$$\begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix} = A$$

Soln

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

=

$$\det\begin{pmatrix} 4-\lambda & 8 & -1 & -2 \\ -2 & -9-\lambda & -2 & -4 \\ 0 & 10 & 5-\lambda & -10 \\ -1 & -13 & -14 & -13-\lambda \end{pmatrix}$$

Using the third row

$$\det = 0 \times (\text{minor}_1) - 10 \times (\text{minor}_2) + (5-\lambda) \times (\text{minor}\,3) - (-10) \times (\text{minor}_4) =$$

$$\det = 0 - 10 \times \begin{pmatrix} 4-\lambda & -1 & -2 \\ -2 & -2 & -4 \\ -1 & -14 & -13-\lambda \end{pmatrix} +$$

$$(5-\lambda)\begin{pmatrix} 4-\lambda & 8 & -2 \\ -2 & -9-\lambda & -4 \\ -1 & -13 & -13-\lambda \end{pmatrix} + 10\begin{pmatrix} 4-\lambda & 8 & -1 \\ -2 & -9-\lambda & -2 \\ -1 & -13 & -14 \end{pmatrix}$$

$$\text{minor}_2 = \begin{pmatrix} 4-\lambda & -1 & -2 \\ -2 & -2 & -4 \\ -1 & -14 & -13-\lambda \end{pmatrix}$$

$$= (4-\lambda)\Big(\big((-2)(-13-\lambda)\big) - \big((-4)(-14)\big)\Big)$$

$$-(-1)\Big(\big((-2)(-13-\lambda)\big) - \big((-4)(-1)\big)\Big)$$

$$+(-2)\Big(\big((-2)(-14)\big) - \big((-2)(-1)\big)\Big)$$

$$= (4-\lambda)\big((26+2\lambda) - (56)\big)$$

$$+\cancel{(1)}\big((26+2\lambda)-4\big) + 52$$

$$= (4-\lambda)(2\lambda-30) + (22+2\lambda) - 52$$

$$= (4-\lambda)(2\lambda-30) + 2\lambda - 30$$

$$\text{minor } 2 = -2\lambda^2 + 40\lambda - 150$$

minor 3 $\begin{pmatrix} 4-\lambda & 8 & -2 \\ -2 & -9-\lambda & -4 \\ -1 & -13 & -13-\lambda \end{pmatrix}$

$$(4-\lambda)\big((-9-\lambda)(-13-\lambda) - (-4)(-13)\big)$$

$$-8\big((-2)(-13-\lambda) - (-4)(-1)\big)$$

$$+(-2)\big((-2)(-13) - (-9-\lambda)(-1)\big) =$$

$$= (4-\lambda)(117 + 9\lambda + 13\lambda + \lambda^2 - 52)$$

$$\cancel{16}\ 4$$

$$-8(26 + 2\lambda - \cancel{9 + \lambda})$$

$$+(-2)(26 - 9 \cancel{+} \lambda)$$

$$= (4-\lambda)(65 + 22\lambda + \lambda^2) - 8(22 + 2\lambda)$$

$$-2(17 - 9)$$

$$= \cancel{-\lambda^3 - 18\lambda^2 + 23\lambda + 260}$$

minor 3 $= -\lambda^3 - 18\lambda^2 + 9\lambda + 50$

Minor 4 = $\begin{pmatrix} 4-\lambda & 8 & -1 \\ -2 & -9-\lambda & -2 \\ -1 & -13 & -14 \end{pmatrix}$

$= (4-\lambda)\left((-9-\lambda)(-14) - (-2)(-13)\right)$

$(-8)\left((-2)(-14) - (-2)(-1)\right)$

$+ (-1)\left((-2)(-13) - (-9-\lambda)(-1)\right)$

$= (4-\lambda)(126 + 14\lambda - 26) + (-208)$

$-1(26 - 9 + \lambda)$

$= (4-\lambda)(100 + 14\lambda) - 208 - 17 + \lambda$

$= (4-\lambda)(100 + 14\lambda) - 225 + \lambda$

Minor 4 = $-14\lambda^2 - 43\lambda + 175$

$$\det(A-\lambda I) = -10(-2\lambda^2+40\lambda-150)$$
$$+(5-\lambda)(-\lambda^3-18\lambda^2+9\lambda+50)$$
$$+10(-14\lambda^2-43\lambda+175)$$

$$= 20\lambda^2-400\lambda+1500+140\lambda^2$$
$$-430\lambda+1750-5\lambda^3-90\lambda^2+45\lambda$$
$$+250+\lambda^4+18\lambda^3+9\lambda^2+50\lambda$$

