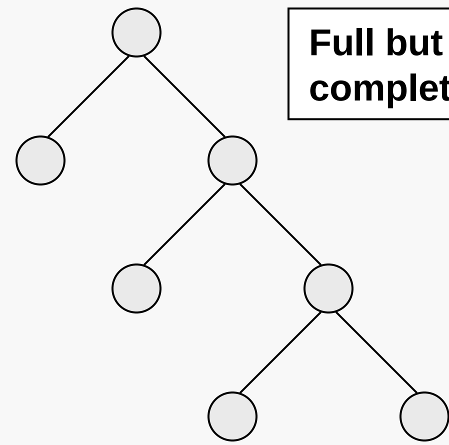
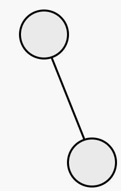
Full and Complete Binary Trees

Here are two important types of binary trees. Note that the definitions, while similar, are logically independent.

<u>Definition</u>: a binary tree T is *full* if each node is either a leaf or possesses exactly two child nodes.

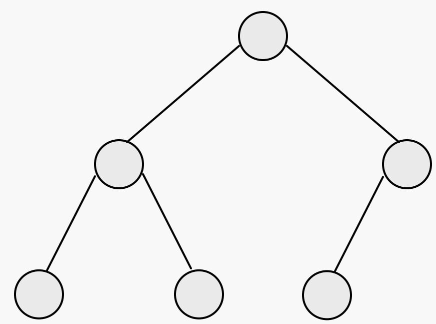


Full but not complete.

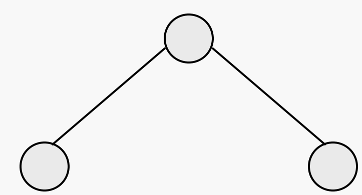


Neither complete nor full.

<u>Definition</u>: a binary tree T with n levels is *complete* if all levels except possibly the last are completely full, and the last level has all its nodes to the left side.



Complete but not full.



Full and complete.

Full Binary Tree Theorem

Theorem: Let T be a nonempty, full binary tree Then:

(a) If T has I internal nodes, the number of leaves is $L = I + 1$.

(b) If T has I internal nodes, the total number of nodes is $N = 2I + 1$.

(c) If T has a total of N nodes, the number of internal nodes is $I = (N - 1)/2$.

(d) If T has a total of N nodes, the number of leaves is $L = (N + 1)/2$.

(e) If T has L leaves, the total number of nodes is $N = 2L - 1$.

(f) If T has L leaves, the number of internal nodes is $I = L - 1$.

Basically, this theorem says that the number of nodes N, the number of leaves L, and the number of internal nodes I are related in such a way that if you know any one of them, you can determine the other two.

Proof of Full Binary Tree Theorem

proof of (a): We will use induction on the number of internal nodes, I. Let S be the set of all integers $I \geq 0$ such that if T is a full binary tree with I internal nodes then T has $I + 1$ leaf nodes.

For the base case, if $I = 0$ then the tree must consist only of a root node, having no children because the tree is full. Hence there is 1 leaf node, and so $0 \in S$.

Now suppose that for some integer $K \geq 0$, every I from 0 through K is in S. That is, if T is a nonempty binary tree with I internal nodes, where $0 \leq I \leq K$, then T has $I + 1$ leaf nodes.

Let T be a full binary tree with $K + 1$ internal nodes. Then the root of T has two subtrees L and R; suppose L and R have I_L and I_R internal nodes, respectively. Note that neither L nor R can be empty, and that every internal node in L and R must have been an internal node in T, and T had one additional internal node (the root), and so $K + 1 = I_L + I_R + 1$.

Now, by the induction hypothesis, L must have I_L+1 leaves and R must have I_R+1 leaves. Since every leaf in T must also be a leaf in either L or R, T must have $I_L + I_R + 2$ leaves.

Therefore, doing a tiny amount of algebra, T must have $K + 2$ leaf nodes and so $K + 1 \in S$. Hence by Mathematical Induction, $S = [0, \infty)$.

QED

Limit on the Number of Leaves

<u>Theorem</u>: Let T be a binary tree with λ levels. Then the number of leaves is at most $2^{\lambda-1}$.

<u>proof</u>: We will use strong induction on the number of levels, λ. Let S be the set of all integers $\lambda \geq 1$ such that if T is a binary tree with λ levels then T has at most $2^{\lambda-1}$ leaf nodes.

For the base case, if $\lambda = 1$ then the tree must have one node (the root) and it must have no child nodes. Hence there is 1 leaf node (which is $2^{\lambda-1}$ if $\lambda = 1$), and so $1 \in S$.

Now suppose that for some integer $K \geq 1$, all the integers 1 through K are in S. That is, whenever a binary tree has M levels with $M \leq K$, it has at most 2^{M-1} leaf nodes.

Let T be a binary tree with K + 1 levels. If T has the maximum number of leaves, T consists of a root node and two nonempty subtrees, say S_1 and S_2. Let S_1 and S_2 have M_1 and M_2 levels, respectively. Since M_1 and M_2 are between 1 and K, each is in S by the inductive assumption. Hence, the number of leaf nodes in S_1 and S_2 are at most 2^{K-1} and 2^{K-1}, respectively. Since all the leaves of T must be leaves of S_1 or of S_2, the number of leaves in T is at most $2^{K-1} + 2^{K-1}$ which is 2^K. Therefore, K + 1 is in S.

Hence by Mathematical Induction, $S = [1, \infty)$.

QED

More Useful Facts

Theorem: Let T be a binary tree. For every $k \geq 0$, there are no more than 2^k nodes in level k.

Theorem: Let T be a binary tree with λ levels. Then T has no more than $2^\lambda - 1$ nodes.

Theorem: Let T be a binary tree with N nodes. Then the number of levels is at least $\lceil \log (N + 1) \rceil$.

Theorem: Let T be a binary tree with L leaves. Then the number of levels is at least $\lceil \log L \rceil + 1$.