MATH 690: Lecture Scribing

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1 Summary

In today's lecture, we covered three parts. We first reviewed the topic on graph denoising and made two remarks on the method we discussed previously. Then we discussed the topic on graph synchronization, with three different examples covered. Finally, we gave a brief introduction the our fifth topic of the course, concentration of measure.

2 Review: Graph Denoising

The problem setting is given graph $G = (V, E)$, with $|V| = n$, and a function $f : v \to R$, $f \in R^n$, and f is smooth on the graph (i.e. $f^T(D - W)f < \delta$). We are given $x = f + \epsilon$, a situation with noise. Our method is to give an estimator, $\tilde{f} = Px$, where $\tilde{f}_i = \sum_{j=1}^n P_{ij}x_j$, where $P = D^{-1}W$ and $\sum_{j=1}^n P_{ij} = 1$.

2.1 Remark One: Spectral Filtering

From our previous discussion, we may write $P = \Psi\Lambda\Phi^T$, and thus $Px = \sum_k \lambda_k \psi_k < \phi_k, x >$, where a generalized version can be written as $\sum_k f(\lambda_k) < \phi_k, x > \psi_k$. The signal-to-noise ratio (SNR) for each k is $SNR_k = \frac{|c_k|^2}{\sigma^2} \to f(\lambda)$. In case that we have more prior knowledge of SNR_k, we can design the filter function $f(\lambda)$ accordingly.

2.2 Remark Two: Optimal Denoising Strategy with Wavelet Shrinkage

P_x: global Fourier Transform, and can we do for local basis?

G : mesh on some continuous domain, $g : [0, 1] \subset R^1$. And we introduce the 'optimal' denoising strategy $\to$ wavelet shrinkage:

$$\begin{aligned}
x &\to c_k(x), wavelet\ transform\ x \sim \sum_k c_k(x)\psi_k \\
&\to change\ c_k\ to\ be\ T_\delta(c_k) = \tilde{c_k}\ to\ get\ rid\ of\ the\ small\ terms \\
&\to \tilde{x} \sim \sum_k \tilde{c_k}\psi_k
\end{aligned} \tag{1}$$

3 Synchronization: $G = (V, E)$

3.1 Z_2 Synchronization

Goal: recover unknown signs ($z_i \in \{1, -1\}$) on V, up to a global sign-flip from pairwise measurement on E:

if $(i, j) \in E$, $G_{ij} = z_i z_j = \pm 1$, positive if $z_i = z_j$ and negative otherwise. A simple example would be friend and enemy relationships.

For A as the adjacency matrix of G, define matrix $G = (zz^T) \odot A$, where $\odot$ is the Hadamard product, defined for A, B, two $n * n$ matrices, where $(A \odot B)_{ij} = A_{ij}B_{ij}$.

If there is no noise and the graph is connected, then we may solve the problem easily by fixing a point to start with, then traverse the graph and assign signs according to the relationship on E.

Situation With Noise: We have

$$G_ij = \begin{cases} z_i z_j,\ (i,j) \in E \\ Bern(\frac{1}{2}),\ o.t. \end{cases} = \begin{cases} z_i z_j,\ A_{ij} = 1 \\ w_{ij},\ A_{ij} = 0 \end{cases} \quad \text{where } w_{ij} = \begin{cases} 1 \text{with probability 0.5},\ i < j, \text{i.i.d} \\ -1\ o.t. \end{cases} \tag{2}$$

Thus, we have $G = (zz^T) \odot A + W \odot (\mathbf{1}\mathbf{1}^T - A)$

The task is for given G, we need to recover z with a global sign flip. The method would be take the first eigenvector of G, v_1, and have $\tilde{z_i} = sign(v_1)_i$

Given A, a random matrix, $i < j$, i.i.d, a matrix that determines the correctness of information.

$$A = \begin{cases} 1 \text{ w.p. } p \\ 0 \text{ w.p. } 1 - p \end{cases} \tag{3}$$

We have pzz^T with rank 1, $\lambda = pn$. And we have $E_A G = zz^T \odot EA + w \odot E(\mathbf{1}\mathbf{1}^T - A) = p(zz^T) + (1-p)W$

For $(1-p)W$, we have $|(1-p)W| \leq O(\sqrt{n})$, since $c\sqrt{n}$ where $c = (1-p)\sqrt{Var(w_{ij})} * 2 \leq (1-p) * 2$

Remark: SDP relaxation of the problem Problem setting:

1. $max_X < G, X >$ s.t. $X \succeq 0$ and $\forall i,\ X_{ii} = 1$

2. $< G, X >= Tr(GX)$, $X^* = zz^T$, and thus $Tr(GX^*) = Tr(Gzz^T) = z^T Gz$

3. constraints: $x_{ii} = z_i^2 = 1$ and $rank(x) = 1$ (relaxed)

3.2 S^1 Synchronization

We assign time $t_i \in [0, 2\pi]$ to each vertex to replace the sign. We observe

$$\theta_{ij} = \begin{cases} t_i - t_j, \text{ w.p. } p \\ Uniform([0, 2\pi]), \text{ w.p. } 1 - p \end{cases} \tag{4}$$

Then for pairwise relationship, we can have $e^{i(t_i - t_j)} = (e^{it_i})(e^{-it_j}) = z_i \bar{z_j}$, where $z_i = e^{it_i}$. Consider the information matrix $z \in C^n$, we have $(zz^*)_{ij} = z_i z_j$, and $G_{ij} = e^{i\theta_{ij}}$, and $G = (zz^*) \odot A + W \odot (\mathbf{11^T} - A)$, where $w_{ij} = e^{ig_{ij}}$, and $g_{ij} \sim U[0, 2\pi]$, i.i.d for $i < j$. And then we may follow the similar analysis as the first case for Z_2.

3.3 $SO(3)$ Synchronization

To recover $R_1, ..., R_n$, unknown rotations (3*3 matrices). Given observations (i, j), $R_i^{-1} R_j$, we have a rank 3 matrices in the form:

$$\begin{bmatrix} R_1^T \\ R_2^T \\ ... \\ R_n^T \end{bmatrix} \begin{bmatrix} R_1 & R_2 & ... & R_n \end{bmatrix}$$

In this case, fraction of correct measurement $P \sim \frac{1}{\sqrt{n}}$, similar to previous examples.

4 Topic 5: Concentration of Measure

Concentrate: Control of the probability of "bad events", the tail part Given X, a random variable, we hope to have $Pr[|X - EX| > \alpha] \leq ?$, controlling the tail. For example, in the previous topic, we may set $x = \lambda_1(W)$