Topic: Fast Randomized SVD

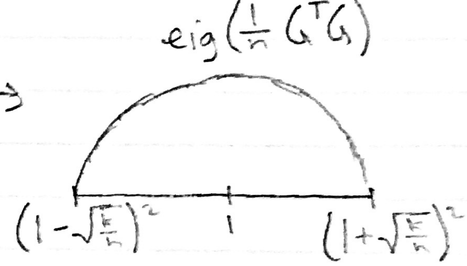
tall Gaussian matrices:

$G \sim N(0,1)$, not symmetric

$k \le m$, denote k singular values as $\sigma_1 \dots \sigma_k$

defn: condition number $\kappa(G) = \frac{\sigma_1}{\sigma_k}$

from M-P: $\begin{cases} \sigma_1 \approx \sqrt{m} + \sqrt{k} \\ \sigma_k \approx \sqrt{m} - \sqrt{k} \end{cases} \longrightarrow$



Thm (Gordon's): $X_{m \times n}$, $X_{ij} \sim$ i.i.d. $N(0,1)$, $m \ge n$

then $\sqrt{m} - \sqrt{n} \le E\,\sigma_n \le E\,\sigma_1 \le \sqrt{m} + \sqrt{n}$

– stronger than last lecture's result, non-asymptotic

Recall Gaussian concentration:

σ_1 or σ_k is 1-Lip function of G

$\rightarrow$ w.h.p. $|\sigma_1 - E\sigma_1| \sim O(1)$ i.e. $Pr(|\sigma_1 - E\sigma_1| > \alpha) \le e^{-\alpha^2/2}$

$\rightarrow$ w.h.p. $\sqrt{m} - \sqrt{k} - t_2 \le \sigma_k \le \sigma_1 \le \sqrt{m} + \sqrt{k} + t_1$, where $t_1, t_2 \sim O(1)$

Thus, $\kappa(G) \le \dfrac{\sqrt{m} + \sqrt{k} + t_1}{\sqrt{m} - \sqrt{k} - t_2}$, where $m \ge k \gg t_1, t_2$

$\rightarrow$ if $m \gg k$, $\kappa(G)$ will be close to 1

Randomized SVD:

Goal: given $A_{m\times n}$, target k-rank factorization

$$A_{m\times n} = B_{m\times k} \cdot C_{k\times n} \quad \text{e.g. } \min_{B,C} \|BC - A\|_{Fro} \rightarrow \text{"k-SVD"}$$

assume $\sigma_1(A) \geq \ldots \geq \sigma_n(A)$, $\sigma_{k+1}(A) = \varepsilon$

Note: classical complexity of k-SVD is $O(mnk)$

Idea: draw $\Omega_{n\times k'}$, Ω_{ij} i.i.d. $N(0,1)$
$\rightarrow$ Compute SVD of $A\Omega$ (can be done $O(mk^2)$)

Note: Ω is well-conditioned, $k' \approx k$
Note: $\text{Range}_k(A\Omega) \approx \text{Range}_k(A)$

Approximate k-SVD algorithm:

Input: $A_{m\times n}$, k

Step 1: draw $\Omega_{n\times l}$, $l = 2k$

$$Y \leftarrow A\Omega$$
$$Y = QR,\ Q_{m\times l},\ QQ^T \approx I_{l\times l}$$

Step 2: Compute SVD of $Q^T A$
i.e. $Q^T A = \tilde{U} S V^T$

Step 3: $U \leftarrow Q\tilde{U}$

Output: $U_{m\times l}$, $S_{l\times l}$, $V_{n\times l}$

Thm: (U, S, V) approximates k-SVD of A

i.e. $\mathbb{E}\|A - USV^T\| \leq \left(1 + 4\sqrt{\frac{2n}{k-1}}\right) \sigma_{k+1}(A)$

Recall: $\sigma_{k+1}(A) = \varepsilon$

Applying concentration argument, we obtain:

$\|A - USV^T\|_{op} \leq c \cdot \sigma_{k+1}(A)$ w.h.p.