Cycle Breaking in Proxy Voting

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1 Introduction

1.1 Introduction to Proxy Voting

1.2 Motivation

Lemma 1.1. *Here is an example Lemma.*

Proof. The proof is trivial. □

[?]

2 Previous Work

2.1 Pseudoforests

3 Our Work

We start with some notation. Let $a \to b$ denote that voter a proxies to voter b. We let $\mathbb{O}$ denote a total order over the set of all voters. Let $G = (V, E)$ denote the graph of proxy votes, where V is the set of all voters, and E is the set of all edges.

3.1 Formalization as Graph Theory

To visualize and analyze a proxy voting system, we can represent the system as a graph, where each voter is a node having weight 1 and each proxy vote is a directed edge pointing from the proxying voter to the voter who is being proxied to. As such, voting power flows between nodes along edge paths; each voter has voting power equal to its own weight of 1, plus the sum of the voting power of the nodes pointing to it. We can formally define the voting power $p(v)$ of voter v recursively:

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Definition 3.1. Voting Power of a voter: For a voter v, $\quad p(v) = 1 + \sum_{\{i \in V | (i,v) \in E\}} p(i)$

Furthermore, we can define the voting power $P(Q)$ of a set Q as follows:

Definition 3.2. Voting Power of a set: For $Q \subseteq V, P(Q)$ is the total number of votes cast from Q.

Proxy voting graphs also have a few special properties. Particularly, each node in the graph can have at most one outgoing edge, or outdegree at most one. Graphs with this property are called **directed pseudoforests**. As a consequence, we know that proxy voting graphs are made up of connected components each of which have at most one cycle.

3.2 Existence of cycles

There are cases where we can guarantee no cycles, even reasonably common cases.

Let us define a competence measure $C_i \in \mathbb{R}$ for $i \in V$, which is the objective measure of voter i's competence in voting. Let the perceived competence measure of voter j perceived by voter i be c_{ij} for $i, j \in V$.

We assume throughout this paper that $i \to j \implies c_{ii} < c_{ij}$, i.e., voter i proxies to voter j only if i perceives j to have greater competence than itself.

At the ideal limit where actual competence equals perceived competence, we have the following monotonicity theorem:

Theorem 3.1. *If $C_i = c_{ji}$ $\forall j, i \in V$, then G contains no cycles.*

Proof. We prove by contradiction

Suppose there exists C_i for $i \in \{1, 2, ..., n\}$ for which nodes i form a cycle $1 \to 2 \to ...n \to 1$. Then $c_{12} < c_{23} < ... < c_{(n-1)n} < c_{n1} \implies C_1 < C_2 < ... < C_n < C_1 \implies C_1 < C_1$. $\Rightarrow\Leftarrow$ $\square$

3.3 Cycle Formation

Let $\mathbb{S}$ denote the set of all possible proxy voting systems. Let $S \in \mathbb{S}$ denote an arbitrary proxy voting system.

Definition 3.3. Choice of Role: S exhibits Choice of Role if all voters in S can choose whether they vote or proxy.

Definition 3.4. Enfranchisement: S exhibits Enfranchisement if $\forall a \in V, P(V) > P(V \setminus \{a\})$

Definition 3.5. Voters Choose: For a voter v, if v voted, let A_v denote the set of candidates for which v voted. If v did not vote, let A_v denote the set of candidates for which v would vote if v were to vote. Let c denote the candidate chosen by the voting rule in S. S exhibits Voters Choose if $c \in \bigcup_{v \in V} A_v$

Lemma 3.2 (Cycles Form)**.** *If no voter votes, then there must exist some cycle.*

Proof. We prove by contradiction: Suppose that G does not contain a cycle. Then G must consist only of trees, which implies $|E| \leq |V| - 1$. However, since nobody voted, everybody proxied, which implies that $|E| = |V|$. $\Rightarrow\Leftarrow$ □

Theorem 3.3. *There exists no system S for which the properties Choice of Role, Enfranchisement, and Voters Choose simultaneously hold.*

