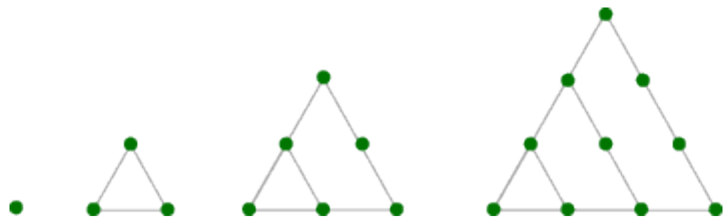
CSC236H: Introduction to the Theory of Computation

Homework 3 Solutions

1. Early members of the Pythagorean Society defined figurate numbers to be the number of dots in certain geometrical configurations.

   (a) The first four triangular numbers are 1, 3, 6, and 10.



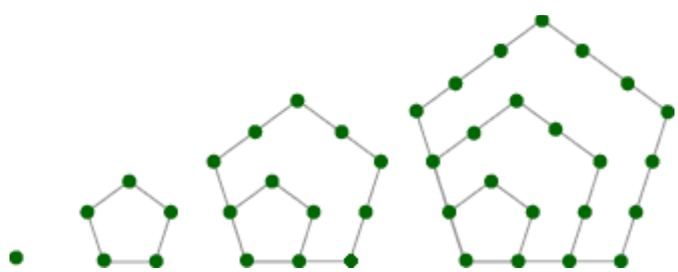
Find a recursive expression for the nth triangular number. Guess (you may use substitution, but note that the answer is very easy, and you have seen it before) the closed form for this function and prove your answer to be correct.

Solution. Let $T(n)$ represent the nth triangular number. By examining the figure, it is easy to observe that each figure is obtained from the previous one by adding an extra row of dots. The number of dots in the new row is exactly the side length of the resulting triangle. Thus we get the recurrence relation:

$$\begin{aligned}&T(1) = 1,\\ &T(n) = T(n-1) + n \qquad n \geq 2.\end{aligned}$$

If we expand this, we get $T(n) = 1 + \cdots + n$. We know from the course notes that the solution to this is $T(n) = n(n+1)/2$ (see the notes for the inductive proof).

   (a) The first four pentagonal numbers are 1, 5, 12, and 22.



Find a recursive expression for the nth pentagonal number. Guess the closed form for this function and prove your answer to be correct.

Solution. Let $P(n)$ represent the nth pentagonal number. Clearly $P(1) = 1$ Examining the figure, we observe that to go from the pentagon of side length i to the next (of side length $i+1$) we add three sets of $i+1$ dots each to three of the sides of the previous pentagon. However, two of the dots are each shared between two out of the three sets of $i+1$ dots that are added. Therefore, the total number of dots added is $3(i+1) - 2 = 3i + 1$. Taking $n = i + 1$ to be the side length of the new pentagon, the total number of added dots will be $3(n-1) + 1 = 3n - 2$.

$$\begin{aligned}&P(1) = 1,\\ &P(n) = P(n-1) + 3n - 2 \qquad n \geq 2.\end{aligned}$$

To guess the answer for this recurrence we use expansion:

$$\begin{aligned}P(n) &= P(n-1) + 3n - 2\\ &= P(n-2) + 3(n-1) - 2 + 3n - 2\\ &= P(n-3) + 3(n-2) - 2 + 3(n-1) - 2 + 3n - 2\\ &= \ldots\\ &= P(1) + 3(2) - 2 + 3(3) - 2 + \cdots + 3(n-1) - 2 + 3n - 2\\ &= 1 + 3(2 + 3 + \cdots + n) - 2(n-1).\end{aligned}$$

Now, recalling that $1+\cdots+n = n(n+1)/2$ we get $P(n) = 1+3(n(n+1)/2-1)-2(n-1) = n(3n-1)/2$.
We now prove this formally by induction on n:
Base case: for $n = 1$, we have $P(n) = 1(3-1)/2 = 1$, as expected.
Inductive hypothesis: we assume that $P(n-1) = (n-1)(3(n-1)-1)/2$.
Induction Step: Using the derived recurrence relation, since $n \geq 2$, we have:

$$P(n) = P(n-1) + 3n - 2 = \frac{(n-1)(3(n-1)-1)}{2} + 3n - 2.$$

Simplifying the right-hand-side algebraically results $n(3n-1)/2$ as desired.