$$= \lambda^4+13\lambda^3-219\lambda^2-835+3500$$
$$=0$$

If $a_1 = 13$, $a_2 = -219$, $a^3 = -835$, $a_4 = 3500$

it becomes $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$

Cubic resowent equation

$$y^3 - a_2y^2 + (a_1a_3 - 4a_4)y + (4a_2a_4 - (a_3)^2$$
$$-a_1^2a_4) = 0$$

$$y^3 - (-219)y^2 + [(13)(-835) - 4(3500)]y$$
$$+ \{4(-219)(3500) - (-835)^2 - (13)^2$$
$$(3500)\} = 0$$

$$y^3 + 219y^2 + (-10,855 - 14,000)y +$$
$$(-3,066,000 - 697,225 - 591,500) = 0$$

$$y^3 + 219y^2 - 24,855y - 4,354,725 = 0$$

To find the cubic equation root, we did some testing to fing y which is the root

$$y_1 = 147.948$$

to find the quartic root we use

$$Z^2 + \tfrac{1}{2}\left(a_1 \pm \sqrt{(a_1)^2 - 4a_2 + 4y}\right)Z +$$
$$\tfrac{1}{2}\left(y_1 \pm \sqrt{(y_1)^2 - 4(a_4)}\right) = 0$$

$$z^2 + \frac{1}{2}\left(13 \pm \sqrt{(13)^2 - (4(-219)) + 4(147.948)}\right) z$$

$$+ \frac{1}{2}\left(147.948 \pm \sqrt{(147.948)^2 - 4(3500)}\right)$$

$$= 0$$

$$z^2 + \frac{1}{2}(13 \pm 40.45) z + \frac{1}{2}(147.948 \pm 88.8) = 0$$

We have 2 equations

$$z^2 + \frac{1}{2}(13 + 40.45) z + \frac{1}{2}(147.948 + 88.86) = 0$$

$$z^2 + \frac{1}{2}(13 - 40.45) z + \frac{1}{2}(147.948 - 88.86) = 0$$

$$z^2 + 26.73 z + 118.40$$

$$z^2 - 13.73 z + 29.54$$

using $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

For equation 1

$$Z = \frac{-26.73 \pm \sqrt{(26.73)^2 - 4(1)(118.40)}}{2}$$

$$Z = \frac{-26.73 \pm \sqrt{240.83}}{2}$$

$$= \frac{26.73 \pm 15.51}{2}$$

$$= \frac{-26.73 + 15.51}{2} \text{ or } \frac{-26.73 - 15.51}{2}$$

$$= -5.61 \text{ or } -21.12$$

For equation 2: $z^2 - 13.73z + 29.54$

$$Z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$Z = \frac{-(13.73) \pm \sqrt{(-13.73)^2 - 4(1)(29.54)}}{2}$$

$$Z = \frac{13.73 \pm \sqrt{70.43}}{2}$$

$$z = \frac{13.73 + 8.39}{2} \quad \text{or} \quad \frac{13.73 - 8.39}{2}$$

$$z = 11.06 \text{ or } 2.67$$

so $\lambda_1 = -21.12$

$\lambda_2 = -5.61$

$\lambda_3 = 2.67$

$\lambda_4 = 11.06$

Finding Eigen vector $\lambda_1 = -21.12$

$$A = \begin{bmatrix} 4 & 8 & -1 & -2 \\ -2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{bmatrix}, \lambda = -21.12$$

To find the eigen vectors

$$(A - \lambda I)v = 0$$

$\lambda = -21.12$, we compute $A - (-21.12)I = A + 21.12I$

$$A + 21.12I = \begin{pmatrix} 4+21.12 & 8 & -1 & -2 \\ -2 & -9+21.12 & -2 & -4 \\ 0 & 10 & 5+21.12 & -10 \\ -1 & -13 & -14 & -13+ \end{pmatrix}$$

$$= \begin{pmatrix} 25.12 & 8 & -1 & -2 \\ -2 & 12.12 & -2 & -4 \\ 0 & 10 & 26.12 & -10 \\ -1 & -13 & -14 & 8.12 \end{pmatrix}$$

$$(A - \lambda I)v = 0$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{pmatrix} 25.12 & 8 & -1 & -2 \\ -2 & 12.12 & -2 & -4 \\ 0 & 10 & 26.12 & -10 \\ -1 & -13 & -14 & 8.12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_1 \div 25.12 \quad (1, 0.318, -0.0398, -0.0796)$

