COMPSCI 527 Homework 2

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Problem 1(a)

$$A_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{q_1} = \frac{\mathbf{a_1}}{|\mathbf{a_1}|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{q_2} = normalized(\mathbf{a_2} - proj_{\mathbf{q_1}}\mathbf{a2}) = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

$$\mathbf{q_3} = normalized(\mathbf{a_3} - proj_{\mathbf{q_1}}\mathbf{a3} - proj_{\mathbf{q_2}}\mathbf{a3}) = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

Problem 1(b)

$$|\mathbf{q_1}| = \sqrt{(\frac{1}{\sqrt{2}})^2 + 0^2 + (\frac{1}{\sqrt{2}})^2} = 1$$

$$|\mathbf{q_2}| = \sqrt{(-\frac{\sqrt{6}}{6})^2 + (\frac{\sqrt{6}}{3})^2 + (\frac{\sqrt{6}}{6})^2} = 1$$

$$|\mathbf{q_2}| = \sqrt{(\frac{\sqrt{3}}{3})^2 + (\frac{\sqrt{3}}{3})^2 + (-\frac{\sqrt{3}}{3}} = 1$$

$$\mathbf{q_1} \cdot \mathbf{q_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} = \mathbf{0}$$

$$\mathbf{q_2} \cdot \mathbf{q_3} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} = \mathbf{0}$$

$$\mathbf{q_1} \cdot \mathbf{q_3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} = \mathbf{0}$$

Problem 1(c)

r is equal to the rank of the matrix A.

Problem 1(d)

Yes, since Gram-Schmidt gives us an orthogonal basis for the column space, it gives us the dimension of the column space, which is equal to the rank.

Problem 1(e)

Q forms a basis for the vector space A. This means that if a solution exists to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ can be expressed as a linear combination of the orthonormal vectors in Q, i.e. if a solution exists to the linear system $Q\mathbf{x}' = \mathbf{b}$, where $\mathbf{x}'$ is a r-by-1 vector of constants (not necessarily satisfying $A\mathbf{x} = \mathbf{b}$), then a solution exists to $A\mathbf{x} = \mathbf{b}$.

Problem 1(f)

$$r_i j = \mathbf{q_i} \cdot \mathbf{a_j}$$

$$r_j j = |\mathbf{a_j'}|$$

Problem 1(g)

$$R = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

Problem 1(h)

$$Q = \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} & \mathbf{q_3} \end{bmatrix}$$

$$R = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}$$

In the third iteration of the `for` loop, the `if` statement fails and therefore r is not incremented. The total number of columns in q is r, so there is one less column in Q. This makes sense, since Q should be an orthogonal matrix whose column space is the same as A; if A has linearly dependent vectors, then the number of column vectors in Q will be reduced.

$r_{jj} = |\mathbf{a'_j}| = 0$, so the last element in the diagonal will be 0.

Problem 1(i)

$$Q_{m\times r}$$

$$R_{r\times n}$$

Problem 1(j)

Problem 1(k)

Problem 2(a)

$$\mathbf{c} = Q^{-1}\mathbf{b}$$

Problem 2(b)

Since Q is an orthogonal matrix, $Q^{-1} = Q^T$. Therefore, we don't require the expensive computation of determining the inverse of Q and can instead take the transpose (which is very efficient).

Problem 2(c)

Since R is a triangular matrix, apply the algorithm of backward substitution pseudo-coded below:

```
x = new arr[n]
for i = n to 1
    x[i] = c[i]
    for j = i + 1 to n
        x[i] = x[i] - x[j] * R[i][j]
    end
    x[i] = x[i] / R[i][i]
end
```

Problem 2(d)

Since some columns of A are linearly independent, the solution has free variables. In order to pick one solution, pick a free variable at random and then back-sbustitute:

Problem 2(e)

Since $leftnull(A) = range(A)^{\perp}$, we can find a basis of R^m by using the identity matrix as A and keeping Q and R. Then we remove the vectors that were already in Q to find a basis for the orthogonal complement of $range(A)$.

```
m = size(Q,1)
[Qn, Rn] = ggs(eye(m), Q, R)
L = Qn(:, m:)
```

Problem 2(f)

Problem 2(g)

The system in eq. 3 admits infinitely many solutions if the rank of R is less than the number of columns in A. In this case, the set of solutions to the system $R\mathbf{x} = \mathbf{c}$ can be defined precisely by the space spanned by null(A). null(A) is a vector space.

