COMPSCI 527 Homework 2

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September 24, 2015

Problem 1(a)

$$A_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{q_1} = \frac{\mathbf{a_1}}{|\mathbf{a_1}|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{q_2} = normalized(\mathbf{a_2} - proj_{\mathbf{q_1}}\mathbf{a2}) = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

$$\mathbf{q_3} = normalized(\mathbf{a_3} - proj_{\mathbf{q_1}}\mathbf{a3} - proj_{\mathbf{q_2}}\mathbf{a3}) = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

Problem 1(b)

$$|\mathbf{q_1}| = \sqrt{(\frac{1}{\sqrt{2}})^2 + 0^2 + (\frac{1}{\sqrt{2}})^2} = 1$$

$$|\mathbf{q_2}| = \sqrt{(-\frac{\sqrt{6}}{6})^2 + (\frac{\sqrt{6}}{3})^2 + (\frac{\sqrt{6}}{6})^2} = 1$$

$$|\mathbf{q_2}| = \sqrt{(\frac{\sqrt{3}}{3})^2 + (\frac{\sqrt{3}}{3})^2 + (-\frac{\sqrt{3}}{3}} = 1$$

$$\mathbf{q_1} \cdot \mathbf{q_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} = \mathbf{0}$$

$$\mathbf{q_2} \cdot \mathbf{q_3} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} = \mathbf{0}$$

$$\mathbf{q_1} \cdot \mathbf{q_3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} = \mathbf{0}$$

Problem 1(c)

r is equal to the rank of the matrix A.

Problem 1(d)

Yes, since Gram-Schmidt gives us an orthogonal basis for the column space when applied to the column vectors in A, it gives us the dimension of the column space, which is equal to the rank.

Problem 1(e)

Q forms a basis for the vector space A. This means that if a solution exists to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ can be expressed as a linear combination of the orthonormal vectors in Q, i.e. if a solution exists to the linear system $Q\mathbf{x'} = \mathbf{b}$, where $\mathbf{x'}$ is a r-by-1 vector of constants (not necessarily satisfying $A\mathbf{x} = \mathbf{b}$), then a solution exists to $A\mathbf{x} = \mathbf{b}$.

Problem 1(f)

$$r_{ij} = \mathbf{q_i} \cdot \mathbf{a_j}$$

$$r_{jj} = |\mathbf{a'_j}|$$

Problem 1(g)

$$R = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

Problem 1(h)

$$Q = \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} & \mathbf{q_3} \end{bmatrix}$$

$$R = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}$$

In the third iteration of the `for` loop, the `if` statement fails and therefore r is not incremented. The total number of columns in q is r, so there is one less column in Q. This makes sense, since Q should be an orthogonal matrix whose column space is the same as A; if A has linearly dependent vectors, then the number of column vectors in Q will be reduced.

$r_{jj} = |\mathbf{a}'_\mathbf{j}| = 0$, so the last element in the diagonal will be 0.

Problem 1(i)

$$Q_{m\times r}$$

$$R_{r\times n}$$

Problem 1(j)

```
function [Q, R] = gs(A)

Q = [];
R = [];

r = 0;
for j = 1:size(A, 2)
    ap = A(:, j);
    for i = 1:r
        R(i, j) = dot(Q(:, i), A(:, j));
        ap = ap - R(i, j).*Q(:, i);
    end
    rjj = norm(ap);
    if rjj > sqrt(eps)
        r = r + 1;
        R(j,j) = rjj;
        Q(:, r) = ap/rjj;
    end
end
```

Problem 1(k)

```
function [Q, R] = ggs(A, Q, R)

if nargin < 3 || isempty(Q) || isempty(R)
    Q = [];
    R = [];
end

[m, n] = size(A);
[r0, n0] = size(R);

if ~isempty(Q) && size(Q, 1) ~= m
    error('A and Q have inconsistent row sizes')
end
```

```
if ~isempty(Q) && size(Q, 2) ~= r0
    error('Q and R have inconsistent sizes')
end

[qr, qc] = size(Q);

% Your code here
r = qc;
for j = 1+qc:n+qc
    ap = A(:, j-qc);
    for i = 1:r
        R(i, j) = dot(Q(:, i), A(:, j-qc));
        ap = ap - R(i, j).*Q(:, i);
    end
    rjj = norm(ap);
    if rjj > sqrt(eps)
        r = r + 1;
        R(j,j) = rjj;
        Q(:, r) = ap/rjj;
    end
end
```

Problem 2(a)

$$\mathbf{c} = Q^{-1}\mathbf{b} = Q^T\mathbf{b}$$

Problem 2(b)

Since Q is an orthogonal matrix, $Q^{-1} = Q^T$. Therefore, we don't require the expensive computation of determining the inverse of Q and can instead take the transpose (which is very efficient).

Problem 2(c)

Since R is a triangular matrix, apply the algorithm of backward substitution pseudo-coded below:

```
x = new arr[n]
for i = n to 1
    x[i] = c[i]
    for j = i + 1 to n
        x[i] = x[i] - x[j] * R[i][j]
    end
    x[i] = x[i] / R[i][i]
end
```

Problem 2(d)

Since some columns of A are linearly independent, the solution has free variables. In order to pick one solution, pick a free variable at random and then back-substitute:

Problem 2(e)

Since $leftnull(A) = range(A)^{\perp}$, we can find a basis of R^m by using the identity matrix as A and keeping Q and R. Then we remove the vectors that were already in Q to find a basis for the orthogonal complement of $range(A)$.

```
[m, n] = size(Q);
[Qn, Rn] = ggs(eye(m), Q, R);
L = Qn(:, n+1:end);
```

Problem 2(f)

Problem 2(g)

The system in eq. 3 admits infinitely many solutions if the rank of A is less than the number of columns in A (i.e. `size(Q,2) < size(A,2)`) and x exists (i.e. `x~=[]`).

