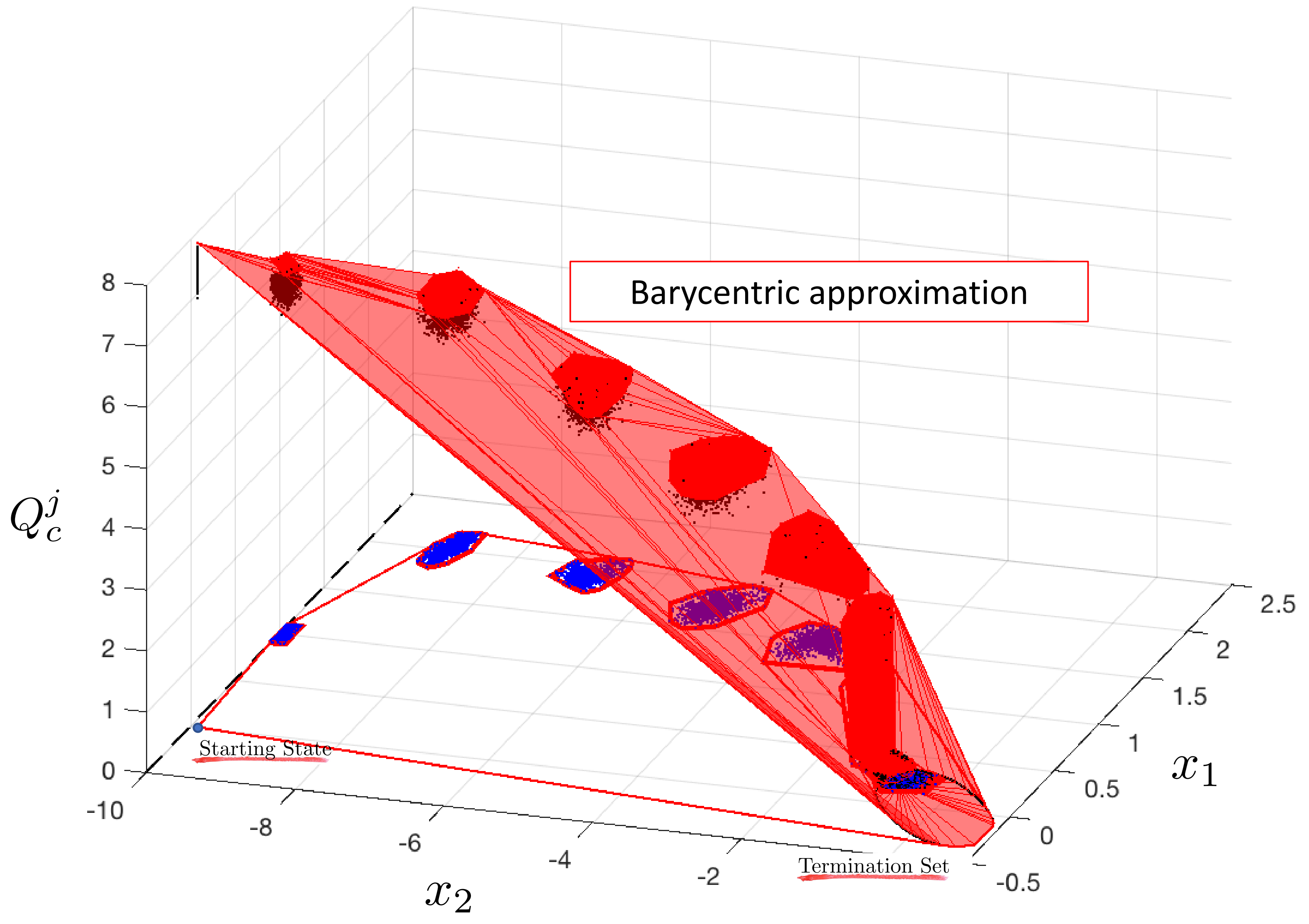

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