Proof. Let us assume that all voters proxy, which is allowed under Choice of Role. We have from Lemma 3.2 that G must contain a cycle. □

3.4 Centrality

As we mentioned in the previous section, one way to deal with with cycles in proxy voting is through **preference elicitation** – that is, querying voters to find out what their true preferences are, and changing their proxy into an actual vote. This leads us to an important question: How do we decide who to query? Clearly, we seek to minimize the number of users queried (this adheres most closely to Choice of Role, as well as minimizing operating costs), which leads us to assume that for each pseudotree in the graph, we should query only one voter — there vote will be allocated to the rest of the pseudotree. Thus, our task is as follows: for one pseudotree, determine the voter best suited to represent the whole tree. We assume that the only information we have is to whom someone proxied – thus, we must determine who to query based solely on the structure of the graph. In order to do this, we make one more assumption: A proxy from A to B represents a vote from A in B's competence. Our task now is to choose the most competent voter in each connected competent — a task for which centrality measures from social network theory are uniquely suited.

A centrality measure intuitively captures the idea of the most important vertex within a graph – in a social network, centrality measures the most influential nodes in the network.[2][3] Centrality is a well established notion, having been used in various network analyses ranging from which people are most likely to spread disease to analyzing dominance in primate communities. Furthermore, centrality has already been successfully used to break cycles in a pseudoforest.[4] We draw upon all of this work to propose the following: Centrality serves as an ideal tool with which to break cycles.

Let A be the adjacency matrix of G. If $C_{Katz}(i)$ denotes the Katz Centrality of a node i, then $C_{Katz}(i)$ is defined as follows:

Definition 3.6. Katz Centrality: $C_{Katz}(i) = \sum_{k=1}^{\infty} \sum_{j=1}^{n} \alpha^k (A^k)_{ji}$

A vector of the Katz Centrality for all nodes in the graph is expressed as follows:

Theorem 3.4. $\overrightarrow{C_{Katz}} = ((I - \alpha A^T)^{-1} - I)\vec{I}$

Theorem 3.5. *$\overrightarrow{C_{Katz}}$ can be computed directly in time $O(n^{2.373})$*

Proof. The fastest known algorithm for matrix inverse runs in time $O(n^2.373)$. Matrix addition can be done in linear time – thus, the inverse takes the longest of all operations in this equation, so the runtime is equal to that of the matrix inverse. □

4 Conclusion

4.1 Future Work

- *Confidence Networks*: Relax the constraints of proxy voting to allow voters to proxy to multiple people. Then, use centrality measures, not as a means of breaking cycles, but as a way to identify experts in a group.

- *Behavioral Model of Voting*: What factors lead people to proxy? Quantify various aspects such as confidence, trust, bias, and knowledge and generate functions that say to whom each voter should proxy.

In this paper, we explored the idea of using centrality measures to break cycles, our reasoning being that a proxy to someone is an implicit vote of confidence in that person's expertise. If we view the graph of proxy votes as a graph where directed edges represent votes of confidence, then centrality is the natural choice to choose the experts in the graph. However, centrality measures do not require that nodes only "proxy" to one other node — having this as a requirement in proxy voting limits the information we can obtain from voters, as it forces the graph to be unnecessarily sparse. One potential way to get around this problem is through the use of *confidence networks*, where voters are allowed to list not just one but multiple other voters in which they feel confident. In this case, centrality measures would be used, not as a way to deal with the unfortunate case of cycles, but rather as way to calculate the relative expertise of voters in a system. There are many different ways this could be used — for instance, votes could be weighted by the calculated expertise of the voters who cast them.

4.2 Impact

In this paper, we explained the motivation behind proxy voting and explored the aspect of cycles — why they form and how to break them. Ultimately, in order to implement a proxy voting system, we need a way to handle cycles. Until now, the only method proposed to deal with cycles involved granting them zero voting-power. By proposing a cycle-breaking method that does not disenfranchise voters in a cycle, we provide allow for the use of proxy voting in situations where voters would be upset if their votes were discarded, such as modern political elections.

References

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