The set of solutions is $\mathbf{n} + \mathbf{x}$, $\forall n \in N$. This is the set of vectors in N, shifted by $\mathbf{x}$, which is an **affine space**.

This is true because of the following theorem, taken verbatim from this link from Stony Brook:

Theorem

Let x_p be a *particular solution* to $Ax = b$. Then the *general solution* to $Ax = b$ is given by $x_p + x_n$ where x_n is a vector in the nullspace of A.

Problem 2(h)

Check that $\mathbf{n} \cdot \mathbf{w} = \mathbf{0}$ $\forall \mathbf{n} \in N$, $\forall \mathbf{w} \in W$. Equivalently, `~any(N.' * W)` (Replace N and W with L and Q respectively, to check for L and Q.)

Problem 2(i)

See next page.

Test case 1: A has rank 1. Product checks **FAILED**. Dimension checks passed.

$$A = \begin{bmatrix} -0.32 & -0.05 \\ -0.55 & -0.08 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.48 \\ 0.54 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.43 \\ 0.07 \end{bmatrix}, \quad R = \begin{bmatrix} 0.63 & 0.10 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.15 \\ -0.99 \end{bmatrix}, \quad W = \begin{bmatrix} -0.99 \\ -0.15 \end{bmatrix}, \quad L = \begin{bmatrix} 0.86 \\ -0.50 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.50 \\ -0.86 \end{bmatrix}$$

Test case 2: A has rank 2. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.93 & -0.56 \\ -0.53 & -0.59 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.00 \\ 0.15 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2.66 \\ -2.66 \end{bmatrix}, \quad R = \begin{bmatrix} 1.07 & 0.78 \\ 0.00 & 0.24 \end{bmatrix}$$

$$N = [\,], \quad W = \begin{bmatrix} -0.86 & 0.51 \\ -0.51 & -0.86 \end{bmatrix}, \quad L = [\,], \quad Q = \begin{bmatrix} -0.87 & 0.50 \\ -0.50 & -0.87 \end{bmatrix}$$

Test case 3: A has rank 2. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} 0.41 & 0.55 \\ -0.34 & -0.19 \\ -0.55 & -0.44 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.23 \\ -0.05 \\ -0.15 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -0.15 \\ 0.53 \end{bmatrix}, \quad R = \begin{bmatrix} 0.76 & 0.69 \\ 0.00 & 0.22 \end{bmatrix}$$

$$N = [\,], \quad W = \begin{bmatrix} 0.60 & -0.80 \\ 0.80 & 0.60 \end{bmatrix}, \quad L = \begin{bmatrix} 0.25 \\ -0.73 \\ 0.64 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.54 & 0.80 \\ -0.44 & 0.53 \\ -0.72 & 0.28 \end{bmatrix}$$

Test case 4: A has rank 1. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.30 & -0.31 \\ -0.23 & -0.24 \\ -0.67 & -0.69 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -0.07 \\ -0.06 \\ -0.16 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.12 \\ 0.12 \end{bmatrix}, \quad R = \begin{bmatrix} 0.77 & 0.80 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.72 \\ -0.69 \end{bmatrix}, \quad W = \begin{bmatrix} -0.69 \\ -0.72 \end{bmatrix}, \quad L = \begin{bmatrix} 0.92 & 0.00 \\ -0.13 & 0.95 \\ -0.37 & -0.32 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.39 \\ -0.30 \\ -0.87 \end{bmatrix}$$

Test case 5: A has rank 1. Product checks **FAILED**. Dimension checks passed.

$$A = \begin{bmatrix} -0.52 & -0.19 & -0.47 \\ -0.54 & -0.20 & -0.49 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.29 \\ 0.34 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.45 \\ 0.16 \\ 0.40 \end{bmatrix}, \quad R = \begin{bmatrix} 0.75 & 0.28 & 0.68 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.70 & 0.00 \\ -0.27 & 0.93 \\ -0.66 & -0.38 \end{bmatrix}, \quad W = \begin{bmatrix} -0.72 \\ -0.26 \\ -0.65 \end{bmatrix}, \quad L = \begin{bmatrix} 0.72 \\ -0.70 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.70 \\ -0.72 \end{bmatrix}$$

Test case 6: A has rank 2. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.62 & -0.79 & -0.77 \\ -0.06 & -0.11 & 0.40 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.23 \\ 0.46 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.20 \\ 0.19 \\ 1.24 \end{bmatrix}, \quad R = \begin{bmatrix} 0.63 & 0.79 & 0.72 \\ 0.00 & 0.03 & -0.47 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.80 \\ -0.60 \\ -0.04 \end{bmatrix}, \quad W = \begin{bmatrix} -0.49 & -0.34 \\ -0.62 & -0.50 \\ -0.61 & 0.79 \end{bmatrix}, \quad L = [\,], \quad Q = \begin{bmatrix} -0.99 & 0.10 \\ -0.10 & -0.99 \end{bmatrix}$$