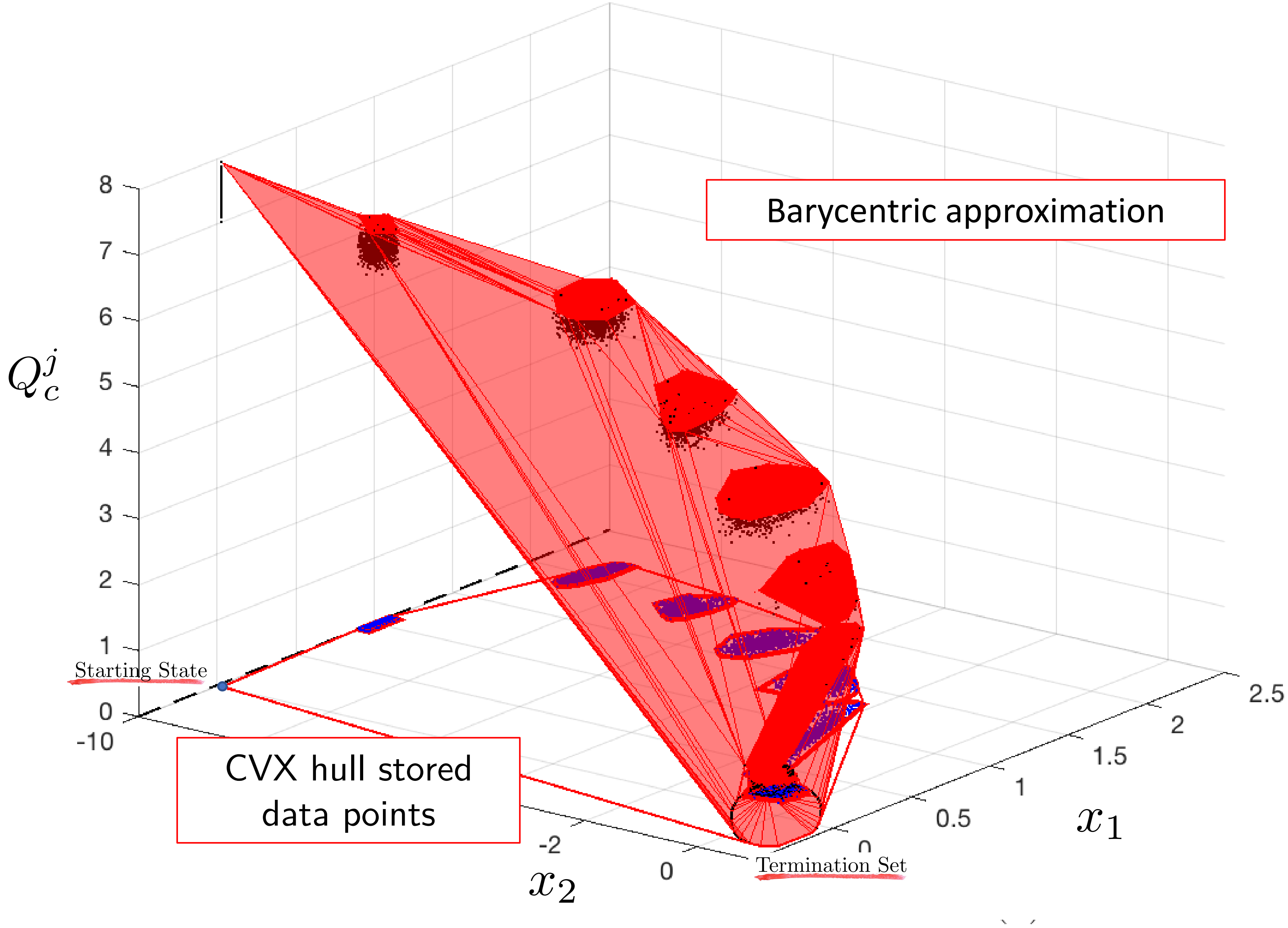

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