

$$A = \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix}$$

Soln

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$\det \begin{pmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -13-\lambda \end{pmatrix}$$

Using the third row

$$\det = 0 \times (\text{minor}_1) - 10 \times (\text{minor}_2) + (5-\lambda) \times (\text{minor}_3) - (-10) \times (\text{minor}_4)$$

$$\det = 0 - 10 \times \begin{pmatrix} 4-\lambda & -1 & -2 \\ -2 & -2 & -4 \\ -1 & -14 & -13-\lambda \end{pmatrix} +$$

$$(5-\lambda) \begin{pmatrix} 4-\lambda & 8 & -2 \\ -2 & -9-\lambda & -4 \\ -1 & -13 & -13-\lambda \end{pmatrix} + 10 \begin{pmatrix} 4-\lambda & 8 & -1 \\ -2 & -9-\lambda & -2 \\ -1 & -13 & -14 \end{pmatrix}$$

$$\text{minor}_2 = \begin{pmatrix} 4-\lambda & -1 & -2 \\ -2 & -2 & -4 \\ -1 & -14 & -13-\lambda \end{pmatrix}$$

$$= (4-\lambda) \left((-2)(-13-\lambda) \right) - ((-4)(-14))$$

$$- (-1) \left((-2)(-13-\lambda) \right) - ((-4)(-1))$$

$$+ (-2) \left((-2)(-14) \right) - ((-2)(-1))$$

$$= (4-\lambda) \left((26+2\lambda) - (56) \right)$$

$$+ 1 \left((26+2\lambda) - 4 \right) + 52$$

$$= (4-\lambda)(2\lambda-30) + (22+2\lambda) - 52$$

$$= (4-\lambda)(2\lambda-30) + 2\lambda - 30$$

$$\text{minor}_2 = -2\lambda^2 + 40\lambda - 150$$

$$\text{minor } 3 \begin{pmatrix} 4-\lambda & 8 & -2 \\ -2 & -9-\lambda & -4 \\ -1 & -13 & -13-\lambda \end{pmatrix}$$

$$(4-\lambda)((-9-\lambda)(-13-\lambda) - (-4)(-13))$$

$$- 8((-2)(-13-\lambda) - (-4)(-1))$$

$$+ (-2)((-2)(-13) - (-9-\lambda)(-1)) =$$

$$= (4-\lambda)(117 + 9\lambda + 13\lambda + \lambda^2 - 52)$$

$$- 8(26 + 2\lambda - \cancel{9} + \cancel{\lambda^4})$$

$$+ (-2)(26 - 9 + \lambda)$$

$$= (4-\lambda)(65 + 22\lambda + \lambda^2) - 8(22 + 2\lambda)$$

$$- 2(17 - 9)$$

$$= -\lambda^3 - 18\lambda^2 + 23\lambda + 260$$

$$\text{minor } 3 = -\lambda^3 - 18\lambda^2 + 9\lambda + 50$$

$$\text{Minor 4} = \begin{vmatrix} 4-\lambda & 8 & -1 \\ -2 & -9-\lambda & -2 \\ -1 & -13 & -14 \end{vmatrix}$$

$$= (4-\lambda) \left((-9-\lambda)(-14) - (-2)(-13) \right)$$

$$+ (-8) \left((-2)(-14) - (-2)(-1) \right)$$

$$+ (-1) \left((-2)(-13) - (-9-\lambda)(-1) \right)$$

$$= (4-\lambda) (126 + 14\lambda - 26) + (-208)$$

$$-1 (26 - 9 + \lambda)$$

$$= (4-\lambda) (100 + 14\lambda) - 208 - 17 + \lambda$$

$$= (4-\lambda) (100 + 14\lambda) - 225 + \lambda$$

$$\text{Minor 4} = -14\lambda^2 - 43\lambda + 175$$

$$\det(A - \lambda I) = -10(-2\lambda^2 + 40\lambda - 150)$$

$$+ (5 - \lambda)(-\lambda^3 - 18\lambda^2 + 9\lambda + 50)$$

$$+ 10(-14\lambda^2 - 43\lambda + 175)$$

$$= 20\lambda^2 - 400\lambda + 1500 + 140\lambda^2$$

$$- 430\lambda + 1750 - 5\lambda^3 - 90\lambda^2 + 45\lambda$$

$$+ 250 + \lambda^4 + 18\lambda^3 + 9\lambda^2 + 50\lambda$$

$$= \lambda^4 + 13\lambda^3 - 219\lambda^2 - 835\lambda + 3500 = 0$$

If $a_1 = 13$, $a_2 = -219$, $a_3 = -835$, $a_4 = 3500$

it becomes $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$

Cubic resolvent equation

$$y^3 - a_2y^2 + (a_1a_3 - 4a_4)y + (4a_2a_4 - (a_3)^2 - a_1^2a_4) = 0$$

$$y^3 - (-219)y^2 + \left[(13)(-835) - 4(3500) \right]y + \left[4(-219)(3500) - (-835)^2 - (13)^2(3500) \right] = 0$$