$R_2 = R_2 + 2R_1 \quad (0, 12.756, -2.0796, -4.15$

$R_4 = R_4 + R_1 \quad (0, -12.682, -14.0398, 8.0404)$

$$\begin{pmatrix} 1 & 0 & 0 & -0.198 \\ 0 & 1 & 0 & -0.366 \\ 0 & 0 & 1 & -0.243 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 - 0.198x_4 = 0 \Rightarrow x_1 = 0.198x_4$$

$$x_2 - 0.366x_4 = 0 \Rightarrow x_2 = 0.366x_4$$

$$x_3 - 0.243x_4 = 0 \Rightarrow x_3 = 0.243x_4$$

x_4 = free parameter

$$v = t\begin{bmatrix} 0.198 \\ 0.366 \\ 0.243 \\ 1 \end{bmatrix}$$

let's make $t = 5$

$$v \simeq \begin{pmatrix} 1 \\ 1.85 \\ 1.23 \\ 5.05 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix}$$

To find the importance of Eigen value

$\lambda_1 = -21.12 \qquad |\lambda_1| = 21.12$

$\lambda_2 = -5.61 \qquad |\lambda_2| = 5.61$

$\lambda_3 = 2.67 \qquad |\lambda_3| = 2.67$

$\lambda_4 = 11.06 \qquad |\lambda_4| = 11.06$

$\Sigma = 40.46$

$$\% \lambda_1 = \frac{21.12}{40.46} \times 100\% \approx 52.1\%$$

$$\% \lambda_2 = \frac{5.61}{40.46} \times 100\% = 13.86\%$$

$$\% \lambda_3 = \frac{2.67}{40.46} \times 100\% = 5.58\%$$

$$\% \lambda_4 = \frac{11.06}{40.46} \times 100\% = 27.33\%$$

$$A = \begin{bmatrix} 4 & 8 & -1 & -2 \\ 2 & -9 & -2 & -4 \\ 0 & 10 & 5 & -10 \\ -1 & -13 & -14 & -13 \end{bmatrix}$$

$\lambda_2 = -5.604$

$$* \; A - \lambda I = \begin{bmatrix} 4-(-5.604) & 8 & -1 & -2 \\ 2 & -9-(-5.604) & -2 & -4 \\ 0 & 10 & 5-(-5.604) & -10 \\ -1 & -13 & -14 & -13-(-5.604) \end{bmatrix}$$

$$= \begin{bmatrix} 9.604 & 8 & -1 & -2 \\ 2 & -3.396 & -2 & -4 \\ 0 & 10 & 10.604 & -10 \\ -1 & -13 & -14 & -7.396 \end{bmatrix}$$

$$* \; (A - \lambda I) v = 0$$

$$= \begin{bmatrix} 9.604 & 8 & -1 & -2 \\ 2 & -3.396 & -2 & -4 \\ 0 & 10 & 10.604 & -10 \\ -1 & -13 & -14 & -7.396 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = 0 \Rightarrow \begin{aligned} &9.604V_1 + 8V_2 - V_3 - 2V_4 = 0 \\ &2V_1 - 3.396V_2 - 2V_3 - 4V_4 = 0 \\ &10V_2 + 10.604V_3 - 10V_4 = 0 \\ &-V_1 - 13V_2 - 14V_3 - 7.396V_4 = 0 \end{aligned}$$

$$\frac{1}{9.604} \Rightarrow \left[\begin{array}{cccc|c} 9.604 & 8 & -1 & -2 & 0 \\ 2 & -3.396 & -2 & -4 & 0 \\ 0 & 10 & 10.604 & -10 & 0 \\ -1 & -13 & -14 & -7.396 & 0 \end{array}\right] \xrightarrow{-5.062} \left[\begin{array}{cccc|c} 1 & 0.833 & -0.104 & -0.208 & 0 \\ 0 & -5.062 & -1.792 & -3.584 & 0 \\ 0 & 10 & 10.604 & -10 & 0 \\ 0 & -12.167 & -14.104 & -7.604 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -0.399 & -0.798 & 0 \\ 0 & 1 & 0.354 & 0.708 & 0 \\ 0 & 0 & 7.064 & -16.08 & 0 \\ 0 & 0 & -9.804 & 0.912 & 0 \end{array}\right] 7.064 \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1.706 & 0 \\ 0 & 1 & 0 & 1.514 & 0 \\ 0 & 0 & 1 & -2.276 & 0 \\ 0 & 0 & 0 & -21.4 & 0 \end{array}\right] -21.4$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$V_1 = V_2 = V_3 = V_3 = 0$$

Thus $\lambda_2 = -5.604$ leads to a zero eigenvector, it may not be an exact eigenvalue.