Problem 2(h)

Problem 2(i)

See next page.

Test case 1: A has rank 1. Product checks **FAILED**. Dimension checks passed.

$$A = \begin{bmatrix} -0.32 & -0.05 \\ -0.55 & -0.08 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.48 \\ 0.54 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.43 \\ 0.07 \end{bmatrix}, \quad R = \begin{bmatrix} 0.63 & 0.10 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.15 \\ -0.99 \end{bmatrix}, \quad W = \begin{bmatrix} -0.99 \\ -0.15 \end{bmatrix}, \quad L = \begin{bmatrix} 0.86 \\ -0.50 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.50 \\ -0.86 \end{bmatrix}$$

Test case 2: A has rank 2. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.93 & -0.56 \\ -0.53 & -0.59 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.00 \\ 0.15 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2.66 \\ -2.66 \end{bmatrix}, \quad R = \begin{bmatrix} 1.07 & 0.78 \\ 0.00 & 0.24 \end{bmatrix}$$

$$N = [\,], \quad W = \begin{bmatrix} -0.86 & 0.51 \\ -0.51 & -0.86 \end{bmatrix}, \quad L = [\,], \quad Q = \begin{bmatrix} -0.87 & 0.50 \\ -0.50 & -0.87 \end{bmatrix}$$

Test case 3: A has rank 2. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} 0.41 & 0.55 \\ -0.34 & -0.19 \\ -0.55 & -0.44 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.23 \\ -0.05 \\ -0.15 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -0.15 \\ 0.53 \end{bmatrix}, \quad R = \begin{bmatrix} 0.76 & 0.69 \\ 0.00 & 0.22 \end{bmatrix}$$

$$N = [\,], \quad W = \begin{bmatrix} 0.60 & -0.80 \\ 0.80 & 0.60 \end{bmatrix}, \quad L = \begin{bmatrix} 0.25 \\ -0.73 \\ 0.64 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.54 & 0.80 \\ -0.44 & 0.53 \\ -0.72 & 0.28 \end{bmatrix}$$

Test case 4: A has rank 1. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.30 & -0.31 \\ -0.23 & -0.24 \\ -0.67 & -0.69 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -0.07 \\ -0.06 \\ -0.16 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.12 \\ 0.12 \end{bmatrix}, \quad R = \begin{bmatrix} 0.77 & 0.80 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.72 \\ -0.69 \end{bmatrix}, \quad W = \begin{bmatrix} -0.69 \\ -0.72 \end{bmatrix}, \quad L = \begin{bmatrix} 0.92 & 0.00 \\ -0.13 & 0.95 \\ -0.37 & -0.32 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.39 \\ -0.30 \\ -0.87 \end{bmatrix}$$

Test case 5: A has rank 1. Product checks **FAILED**. Dimension checks passed.

$$A = \begin{bmatrix} -0.52 & -0.19 & -0.47 \\ -0.54 & -0.20 & -0.49 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.29 \\ 0.34 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.45 \\ 0.16 \\ 0.40 \end{bmatrix}, \quad R = \begin{bmatrix} 0.75 & 0.28 & 0.68 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.70 & 0.00 \\ -0.27 & 0.93 \\ -0.66 & -0.38 \end{bmatrix}, \quad W = \begin{bmatrix} -0.72 \\ -0.26 \\ -0.65 \end{bmatrix}, \quad L = \begin{bmatrix} 0.72 \\ -0.70 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.70 \\ -0.72 \end{bmatrix}$$

Test case 6: A has rank 2. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.62 & -0.79 & -0.77 \\ -0.06 & -0.11 & 0.40 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.23 \\ 0.46 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.20 \\ 0.19 \\ 1.24 \end{bmatrix}, \quad R = \begin{bmatrix} 0.63 & 0.79 & 0.72 \\ 0.00 & 0.03 & -0.47 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.80 \\ -0.60 \\ -0.04 \end{bmatrix}, \quad W = \begin{bmatrix} -0.49 & -0.34 \\ -0.62 & -0.50 \\ -0.61 & 0.79 \end{bmatrix}, \quad L = [\,], \quad Q = \begin{bmatrix} -0.99 & 0.10 \\ -0.10 & -0.99 \end{bmatrix}$$