$$y^3 + 219y^2 + (-10,855 - 14,000)y + (-3,066,000 - 697,225 - 591,500) = 0$$

$$y^3 + 219y^2 - 24,855y - 4,354,725 = 0$$

To find the cubic equation root, we did some testing to find y which is the root

$$y_1 = 147.948$$

to find the quartic root we use

$$z^2 + \frac{1}{2} \left(a_1 \pm \sqrt{(a_1)^2 - 4a_2} \right) z +$$

$$\frac{1}{2} \left(y_1 \pm \sqrt{(y_1)^2 - 4(a_4)} \right) = 0$$

$$z^2 + \frac{1}{2} (13 \pm \sqrt{(13)^2 - (4(-219)) + 4(147.948)})z + \frac{1}{2} (147.948 \pm \sqrt{(147.948)^2 - 4(3500)}) = 0$$

$$z^2 + \frac{1}{2} (13 \pm 40.45)z + \frac{1}{2} (147.948 \pm 88.86) = 0$$

We have 2 equations

$$z^2 + \frac{1}{2} (13 + 40.45)z + \frac{1}{2} (147.948 + 88.86) = 0$$

$$z^2 + \frac{1}{2} (13 - 40.45)z + \frac{1}{2} (147.948 - 88.86) = 0$$

$$z^2 + 26.73z + 118.40$$

$$z^2 - 13.73z + 29.54$$

$$\text{using } z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For equation 1

$$Z = \frac{-26.73 \pm \sqrt{(26.73)^2 - 4(1)(118.40)}}{2}$$

$$Z = \frac{-26.73 \pm \sqrt{240.83}}{2}$$

$$= \frac{-26.73 \pm 15.51}{2}$$

$$= \frac{-26.73 + 15.51}{2} \text{ or } \frac{-26.73 - 15.51}{2}$$

$$= -5.61 \text{ or } -21.12$$

For equation 2: $Z^2 - 13.73Z + 29.54$

$$Z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$Z = \frac{-(13.73) \pm \sqrt{(-13.73)^2 - 4(1)(29.54)}}{2}$$

$$Z = \frac{13.73 \pm \sqrt{70.43}}{2}$$

$$Z = \frac{13.73 + 8.39}{2} \text{ or } \frac{13.73 - 8.39}{2}$$

$$Z = 11.06 \text{ or } 2.67$$

$$\text{so } \lambda_1 = -21.12$$

$$\lambda_2 = -5.61$$

$$\lambda_3 = 2.67$$

$$\lambda_4 = 11.06$$

Finding Eigen vector $\lambda_1 = -21.12$

$$A = \begin{bmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{bmatrix}, \lambda = -21.12$$

To find the eigen vectors

$$(A - \lambda I)v = 0$$

$\lambda = -21.12$, we compute $A - (-21.12)I = A + 21.12I$

$$A + 21 \cdot 121 = \begin{pmatrix} 4 + 21 \cdot 12 & 8 & -1 & -2 \\ -2 & -9 + 21 \cdot 12 & -2 & -2 \\ 0 & 10 & 5 + 21 \cdot 12 & -4 \\ -1 & -13 & -14 & -13 \end{pmatrix}$$

$$= \begin{pmatrix} 25 \cdot 12 & 8 & -1 & -2 \\ -2 & 12 \cdot 12 & -2 & -4 \\ 0 & 10 & 26 \cdot 12 & -10 \\ -1 & -13 & -14 & 8 \cdot 12 \end{pmatrix}$$

$$(A - \lambda I)v = 0$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \begin{pmatrix} 25 \cdot 12 & 8 & -1 & -2 \\ -2 & 12 \cdot 12 & -2 & -4 \\ 0 & 10 & 26 \cdot 12 & -10 \\ -1 & -13 & -14 & 8 \cdot 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$R_1 \div 25 \cdot 12 \quad (1, 0.318, -0.0398, -0.0796)$$

$$R_2 = R_2 + 2R_1 \quad (0, 12.756, -2.0796, -4.15)$$

$$R_4 = R_4 + R_1 \quad (0, -12.682, -14.0398, 8.0404)$$

$$\begin{pmatrix} 1 & 0 & 0 & -0.198 \\ 0 & 1 & 0 & -0.366 \\ 0 & 0 & 1 & -0.243 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 - 0.198x_4 = 0 \Rightarrow x_1 = 0.198x_4$$

$$x_2 - 0.366x_4 = 0 \Rightarrow x_2 = 0.366x_4$$

$$x_3 - 0.243x_4 = 0 \Rightarrow x_3 = 0.243x_4$$

$$x_4 = \text{free parameter}$$

$$V = \begin{bmatrix} 0.198 \\ 0.366 \\ 0.243 \\ 1 \end{bmatrix}$$

$$\text{let's make } t = 5$$

$$V \approx \begin{pmatrix} 1 \\ 1.85 \\ 1.23 \\ 5.05 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix}$$

To find the importance of Eigen value

$$\lambda_1 = -21.12 \quad |\lambda_1| = 21.12$$

$$\lambda_2 = -5.61 \quad |\lambda_2| = 5.61$$

$$\lambda_3 = 2.67 \quad |\lambda_3| = 2.67$$

$$\lambda_4 = 11.06 \quad |\lambda_4| = 11.06$$

$$\Sigma = 40.46$$

$$\% \lambda_1 = \frac{21.12}{40.46} \times 100\% = 52.1\%$$

$$\% \lambda_2 = \frac{5.61}{40.46} \times 100\% = 13.86\%$$

$$\% \lambda_3 = \frac{2.67}{40.46} \times 100\% = 5.58\%$$

$$\% \lambda_4 = \frac{11.06}{40.46} \times 100\% = 27.33\%$$