2. Define the function f as follows:

- $f(1) = 1$
- $f(2) = 5$
- $f(n+1) = 5f(n) - 6f(n-1)$

(a) Compute $f(3)$ and $f(4)$. **Solution.**

$$\begin{aligned} f(3) &= 5f(2) - 6f(1) \\ &= 5 \times 5 - 6 \times 1 \\ &= 19 \end{aligned}$$

$$\begin{aligned} f(4) &= 5f(3) - 6f(2) \\ &= 5 \times 19 - 6 \times 5 \\ &= 65 \end{aligned}$$

(b) Use strong induction to prove that $f(n) = 3^n - 2^n$ for every positive integer n. **Solution.** Let $P(n)$ be the proposition that $f(n) = 3^n - 2^n$ for every positive integer n. We will prove it by strong induction over n.
Base: The base cases are when $n = 1$ and $n = 2$. When $n = 1$, we have $3^1 - 2^1 = 3 - 2 = 1$ and when $n = 2$, we have $3^2 - 2^2 = 9 - 4 = 5$ which match the given values of $f(1)$ and $f(2)$.
Inductive hypothesis: We assume that $P(j)$ holds for $1 \leq j \leq k$.
Inductive Step: Assuming the above inductive hypothesis, we prove that $P(k+1)$ holds. We have:

$$\begin{aligned} f(k+1) &= 5f(k) - 6f(k-1) \\ &= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \\ &= 5 \times 3^k - 5 \times 2^k - 2 \times 3 \times 3^{k-1} + 2 \times 3 \times 2^{k-1} \\ &= 5 \times 3^k - 5 \times 2^k - 2 \times 3^k + 3 \times 2^{k-1} \\ &= 3 \times 3^k - 2 \times 2^k \\ &= 3^{k+1} - 2^{k+1}. \end{aligned}$$

Thus we have shown that $P(k+1)$ holds. This completes the induction implying that $P(n)$ holds for all $n \geq 1$.

3. Define a set $M \subseteq \mathbb{Z}^2$ as follows

(1) $(3, 2) \in M$
(2) If $(x, y) \in M$, then $(3x - 2y, x) \in M$

Use structural induction to prove that elements of M always have the form $(2^{k+1} + 1, 2^k + 1)$, where k is a natural number. (The point of this problem is to learn how to use structural induction, so you may not rephrase this into a normal proof by induction on k.)
Solution. Base: $3 = 2^{0+1} + 1$ and $2 = 2^0 + 1$. So, The relationship holds for $(3, 2)$, which is the only base element of the set.
Inductive Hypothesis: Assume that for some $(x, y) \in M$, $x = 2^{k+1} + 1$, and $y = 2^k + 1$.
Inductive Step: We must show that the property holds for $(3x - 2y, x)$ (which is the only recursive rule that builds new objects for the set). In other words, we must show that $3x - 2y = 2^{m+1} + 1$, and $x = 2^m + 1$ for some integer m.

Based on the induction hypothesis we have:

$$3x - 2y = 3(2^{k+1} + 1) - 2(2^k + 1) = 3 \times 2^{k+1} + 3 - 2^{k+1} - 2 = 2 \times 2^{k+1} + 1 = 2^{k+1} + 1.$$

Now if we choose m to be $k+1$, we have $3x - 2y = 2^{k+2} + 1 = 2^{m+1} + 1$ and $x = 2^{k+1} + 1 = 2^m + 1$. So, we have proved the claim.

4. The Fibonacci trees T_k are a special sort of binary trees that are defined as follows.

   Base: T_1 and T_2 are binary trees with only a single vertex.

   Induction: For any $n \geq 3$, T_n consists of a root node with T_{n-1} as its left subtree and T_{n-2} as its right subtree.

   Use structural induction to prove that the height of T_n is $n-2$, for any $n \geq 2$. (Again, use structural induction rather than looking for an explicit induction variable n.)

   Solution. Proof by structural induction.
   Base: We have to show the base case for the base elements of the set. These will not be T_1 and T_2 since the condition of the problem states $n \geq 3$. T_3 is a Fibonacci tree with a three nodes and two edges and therefore has height 1. In this case $n-2$ is also equal to 1, as expected. T_4 is a Fibonacci tree with a five nodes and four edges and has height 2. In this case $n-2$ is also equal to 2, as expected. You need to discuss the basis for both cases since the inductive definition always refers to two smaller size trees.
   Induction Hypothesis: Suppose that the claim is true for smaller trees (that is T_{n-1} and T_{n-2}). We will show that it will then be true for bigger trees constructed using the recursive definition. Induction Step: We need to show that the claim is also true for T_n. T_n consists of a root node with children T_{n-1} and T_{n-2}. By the inductive hypothesis, the claim holds for T_{n-1} and T_{n-2}, so we know that the height of T_{n-1} is $n-3$ and the height of T_{n-2} is $n-4$ (note that $n \geq 5$ since we already covered the cases of 3 and 4 as the basis). Now, we know that the height of any binary tree is one more than the maximum height of any of its children. So we have

   $$\text{height}(T_n) = \max(\text{height}(T_{n-1}), \text{height}(T_{n-2})) + 1.$$

   Substituting the heights of T_{n-1} and T_{n-2} we find that the height$(T_n) = (n-3) + 1 = n-2$, and this proves the claim.